02 July 2021 10:38

Q1: $X_t = \int^t s \cdot dW_s$ where W is a Brownian motion.

↳ This notation is used in all the questions.

Write an expression for dX_t.

$dX_t = t\,dW_t$

Q2: Apply Itô's formula to

$$X_t = t^2 W_t - 2\int_0^t s\,W_s\,ds$$

and show that X_t is a martingale.

- $A = \underbrace{t^2 W_t}_{t}$
- $B = \int_0^t s\,W_s\,dW_s$

$A: d(t^2 W_t) = 2t W_t\,dt + t^2 dW_t + (\;)\underbrace{dt \cdot dW_t}_{=0}$

$= 2t W_t\,dt + t^2 dW_t$

$B: d\left(\int_0^t s W_s\,ds\right) = t W_t\,dt$

$dX_t = d(t^2 W_t) - 2d\left(\int_0^t s W_s\,ds\right)$

$= \cancel{2t W_t\,dt} + t^2 dW_t - \cancel{2t W_t\,dt}$

$\boxed{dX_t = t^2 dW_t}$ $\quad X_t = \int_0^t s^2 dW_s$

$f(x) = g(x) + h(x)$

$df(x) = dg(x) + dh(x)$

Q3: Apply Itô's formula to

$$\boxed{X(t) = W_1(t)\,W_2(t)}$$

and show that $X(t)$ is a martingale. W_1 & W_2 are two independent Brownian motions

$dX(t) = W_2(t)\,dW_1(t) + W_1(t)\,dW_2(t)$

$\quad + dW_1(t) \cdot dW_2(t)$

$dW_1(t) \cdot dW_2(t) = 0$

$= W_2(t)\,dW_1(t) + W_1(t)\,dW_2(t)$

$X(t) = \int_0^t W_2(s)\,dW_1(s) + \int_0^t W_1(s)\,dW_2(s)$

$Y = aX_1 + bX_2$ $\quad X_1, X_2$: random variable; a, b: constants

- $E[Y] = E[aX_1 + bX_2] = E[aX_1] + E[bX_2]$

$$= a\mathbb{E}[X_1] + b\,\mathbb{E}[X_2]$$

\# $Y = aX_1$

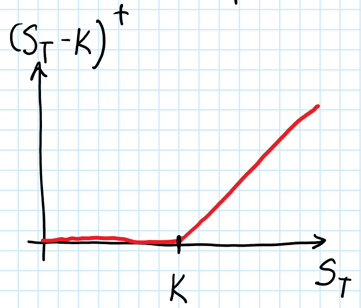
$$\text{var}(Y) = a^2\,\text{var}(X_1)$$

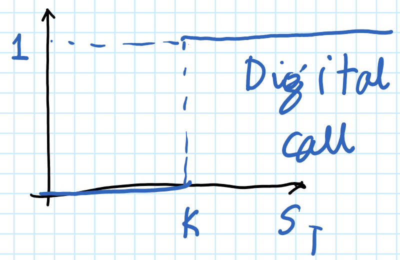
\# $Y = aX_1 + bX_2$

$$\text{var}(Y) = a^2\,\text{var}(X_1) + b^2\,\text{var}(X_2) + 2ab\,\text{cov}(X_1, X_2)$$

call option payoff $= \max(S_T - K, 0)$

digital call option payoff $= \begin{cases} 1, & S_T \geq K \\ 0, & \text{otherwise} \end{cases}$





$$C(0) = \mathbb{E}\left[e^{-rT}(S_T - K)^+\right] \quad \text{(risk-neutral pricing)}$$

$$0 = \mathbb{E}\left[e^{-rT}\left((S_T-K)^+ - P\right)\mathbb{1}_{\{S_T \geq K\}}\right]$$

$$= \underbrace{\mathbb{E}\left[e^{-rT}(S_T-K)^+\mathbb{1}_{\{S_T \geq K\}}\right]} - \mathbb{E}\left[e^{-rT}P\,\mathbb{1}_{\{S_T \geq K\}}\right]$$

$$0 = \text{call}(T, K) - P\,\text{Digital}(T, K)$$

$$\Rightarrow \boxed{P = \frac{\text{call}(T,K)}{\text{Digital}(T,K)}}$$

$P \geq \text{call}(T,K)$?

$\text{Digital}(T,K) \leq 1$

Pay-later option:

Payoff: $\begin{cases} (S_T-K)^+ - P, & \text{if } S_T \geq K \\ 0, & \text{if } S_T < K \end{cases}$

call option payoff: $\begin{cases} (S_T-K)^+, & \text{if } S_T \geq K \\ \ldots \end{cases}$

- payoff : $\begin{cases} (S_T - K)^+ , & \text{if } S_T \geq K \\ 0 , & \text{if } S_T < K \end{cases}$

- Digital option payoff : $\begin{cases} 1 , & \text{if } S_T \geq K \\ 0 , & \text{if } S_T < K \end{cases}$

$$P = \frac{\text{call}(T, K)}{\text{Digital}(T, K)}$$

Static portfolio = Long 1 call(T,K) + Short P Digital(T,K)

value of static portfolio = call$(T,K) - P \times$ Digital(T,K)

$$= \text{call}(T,K) - \frac{\text{call}(T,K)}{\cancel{\text{Digital}}(T,K)} \times \cancel{\text{Digital}}(T,K)$$

$$= 0$$

$$\Delta_{\text{put later}} = \Delta_{\text{call}}(T,K) - P \times \Delta_{\text{Digital}}(T,K)$$

$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$ (Tower rule of conditional expectation)

$$\mathbb{E}\left[e^{-r(\theta - t)} S_\theta \middle| \mathcal{F}_t\right] = S_t$$

discounted prices of tradable assets are martingales