Prices and Hedge Ratios of Average Exchange Rate Options

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In this paper an analytic approximation for the value of average (or Asian) exchange rate options is derived. The approximation formula is of the Black-Scholes type in which the striking price has been adjusted. It is found that especially in terms of implied volatility estimation errors, the approximation is very accurate. Analytic expressions for several hedge parameters are also derived.

I. INTRODUCTION

The objective of this paper is to give a fast numerical method to calculate prices and hedge ratios of average exchange rate options, which can be used by traders not only for pricing but also for hedging their positions.

Recently, banks introduced over-the-counter options on currencies, based on average exchange rates over some period of time.[1] For example, call options on average exchange rates entitle the holder to the amount by which the average of the exchange rates (over a number of prespecified dates) exceeds the exercise price of the option. The number of dates depends on the contract. For instance, there are options where the average is taken over the fixings of the exchange rate on the last trading day of each month of the year, resulting in an average of 12 exchange rates.[2] Typically, these options are European, i.e., they cannot be exercised before maturity.

Since these currency option contracts are tailored to the needs of the banks' clients, the banks not only need to know the price of the options, but they must also develop a fair understanding of the risks involved. In particular, the hedge ratio and the other sensitivity parameters of an average rate call option, such as Γ, Λ, and Θ, are of paramount interest to them. In this paper, analytic formulas for some of these parameters are given, and how other parameters can be calculated is set forth. These methods enable currency portfolio managers to assess how a particular contract fits within their existing positions and to what extent extra hedging is necessary.

Banks launched these options to enhance the possibilities for international clients to hedge a series of foreign currency positions with different maturities. For example, a U.S. firm having liabilities in Deutsche marks at the end of each month might use the average value currency

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options with averages over the last days of months to counter the downside of its currency risk. Of course, the firm can also achieve this by buying a series of currency options that mature each month. This kind of insurance, however, is more expensive since such a series can be seen as an index of options on currencies, while the average exchange rate option can be viewed as an option on an index of exchange rates.

The index in this case is not over exchange rates for a number of different currencies, but over a number of different dates for the same currencies. As is well known (Cox and Rubinstein 1985), an index of options is more expensive than an option on the index. Of course, the payoff of the series of options is generally higher, but the firm might not want to pay the extra premium for these higher cash flows, since the average exchange rate option already gives enough protection.

A brief review of the literature follows in the next section. In the third section, the theoretical model for average exchange rate options is developed and approximation formulas for the value of these options (under the assumption that the averaging period has yet to start) are derived. In Section IV, the values from the formulas developed in Section III are compared with Monte Carlo simulated values to assess the accuracy of the approximation. In Section V, the results are extended to the general case: designating the point in the time interval over which the average is taken. In Section VI, analytic formulas for the hedge ratios and other derivatives of the option value are derived for the benefit of currency portfolio managers in estimating exchange rate risk exposure. Section VII concludes the paper.

II. DERIVATION OF METHODOLOGY

In this study, an analytic model that approximates the value of an arithmetic average rate call option is derived by using the formula for the geometric average rate call option adjusted for exercise price. Additionally, an upper bound for the approximation error is also derived. While approximations for the price of securities can always be given, the practical value of these approximations depends on their accuracy.

To assess accuracy, the methodology suggested by Figlewski (1989) is used. Figlewski studies deviations between market and theoretical prices of a standard call option that are necessary to enable traders to make arbitrage profits. For example, Figlewski shows that for a one-month, at-the-money call with a Black-Scholes value of 2.05, the price could be anywhere between 1.74 and 2.35 before an arbitrageur can make a sure profit. In relation to the deviations required for arbitrage profits, the approximation error is very small, and it can be concluded that the approximation is very accurate.

Latane and Rendleman (1976) and Rubinstein (1985) also find that estimates of implied volatilities have a large variance. Concurrently, the Black-Scholes call option value's sensitivity to its volatility parameter is well documented. Hence in practice, due to the uncertainty about volatility, there will be also much uncertainty about the correct theoretical value of an option.

Compared with this uncertainty, the value for an average exchange rate option given by the analytic approximation formula in this paper is very close to the value that can be derived using Monte Carlo simulation. In most of the relevant cases, the difference is below the 1% level. This is an important figure, since it will be demonstrated that mistakes in estimation of volatility of 0.01 lead to a larger difference between actual and calculated values.

Kemna and Vorst (1990) analyze options based on average asset values and show that an analytic formula (such as the Black-Scholes model) does not exist for this kind of option.[3] They give examples of average asset value options as implicit parts of commodity-linked bond contracts, and price such options using Monte Carlo simulation and a variance reduction method. They conclude that whenever the price is based on the geometric average (instead of the usual arithmetic average) an analytic pricing formula can be derived that gives a lower bound for the value of a call option based on the arithmetic average.

Ruttiens (1989) uses this method to find values for currency options on average exchange rates by taking the convenience yield on the foreign currency (the foreign interest rate) into account. Regardless, it should be remembered that call options on a *geometric* average only give lower bounds for call options on an *arithmetic* average.

Also Levy (1992) and Turnbull and Wakeman (1991) derive approximation formulas. However, these formulas are different from each other and different from the formula used in this study. Furthermore, Bergman (1981) studies average rate options, but only considers options with a zero striking price. None of these papers give easy analytic approximations of the hedge ratios, ever so important in risk management. Bouaziz, Briys and Crouhy (1991) study options where the striking price is equal to the arithmetic average and hence is not fixed. In the latter study, just as in this one, not only is an easily calculated approximation formula for this option derived, but also an upper bound for the approximation error is found.

III. AVERAGE EXCHANGE RATE OPTIONS

A perfect currency market which is open continuously and in which no transaction costs are incurred is assumed. Let $S(t)$, or S, be the exchange rate of the foreign currency over the domestic currency at time t. It is also assumed that the exchange rate S(t) can be expressed by the usual stochastic differential equation:

$$dS(t) = \alpha S(t) + \sigma S(t) dW(t) \tag{1}$$

in which $W(t)$ is a Wiener process and α (drift) and σ (volatility) are constants. The intertemporal uncertainty is modelled through a filtered probability space (Ω, $\mathbf{f}$, F, P) where Ω is the set of all possible states of the world which can exist at the maturity date T of the option. $F=\{\mathbf{f}(t), t\in[0,T]\}$ is the filtration generated by the Wiener process $W(t)$. Hence the asset price $S(t)$ is adapted to $\mathbf{f}(t)$. The riskless interest rates in the country of the foreign currency, r_f, and in the country of the domestic currency, r, are also assumed to be constants. First, attention is focused on the special case of an average exchange rate call option, prior to the start of the averaging period. This means that there are no observations of exchange rates available to be included in the average. These are so-called forward starting options. (A more general case will be considered in Section V.)

Let t be the actual date and $t_1 < ... < t_n = T$ the dates over which averaging will take place. Then, the final payoff of the average rate call option with striking price X at maturity T is:

$$\text{Max}\ \{A(T) - X, 0\}$$

with

$$A(T) = \frac{1}{n} \sum_{i=1}^{n} S(t_i).$$

Since the average $A(T)$ is a deterministic function of the exchange rates only, a riskless hedge for the option can be created by eliminating the exchange-rate-induced risk. (See Ingersoll 1987 or Kemna and Vorst 1990.) Hence, the value C of the option is equal to its expected value with respect to an equivalent measure for which the discounted price process is a martingale; i.e.,

$$C = e^{-r(T-t)} \hat{E}\,[Max\{A(T) - X,0\} \mid \mathbf{f}(t)],$$

where $\hat{E}$ denotes the expectation with respect to the equivalent martingale measure. The exchange rate process is

$$dS(t) = (r - r_f)S(t)dt + \sigma S(t)d\tilde{W}(t) \tag{2}$$

and $\tilde{W}$ is the Brownian motion under the new probability measure.

From Equation (2) it can be deduced that each individual exchange rate $S(t_i)$ is lognormally distributed and hence $A(T)$ is a sum of lognormally distributed variables. Accordingly, $A(T)$ is no longer lognormally distributed.[4] For all practical purposes, an analytic approximation for the call value C that is both accurate and easy to calculate must be found. For a possible way to develop such an approximation another kind of a average rate option must be considered. This option is based on the geometric average $G(T)$ of the exchange rates;[5] i.e.,

$$G(T) = [\prod_{i=1}^{n} S(t_i)]^{1/n}.$$

The final payoff at maturity of a call option on the geometric average exchange rate is:

$$\text{Max } \{G(T) - X,0\}.$$

If C^G denotes the value of this geometric average rate option, then

$$C^G = e^{-r(T-t)} \hat{E}\,[\text{Max}\{G(T) - X,0\} \mid \mathbf{f}(t)], \tag{3}$$

Once again the expectation is taken with respect to Equation (2). Because $G(T)$ is a product of lognormally distributed variables, it is also lognormally distributed with mean

$$M = \hat{E}\,\{\ln(G(T))\} = \ln(S(t)) + \frac{1}{n} \sum_{i=1}^{n} (r - r_f - \sigma^2/2)(t_i - t) \tag{4}$$

and variance:

$$V = Var\{\ln(G(T))\} = \frac{\sigma^2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min\,(t_i - t, t_j - t).$$

Proofs of these formulas are given in the appendix along with the more compact form in the case when the points t_i are equally spaced over time (weekly or monthly averages).

Following Jarrow and Rudd (1983, pp. 92-95), for the expression in Equation (3)

$$C^G = e^{-r(T-t)}\left\{e^{M+V/2}N\left[\frac{M-\ln(X)+V}{\sqrt{V}}\right]-XN\left[\frac{M-\ln(X)}{\sqrt{V}}\right]\right\} \tag{6}$$

is found, where N is the cumulative, standard normal distribution function. Equation (6) is an explicit expression for the value of a geometric average exchange rate call option.

To value the standard arithmetic average rate call option, one must note that a geometric average is always less than an arithmetic average; i.e. $G(T) \le A(T)$ for all possible outcomes of the exchange rate process $S(t)$. Hence, C^G is a lower bound for the option value C.

Also,

$$\mathrm{Max}\{A(T)-X,0\} \le \mathrm{Max}\{G(T)-X,0\}+A(T)-G(T).$$

Consequently,[6]

$$C \le C^G + e^{-r(T-t)}\{\hat{E}A(T)-\hat{E}G(T)\}, \tag{7}$$

where $\hat{E}A(T)$ and $\hat{E}G(T)$ are shorthand expressions for $\hat{E}(A(T)\,|\,\mathbf{f}(t))$ and $\hat{E}(G(T)\,|\,\mathbf{f}(t))$, respectively. Thus, Equation (7) is the upper bound for C because $\hat{E}A(T)$ and $\hat{E}G(T)$ can be calculated (as shown in the appendix).

$$\hat{E}A(T) = \frac{S(t)}{n}\sum_{i=1}^{n} e^{(r-r_f)(t_i-t)} \tag{8}$$

$$\hat{E}G(T) = e^{M+V/2} \tag{9}$$

The upper bound can be construed as a correction of the theoretical price for the difference between the expectation of the arithmetic average and the geometric average. The proposed approximation purports to correct the exercise price X for the difference between these two expectations; i.e.,

$$C \approx e^{-r(T-t)}\hat{E}[\mathrm{Max}\{G(T)-(X-EA(T)+EG(T)),0\}\mathbf{f}(t)]. \tag{10}$$

Which according to Equation (6), becomes

$$C \approx \tilde{C} = e^{-r(T-t)}\left\{e^{M+V/2}N\left[\frac{M-\ln(X')+V}{\sqrt{V}}\right]-X'N\left[\frac{M-\ln(X')}{\sqrt{V}}\right]\right\} \tag{11}$$

with

$$X' = X-(\hat{E}A(T)-\hat{E}G(T)).$$

Roughly stated, Equation (10) only corrects for the difference in expectations if the option ends up in the money.

It immediately follows from Equation (10) that $\tilde{C}$ also lies between the lower and upper bound. Hence, the approximation error is at most the difference between the upper and lower bound; i.e.,

$$|C - \tilde{C}| \leq e^{-r(T-t)} \{\hat{E}A(T) - \hat{E}G(T)\}.$$

It should be noted that the value of the upper bound is mostly determined by the volatility and the length of the averaging period.

So far the focus of this paper has been on the prices of call options of average exchange rates without considering put options. From the put-call parity of average exchange rate options, put values can be easily approximated. Accordingly,

$$\text{Max}\{X - A(T), 0\} + A(T) = \text{Max}\{A(T) - X, 0\} + X \tag{12}$$

where the first term on the left-hand side of Equation (12) is the payoff, at maturity, of an average exchange rate put option.

The claim $A(T)$ can be replicated by the following strategy: for each averaging point t_i, buy $e^{-r(T-t)}e^{(r-r_f)(t_i-t)}/n$ units of the foreign currency at time t and lend this money at the foreign riskless rate with maturity date, t_i. At date t_i, change the units plus accrued interest into the domestic currency and lend this amount at the domestic riskless rate. The value at time T, including accrued interest, is $S(t_i)/n$. Hence, the total portfolio that replicates $A(T)$, and its initial cost, is

$$e^{-r(T-t)} \frac{S(t)}{n} \sum_{i=1}^{n} e^{(r-r_f)(t_i-t)}$$

Then, the put price P is

$$P = C + e^{-r(T-t)}X - e^{-r(T-t)}\hat{E}A(T) \tag{13}$$

according to Equation (8).

If the put-call parity holds, then the approximation for the call also gives an approximation for the put. One can easily verify that if the value of the put is approximated directly by a formula similar to Equation (10); i.e.,

$$P = e^{-r(T-t)}\hat{E}[\text{Max}\{X - EA(T) + EG(T) \; - G(T), 0\}\mathrm{f}(t)].$$

then identical results to Equation (13) are obtained.

IV. NUMERICAL ANALYSIS

In Table 1 the values for the approximation formula $\tilde{C}$ of the arithmetic average exchange rate call options are given, with a weekly average over 0.5 years and averaging starting after 0.5 months. Accordingly $n = 27$, $t = 0$, $t_1 = 1/24$, $t_i = t_1 + (i - 1)/52$; $T = t_{27} = 1/24 + 1/2$. The spot exchange rate, S_i, is 2, the foreign interest rate, r_f, is 0.08, the domestic interest rate r, has two different values 0.06 and 0.10, the contract is on 10,000 units of the foreign currency and the

Table 1
Bounds and Approximations for Weekly Average Exchange Rate Call Option Values for 10.000 Units of Foreign Currency*

Volatility	*Exercise Price*	*Interest Rate*	*Lower Bound*	*Upper Bound*	*Approx. Value*	*Monte Carlo Value*
0.10	1.9	0.06	918.92	927.32	925.94	925.01 (0.21)
0.10	2.0	0.06	291.66	300.06	295.34	295.13 (0.20)
0.10	2.1	0.06	48.70	57.10	49.61	50.05 (0.16)
0.10	1.9	0.10	1092.18	1100.49	1099.59	1098.89 (0.24)
0.10	2.0	0.10	397.05	405.37	401.54	401.66 (0.24)
0.10	2.1	0.10	79.56	87.88	80.93	82.04 (0.23)
0.20	1.9	0.06	1168.12	1201.45	1190.54	1188.60 (0.88)
0.20	2.0	0.06	626.45	659.78	641.53	641.62 (0.87)
0.20	2.1	0.06	291.26	324.58	299.76	302.25 (0.85)
0.20	1.9	0.10	1302.63	1335.63	1326.32	1324.95 (0.96)
0.20	2.0	0.10	725.67	758.66	742.29	743.07 (0.97)
0.20	2.1	0.10	352.18	385.18	362.03	365.24 (0.96)
0.50	1.9	0.06	2031.73	2238.62	2140.51	2148.18 (6.28)
0.50	2.0	0.06	1576.39	1783.28	1666.44	1679.69 (6.25)
0.50	2.1	0.06	1204.56	1411.45	1277.41	1298.18 (6.24)
0.50	1.9	0.10	2122.63	2327.46	2234.54	2244.52 (6.61)
0.50	2.0	0.10	1657.65	1862.48	1750.99	1766.02 (6.59)
0.50	2.1	0.10	1275.05	1479.89	1351.16	1374.20 (6.60)

Note: $t_1 - t = 1/24$, $t_i - t_{i-1} = 1/52$, $n = 27$; $S(t) = 2$; $r_f = 0.08$
*Standard deviations in parentheses for Monte Carlo values.

contract prices are in units of the second currency. In Table 1 the lower bound, C^G, and the upper bound, as derived by Equation (7) for the average exchange rate options, are also given. Finally, one should note that the Monte Carlo (MC) simulated values for C (where the value of C^G as given by Equation (6) is used as a control variable to reduce the variance of the simulation) are given. This procedure is described in Kemna and Vorst (1990) and is a simulation of the differences in payoffs between the arithmetic average rate option and the geometric average rate option.

Every Monte Carlo simulation is based on 1,000 runs. The standard deviations, as given in Table 1, are very small due to the variance reduction technique. Hence the Monte Carlo

Table 2
Bounds and Approximations for Monthly Average Exchange Rate Call Option Values for 10.000 Units of Foreign Currency.*

Volatility	*Exercise Price*	*Interest Rate*	*Lower Bound*	*Upper Bound*	*Approx. Value*	*Monte Carlo Value*
0.10	1.9	0.06	913.74	923.01	921.53	920.15 (0.24)
0.10	2.0	0.06	283.77	293.04	287.82	287.38 (0.23)
0.10	2.1	0.06	44.96	54.23	45.92	46.43 (0.20)
0.10	1.9	0.10	1088.15	1097.33	1096.38	1095.44 (0.28)
0.10	2.0	0.10	389.18	398.36	394.15	394.16 (0.29)
0.10	2.1	0.10	74.64	83.82	76.11	76.93 (0.26)
0.20	1.9	0.06	1152.65	1189.38	1177.50	1173.62 (1.04)
0.20	2.0	0.06	609.74	646.48	626.34	625.67 (1.04)
0.20	2.1	0.06	277.60	314.33	286.82	288.51 (1.01)
0.20	1.9	0.10	1288.14	1324.51	1314.41	1311.65 (1.14)
0.20	2.0	0.10	709.00	745.37	727.34	727.72 (1.16)
0.20	2.1	0.10	337.63	374.00	348.34	351.27 (1.15)
0.50	1.9	0.06	1982.99	2210.88	2103.17	2107 (7.49)
0.50	2.0	0.06	1528.28	1756.17	1627.36	1642.95 (7.51)
0.50	2.1	0.06	1158.86	1386.74	1238.59	1262.91 (7.47)
0.50	1.9	0.10	2074.49	2300.12	2198.23	2204.18 (7.90)
0.50	2.0	0.10	1609.70	1835.33	1712.53	1729.76 (7.93)
0.50	2.1	0.10	1229.08	1454.70	1312.49	1340.39 (7.92)

Note: $t_1-t = 1/24$, $t_i-t_{i-1} = 1/12$, $n = 7$; $S(t) = 2$; $r_f = 0.08$
*Standard deviations in parentheses for Monte Carlo values.

simulated values can be considered accurate in estimating the true value of the average exchange rate option.

As expected, it can be seen from Table 1 that higher domestic interest rates and higher volatilities lead to higher average rate call option prices. More importantly, the approximation formula gives prices very close to the Monte Carlo simulated values. In almost all cases with volatilities less than 20%, the approximation error is well below 1%; in many cases, even below the 0.1% level. For a volatility of 50%, the approximation is less accurate. Nonetheless, the approximation error is never over 2%.

For the currencies for which these average rate options are traded in the over-the-counter market, the volatility usually is well below 20%. Only for average rate options on commodities

Table 3
Implied Volatilities for the Monte Carlo Simulated (*MC*) in the Approximation Formula ($\tilde{C}$)

Averaging Frequency	*Volatility*	*Exercise Price*	*Approximation*	*Monte Carlo Value*	*Implied Volatility*
weekly	0.10	1.9	1011.93	1011.15	0.0996
weekly	0.10	2.0	345.96	345.95	0.1000
weekly	0.10	2.1	63.74	64.53	0.1004
monthly	0.10	1.9	1008.12	1007.02	0.0993
monthly	0.10	2.0	338.44	338.30	0.1000
monthly	0.10	2.1	59.48	60.11	0.1003
weekly	0.20	1.9	1257.75	1255.95	0.1994
weekly	0.20	2.0	690.86	691.99	0.2001
weekly	0.20	2.1	329.85	332.72	0.2009
monthly	0.20	1.9	1245.25	1242.04	0.1988
monthly	0.20	2.0	675.74	675.61	0.2000
monthly	0.20	2.1	316.51	381.75	0.2008
weekly	0.50	1.9	2187.36	2196.28	0.5028
weekly	0.50	2.0	1708.48	1722.76	0.5043
weekly	0.50	2.1	1314.01	1335.96	0.5066
monthly	0.50	1.9	2150.52	2155.41	0.5016
monthly	0.50	2.0	1669.69	1686.07	0.5050
monthly	0.50	2.1	1275.24	1301.25	0.5080

might volatilities exceed 50%. In the latter case, it is very hard to specify volatility exactly, and as is explained below, this leads to even larger pricing errors.[7]

In Table 2, the prices for average rate options averaged monthly instead of weekly over half a year are presented. Thus, there are seven observations instead of 27. Again, it can be seen that the approximation formula is very accurate. However, due to the larger intervals between two consecutive averaging dates, the standard deviation of the Monte Carlo simulated values is larger.[8]

To further assess the accuracy of the formula, in Table 3 the implied volatility of the *MC* simulated values for the formula $\tilde{C}$ for selected parameters is shown. For example, with $\sigma = 0.1, K = 1.9$ $r = 0.06$ (the first line of Table 2) the implied volatility of the *MC*-value is 0.0996. This means that if in formula (11) the volatility is equal to 0.0996 instead of 0.1, the average rate option price would fall to 1011.15 (the MC simulated value for a volatility of 0.1) from 1011.93 (the approximation value). Hence, a small change in volatility leads to the same error as the approximation formula with respect to the *MC* value.

As is clear from Table 3, in all other cases the implied volatilities are quite close to the actual volatilities. Even in the cases where the volatility is 50%, small deviations in the estimation of the actual volatility lead to larger errors than the approximation formula. For example, a volatility estimate of 51% instead of 50% already gives larger errors. Since accurate estimation of commodity volatilities is difficult, it can be concluded that the results here are quite satisfactory. Given the usual uncertainty about true volatility, the approximation can be considered accurate.

V. EXTENSION TO POINTS IN THE AVERAGING INTERVAL

In the previous section, the value of average rate options where the averaging period has not yet begun was considered. In this section, the more general case (when actual time t lies between t_1 and T) is shown.

Let $t_m \le t < t_{m+1}$ for some $1 \le m < n$. Hence at time t the values of $S(t_1),\ldots,S(t_m)$ are known and revealed in $\mathbf{f}(t)$. Define

$$B(t) = \frac{1}{m}\sum_{i=1}^{m} S(t_i)$$

and

$$D(t,T) = \frac{1}{n-m}\sum_{i=m+1}^{n} S(t_i).$$

$B(t)$ is the average of the first m exchange rates and is known at time t, while $D(t,T)$ is the unknown average over future time periods. It is clear that

$$A(T) = \frac{m}{n}B(t) + \frac{n-m}{n}D(t,T).$$

Hence,

$$\text{Max}\,\{A(T) - X, 0\} = \frac{n-m}{n}\,\text{Max}\,\{D(t,T) - \frac{n}{n-m}(X - \frac{m}{n}B(t)), 0\}.$$

Since none of the values that form the average $D(t,T)$ are known, the factor

$$\text{Max}\,\{D(t,T) - \frac{n}{n-m}(X - \frac{m}{n}B(t)), 0\}.$$

can be viewed as the payoff of an average exchange rate option, where the averaging has yet to begin, with exercise price equal to

$$X'' = \frac{n}{n-m}(X - \frac{m}{n}B(t)).$$

The value of this option can be approximated using the method developed in the previous section and then multiplying the result by $(n-m)/n$ for this general case.

X'' must be strictly positive since otherwise the option has a negative exercise price. If $X'' \le 0$, it follows that $A(T) > X$ with probability one and the value of the call is

$$C = e^{-r(T-t)}\{\hat{E}A(T) - X\} = e^{-r(T-t)}\left\{\frac{m}{n}B(t) + \frac{n-m}{n}\hat{E}(D(t,T)) - X\right\}$$

where $\hat{E}D(t,T))$ is the expectation of an arithmetic average that can be calculated the same way as $\hat{E}A(T)$.

VI. HEDGE RATIOS

As has been stated earlier, a bank that issues options on an over-the-counter basis will not only be interested in a fair value for these options, but also in the hedge ratio, Δ, and the other derivatives such as Γ and Λ. This is in order to assess how a new contract fits into the existing portfolio of over-the-counter and regular currency options.

Applying the results of the previous section to rewrite options that are already running as options for which the averaging period has not yet begun, only the hedge ratio and other derivatives for average rate options for which the averaging period has not yet begun must be found.

Since a riskless hedge for a short call can be established by eliminating the exchange rate risk, the hedge ratio Δ is equal to the derivative of the call price with respect to the exchange rate: $\partial C/\partial S$. (See also Kemna and Vorst 1990). $\partial C/\partial S$ will be approximated by $\partial\tilde{C}/\partial S$. From Equation (11) it follows that

$$\Delta = \frac{\partial\tilde{C}}{\partial S} = \frac{\partial\tilde{C}}{\partial M}\bullet\frac{\partial M}{\partial S} + \frac{\partial\tilde{C}}{\partial X'}\bullet\frac{\partial X'}{\partial S}. \tag{14}$$

Explicit formulas for all factors on the right hand side of Equation (14) can be easily derived:

$$\frac{\partial\tilde{C}}{\partial M} = e^{-r(T-t)}e^{M+1/2}N\left[\frac{M - \ln(X') + V}{\sqrt{V}}\right] \tag{15}$$

$$\frac{\partial M}{\partial S} = \frac{1}{S} = \frac{1}{S(t)} \tag{16}$$

$$\frac{\partial\tilde{C}}{\partial X'} = -e^{-r(T-t)}N\left[\frac{M - \ln(X')}{\sqrt{V}}\right] \tag{17}$$

$$\frac{\partial X'}{\partial S} = -\{\hat{E}A(T) - \hat{E}G(T)\}/S. \tag{18}$$

To calculate Γ, Equation (14) must be differentiated also with respect to S, resulting in

$$\Gamma = \frac{\partial^2\tilde{C}}{(\partial S)^2} = \frac{\partial^2\tilde{C}}{(\partial M)^2}\left[\frac{\partial M}{\partial S}\right]^2 + 2\frac{\partial^2\tilde{C}}{\partial M\partial X'}\left[\frac{\partial M}{\partial S}\frac{\partial X'}{\partial S}\right] + \frac{\partial^2\tilde{C}}{(\partial X')^2}\bullet\left[\frac{\partial X'}{\partial S}\right]^2 + \frac{\partial\tilde{C}}{\partial M}\bullet\frac{\partial^2 M}{\partial S^2}$$

since $\partial^2X'/\partial M^2 = 0$. The factors not already described in (15)–(18) are $\partial^2\tilde{C}/\partial M^2$, $\partial^2\tilde{C}/\partial M\partial X'$, $\partial^2\tilde{C}/(\partial X')^2$ and $\partial^2 M/\partial S^2$. It is straightforward to find explicit expressions for these factors by further differentiating equations (15)–(18). The same method can be applied to find Λ and Θ.

Table 4
Hedge Ratios Calculated Monte Carlo Simulation and Analytic Approximation

Volatility	*Excercise Price*	*Interest Rate*	*Monthly Monte Carlo*	*Monthly Approx. Value*	*Weekly Monte Carlo*	*Weekly Approx. Value*
0.10	1.9	0.06	0.8207	0.8211	0.8185	0.8149
0.10	2.0	0.06	0.4392	0.4384	0.4405	0.4379
0.10	2.1	0.06	0.1087	0.1080	0.1138	0.1166
0.10	1.9	0.10	0.8625	0.8641	0.8579	0.8568
0.10	2.0	0.10	0.5346	0.5324	0.5337	0.5323
0.10	2.1	0.10	0.1636	0.1627	0.1688	0.1707
0.20	1.9	0.06	0.6848	0.6834	0.6814	0.6776
0.20	2.0	0.06	0.4728	0.4686	0.4337	0.4706
0.20	2.1	0.06	0.2726	0.2717	0.2773	0.2775
0.20	1.9	0.10	0.7192	0.7167	0.7153	0.7122
0.20	2.0	0.10	0.5181	0.5147	0.5180	0.5131
0.20	2.1	0.10	0.3139	0.3161	0.3179	0.3178
0.50	1.9	0.06	0.5886	0.6007	0.5959	0.5999
0.50	2.0	0.06	0.5036	0.5126	0.5090	0.5139
0.50	2.1	0.06	0.4235	0.4273	0.4280	0.4305
0.50	1.9	0.10	0.6100	0.6139	0.6084	0.6127
0.50	2.0	0.10	0.5174	0.5278	0.5206	0.5285
0.50	2.1	0.10	0.4392	0.4433	0.4383	0.4460

Note: $t_1 - t = 1/24$, $S(t) = 2$;, $r_f = 0.08$

The hedge ratio as given by formula (14) is the derivative of the approximation formula. Thus (14) is at best an approximation of the true hedge ratio. However, since there is no analytic expression for the average rate option price, an analytic expression for the true hedge ratio cannot be found.

The hedge ratio can be calculated by Monte Carlo simulation using the same set of paths for two values of the exchange rate.[9] In Table 4 the hedge ratios by formula (14) and the finite difference approximation of the Monte Carlo simulated values are given. The last values are obtained by calculating the difference between the Monte Carlo values for $S = 2.001$ and $S = 1.999$ and dividing this difference by 0.002.

From Table 4, it is evident that Equation (14) is a good approximation of the hedge ratio and that differences occur only in the third decimal. Therefore, a position in an average rate call option can be accurately hedged with formula (14). Also, there is no reason to assume that one cannot expect the same accuracy for other values, such as Λ and Θ.

VII. CONCLUSIONS

In this paper, the valuation of options based on average exchange rates is studied. Using the standard Black-Scholes model, it is shown that there is no analytic expression for the value of these options. This is largely due to the fact that averages of lognormally distributed variables are not lognormally distributed. Hence, numerical techniques must be used to approximate the price of these options. By comparing the arithmetic average of the exchange rates with the geometric average, analytic lower and upper bounds for the value of an average exchange rate call option can be derived. An accurate approximation of the option value can be found. Analytic approximations of the hedge ratio and other implied variables are also derived.

The methods discussed in this paper are not only applicable to options based on average exchange rates, but also for averaging over other security or commodity prices. Further, the proposed approximation procedure can be extended to various kinds of options; for example, an option which depends on the difference between the average and the final spot price.

APPENDIX

Derivation of Equations (4), (5) and (8)

It follows from Equation (1) (see Arnold 1974) that $\{\ln(S(t_1)/S(t)), \ldots, \ln(S(t_n)/S(t))\}^T$ is multivariate normally distributed with mean $\{(r - r_f - \sigma^2/2)(t_n - t)\}^T$ and covariance matrix

$$\sigma^2 \times \begin{bmatrix} t_1 - t & t_1 - t & \cdots & t_1 - t \\ t_1 - t & t_2 - t & \cdots & t_2 - t \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ t_1 - t & t_2 - t & \cdots & t_n - t \end{bmatrix}$$

$$\ln(G(T)) = \frac{1}{n} \sum_{i=1}^{n} \ln(S(t_i))$$

and hence

$$\hat{E}\{\ln(G(T))\} = \ln(S(t)) + \frac{1}{n}(r - r_f - \sigma^2/2) \sum_{i=1}^{n} (t_i - t) \tag{4}$$

and

$$Var \ln(G(T)) = \frac{\sigma^2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min(t_i - t, t_j - t). \tag{5}$$

It also follows from the normality of $\ln(S(t_i))$ and the value of its mean and variance that $ES(t_j) = e^{(r-r_f)(t_j-t)}S(t)$. This leads to Equation (8).

Compact Expressions for M and V

It is now assumed that the t_i are equally spaced over time. This means that there is some $h > 0$ such that $h = t_{i+1} - t_i$ for all i. Hence $h = (T - t_1)/(n - 1)$.

An example is a weekly average, where $h = 1/52$, or a monthly average where $h = 1/12$. To be exact all months should have the same number of days. This is clearly not the case. However the formula is merely an approximation, and therefore this small aberration of months with different numbers of days is negligible. Also, in the case of weekly averages, one should account for holidays. Once again this does not seem to lead to serious differences. For daily averages, h should be 1 over the number of trading days within a year; however, because of weekends, not all trading days are equally spaced over time.

If these small discrepancies can be neglected, (4), (5) and (8) can be given, using summation formulas, in the more compact forms:

$$M = \ln(S(t)) + (r - r_f - \sigma^2/2)\{(t_1 - t) + (T - t_1)/2\} \tag{A1}$$

$$V = \sigma^2\{(t_1 - t) + (T - t_1)(2n - 1)/6n\} \tag{A2}$$

$$\hat{EA}(T) = \frac{S(t)}{n} E^{(r-r_f)(t_1-t)} \left\{\frac{1 - e^{(r-r_f)nh}}{1 - e^{(r-r_f)h}}\right\} \tag{A3}$$

where the last formula is equal to $S(t)$ if $r = r_f$.

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Notes

1. Examples are Banque Indosuez and Citibank.
2. AB Svensk Exportkredit, a Swedish export credit corporation, issued average exchange rate options of the Duetsche mark against the dollar, where the average is taken over the first fixings on all trading days during a year. They have a similar contract for the yen against the dollar.
3. Eric Reiner (1991) and Marc Yor (1992) have developed explicit solutions for options on averages with continuous fixings in terms of Laplace transformations. In this paper, by contrast, discrete fixings are considered. Furthermore, to explicitly calculate option values using their formulae requires numerical approximations of complex integrals.
4. This implies that a Black-Scholes type of formula for average rate options does not exist. As stated in Note 3, explicit solutions for the continuous average case exist.
5. These options are also discussed by Conze and Viswanathan (1991a)
6. Conze and Viswanathan (1991b) derive similar results.
7. The approximation is compared with the Monte Carlo simulated value, which is itself only an approximation. However, given the very small standard deviations, it can be stated that the Monte Carlo simulated values are accurate, especially for volatilities less than 20%. The advantage of the approximation is that it can be easily calculated, even with a simple spreadsheet, in contrast to the Monte Carlo simulated value.

8. It is well known that the value of an average rate option is always smaller than the value of a comparable standard option. From Tables 1 and 2, it can be inferred that if more averaging points in the same time period are taken, the option price is increasing. This seems to contradict the fact that averaging leads to a lower option value. To understand this occurrence one has to split the effect of averaging in two parts. First, because of the averaging of prices of the underlying asset, earlier dates are taken into account, and these typically have a lower volatility. This reduces the volatility of the average. Second, if the first day in the averaging is fixed,with more observations in the averaging, these first dates have a lower weight and hence a higher volatility for the average results.

9. The same set of paths means with the same relative changes in exchange rates that differ only in their starting exchange rate. Hence, in an absolute sense, the paths are completely different.

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