Lab Week 3 - Delta Hedging

For a positive integer n, we denote $\Delta \mathrm{T} := 2^{-n}\mathrm{T}$, $t_i^n = i\Delta\mathrm{T}$, $i = 0, 1, \ldots, 2^n$. Consider a Brownian motion $\mathrm{W}(t)$. Recall that $\mathrm{W}(t)$ is normally distributed with mean 0 and variance t. Thus, in distribution $\mathrm{W}(t)$ is same as a standard Gaussian random variable $\mathrm{X}\sqrt{t}$. We know that the solution of the Black-Scholes model with drift parameter μ and volatility $\sigma > 0$ is given as

$$\mathrm{S}(t) = \mathrm{S}(0)\mathrm{e}^{\left(\mu-\frac{1}{2}\sigma^2\right)t+\sigma\mathrm{W}(t)}, \quad t \geq 0.$$

Under the Black-Scholes model, the price of a European call option with maturity T and strike K is given as

$$\mathrm{C}(t, \mathrm{S}(t), \mathrm{T}, \mathrm{K}) = \mathrm{S}(t)\Phi\Big(d_1(t, \mathrm{K})\Big) - \mathrm{K}\mathrm{e}^{-r(\mathrm{T}-t)}\Phi\Big(d_2(t, \mathrm{K})\Big),$$

$$d_1(t, \mathrm{K}) = \frac{\ln\left(\frac{\mathrm{S}(t)}{\mathrm{K}\mathrm{e}^{-r(\mathrm{T}-t)}}\right) + \frac{1}{2}\sigma^2(\mathrm{T}-t)}{\sigma\sqrt{\mathrm{T}-t}},$$

$$d_2(t, \mathrm{K}) = \frac{\ln\left(\frac{\mathrm{S}(t)}{\mathrm{K}\mathrm{e}^{-r(\mathrm{T}-t)}}\right) - \frac{1}{2}\sigma^2(\mathrm{T}-t)}{\sigma\sqrt{\mathrm{T}-t}}.$$

In the above r is the risk-free rate of interest and $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

As seen in the lecture, the optimal hedging strategy for European call option under the Black-Scholes model suggests to hold $\Delta(t)$ shares of the underlying asset (stock) at each time t where

$$\Delta(t, \mathrm{K}) = \Phi(d_1(t, \mathrm{K})).$$

1. Write a program which produces a sample of $\mathrm{N} = 100$ copies of the discrete path $\{\mathrm{S}(t_i^n), i = 0, 1, \ldots, 2^n\}$. Take $\mathrm{T} = 1, \mathrm{S}(0) = 100, \sigma = 0.3, r = 0.05, \mu = 0.02$ and $n = 5, 8, 10$. Comment on the results about the choice of n.

2. Next, denote the discounted value of the hedging portfolio at time T by $\mathrm{e}^{-r\mathrm{T}}\mathrm{X}^n(\mathrm{T}, \mathrm{K})$ which is given as below

$$\mathrm{e}^{-r\mathrm{T}}\mathrm{X}^n(\mathrm{T}, \mathrm{K}) := \mathrm{C}(0, \mathrm{S}(0), \mathrm{T}, \mathrm{K}) + \sum_{i=1}^{2^n} \Delta(t_{i-1}^n, \mathrm{K})\Big(\mathrm{e}^{-rt_i^n}\mathrm{S}(t_i^n) - \mathrm{e}^{-rt_{i-1}^n}\mathrm{S}(t_{i-1}^n)\Big).$$

The first term denotes the option price received by selling one unit of it. The second sum denotes the change in the value of the stock's holding given the readjustments made in Δ at time $t_i^n, i = 0, 1, \ldots, 2^n - 1$.

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(a) Simulate a sample of $N = 1000$ copies of $X^n(T, K)$. Use different values of $K \in \{100 \pm i, i = 0, 1, 2, \ldots, 20\}$ and $n = 5, 8, 10$.

(b) Compute the corresponding Profit and Loss (PnL)

$$PL^n(T, K) := X^n(T, K) - \max(S(T) - K, 0).$$

(OPTIONAL) For each value of $K \in \{100 \pm i, i = 0, 1, 2, \ldots, 20\}$ and $n = 5, 8, 10$, compute the sample mean and variance of $PL^n(T, K)$ and provide the corresponding plots with respect to the number of steps n and the strike K.

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