Four fundamental subspaces.

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

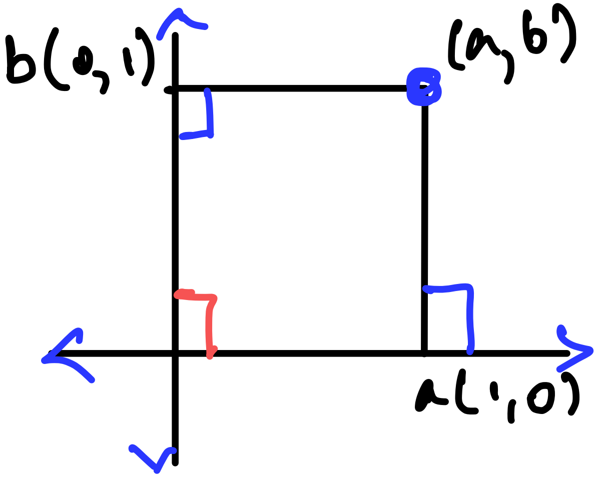
$$T(X) = \underset{m\times n}{A} X_{n\times 1}.$$

$$\mathbb{R}^n = \underset{\substack{\downarrow \\ \text{rank of} \\ A = r}}{R(A)} \oplus \underbrace{N(A)}_{\substack{n-r \\ \text{dimension}}}.$$

$X \in \mathbb{R}^n$

$\Rightarrow X = X_r + X_n$; (unique representation)

$X_r \in R(A)$, $X_n \in N(A)$



$$(a,b) = a(1,0) + b(0,1)$$

[unique representation using standard basis].

Consider a consistent system of equations $AX=b$ (Equivalently, $AX=b$ and $b \in C(A)$).

$$AX = A(X_r + X_n)$$

$$= AX_r + \underbrace{AX_n}_{0}$$

0 (as $X_n \in N(A)$)

$$= AX_r$$

Take $y \in N(A)$.

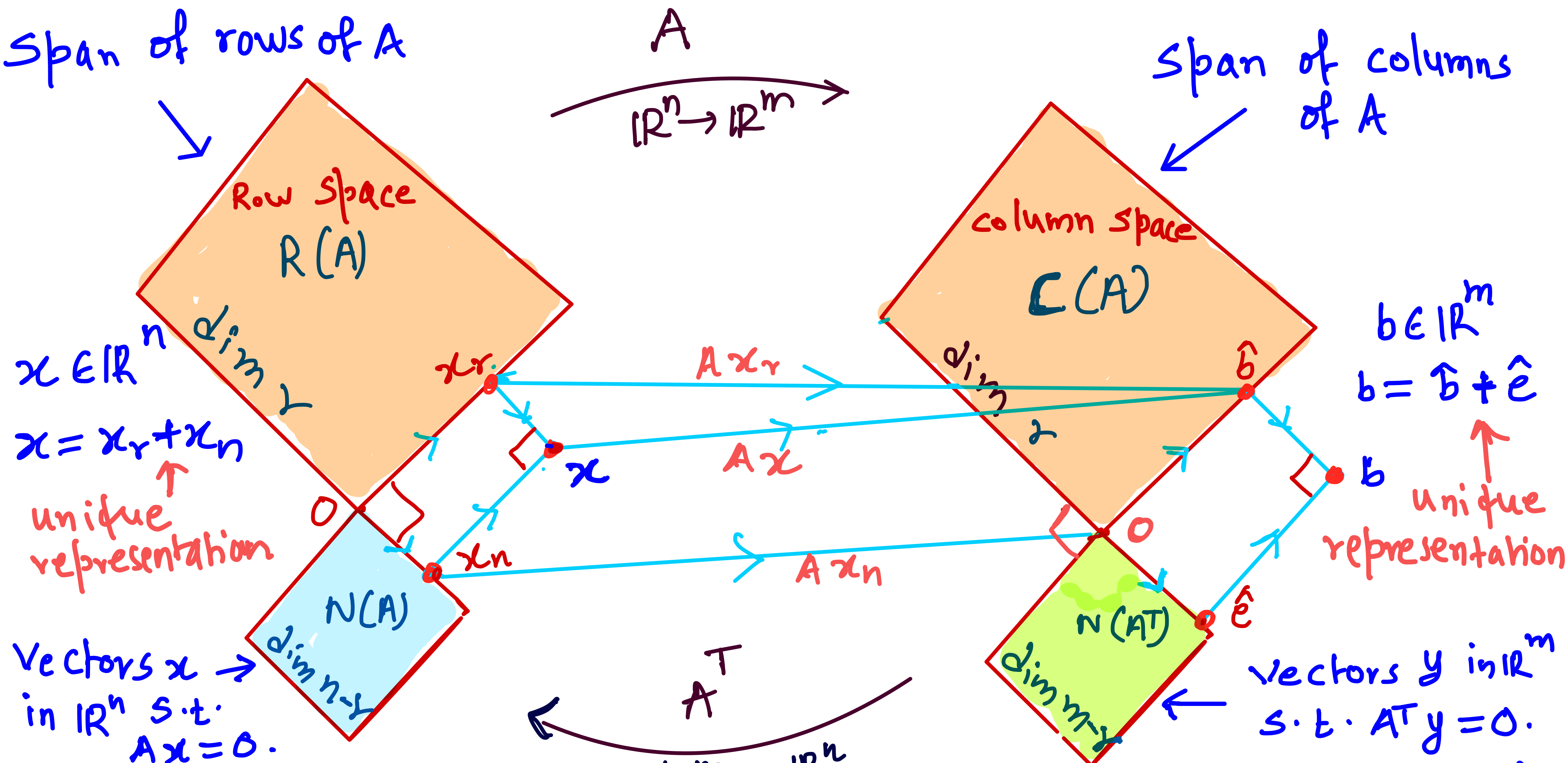
Then if x is a solution, then $x+y$ is a solution.

$$A(x+y) = Ax + Ay$$
$$= Ax + 0$$
$$= Ax \quad \text{and } x \text{ is a solution}$$

$\Rightarrow x+y$ is also a solution.

$\Rightarrow$ If $\dim(N(A)) \geq 1$ then there are infinitely many solutions whenever $Ax=b$ is consistent.

The Four Fundamental subspaces given $A_{m\times n}$:



$\mathbb{R}^n = R(A) \oplus N(A)$

$\mathbb{R}^m = C(A) \oplus N(A^T)$

$\hat{b} \to$ Projection of b on C(A); $\hat{e} = b - \hat{b} \to$ error vector; N(A) → called Null space

$N(A^T) \to$ called Left Null space

If $A = A^T$ (that is, A is symmetric) then four fundamental subspaces reduces to just two fundamental subspaces.

$$A_{n\times n} = A^T_{n\times n} \Rightarrow \begin{array}{c} N(A) = N(A^T) \\ \text{and} \\ R(A) = C(A). \end{array}$$

$$T : \mathbb{R}^n \to \mathbb{R}^n$$

$$T(x) = Ax = b$$

$$\mathbb{R}^n = C(A) \oplus N(A)$$

The direct sum separates signal from noise.

$C(A)$: give directions which explain b (features from A which explain b)

$N(A)$: has directions which cannot explain b where (parts of b where A has no influence on b). called 'Noise' or 'Error'.

Any $x \in \mathbb{R}^n$, $x = x_r + x_n$ where $x_r \in C(A)$

$x_n \in N(A)$

$Ax = b$

- has unique solution if $N(A) = \{0\}$ (i.e. A has full rank $= n$) and $b \in C(A)$.
- has infinitely many solutions if $\dim(N(A)) \geq 1$ and the system is consistent (Equivalently, $Ax = b$, $b \in C(A)$ and $\text{rank}(A) < n$)
- No solution when $b \notin C(A)$ i.e. when $\text{rank}[A] \neq \text{rank}[A|b]$

In this case, we found least squares solution whenever $\text{rank}[A] = n$ (whenever A had n independent columns).

Any $x \in \mathbb{R}^n$, $x = x_r + x_n$ where $x_r \in C(A)$

(data)

$x_n \in N(A)$

x_r called 'signal'

x_n called 'noise'

* covariance matrix in statistics is always symmetric
  → used to separate explained variance (signal) from unexplained varince (Noise).

* Algorithms like PCA, regression, finite element methods etc. rely on this separation of signal from noise.

Least square formula.

$$A^T A x = A^T b.$$

$$\Rightarrow x = (\underbrace{A^T A}_{\text{symmetric}})^{-1} A^T b$$

Theorem: Let $A_{m \times n}$ be a tall matrix (that is, $m \geq n$) with number of independent columns $= n$.

Then (i). $A^T A$ is an invertible matrix

(ii). left inverse of A exists.

Proof: (i). Form a matrix $\underset{n\times n}{B} = \underset{n\times m}{A^T}\underset{m\times n}{A}$.

$$
\begin{aligned}
\text{Then } B^T &= (A^T A)^T \\
&= A^T (A^T)^T \\
&= A^T A \\
&= B
\end{aligned}
$$

$\Rightarrow B$ is symmetric.

Take T to be the linear transformation given by B.

Claim: T is one-one and onto.

Take $T(X) = 0$

$$\Rightarrow BX = 0$$

$$\Rightarrow (A^T A)X = 0$$

$$\Rightarrow A^T(AX) = 0$$

$$\Rightarrow AX \in N(A^T) \quad \text{—— (1)}$$

We know that,

$$AX \in C(A) \quad \text{—— (2)}$$

By (1) & (2),

$$AX \in N(A^T) \cap \underbrace{C(A)}_{R(A^T)} = N(A^T) \cap R(A^T)$$

$$= \{0\}$$

$$\Rightarrow AX = 0$$

$$\Rightarrow X \in N(A)$$

Given : columns of A are independent &

$AX = 0$.

If $X \neq 0$ then we get a linear combination of columns of A equal to 0 and one of the scalars not zero.

$\Rightarrow$ columns of A won't be independent (Contradiction to assumption).

$\therefore AX = 0 \Rightarrow X = 0$ (otherwise, columns of A will be dependent.)

$$\begin{array}{c} A \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \\ c_1 \quad c_2 \end{array} \begin{array}{c} X \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \ \end{array} \begin{array}{c} = \\ = \\ \ \end{array} \begin{array}{c} b. \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \ \end{array}$$

Eg:

(we have already solved this system by finding left inverse of A).

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad X = (x_1, x_2) \neq 0$$

$\therefore AX = 0$

$\Rightarrow X = 0$

$\Rightarrow T$ is one-one.

$\Rightarrow \ker T = \{0\}$

$\therefore \dim \ker T = 0.$

To <u>prove</u>: T is onto.

By Rank-Nullity theorem,

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\Rightarrow \text{rank}(T) + 0 = n \quad (\text{as } \operatorname{Ker} T = \{0\})$$

$$\Rightarrow \text{rank}(T) = n$$

$$\Rightarrow \dim(\operatorname{Im}(T)) = n$$

$$\Rightarrow T \text{ is onto.}$$

$\therefore$ T is one-one and onto $\Rightarrow T^{-1}$ exists.

$\Rightarrow B^{-1}$ exists.

$\Rightarrow (A^TA)^{-1}$ exists.

(ii). Let F be the left inverse of A.

Then $FA = I$.

exists?

The left inverse of A is given by

$F = (A^TA)^{-1}A^T$; which exists if $(A^TA)^{-1}$ exists.

As columns of A are independent, by (i),

$(A^T A)^{-1}$ exists.

$\Rightarrow$ left inverse of A exists.

Q:

Take a symmetric matrix A. Show that all its eigen values are real.