ESSCA - Quantitative finance on VBA

Corporate bonds and Merton model

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1. About corporate Bonds:

A corporate bond is a tradable debt security that represents a ”promise on the part of the issuer (i.e. the borrower or the corporation) to make one or more payment(s) to the holder (i.e. the lender or the investor) at a specified future date or dates. It usually carries a specific rate of interest (the coupon) and/or are sold at a discount to the amount that will be repaid at maturity”. (Source: European Central Bank, 2004, Annual Report: 2004, ECB, Frankfurt, Glossary).

Table 1. summarizes the main features of the corporate bond contract and sheds light, when it is the case, on the features that vary during the life of the bond.

Table 1. Main features of a corporate bond contract:

N:[1]	Feature	Description
	Issuer	Designation of the corporation that issues the bond contract.
A	Amount	The total amount of money borrowed by the corporation.
D	Nominal, face or par value	The total amount of the debt contracted by the corporation is generally divided into a certain number of units called nominal value (similarly face or par value). The issuer of the bond pays interest to the holder on the base of this nominal value and, unless otherwise specified, it is the amount of that has to be repaid by the issuer at the end of the bond contract (i.e. the maturity).
P	Bond price	The actual value at which the bond is sold. At issuance, the bond price (issue price) may be fixed to be higher than the face value (bond issued at a premium), lower than the face value (bond issued at a discount) or equal to the face value (bond issued at par). Afterwards, the bond is sold on secondary markets at different prices which are usually presented as a percentage of the nominal value. The difference between the price of the bond and its nominal value provides the bondholder a return that makes purchasing the bond sometimes worthwhile even if the bond pays no interest.

[1] This column contain a notation of each feature that will used in subsequent formulations.

	Date of issue	The date on which interest on the bond begins to accrue.
	Settlement date	The date on which the trade of the bond must be settled. It may be different from the date of issue.
	Normal redemption date	This is the date on which the bond (seen here as a loan) is amortized; the bond is hence said to be redeemed. Similar to bank loans, redemption can be at maturity; in equal slices (constant amortization); or in fixed instalments. The terms of the bond contract may also include options for the issuer to make earlier redemptions. Many other financial innovations exist for redemption methods, such as redemption by lottery[2].
T	Maturity	Corresponds to the period of time for which the bond remains outstanding. It goes from the issue date to the final redemption date where the principal is repaid with interest. It is generally superior to one year.
C	Coupon	This is the periodic interest that the bond holder (i.e. the lender) receives between the date of the issue and the maturity. It is generally presented in term of a coupon rate (or nominal rate) and is calculated on the base of the nominal value. A corporate bond may pay no coupons and it is then called a zero-coupon bond (or a discount bond). The coupon rate can be constant throughout the life of the bond (i.e. fixed rate bond) or floating on the base of another reference rate (i.e. floating rate bond).
Y	Yield to maturity	The yield to maturity is the rate of return earned by an investor who buys the bond assuming that he will hold it until maturity. It depends mainly on the price of the bond, the coupon rate (if any) and naturally on the time left to maturity. The yield to maturity varies hence throughout the life of the bond.
S	Credit Spread	The credit spread is the difference between the yield to maturity of the bond and that on a benchmark bond used by the market. In the euro area, the benchmark for long-term debt is most often the German government bond yield or the Interest Rate Swap (IRS) rate. Along with the yield to maturity, the spread is a key aspect for bonds in general and corporate bonds in particular[3].

[2] Some bonds issues contain a "sinking-fund" provision that means that a certain portion of the issue must be retired each specified period. The bonds retired are sometimes selected by lottery.

[3] The spread and yield to maturity will be considered with more details in what follows.

R	Rating	The rating or credit rating is an evaluation of the ability of the issuer to pay back its debt obligation provided by an "independent" Rating Agency. These are mainly: Standard and Poor's, Fitch and Moody's. These rating agencies can decide to rate a specific issue or to give an absolute rating for the issuer (rating given to first-ranking debt). In Europe, rating agencies generally rate companies at their request, which enables them to access privileged information (e.g. medium-term plans, contacts with management). Rating agencies check the debtor's ability to pay back its debt and the likelihood of default, and attribute a rating ranging from AAA, for the debtors with the highest ability to meet their debt obligation, to C for the most risky debtors; D rating is attributed for entities in Default (in this thesis we use the rating system provided by Standard & Poor's as the benchmark rating system.). Broadly, ratings between AAA and BBB- are referred to as Investment-grade (also high grade), and those between BB+ and D are referred to as High-yield (also speculative grade or non-investment grade). Investment grade entities have higher ability to meet their debt obligations and consist thus of a more secure investment compared to High-yield bonds.
	Guarantee	The principal and interest payments of the bond can be guaranteed by the issuer of the bond, by a collateral (such as property, equipment, or other assets that the company owns) or by a third party such as the parent company of the issuer. Corporate bonds that have no guarantee pledged to them are often called unsecured or "debentures".
I	Indenture and covenants	The bond indenture specifies the legal terms of the bond contract while the bond covenants precise the rights of the bondholder and the obligations of the issuer (for instance the recovery rules of the capital of the bondholder in the case of default of the issuer). The obligations of the issuer may include maintaining certain financial performance, providing financial statements, interdiction from taking certain actions while the bond is active such as selling the company or merging it with another company. After the issuance of the bond, these terms can be hardly modified during the life of the bond[4]. Hence, in order to have more freedom, many corporate issuers prefer to issue bonds without covenants.
	Optional and Special features	The corporate bond contract may specify other special features such as optionality (a corporate bond can be callable or puttable), convertibility (the bond can be converted into shares of the issuing company), exchangeability (the bond can be exchanged into shares of any other) etc. Many financial innovations exist nowadays with regard to the special features of the bond.
	Listing	Market place of the bond issuance

[4] This requires at least an approval by the majority of the bond holders.

2. Basic valuation relations:

Theoretically, the value or price of a financial instrument is widely perceived as equal to the present value of its expected future cash flows. The price of a corporate bond can be similarly expressed in terms of its future cash flows, which are the face value of the bond D, and the coupons C in the case of a coupon-paying bond. Assuming a continuous compounding, the price or the present value of a zero-coupon corporate bond at a certain date t can be expressed as follows:

$$P\ t,T\ = D\ e^{-Y(t,T)\ T-t} \tag{1}$$

Where $Y(t,T)$ is the continuously compound yield to maturity, and $T-t$ is the time left until the maturity of the bond.

For a coupon-paying bond, the price will naturally take into account the discounted sum of the future coupons that the bond will pay[5]. Assuming that the bond pays a stream of fixed coupons C at times $\{t_i\}$, where $t_1 < t_2 < \cdots < t_n$ and $t_n = T$, the price of coupon-paying corporate bond can be expressed by:

$$P_c\ t,T\ = \sum_{i=1}^{N} C\, e^{-Y(t,T)\ t_i-t} + De^{-Y(t,T)\ T-t} \tag{2}$$

When comparing corporate bond investments, investors use usually the "yield to maturity" as a measure of the effective return of the bond. The yield to maturity can be indeed perceived as the rate of return (or interest rate) that accounts for the present value of all the future cash flows of the bond. It considers hence the current price of the bond, its face value, the coupons (if any), and the time left until the bond matures. Rewriting from equation (11), the following expression can be provided for the yield to maturity of a discount corporate bond:

$$Y(t,T) = -\frac{\ln\left(\frac{P(t,T)}{D}\right)}{T-t} \tag{3}$$

Where ln() is the natural logarithm function[6]. However, for a coupon-paying bond it is not generally possible to provide closed form formulas for the yield to maturity in terms of the bond's price. Numerical techniques need hence to be used in order to approximate the yield $Y(t,T)$ that takes into account the price of the bond provided in equation (2)[7].

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[5] We concentrate here on the valuation of zero-coupon bonds.

[6] Equation (3) shows the existence of a negative relation between a bond's yield and price over time.

[7] Two main assumptions are made when measuring the yield to maturity of a bond: first, that the bond will be held until maturity, and second that the received coupons (if any) will be reinvested at an interest rate equal to the yield to maturity.

Furthermore, by purchasing corporate bonds, investors bear generally higher levels of risk on the return of their investment compared to a typical treasury bond, which is considered theoretically to be risk-free[8]. This risk difference is captured by the notion of credit spread, which can be formulated as follows:

$$S(t,T) = Y(t,T) - r(t,T) \tag{4}$$

Where $S(t,T)$ is the credit spread of a corporate bond of maturity T at a certain date t; $Y(t,T)$ the yield to maturity of the bond defined in equation (3), and $r(t,T)$ the yield of a benchmark government bond of equivalent maturity. Traditionally, credit spread has been viewed as reflecting the higher default risk of a corporate issuer compared to a government issuer. In what follows, we use the formulas provided in this sub-section (equations 1-4) to derive and analyze the credit spreads of the Merton model for corporate bond pricing.

3. Merton Model:

In its former corporate bond pricing model, Merton (1974) assumes first that the firm issuing the bond has a simple capital structure composed by assets V, equities E and a single Zero-coupon bond with face value D and maturity T. Respecting the equality of the assets and liabilities, the capital structure of the firm in this setup can be written as follows:

$$V = E + D \tag{5}$$

Since the value of the firm's assets is not observable, Merton (1974) assumes afterwards that the evolution of the firm's assets over time V_t follows a Geometric Brownian motion with respect to the following stochastic differential equation[9]:

$$dV_t = \ r - \kappa \ V_t \ dt + \sigma_v V_t \ d\widehat{\mathbb{W}}_t \tag{6}$$

Where r is the constant risk-free interest rate; κ the constant pay-out rate, σ_v the constant volatility of the of the firm's assets; $\widetilde{\mathbb{W}}_t$ a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$[10]; and $V_0 > 0$ the initial value of the firm's assets. Figure I.1 shows how the firm value evolves under this assumption.

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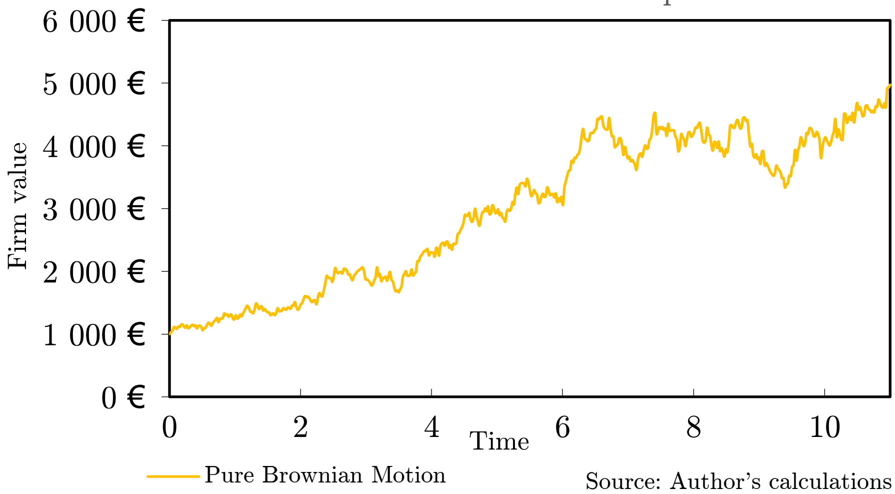
[8] Government bonds are considered to be theoretically default-risk-free because, as opposed to a corporate issuer, the government has the possibility to raise taxes or create additional currency in order to redeem its bonds at maturity. However, it is important to note that there has been cases where a government has defaulted on his debt, such as the Russian government in 1998 or the Greek government since 2010.

[9] A stochastic process (i.e. a system of random values that evolves over time) V_t is said to follow a geometric Brownian motion if it satisfies the following stochastic differential equation: $V_t = \mu V_t \ dt + \sigma_v V_t \ d\mathbb{W}_t$, where μ is called the percentage drift and σ the percentage volatility; $\mu V_t \ dt$ is called the trend of the stochastic differential equation while $\sigma_v V_t \ d\mathbb{W}_t$ is the random diffusion trajectory.

[10] The standard Brownian motion (or Wiener process) $\mathbb{W}_t$ is a continuous-time stochastic process that satisfies the following characteristics:

Figure I.1 - Firm value path under the geometric Brownian motion assumption



Source: Author's calculations

Assuming that the firm value follows a geometric Brownian motion with a drift $r - \kappa$ and a volatility σ_v amounts to saying, on the one hand, that the average change in the firm's value is an increasing function of the risk-free rate r and a decreasing function of the pay-out rate κ; and on the other hand, that the firm faces unpredictable random events that affect its assets' value (following a normal distribution), with a constant volatility rate σ_v. This constitutes one of the key assumptions of the Merton (1974) model since it allows a continuous time modelling of the random values of the firm's assets, and hence, the use of the stochastic calculus techniques employed in the Black & Scholes (1973) option pricing framework.

Merton (1974) assumes afterwards that the ability of the firm to redeem its debt (i.e. the zero-coupon bond) is determined by the value of the firm's assets at maturity T (i.e. the value of the geometric Brownian motion at the maturity of the zero-coupon bond). If at time T the value of the firm's assets V_T is superior to the face value of the debt D, the firm has the capacity to pay its creditors (i.e. the bond holders), and it does so by paying the principal amount of the debt D (the firm's equity here is equal to the remaining value of assets, that is $V_T - D$)[11]. If however at time T the value of the firm's assets V_T is inferior to the face value of the debt, the firm cannot redeem its debt and default occurs. In this situation,

- $\mathbb{W}_t$ has independent increments: this means that any increment of the Brownian motion (i.e. the difference between the realizations of the Brownian motion at two successive times) is independent from all the other realizations of the Brownian motion.

- $\mathbb{W}_t$ has stationary and Gaussian increments: this means that the increments of the Brownian motion follows a Gaussian distribution with a zero mean and a variance equal to the time interval of the increment (i.e. for times $0 \leq t_1 \leq t_2$ the increment of the Brownian motion $\mathbb{W}_{t2} - \mathbb{W}_{t1} \sim N(0, t_2 - t_1)$)

- $\mathbb{W}_t$ is almost surely continuous: this means that realizations of the Brownian motion have a probability of 1 of being continuous.

- For a standard Brownian motion the initial value of the process $\mathbb{W}_{t_0}$ is equal to zero.

See for instance J. Hull (2012), page 333.

[11] Hence in the Merton (1974) setup, D is assumed to be the default boundary.

the creditors take the residual value of the firm V_T (the firm's equity here is null). In this setup the value of the corporate bond contract at maturity corresponds to the value paid by the firm to its creditors which can be, either the face value of the debt D, or the residual value of the firm's assets V_T, that is formally:

$$P(T,T) = V_T \ \mathbf{1}_{\{V_T<D\}} + D \ \mathbf{1}_{\{V_T \geq D\}} \tag{7}$$

With $\mathbf{1}_{\{\ \}}$ is the indicator function of the events $V_T < D$ and $V_T \geq D$ where the firm pays, respectively, V_T or D. In this framework, the major contribution of Merton (1974) was by viewing the payoff of the creditors at maturity (i.e. the corporate bond contract) as being similar to the situation where creditors sell a European put option written on the assets of the firm, with a strike price equal to the face value of the bond D and with maturity T[12]. That is formally:

$$P \ T,T \ = \min \ V_T, D \ = D - \max(D - V_T, 0) \tag{8}$$

Hence in the Merton (1974) setup, the pricing of the corporate bond is reduced to the framework of European options pricing initiated by Black-Scholes (1973). Discounting the right hand side of equation (8) at a date t between the issuance of the bond t_0 and its maturity T, the price of the risky corporate bond can be obtained by:

$$P \ t,T \ = e^{-r \ T-t} \ D - Put_{B\&S}(t,T) \tag{9}$$

Where $e^{-r \ T-t}$ is the continuous discount factor that takes into account the date of the pricing t, the maturity T, and the constant risk-free rate r[13]; and $Put_{B\&S}(t,T)$ is the payoff of a put option with strike D and maturity T given by Black-Scholes (1973) as follows:

$$Put_{B\&S} \ t,T \ = \ D \ e^{-r \ T-t} \ \ N \ -d_2 \ - V_t \ e^{-\kappa \ T-t} \ \ N \ -d_1 \tag{10}$$

[12] In other words, in Merton (1974) setup, the corporate bond investors (i.e. the creditors of the firm) can be viewed as selling a put option to the firm with a strike price D and maturity T. By doing so, they give the firm the right, but not the obligation, to exercise a put option according to the value of the firm's assets at maturity. Hence:

- If at maturity $V_T \geq D$: the firm doesn't exercise its put option (which means that the payoff of the put option $\max\,(D - V_T, 0)$ is equal to zero) and that the entity $D - \underbrace{\max\,(D - V_T, 0)}_{0}$ is equal to D; that's what investors receive at maturity in this case.

- If however at maturity $V_T < D$: the firm exercise its put option (which means that the payoff of the put option $\max\,(D - V_T, 0)$ is equal to $D - V_T$), and the entity $D - \underbrace{\max \ D - V_T, 0}_{(D-V_T)}$ is equal to V_T; that's what investors receive at maturity in this case.

[13] Using a flat term structure where the risk-free interest rate r is constant and known with certainty at all times consist of another simplifying assumption of the Merton (1974) model.

Where:

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{V_t}{D}\right) + \left(r - \kappa + \frac{\sigma_v{}^2}{2}\right) \ T - t}{\sigma\sqrt{T - t}} \\
d_2 &= \frac{\ln\left(\frac{V_t}{D}\right) + \left(r - \kappa - \frac{\sigma_v{}^2}{2}\right) \ T - t}{\sigma\sqrt{T - t}} \\
&= d_1 - \sigma\sqrt{T - t}
\end{aligned} \tag{11}
$$

And $N(\ )$ denotes the standard Gaussian cumulative distribution function given by:

$$
N \ x \ = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-\frac{t^2}{2}} \ dt \qquad \forall \ x \in \mathbb{R} \tag{12}
$$

Replacing back in equation (9), the value of the corporate bond in the Merton (1974) framework is found to be equal to[14]:

$$
\begin{aligned}
P_{Merton} \ t, T \ &= e^{-r \ T-t} \ D - \left(D \ e^{-r \ T-t} \ \ N \ -d_2 \ - V_t e^{-\kappa \ T-t} \ \ N \ -d_1 \ \right) \\
&= D \ e^{-r \ T-t} \ \ N \ d_2 \ + V_t e^{-\kappa \ T-t} \ \ N \ -d_1
\end{aligned} \tag{13}
$$

Probability of Default

Given the price of the corporate bond, it is possible to calculate a firm's probability of default according to the Merton Distance to Default model. The first step is calculating Distance to Default:

$$
DD = \frac{\ln\left(\frac{V_t}{D}\right) + \left(r - \kappa + \frac{\sigma_v{}^2}{2}\right) \ T - t}{\sigma\sqrt{T - t}}
$$

If we assume that the expected frequency of default follows a normal distribution (which is not the best assumption if we want to calculate the true probability of

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[14] The price of the risky debt in Merton (1974) model can be indeed derived in three different ways:

- By using the formulation of a European put option as provided here;
- By using the put-call parity: the firm's equity can be perceived as a European call option on the firm's assets; hence through the Black-Scholes formulations of a European call and the capital structure equality given in equation (5), the same results of equation (13) can be found;
- By using risk-neutral valuation techniques (see for instance S. Sherve, 2004).

default, but may suffice for simply rank ordering firms by credit worthiness), then the probability of default is according to the Merton model given by:

$$\boldsymbol{PD} = P\left(V_T < D\right) = N\left(-DD\right) = 1 - N(DD)$$