CS240 Algorithm Design and Analysis

Lecture 7

Network Flow

Quan Li
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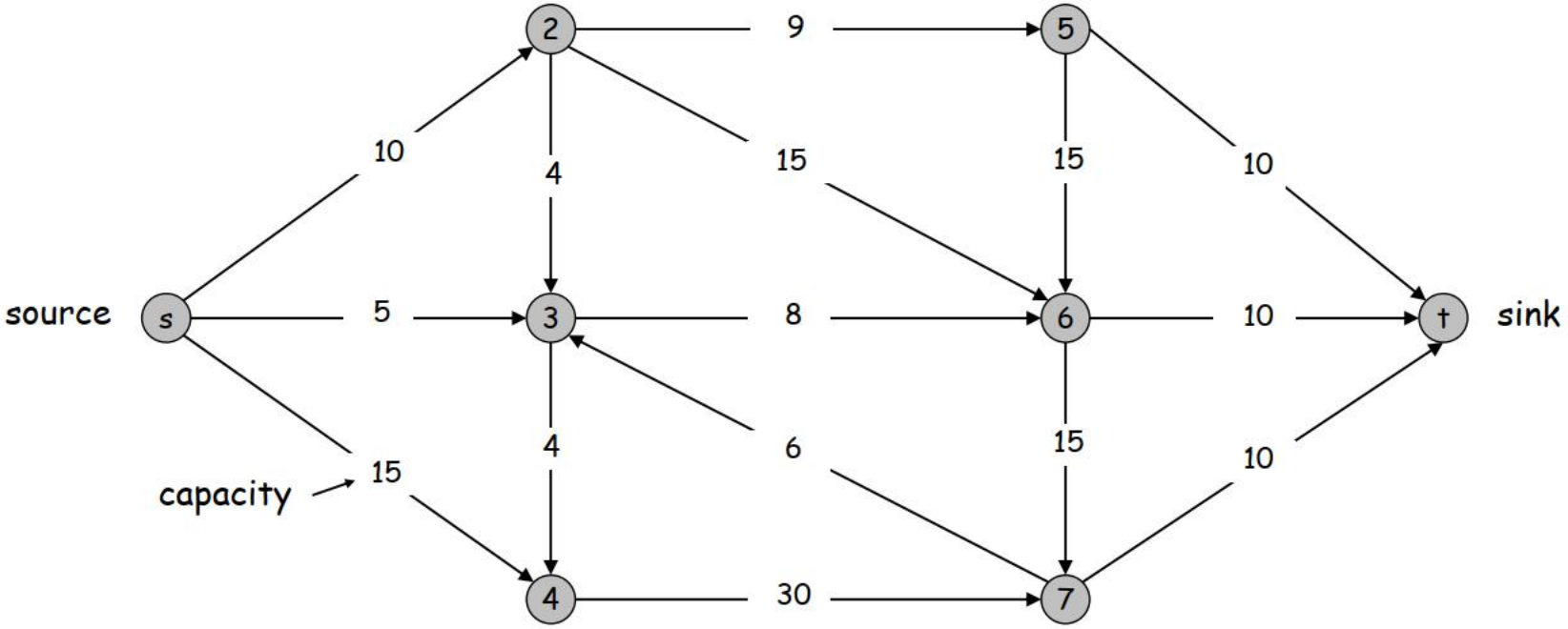
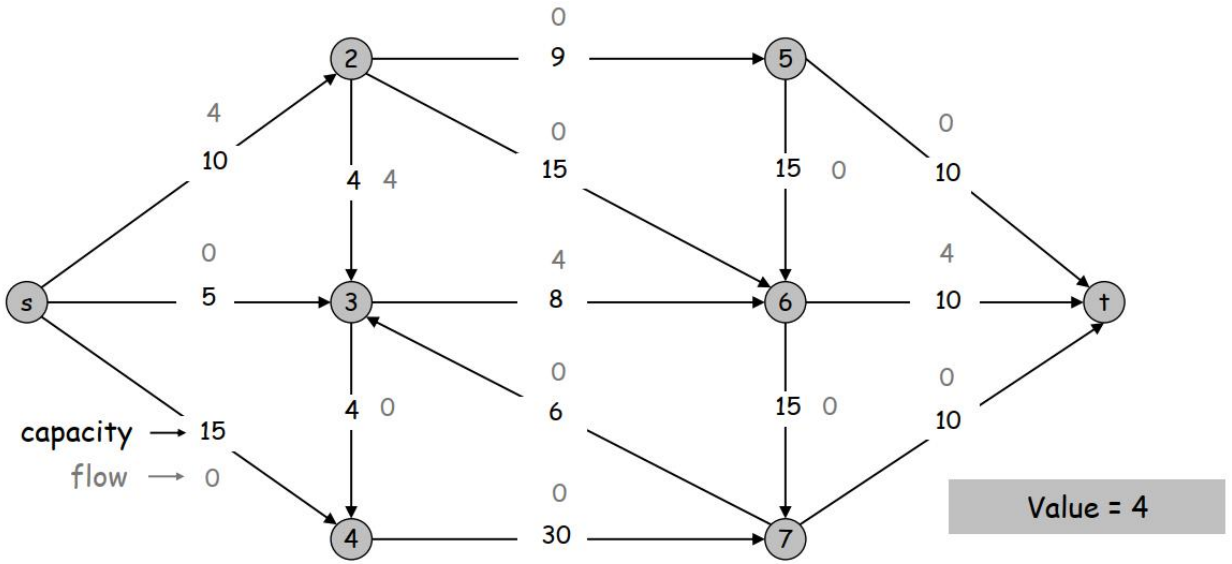
Flows

- Flow network
  - Abstraction for material **flowing** through the edges
  - G = (V, E) = directed graph
  - Two distinguished nodes: s = source, t = sink
  - c(e) = nonnegative capacity of edge e



Flows

- **Def.** An **s-t flow** is a function that satisfies:
  - For each e ∈ E: 0 <= f(e) <= c(e)(capacity)
  - For each v ∈ V − {s, t}: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)
- **Def.** The value of a flow f is : $v(f) = \sum_{e \text{ out of } s} f(e)$



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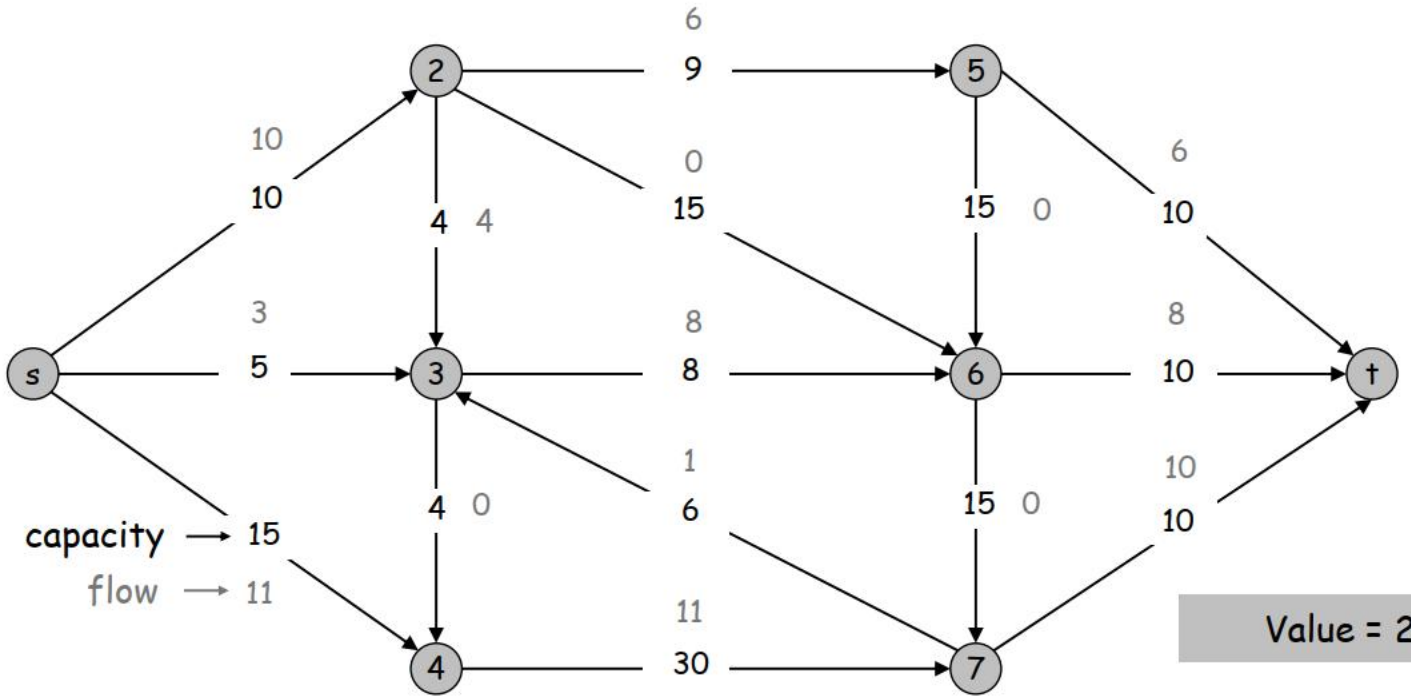
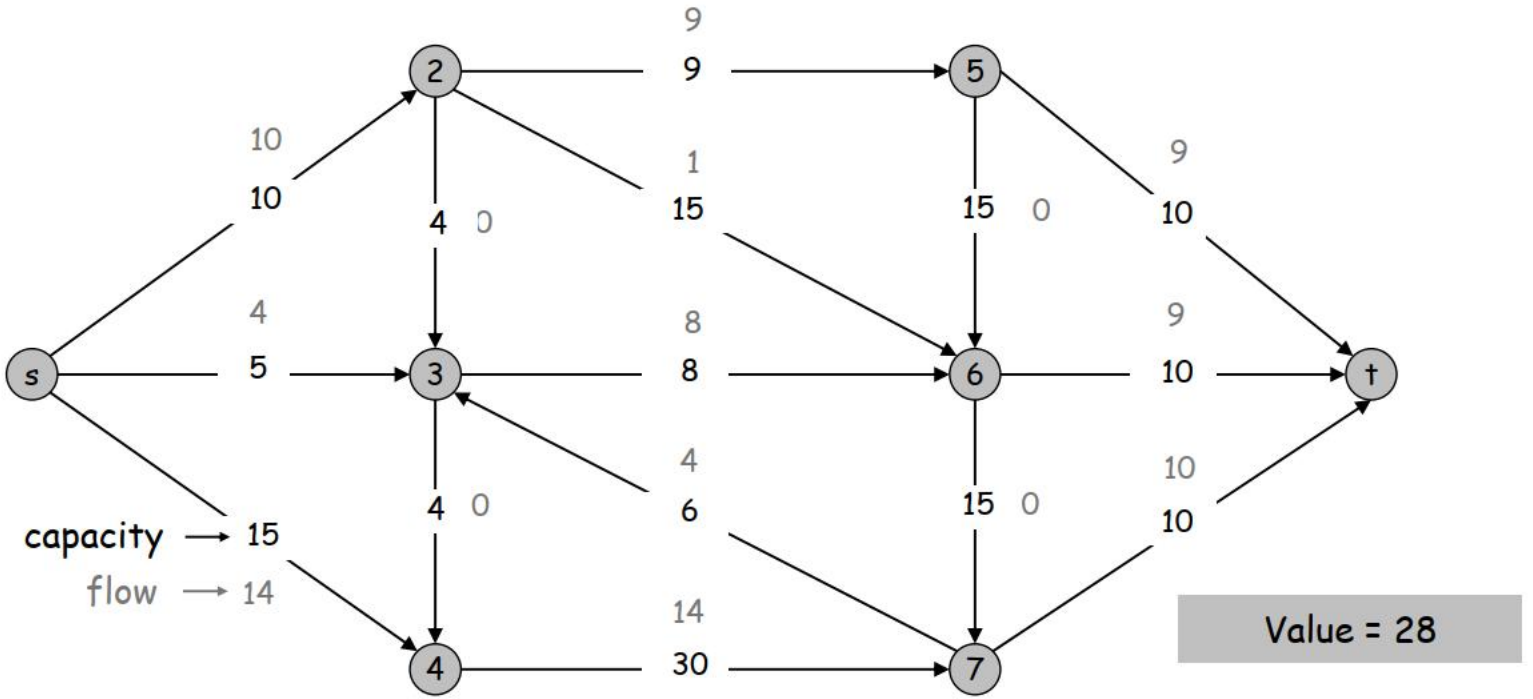


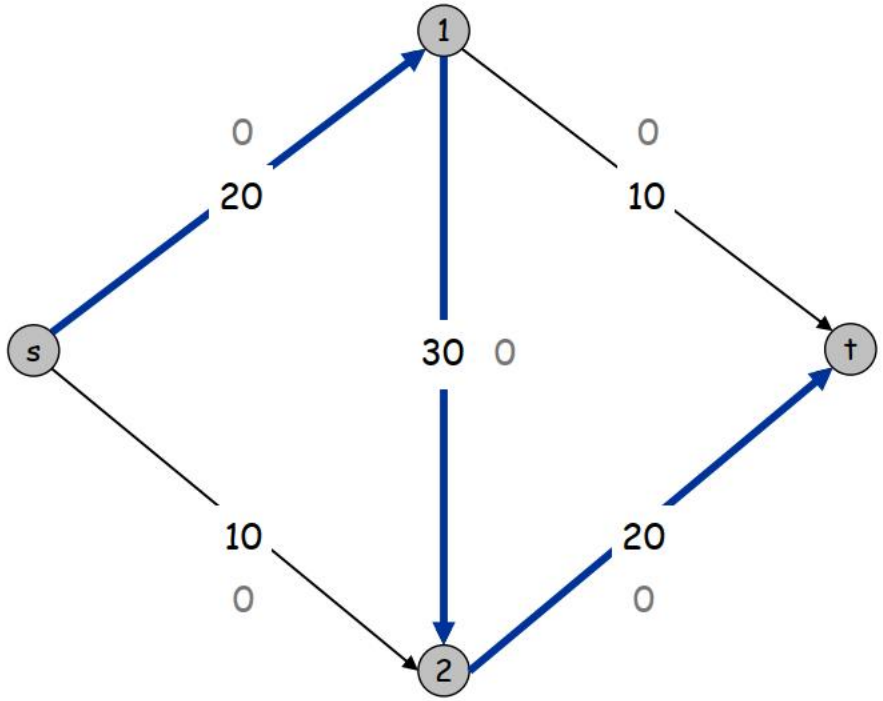
Maximum Flow Problem

- **Max flow problem.** Find s-t flow of maximum value



Towards a Max Flow Algorithm

- **Greedy algorithm**
  - Start with f(e) = 0 for all edge e ∈ E
  - Find an s-t path P where each edge has f(e) < c(e)
  - Augment flow along path P
  - Repeat until you get stuck



Flow value = 0

Towards a Max Flow Algorithm

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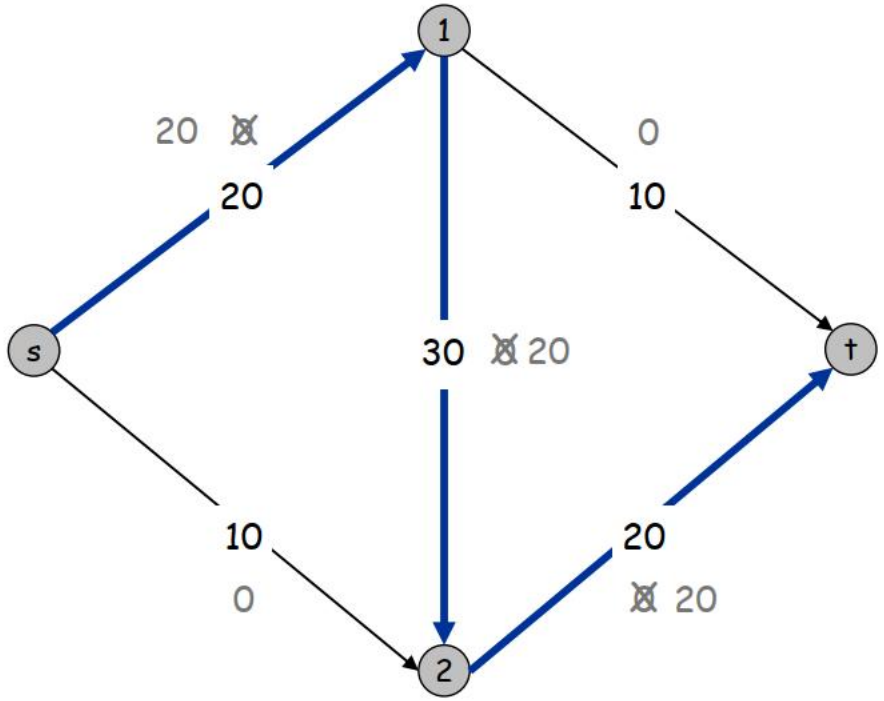

Flow value = 20

Towards a Max Flow Algorithm

- **Greedy algorithm**
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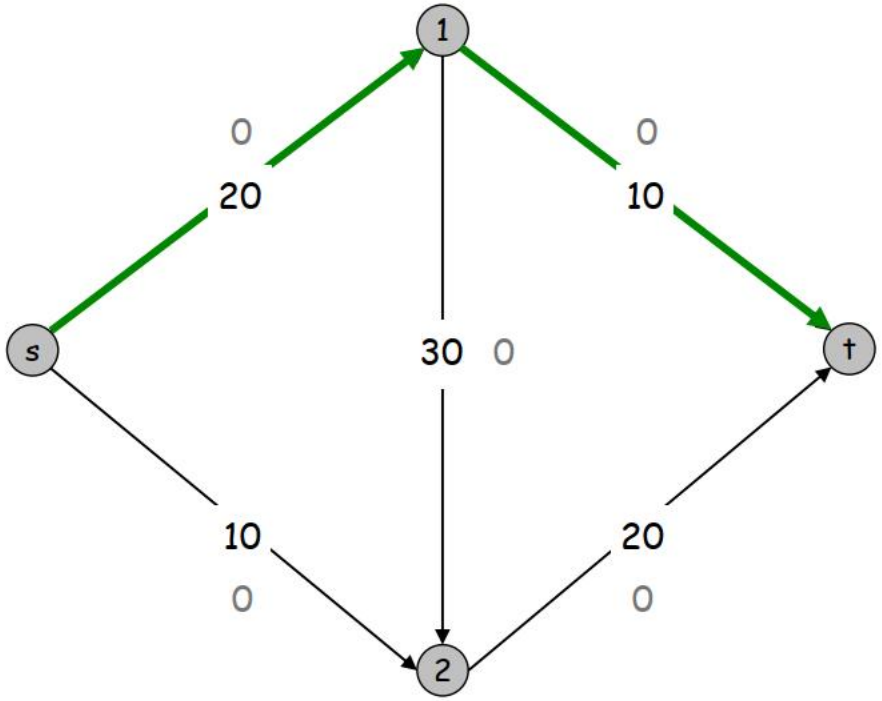

Flow value = 0

Towards a Max Flow Algorithm

- **Greedy algorithm**
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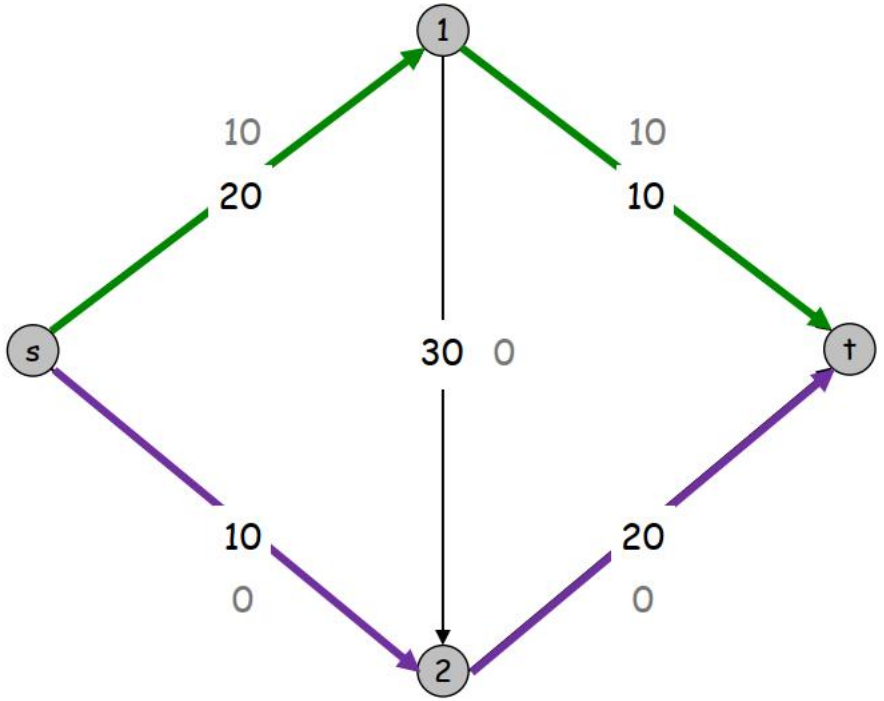

Flow value = 10

Towards a Max Flow Algorithm

- **Greedy algorithm**
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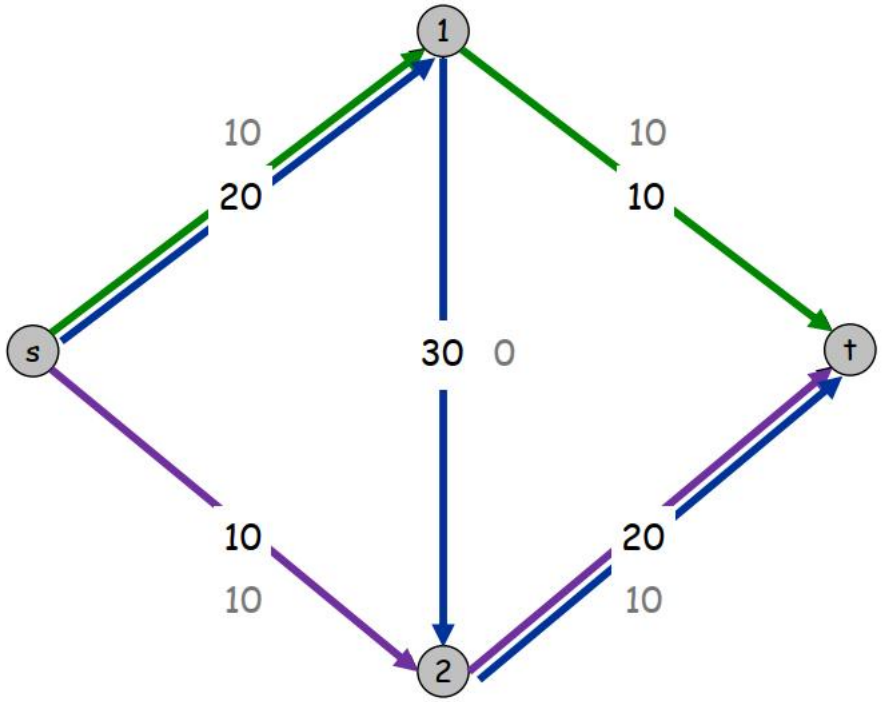

Flow value = 20

Towards a Max Flow Algorithm

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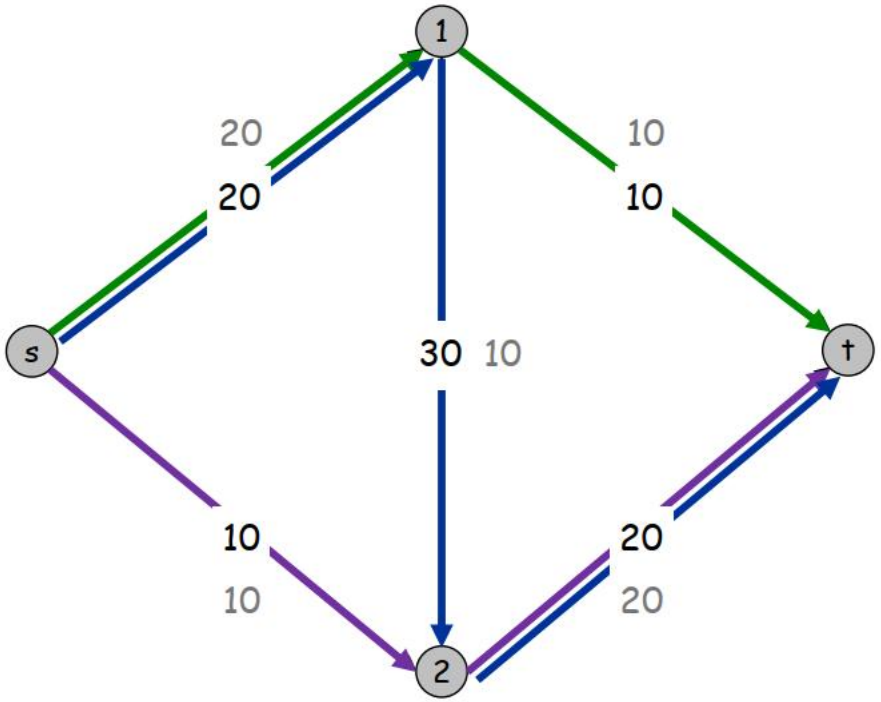
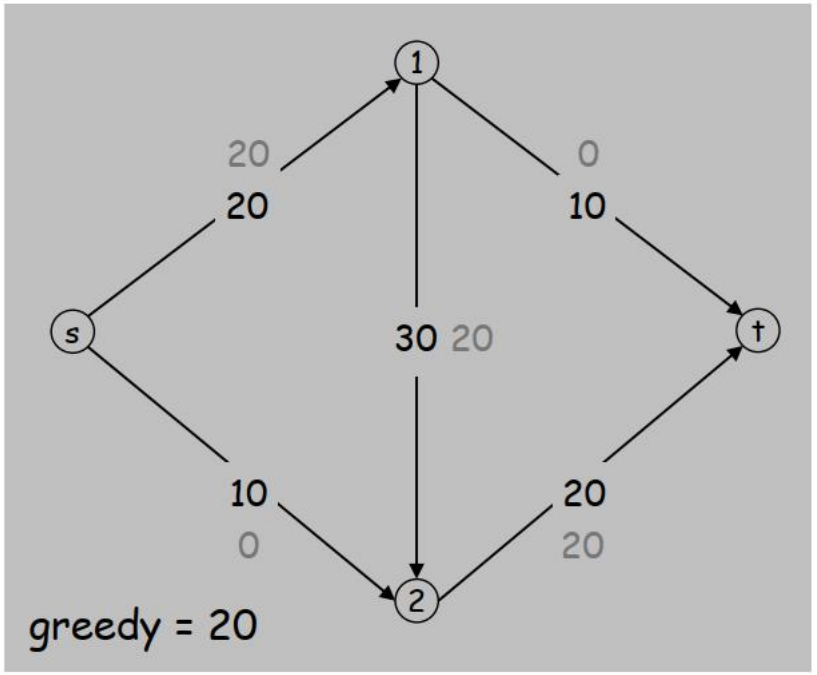


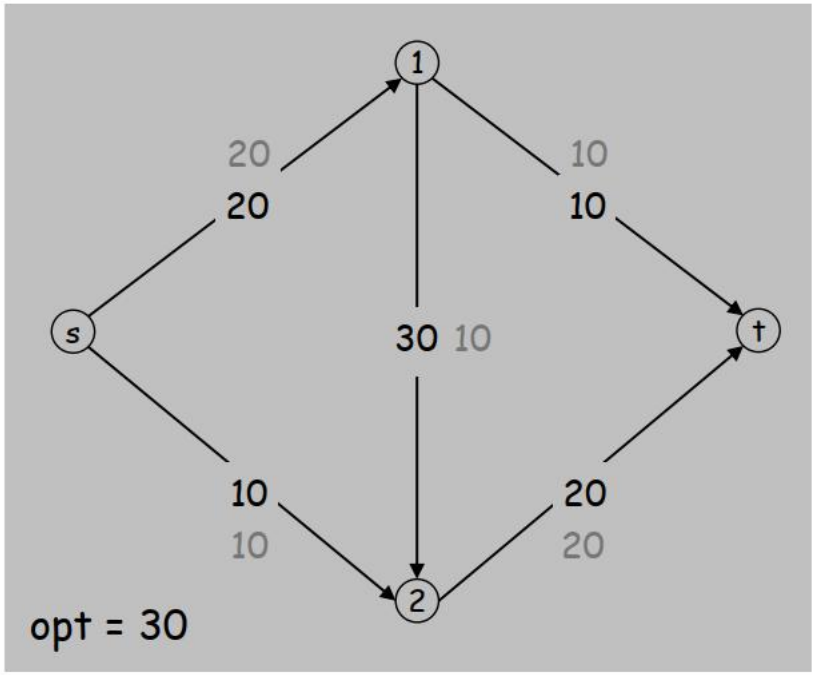
Flow value = 30

Towards a Max Flow Algorithm

- **Greedy algorithm**
  - Start with f(e) = 0 for all edge e ∈ E
  - Find an s-t path P where each edge has f(e) < c(e)
  - Augment flow along path P
  - Repeat until you get **stuck**

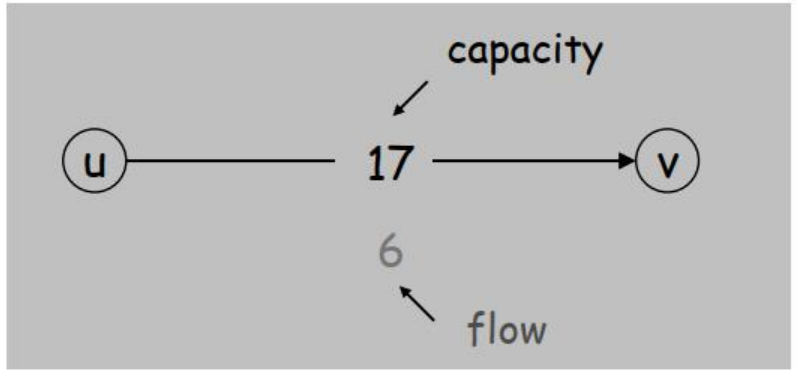
locally optimality ⇏ globally optimality





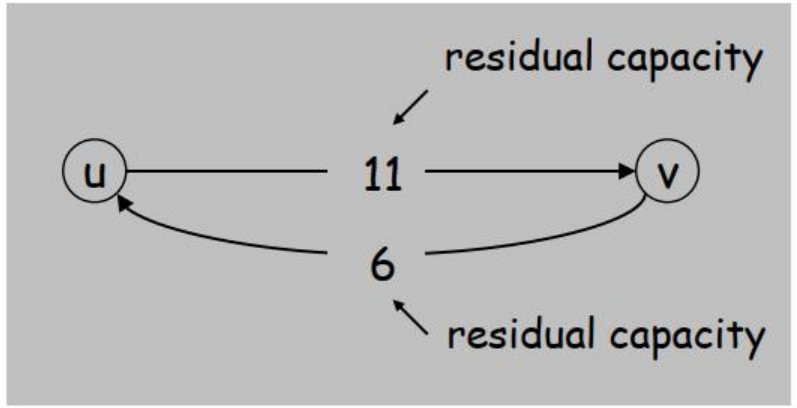
Residual Graph

- **Original edge:** e = (u, v) ∈ E
  - Flow f(e), capacity c(e)



- **Residual edge**
  - "Undo" flow sent
  - e = (u, v) and e^R = (v, u)
  - Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$



- **Residual graph:** $G_f = (V, E_f)$
  - Residual edges with positive residual capacity
  - $E_f = \{e: f(e) < c(e)\} \cup \{e^R: f(e) > 0\}$

Augmenting Path

- **Augmenting path:** a simple s-t path P in the residual graph G_f
- **Bottleneck capacity** of an augmenting path P is the minimum residual capacity of any edge in P

```
Augment(f, c, P) {
    b ← bottleneck(P)
    foreach e ∈ P {
        if (e ∈ E) f(e) ← f(e) + b      forward edge
        else       f(e^R) ← f(e^R) - b  reverse edge
    }
    return f
}
```

- **Claim:** After augmentation, f is sill a flow

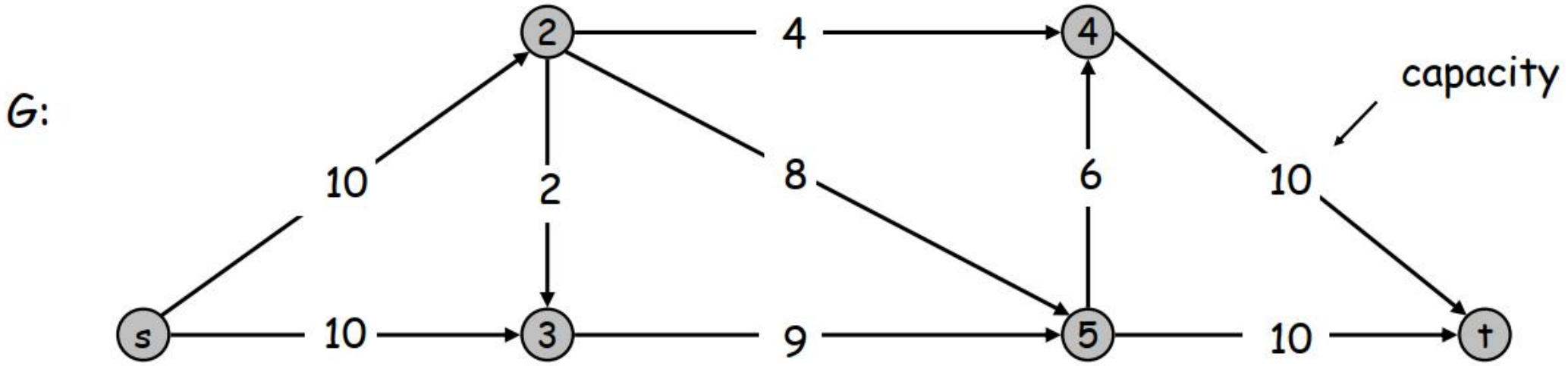
Ford-Fulkerson Algorithm

- **Ford-Fulkerson Algorithm**
  - Start with f(e) = 0 for all edge $e \in E$
  - Find an augmenting path P in the residual graph G_f
  - Augment flow along path P
  - Repeat until you get stuck

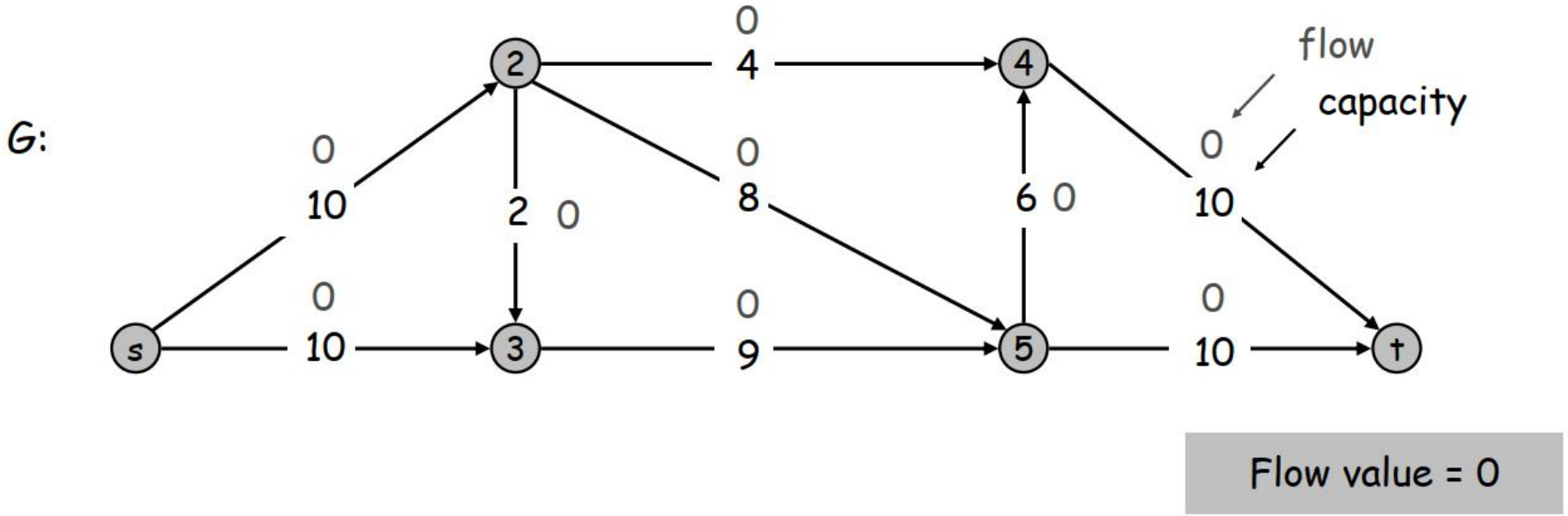
```
Ford-Fulkerson(G, s, t, c) {
    foreach e ∈ E  f(e) ← 0
    G_f ← residual graph

    while (there exists augmenting path P) {
        f ← Augment(f, c, P)
        update G_f
    }
    return f
}
```

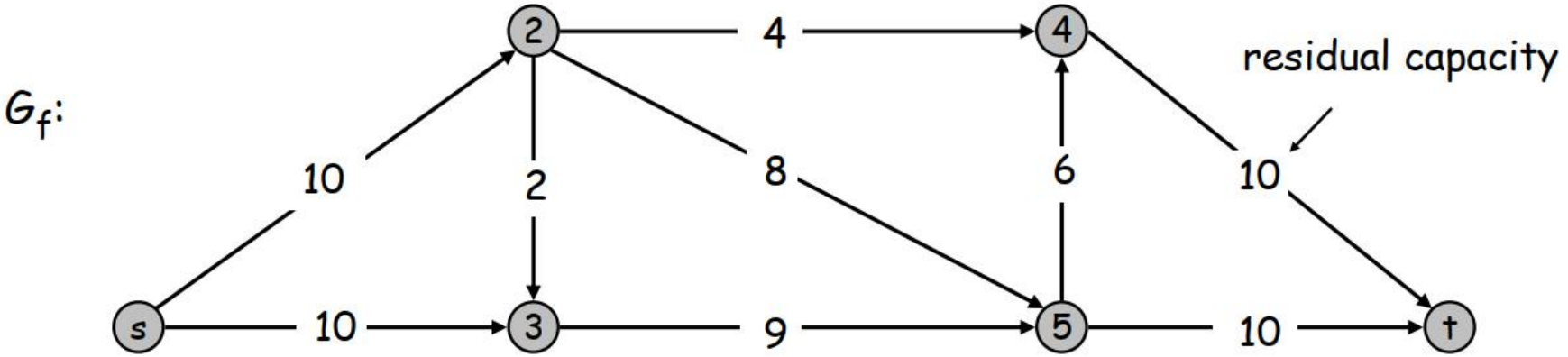
Ford-Fulkerson Algorithm



Ford-Fulkerson Algorithm



Flow value = 0



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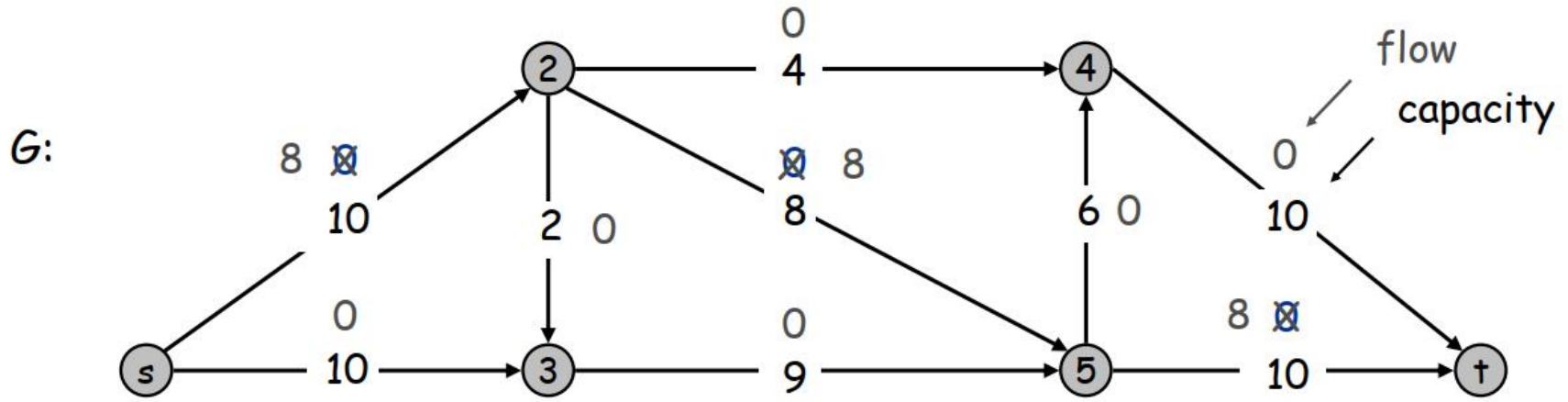

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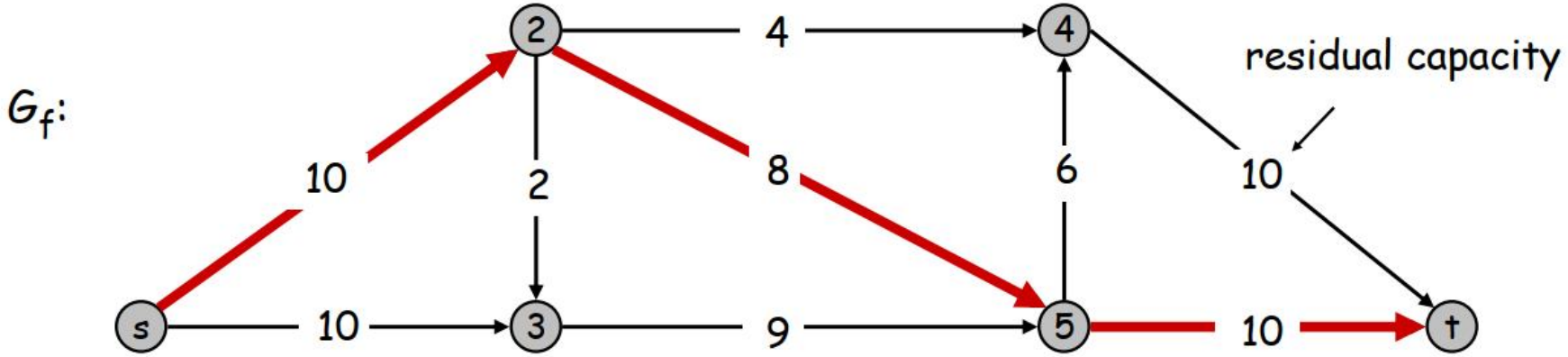

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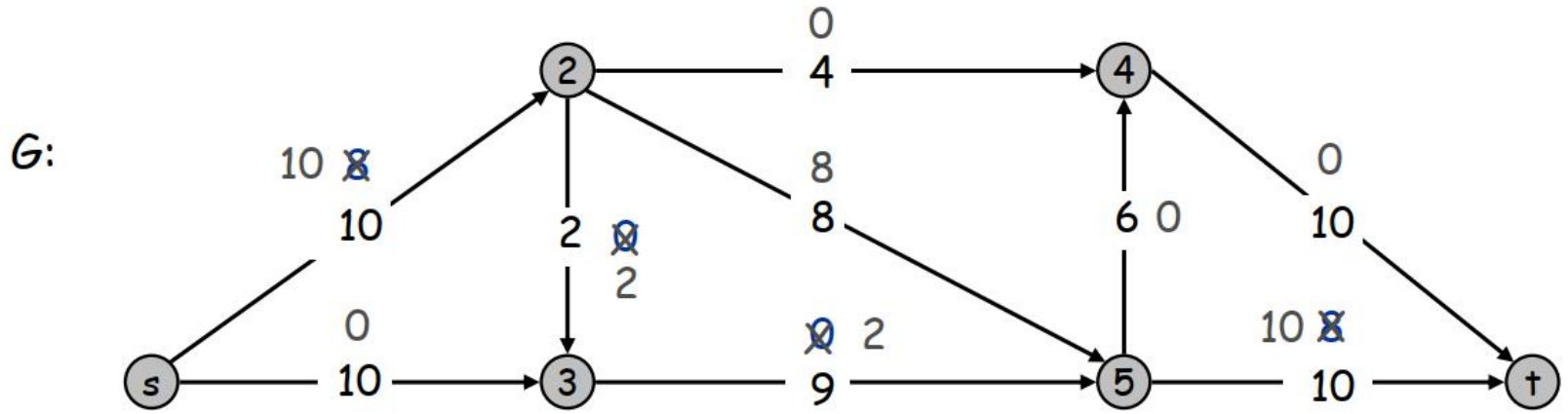
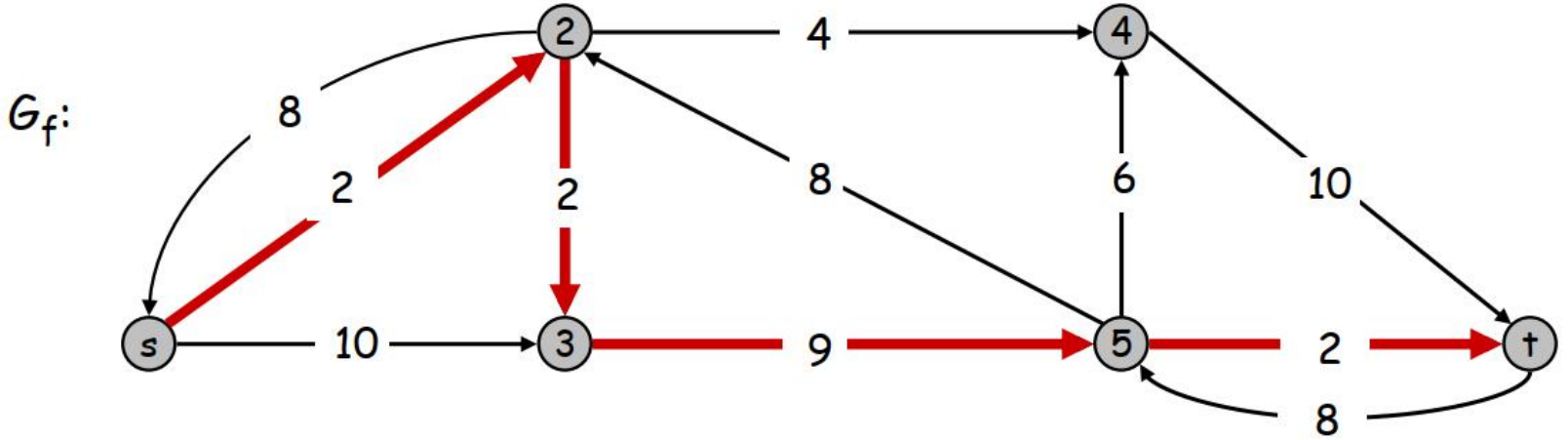


Flow value = 8



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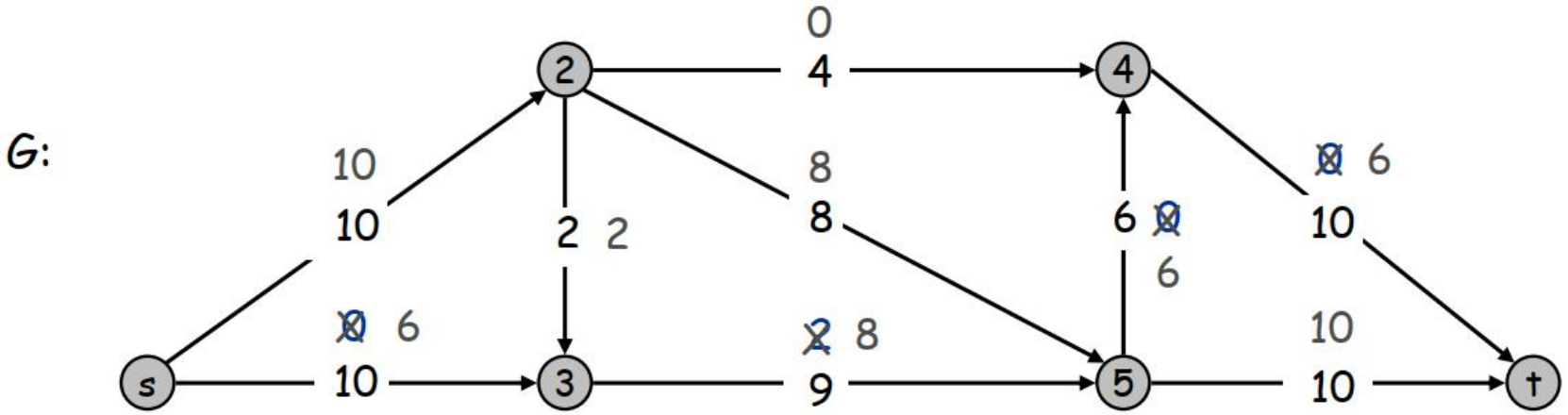
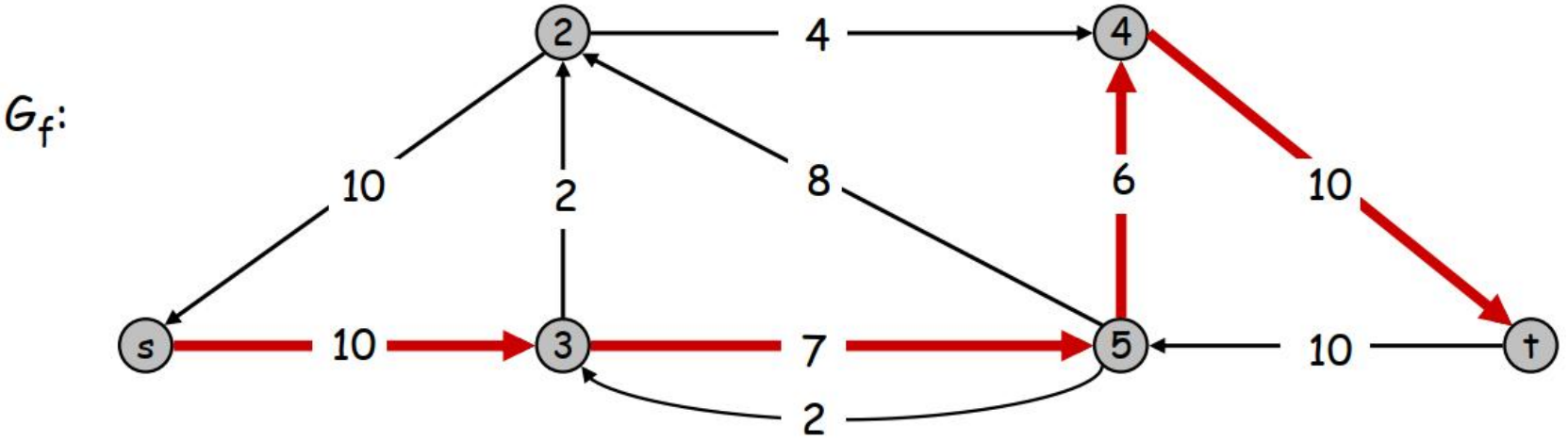
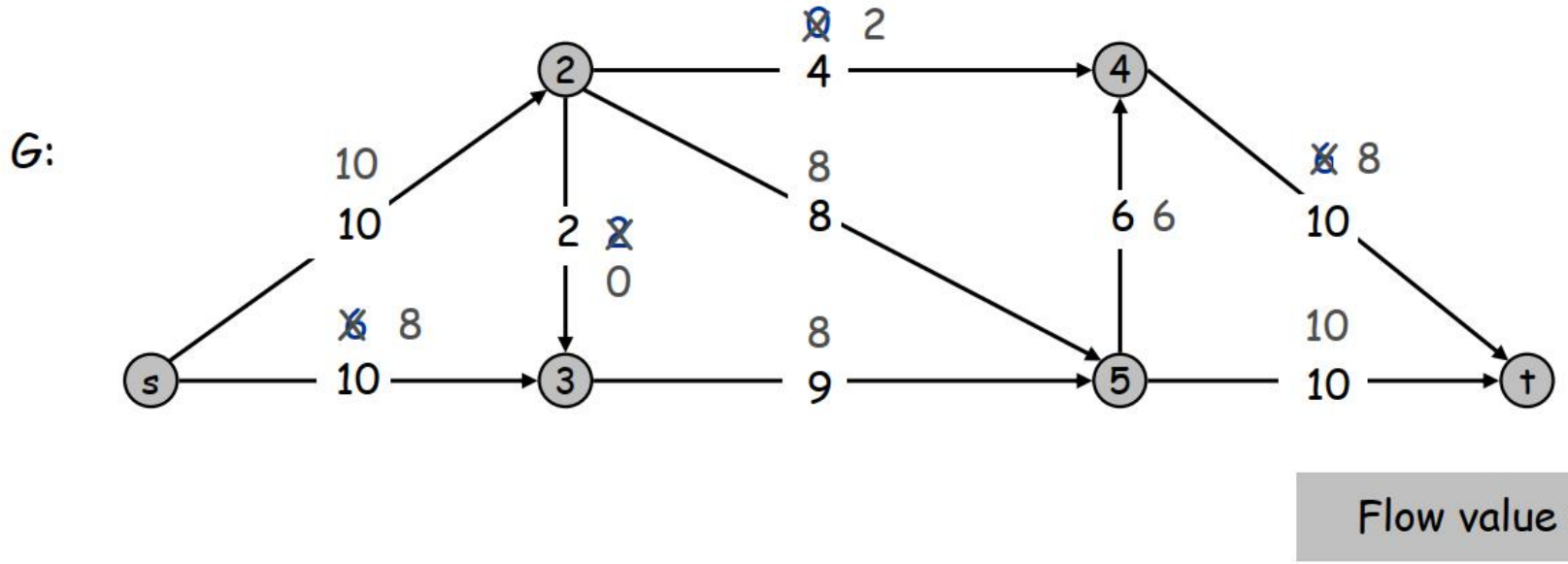


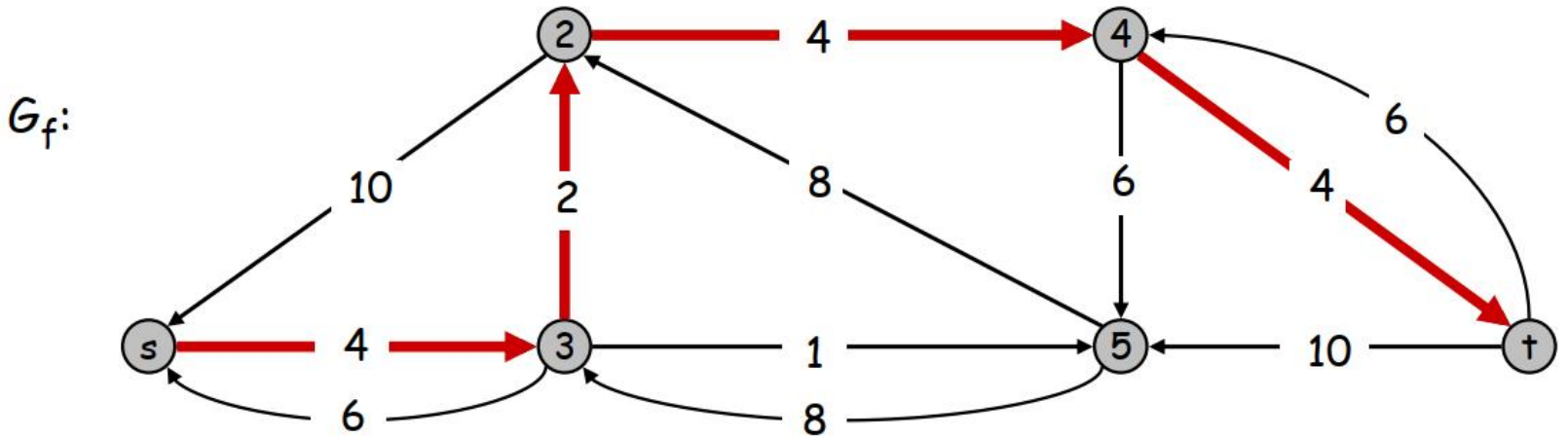
Flow value = 10



Ford-Fulkerson Algorithm

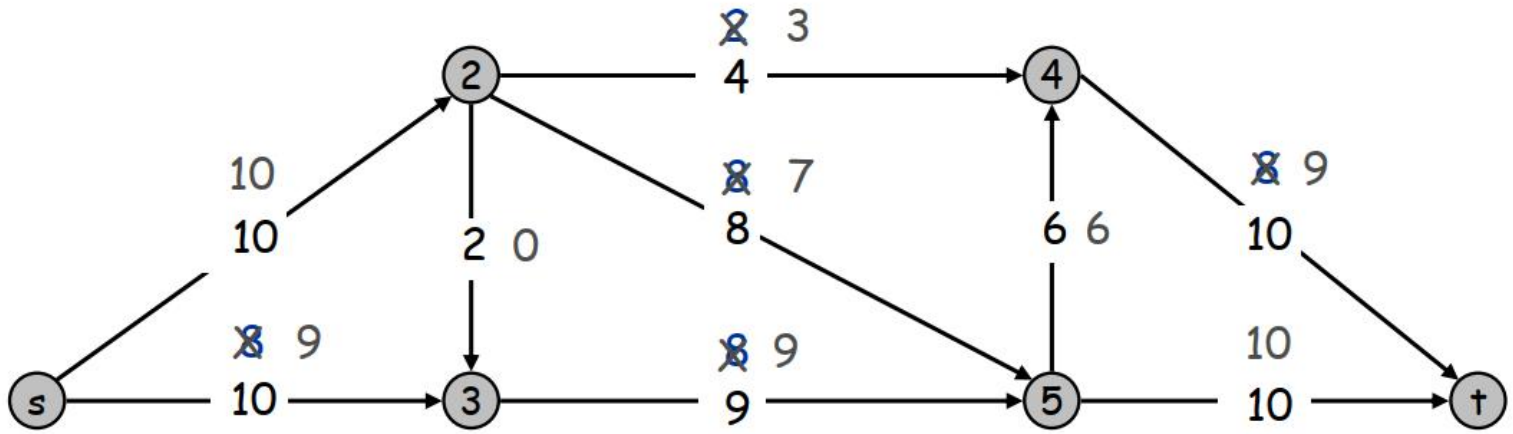


Flow value = 16



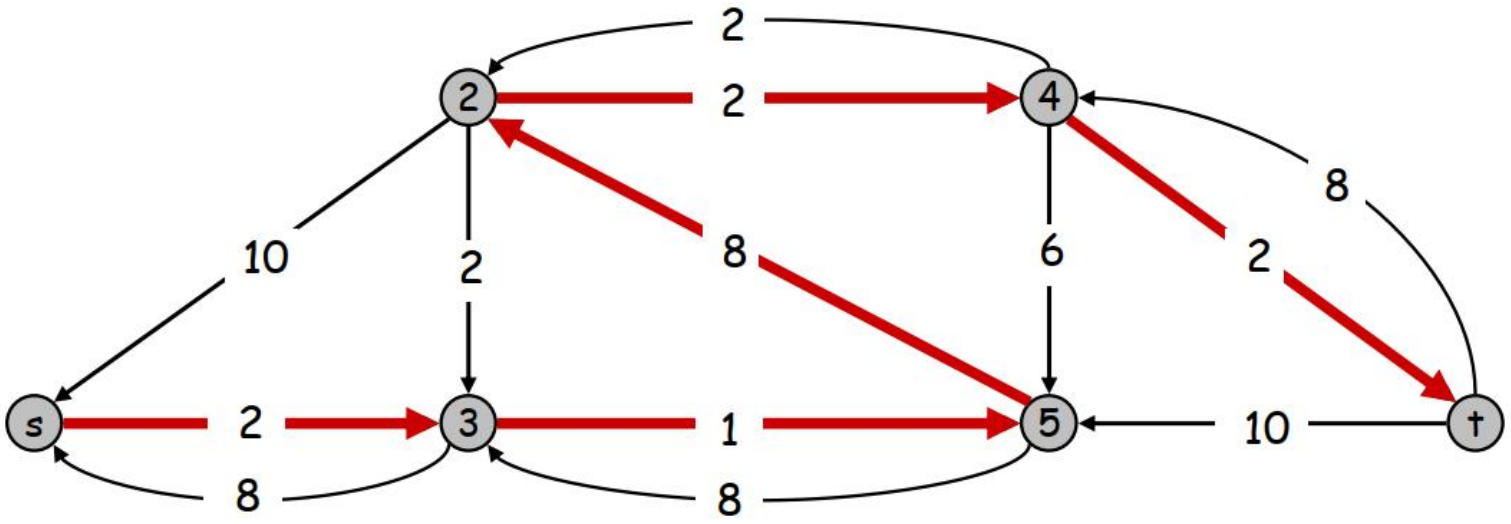
Ford-Fulkerson Algorithm

G:



Flow value = 18

G_f:



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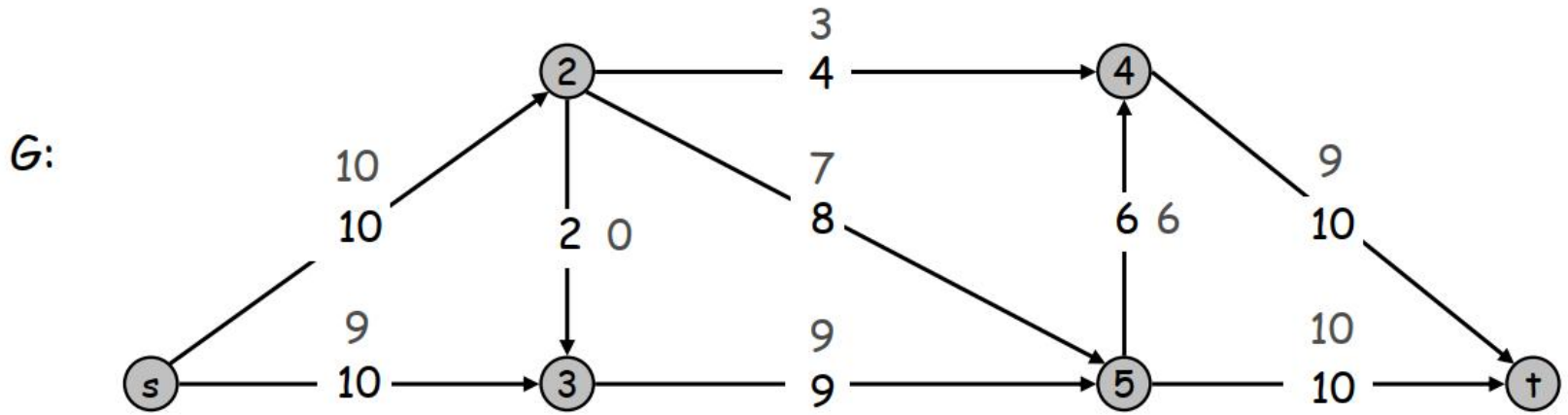
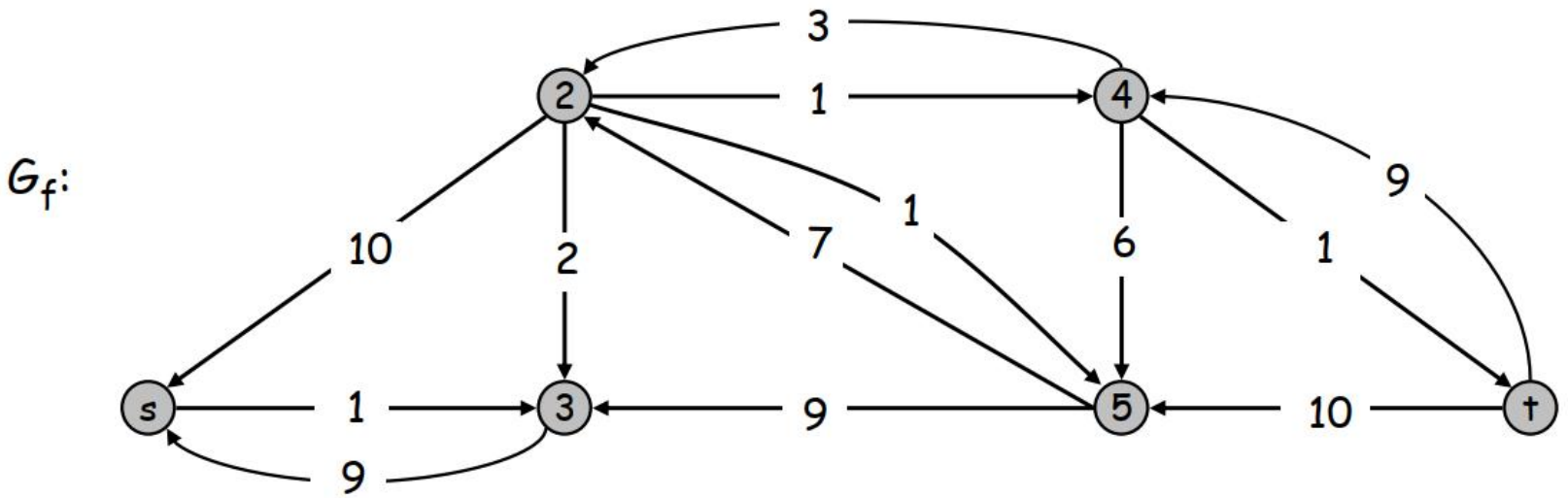


Flow value = 19

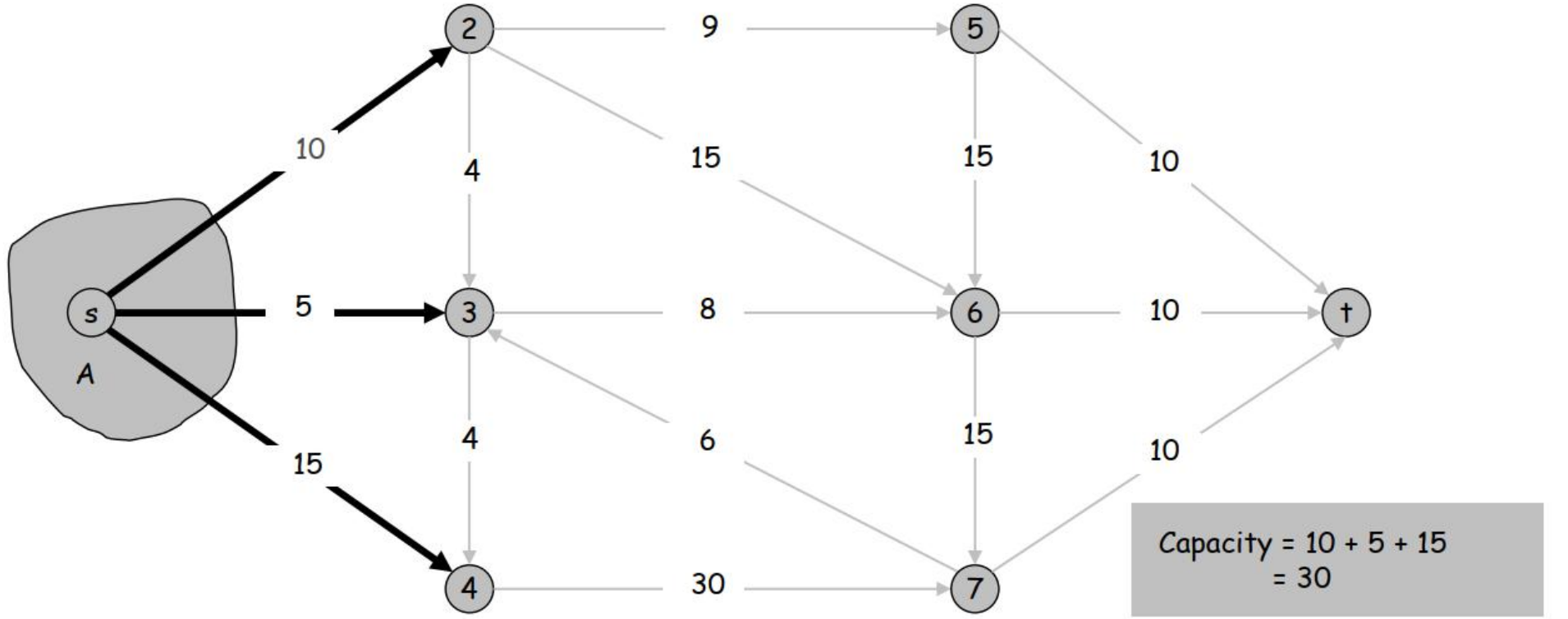


Max-flow and Min-Cut

Cuts

- **Def.** An s-t cut is a partition (A, B) of V with $s \in A$ and $t \in B$

- **Def.** The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



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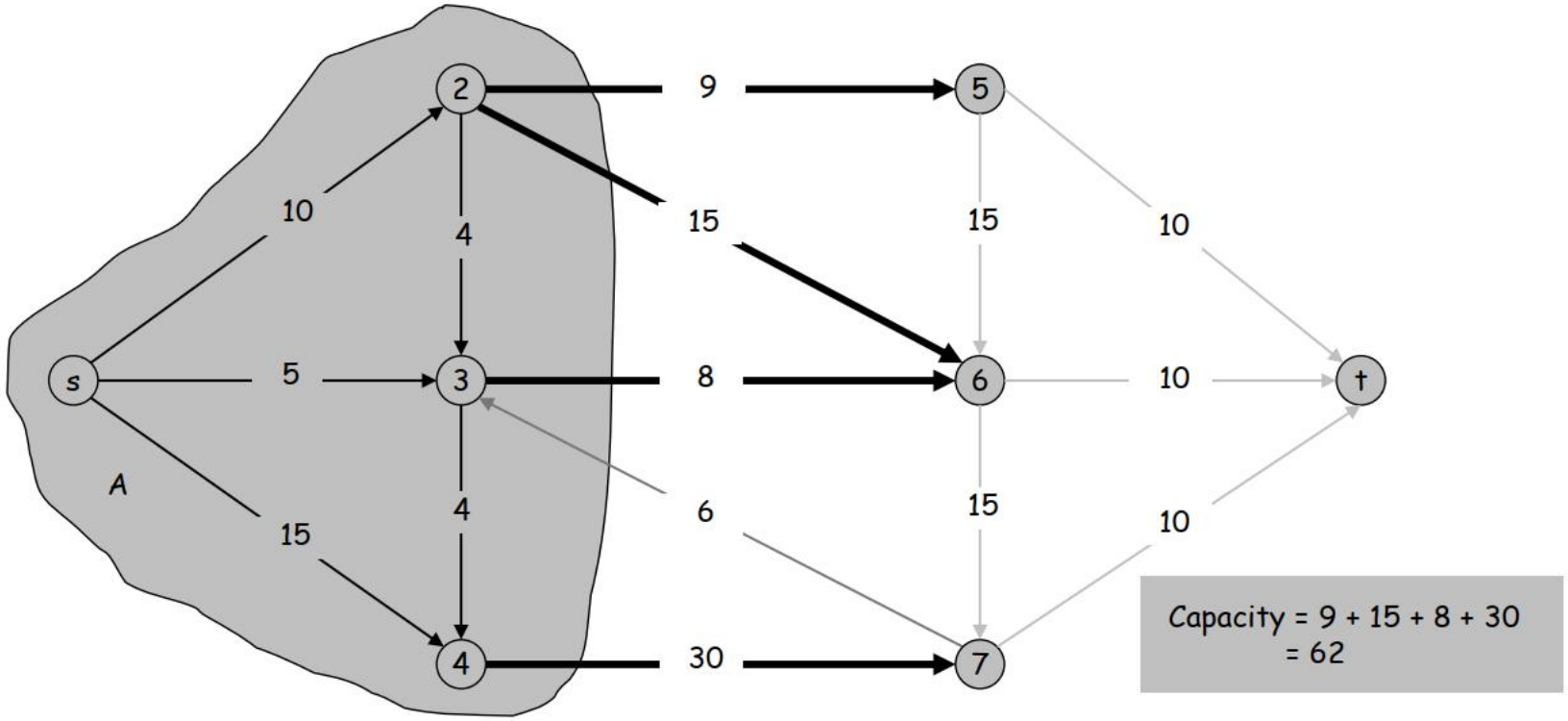
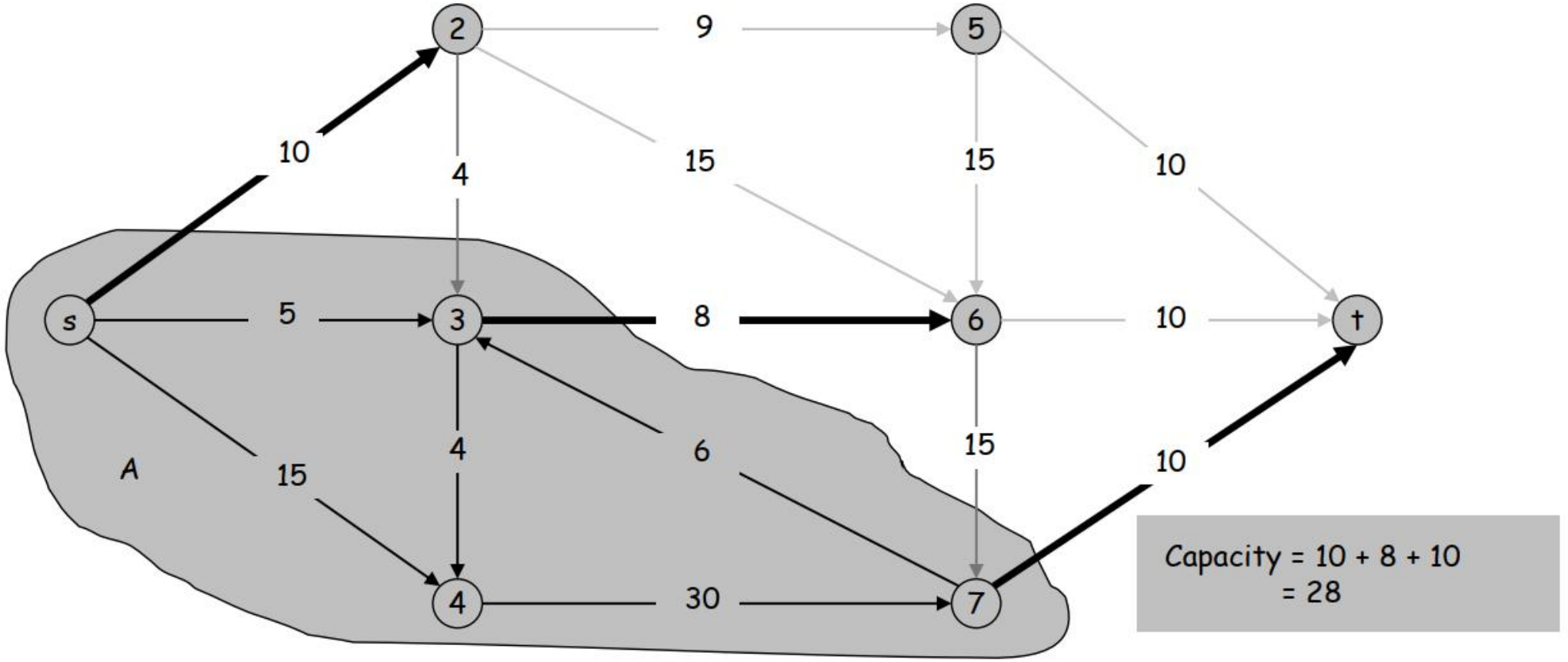


Minimum Cut Problem

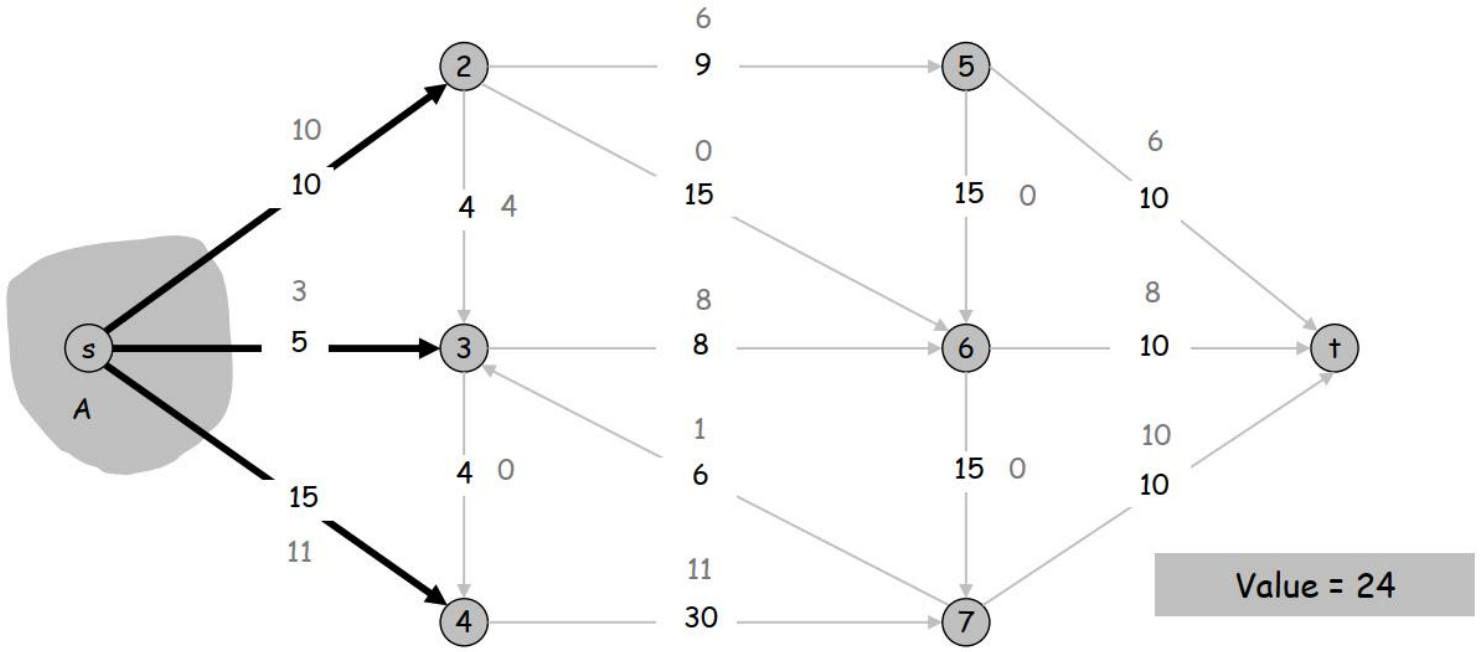
- **Min s-t cut problem.** Find an s-t cut of minimum capacity



Flows and Cuts

- **Flow value lemma.** Let f be any flow, and let (A, B) be any s-t cut
- Then, the net flow sent across the cut is equal to the amount leaving s

$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to A}} f(e) \; = \; v(f)$$



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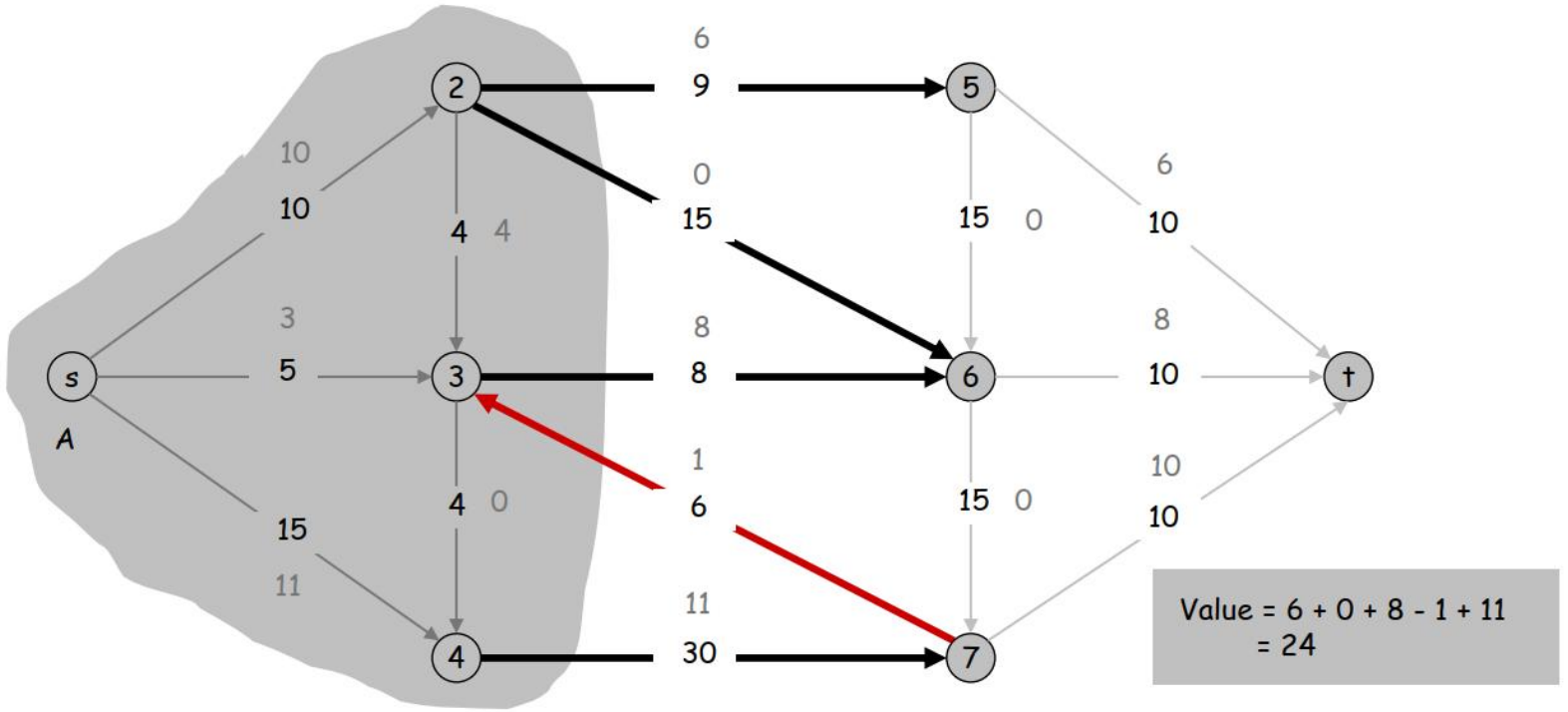

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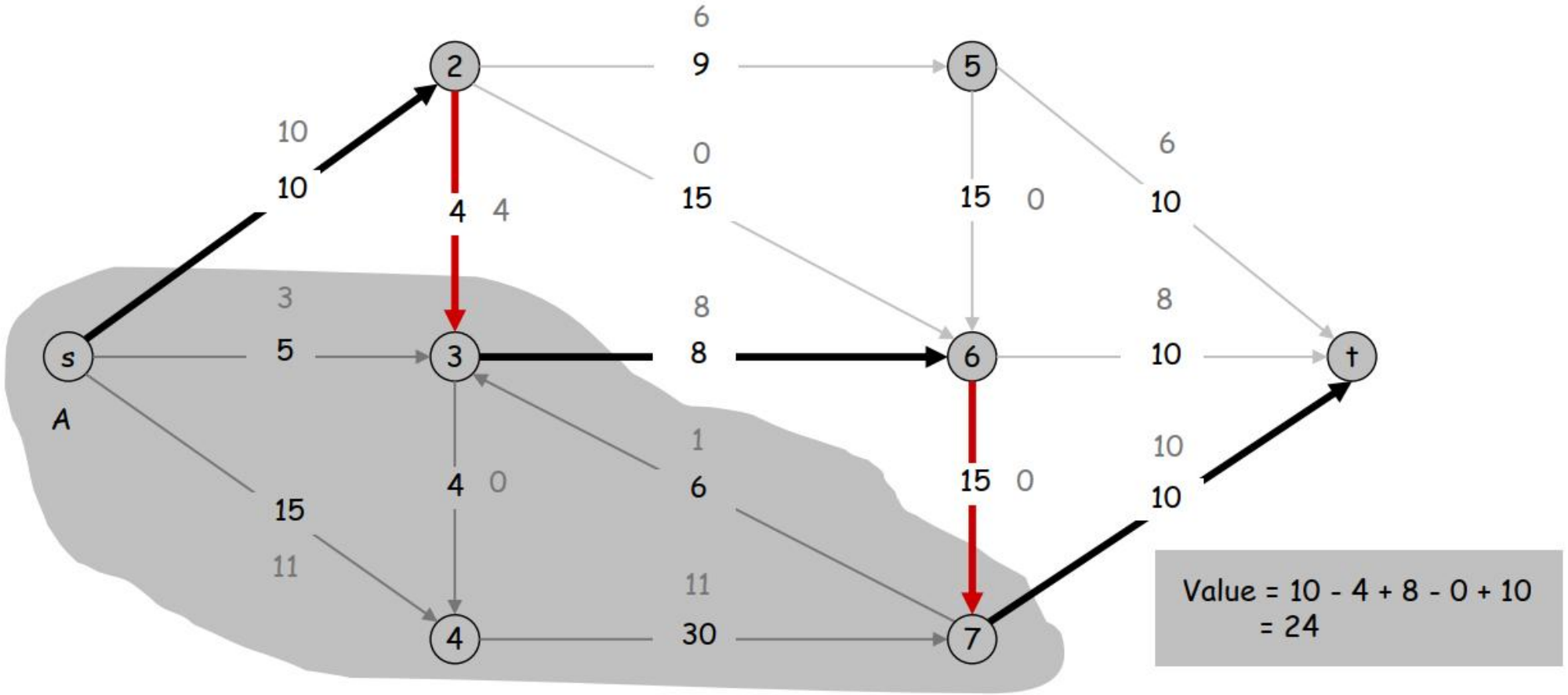


Flows and Cuts

- **Flow value lemma.** Let f be any flow, and let (A, B) be any s-t cut. Then,

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

by flow conservation, all terms except v = s are 0 $\longrightarrow$

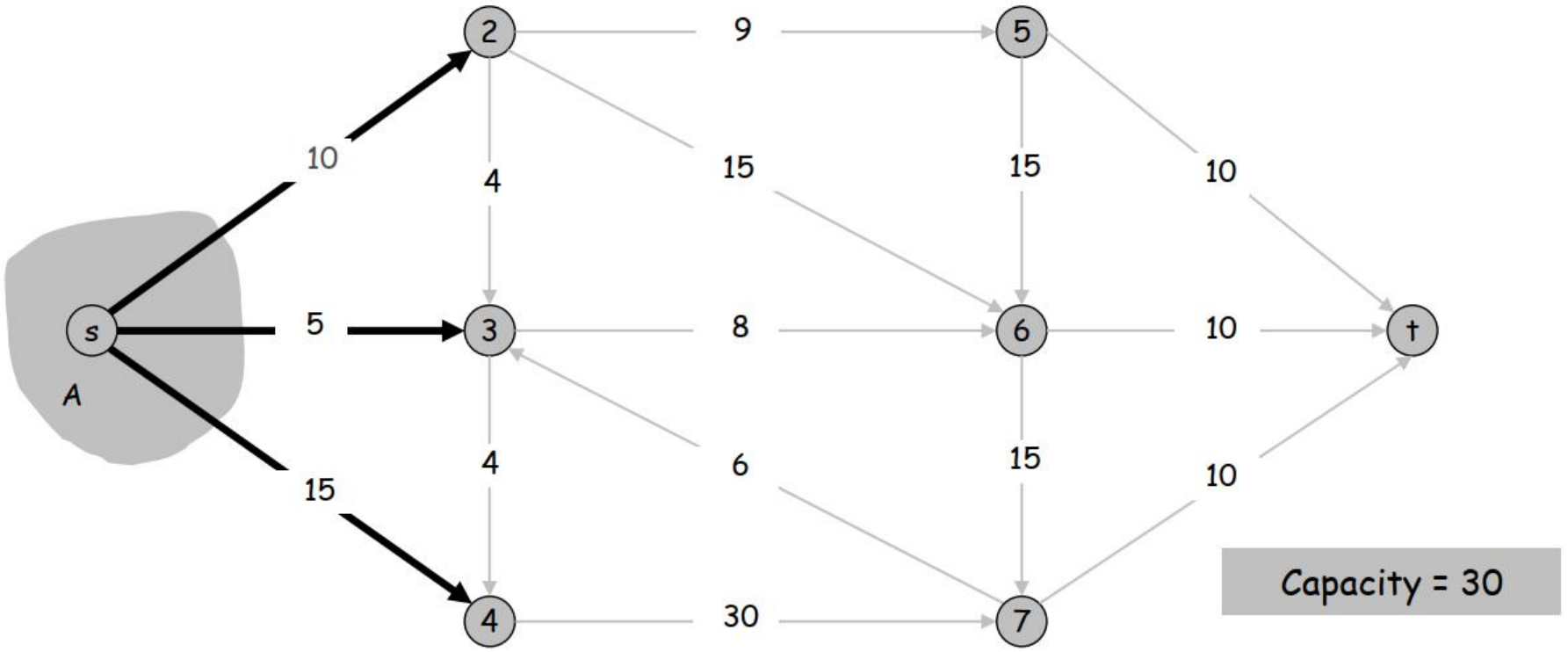
$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to v}} f(e) \right)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e).$$

Flows and Cuts

- **Weak duality.** Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut

Cut capacity = 30 ⇒ Flow value ≤ 30



Flows and Cuts

- **Weak duality.** Let f be any flow. Then, for any s-t cut (A, B) we have v(f) <= cap(A, B)
- Pf.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&\le \sum_{e \text{ out of } A} f(e) \\
&\le \sum_{e \text{ out of } A} c(e) \\
&= \operatorname{cap}(A, B) \quad \blacksquare
\end{aligned}
$$

Max-Flow Min-Cut Theorem

- **Augmenting path theorem.** Flow f is a max flow iff there are no augmenting paths
- **Max-flow min-cut theorem.** [Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut
- **Proof strategy.** We prove both simultaneously by showing the equivalence of the following three conditions for any flow f:

  (i) There exists a cut (A, B) such that v(f) = cap(A, B)

  (ii) Flow f is a max flow

  (iii) There is no augmenting path relative to f

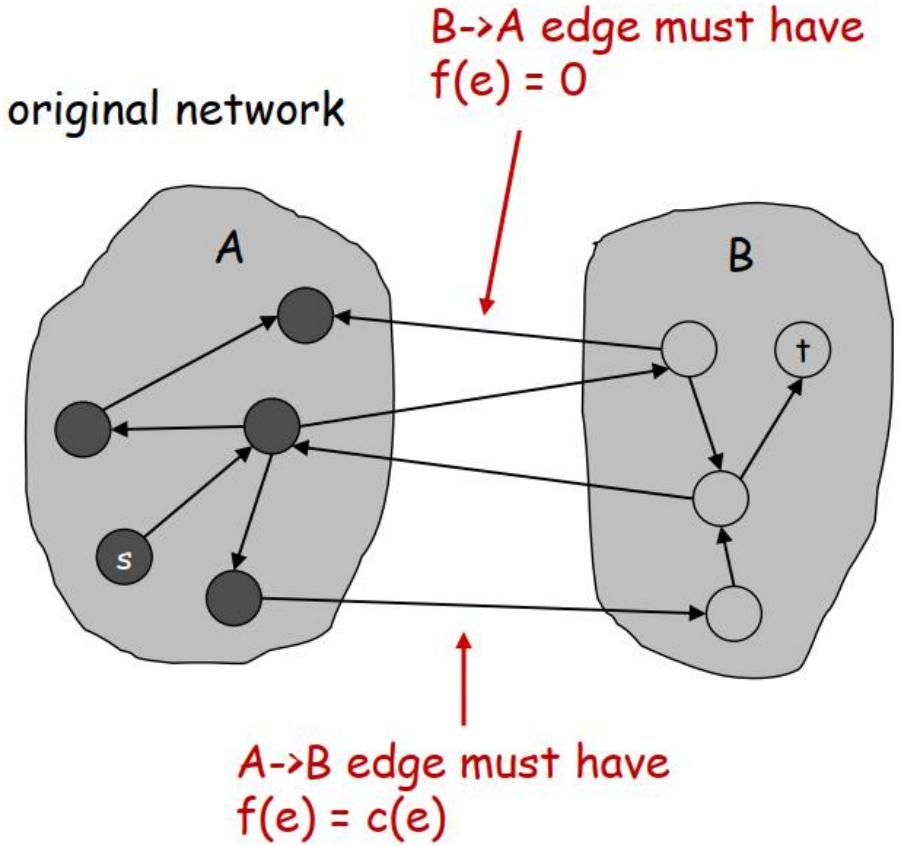
- (i) → (ii) This was the corollary to weak duality lemma

- (ii) → (iii) We show contrapositive
  - If there exists an augmenting path, then we can improve f by sending flow along path

Proof of Max-Flow Min-Cut Theorem

- (iii) → (i)
  - Let f be a flow with no augmenting paths
  - Let A be set of vertices reachable from s in residual graph
  - By definition of A, s ∈ A
  - By definition of f, t ∉ A

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\
&= \sum_{e \text{ out of } A} c(e) \\
&= cap(A, B) \quad \blacksquare
\end{aligned}
$$



Choosing Good Augmenting Paths

Running Time

- **Assumption.** All capacities are integers between 1 and C

- **Invariant.** Every flow value f(e) and every residual capacities $c_f(e)$ remains an integer throughout the algorithm

- **Integrality theorem.** If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer

  Pf. Since algorithm terminates, theorem follows from invariant

- **Theorem.** The algorithm terminates in at most $v(f^*) \leq nC$ iterations.

  Pf. Each augmentation increase value by at least 1

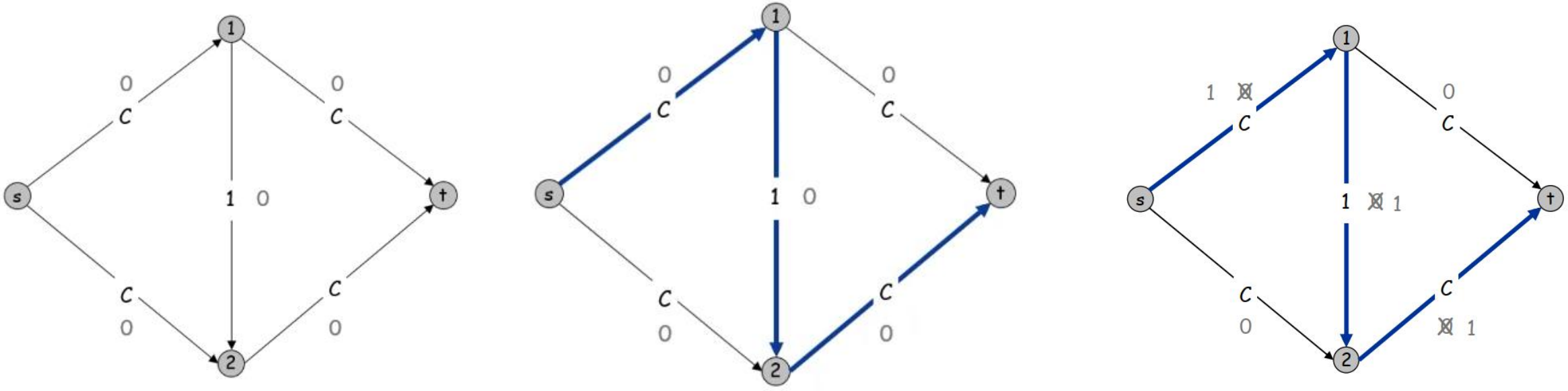
- **Corollary.** Running time of Ford-Fulkerson is O(mnC) ← Polynomial?

Ford-Fulkerson: Exponential Number of Augmentations

- **Generic Ford-Fulkerson algorithm is not polynomial in input size?**

m, n, and log C

- **An example:** If max capacity is C, then the algorithm can take >= C iterations



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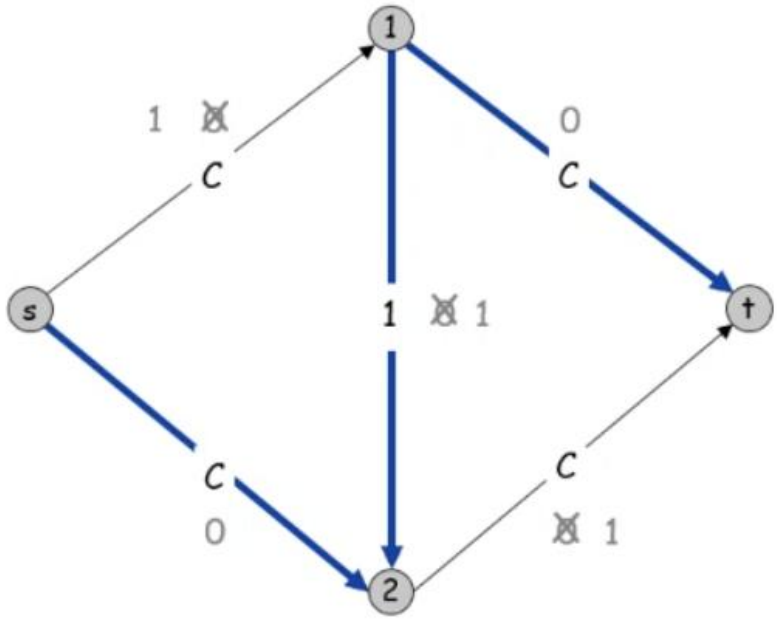

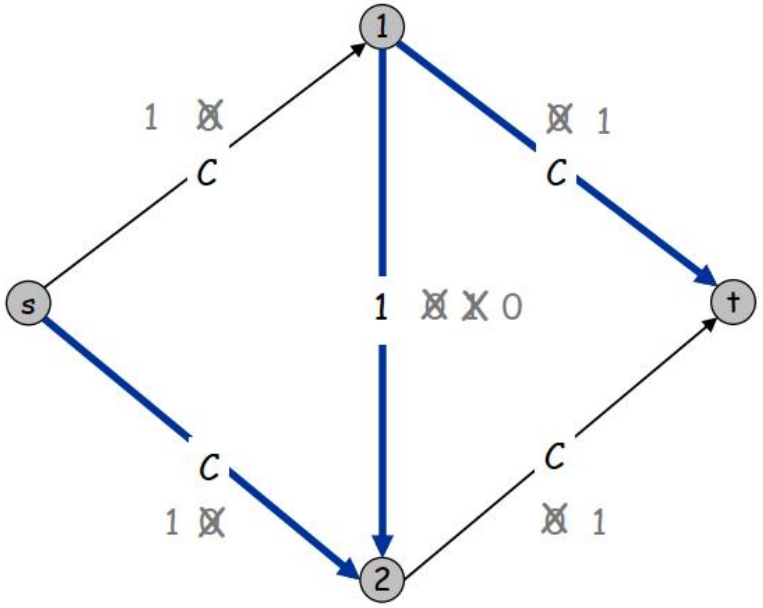

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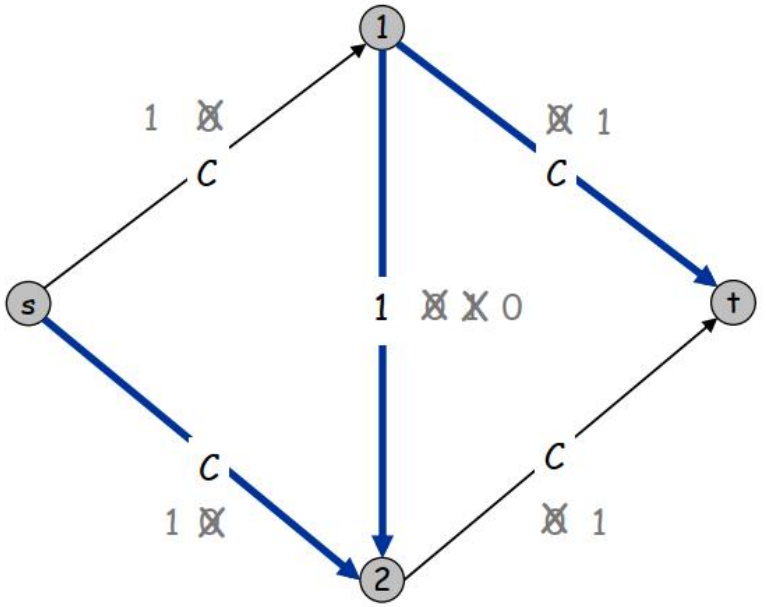


Each augmenting path sends only 1 unit of flow (#augmenting paths = 2C)

Choosing Good Augmenting Paths

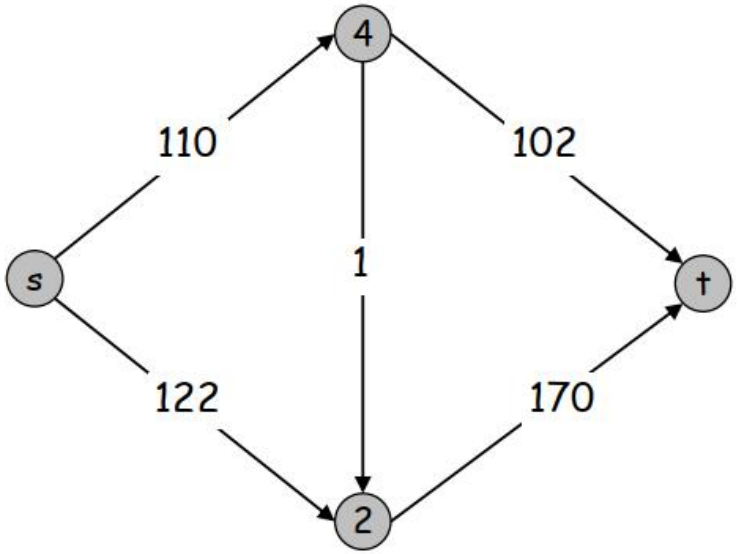
- **Use care when selecting augmenting paths**
  - Some choices led to exponential algorithms
  - Clever choices lead to polynomial algorithms
  - (If capacities are irrational, algorithm not guaranteed to terminate!)

- **Goal: choose augmenting paths so that:**
  - Can find augmenting paths efficiently
  - Few iterations

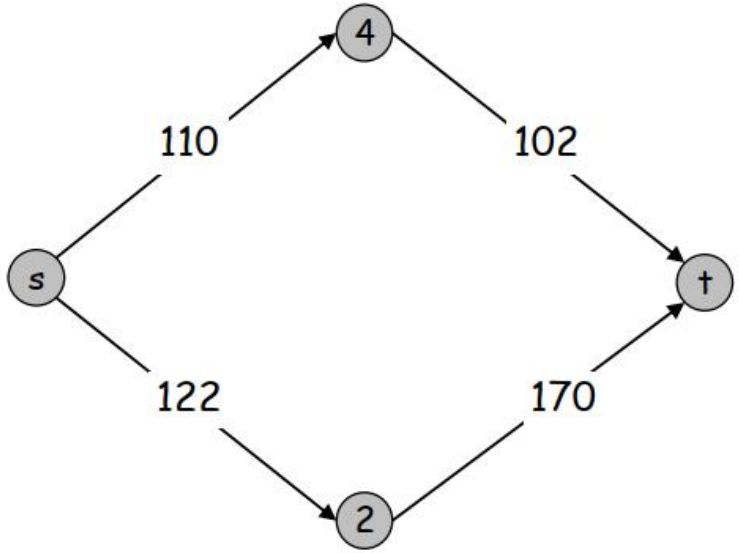
- **Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]**
  - Max bottleneck capacity
  - Fewest number of edges
  - **Sufficiently large bottleneck capacity**

Capacity Scaling

- **Intuition.** Choosing path with high bottleneck capacity
  - Maintain scaling parameter Δ
  - Let the **Δ-residual graph $G_f(\Delta)$** be the subgraph of the residual graph consisting of only arcs with capacity at least Δ



G_f



$G_f(100)$

Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {
    foreach e ∈ E   f(e) ← 0
    Δ ← largest power of 2 ≤ C

    while (Δ ≥ 1) {
        G_f(Δ) ← Δ-residual graph
        while (there exists augmenting path P in G_f(Δ)) {
            f ← augment(f, c, P)
            update G_f(Δ)
        }
        Δ ← Δ / 2
    }
    return f
}
```

Capacity Scaling: Correctness

- **Assumption.** All edge capacities are integers between 1 and C
- **Integrality invariant.** All flow and residual capacity values are integral
- **Correctness.** If the algorithm terminates, then f is a max flow
- **Pf.**
  - By integrality invariant, when $\Delta = 1 \rightarrow G_f(\Delta) = G_f$
  - Upon termination of $\Delta = 1$ phase, there are no augmenting paths

Capacity Scaling: Running Time

- **Lemma 1.** The outer while loop repeats $1 + \lfloor \log_2 C \rfloor$ times
- Pf. Initially $C/2 < \Delta <= C$. Δ decreases by a factor of 2 each iteration

- **Lemma 2.** Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow is at most $v(f) + m\Delta$ ← proof on next slide

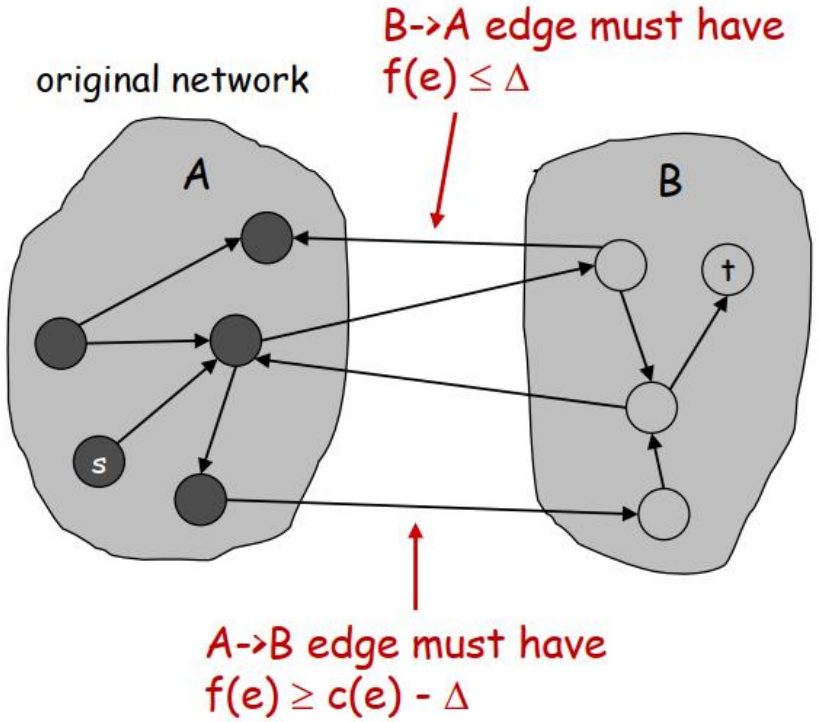
- **Lemma 3.** There are at most 2m augmentation per scaling phase.
  - Let f be the flow at then end of the previous scaling phase
  - L2 → $V(f^*) <= v(f) + m(2\Delta)$
  - Each augmentation in a Δ-phase increases v(f) by at least Δ

- **Theorem.** The scaling max-flow algorithm finds a max flow in O(mlogC) augmentations. It can be implemented to run in $O(m^2 \log C)$ time

Capacity Scaling: Running Time

- **Lemma 2.** Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow is at most v(f) + mΔ
- **Pf.** (almost identical to proof of max-flow min-cut theorem)
  - We show that at the end of a Δ-phase, there exists a cut (A, B) such that cap(A, B) <= v(f) + mΔ
  - Choose A to be the set of nodes reachable from s in $G_f(\Delta)$
  - By definition of A, s ∈ A
  - By definition of f, t ∉ A

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\
&\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
&= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\
&\geq cap(A, B) - m\Delta \qquad \blacksquare
\end{aligned}
$$



Next Time:
Network Flow (Cont.)