Estimation

Sometimes a sample from a random variable X may be used for the purpose of estimating one or several (unknown) parameters associated with the probability distribution of X.

Consider the following example:

A manufacturer has supplied ~~to~~ us with 10,000 small rivets. A soundly riveted joint requires that a rivet ~~to~~ fits properly into its machine hole and consequently some trouble will be experienced when the rivet is (defective) burred. Before accepting this shipment, we want to have some idea about the magnitude of p, the proportion of defective (ie, burred) rivets.

We proceed as follows:

We inspect n rivets chosen at random from the lot.

Define the random variable X_i.

$$X_i = \begin{cases} 1 & \text{if } i\text{th item is defective} \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1, 2, \ldots n$.

Hence, we may consider $X_1, X_2, \ldots, X_n$ to be a sample from the random variable X whose distribution is given by $P(X=1) = p$, $P(X=0) = 1-p$

Now the probability distribution of X depends on the unknown parameter p. The sample $X_1, X_2, \ldots, X_n$ estimate the value of p.

So we have to find a Statistics $H(X_1, X_2, \dots X_n)$ as a (point) estimate of p.

Definition: Let X be a random variable with some probability distribution depending on an unknown parameter θ.
Let $X_1, X_2, \dots, X_n$ be a sample of X and let $x_1, x_2, \dots, x_n$ be the corresponding sample values. If $g(X_1, X_2, \dots, X_n)$ is a function of the sample to be used for estimating θ then 'g' is called an $\boxed{\text{estimator}}$ of θ.
The value which g assumes, that is $g(x_1, x_2, \dots, x_n)$ will be referred as estimate of θ.

Written as $\hat{\theta} = g(x_1, x_2, \dots, x_n)$.

Note: (i) $\hat{\theta}$, the estimate of θ (that is, estimator $g(X_1, \dots$

(ii) $E(\hat{\theta})$ = the mean $E[g(X_1, X_2, \dots, X_n)]$.

Definition: Let $\hat{\theta}$ be an estimate for the unknown parameter θ associated with the distribution of a random variable X. Then $\hat{\theta}$ is called an unbiased estimator (or unbiased estimate) for θ if $E(\hat{\theta}) = \theta$ for all θ.

Example: If $\bar{X}$ is the sample mean and μ is the ~~popu~~ pop mean then $E(\bar{X}) = \mu$
Therefore $\bar{X}$ is an unbiased estimator for μ.

Definition:

Example: Show that the sample variance s^2 is not an unbiased estimator for σ^2.

Solution:

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \qquad (\text{refer: } --)$$

Consider

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}\left[(x_i-\mu)+(\mu-\bar{x})\right]^2$$

$$= \sum_{i=1}^{n}\left[(x_i-\mu)^2 + 2(x_i-\mu)(\mu-\bar{x}) + (\mu-\bar{x})^2\right]$$

$$= \sum_{i=1}^{n}(x_i-\mu)^2 + 2(\mu-\bar{x})\sum_{i=1}^{n}(x_i-\mu) + n(\mu-\bar{x})^2$$

But

$$\sum_{i=1}^{n}(x_i-\mu) = \sum_{i=1}^{n}x_i - n\mu = n\bar{x} - n\mu = -n(\mu-\bar{x})$$

Therefore

$$\sum_{i=1}^{n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}(x_i-\mu)^2 - 2n(\mu-\bar{x})^2 + n(\mu-\bar{x})^2$$

$$= \sum_{i=1}^{n}(x_i-\mu)^2 - n(\mu-\bar{x})^2$$

$$E(s^2) = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n}E\left[\sum_{i=1}^{n}(x_i-\mu)^2 - n(\mu-\bar{x})^2\right] = \frac{1}{n}\left[\underbrace{E\{(x_i-\mu)^2}_{\sigma^2} - \underbrace{nE(\bar{x}-\mu)^2}_{\frac{\sigma^2}{n}}\right]$$

$$= \frac{1}{n}\left[n\sigma^2 \cancel{+} - n\frac{\sigma^2}{n}\right] = \frac{n-1}{n}\sigma^2 \neq \sigma^2 \cancel{= E(x)}$$

Therefore s^2 is not an unbiased estimator for σ^2

$\frac{1}{n}\{(n-1)\sigma^2\}$

Remark; $\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{x})^2$ is an unbiased estimator for σ^2.

Verification: $E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{x})^2\right] = \frac{1}{n-1}\left[n\sigma^2 - n\frac{\sigma^2}{n}\right] = \sigma^2$.

(from above ex.)

Example; Show that the mean $\bar{X}$ of a random sample of size 'n' from a distribution having pdf.

$$f(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & , 0 < x < \infty, \quad 0 < \theta < \infty \\ 0 & \text{elsewhere} \end{cases}$$

exponential dis. with $\lambda = \frac{1}{\theta}$.

is an unbiased estimator of θ and has variance $\frac{\theta^2}{n}$.

(show that $\lim_{n\to\infty} V(\bar{x}) = 0$)

Solution; $E(X) = \theta$ (since $E(X) = \frac{1}{\lambda} = \frac{1}{1/\theta} = \theta$).

$E(X^2) = \frac{1}{\theta}\int_0^\infty x^2 e^{-x/\theta}\,dx = 2\theta^2$

$V(X) = 2\theta^2 - \theta^2 = \theta^2$.

$E(\bar{X}) = \mu = \theta$

Therefore $\bar{X}$ is an unbiased estimator of θ.

$V(\bar{x}) = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$,

$\lim_{n\to\infty} V(\bar{x}) = \lim_{n\to\infty}\frac{\theta^2}{n} = 0$.

problem; Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution with mean zero and variance) θ, $0<\theta<\infty$

Show that $\frac{\sum X_i^2}{n}$ is an unbiased estimator of θ and has variance $\frac{2\theta^2}{n}$.

Solution: Given $X \sim N(0, \theta)$, $E(X) = 0$, $V(X) = \theta = E(X^2)$

(Since $V(X) = E(X^2) - (E(X))^2 = E(X^2) - 0$.)

$$E\left(\frac{\sum x_i^2}{n}\right) = \frac{1}{n} E\left(\sum x_i^2\right) = \frac{1}{n} \cdot n\theta = \theta.$$

Therefore $\dfrac{X^2}{\theta} \sim \chi^2(1)$

(Reason: If $z = \dfrac{X-\mu}{\sigma}$ then $z^2 = \dfrac{(X-\mu)^2}{\sigma^2} \sim \chi^2(1)$.) .

Here $\mu = 0$ and $\sigma^2 = \theta$)

$V(X^2/\theta) = 2 \cdot 1 = 2.$ $\quad$ ($\because$ 2. (deg. of freedom)) Since χ^2-dis with d.f. 1. $\frac{X^2}{\theta}$ is

i.e., $\dfrac{1}{\theta^2} V(X^2) = 2$

$\Rightarrow V(X^2) = 2\theta^2$

Therefore $V\left(\dfrac{\sum x_i^2}{n}\right) = \dfrac{1}{n^2} V\left(\sum x_i^2\right) = \dfrac{1}{n^2} \sum 2\theta^2 = \dfrac{1}{n^2} \cdot n \cdot 2\theta^2$

$= \dfrac{2\theta^2}{n}.$

Definition: Let $\hat{\theta}$ be an unbiased estimate of θ; We say that $\hat{\theta}$ is an unbiased minimum variance estimate of θ if for all estimates θ^* such that $E(\theta^*) = \theta$, we have

$V(\hat{\theta}) \leq V(\theta^*)$ for all θ.

That is, among all unbiased estimates of θ, $\hat{\theta}$ has the smallest variance.

Definition: Let $\hat{\theta}$ be an estimate (based on a sample $X_1, \dots, X_n$) of the parameter θ. We say that $\hat{\theta}$ is a consistent estimate of θ if

$$\lim_{n \to \infty} pr\left[|\hat{\theta} - \theta| > \epsilon\right] = 0 \quad \text{for all } \epsilon > 0.$$

$P\{|X-\mu| \geq \epsilon\} \leq \dfrac{\sigma^2}{\epsilon^2}$ $\quad$ $P(|X-\mu| < k\sigma) \geq 1 - 1/k^2$ $\quad$ $P(|X-\mu| \geq k\sigma) \leq 1/k^2$ $\quad$ $k\sigma = \epsilon > 0$

Chebychev's inequality Other form: Let X be a r.v. with $E(X) = \mu$ and let c be any real number. Then, if $E(X-c)^2$ is finite and ϵ be positive number then

$$pr\left[|X-c| \geq \epsilon\right] \leq \frac{1}{\epsilon^2} E(X-c)^2.$$

(Equivalently: $\lim_{n\to\infty} pr[|\hat{\theta}-\theta| \le \epsilon] = 1$ for all $\epsilon > 0$).

Example: Show that $\bar{X}$ is a consistent estimate of μ.

Solution: By Chebyshev's inequality.

$$pr\{|X-c| \geq \epsilon\} \leq \frac{1}{\epsilon^2} E(X-c)^2 \quad \text{(by above)}$$

put $c=\mu$ and $X=\bar{X}$

$$pr\{|\bar{X}-\mu| \geq \epsilon\} \leq \frac{1}{\epsilon^2} E(\bar{X}-\mu)^2$$

$$= \frac{1}{\epsilon^2} V(\bar{X})$$

$$= \frac{1}{\epsilon^2} \cdot \frac{\sigma^2}{n} \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore $\bar{X}$ is a consistent estimate of μ.

Theorem: Let $\hat{\theta}$ be an estimate of θ based on a sample of size n. If $\lim_{n\to\infty} E(\hat{\theta}) = \theta$ and if $\lim_{n\to\infty} V(\hat{\theta}) = 0$, then $\hat{\theta}$ is a consistent estimate of θ.

proof: By Chebyshev's inequality.

$$pr[|\hat{\theta}-\theta| \geq \epsilon] \leq \frac{1}{\epsilon^2} E[\hat{\theta}-\theta]^2$$

$$= \frac{1}{\epsilon^2} E[\hat{\theta} - E(\hat{\theta}) + E(\hat{\theta}) - \theta]^2$$

$$= \frac{1}{\epsilon^2} E\{[\hat{\theta}-E(\hat{\theta})]^2 + 2[\hat{\theta}-E(\hat{\theta})][E(\hat{\theta})-\theta] + [E(\hat{\theta})-\theta]^2\}$$

$$= \frac{1}{\epsilon^2}\{\text{var}\,\hat{\theta} + 0 + [E(\hat{\theta})-\theta]^2\}.$$ (Since $\hat{\theta}$ is an estimate of θ, we have $E(\hat{\theta}$

taking $\lim n \to \infty$ and using the hypo., we have that

$\lim_{n \to \infty} pr[|\hat{\theta} - \theta| \geq \epsilon] \leq 0$ and thus equal to 0.

Definition: We say that $\hat{\theta}$ is a <u>best linear unbiased</u> estimate of θ if:

(i) $E(\hat{\theta}) = \theta$

(ii) $\hat{\theta} = \sum_{i=1}^{n} a_i x_i$. That is, $\hat{\theta}$ is a linear function of the sample.

(iii) $v(\hat{\theta}) \leq v(\theta^*)$ where θ^* is any other estimate of θ satisfying (i) & (ii).

Definition: Let $X_1, X_2, \ldots, X_n$ be a random sample from the random variable X that has pdf $f(x,\theta)$ and let $x_1, x_2, \ldots, x_n$ be the sample values. We define the <u>likelihood function</u> L as the following function of the sample and θ (unknown parameter)

$$L(X_1, X_2, \ldots X_n; \theta) = f(X_1;\theta) f(X_2;\theta) \cdots f(X_n;\theta)$$

Definition: The <u>maximum likelihood estimate</u> of θ, say $\hat{\theta}$, based on a random sample $X_1, X_2, \ldots, X_n$ is that value of θ which maximizes $L(X_1, X_2, \ldots, X_n; \theta)$, considered as a function of θ for a given sample $X_1, \ldots, X_n$, where L is defined above.

Denoted by MLE.

Note: To determine MLE, we consider $\log x$ (the increasing function of x) $\log L(X_1, X_2, \ldots, X_n; \theta)$ will attain maximum value for the same value of θ as that of $L(X_1, X_2, \ldots X_n; \theta)$. So for max. we will equate $\frac{\partial}{\partial\theta} \log L(X_1, X_2, \ldots, X_n; \theta) = 0$.

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Problem: Let $X_1, X_2, \ldots, X_n$ represent a random sample from each of the distributions having the probability density functions. Find MLE $\hat{\theta}$ of θ.

(i) $$f(x,\theta) = \begin{cases} \dfrac{\theta^{x} e^{-\theta}}{x!}, & x = 0, 1, 2, \ldots \quad 0 \le \theta \le \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Sol:

$$L(x_1, x_2, \ldots, x_n, \theta) = f(x_1,\theta) \cdots f(x_n,\theta)$$

$$= \frac{\theta^{x_1} e^{-\theta}}{x_1!} \cdot \frac{\theta^{x_2} \cdot e^{-\theta}}{x_2!} \cdots \frac{\theta^{x_n} \cdot e^{-\theta}}{x_n!}$$

$$= \frac{\theta^{x_1 + x_2 + \cdots + x_n} \cdot e^{-n\theta}}{(x_1!)(x_2!)\cdots(x_n!)}$$

$$\log L = (x_1 + x_2 + \cdots + x_n)\log\theta - n\theta - \log\{(x_1!)(x_2!)\cdots(x_n!)\}$$

For MLE

$$\frac{\partial}{\partial\theta}(\log L) = 0.$$

i.e., $$(x_1 + x_2 + \cdots + x_n)\frac{1}{\theta} - n = 0$$

$$\Rightarrow \theta = \frac{x_1 + x_2 + \cdots + x_n}{n} = \bar{x}$$

Therefore MLE of θ is $\hat{\theta} = \bar{x}$

Interval Estimation:

Definition: An interval with one or both of its end points as random variables is called a random interval

Ex: $(X, 2X)$, $(2, \bar{X})$

Confidence interval: Let θ be an unknown parameter to be determined by a random sample $(X_1, X_2, \ldots, X_n)$ of size n. The confidence interval for the parameter θ is a random interval containing the parameter with high probability, say $1-\alpha$. $(1-\alpha)$ is called confidence coefficient.

Suppose that $\Pr\{H_1(X_1, \ldots, X_n) < \theta < H_2(X_1, \ldots, X_n)\} = 1-\alpha$

then the interval $(H_1(X_1, \ldots, X_n), H_2(X_1, \ldots, X_n))$ is the $(1-\alpha) \times 100\%$ confidence interval for θ

Note: Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution $N(\mu, \sigma^2)$. Then

(i) $Z = \dfrac{(\bar{X}-\mu)\sqrt{n}}{\sigma} \sim N(0,1)$

(ii) $V = \dfrac{nS^2}{\sigma^2} \sim \chi^2(n-1)$

(iii) $\dfrac{\sqrt{n-1}\,(\bar{x}-\mu)}{S} \sim T(n-1)$

(iv) $\dfrac{\sum_{i=1}^{n}(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$

Confidence Interval for Mean: (when σ^2 is known)

Consider $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Then $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

Suppose that we require an $(1-\alpha)100\%$ confidence interval, then find 'a' such that $pr(-a < Z < a) = 1-\alpha$.

$$2\Phi(a) - 1 = 1-\alpha$$

$$\Rightarrow 2\Phi(a) = 2-\alpha$$

$$\Rightarrow \Phi(a) = \frac{1}{2}(2-\alpha) = 1 - \frac{\alpha}{2}.$$

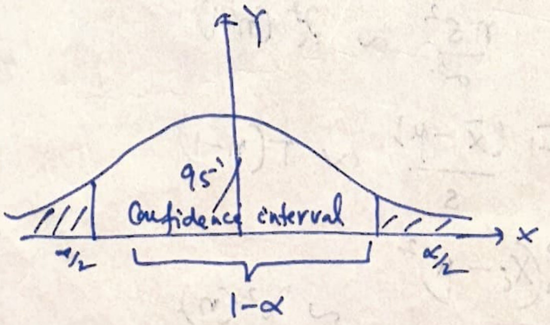
We can find 'a' from the table.

Consider $-a < Z < a = -a < \frac{(\bar{X}-\mu)\sqrt{n}}{\sigma} < a$

$$= -a\sigma < (\bar{X}-\mu)\sqrt{n} < a\sigma$$

$$= \bar{X} - \frac{a\sigma}{\sqrt{n}} < \mu < \bar{X} + \frac{a\sigma}{\sqrt{n}}$$

Therefore the interval $\left(\bar{X} - \frac{a\sigma}{\sqrt{n}}, \bar{X} + \frac{a\sigma}{\sqrt{n}}\right)$ is a $(1-\alpha)100\%$ confidence interval for μ, when σ is known.



Problem: Let the observed value of the mean $\bar{X}$ of a random sample of size 20 from a distribution that is $N(\mu, 80)$ be 81.2. Find a 95% Confidence interval for μ.

Solution: Given $\bar{X} = 81.2$, $n = 20$, $\sigma^2 = 80$.

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) = N(\mu, 4)$$

Let $Z = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \dfrac{\bar{x}-\mu}{2}$.

$$pr(-a < Z < a) = 0.95$$

$$\phi(a) - (1-\phi(a)) = 0.95$$

$$\Rightarrow 2\phi(a) = 1+0.95 = 1.95$$

$$\Rightarrow \phi(a) = 0.975$$

$\Rightarrow a = 1.96$. (from the table).

Therefore the confidence interval for μ, given $\sigma^2 = 80$ is $\left(\bar{X} - \dfrac{a\sigma}{\sqrt{n}}, \bar{X} + \dfrac{a\sigma}{\sqrt{n}}\right)$

$$= (81.2 - 1.96 \times 2, 81.2 + 1.96 \times 2) = (77.28, 85.12)$$

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Problem: Let $\bar{X}$ be the mean of a random sample of size n from a distribution that is $n(\mu, 9)$. Find n such that $pr(\bar{X}-1 < \mu < \bar{X}+1) = 0.90$, approximately.

Solution: $pr(\bar{X}-1 < \mu < \bar{X}+1) = 0.90 \Rightarrow pr(-1-\bar{X} < -\mu < 1-\bar{X}$

$$\Rightarrow pr(-1 < \bar{X}-\mu < 1) = 0.90$$

$$\Rightarrow pr\left(\frac{-1}{\sigma/\sqrt{n}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{1}{\sigma/\sqrt{n}}\right) = 0.90$$

$$\Rightarrow 2\Phi(\sqrt{n}/\sigma) = 1.90$$

$$\Rightarrow \Phi(\sqrt{n}/\sigma) = 0.95$$

$$\Rightarrow \sqrt{n}/\sigma = 1.65$$

But $\sigma^2 = 9$, $\sigma = 3$.

$\sqrt{n} = 1.65 \times 3 = 4.95 \sim 5 \Rightarrow \underline{n \simeq 25}$.

(from the table)

Problem: Let a random sample of size 17 from the normal distribution $n(\mu, \sigma^2)$ yield $\bar{x} = 4.7$ and $s^2 = 5.76$. Determine a 90 percent confidence interval for μ.

Note: Confidence interval when σ^2 is unknown:

Consider the statistic

$$T = \frac{(\bar{X}-\mu)\sqrt{n-1}}{S} \sim T(n-1)$$

Find 'a' such that

$$pr\{-a < T < a\} = 1-\alpha, \text{ the given probability}$$

$$\Rightarrow pr\left\{-a < \frac{(\bar{x}-\mu)\sqrt{n-1}}{s} < a\right\} = 1-\alpha$$

$$\Rightarrow pr\left\{\frac{-as}{\sqrt{n-1}} < \bar{x}-\mu < \frac{as}{\sqrt{n-1}}\right\} = 1-\alpha$$

$$\Rightarrow pr\left\{\frac{as}{\sqrt{n-1}} > -(\bar{x}-\mu) > \frac{-as}{\sqrt{n-1}}\right\} = 1-\alpha$$

(monotone)

$$\Rightarrow pr\left\{\frac{as}{\sqrt{n-1}} > \mu-\bar{x} > \frac{-as}{\sqrt{n-1}}\right\} = 1-\alpha$$

$$\Rightarrow pr\left\{\frac{-as}{\sqrt{n-1}} < \mu-\bar{x} < \frac{as}{\sqrt{n-1}}\right\} = 1-\alpha$$

ie; $pr\left\{\bar{x} - \frac{as}{\sqrt{n-1}} < \mu < \bar{x} + \frac{as}{\sqrt{n-1}}\right\} = 1-\alpha.$

Therefore $\left(\bar{x}-\frac{as}{\sqrt{n-1}}, \bar{x}+\frac{as}{\sqrt{n-1}}\right)$ is the confidence interval for μ when σ^2 is not known.
(or σ)

Solution for (*): To find 'a' such that

$$pr(-a < T < a) = 0.90$$

$$\Rightarrow 2\phi(a) = 1.90$$

$$\Rightarrow \phi(a) = 0.95 \Rightarrow a \cong 1.746 \text{ (~~approx.~~)}$$

(Refer: T-dist.. table with $n-1 = 16$ df. corre-.. 0.95)

The confidence interval is

$$\left(\bar{x}-\frac{as}{\sqrt{n-1}}, \bar{x}+\frac{as}{\sqrt{n-1}}\right)$$ where $s = \sqrt{5.76} = 2.4$

$n = 17$, $a = 1.75$, $\bar{x} = 4.7$

$$= \left(4.7 - \frac{(1.746)(2.4)}{\sqrt{16}} ,\ 4.7 + \frac{(1.746)(2.4)}{\sqrt{16}} \right)$$

$$\cong (3.7, 5.7).$$

Problem: A random sample of size 15 from the normal distri[bution] $n(\mu, \sigma^2)$ yields $\bar{x} = 3.2$ and $s^2 = 4.24$. Determi[ne] a 90 percent confidence interval for μ (σ^2 unknown)

Solution: $pr\left\{ \bar{x} - \frac{as}{\sqrt{n-1}} < \mu < \bar{x} + \frac{as}{\sqrt{n-1}} \right\} = 1-\alpha$

Find 'a' such that $pr(-a < T < a) = 0.90$

$$\Rightarrow 2\Phi(a) = 1.90$$

$$\Rightarrow \Phi(a) = 0.95$$

$$\Rightarrow a \cong 1.75$$

The confidence interval is $\left(\bar{x} - \frac{as}{\sqrt{n-1}} , \bar{x} + \frac{as}{\sqrt{n-1}} \right)$

where $\bar{x} = 3.2$, $s = \sqrt{4.24}$, $a = 1.75$, $n = 15$

$$\left(3.2 - \frac{(1.75)(\sqrt{4.24})}{\sqrt{14}} ,\ 32 + \frac{(1.75)(\sqrt{4.24})}{\sqrt{14}} \right)$$

$$= (2.68, \cancel{9.68}\ 4.17) \quad \text{(verify?)}$$

Problem: Suppose that 10, 12, 16, 19 is a sample taken from a normal distribution with variance $\sigma^2 = 6.25$. Find a 95% confidence interval for the mean μ.

Solution: $\bar{X} = \frac{10+12+16+19}{4} = 14.25$.

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \cong N(\mu, 1.5625)$

Let $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-\mu}{1.25}$

Find 'a' such that $\Pr(-a < Z < a) = 0.95$

$2\phi(a) = 1.95$

$\phi(a) = 0.975$

$\Rightarrow a = 1.96$ (from normal dist..)

Therefore the confidence interval for μ is

$\left(\bar{X} - \frac{a\sigma}{\sqrt{n}}, \bar{X} + \frac{a\sigma}{\sqrt{n}}\right)$ where $\bar{X} = 14.25$

$\sigma = \sqrt{6.25} = 2.5$

$n = 4$

ie, $\left(14.25 - \frac{1.96 \times 1.5625}{2}, 14.25 + \frac{1.96 \times 1.5625}{2}\right)$

$= \left(\cancel{12.71875}, \cancel{15.78125}\right)$ $(11.8, 16.7)$

(ii) $f(x, \theta) = \begin{cases} \theta x^{\theta-1} , & 0<x<1 , \quad 0<\theta<\infty \\ 0 & \text{elsewhere.} \end{cases}$

$L(x_1, x_2, \ldots, x_n ; \theta) = \theta x_1^{\theta-1} \ldots \theta x_n^{\theta-1}$

$= \theta^n x_1^{\theta-1} \cdot x_2^{\theta-1} \ldots x_n^{\theta-1}$

for MLE, $\frac{\partial}{\partial\theta} \log L = 0$

now ie, $\log L = n \log \theta + (\theta-1)\log x_1 + \ldots + (\theta-1) \log x_n$.

$\frac{\partial}{\partial\theta} \log L = \frac{n}{\theta} + \log x_1 + \ldots + \log x_n$

$\frac{\partial}{\partial\theta} \log L = 0 \Rightarrow \frac{1}{L}\frac{\partial L}{\partial\theta} = 0 \Rightarrow \frac{\partial L}{\partial\theta} = 0$

$\Rightarrow \frac{n}{\theta} + \log x_1 + \ldots + \log x_n = 0$

MLE of θ ie, $\hat{\theta} = \frac{-n}{\log(x_1 \cdots x_n)}$

(iii) $f(x;\theta) = \begin{cases} \left(\frac{1}{\theta}\right) \cdot e^{\frac{-x}{\theta}} , & 0<x<\infty, \quad 0<\theta<\infty \\ 0 & \text{elsewhere} \end{cases}$

$L(x_1, x_2, \ldots, x_n ; \theta) = \frac{1}{\theta} e^{\frac{-x_1}{\theta}} \ldots \frac{1}{\theta} e^{\frac{-x_n}{\theta}}$.

$= \frac{1}{\theta^n} \cdot e^{-\frac{1}{\theta}(x_1+x_2+\ldots+x_n)}$

$\log L = \log\left(\frac{1}{\theta}\right)^n + \log\left(e^{-\frac{1}{\theta}(x_1+\ldots+x_n)}\right)$

$= n \log\left(\frac{1}{\theta}\right) + \left(-\frac{1}{\theta}(x_1+\ldots+x_n)\right)$

for MLE $\frac{\partial}{\partial\theta}(\log L) = 0$ ie,

$n \cdot \frac{1}{(1/\theta)} \cdot \frac{-1}{\theta^2} + \frac{1}{\theta^2}(x_1+\ldots+x_n) = 0$

$$\Rightarrow \quad -n\theta \cdot \frac{1}{\theta^2} + \frac{1}{\theta^2}(x_1 + \cdots + x_n) = 0$$

$$\Rightarrow \quad n\theta = x_1 + \cdots + x_n$$

$$\Rightarrow \quad \theta = \frac{x_1 + \cdots + x_n}{n} = \bar{X}.$$

Therefore the MLE of θ ; $\hat{\theta} = \bar{X}$.

Problem: Suppose that the random variable X is normally distributed with expectation μ and variance 1. Find the MLE of μ.

Solution: Given $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$ (since $\sigma^2 = 1$)

If $(X_1, \cdots, X_n)$ is a random sample from X. The likelihood function of this sample is

$$L(X_1, X_2, \cdots, X_n, \mu) = f(x_1, \mu) \cdots f(x_n, \mu)$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_1-\mu)^2} \cdots \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_n-\mu)^2}$$

$$= \frac{1}{(2\pi)^{n/2}} \cdot e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2$$

for MLE $\frac{\partial}{\partial\mu}(\log L) = 0 \Rightarrow 0 - \frac{1}{2}\sum_{1}^{n} 2(x_i-\mu)(-1) = 0$

i.e., $\sum_{i=1}^{n}(x_i - \mu) = 0$

Therefore $\mu = \frac{\sum x_i}{n} = \bar{X}$; which maximizes L.

Example: Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution that is $n(\theta_1, \theta_2)$, $-\infty < \theta_1 < \infty$, $0 < \theta_2 < \infty$. Find the MLE for θ_1 and θ_2.

Solution: Given $X \sim n(\theta_1, \theta_2)$. i.e., $\mu = \theta_1$, $\sigma^2 = \theta_2$.

$$f(x, \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2\theta_2}(x-\theta_1)^2} \quad (\text{Since } \sigma = \sqrt{\theta_2})$$

Therefore $L(x_1, x_2, \ldots, x_n; \theta)$

$$= \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2\theta_2}(x_1-\theta_1)^2} \cdots \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2\theta_2}(x_n-\theta_1)^2}$$

$$= \frac{1}{(2\pi\theta_2)^{n/2}} \cdot e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}$$

$$\log L = -\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2 - \frac{n}{2}\log 2\pi - \frac{n}{2}\log\theta_2$$

for MLE, $\frac{\partial}{\partial\theta_1}(\log L) = 0 \Rightarrow -\frac{1}{\cancel{2}\theta_2}\sum_{i=1}^{n}\cancel{2}(x_i-\theta_1) - 0 - 0 = 0.$

i.e., $\Rightarrow \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i-\theta_1) = 0$

$$\Rightarrow \boxed{\theta_1 = \frac{\sum x_i}{n} = \bar{x}.}$$

Also $\frac{\partial}{\partial\theta_2}(\log L) = 0 \Rightarrow \frac{1}{2\theta_2^2}\sum(x_i-\theta_1)^2 - \frac{n}{2\theta_2} = 0$

$$\Rightarrow \frac{\sum(x_i-\theta_1)^2}{\theta_2} = n$$

$$\Rightarrow \theta_2 = \frac{\sum(x_i-\theta_1)^2}{n} = \frac{\sum(x_i-\bar{x})^2}{n} = s^2.$$