Lab Week 10 - Local volatility and the Dupire formula

We denote by $C(T;k)$ the time price of a European call option with maturity T and strike K. We shall denote C_T, C_K and C_{KK} the corresponding partial derivative with respect to T, with respect to K, and the second partial derivative with respect to K, respectively. We recall the Dupire formula

$$\sigma^2(T,K) = 2\frac{C_T(T,K) + rKC_K(T,K)}{K^2 C_{KK}(T,K)}. \tag{1}$$

In terms of the implied volatilities $I(T,K)$, we can obtain the inversion of the Black-Scholes formula C^{BS}, that is, $C(T,K) = C^{BS}(T,K,I(T,K))$. The Dupire formula then reduces to

$$\sigma^2(T,K) = \frac{\frac{I}{T} + 2I_T + 2rKI_K}{K^2\left(\frac{1}{K^2 IT} + 2\frac{d_+}{KI\sqrt{T}}I_K + \frac{d_+ d_-}{I}I_K^2 + I_{KK}\right)}, \tag{2}$$

where $d_\pm$ is the standard functions used in the Black-Scholes formula, and the subscripts indicate again the partial derivatives with respect to T and K.

The file `optionprices.txt` contains call prices $C(T_i,K_j)$ for a spot price $S_0 = 100$, spot interest rate $r = 0$, maturities $T_i := i\,2^{-n}T, n = 8, T = 0.9, i = 0,\dots,2^n$, and strikes $K_j := 80 + j10^{-1}, j = 0,\dots,400$.

1. Provide an approximation of the Dupire local volatility function by using the Dupire formula (1). Comment on the encountered numerical difficulties, if any.

2. We next turn to an alternative approximation method of the Dupire local volatility function.

   (a) Deduce from the provided data the corresponding implied volatilities $I(T_i,K_j), i = 0,\dots,2^n$ and $j = 0,\dots,400$.

   (b) Provide an alternative approximation $\hat{\sigma}^2(T_i,K_j)$ of the Dupire local volatility function by using the Dupire formula (2).

   (c) Build a MATLAB function which produces a linear interpolation in the variables (T,K) of the points $T_i\hat{\sigma}^2(T_i,K_j), i = 0,\dots,2^n$ and $j = 0,\dots,400$.

(OPTIONAL) We finally verify numerically the validity of the Dupire formula. Consider the local volatility model $d\hat{S}(t) = \hat{S}(t)\hat{\sigma}(t,S(t))dB(t)$ where B is a standard Brownian motion under the risk-neutral measure $\hat{\mathbb{Q}}$.

   (a) By using an Euler discretization scheme for the process $\hat{S}$, provide Monte Carlo approximations of $\hat{C}(T_i,K_j) := \mathbb{E}[(\hat{S}(T_i) - K_j)^+], i = 0,\dots,2^n$ and $j = 0,\dots,400$.

   (b) Compare the data $\hat{C}(T_i,K_j)$ to the initial data $C(T_i,K_j)$.

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