The Weight of Nonstrongly Complete Profinite Groups

Tamar Bar-On*

February 16, 2021

Abstract

We compute the local weight of the completion of a nonstrongly complete profinite group, and conclude that if a profinite group is abstractly isomorphic to its own profinite completion, then they are equal. The local weight of all the groups in the tower of completions is computed as well.

1 Introduction

We write $H \trianglelefteq_f G$ to say that H is a normal subgroup of finite index in G. The profinite completion of an abstract group G is the projective limit of finite quotients

$$\hat{G} = \lim_{\leftarrow H \trianglelefteq_f G} G/H,$$

endowed with the limit topology. There is a canoical homomorphism $i : G \to \hat{G}$. In fact, the profinite completion is a universal object in the following sense: For every homomorphism $\varphi : G \to H$, where H is a profinite group, there is a unique continuous homomorphism $\hat{\varphi} : \hat{G} \to H$ such that $\varphi = \hat{\varphi} \circ i$. The completion $\hat{G}$ itself is obviously profinite, namely, it is an inverse limit of directed system of finite groups.

We say that a group G is *residually finite* if $\bigcap_{H \trianglelefteq_f G} H = \{e\}$. The canonical homomorphism $i : G \to \hat{G}$ is injective if and only if G is residually finite.

Let G be a profinite group. Regarding G as an abstract group, it also has a profinite completion. Moreover, every profinite group is residualy finite, so $i : G \to \hat{G}$ is injective. We say that a profinite group G is *strongly complete* (or *rigid* in some papers, such as [3]) if G is equal to its own profinite completion, i.e, if the natural embedding $i : G \to \hat{G}$ is an isomorphism. One easily sees that the following conditions are equivalent for a profinite group G:

- G is strongly complete.
- Every subgroup of finite index in G is open.

*This research was supported by the Israel Science Foundation (grant No. 630/17).

- Every normal subgroup of finite index in G is open.
- Every homomorphism $\varphi : G \to H$ onto a finite group is continuous.

Strongly complete profinite groups, or more generally, the connection between a profinite group to its profinite completion, have been the focus of many papers, such as [4], [3], [10] and [5]. In [7] the authors proved the following useful result:

Proposition 1. *The following conditions are equivalent for a profinite group G:*

- *G is strongly complete.*
- *For every n, there are finitely many subgroups of index n.*
- *There are at most countably many subgroups of finite index.*

Theorem 2. *considered as an abstract group, a nonstrongly complete profinite group is never isomorphic to its profinite completion.*

The paper is organized as follows: In the introduction we give some background on profinite groups and profinite completion, In Chapter 2 We prove Theorem 2. In chapter 3 we present the infinite tower of profinite completions and compute the number of open subgroups in each level. The computation is based on computation of the cardinality of a certain set of ultrafilters satisfying some conditions. In chapter 4 we give a simpler proof for the abelian case.

In order to prove theorem 2, we should be familiar with an invariant of profinite groups called the *local weight*.

Definition 3. Let G be an infinite profinite group. The *local weight* of G (sometimes referred to as the *rank*), which is denoted by $\omega_0(G)$, is the cardinality of the set of all open subgroups of G.

let us recall some properties of the local weight.

Proposition 4 ([2, chapter 17])**.** *Let G be an infinite profinite group.*

1. *$\omega_0(G)$ is equal to the cardinality of the set of all open normal subgroups of G.*
2. *If A is a profinite group, and $\varphi : G \to A$ a continuous epimorphism, then $\omega_0(A) \leq \omega_0(G)$.*
3. *If $H \leq_c G$ is a closed subgroup then $\omega_0(H) \leq \omega_0(G)$. In addition, if H is open, then $\omega_0(H) = \omega_0(G)$.*

There is a strong connection between the number of finite index subgroups of G and the local weight of $\hat{G}$, which arises from the following proposition:

Proposition 5 ([6, Proposition 3.2.2])**.** *Let G be a residually finite group. One may identify G with its image in $\hat{G}$ and assume $G \subseteq \hat{G}$. Then there is a one-to-one correspondence*

$$\{O \mid O \leq_o \hat{G}\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{N \mid N \leq_f G\}$$

defined by

$$\Phi(O) = O \cap G,$$

and

$$\Psi(N) = \bar{N},$$

where $\bar{N}$ is the topological clousre of N. Moreover, for every $N \leq_f G$, $G/N \cong \hat{G}/\bar{N}$.

Corollary 6. *Let G be a profinite group. Then $\omega_0(\hat{G})$ is the cardinality of the set of all finite index subgroups in G.*

2 The weight of a profinite group

A fundemental example of nonstrongly complete profinite group is given in [6], chapter 4.

Example 7. Let T be a finite group, and I an index set. We define $G = \Pi_I T$. Obviously, G is a profinite group with the product topology. Every continuous projection to a finite quotient splits through some $\Pi_J T$ where J is a finite subset of I, therefore $\omega_0(G) = |I|$. To every nonprincipal ultrafilter $\mathcal{F}$ on I one can associate the subgroup $H_{\mathcal{F}} = \{(t_i) : \{i \in I : t_i = 1\} \in \mathcal{F}\}$. It if easily checked that for every $\mathcal{F}$, $H_{\mathcal{F}}$ is a dense subgroup which satisfies $G/H_{\mathcal{F}} \cong T$, where each $t \in T$ corresponds to its diagonal image $(t)_{i \in I}$. Moreover, by [8], section 111, there are $2^{2^{|I|}}$ ultrafilters on I. So, G has at least $2^{2^{\omega_0(G)}}$ subgroups of finite index. this is actually the accurate amount of finite index subgroups, since $|G| = |\prod_I T| = 2^|I| = 2^{\omega_0(G)}$, and the number of finite index subgroups can not exceed the number of maps from G to a finite set, which is equal to $2^{|G|}$.

We are now ready for the main theorem.

Theorem 8. *Let G be a nonstrongly complete profinite group. Then $\omega_0(\hat{G}) = 2^{2^{\omega_0(G)}}$.*

Proof. We first prove that $\omega_0(\hat{G}) \geq 2^{2^{\omega_0(G)}}$. By Proposition 1, if a profinite group has only countably many subgroups of finite index, then it is strongly complete. So, G must have uncountably many subgroups of finite index.

We can now assume that $\omega_0(G) = \mathbf{m} > \aleph_0$.

Since there are only countably many finite groups, there is some finite group K such that there are $\mathbf{m}$ distinct open subgroups $H \trianglelefteq_o G$ for which $G/H \cong K$. We choose K so that no proper quotient of K has this property. Let

$$\mathcal{M} = \{H \trianglelefteq_o G \mid G/H \cong K\}$$

and

$$M = \bigcap_{H \in \mathcal{M}} H.$$

Denote by $\ell(K)$ the length of a composition series of K. We prove the claim by induction on $\ell(K)$.

When $\ell(K) = 1$, K is a simple group. According to lemma 8.2.2 in [6],

$$G/M \cong \Pi_{i\in I} K,$$

the product of $|I|$ copies of K for some index set I. Note that

$$\omega_0(G/M) \geq \mathbf{m},$$

since the H_i/M for all $H_i \in \mathcal{M}$ are distinct open subgroups of G/M. On the other hand, by Proposition 4,

$$\omega_0(G/M) \leq \omega_0(G) = \mathbf{m},$$

so

$$\omega_0(G/M) = \mathbf{m}.$$

Now, by the fundemental Example 7,

$$\omega_0(\Pi_{i\in I} K) = |I|,$$

so $|I| = \mathbf{m}$. Thus, applying the fundemental example again, G/M has $2^{2^{\mathbf{m}}}$ subgroups of finite index. Taking their preimages, we get that G has $2^{2^{\mathbf{m}}}$ subgroups of finite index, so

$$\omega_0(\hat{G}) \geq 2^{2^{\mathbf{m}}}.$$

Assume that $\ell(K) = n > 1$ and that the chain holds for all finite groups shorter than K. Choose a simple quotient S of K. Thus, S is a simple quotient of G as well. By minimality of K there are less than $\mathbf{m}$ open normal subgroups $U \trianglelefteq_o G$ such that $G/U \cong S$. Hence, we can choose such a subgroup U that contains $\mathbf{m}$ different subgroups from $\mathcal{M} = \{H \trianglelefteq_o G | G/H \cong K\}$. This U is a profinite group, with $\omega_0(U) = \mathbf{m}$, by Proposition 4. In addition, there is some proper subgroup $K' \triangleleft K$ such that

$$\forall H \in \mathcal{M}, \quad U/H \cong K'.$$

Notice that $\ell(K') < \ell(K)$ since K' is a proper normal subgroup of K. Choose a quotient K'' of K' which is minimal with respect to the property that it can be realized as a quotient of U in $\mathbf{m}$ different ways. We get that

$$\ell(K'') \leq \ell(K') < \ell(K)$$

so we can apply the induction hypothesis to the profinite group U. Thus, U has $2^{2^{\mathbf{m}}}$ finite index subgroups, and since U has finite index in G, each one of them is of finite index in G. In conclusion,

$$\omega_0(\hat{G}) \geq 2^{2^{\mathbf{m}}}.$$

We now prove that $\omega_0(\hat{G}) \leq 2^{2^{\omega_0(G)}}$:

Denote $\omega_0(G) = \mathbf{m}$. By the explicit construction of the inverse limit, we have $G \subseteq \Pi_{U \trianglelefteq_o G} G/U$. Since G has $\mathbf{m}$ open normal subgroups, and for each subgroup the quotient is finite, we get that $|G| \leq 2^{\mathbf{m}}$. So, the number of finite index normal subgroups is equal to the number of epimorphisms from G to finite groups. Since there are countably many finite groups, we get that the number of such epimorphisms is at most $2^{|G|} \leq 2^{2^{\mathbf{m}}}$. $\square$

Corollary 9. *Let G be a nonstrongly complete profinite group. Then $G \ncong \hat{G}$.*

We also got a result about the connection between the cardinality of G and its local weight.

Proposition 10. *Let G be an infinite profinite group. Then, $|G| = 2^{\omega_0(G)}$.*

Proof. Denote $\omega_0(G) = \mathbf{m}$. We already saw in the proof of Theorem 8 that $|G| \leq 2^{\mathbf{m}}$.

For the other direction, assume first that G is not strongly complete. As we showed in the proof of Theorem 8, G admits a subgroup U that projects on $\Pi_{i \in I} H$ where H is some finite group, and I is a set of indices of cardinality $\mathbf{m}$. Thus, $|G| \geq |U| \geq |\Pi_{i \in I} H| = 2^{\mathbf{m}}$.

Now assume that G is strongly complete. By proposition 1, $\omega_0(G) = \aleph_0$. Hence, by [6], corollary 2.6.6, $G = \lim_{\leftarrow n \in \omega} G_n$ is the inverse limit of an inverse system of finite groups indexed by ω, such that all the maps in the system are proper projections. Thus, for every n and every $g \in G_n$, g admits at least 2 sources in G_{n+1}. So, $|G| \geq 2^{\aleph_0}$. $\square$

3 The tower of profinite completions

Now we would like to generalize this result for the whole tower of completions.

Let G be a nonstrongly complete profinite group. In [1] we studied the following construction. We define an ascending chain over the ordinals by setting:

$$G_0 = G,$$

- For a successor ordinal $\alpha = \beta + 1$.

$$G_\alpha = \widehat{G_\beta},$$

  with the natural map $\varphi_{\beta\alpha} : G_\beta \to G_\alpha$, and such that for every $\gamma < \beta, \varphi_{\gamma\alpha} = \varphi_{\beta\alpha} \circ \varphi_{\gamma\beta}$;

- For a limit ordinal α, the system of maps $\{\varphi_{\gamma\beta}\}_{\gamma<\beta<\alpha}$ is compatible, so we can define $H_\alpha = \lim_{\to}\{G_\beta, \varphi_{\gamma\beta}\}$ with the natural maps $\varphi'_{\beta\alpha} : G_\beta \to H_\alpha$, and then $G_\alpha = \widehat{H_\alpha}$ with the natural map $\psi_\alpha : H_\alpha \to G_\alpha$; we also define, for every $\beta < \alpha$, $\varphi_{\beta\alpha} = \psi_\alpha \circ \varphi'_{\beta\alpha}$.

We proved that the chain never terminates, and moreover, that all the maps are injective. So, we may assume that for every $\beta < \alpha$, $G_\beta < G_\alpha$. Now we would like to calculate $\omega_0(G_\alpha)$ for all α. Theorem 8 gives the computation for the successor case.

Remark 11. For every ordinal α, $\omega_0(G_{\alpha+1}) = 2^{2^{\omega_0(G_\alpha)}}$.

We are left with finding $\omega_0(G_\alpha)$ where α is a limit ordinal. Recall that by proposition 5, $\omega_0(G_\alpha)$ is the number of subgroups of finite index in H_α.

Lemma 12. *Let α be a limit ordinal. Then*

$$\omega_0(G_\alpha) \leq \Pi_{\beta<\alpha}\omega_0(G_\beta).$$

Proof. For any group A we denote by FI(A) the set of finite index normal subgroups in A. As we pointed out, if α is a limit ordinal, then $\omega_0(G_\alpha) = |FI(H_\alpha)|$. Now consider the map $\theta : FI(H_\alpha) \to \Pi_{\beta<\alpha}FI(G_\beta)$ defined by $\theta(U) = \Pi_{\beta<\alpha}(U \cap G_\beta)$. This map is obviously one-to-one, so

$$\omega_0(G_\alpha) \leq \Pi_{\beta<\alpha}|FI(G_\beta)| = \Pi_{\beta<\alpha}2^{2^{\omega_0(G_\beta)}} = \Pi_{\beta<\alpha}\omega_0(G_{\beta+1}) = \Pi_{\beta<\alpha}\omega_0(G_\beta).$$

The second equality is due to Corollary 6 and Theorem 8. $\square$

Now we are going to prove that for every nonstrongly complete profinite group,

$$\omega_0(G_\alpha) = \Pi_{\beta<\alpha}\omega_0(G_\beta).$$

We will do so in several steps.

Lemma 13. *Let G be a nonstrongly complete profinite group, and $U \leq_o G$. Then, U is nonstrongly complete too, and $\omega_0(G_\alpha) = \omega_0(U_\alpha)$ for every ordinal α.*

Proof. First We show that U is nonstrongly complete. Let $H \leq_f G$ be a finite-index subgroup which is not open. Then, $H \cap U \leq_f U$. In addition, $U \cap H$ is not open in G since otherwise $int(H) \neq \emptyset$, implies that H is open. Consequently, since U is open in G, $H \cap U$ is not open in U. In order to prove the lemma, by Proposition 4, it is enough to prove that for all α, $U_\alpha \leq_o G_\alpha$. We prove it by transfinite induction. Successor ordinal: Assume that $U_\alpha \leq_o G_\alpha$. $U_{\alpha+1} = \widehat{U_\alpha}$, $G_{\alpha+1} = \widehat{G_\alpha}$. By [6], Proposition 3.2.2, $\overline{U_\alpha} \leq_o G_{\alpha+1}$, and $[G_\alpha : U\alpha] = [G_{\alpha+1} : \overline{U_\alpha}]$. By [6] Lemma 3.2.4, $\overline{U_\alpha} = \hat{i}[U_\alpha]$ where $\hat{i}$ is the lifting of the inclusion map $i : U_\alpha \to G_\alpha$. Finally, notice that since by the induction hypothesis $U_\alpha \leq_o G_\alpha$, then every finite-index subgroup if U_α is also a finite-index subgroup of G_α, so the profinite topology of G_α induces on U_α its full profinite topology, and thus by [6] Lemma 3.6.2, $\hat{i}[U_\alpha] = \widehat{U_\alpha}$. In conclusion, $U_{\alpha+1} \leq G_{\alpha+1}$ and $[G_\alpha : U_\alpha] = [G_{\alpha+1} : U_{\alpha+1}]$.

Limit ordinal: Now assume that β is a limit ordinal such that for every $\gamma < \alpha < \beta$ $U_\alpha \leq_o G_\alpha$ and in addition $[G_\alpha : U_\alpha] = [G_\gamma : U_\gamma]$. Then $\lim_{\to\alpha<\beta}U_\alpha$ is a subgroup of $\lim_{\to\alpha<\beta}G_\alpha$ of the same index, and thus by [6], Proposition 3.2.2, $U_\beta = \widehat{\lim_{\to\alpha<\beta}}U_\alpha \leq_o \widehat{\lim_{\to\alpha<\beta}}G_\alpha = G_\beta$ of the same index. $\square$

Remark 14. Let G be a nonstrongly complete profinite group. In order to compute $\omega_0(G_\alpha)$ for a limit ordinal α, we may assume G has a quotient of the form $\prod_{\omega_0(G)} S$, for some finite simple group S.

Proof. By the proof of Theorem 8, every nonstrongly complete profinite group of $\omega_0(G) = \mathbf{m} > \aleph_0$ has an open subgroup U with a quotient of the form $\prod_{\mathbf{m}} S$ where S is some finite simple group. By Lemma 13, we can replace G by U. Notice that in order to compute $\omega_0(G_\alpha)$ where α is a limit ordinal, we may assume that the tower begins with $\hat{G}$. Thus, in the case $\omega_0(G) = \aleph_0$ we can replace G by $\hat{G}$ and get that $\omega_0(G) = \mathbf{m} > \aleph_0$. □

Proposition 15. *Let $G \to H$ be an epimorphism of nonstrongly complete profinite groups. Then, for every ordinal α, there are epimorphisms $\phi_\alpha : G_\alpha \to H_\alpha$ which are compatible $\varphi_{\alpha\beta}$.*

Proof. We will prove it by transfinite induction. Let α be an ordinal such that for each $\beta < \alpha$ the claim holds. First case: $\alpha = \gamma + 1$. then there is an epimorphism $\phi_\gamma : G\gamma \to H_\gamma$. By [6] Proposition 3.2.5, there is an epimorpism $\hat{\phi} : G_{\gamma+1} = \widehat{G_\gamma} \to \widehat{H_\gamma} = H_{\gamma+1}$ which is compatible with the natural homomorphisms from every group to its profinite completion. Second case: α is a limit ordinal. We can define ϕ_α to be the direct limit $\lim_{\to\beta<\alpha} \phi_\beta$. □

Corollary 16. *Let $G \to H$ be an epimorphism of nonstrongly complete profinite groups. Then, for every ordinal α, $\omega_0(G_\alpha) \geq \omega_0(H_\alpha)$.*

Proof. By proposition 4 it is enough to prove that H_α is a quotient of G_α. So, by Proposition 15, we are done. □

Corollary 17. *It is enough to prove the computation for nonstrongly complete profinite groups of the form $\prod_{\mathbf{m}} S$ for a finite simple group S.*

Proof. By Remark 14 we may assume G has a quotient of the form $H = \prod_{\mathbf{m}} S$, where $\mathbf{m} = \omega_0(G)$. Recall that by Example 7, $\omega_0(H) = \mathbf{m}$. Assume that we prove that for every limit ordinal α, $\omega_0(H_\alpha) = \prod_{\beta<\alpha} \omega_0 H_\beta$. We will prove by transfinite induction that in that case, for every ordinal α, $\omega_0(G_\alpha) = \omega_0(H_\alpha)$.

$\alpha = 0$: $\omega_0(H) = \mathbf{m} = \omega_0(G)$.

$\alpha = \beta + 1$: $\omega_0(H_{\beta+1}) = 2^{2^{\omega_0(H_\beta)}} = 2^{2^{\omega_0(G_\beta)}} = \omega_0(G_{\beta+1})$.

α is a limit ordinal: By Proposition 15, $\omega_o(G\alpha) \geq \omega_0(H_\alpha)$. But by the assumption, $\omega_0(H_\alpha) = \prod \beta < \alpha \omega_0(H_\beta)$. By induction hypothesis, $\omega_0(H_\beta) = \omega_0(G_\beta)$ for all $\beta < \alpha$. So,

$$\omega_0(G_\alpha) \geq \omega_0(H_\alpha) = \prod_{\beta<\alpha} \omega_0(H_\beta) = \prod_{\beta<\alpha} \omega_0(G_\beta) \geq \omega_0(G_\alpha).$$

□

In order to prove our claim we need some propositions about ultrafilters and the subgroups they define.

Proposition 18 ([9], Theorem 7.6)**.** *Let X be a set of cardinality $\mathbf{m}$. Denote by $\mathcal{A}$ the set of all subsets of X for which $|X^c| < \mathbf{m}$. It has the finite intersection property and thus contained in a filter. The number of ultrafilters containing $\mathcal{A}$ equals to $2^{2^{\mathbf{m}}}$.*

Let I be a set. We denote the set of all ultrafilters on I by βI.

Lemma 19. *Let I be an infinite set of cardinality $\mathbf{m}$, and let $\mathcal{C}$ be the set of all subgroups of $\prod_{\mathbf{m}} S$, where S is a finite simple group, of the form $H_{\mathcal{F}}$ for $\mathcal{F} \in \beta I$ as defined in Example 7. Then $\mathcal{C}$ forms a subbase for a topology on G, contained in the profinite topology. Denote the completion of G with respect to this topology by $G_{\hat{\mathcal{C}}}$. Then $G_{\hat{\mathcal{C}}} \cong \prod_{\beta I} S$ is the product of $2^{2^{\mathbf{m}}}$ copies of S.*

Proof. By [8] section 111, there are $2^{2^{\mathbf{m}}}$ nonprincipal ultrafilters on a set of cardinality $\mathbf{m}$. The only thing we need to show is that for finitely many ultrafilters $\mathcal{F}_1, ..., \mathcal{F}_n$, $G/(H_{\mathcal{F}_1} \cap ... \cap H_{\mathcal{F}_n}) \cong \prod_n S$. We prove this by induction on n. Assume that the claim holds for n, and consider $G/(H_{\mathcal{F}_1} \cap ... \cap H_{\mathcal{F}_{n+1}})$. Clearly, the natural morphism

$$G/(H_{\mathcal{F}_1} \cap ... \cap H_{\mathcal{F}_n}) \to G/(H_{\mathcal{F}_1} \cap ... \cap H_{\mathcal{F}_n}) \times G/H_{\mathcal{F}_{n+1}} \cong \prod_n S \times S$$

is injective. So the only thing left to show is that it is onto. Since $G/H_{\mathcal{F}_1}$ is simple, it is enough to show that $H_{\mathcal{F}_1} \cap ... \cap H_{\mathcal{F}_{n+1}}$ is a proper subset of $H_{\mathcal{F}_1} \cap ... \cap H_{\mathcal{F}_n}$. Indeed, since $\mathcal{F}_{n+1}$ is diffrerent than $\mathcal{F}_1, ..., \mathcal{F}_n$, for every $i \in \{1, ..., n\}$ there is some $B_i \subseteq I$ such that $B_i \in \mathcal{F}_i$ and $B_i^c \in \mathcal{F}_{n+1}$. So $B = B_1 \cup \cdots \cup B_n \in \mathcal{F}_1 \cap \cdots \cap \mathcal{F}_n$, while $B^c \in \mathcal{F}_{n+1}$. Hence, taking an element $g \in G$ which satisfies $g_i = e, (\forall i \in B)$ and $g_j \neq e(\forall j \in B^c)$, we get that $g \in H_{\mathcal{F}_1} \cap ... \cap H_{\mathcal{F}_n}$ while $g \notin \mathcal{F}_{n+1}$. In conclusion, since that epimorphism between the quotient groups are precisely the natural epimorphisms $\prod_{I'} S \to \prod_{J'} S$ where $J' \subset I'$, we get by [6] Exercise 1.1.4 that $G_{\hat{\mathcal{C}}} \cong \prod_{2^{2^{\mathbf{m}}}} S$. □

Proposition 20. *Let X be a set of cardinality $2^{\mathbf{n}}$, and let $Y \subseteq P(X)$ be a set of cardinality $\mathbf{n}$, closed under finite intersections, such that for each $A \in Y$, $|A| = 2^{\mathbf{n}}$. Then, there are $2^{2^{\mathbf{m}}}$ ultrafilters $\mathcal{F}$ on X consists of sets of cardinality $2^{\mathbf{n}}$ which contains Y.*

Proof. We construct a set of subsets of X, $\{C_i, D_i\}_{i<2^{\mathbf{n}}}$ such that for every $i < 2^{\mathbf{n}}$, $C_i \cap D_i = \emptyset$ and for every disjoint finite subsets $I, J \subseteq 2^{\mathbf{n}}$, not both empty, $(\bigcap_{i\in I} C_i) \cap (\bigcap_{j\in J} D_j) \cap B$ is of cardinality $2^{\mathbf{n}}$ for every $B \in Y$. Let us look at the following set of triples (B, I, J, α) such that $B \in Y$, I and J are disjoint finite subsets of $2^{\mathbf{n}}$ not both empty, and $\alpha < 2^{\mathbf{n}}$ is an ordinal. Since this set is of cardinality $2^{\mathbf{n}}$, we can index it by $\beta < 2^{\mathbf{n}}$. We construct the required subsets by recursion. Suppose $C_{i,\gamma}$ and $D_{i,\gamma}$ were defined for every $\gamma < \beta$, and their cardinality is less or equal then γ, and only less than $2^{\mathbf{n}}$ of them are not empty. Now look at the triple (B, I, J, α) corresponding to the ordinal β. If $i \in I$, define $C_{i,\beta}$ to be $\bigcup_{\gamma<\beta} C_{i,\gamma} \cup x$ where $x \in B \setminus ((\bigcup_{i<2^{\mathbf{n}},\gamma<\beta} C_{i,\gamma}) \cup (\bigcup_{i<2^{\mathbf{n}},\gamma<\beta} D_{i,\gamma}))$, otherwise $\bigcup_{\gamma<\beta} C_{i,\gamma}$. Likewise, If $i \in J$, define $D_{i,\beta}$ to be $\bigcup_{\gamma<\beta} D_{i,\gamma} \cup y$ where $x \neq y \in B \setminus ((\bigcup_{i<2^{\mathbf{n}},\gamma<\beta} C_{i,\gamma}) \cup (\bigcup_{i<2^{\mathbf{n}},\gamma<\beta} D_{i,\gamma}))$, otherwise $\bigcup_{\gamma<\beta} D_{i,\gamma}$. Notice that since the number of subset $C_{i,\gamma}, D_{i,\gamma}$ for $\gamma < \beta$ which are not empty is less than $2^{\mathbf{n}}$, and that the cardinaly of each one of them is less or equal then γ and $\beta < 2^{\mathbf{n}}$, then $(\bigcup_{i<2^{\mathbf{n}}}, \gamma < \beta C_{i,\gamma}) \cup s(\bigcup_{i<2^{\mathbf{n}},\gamma<\beta} D_{i,\gamma})$ is of cardinality less then β which in turn is less than $2^{\mathbf{n}}$. Thus, Using the fact that $|B| = 2^{\mathbf{n}}$, it is possible to find such elements $x, y \in B$. In addition, it is easy to see that $C_{i,\beta}$ and $D_{i,\beta}$ satisfy the properties in the recursion hypothesis. Define $C_i = \bigcup_{\beta<2^{\mathbf{n}}} C_{i,\alpha}$ and $D_i = \bigcup_{\beta<2^{\mathbf{n}}} D_{i,\alpha}$. By the construction they satisfy the required propertis. Now, denote $\mathcal{A}$ the set of all subsets $B \subseteq X$ for which $|B^c| < 2^{\mathbf{n}}$. For each $I \subseteq 2^{\mathbf{n}}$ (not necessarily finite) define the following sets: $A_I = Y \cup \{C_i\}_{i\in I} \cup \{D_j\}_{j\notin I} \cup \mathcal{A}$. Since all the finite intersections os subsets from A, $\{C_i\}_{i\in I}$ and $\{D_j\}_{j\notin I}$ are of cardinality $2^{\mathbf{n}}$, then so are the finite intersection with elements from $\mathcal{A}$. Thus, for all $I \subseteq 2^{\mathbf{n}}$, A_I is contained in some ultrafilter. In addition, if $I \neq J$ then without loss of generality there is some $i \in I \setminus J$. Thus $C_i \in A_I$ and $D_i \in A_J$ are disjoint, so A_I and A_J can not be contained in the same ultrafilter. In conclusion, there are $2^{2^{\mathbf{n}}}$ different ultrafilters containing Y, each of then consists of subsets of maximal cardinality. $\square$

Theorem 21. *Let S be a finite simple group, and let $G = \prod_{\mathbf{m}} S$. Denote by $\{G_\alpha\}_\alpha$ the infinite tower of profinite completions of G. Then there exists an infinite chain of profinite groups $\{K_\alpha\}_\alpha$, with morphisms $\epsilon_{\beta\alpha} : K_\beta \to K_\alpha$ which satisfy:*

1. *$K_0 = G$.*
2. *For every ordinal β, $K_{\beta+1} = \prod_{2^{2^{\omega_0(K_\beta)}}} S$*
3. *For every limit ordinal α, $K_\alpha = \prod_{\mathbf{n}} S$, when $\mathbf{n} = \Pi_{\beta<\alpha}\omega_0(K_\beta)$*
4. *For every $\beta < \alpha$ the maps $\epsilon_{\beta\alpha} : K_\beta \to K_\alpha$ are compatible.*
5. *For every ordinal α there are epimorphisms $\eta_\alpha : G_\alpha \to K_\alpha$ which are compatible with the homomorhisms $\varphi_{\beta\alpha} : G_\beta \to G_\alpha$ and $\epsilon_{\beta\alpha} : K_\beta \to K_\alpha$.*
6. *Every K_α admits a set of normal subgroups $\mathbb{A}_\alpha$ such that:*
   - $|\mathbb{A}_0| = 2^{2^{\mathbf{m}}}$.
   - *For every $N \in \mathbb{A}_\alpha$, N is a subgroup of the form $H_{\mathcal{F}}$ where $\mathcal{F}$ is an ultrafilter on $\omega_0(K_\alpha)$ all of whose elements have maximal cardinality.*
   - *For every $N \in \mathbb{A}_\alpha$, $K_\alpha/N \cong S$.*
   - *For every $\beta < \alpha$ and $N \in \mathbb{A}_\alpha$, $N \cap K_\beta \in \mathbb{A}_\beta$.*
   - *For every ordinal α and $N \in \mathbb{A}_\alpha$ there are $2^{\omega_0(K_{\alpha+1})}$ subgroups N' in $\mathbb{A}_{\alpha+1}$ which satisfy $N' \cap G_\alpha \in \mathbb{A}_\alpha$.*
   - *For every limit ordinal α and a series $(N_\beta)_{\beta<\alpha}$ of normal subgroups, $N_\beta \in \mathbb{A}_\beta$, such that if $\gamma < \beta$ then $N_\beta \cap K_\gamma = N_\gamma$, there are $\Pi_{\beta<\alpha}\omega_0(K_\beta)$ subgroups $N_\alpha \in \mathbb{A}_\alpha$ whose intersection with each $K_\beta, \beta < \alpha$ is equal to N_β.*

Proof. Let α be an ordinal and assume that for all $\beta < \alpha$ such groups exist. First case: $\alpha = 0$. Define $\mathbb{A}_0$ to be set of all subgroups of the form $H_{\mathcal{F}}$ for $\mathcal{F}$ an ultrafilter on $\mathbf{m}$ all of whose elements have cardinaily $\mathbf{m}$. By Proposition 18 $|\mathbb{A}|_0 = 2^{2^{\mathbf{m}}}$.

Second case: $\alpha = \beta + 1$ for some ordinal β. By the induction hypothesis, $K_\beta \cong \prod_{\mathbf{n}} S$ for some cardinal $\mathbf{n}$. Let $\mathcal{C}$ be the set of all subgroups of the form $H_\mathcal{F}$ where $\mathcal{F}$ is an ultrafilter on $\omega_0(K_\beta)$. Define $K_{\beta+1} = K_{\beta,\hat{\mathcal{C}}}$. Then, by Lemma 19, $K_{\beta+1} \cong \prod_{2^{2^{\mathbf{n}}}} S$. By definition of the completion with respect to some directed system of of subgroups, there is a natural homomorphism $\epsilon_{\beta,\alpha} : K_\beta \to K_{\beta,\hat{\mathcal{C}}} = K_\alpha$. Define $\epsilon_{\gamma,\alpha}$ for all $\gamma < \beta$ to be $\epsilon_{\beta,\alpha} \circ \epsilon_{\gamma,\beta}$.

Since $\mathcal{C}$ is a subtopology of the profinite topology, then there is a natural epimorphism $\xi_\beta : \widehat{K_\beta} \to K_{\beta,\hat{\mathcal{C}}}$ which commutes with the natural homomorphism from K_β to its completions. By induction hypothesis there is a natural epimorphism $\eta : G_\beta \to K_\beta$. By [6] Proposition 3.2.5, it can by lifted to a compatible epimorphism $\hat{\eta_\beta}; G_\alpha = \widehat{G_\beta} \to \widehat{K_\beta}$. So, define $\eta_{\beta+1}$ to be $\xi_\beta \circ \widehat{\eta_\beta}$. Let $N \in \mathcal{A}_\beta$. By the induction hypothesis, N has the form $H_\mathcal{F}$ for some ultrafilter on $\mathbf{n}$, such that all of whose elements have cardinality $\mathbf{n}$. Recall that for each ultrafilter $\mathcal{F}$, there is a given isomorphism $\prod_{\mathbf{n}} S/H_\mathcal{F} \cong S$. We use it to identify the quotient with S. Use this isomorphisms to identify all the quotient groups with the same group. for all $g \in K_\beta$ define A_g to be the set of all ultrafilters $\mathcal{F}'$ for which the image of g in the natural epimorphism $\prod_{\mathbf{n}} S \to \prod_{\mathbf{n}} S/H_{\mathcal{F}'}$ is equal to the image of g in the natural epimorphism $\prod_{\mathbf{n}} S \to \prod_{\mathbf{n}} S/H_\mathcal{F}$. There is a set of indexes $A \in FF$ determining the image of g in $\prod_{\mathbf{n}} S/H_\mathcal{F}$, so in fact $A_g = \{\mathcal{F}' : A \in \mathcal{F}'\}$. Notice that since $\mathcal{F}$ contains only sets of maximal cardinality, $|A| = \mathbf{n}$, so the number of ultrafilters containing A is equal to the number of ultrafilters on A, which is $2^{2^{\mathbf{n}}}$. In addition, for each $g_1 \neq g_2 \in K_\beta$ there is some $g_3 \in K_\beta$ for which $A_{g_1} \cap A_{g_2} = A_{g_3}$: Indeed, Let A and B be the subsets determining the images of g_1 and g_2 correspondingly. Then $A_{g_1} \cap A_{g_2}$ equals to the set of all ultrafilters containing $A \cap B$. So define g_3 to be e on $A \cap B$ and x on $(A \cap B)^c$ for some $e \neq x \in S$.

Let $\beta\mathbf{n}$ denote the set of all ultrafilters on $\mathbf{n}$. Then the set of the subsets of the form A_g for $g \in K_\beta$ is a set of $2^{\mathbf{n}}$ subsets of the maximal cardinality, closed under finite intersections. Thus by Proposition 20, there are $2^{\omega_0(K_\alpha)}$ ultrafilters containing all these sets, which contain only subsets of maximal cardinality. Denote this collection by $\mathbb{A}_{\mathcal{F},\alpha}$. For each $\mathcal{G} \in \mathbb{A}_{\mathcal{F},\alpha}$, the subgroup $H_\mathcal{G}$ of K_α satisfies that its intersection with K_β is equal to $H_\mathcal{F}$ and $K_\alpha/H_\mathcal{G} \cong S$. Define $\mathbb{A}_\alpha = \bigcup_{\mathcal{F} \in \mathbb{A}_\beta} \mathbb{A}_{\mathcal{F},\alpha}$.

Third case: Let α be a limit ordinal. Define K'_α to be the direct limit of $\{K_\beta, \epsilon_{\gamma,\beta}\}$. Denote by $\mathcal{C}_\beta$ the set of all subgroups N of the form $\lim_\rightarrow N_\beta$ where $\{N_\beta\}$ is a compatible set of normal subgroups as described in condition 6. Then, $K'_\alpha/N \cong S$, and for $N_1, ..., N_l$ different subgroups of this form $K'_\alpha/N_1 \cap ... \cap N_l \cong \prod_l S$, since the is a natural monomorphism $K'_\alpha/N_1 \cap ... \cap N_l \to \prod_l S$, and if we take β to be an ordinal for which the chains differ from each other then any element on $\prod_l S$ has an origin in K_β. Define K_α to be the completion of K'_α with regard to this set of subgroups. Then, $K_\alpha = \prod_{\mathbf{n}} S$, when $\mathbf{n} = \Pi_{\beta<\alpha}\omega_0(K_\beta)$. Define $\epsilon_{\beta,\alpha}$ to be the compositions of the natural maps to the direct limit, and from the direct limit to its completion, and η_α to be the lifting of $\lim_{\rightarrow\beta<\alpha}\eta_\beta$ to the corresponding completions. For each subgroup N and $g \in K'_\alpha$ of this form, let A_g be the set of all such subgroups N' such that the image of g in K'_α/N' is equal to its image in K'_α/N. Since g is an element in the direct limit, it comes from some K_β. In addition, N is the direct limit of a compatible chain $\{N_\beta\}_{\beta<\alpha}$ Thus, $N' = \lim_{\to\beta<\alpha} N'\beta$ satisfies the condition iff $N'_{\beta+1}$ is a lifting of N_β in $K_{\beta+1}$. So the number of such subgroups equals to ω_{K_α}. In addition, by the same argument we can show the the set of all A_g is closed under finite intersections. Thus by a similar way of the successor case we can show there are $2^{\omega_0(K_\alpha)}$ subgroups extending any such $N \trianglelefteq K'_\alpha$ and satisfy all the conditions. Define $\mathbb{A}$ to be the set of all such extensions. □

Theorem 22. *Let G be a nonstrongly complete profinite group. Then for every limit ordinal $\alpha \neq 0$:*

$$\omega_0(G_\alpha) = \Pi_{\beta<\alpha}\omega_0(G_\beta).$$

Proof. By Corollary 17, it is enough to prove the claim for $G = \prod_{\mathbf{m}} S$. By Theorem 21, there is a series of subgroups K_α as described in the theorem. We will prove by transfinite induction that $\omega_0(G_\alpha) = \omega_0(K_\alpha)$. So, assume this equality holds for every $\beta < \alpha$. For $\alpha = 0$ the claim is clear. For $\alpha = \beta + 1$, $\omega_0(G_\alpha) = 2^{2^{\omega_0(G_\beta)}}$. Since $\mathcal{C}$ has $2^{2^{\omega_0(K_\beta)}}$ subgroups, we have that

$$\omega_0(K_\alpha) = 2^{2^{\omega_0(K_\beta)}} = 2^{2^{\omega_0(G_\beta)}} = \omega_0(G_\alpha).$$

For α a limit ordinal, we already know that $\omega_0(G_\alpha) \leq \prod_{\beta<\alpha}\omega_0(G_\beta)$. By proposition 15 we get that $\omega_0(G_\alpha) \geq \omega_0(K_\alpha)$. K_α is defined as the completion oa K'_α with respect to the set $\mathcal{C}$ of subgroups, thus $\omega_0(K_\alpha) = |\mathcal{C}$. By the building of $\mathcal{C}$,

$$|\mathcal{C}| = 2^{2^{\mathbf{m}}} \prod_{0\neq\beta<\alpha} 2^{\omega_0(K_0)} = 2^{2^{\mathbf{m}}} \prod_{0\neq\beta<\alpha} 2^{\omega_0(G_0)} = \prod_{\beta<\alpha} 2^{2^{\omega_0(G_\alpha)}} = \prod_{\beta<\alpha}\omega_0(G_\beta).$$

□

Actually, we can calculate $\Pi_{\beta<\alpha}\omega_0(G_\beta)$ explicitly.

Lemma 23 ([9, Lemma 5.9])**.** *If λ is an infinite cardinal and $\langle\kappa_\beta : \beta < \lambda\rangle$ is a nondecreasing sequence of nonzero cardinals, then*

$$\Pi_{\beta<\lambda}\kappa_\beta = \sup_\beta\{\kappa_\beta\}^\lambda.$$

Definition 24. Let κ be a cardinal. We say that κ is a strong limit cardinal if for all $\lambda < \kappa$, $2^\lambda < \kappa$.

In our case, the limit ordinal α is not necessarily a cardinal, but we may replace it by its cofinality. So, one may take $\lambda = cof(\alpha)$, and $\kappa_\beta = \omega_0(G_{\psi(\beta)})$, where $\psi : \lambda \to \alpha$ is an order preserving cofinal map. Notice that $\sup_\beta \kappa_\beta$ is a strong limit cardinal with cofinality $cof(\alpha) = \lambda$. Thus we can use the following proposition:

Proposition 25 ([9, pg. 58, Equation 5.23])**.** *Let κ be a strong limit cardinal. Then $\kappa^{cof(\kappa)} = 2^\kappa$.*

Corollary 26. *We have that*

$$\omega_0(G_\alpha) = \prod_{\beta<\alpha} \omega_0(G_\beta) = 2^{\sup_{\beta<\alpha}\{\omega_0(G_\beta)\}}$$

Remark 27. Let G and G' be two nonstrongly complete profinite groups. Then there exists some ordinal α such that for all $\beta \geq \alpha$, $\omega_0(G_\beta) = \omega_0(G'_\beta)$.

Proof. Denote $\mathbf{m_0} = \omega_0(G)$ and $\mathbf{n_0} = \omega_0(G')$. It is enough to prove that there exists α such that $\omega_0(G_\alpha) = \omega_0(G'_\alpha)$, since each successor level depends only on the previous level, and any limit level does not depend on initial segments. Assume $\mathbf{n_0} > \mathbf{m_0}$. Since the series of local weights is a strongly ascending series over the ordinals, it is not bounded. So, denote by α_0 the minimal ordinal for which $\omega_0(G_{\alpha_0}) \geq \mathbf{n_0}$. Now continue by recursion: Assume α_n is defined. Define α_{n+1} to be the minimal ordinal for which $\omega_0(G_{\alpha_{n+1}}) \geq \omega_0(G'_{\alpha_n})$. Eventually, $\alpha = \sup\{\alpha_n\}_{n\in\omega}$ is the desired ordinal. ☐

4 The Abelian Case

If G is a nonstrongly complete profinite **abelian** group, then we can offer a simpler proof for the local weight of G_α where α is a limit ordinal.

By remark 14, we may assume G has a quotient of the form $\prod_{\omega_0(G)} S$ for some finite simple group. Since G is abelian, then so is S.

In conclusion, there is a finite simple abelian group S on which G projects continuously in $\omega_0(G)$ different way.

Lemma 28. *Let G be a nonstrongly complete profinite group, and S a finite simple group which appear as a quotient of G in $\mathbf{n}$ different ways for some infinite cardinality $\mathbf{n}$, then G has $2^{2^{\mathbf{n}}}$ dense normal subgroups H_i such that $G/H_i \cong S$.*

Proof. By the proof ot Theorem 8, Since G has $\mathbf{n}$ projections on S, then G projects on $\prod_{\mathbf{n}} S$. By Example 7, $\prod_{\mathbf{n}} S$ has $2^{2^{\mathbf{n}}}$ non continuous epimorphisms on S. Compose them with the epimorphism from G to $\prod_{\mathbf{n}} S$, we get $2^{2^{\mathbf{n}}}$ non-continuous epimorphisms from G to S. For any such epimorphism, the kernel is a subgroup H of finite index which is not open, and thus not closed, and for which $G/H \cong S$. Thus, there are no normal subgroups between H and G. But since H is normal, so is $\overline{H}$. So $\overline{H} = G$. I.e, H is dense. ☐

Lemma 29. *Let G be a nonstrongly complete profinite group. Then G is a continuous semi-directed component of $\hat{G}$.*

Proof. Look at the identity map $G \to G$. By the universal property of profinite completion we get the following diagram:

$$\begin{tikzcd} \hat{G} \arrow[dr, "\widehat{id}"] & \\ G \arrow[u, "i"] \arrow[r, "id"'] & G \end{tikzcd}$$

Thus $\hat{G} \cong G \ltimes K$ for $K = \ker \widehat{id}$. $\square$

Remark 30. If G is abelian then so are all its finite quotients, and thus $\hat{G}$ is abelian as inverse limit of abelian groups.

Corollary 31. *If G is abelian then $\hat{G} \cong G \times K$ for $K = \ker \widehat{id}$.*

Remark 32. Let G be a nonstrongly complete profinite group, and denote by $K = \ker \widehat{id}$. Then, being the kernel of the natural epimorphism $\lim_{\leftarrow U \trianglelefteq_f G} G/U \to \lim_{\leftarrow U \trianglelefteq_o G} G/U$, and since for every $U \trianglelefteq_f G$ the clousre $\overline{U} \trianglelefteq_o G$ we get that $K = \lim_{\leftarrow U \trianglelefteq G} \overline{U}/U$.

From now on we denote K for $\ker(\widehat{id} : \widehat{G} \to G)$, and K_α for $\ker(\widehat{id} : \widehat{G_\alpha} \to G_\alpha)$

Corollary 33. *If H is a normal dense subgroup of G, then K projects naturally on G/H. Moreover, if $H_1 \neq H_2$, then the natural projections $K \to G/H_1$, $K \to G/H_2$ are different.*

Proof. Being the inverse limit of the directed system of finite groups $\{\overline{U}/U\}_{U \trianglelefteq G}$, every group in the system corresponding to a continuous quotient of K. So each dense normal subgroup H of G corresponding to a continuous epimorphism $K \to \overline{H}/H = G/H$. $\square$

Proposition 34. *Let $\mathbf{n}$ be an infinite cardinality, and G be a nonstrongly complete profinite abelian group, which projects on some finite simple group S in $\mathbf{n}$ different ways for which the kernel is a dense subgroup. Then each projection $G \to S$ has $2^{2^{\mathbf{n}}}$ different liftings to $\hat{G}$.*

Proof. By Corollary 33, K has $\mathbf{n}$ different projections on S. So, by Lemma 28 K has $2^{2^{\mathbf{n}}}$ abstract epimorphisms $\psi_i : K \to S$.

Let $f : G \to S$ be some projection. Recall that since G is abelian then so is S. By Corollary 31, $\hat{G} = G \times K$. Since S is abelian we can define epimorphisms $(f, \psi_i) : G \times K \to S$, which clearly extend f. $\square$

Proposition 35. *Let G be nonstrongly complete profinite abelian group which has $\omega_0(G)$ projections on some finite simple group S. Consider $\{G_\alpha\}$ to be the chain of profinite completions as described in the beginning of the previous section. Then:*

- $\omega_0(G_\alpha) = \prod_{\beta<\gamma<\alpha} 2^{2^{\omega_0(G_\gamma)}}$ *for all limit ordinal α.*

for each α there is a set $\mathbb{A}_\alpha$ of not necessarily open subgroups H_i which satisfy:

1. *For each $H_i \in \mathbb{A}_\alpha$, $G_\alpha/H_i \cong S$*
2. *For each $\beta < \alpha$ and $H_i \in \mathbb{A}_\alpha$, $H_i \cap G_\beta \in \mathbb{A}_\beta$.*
3. *For each limit ordinal α, all the subgroups in $\mathbb{A}_\alpha$ are open.*

4. *For each $\beta < \alpha$ and $H \in \mathbb{A}_\beta$ there are $\mathbf{n}_\alpha$ subgroups $H' \in \mathbb{A}_\alpha$ for which $H' \cap G_\beta = H$ where*

$$\mathbf{n}_\alpha = \begin{cases} 2^{2^{\omega_0(G_\alpha)}} & \alpha \text{ is a successor ordinal} \\ \omega_0(G_\alpha) & \alpha \text{ is a limit ordinal} \end{cases}$$

5. *For each α $|\mathbb{A}_\alpha| = \mathbf{n}_\alpha$*

Proof. We prove the claim by transfinite induction.

For $\alpha = 0$ take $\mathbb{A}_0 = \{\ker f_i \mid f_i : G \to S \text{ is an epimorphism}\}$. By assumption G has $\omega_0(G)$ different continuous projections on S. so by Lemma 28 $|\mathbb{A}_0| = 2^{2^{\omega_0(A)}}$.

Let $\alpha = \beta + 1$ and assume $\mathbb{A}_\beta$ is defined. By induction hypothesis $|\mathbb{A}_\beta| = 2^{2^{\omega_0(G_\beta)}}$ or $\omega_0(G_\beta)$. In the first case, since the number of open subgroups of G_β is equal to $\omega_0(G_\beta)$ by definition, then $2^{2^{\omega_0(G_\beta)}}$ are not open, and thus since S is simple they are dense. In the second case we can use Lemma 28 to conclude that G_β has $2^{2^{\omega_0(G_\beta)}}$ dense subgroups H for which $G/H \cong S$. Thus by Proposition 34 each subgroup $H \in \mathbb{A}_\beta$ has a set $\mathbb{A}_{\beta+1,H}$ of cardinality $2^{2^{2^{2^{\omega_0(G_\beta)}}}} = 2^{2^{\omega_0(G_\alpha)}}$ consisting of normal subgroups $H' \trianglelefteq G_\alpha$ for which $G/H' \cong S$ and $H' \cap G_\beta = H$. Define $\mathbb{A}_\alpha = \bigcup_{H \in \mathbb{A}_\beta} \mathbb{A}_{\beta+1,H}$.

Let α be a limit ordinal. For each series $(H_\beta)_{\beta<\alpha}$ of normal subgroups from $\mathbb{A}_\beta$ such that for every $\gamma < \beta$ $H_\beta \cap G_\gamma = H_\gamma$, the union is a normal subgroup $H \trianglelefteq \lim_{\to \beta<\alpha} G_\beta$ for which the quotient is isomorphic to S and the intersection with G_β is equal to H_β. By induction hypothesis there are $\prod_{\beta<\gamma<\alpha} 2^{2^{\omega_0(G_\gamma)}}$ such serieses. Define $\mathbb{A}_\alpha$ to be the closures of all this subgroups in $\widehat{\lim_{\to \beta<\alpha}} G_\beta = G_\alpha$.

We get that G_α has at least $\lim_{\to \beta<\alpha} G_\beta$ open subgroups. On the other hand by Lemma 12 $\omega_0(G_\beta) \leq \prod_{\gamma<\beta} \omega_0(G_\beta)$, so $\omega_0(G_\beta) = \prod_{\gamma<\beta} \omega_0(G_\gamma)$.

□

References

[1] Tamar Bar-On. The tower of profinite completions. *Journal of group theory*, 21(6):1065– 1072, 2018.

[2] Michael D Fried and Moshe Jarden. *Field arithmetic*, volume 11. Springer Science & Business Media, 2006.

[3] Nikolay Nikolov. Algebraic properties of profinite groups. *arXiv preprint arXiv:1108.5130*, 2011.

[4] Nikolay Nikolov and Dan Segal. On finitely generated profinite groups, i: strong completeness and uniform bounds. *Annals of mathematics*, pages 171–238, 2007.

[5] Andrew Pletch. Strong completeness in profinite groups. *Pacific Journal of Mathematics*, 97(1):203–208, 1981.

[6] Luis Ribes and Pavel Zalesskii. *Profinite groups*. Springer, 2000.

[7] MG Smith and JS Wilson. On subgroups of finite index in compact hausdorff groups. *Archiv der Mathematik*, 80(2):123–129, 2003.

[8] Lynn Arthur Steen, J Arthur Seebach, and Lynn A Steen. *Counterexamples in topology*, volume 18. Springer, 1978.

[9] Jech Thomas. Set theory: Third millennium edition. *Springer Monographs in Mathematics*, page 642, 2003.

[10] John S Wilson et al. A note on the finite images of profinite groups. *Illinois Journal of Mathematics*, 47(1-2):567–570, 2003.