J. Group Theory, Ahead of Print
DOI 10.1515/jgth-2018-0024

The tower of profinite completions

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Communicated by Pavel A. Zalesskii

Abstract. We show that the tower of profinite completions of a nonstrongly complete profinite group continues indefinitely.

Every group has a profinite completion, which is a profinite group. However, the profinite completion $\widehat{G}$ of a profinite group G may properly contain G. A profinite group is said to be *strongly complete* if $\widehat{G} = G$, equivalently if every subgroup of finite index is open. Strong completeness of profinite groups was studied by several authors, including Pletch in [2], Ribes and Zalesskii in [3], and Wilson in [4]. Nikolov and Segal proved in [1] that every finitely generated profinite group is strongly complete. However, nonstrongly complete profinite groups do exist, e.g., [3, p. 131].

Recall that the profinite completion of a group G is the inverse limit

$$\widehat{G} = \varprojlim G/N,$$

where N runs over the finite index normal subgroups in G. The completion is equipped with a homomorphism $G \to \widehat{G}$ defined by $g \mapsto (gN)$. This homomorphism is injective if and only if G is residually finite.

Let G be a profinite group. Then G is residually finite, and hence there is an embedding $G \hookrightarrow \widehat{G}$. Since $\widehat{G}$ is again profinite, there is an embedding

$$\widehat{G} \hookrightarrow \widehat{\widehat{G}},$$

and so on. We thus define $G_0 = G$ and set

$$G_{i+1} = \widehat{G_i}.$$

We thus have a chain

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow \cdots,$$

whose direct limit

$$H = \varinjlim G_i$$

is the union of all the groups G_i.

This research was supported by the Israel Science Foundation (Grant No. 03671/).

This direct limit is not necessarily a profinite group, but we can take its profinite completion, $\widehat{H}$, and then the chain of completions again, and so on. By transfinite induction we define the groups G_α and morphisms $\varphi_{\beta\alpha}$ (for $\beta < \alpha$) as follows:

- For a successor ordinal $\alpha = \beta + 1$,
$$G_\alpha = \widehat{G_\beta},$$
with the natural map $\varphi_{\beta\alpha} : G_\beta \to G_\alpha$, and for every $\gamma < \beta, \varphi_{\gamma\alpha} = \varphi_{\beta\alpha} \circ \varphi_{\gamma\beta}$.
- For a limit ordinal α, the system of maps $\{\varphi_{\gamma\beta}\}_{\gamma<\beta<\alpha}$ is compatible, so we can define $H_\alpha = \varinjlim\{G_\beta, \varphi_{\gamma\beta}\}$ with the natural maps $\varphi'_{\beta\alpha} : G_\beta \to H_\alpha$, and then $G_\alpha = \widehat{H_\alpha}$ with the natural map $\psi_\alpha : H_\alpha \to G_\alpha$; we also define, for every $\beta < \alpha, \varphi_{\beta\alpha} = \psi_\alpha \circ \varphi'_{\beta\alpha}$.

Obviously, if G is strongly complete, then $G_\alpha = G$ for all α. But what happens if G is not strongly complete? It turns out that the chain *never* terminates, and not just that, but all the maps are (proper) inclusions. We prove:

Theorem 1. *For all $\alpha < \beta$, the natural map $\varphi_{\alpha\beta} : G_\alpha \to G_\beta$ is a proper embedding.*

In order to prove this, we need a few more facts.

Lemma 2. *Every profinite quotient group of a strongly complete group is strongly complete.*

Proposition 3. *There are continuous epimorphisms $f_\beta : G_\beta \to G_0 = G$ such that $f_\beta = f_\alpha \circ \varphi_{\beta\alpha}$ for every $\beta < \alpha$.*

Proof. We will prove this by transfinite induction. Assume that for all $\alpha < \beta$ there are continuous epimorphisms $f_\alpha : G_\alpha \to G$, such that $f_\gamma = f_\alpha \circ \varphi_{\gamma\alpha}$ for every $\gamma < \alpha$.

First case: β is a successor ordinal, so that $\beta = \alpha + 1$ for some α, and $G_\beta = \widehat{G_\alpha}$. By assumption, there is a continuous epimorphism $f_\alpha : G_\alpha \to G$, and G is a profinite group, so by definition of the profinite completion there is a continuous homomorphism $\widehat{f_\alpha}$ such that the following diagram commutes:

$$\begin{tikzcd} \widehat{G_\alpha} \arrow[dr, "\widehat{f_\alpha}"] & \\ G_\alpha \arrow[u, hook] \arrow[r, "f_\alpha"'] & G. \end{tikzcd}$$

Now $\widehat{f}$ is onto because $G = f_\alpha(G_\alpha) \subseteq \widehat{f_\alpha}(\widehat{G_\alpha})$, and we can take $f_\beta = \widehat{f_\alpha}$.

Second case: β is a limit ordinal, so $G_\beta = \widehat{H_\beta}$ for

$$H_\beta = \varinjlim_{\alpha<\beta} G_\alpha.$$

By assumption, there are compatible epimorphisms $f_\alpha : G_\alpha \to G$. By definition of the direct limit there is an epimorphism $f : H_\beta \to G$ such that $f \circ i_\alpha = f_\alpha$, where i_α denotes the inclusion $i_\alpha : G_\alpha \to H_\beta$. By the definition of the profinite completion, since G is a profinite group, there is a continuous epimorphism $\widehat{f} : \widehat{H} \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
\widehat{H} \arrow[dr, "\widehat{f}"] & \\
H \arrow[u] \arrow[r, "f"'] & G.
\end{tikzcd}$$

So, $f_\beta = \hat{f}$ is compatible with f_α for every $\alpha < \beta$ □

Corollary 4. *For every ordinal α, G_α is nonstrongly complete.*

Proof. By Proposition 3 there is a continuous epimorphism $f_\alpha : G_\alpha \to G$. But G is nonstrongly complete, so by Lemma 2 the groups G_α are nonstrongly complete for all α. □

The final proposition that we need is:

Proposition 5. *For all $\alpha < \beta$, the map $\varphi_{\alpha\beta} : G_\alpha \to G_\beta$ is an embedding.*

Proof.[1] By transfinite induction on β. Assume that $\varphi_{\gamma\alpha} : G_\gamma \to G_\alpha$ is an embedding for every $\gamma < \alpha < \beta$.

First case: β is a successor ordinal, so $\beta = \alpha + 1$ for some α. By definition, we have $G_\beta = \widehat{G_\alpha}$. Since G_α is a profinite group, the map $\varphi_{\alpha\beta} : G_\alpha \to G_\beta$ is an embedding. But for every $\gamma < \alpha$ we have that $\varphi_{\gamma\alpha}$ is an embedding by the induction hypothesis, so $\varphi_{\gamma\beta} = \varphi_{\alpha\beta} \circ \varphi_{\gamma\alpha}$ is an embedding as well.

Second case: β is a limit ordinal. Hence, $G_\beta = \widehat{H_\beta}$ for

$$H_\beta = \varinjlim_{\alpha<\beta} G_\alpha.$$

[1] This proof for limit ordinal was suggested to us by Dan Segal. The claim can also be proved by lifting any continuous map onto a finite group, from G_α to G_β, for all $\beta > \alpha$, as we have done in Proposition 3.

By the induction hypothesis, all the maps $G_\gamma \to G_\delta$ for $\gamma < \delta < \beta$ are injective, and thus H is the direct limit of an injective limit, so $G_\alpha \hookrightarrow H_\beta$ for all $\alpha < \beta$. The only thing left to show is that $H_\beta \hookrightarrow \widehat{H_\beta}$. Equivalently: that H_β is residually finite.

We first assume that for all $\alpha < \beta$ and for every $N_\alpha \trianglelefteq G_\alpha$ of finite index, there is a normal subgroup $N_\beta \trianglelefteq H_\beta$ of finite index such that $N_\beta \cap G_\alpha = N_\alpha$.

Under this assumption, let $a \in \bigcap_{N \in A} N$, where $\mathcal{A}$ is the set of all normal subgroups of finite index in H. The fact that $a \in H$ means that $a \in G_\alpha$ for some $\alpha < \beta$. So, $a \in G_\alpha \cap (\bigcap_{N \in \mathcal{A}} N) = \bigcap_{N \in \mathcal{A}} (N \cap G_\alpha)$. Clearly, for every $N \in \mathcal{A}$, $N \cap G_\alpha$ is a normal subgroup of finite index in G_α. Moreover, by assumption, every normal subgroup of finite index in G_α is of the form $N \cap G_\alpha$ for some $N \in \mathcal{A}$. So, this is exactly the intersection of all normal subgroups of finite index in G_α, which, since G_α is a profinite group, is equal to $\{e\}$. In conclusion, $a = e$.

It remains to prove that our assumption on normal subgroups of G_α always holds. We will prove a bit more. For simplicity, let $G_\beta = H_\beta$. Let $N_\alpha \trianglelefteq_f G_\alpha$. Then for all $\beta > \alpha$ there exists $N_\beta \trianglelefteq_f G_\beta$ such that for all $\alpha \leq \delta < \beta$,

$$N_\beta \cap G_\delta = N_\delta, \quad N_\beta G_\delta = G_\beta.$$

The proof will be by transfinite induction. Suppose that N_γ is defined for all $\gamma < \beta$.

Case 1: $\beta = \gamma + 1$. Let N_β be the closure of N_γ in $G_\beta = \widehat{G_\gamma}$. Then, by [3, Proposition 3.2.2], $N_\beta \trianglelefteq_f G_\beta$ and $N_\beta \cap G_\gamma = N_\gamma$. Since N_β is open of finite index, so is $N_\beta G_\delta$. Thus, $N_\beta G_\delta$ is closed. But G_δ is dense in G_β, so $N_\beta G_\delta = G_\beta$. Moreover, if $\delta < \gamma$, then

$$N_\beta \cap G_\delta = N_\gamma \cap G_\delta = N_\delta$$

and

$$G_\beta = N_\beta G_\gamma = N_\beta N_\gamma G_\delta = N_\beta G_\delta.$$

Case 2: β is a limit ordinal. Let $N_\beta = \bigcup_{\alpha \leq \gamma < \beta} N_\gamma$. Then for all $\delta < \beta$ we have

$$N_\beta \cap G_\delta = \left(\bigcup_{\alpha \leq \gamma < \beta} N_\gamma \right) \cap G_\delta = N_\delta.$$

Moreover, $N_\beta G_\delta$ contains $\bigcup_{\alpha \leq \gamma < \beta} N_\gamma G_\delta = \bigcup_{\alpha \leq \gamma < \beta} G_\delta = G_\beta$, and so we have $N_\beta G_\delta = G_\beta$. It is also clear that $N_\beta \trianglelefteq G_\beta$. And eventually,

$$G_\beta / N_\beta = G_\alpha N_\beta / N_\beta \cong G_\alpha / N_\beta \cap G_\alpha = G_\alpha / N_\alpha$$

Hence, $[G_\beta : N_\beta] = [G_\alpha : N_\alpha] < \infty$. □

Now we can prove Theorem 1.

Proof of Theorem 1. Let $\alpha < \beta$ be ordinals. Clearly $\alpha < \beta$ implies $\alpha + 1 \leq \beta$, so the map $\varphi_{\alpha\beta}$ from G_α to G_β is equal to the composition $\varphi_{(\alpha+1)\beta} \circ \varphi_{\alpha(\alpha+1)}$. By Proposition 5, the maps $\varphi_{\alpha(\alpha+1)}, \varphi_{(\alpha+1)\beta}$ are embeddings. By Corollary 4, G_α is nonstrongly complete, so $\varphi_{\alpha(\alpha+1)} : G_\alpha \to \widehat{G} = G_{\alpha+1}$ is a proper embedding. In conclusion, $\varphi_{\alpha\beta}$ is a proper embedding. □

Another natural sequence to look at, is the *inverse tower of profinite completions*. Let G be a nonstrongly complete profinite group. Let

$$G_0 = G.$$

We build the next sequence by transfinite induction: Assume that for all $\gamma < \alpha$, G_γ is defined, and that for all $\delta < \gamma$ there are compatible epimorphisms $\psi_{\gamma,\delta}$. For a successor ordinal $\alpha = \beta + 1$ let

$$G_\alpha = \widehat{G_\beta}.$$

There is a real epimorphism (i.e., the kernel is nontrivial) $\psi_{\alpha,\beta} : G_\alpha \to G_\beta$, as we showed in Proposition 3. For $\gamma < \alpha$ define $\psi_{\alpha,\gamma} = \psi_{\beta,\gamma} \circ \psi_{\alpha,\beta}$. For a successor ordinal α, define

$$G_\alpha = \varprojlim_{\beta<\alpha} G_\beta.$$

Notice that G_α is a profinite group, as an inverse limit of profinite groups. In addition, by definition of the inverse limit, there are compatible epimorphisms $\psi_{\alpha,\beta}$ for all $\beta < \alpha$.

Proposition 6. *For all α, G_α is nonstrongly complete.*

Proof. The proof follows from Lemma 2 and the fact that for all α there is an epimorphism $G_\alpha \to G$. □

Corollary 7. *This chain also never terminates.*

Remark 8. It is natural to ask what is the connection between these two chains. Are they equal? The answer is that they can never be equal. We show this in the next theorem.

Theorem 9. *Let G be a nonstrongly complete profinite group, and let $\{G_n\}$ be the series of completions as we defined over the set ω. For all $n < m$ there are injections $\varphi_{nm} : G_n \to G_m$ and projections $\psi_{mn} : G_m \to G_n$ that satisfy*

$$\psi_{mn} \circ \varphi_{nm} = \mathrm{Id}.$$

We claim that the ω-th element of the "direct limit series" is different from the ω-th element of the "inverse limit series", i.e.,

$$\varprojlim_{n\in\omega} G_n \neq \widehat{\varinjlim_{n\in\omega} G_n}.$$

More specifically, for the natural homomorphism $i : \varinjlim G_n \to \varprojlim G_n$, *there is a homomorphism* $\alpha : \varinjlim G_n \to H$ *onto a finite group, which cannot be lifted to a continuous homomorphism* $\hat{\alpha} : \varprojlim G_n \to H$.

Proof. First, we need to describe the natural map $i : \varinjlim G_n \to \varprojlim G_n$. For all $n \in \omega$ and $m \geq n$ there is a map $\varphi_{nm} : G_n \to G_m$. These maps can be viewed as compatible maps into the inverse system $\{G_m, \psi_{m'm}\}_{n\leq m\leq m'\in\omega}$. This follows from the commutativity of the following diagram:

$$\begin{tikzcd} G_{m'} \arrow[r, "\mathrm{Id}"] & G_{m'} \arrow[d, "\psi_{m'm}"] \\ G_n \arrow[u, "\varphi_{nm'}"] \arrow[r, "\varphi_{nm}"] & G_m, \end{tikzcd}$$

which follows from the equation

$$\psi_{m'm} \circ \varphi_{nm'} = \psi_{m'm} \circ \varphi_{mm'} \circ \varphi_{nm} = \mathrm{Id} \circ \varphi_{nm} = \varphi_{nm}.$$

So, by the definition of the inverse limit, there is a map

$$i_n : G_n \to \varprojlim_{m\geq n} G_m,$$

which is equal to $\varprojlim G_m$, since the set $\{m \in \omega : m \geq n\}$ is cofinal in ω. This map satisfies

$$\varphi_{nm} = \psi_m \circ i_n.$$

Moreover, for all $n \leq m \in \omega$ the maps i_n and i_m are compatible as maps from the directed system $\{G_n, \varphi_{m'm''}\}_{m\leq m'\leq m''\in\omega}$, which is equal to $\varinjlim G_n$, i.e.,

$$i_n = i_m \circ \varphi_{nm}.$$

This follows from the commutativity of the following diagrams:

$$\begin{tikzcd} G_m \arrow[r, "\varphi_{mm'}"] & G_{m'} \\ G_n, \arrow[u, "\varphi_{nm}"] \arrow[ur, "\varphi_{nm'}"] & \end{tikzcd}$$

i.e., for all $m \leq m'$,

$$\varphi_{nm'} = \varphi_{mm'} \circ \varphi_{nm},$$

and thus

$$\varprojlim \varphi_{nm'} = (\varprojlim \varphi_{mm'}) \circ \varphi_{nm}.$$

In other words, $i_n = i_m \circ \varphi_{nm}$.

In conclusion, by the definition of the direct limit, there is a map

$$i : \varinjlim G_n \to \varprojlim G_n$$

which satisfies

$$i_n = i \circ \varphi_{n\omega}.$$

Actually, we can give a precise description of this map. It is known that $\varprojlim G_n$ can be expressed as the subgroup of $\Pi_{n\in\omega} G_n$ of all the tuples

$$\{(g_k)_{k\in\omega} : \psi_{mn}(g_m) = g_n \text{ for all } n \leq m \in \omega\}.$$

Let $g \in \varinjlim G_n$. Then there is some index $n \in \omega$ such that $g \in G_n$. One can easily verify that $i(g) = (g_k)_{k\in\omega}$, where

$$g_k = \begin{cases} \varphi_{nk}(g) & \text{for all } k \geq n, \\ \psi_{nk}(g) & \text{for all } k < n, \end{cases}$$

which is clearly in $\varprojlim G_n$.

By assumption, the group $G_0 = G$ is not strongly complete. So, there is an epimorphism

$$\alpha_0 : G \to H$$

onto a finite group, which is not continuous.

We can lift this epimorphism to noncontinuous compatible epimorphisms

$$\alpha_n : G_n \to H$$

in the following way: Define

$$\alpha_n = \alpha_0 \circ \psi_{n0}.$$

It is easy to see that these homomorphisms are compatible. We need to explain why they are still continuous. Well, recall that $\psi_{n0} : G_n \to G_0$ is a quotient map, as a continuous map from a compact space to an Hausdorff space. There is an open set $O \subseteq H$ such that $\alpha_0^{-1}[O] \subseteq G$ is not open. If $\alpha_n^{-1}[O] = \psi_{n0}^{-1}[\alpha_0^{-1}[O]]$ is open, then $\alpha_0^{-1}[O]$ is open, by the definition of quotient map. This is a contradiction.

The maps $\{\alpha_n\}$ define an epimorphism

$$\alpha : \varinjlim G_n \to H.$$

We would like to lift α to a continuous homomorphism

$$\hat{\alpha} : \varprojlim G_n \to H$$

such that $\alpha = \hat{\alpha} \circ i$. Assume by contradiction that such a homomorphism exists.

It is known that if $K = \varprojlim K_n$ is an inverse limit of profinite spaces, and there is a continuous map from K to a finite (discrete) space, then it splits through one of the K_n. So, $\hat{\alpha}$ must split through some G_n, i.e., there is some $n \in \omega$ and a map $f : G_n \to H$ such that

$$\hat{\alpha} = f \circ \psi_n.$$

But, since $\hat{\alpha}$ is a lifting of α, we have

$$\alpha_n = \alpha \circ \varphi_{n\omega} = \hat{\alpha} \circ i \circ \varphi_{n\omega} = \hat{\alpha} \circ i_n = f \circ \psi_n \circ i_n.$$

Notice that since for all $m > n$, $\psi_{mn} \circ \varphi_{nm} = \mathrm{Id}$, we have

$$\psi_n \circ i_n = \mathrm{Id}.$$

So, $f = \alpha_n$.

Hence, $\hat{\alpha} = \alpha_n \circ \psi_n$ is a composition of a quotient map by a noncontinuous map, which, as we have already shown, is never continuous. Thus, we cannot lift $\alpha : \varinjlim G_n \to H$ to a continuous map

$$\hat{\alpha} : \varprojlim G_n \to H.$$

In conclusion, $\varprojlim G_n \neq \widehat{\varinjlim G_n}$. □

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Received April 3, 2017; revised June 4, 2018.

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