MATHEMATICS FOR COMPUTING

WEEK 8

WHY DO WE NEED AN INVERSE?

Exploration Consider the set of real numbers, and say that we have the equation

$$3x = 2$$

and we want to solve for x. What do we do?

We multiply both sides of the equation by $\frac{1}{3}$ to obtain

$$\frac{1}{3}(3x) = \frac{1}{3}(2) \implies x = \frac{2}{3}.$$

multiplicative inverse of 3 since $\frac{1}{3}(3) = 1$

Now, consider the linear system

$$\begin{aligned} 3x_1 - 5x_2 &= 6 \\ -2x_1 + 3x_2 &= -1 \end{aligned}$$

Notice that we can rewrite equations as

$$\underbrace{\begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ -1 \end{bmatrix}}_{\vec{b}}$$

WHY DO WE NEED AN INVERSE?

$$\underbrace{\begin{bmatrix} \quad ? \quad \end{bmatrix}\left(\begin{bmatrix}3 & -5\\-2 & 3\end{bmatrix}}_{\text{want this equal to identity matrix, } I}\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix} \quad ? \quad \end{bmatrix}\begin{bmatrix}6\\-1\end{bmatrix}$$

$$\begin{bmatrix}-3 & -5\\-2 & -3\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-3 & -5\\-2 & -3\end{bmatrix}\begin{bmatrix}6\\-1\end{bmatrix} = \boxed{\begin{bmatrix}-13\\-9\end{bmatrix}}$$

Given that

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

BRAIN FOOD

Find AB

Find BA

Compare the answers

DETERMINANT OF 2*2 MATRICES

- For a matrix A, its determinant denoted as $|A|$ or $\det(A)$
- Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ then

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = +ad - bc$$

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DETERMINANT OF 2*2 MATRICES

1) $A = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix}$ then $|A| = \begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 5 = -6$

2) $A = \begin{vmatrix} 6 & 5 \\ 2 & 4 \end{vmatrix}$ $|A| = \begin{vmatrix} 6 & 5 \\ 2 & 4 \end{vmatrix} = 6 \times 4 - 2 \times 5 = 14$

3) $A = \begin{vmatrix} 6 & 12 \\ 2 & 4 \end{vmatrix}$ $|A| = \begin{vmatrix} 6 & 12 \\ 2 & 4 \end{vmatrix} = 6 \times 4 - 2 \times 12 = 0$

PROPERTIES

Following properties are true for determinants of <u>*any*</u> order.

1. If every element of a row (column) is zero, then $|A| = 0$.

2. $|A^T| = |A|$

3. $|AB| = |A||B|$

BRAIN FOOD

Given that $A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix}$

Find

$i)\ |A|$

$ii)\ |B|$

$iii)\ |A^T B|$

$iv)\ |BA|$

$v)\ |AB|\, A^T$

INVERSE OF A 2 X 2 MATRIX

- Given a matrix $A = \begin{bmatrix} a & b \\ C & d \end{bmatrix}$ then det(A) can be found as:

$$\det(\mathbf{A}) = ad - bc$$

- If det(A) != 0, then A is invertible.

$$\boxed{\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

TEST YOUR KNOWLEDGE

Find the inverse of the matrix, $\mathrm{B} = \begin{bmatrix} -3 & 6 \\ 2 & -5 \end{bmatrix}$.

Find the inverse of the matrix, $\mathrm{B} = \begin{bmatrix} -4 & -8 \\ -5 & -5 \end{bmatrix}$.

DETERMINANT OF A 3 X 3 MATRIX USING THE DIAGONAL METHOD

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \left[\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right] \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$\Rightarrow |\mathrm{A}| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

BRAIN FOOD

Find the determinants

$$\begin{pmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & -1 \\ 2 & 0 & 7 \end{pmatrix}$$

DETERMINANT OF 3RD ORDER MATRICES – ROW/COLUMN EXPANSION METHOD

1. Select a row or a column for the expansion
2. Find Minors for every element of the selected row(column)
3. Find cofactors and then the determinant
4. Used to find 3x3 or higher order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

1. Select a row or a column for the expansion

1st row $\longrightarrow$ $a_1 \quad b_1 \quad c_1$

2. Find Minors for every element of the selected row(column)

Minor of $a_1 = M_{a_1} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ $\longleftarrow$ $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$M_{b_1} = ?$$

$$M_{c_1} = ?$$

3. Find cofactors and then the determinant

Cofactors of each element of the selected row

Generally for an element a_{ij}, its cofactor is denoted as $C_{a_{ij}}$ and

Defined as $C_{a_{ij}} = (-1)^{ij} M_{a_{ij}}$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

EXAMPLE

Expanding by 1st row

$$\begin{vmatrix} 2 & 1 & 3 \\ 6 & 0 & -2 \\ -1 & 4 & 5 \end{vmatrix} = 2\begin{vmatrix} 0 & -2 \\ 4 & 5 \end{vmatrix} - 1\begin{vmatrix} 6 & -2 \\ -1 & 5 \end{vmatrix} + 3\begin{vmatrix} 6 & 0 \\ -1 & 4 \end{vmatrix}$$

$$= 2(0-(-8)) - 1(30-2) + 3(24-0) = 16 - 28 + 72 = 60$$

Expanding by 3rd column

$$\begin{vmatrix} 2 & 1 & 3 \\ 6 & 0 & -2 \\ -1 & 4 & 5 \end{vmatrix} = 3\begin{vmatrix} 6 & 0 \\ -1 & 4 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} + 5\begin{vmatrix} 2 & 1 \\ 6 & 0 \end{vmatrix}$$

$$= 3(24-0) + 2(8-(-1)) + 5(0-6) = 72 + 18 - 30 = 60$$

COFACTOR MATRIX

Cofactor matrix of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Find the cofactor for each element of A (See the pattern of the sign of minors)

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24 \qquad A_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5 \qquad A_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12 \qquad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3 \qquad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2 \qquad A_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5 \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$$

COFACTOR MATRIX

Cofactor matrix of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$ is $\begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$

Place signs $(-1)^{i+j}$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

ADJOINT MATRIX

Adjoint matrix of the matrix A is denoted as adj (A) and defined as

$$\text{adj}(A)=(\text{Cofactor Matrix of } A)^T$$

E.g. What is the adjoint matrix of the matrix

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

BRAIN FOOD

For the given matrix find,

a) Determinant using diagonal method

b) The matrix of minors

c) The matrix of cofactors

d) The adjoint matrix

INVERSE OF A MATRIX

- You may have realised that we have not divided a matrix by a matrix – there is no useful way of doing this.
- However, in some circumstances, multiplying by an inverse matrix suffices in situations where we want to divide.
- Inverse of the matrix A denoted as A^{-1}
- If $AB = BA = I$ then $A = B^{-1}$ and $B = A^{-1}$

INVERSE OF A MATRIX

Consider two matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

then

$$AB = BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ans: Note that

Can you show the details?

INVERSE OF AN n x n MATRIX

Let A be a square matrix with $|A| \neq 0$. A is invertible with

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A)$$

A square matrix A is invertible if and only if $|A| \neq 0$.

INVERSE OF 3*3 MATRICES

Inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{|A|} \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}^T = \frac{1}{22} \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12/11 & -6/11 & -1/11 \\ 5/22 & 3/22 & -5/22 \\ -2/11 & 1/11 & 2/11 \end{bmatrix}$$

BRAIN FOOD

Find the inverse of the given matrices

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 4 & -2 \\ 1 & -3 & 5 \end{bmatrix}$$

- Consider the system of equations with two unknowns

$$2x + 5y = 1$$
$$x + 3y = 4$$

BRAIN FOOD

a) Represent the system in matrix form AX=B

b) Find the inverse of the matrix A found above

c) Solve the system for x and y using the inverse of A

QUESTIONS?

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