9-6 Probability Theory

Friday, September 6, 2019 12:37 PM

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9-6: Probability and Set Theory
CSCI 3022 Fall 19

Probability Fall 2019 1 / 22

Announcements and Reminders

+ a few hours...

- Homework 1 due a week from now
- Where is your Titanic data coming from? – use Canvas.
- Quizlet01 over the weekend, due Monday – also Canvas

Overview: Probability

Many aspects of the world seem random and unpredictable.

1. Are we tall or short?
2. Do we have Mom's eyes or Dad's?
3. Is the hurricane going to hit Alabama?
4. Which team will win the NFC North?
5. How long until the Stampede bus shows up?
6. Which grocery store line should I get in?

One main objective of statistics/data science is to help make good decisions under conditions of uncertainty.

Overview: Probability

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One main objective of statistics/data science is to help make good decisions under conditions of uncertainty.

Overview: Definitions

Definition: *Set*
A *set* is a collection of objects.

Definition: *Probabilistic Process* - coin
A *probabilistic process* is system/experiment whose outcomes are uncertain.

Definition: *Outcome* - H or T
An *outcome* is a possible result of a probabilistic process .

Definition: *Sample Space*
A *sample space* (denoted $\boxed{\Omega}$) of a probabilistic process is the set of all possible outcomes of that process. $\{H, T\}$

Discrete vs. Continuous

Sets can contain many types of objects, both discrete and continuous. Our associates mathematics will shift accordingly.

Discrete (Structures)

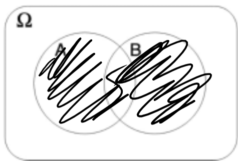
1. Math: summation, counting, sorting
2. Sets: ~~times, intervals~~ $\{H, T\}$, counts of observations

Continuous

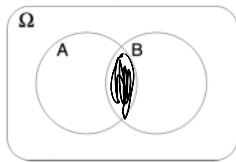
1. Math: integrals, derivatives, smooth functions
2. Sets: times, intervals

Basic Set Operations

Complement;
A^C;
"Not"

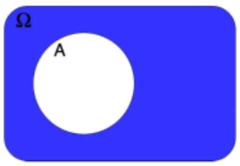


Union;
$A \cup B$;
"Or"



Intersection;
$A \cap B$;
"And"

Basic Set Operations



Complement;
A^C;
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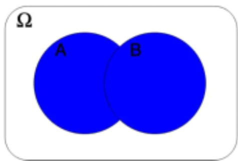
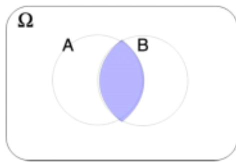
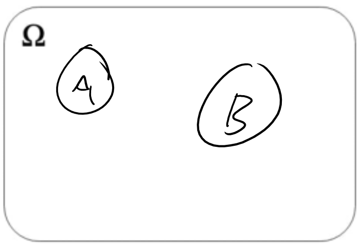


Union;
$A \cup B$;
"Or"



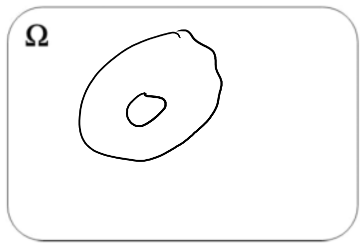
Intersection;
$A \cap B$;
"And"

Basic Set Definitions



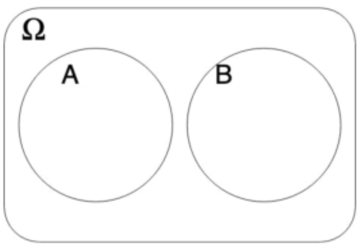
Mutually Exclusive ;

disjoint

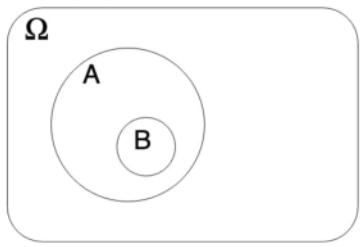


Subset

Basic Set Definitions

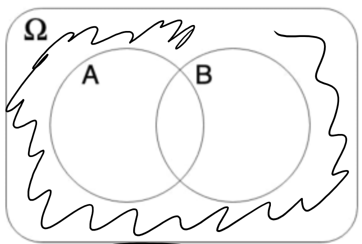


Mutually Exclusive;
$(A \cap B) = \emptyset$;
"If A, not B; If B, not A."



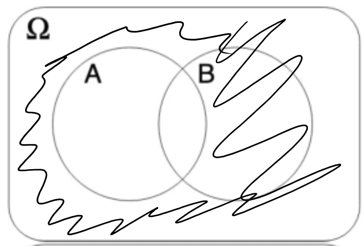
Subset;
$A \supseteq B$;
$A \supset B$;
↳ strict

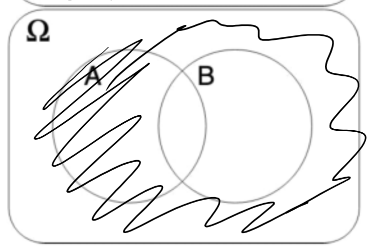
DeMorgan's Laws



"neither A nor B"
compared to:
"not A"
"not B"

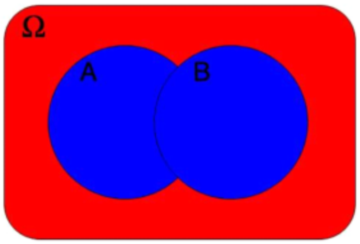
$(A \cup B)^c$



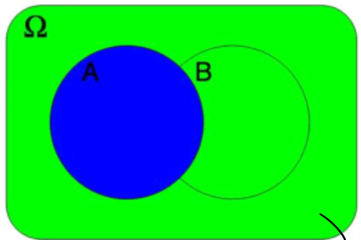


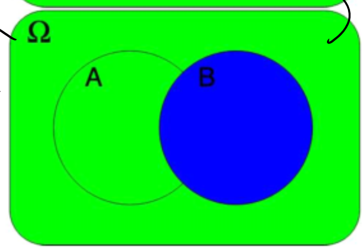
$A^c \cap B^c$

DeMorgan's Laws



"neither A nor B;"
$(A \cup B)^C =$
$A^C \cap B^C$;
"not A AND not B"





red

intersect green

Some Sample Spaces

Describe sample spaces for:

Process is the coin

1. Tossing a coin twice

$\{HH, TT, HT, TH\}$ $\qquad |\{HH, TT, HT, TH\}| = 4$

2. Selecting a card from a deck

$\{2\diamond, 3\diamond, \dots\}$ $\qquad |\{\cdot\}| = 52$

3. Measuring your commute time on a particular morning

$\{t : t \in [d/c, \infty)\}$

speed of light

Some Sample Spaces

Describe sample spaces for:

1. Tossing a coin twice
   $\{HH, HT, TH, TT\}$
2. Selecting a card from a deck
   $\{2\clubsuit, 2\spadesuit, 2\diamondsuit, 2\heartsuit, 3\clubsuit \ldots\}$
3. Measuring your commute time on a particular morning
   $\{t : t \in (0, T]\}$ where T is... infinity? The maximum reasonable time it *could* take?

Event

Definition: *Event*
An *event* is any collection (subset) of outcomes from the sample space.

An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcome.

When an experiment is performed, an event A is said to *occur* if the resulting experimental outcome is contained in A.

Events

Example: Suppose that we flip a coin 3 times.

Sample space: {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Some Possible Event(s):

E_1 : the event that we see the same flip all 3 times. $E_1 = \{HHH, TTT\}$.

E_2 : the event that flip # 2 is heads.

E_2, {?H?}, 4 possible ways

What outcomes or elements(s) of Ω are in $E_1 \cap E_2$?

$E_1 \cap E_2 = \{HHH\}$.

Events

Example: Suppose that we flip a coin 3 times.

Sample space:
$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

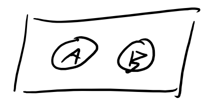
Some Possible Event(s):
E_1 : the event that we see the same flip all 3 times.
E_2 : the event that flip # 2 is heads.

What outcomes or elements(s) of Ω are in $E_1 \cap E_2$? : just $\{HHH\}$

Probability Axioms

Definition: *Probability*
Probability is a function that takes in sets (and later, we'll see, random variables) and outputs numbers according to the following rules:

1. *Non-negativity:* $P(A) \geq 0$ for all $A \in \Omega$.
2. *Unity:* $P(\Omega) = 1$
3. *σ-additivity:* If A and B are disjoint events.



$$P(A) + P(B) = P(A \cup B).$$

OR

if $A \cap B = \emptyset$

Probability Axioms

Definition: *Probability*
Probability is a function that takes in sets (and later, we'll see, random variables) and outputs numbers according to the following rules:

1. *Non-negativity:* For every $A \in \Omega$, $P(A) \geq 0$.
2. *Unity:* Given a sample space Ω, $P(\Omega) = 1$.
3. σ*-additivity:* If A and B are disjoint (mutually exclusive) sets, $P(A \cup B) = P(A) + P(B)$.

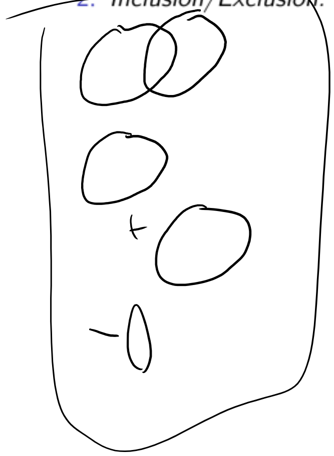
Probability Theorems

The axioms of probability give us a couple of important results.

1. *Complementation:* $P(A^c) = 1 - P(A)$

2. *Inclusion/Exclusion:* What is $P(A \cup B)$?

$= P(A) + P(B) - P(A \cap B)$



Probability Theorems

The axioms of probability give us a couple of important results.

1. *Complementation:* $P(A^C) = 1 - P(A)$.
   Proof:
   $P(\Omega) = 1$
   $A^c \cup A = \Omega$
   $A^c \cap A = \emptyset$
   $P(A^c \cup A) = 1 = P(A^c) + P(A)$
2. *Inclusion/Exclusion:* What is $P(A \cup B)$?
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   Proof:

Prob

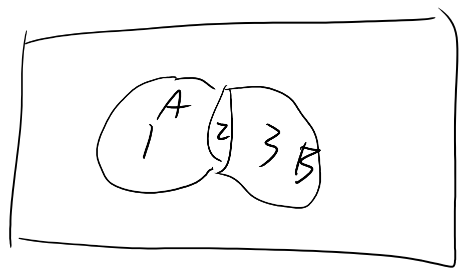
$A \cup B$: 1 $A \cap B^c \Rightarrow P(A) - P(A \cap B)$

2 $A \cap B \Rightarrow P(A \cap B)$

3 $A^c \cap B \Rightarrow P(B) - P(A \cap B)$

sum Σ_i

$= P(A) + P(B) - P(A \cap B)$



Probability Theorems

The axioms of probability give us a couple of important results.

1. *Complementation:* $P(A^C) = 1 - P(A)$.
   Proof: From unity, $P(\Omega) = 1$, and $\Omega = A \cup A^C$, which are disjoint sets. So $P(\Omega) = P(A \cup A^C) = P(A) + P(A^C) = 1$.

2. *Inclusion/Exclusion:* What is $P(A \cup B)$?
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   Proof: $A \cup B$ is "A or B," which can happen 3 disjoint ways:

   0.1 "A not B;" or $A \cap B^C$, with probability $P(A) - P(A \cap B)$;

   0.2 "B not A;" or $B \cap A^C$, with probability $P(B) - P(A \cap B)$;

   0.3 "both;" with probability $P(A \cap B)$;

   Summing these 3 probabilities gives the desired result.

Probability Theorems

The axioms of probability give us a couple of important results.

1. *Complementation:*

2. *Inclusion/Exclusion:* What is $P(A \cup B)$?
   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   Proof:

   This idea works for more than 2 sets!

Probabilities on Random Variables

Let $X = \#$ of heads in three tosses of a fair coin. X is a *random variable:* it maps events (a count of heads) into real numbers through probabilities.

What is the underlying probabilistic process?

its a (fair) coin

What is the sample space?

Same as before: 8 outcomes.

What are the possible values for X?

$X = \{0, 1, 2, 3\}$.

What is the probability that X is equal to 1: $P(X = 1)$?

$P(\{HTT\} \cup \{THT\} \cup \{TTH\}) = 3/8$

set w/ 3 outcomes

all outcomes

event	X	Prob
{HHH	$X=3$	1/8
HHT	$X=2$	1/8
HTH	$X=2$	1/8
THH	$X=2$	.
TTH	$X=1$	.
THT	$X=1$	.
HTT	$X=1$	
TTT}	$X=0$	1/8

Probabilities on Random Variables

Let $X = \#$ of heads in three tosses of a fair coin. X is a *random variable:* it maps events (a count of heads) into real numbers through probabilities.

What is the underlying probabilistic process?
Flipping a fair coin.

What is the sample space?
The same 8 flip-outcomes as before.

What are the possible values for X?
$X \in \{0, 1, 2, 3\}$

What is the probability that X is equal to 1: $P(X = 1)$?
If all outcomes are equally likely in a set, we can arrive at this by counting the elements of Ω in X compared to all of Ω: or $\boxed{\frac{|X|}{|\Omega|} = 3/8}$

$q = 1-p$

$\{H, T\}$

$\{H, T\} \otimes \{H, T\} = \{HH, HT, TH, TT\}$.

no.

Probabilities on Random Variables

Our coin is *unfair*, and comes up heads p proportion of the time. What is the probability that I flip a biased coin twice and both flips come up heads?

Sample space for one flip:

Sample space for both flips (a product of sample spaces!):

Should the probability of the second flip change based on the result of the first?

Probabilities on Random Variables

Our coin is *unfair*, and comes up heads p proportion of the time. What is the probability that I flip a biased coin twice and both flips come up heads?

Sample space for one flip: $\{H, T\}$

Sample space for both flips (a product of sample spaces!): $\{HH, HT, TH, TT\}$

Should the probability of the second flip change based on the result of the first?
Not usually: we call these *independent*... Not everything is independent! **Idea:** two or more trials are *independent* if they don't affect each other

Independence and Probabilities

Our coin is *unfair*, and comes up heads p proportion of the time. What is the probability that I flip a biased coin twice and both flips come up heads?

If two outcomes are independent, probabilities on their intersection ("and") becomes a product.

Result: What are $P(\{HH\})$ and $P(\{TT\})$?

$P(\{HH\}) = P(\{flip\ 1 = H\} \cap \{flip\ 2 = H\}) = P(flip\ 1 = H) \cdot P(flip\ 2 = H)$

$p \cdot p = p^2$

If two outcomes are disjoint, probabilities on their union ("or") becomes a sum.

Result: What is $P(\{HT\}$ **OR** $\{\cancel{HT}\ TH\})$?

$P(\{HT\}) + P(\{TH\}) = p(1-p) + (1-p) \cdot p$

$P(\{TT\}) = P(\{T\})^2 = (1-p)^2$

Sanity check: did we just add up to 1?

$1 = p^2 + (1-p)^2 + 2p(1-p) = (p + (1-p))^2$

Independence and Probabilities

Our coin is *unfair*, and comes up heads p proportion of the time. What is the probability that I flip a biased coin twice and both flips come up heads?

If two outcomes are independent, probabilities on their intersection ("and") becomes a product.

Result: What are $P(\{HH\})$ and $P(\{TT\})$?
A: $p \cdot p$ and $q \cdot q = (1-p)^2$

If two outcomes are disjoint, probabilities on their union ("or") becomes a sum.
Result: What is $P(\{HT\}$ **OR** $\{HT\})$?
A: $P(\{HT\} = pq$ PLUS $P(\{TH\} = qp$

Sanity check: did we just add up to 1?

Counting outcomes

Finally, what is the probability of I flip our biased coin five times and get *exactly* one heads?

$P(\{HTTTT\} \text{ OR } \{THTTT\} \text{ OR } \cdots \})$

$P(\{HTTTT\}) + P(\{THTTT\}) + \cdots$

$P(H) \cdot P(T)^4 + P(T) \cdot P(H) \cdot P(T)^3 + \cdots$

$5 \cdot P(\{H\}) \cdot P(\{T\})^4 = 5pq^4 = 5p(1-p)^4$

Counting outcomes

Finally, what is the probability of I flip our biased coin five times and get *exactly* one heads?

This is the set of events $\{HTTTT, THTTT, TTHTT, TTTHT, TTTTH\}$.

Counting outcomes

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Each is composed of 5 independent flips, so the probability of any one of these events is the product pq^4

Counting outcomes

Finally, what is the probability of I flip our biased coin five times and get *exactly* one heads?

This is the set of events $\{HTTTT, THTTT, TTHTT, TTTHT, TTTTH\}$.

Each is composed of 5 independent flips, so the probability of any one of these events is the product pq^4

Each outcome is disjoint/exclusive, so the full cumulative probability is the sum of 5 of these: $5pq^4$

Moving Forward

Suppose we have a coin and we don't know if it's biased... what could we do? (nb04, lecture next week to come!)

1) flip coin A lot.
2) how often did H occur.

estimate true p.