COMP3143
Data Structures and Algorithms

AVL-Trees (Part 1: Single Rotations)

Balance Binary Search Tree

- Worst case height of binary search tree: N-1
  - Insertion, deletion can be O(N) in the worst case
- We want a tree with small height
- Height of a binary tree with N node is at least $\Theta(\log N)$
- Goal: keep the height of a binary search tree O(log N)
- Balanced binary search trees
  - Examples: AVL tree, red-black tree

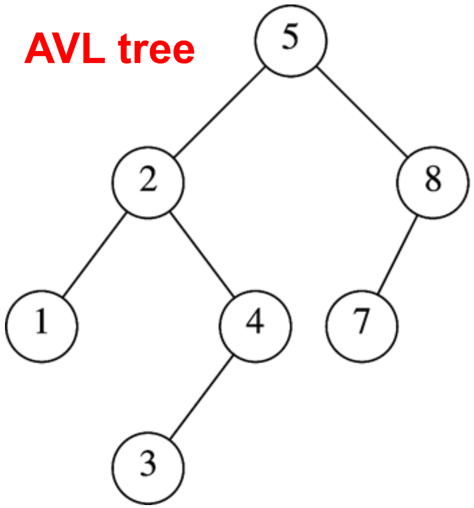
Balanced Tree?

- Suggestion 1: the left and right subtrees of root have the same height
  - But the left and right subtrees may be linear lists!
- Suggestion 2: every node must have left and right subtrees of the same height
  - Only complete binary trees satisfy
  - Too rigid to be useful
- Our choice: for each node, the height of the left and right subtrees can differ at most 1

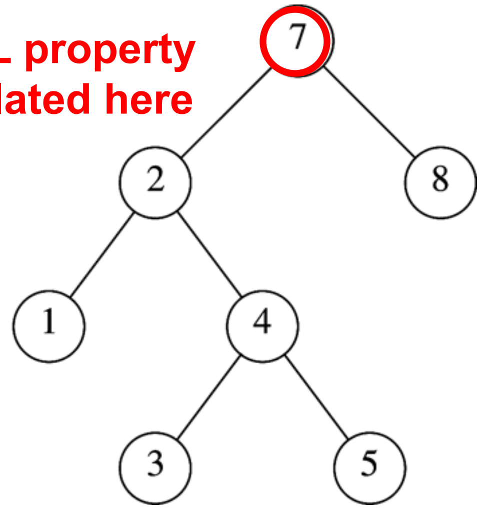
AVL Tree

- An AVL tree is a binary search tree in which
  - for *every* node in the tree, the height of the left and right subtrees differ by at most 1.
- Height of subtree: Max # of edges to a leaf
- Height of an empty subtree: -1
  - Height of one node: 0

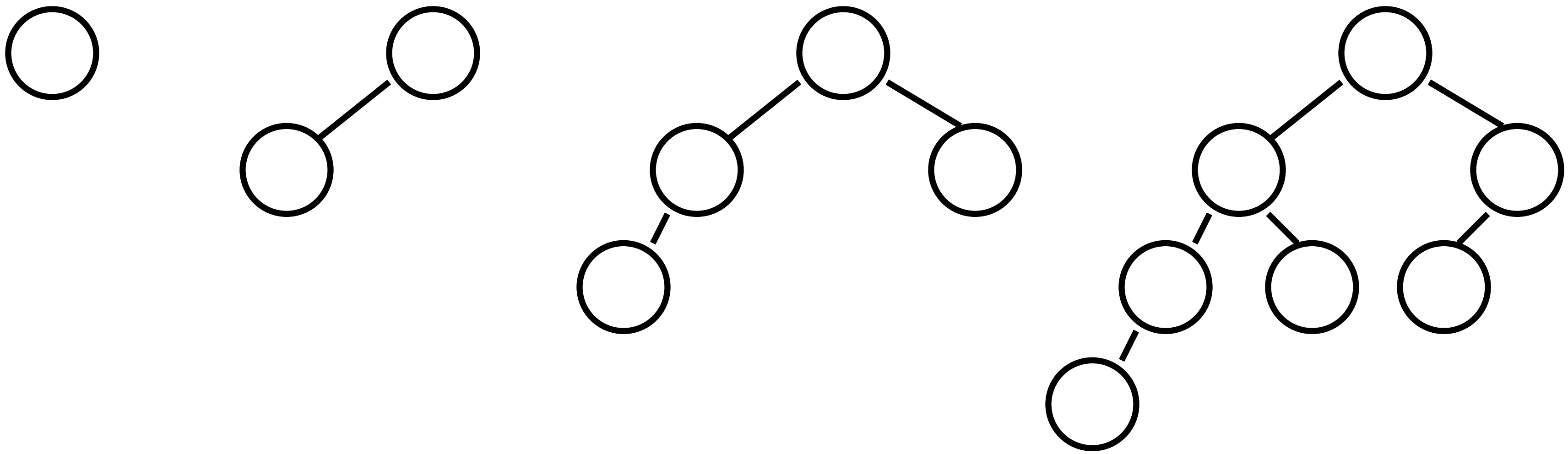
AVL tree



AVL property violated here



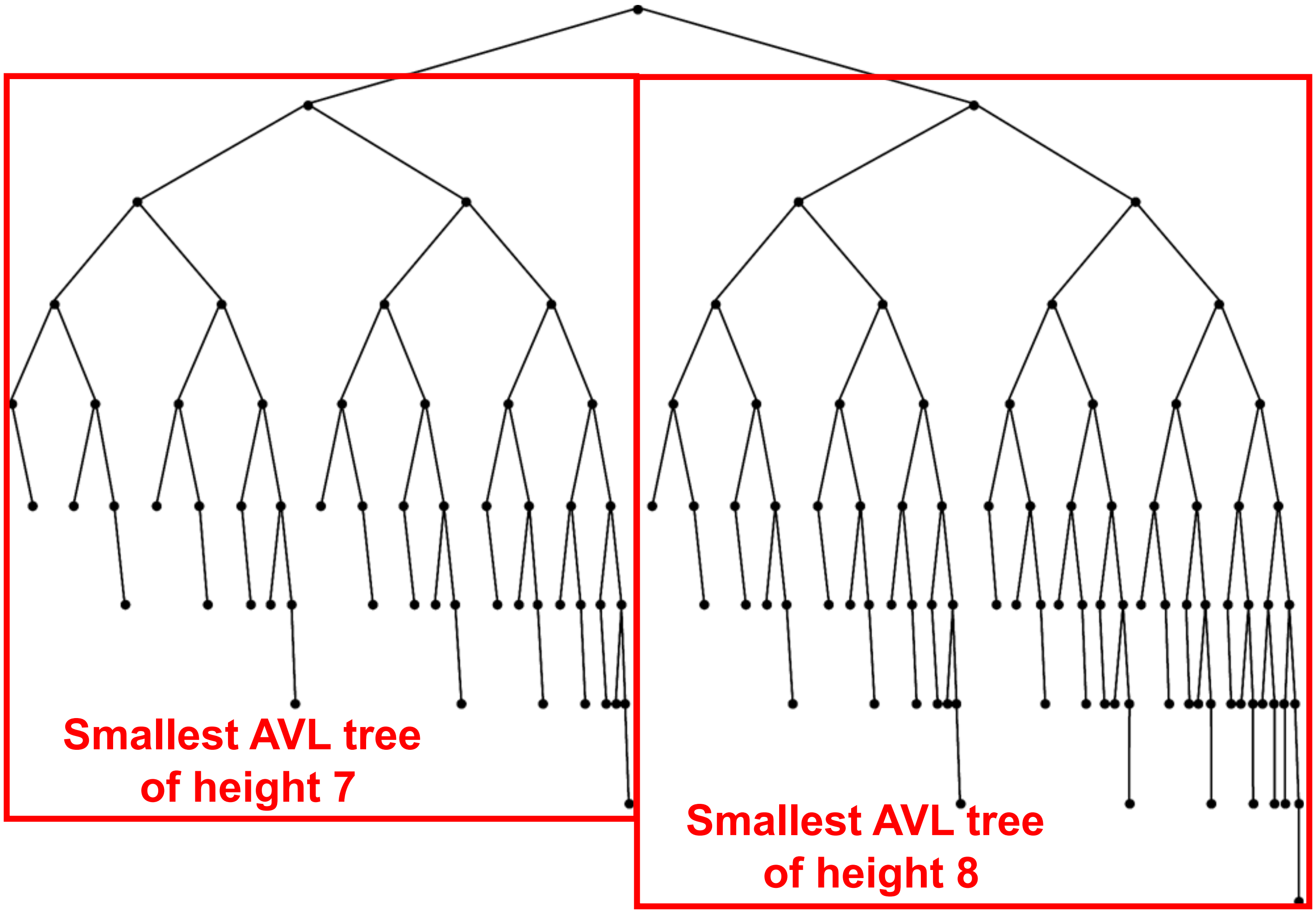
AVL Tree with Minimum Number of Nodes



$N_0 = 1$ $N_1 = 2$ $N_2 = 4$ $N_3 = N_1 + N_2 + 1 = 7$

height of left=?

Height right=?



Smallest AVL tree of height 9

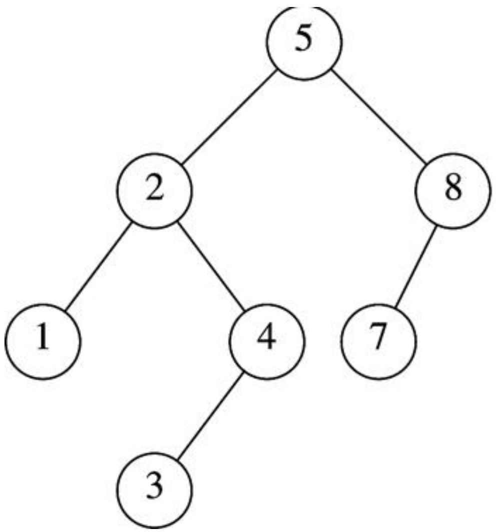
Height of AVL Tree

- Denote N_h the minimum number of nodes in an AVL tree of height h
- $N_0=0$, $N_1=2$ (base)
  $N_h= N_{h-1} + N_{h-2} +1$ (recursive relation)
- $N > N_h= N_{h-1} + N_{h-2} +1$

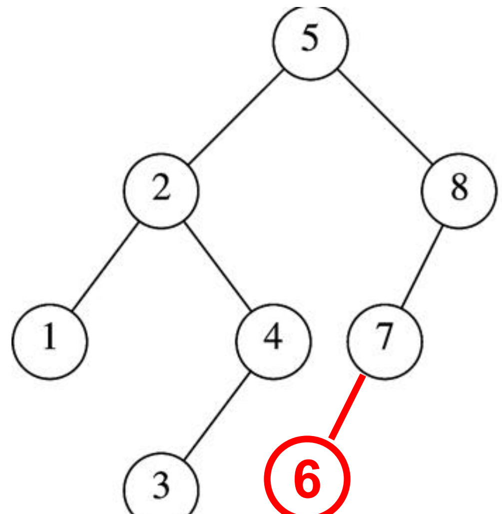
  $>2 N_{h-2} >4 N_{h-4} >\ldots>2^i N_{h-2i}$
- If h is even, let $i=h/2-1$. The equation becomes $N>2^{h/2-1}N_2$
  $\Rightarrow N>2^{h/2-1}x4 \Rightarrow h=O(\log N)$
- If h is odd, let $i=(h-1)/2$. The equation becomes $N>2^{(h-1)/2}N_1$
  $\Rightarrow N>2^{(h-1)/2}x2 \Rightarrow h=O(\log N)$
- Thus, many operations (i.e. searching) on an AVL tree will take O(log N) time

Insertion in AVL Tree

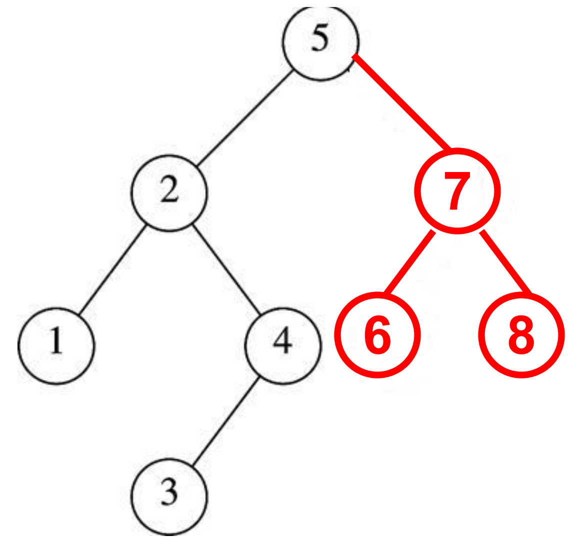
- Basically follows insertion strategy of binary search tree
  - But may cause violation of AVL tree property
- Restore the destroyed balance condition if needed



Original AVL tree



Insert 6
Property violated

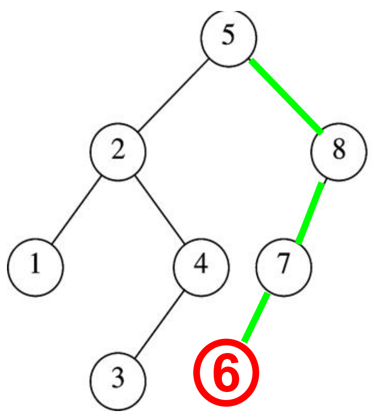


Restore AVL property

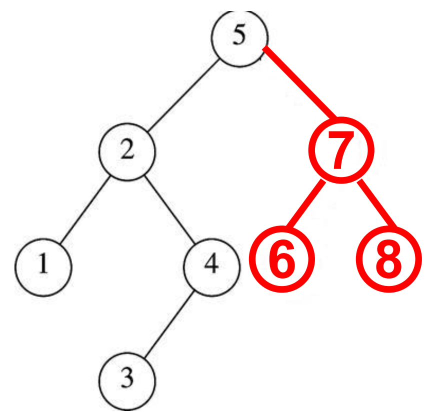
Some Observations

- After an insertion, only nodes that are on the path from the insertion point to the root might have their balance altered
  - Because only those nodes have their subtrees altered

- Rebalance the tree at the deepest such node guarantees that the entire tree satisfies the AVL property



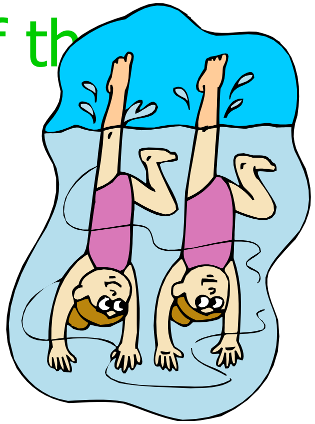
Node 5,8,7 might
have balance altered



Rebalance node 7
guarantees the whole tree be AVL

Different Cases for Rebalance

- Denote the node that must be rebalanced α
  - Case 1: an insertion into the left subtree of the left child of α
  - Case 2: an insertion into the right subtree of the left child of α
  - Case 3: an insertion into the left subtree of the right child of α
  - Case 4: an insertion into the right subtree of the right child of α
- Cases 1&4 are mirror image symmetries with respect to α, as are cases 2&3



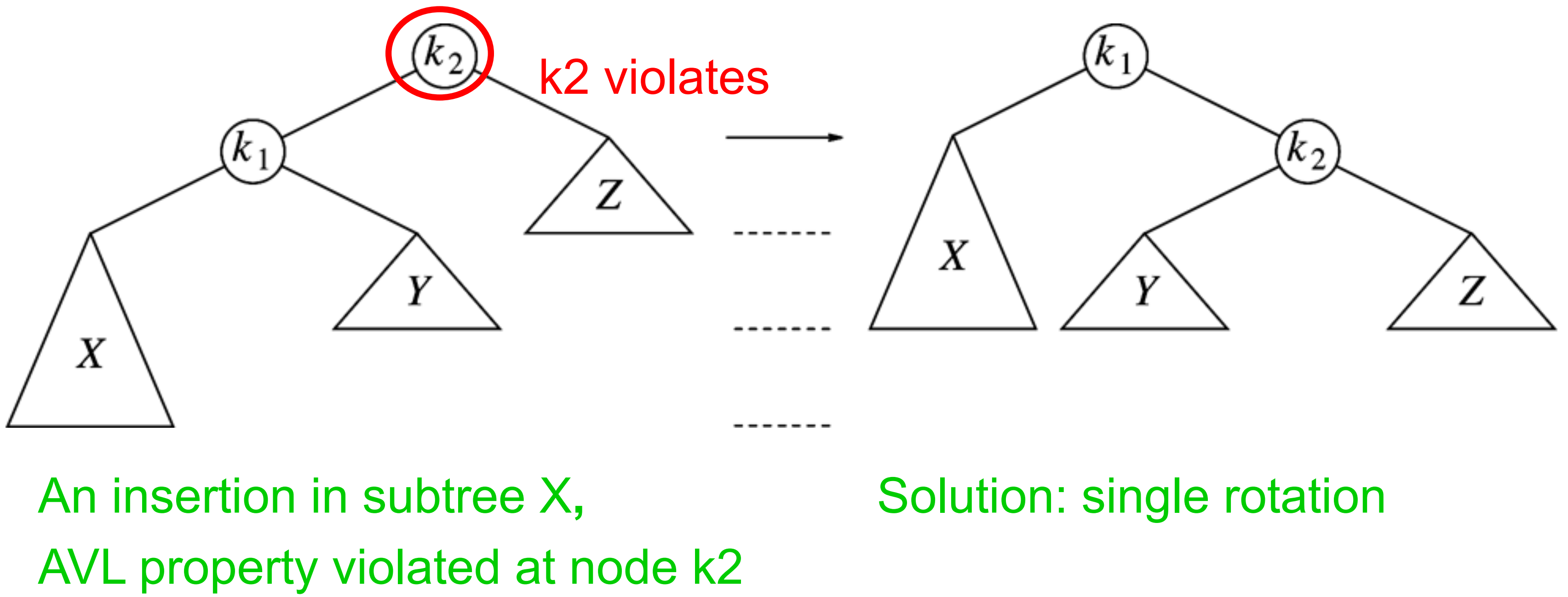
Rotations

- Rebalance of AVL tree are done with simple modification to tree, known as rotation
- Insertion occurs on the "outside" (i.e., left-left or right-right) is fixed by single rotation of the tree
- Insertion occurs on the "inside" (i.e., left-right or right-left) is fixed by double rotation of the tree

Insertion Algorithm

- First, insert the new key as a new leaf just as in ordinary binary search tree
- Then trace the path from the new leaf towards the root. For each node x encountered, check if heights of left(x) and right(x) differ by at most 1
  - If yes, proceed to parent(x)
  - If not, restructure by doing either a single rotation or a double rotation
- Note: once we perform a rotation at a node x, we won't need to perform any rotation at any ancestor of x.

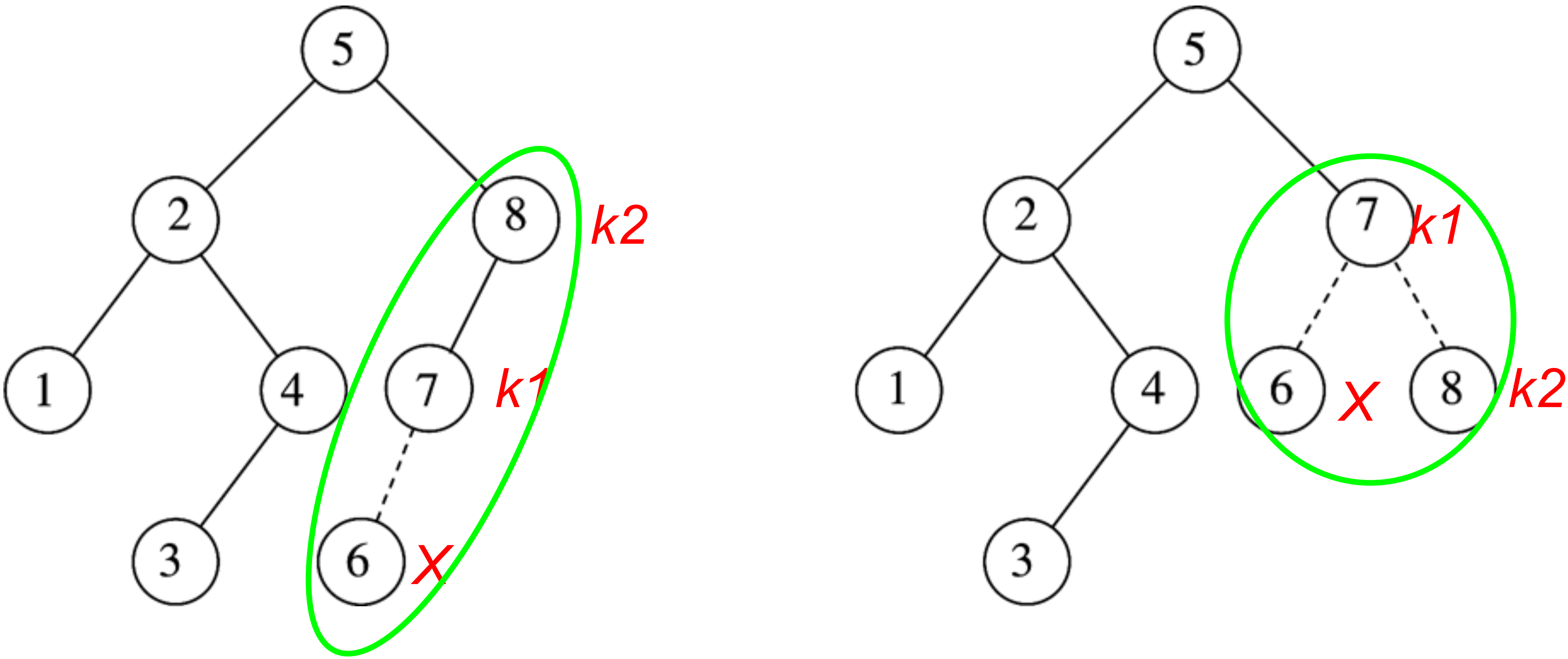
Single Rotation to Fix Case 1(left-left)



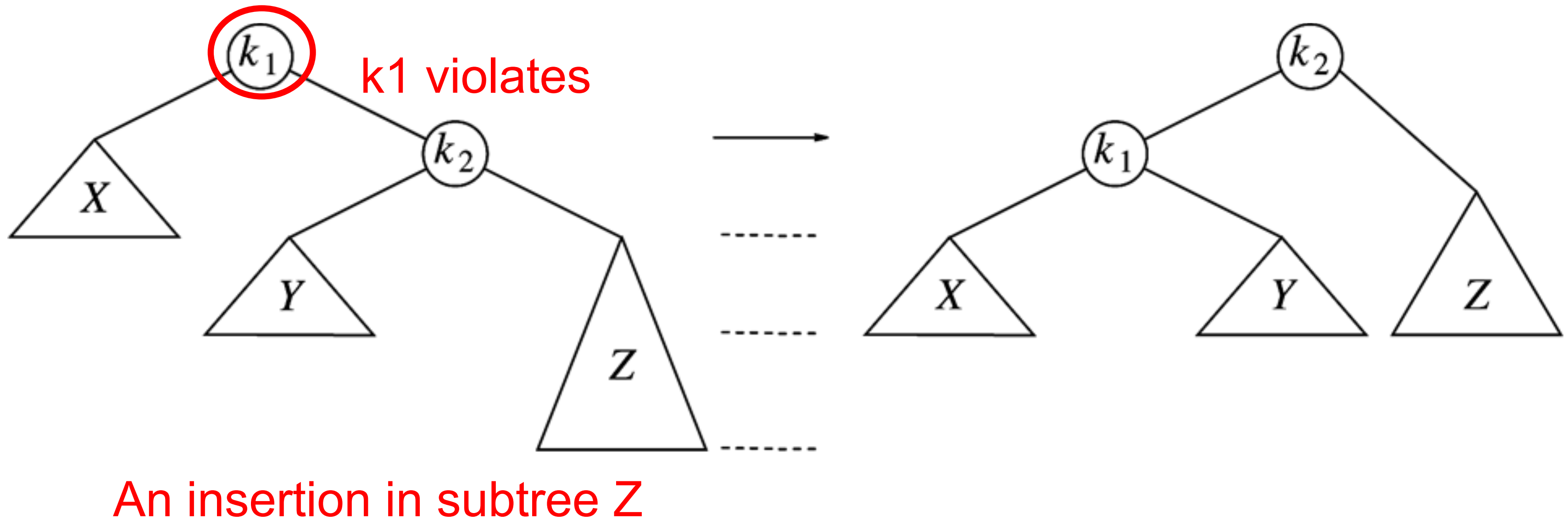
AVL-property quiz:

1. Can Y have the same height as the new X?
2. Can Y have the same height as Z?

Single Rotation Case 1 Example



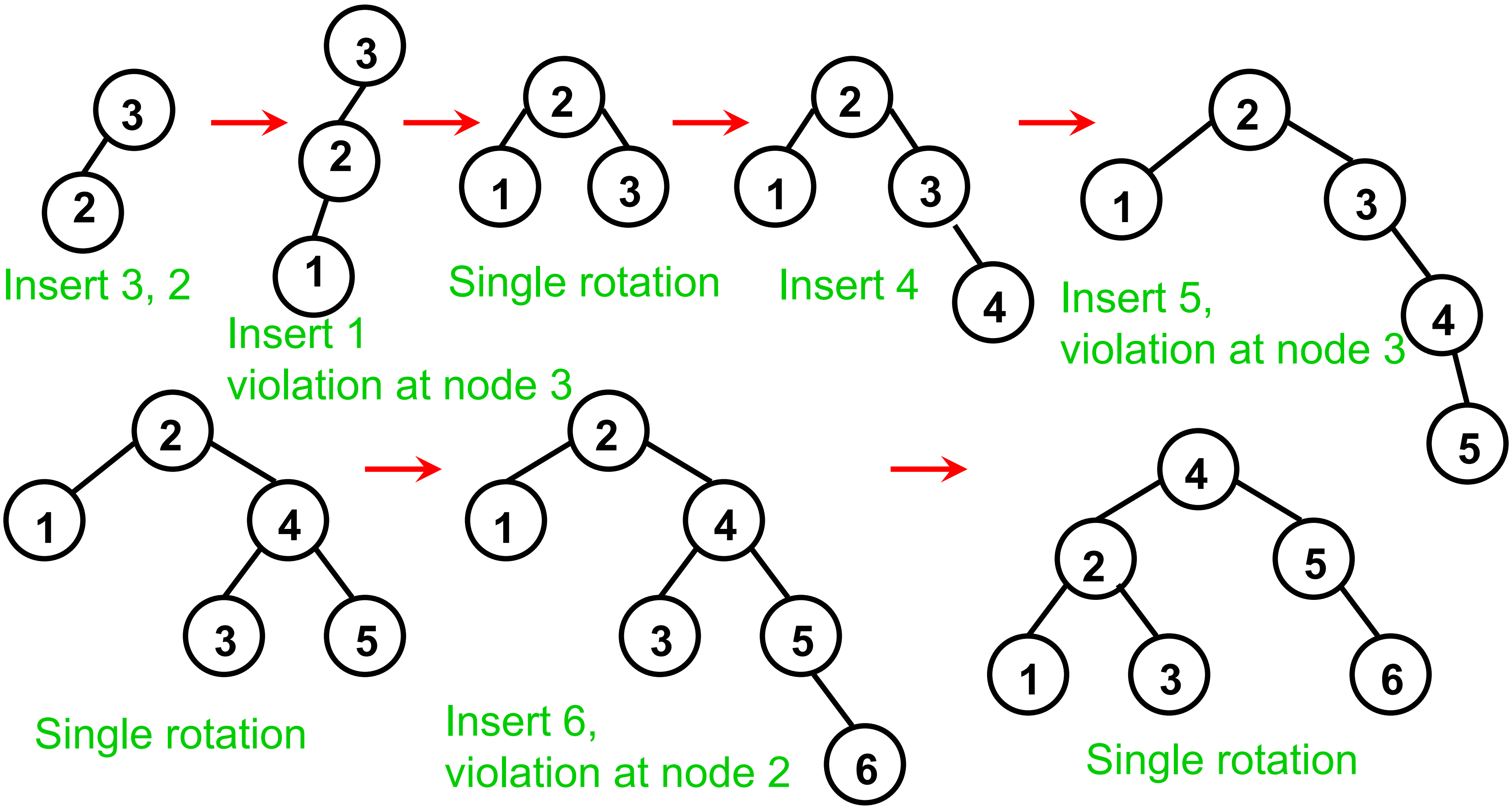
Single Rotation to Fix Case 4 (right-right)



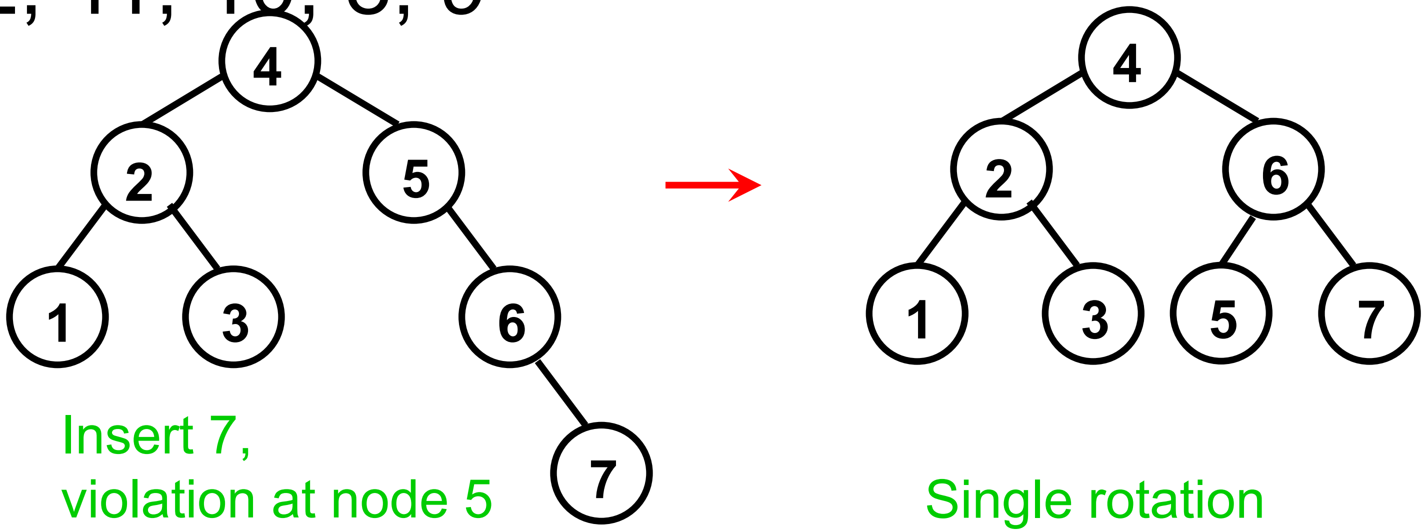
- Case 4 is a symmetric case to case 1
- Insertion takes O(Height of AVL Tree) time, Single rotation takes O(1) time

Single Rotation Example

- Sequentially insert 3, 2, 1, 4, 5, 6 to an AVL Tree

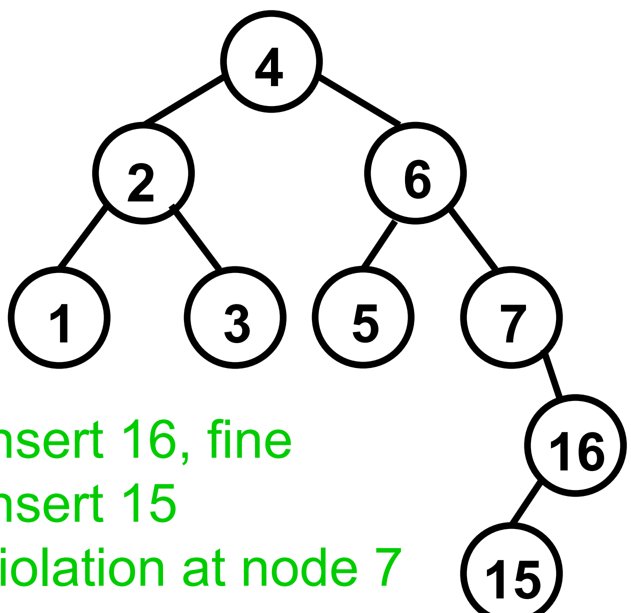


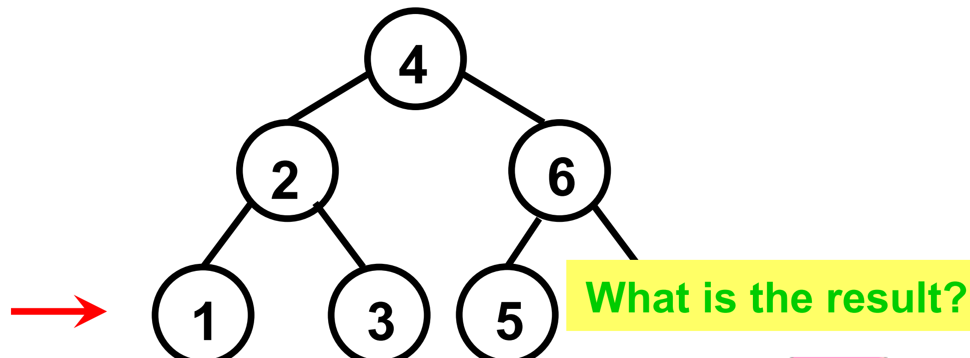
- If we continue to insert 7, 16, 15, 14, 13, 12, 11, 10, 8, 9



Insert 7,
violation at node 5

Single rotation

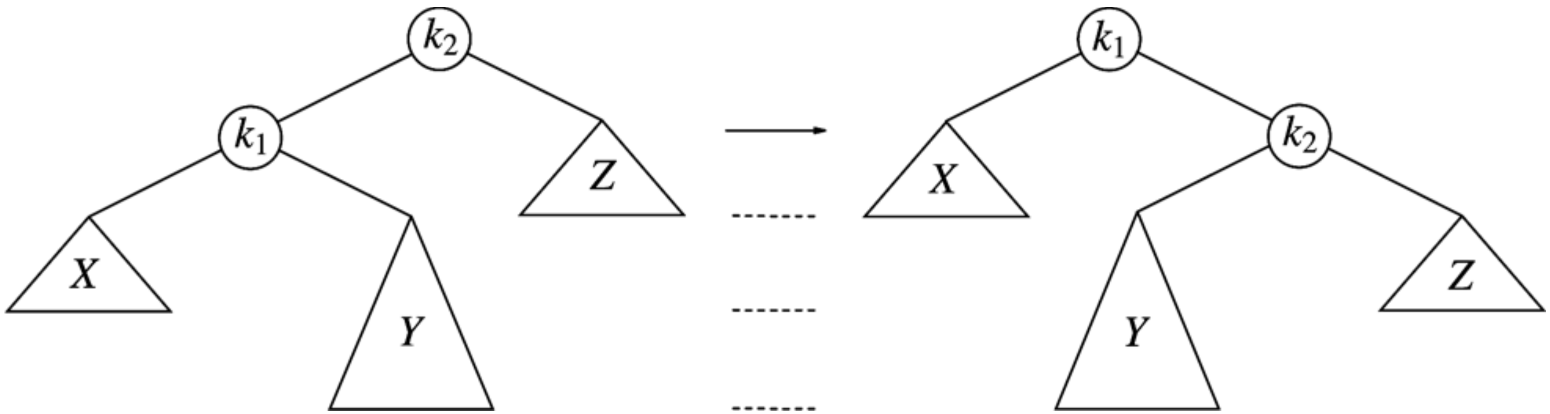




Insert 16, fine
Insert 15
violation at node 7

Single rotation
But….
Violation remains

Single Rotation Fails to fix Case 2&3



Case 2: violation in k2 because of insertion in subtree Y

Single rotation result

- Single rotation fails to fix case 2&3
- Take case 2 as an example (case 3 is a symmetry to it)
  - The problem is subtree Y is too deep
  - Single rotation doesn't make it any less deep

Single Rotation Fails

- What shall we do?
- We need to rotate twice
  - Double Rotation