MATHEMATICS FOR COMPUTING

WEEK 5:1

LOGIC ENGINEERING IN CPL

- To build logic we must define
  - Syntax
  - Semantics
  - Provide the proof theory
  - Establish the correctness of the construction

SYNTAX

- Alphabet of Logic
  - A set of atomic propositions: p, q, r, p1, q1, r1,….
  - A set of logical operators: ¬, ∧,⇒, ∨
  - Technical symbols: ()

WELL FORMED FORMULAE

- **Definition 1**
  - Any atomic proposition is wff
  - If A and B are wff then $\neg A$, $A \wedge B$, $A \vee B$, $A \Rightarrow B$ are wff
  - Nothing else is a wff

- A proposition should be a wff to enable reasoning properly.

SATISFIABILITY AND VALIDITY

- **Definition 2**
  - Formula A is satisfiable if there is an interpretation of its atomic propositions which makes A true.
  - Alternatively, if no interpretation of atomic propositions makes A true then A is unsatisfiable.
- **Definition 3**
  - Formula A is valid if every interpretation of its atomic propositions makes it true.
  - $\models A$ notation is used to denote the fact that A is valid.

LOGICAL CONSEQUENCE

- **Definition 4**
  - Formula B is a logical consequence of a knowledge base $A_1, A_2, A_3, \ldots A_n$ if the following formula is valid:
    - $(A_1 \wedge (A_2 \wedge (A_3 \wedge \ldots A_n))) \Rightarrow B$
  - $A_1, A_2, A_3, \ldots A_n \models B$ is used to denote logical consequence of B

PROOF STRATEGIES

AXIOMATIC APPROACH

- An axiom is a proposition formally accepted without demonstration, proof, or evidence as one of the starting-points for the systematic derivation of an organized body of knowledge.
- From all valid formulae we choose a set of formulae and assume they do not need proofs.
- Next, we formulate the proof technique such that all other valid formulae can be proven from axioms using corresponding rules.

AXIOMS IN CPL

- Definition 6 [Logic CPLAx-axiomatic formulation of CPL]

  Ax.1 $(p \Rightarrow (q \Rightarrow p))$

  Ax.2 $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

  Ax.3 $(\neg p \Rightarrow \neg q) \Rightarrow (q \Rightarrow p)$

RULES OF INFERENCE

- **Substitution**: Let A be a formula of CPL and p a propositional variable in A. Let $A(p/B)$ be a result of substituting all occurrences of p in A by a formula B.
- Then the following rule can be carried out:

If A then $A(p/B)$

$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$	Ax.2
$(s \Rightarrow (q \Rightarrow r)) \Rightarrow ((s \Rightarrow q) \Rightarrow (s \Rightarrow r))$	from 1, Substitution p/s,

RULES OF INFERENCE

- **Modus Ponens (Implication Elimination)**

From A and A ⇒ B follows B

- It can be summarized as "P implies Q and P is asserted to be true, therefore Q must be true.“

SEARCHING FOR PROOFS

Breadth-First Search

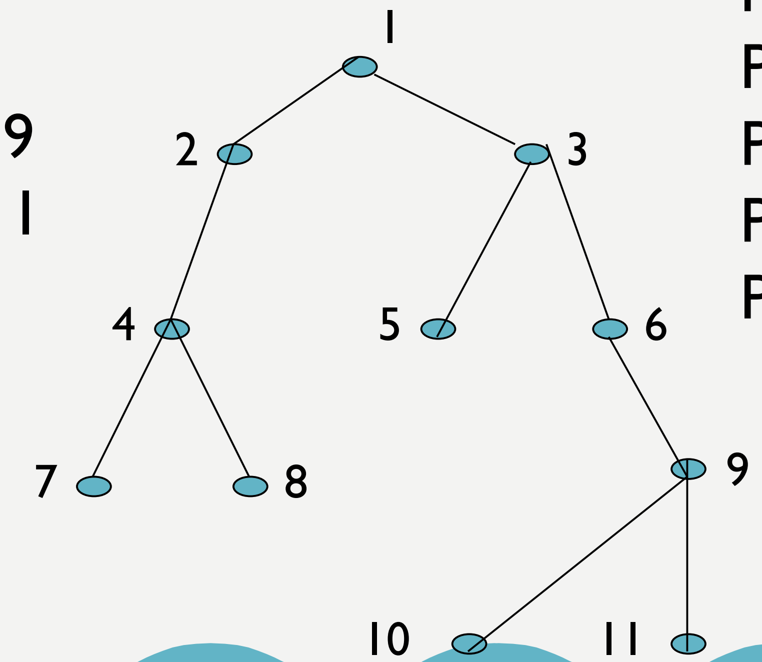
Here we explore the tree in order of levels

Level 1 – nodes 2 and 3

Level 2 – nodes 4, 5, 6

Level 3 – nodes 7, 8 and 9

Level 4 – nodes 10 and 11



Depth-First Search

Here we explore the tree in order of paths

Path 1 – nodes 1, 2, 4, 7

Path 2 – nodes 1, 2, 4, 8

Path 3 – nodes 1, 3, 5

Path 4 – nodes 1, 3, 6, 9, 10

Path 5 – nodes 1, 3, 6, 9, 11

FORWARD CHAINING

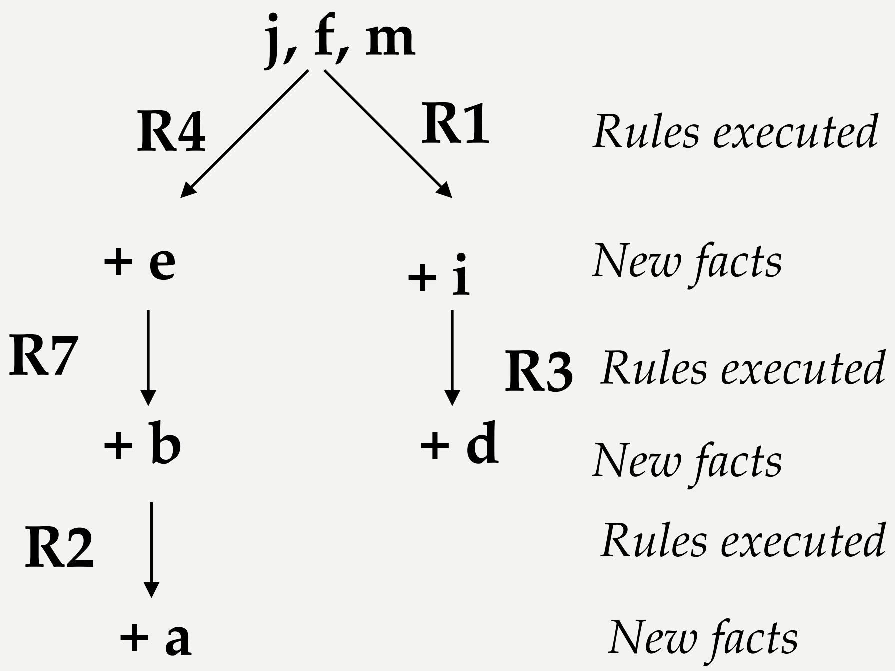
- Match the FACTS contained in the knowledge base to left hand sides of rules
- Apply the rules and instantiate new (temporary) facts
- Repeat until: the goal is achieved or until no more rules fire
- Operates as a **breadth-first search**

FORWARD CHAINING

KNOWLEDGE BASE

Rules	Initial facts
R1: m⇒i	j, f, m
R2: b⇒a	
R3: i⇒d	
R4: j⇒e	
R5: k⇒e	
R6: g&h⇒c	
R7: e&f⇒b	
R8: c&d⇒a	
R9: l⇒i	

State-space Representation



BACKWARD CHAINING

- Match the GOAL to right hand side of rules
- Set up left side of rule as sub-goals
- Repeat until: all sub-goals match directly with known facts (goal is true) or some sub-goals fail to match (goal is false)
- Backward chaining is usually implemented as **depth-first search**

BACKWARD CHAINING

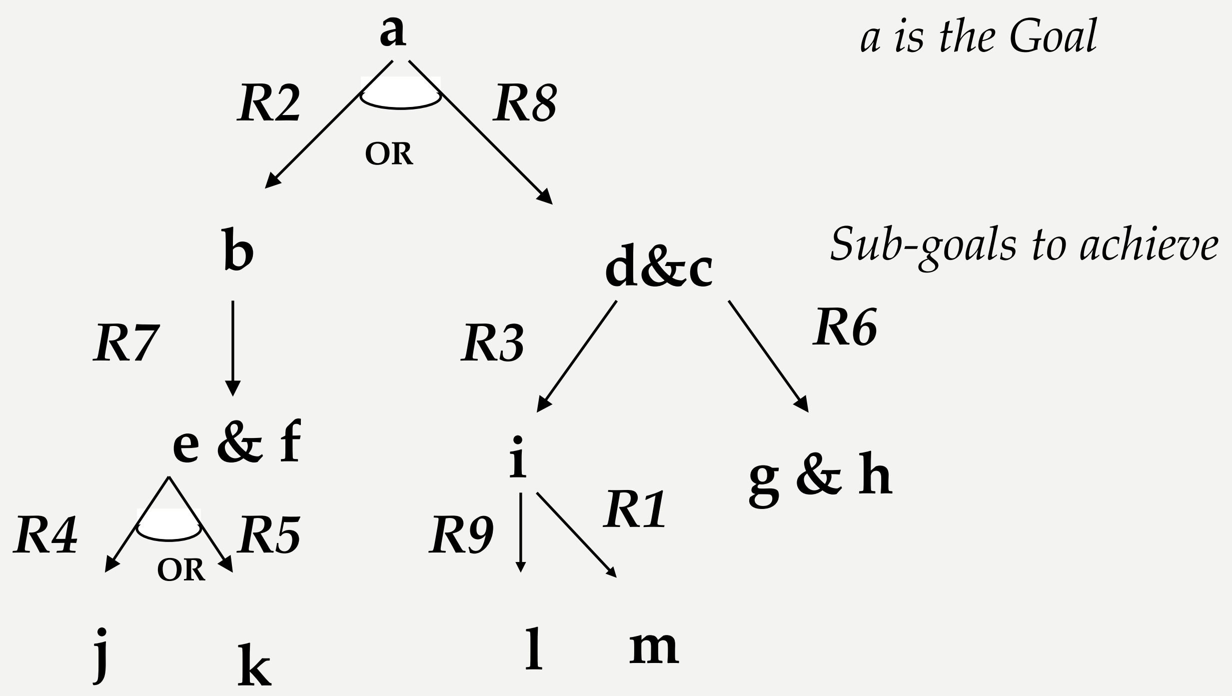
Rules

R1: m⇒i
R2: b⇒a
R3: i⇒d
R4: j⇒e
R5: k⇒e
R6: g&h⇒c
R7: e&f ⇒b
R8: c&d⇒a
R9: l⇒i

Initial facts

j, f, m

Problem-space Representation



PROVING VALIDITY BY CONTRADICTION

Consider Axiom 1. $p \Rightarrow (q \Rightarrow p)$

Goal: proof that Axiom 1 is valid but not by constructing the full truth table

1. Assume the contrary - $p \Rightarrow (q \Rightarrow p)$ is not valid.
2. It should be the case then that p is true but $q \Rightarrow p$ is false
3. Since $q \Rightarrow p$ is false q should be true but p false
4. Contradiction:
5. Therefore our assumption that $p \Rightarrow (q \Rightarrow p)$ is not valid is wrong!
6. Therefore, $p \Rightarrow (q \Rightarrow p)$ is valid

QUESTIONS?

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