Lec 13:-

Complex No.s

A complex no. z is an ordered pair (x,y) of real nos x & y.

$i = (0,1)$

$x = Re(z)$ & $y = Im(z)$.

$z_1 = z_2 \iff Re\, z_1 = Re\, z_2$ & $Im\, z_1 = Im\, z_2$.

Sum: $z_1 + z_2 = (x_1+x_2, y_1+y_2)$ Add componentwise (commutes)

Product $z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_2 y_1 + x_1 y_2) = z_2 z_1$ (commutes)

In particular, if $y_1 = y_2 = 0$, then we get

$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$

$(x_1, 0) \times (x_2, 0) = (x_1 x_2, 0)$

So, these are rightly the extensions of real nos.

~~$z = x$ For convenience, we write~~ $z = x+iy$ (x,y) which is nothing but $(x,0) + (0,1)(y,0)$.

as $(0,1)(y,0) = (0, y)$

~~This is~~ If we denote i as $(0,1)$, we can rewrite $z = x+iy$.

(Note: ~~where~~ $(0,1)(0,1) = (-1,0) = -1$)

Associative laws : (i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

Thus, $z = x+iy = x+yi$

Additive identity: $(0,0) = 0$. **Inverse** : $-z$

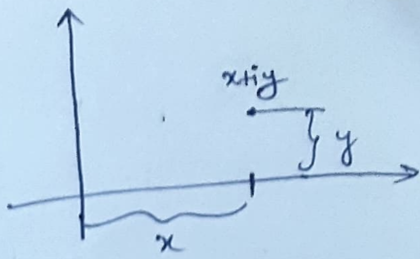
Multiplicative identity : $z.1 = 1.z = z$. **Inverse** : $\frac{1}{z}$ (for $z \neq 0$)

Now, subtraction is also defined as $z_1 + (-z_2)$.

division $z_1 \times (z_2)^{-1}$ $(z_2 \neq 0)$

Complex Plane (Argand Plane)

$z = x+iy$ or (x,y) is



Complex conjugate of $z = x+iy$ is denoted as $\bar{z}$

& defined as $\bar{z} = x-iy$

It is the reflection of z wrt Real axis.

Properties :-

$z\bar{z} = x^2+y^2$

$z+\bar{z} = 2x = 2\,\mathrm{Re}\,z$

$z-\bar{z} = 2i\,\mathrm{Im}\,z$

$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$.

~~Arg~~ Complex Conjugation commutes wrt $+, -, \times, \div$.

Ex. 1) $z = x+iy$ is purely imaginary

$\iff \bar{z} = -z$

Polar co-ordinates

Instead of writing complex nos z interms of xy co-ordinates, we also employ polar co-ordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

r - absolute value or modulus of z, denoted $|z|$.

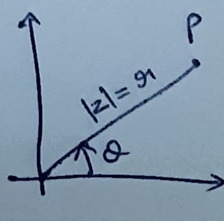
$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$.

θ - argument of z $= \arg z$ & $\tan\theta = \frac{y}{x}$ if $z \neq 0$

directed angle from the X axis to OP

ie, +ve in the counterclockwise sense

(θ - measured in radians)



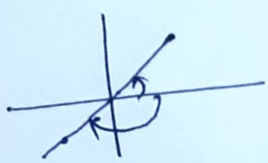
θ is undefined if $z=0$.

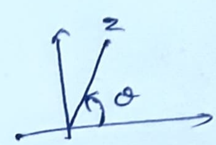
~~Since θ~~ Let $z\neq 0$. ~~And~~ ~~Arg~~ $\arg(z)$ is determined only upto integer multiples of 2π as Sin & Cos are 2π periodic. So, we define <u>Principal value of argument of z</u> ~~Arg z~~ so that unique value $\operatorname{Arg} z$ such ~~Arg z~~ that $-\pi < \operatorname{Arg} z \leq \pi$. Thus,

$$\arg z = \operatorname{Arg} z + 2n\pi, \quad n \in \mathbb{Z}$$

<u>Note</u>:- If z is a $-$ve real no., $\operatorname{Arg} z = \pi$ (& not $-\pi$)

Eg $\operatorname{Arg}(1+i) = \frac{\pi}{4}$, but $\operatorname{Arg}(-1-i) = -\frac{3\pi}{4}$ & not $\frac{5\pi}{4}$





<u>Euler's formula</u>

$e^{i\theta}$ or $\exp(i\theta) = \cos\theta + i\sin\theta$ are points on the unit circle $|z|=1$.

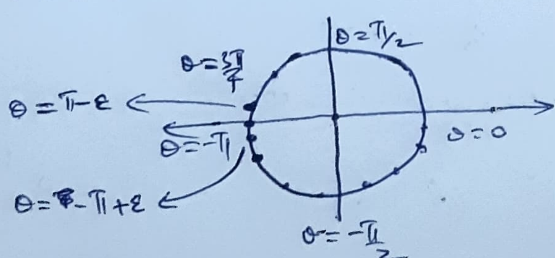
<u>Exponential form</u>: $z = re^{i\theta}$

Let ~~$z_1 = r_1 \cos\theta_1$ & $z_2 = r_2\cos\theta_2$~~ $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

~~Multiplication~~

~~$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$~~

The equation $z = re^{i\theta}$ $-\pi < \theta \leq \pi$ ~~ref~~ is the parametric representation of the circle $|z| = r$, centered at origin with radius $r>0$. As θ increases from $-\pi$ to π, the point z starts from the $-$ve real axis & traverses the circle once in the CCW direction.

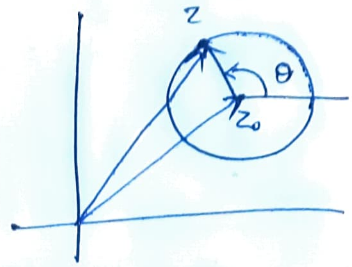


$$(-\pi, \pi] \longrightarrow \mathbb{C}$$
$$\theta \longmapsto re^{i\theta}$$

More generally, any pt z on the circle centered at z_0 with radius r has the parametric representation

$$z = z_0 + re^{i\theta} \quad , \quad -\pi < \theta \leq \pi,$$

$$\text{or} \quad |z - z_0| = r$$



Eg 1) $e^{\pi i} = -1$ $\qquad e^{2\pi i} = 1 = e^{-4\pi i}$

$e^{-i\pi/2} = -i$

Exercises

1) $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

$LHS = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$= e^{i(\theta_1+\theta_2)}.$

~~2) $z = re^{i\theta} \Rightarrow z^{-1} = \frac{1}{r}e$~~ More generally,

$(r_1 e^{i\theta_1}) \cdot (r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}.$

2) If $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$, then $(z_2 \neq 0)$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \Rightarrow \frac{1}{z} = \frac{1}{r} e^{-i\theta}.$

3) $z^n = r^n e^{in\theta}$ $\quad \forall n \geq 1$

Prove using mathematical induction.

Further $z^n = r^n e^{in\theta}$ $\forall n \leq 0$ (if $z \neq 0$) by applying the above on $\frac{1}{z}$ $(z \neq 0)$.

Ex 4) Show that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad \forall n \in \mathbb{Z}$$

Use 3 with $r = 1$ or use induction to prove independently.

Lec 14 : Powers & roots

~~Let z~~ Recall that $\underset{(z\neq 0)}{z} = x+iy$ (in terms of x & y co-ordinates)

$= r(\cos\theta + i\sin\theta)$ (in terms of polar co ordinates)

$= re^{i\theta}$ (in terms of exponential form)

where $r = |z|$ & θ is any real no. that satisfies $\tan\theta = \frac{y}{x}$ if $x\neq 0$

~~& if $x=0$, $\theta = \begin{cases} \pm\frac{\pi}{2} + 2n\pi & \text{if } y>0 \\ -\frac{\pi}{2}+2n & \text{if } y<0 \end{cases}$~~

& if $x=0$, θ is of the form

$\frac{\pi}{2} + 2n\pi$ if $y>0$

$-\frac{\pi}{2} + 2n\pi$ if $y<0$.

* In terms of z, $r = |z|$ & $\theta \in \arg z$.

Now,

$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$, $z_2 \neq 0$

~~In particular, $z^{-1} = r^{-1}e^{-i\theta}$, $z\neq 0$~~

<u>Lemma</u> :- For every $n\in\mathbb{Z}$, $\underset{(z\neq 0)}{z^n} = r^n e^{in\theta}$. Here $z = re^{i\theta}$, $z\neq 0$.
For $k\in\mathbb{N}$, $z^{-k} := (z^{-1})^k$.

<u>Proof</u> :- ~~let~~ for $n\geq 1$, we prove using mathematical induction.

~~Sps~~ $z^m = r^m e^{im\theta}$ for $m\geq 1$, then

$z^{m+1} = z\cdot z^m = re^{i\theta}\cdot r^m e^{im\theta}$

$= r^{m+1} e^{i(m+1)\theta}$

Thus, $z^n = r^n e^{in\theta}$ $\forall\, n\geq 1$ & by verification for $n=0$.

Now, consider $\frac{1}{z}$ ~~then~~ & ~~z^m~~ any $m<0$. Then,

$z^m = (z^{-1})^{|m|} = (r^{-1}e^{-i\theta})^{|m|} = r^m e^{im\theta}$

True for every $m<0$.

Exponential form can help in finding powers of complex nos.

Eg 1. $(\sqrt{3}+i)^7 = 2^7\left(\frac{\sqrt{3}}{2}+i\times\frac{1}{2}\right)^7 = 2^7\left(e^{i\pi/6}\right)^7 = 2^7 e^{\pi i} e^{\pi i/6}$

$= -2^7 e^{\pi i/6}$.

Arguments of products & quotients

If $z_j = r_j e^{i\theta_j}$ for $j = 1, 2$, then

$\arg(z_1 z_2) = \arg z_1 + \arg z_2$. Here, $\arg z_j = \{\theta_j + 2m_j\pi \mid m_j \in \mathbb{Z}\}$.

Proof:- Note: $\theta_1 \in \arg z_1$ & $\theta_2 \in \arg z_2$. We prove LHS $\subseteq$ RHS & RHS $\subseteq$ LHS

Now, let $\theta \in \arg(z_1 z_2) \Rightarrow z_1 z_2 = |z_1 z_2| e^{i\theta} = r_1 r_2 e^{i\theta}$

But $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} \Rightarrow \theta_1 + \theta_2 = \theta + 2n\pi$ for some $n \in \mathbb{Z}$

$\Rightarrow \theta = \underbrace{(\theta_1 - 2n\pi)}_{\in \arg z_1} + \underbrace{(\theta_2)}_{\in \arg z_2}$

$\Rightarrow \theta \in \arg z_1 + \arg z_2$

IIIly, let $\alpha_j \in \arg z_j \Rightarrow \alpha_1 = \theta_1 + 2n_1\pi$ & $\alpha_2 = \theta_2 + 2n_2\pi$

$\Rightarrow \alpha_1 + \alpha_2 = \theta_1 + \theta_2 + 2\pi(n_1 + n_2)$

$\in \arg(z_1 z_2)$

as $r_1 r_2 e^{i(\theta_1+\theta_2)} = z_1 z_2$

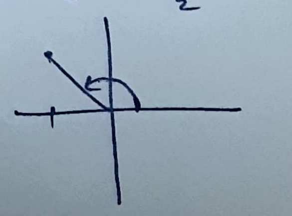
Qn 1) Is $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \mathrm{Arg}(z_1 z_2)$?

(Here, $-\pi < \mathrm{Arg}\, z \le \pi$). If $z_1 = e^{i\pi} = -1$ & $z_2 = e^{i\pi/2} = i$, then

$\mathrm{Arg}\, z_1 = \pi$ & $\mathrm{Arg}\, z_2 = \frac{\pi}{2} \Rightarrow \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \frac{3\pi}{2}$.

but $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(-i)$

$= -\frac{\pi}{2}$



$\boxed{\text{Ex 1) } z \neq 0, \ \arg z^{-1} = -\arg z}$

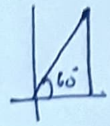
From prev. lemma, if $z_2 \neq 0$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

True if $\mathrm{Re}(z_1) > 0$ & $\mathrm{Re}(z_2) > 0$.

Eg 2) Let $z = \dfrac{-2}{1+\sqrt{3}i}$. find Arg z. ③

~~Note~~ Let $z_1 = -2$ & $z_2 = 1+\sqrt{3}i$



$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

$$\arg z_1 = \{\pi + 2n_0\pi \mid n_0 \in \mathbb{Z}\}$$

$$-\arg z_2 = \{-\tfrac{\pi}{3} + 2m\pi \mid m \in \mathbb{Z}\}$$

$$\Rightarrow \cancel{\text{...}} = \{\tfrac{2\pi}{3} + 2l\pi \mid l \in \mathbb{Z}\}$$

Clearly $\text{Arg}\left(\frac{z_1}{z_2}\right)$ is the unique elt of $\arg\left(\frac{z_1}{z_2}\right)$ s.t $-\pi < \text{Arg}\left(\frac{z_1}{z_2}\right) \le \pi$

$$\Rightarrow \text{Arg } z = \frac{2\pi}{3}$$

$\sqrt{\ } : \mathbb{R}^+ \to \mathbb{R}^+$

Lec 15

Roots of complex no.s

Let $r_0 > 0$. Then, n^{th} root of r_0 ($\sqrt[n]{r_0}$) is that unique + no. r s.t $r^n = r_0$. For eg. square root of 3 ~~is not~~ cannot be taken as $-\sqrt{3}$.

Let $z_0 = r_0 e^{i\theta_0}$ where $z_0 \neq 0$ & we aim to find the n^{th} roots of z_0. ie, all ~~a nonzero~~ z s.t $z^n = z_0$

($\because |z^n| = z_0$, it follows that $z \neq 0$ & hence, $z = re^{i\theta}$)

ie, $r^n e^{in\theta} = r_0 e^{i\theta_0}$

$\therefore r^n = r_0$ & $n\theta = \theta_0 + 2k\pi$ ~~, $k \in \mathbb{Z}$~~. for some $k \in \mathbb{Z}$.

ie, $r = \sqrt[n]{r_0}$ & $\theta = \dfrac{\theta_0}{n} + \dfrac{2k\pi}{n}$, $k \in \mathbb{Z}$. — ①

(The unique +ve ~~eq.~~ n^{th} root of +ve r_0.)

$\therefore$ Any z ~~of the~~ $= re^{i\theta}$ where r & θ are as in ① ~~is~~ surely satisfies $z^n = z_0$ ~~as~~ & hence, all the complex nos. of the form $z = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}$ where $k \in \mathbb{Z}$

are the n^{th} roots of unity.

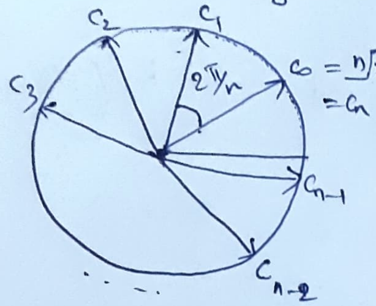
But many of them repeat. For eg,

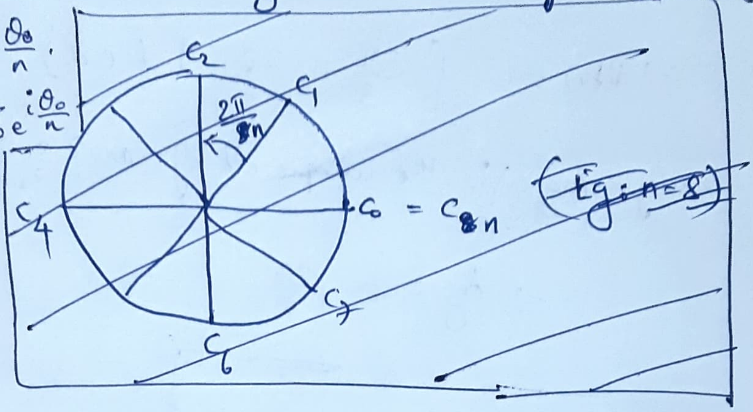
$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} = c_{k+n} = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n} + 2\pi\right)}$$

Thus, the distinct n^{th} roots of z_0 are given by

$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)} \quad \text{where} \quad 0 \le k \le n-1.$$

There are n many distinct n^{th} roots of z_0. We can plot them on the circle $|z| = \sqrt[n]{r_0}$ at about the origin & they are equally distributed ~~angular~~ ~~in terms of angles~~. by an angle $\frac{2\pi}{n}$. starting with $\frac{\theta_0}{n}$.





$\forall\, 0 \le k \le n-1$, length of $c_k = \sqrt[n]{r_0}$

When $n \ge 3$, the roots lie at the vertices of a regular n gon inscribed in that circle. We define the following symbol for the set of n^{th} roots of z_0 :- $(z^{1/n})$

Let $z_0^{1/n} = \{ c_k \mid 0 \le k \le n-1 \}$

If $z_0 \in \mathbb{R}^+$, ie $z_0 = r_0$, then

$(r_0)^{1/n}$ denotes the whole set of n roots of r_0

& $\sqrt[n]{r_0}$ denotes the unique +ve n^{th} root of r_0.

For eg. $(2)^{1/3}$ & $\sqrt[3]{2}$ are different for us from now ~~again~~ onwards

like $\arg i$ & $\frac{\pi}{2}$ are $\qquad \sqrt[3]{2} \in (2)^{1/3}$.

$\sqrt[3]{2} \in (2)^{1/3}$ like $\frac{\pi}{2} \in \arg i$

In the expression

$$c_k = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}$$

if $-\pi < \theta_0 \le \pi$, then c_0 is called the principal (n^{th}) root of z_0.

Thus, if $z_0 \in \mathbb{R}^+$, ie, $z_0 = r_0$, then

$$c_0 = \sqrt[n]{r_0}$$

ie, the nth root of r_0 is the principal (n^{th}) root of r_0.

Consider

$$c_k = \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n}}\, e^{i\frac{2k\pi}{n}}$$

Let $\omega_n = e^{i\frac{2\pi}{n}}$. Then

$$c_k = \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n}} (\omega_n)^k .$$

Further,

$$c_0 = \sqrt[n]{r_0}\, e^{i\frac{\theta_0}{n}}$$

$$\Rightarrow \boxed{c_k = c_0 (\omega_n)^k} \quad \text{for each } 0 \le k \le n-1.$$

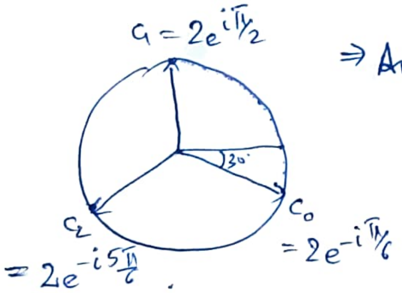
Eg 3) Find all cube roots of $(-8i)$.

$z_0 = -8i = r_0 e^{i\theta_0}$. Here, $r_0 = 8$ & $e^{i\theta_0} = -i$
for simplicity, we choose $\theta_0 = -\frac{\pi}{2}$.

$$\Rightarrow z_0 = 8 \times e^{-i\pi/2}$$

Note $c_k = c_0 (\omega_3)^k$ where $c_0 = (r_0)^{1/3} e^{i\frac{\theta_0}{3}}$
$= 2e^{-i\pi/6}$.

$$\Rightarrow \text{Ans} = \left\{ 2e^{-i\pi/6},\ 2e^{-i\pi/6} e^{i2\pi/3},\ 2e^{-i\pi/6} e^{i4\pi/3} \right\}.$$



ie, $c_k = 2e^{-i\pi/6} \left(e^{i2\pi/3}\right)^k$; $k = 0, 1, 2$

$$\text{Ans} = \{ c_0,\ c_0\omega_3,\ c_0(\omega_3)^2 \}.$$

Exercise: a) n^{th} root of unity, ie 1 are $\{1, \omega_n, (\omega_n)^2, \dots, (\omega_n)^{n-1}\}$
$= \{ e^{2\pi/n \times k} \mid 0 \le k \le n-1 \}$.

b)

Q2) ~~Show that if c is~~ Establish the identity

$$1+z+z^2+\dots+z^n = \frac{1-z^{n+1}}{1-z}, \quad z \neq 1.$$ (6)

& use it to derive Lagrange trigonometric identity

$$1+\cos\theta+\cos 2\theta+\dots+\cos n\theta = \frac{1}{2}+\frac{\sin\left[\frac{(2n+1)\theta}{2}\right]}{2\sin\left(\frac{\theta}{2}\right)}, \quad 0<\theta<2\pi$$

Q3) Show that if c is any n^{th} root of unity other than unity itself, (1)

then
$$1+c+c^2+\dots+c^{n-1} = 0.$$

Q4) Prove that the usual formula solves the quadratic eqn

$$az^2+bz+c=0 \qquad a\neq 0$$

when the coefficients a, b & c are complex nos. Specifically, by completing the square on the left-hand side, derive the quadratic formula

$$z = \frac{-b \pm (b^2-4ac)^{1/2}}{2a}$$

where both square roots are to be considered when $b^2-4ac\neq 0$.

Q5) Find the four zeros of the poln. z^4+4 & ~~exact~~ factorise it into quadratic factor with real coefficients.

Important formulas for $A, B \in \mathbb{R}$, $\theta \in \mathbb{R}$,

1) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

2) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

3) $\cos A+\cos B = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

4) $\cos A-\cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

5) $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$ & $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$.

Use 5 to prove 3 & 4.

6) $\sin A-\sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

7) $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$. (7)

Q2 : Soln

a) Let $S(z) = 1 + z + z^2 + \dots + z^n$

$\Rightarrow z\,S(z) = z + z^2 + \dots + z^n + z^{n+1}$

$\Rightarrow (1-z)\,S(z) = 1 - z^{n+1} \Rightarrow S(z) = \dfrac{1-z^{n+1}}{1-z}, \quad z \neq 0$

b) $z = e^{i\theta} \rightarrow$

$$1 + e^{i\theta} + \dots + e^{in\theta} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = \frac{\left[1-e^{i(n+1)\theta}\right]\left[1-e^{-i\theta}\right]}{|1-\cos\theta - i\sin\theta|^2}$$

$$= \frac{1 - e^{i(n+1)\theta} - e^{-i\theta} + e^{in\theta}}{(1-\cos\theta)^2 + (\sin\theta)^2}$$

~~b) $z = e^{i\theta} \Rightarrow$~~

$\Rightarrow 1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta$

$$= \frac{1 - \cos((n+1)\theta) - \cos\theta + \cos n\theta}{(1-\cos\theta)^2 + (\sin\theta)^2}$$

~~$1-\cos\theta = 2\sin^2\theta/2 \Rightarrow \text{denom} = 4\sin^4\theta$~~

$(1-\cos\theta)^2 + \sin^2\theta = 2 - 2\cos\theta = 4\sin^2\theta/2$

$$\therefore \sum_{k=0}^{n} \cos(k\theta) = \frac{2\sin^2\theta/2 + \cos n\theta - \cos((n+1)\theta)}{4\sin^2\theta/2}$$

$$= \frac{1}{2} + \frac{-2\sin\left(\frac{n\theta + (n+1)\theta}{2}\right)\sin\left(\frac{n\theta - (n+1)\theta}{2}\right)}{4\sin^2\theta/2}$$

$$= \frac{1}{2} + \frac{\sin\left[\frac{(2n+1)\theta}{2}\right]}{2\sin\theta/2} \quad ; \quad 0 < \theta < 2\pi$$

Q3) If $c \in (1)^{1/n} \setminus \{1\}$, then

$$1+c+\dots+c^{n-1} = \frac{1-c^n}{1-c} = 0$$

Q4) $$az^2+bz+c = a\left(z^2+\frac{b}{a}z+\frac{c}{a}\right) = a\left(\left(z+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}\right)$$

$$= a\left[\left(z+\frac{b}{2a}\right)^2 - \left(\frac{b^2-4ac}{4a^2}\right)\right] = 0$$

$$a \neq 0 \Rightarrow \left(z+\frac{b}{2a}\right) \in \left(\frac{b^2-4ac}{4a^2}\right)^{1/2}$$

$$\therefore z \in -\frac{b}{2a} + \left(\frac{b^2-4ac}{4a^2}\right)^{1/2}$$

Lec 16:

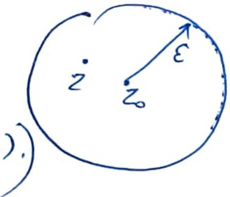
Regions in the Complex plane

Let $\varepsilon > 0$ & $z_0 \in \mathbb{C}$. ~~Then~~ and $S \subseteq \mathbb{C}$. Then,

Defn 1) ε-neighbourhood (of z_0)

$$= \{z \in \mathbb{C} \mid |z-z_0| < \varepsilon\}$$

(written also as $B(z_0, \varepsilon)$ or $B_\varepsilon(z_0)$).



2) deleted neighbourhood / punctured disk

$$= \{z \in \mathbb{C} \mid 0 < |z-z_0| < \varepsilon\}$$

3) Interior pt of S

A pt z_0 in $\mathbb{C}$ is said to be an interior point of S whenever there is some neighbourhood of z_0 that contains only points of S, ie, $\exists\ \varepsilon > 0$ s.t $B(z_0, \varepsilon) \subseteq S$.

4) Exterior pt of S

... whenever there is some nbhd of z_0 that contains no points of S

ie, $\exists\ \varepsilon > 0$ s.t $B(z_0, \varepsilon) \subseteq S^c$.

5) If z_0 is neither an interior point nor an ~~exter~~ exterior point, then it is called a <u>boundary point</u> of S.

Hence, equivalently, a point all of whose neighbourhoods contain atleast one point in S & atleast one point in S^c is called a boundary point.

6) <u>Boundary of S</u>, ∂S

The collection of all boundary points is called the <u>boundary of S</u>.

7) ~~A set is called open if it contains~~ <u>Interior of S</u>, $int(S)$

The collection of all interior points of S.

8) A set S is <u>open</u> if every pt in S is an interior point of S.

9) A set S is <u>closed</u> if its complement S^c (in $\mathbb{C}$) is open.

10) A point $z \in \mathbb{C}$ is said to be a <u>limit point</u> (accumulation point) of S if $\exists$ ~~$z_n \in S$ such~~ a sequence of points $\{z_n\}_{n \geq 1}$ ($z_n \in S$) such that $z_n \neq z$ & $\lim_{n \to \infty} z_n = z$. (Sequential defn) (ii) equivalently, if each deleted neighbourhood of z contains one pt of S. (Ex 1)

<u>Exercise 2)</u> S is closed $\iff$ S contains all its limit points.

11) Closure of S, ie $\bar{S}$ is the union of S & its limit points.

<u>Exercise 3)</u> $\partial S = \bar{S} \setminus int(S)$

12) A set S is <u>bounded</u> if $\exists$ $R > 0$ such that $|z| < R$ $\forall z \in S$.

if no such $R > 0$ exists, then S is unbounded.

13) An open set $S \subseteq \mathbb{C}$ is ~~connected~~ disconnected if there exists two disjoint non-empty open sets S_1 & S_2 such that

$$S = S_1 \cup S_2$$

An open set is <u>connected</u> if it is not disconnected.

14) A connected open set in $\mathbb{C}$ is called a <u>region</u> or sometimes a <u>domain</u>.

Theorem

If z_0 is a limit point of S, then every neighbourhood of z_0 contains infly many pts of S.

Corollary

A finite set has no limit points.

~~Functi~~ Ex 2) S is closed $\Leftrightarrow$ S contains all its limit points.

"$\Rightarrow$" Let z_0 be a limit pt of S. Then, $\exists$ $\varepsilon > 0$ s.t

$$\{0 < |z - z_0| < \varepsilon\} \cap S \neq \phi$$

~~If~~ Spz $z_0 \notin S$, then $z_0 \in S^c$ $\Rightarrow$ z_0 is an int pt of S^c as S^c is open.

$\Rightarrow \exists$ $\varepsilon > 0$ st $B(z_0, \varepsilon) \subseteq S^c$

$\therefore$ $B(z_0, \varepsilon) \cap S = \phi$

$\rightarrow\leftarrow$ as z_0 is a limit pt of S.

$\Leftarrow$ Let S contains all its limit pts. Then, we prove S^c is open.

On the contrary, if S^c was not open, $\exists$ $z_0 \in S^c$ which is not an interior pt of S^c $\Rightarrow$ $\forall \varepsilon > 0$, $B(z_0, \varepsilon) \cap S \neq \phi$

~~$\therefore z_0 \notin S$, z_0~~ $\therefore$ z_0 has the property that every deleted nbhd $B^{del}(z_0, \varepsilon) \cap S \neq \phi$ as $z_0 \notin S$. Hence, z_0 is a limit pt of S

But then $z_0 \in S$ as S contains all its limit pts.

$\therefore$ $z_0 \in S$ & $z_0 \in S^c$ $\leftarrow\rightarrow$.

Functions of a complex variable $S \subseteq \mathbb{C}$

~~Let $w = f(z)$~~ A function f defined on S is a rule that assigns to each $z \in S$ a complex no. w, ie, $w = f(z)$

Spz that $w = u + iv$ is the value of f at $z = x + iy$, so that

$$u + iv = f(x + iy).$$

Here, each real no. u & v depend on x & y.

$$\text{ie, } f(z) = u(x, y) + iv(x, y).$$

If z is written as $re^{i\theta}$, then, $u + iv = f(re^{i\theta})$

& $u = u(r, \theta)$ & $v = v(r, \theta)$. & $f(z) = u(r, \theta) + iv(r, \theta)$.

Functions on the complex plane

Let f be a function defined ~~on a set~~ at all points z in some deleted neighbourhood of z_0.

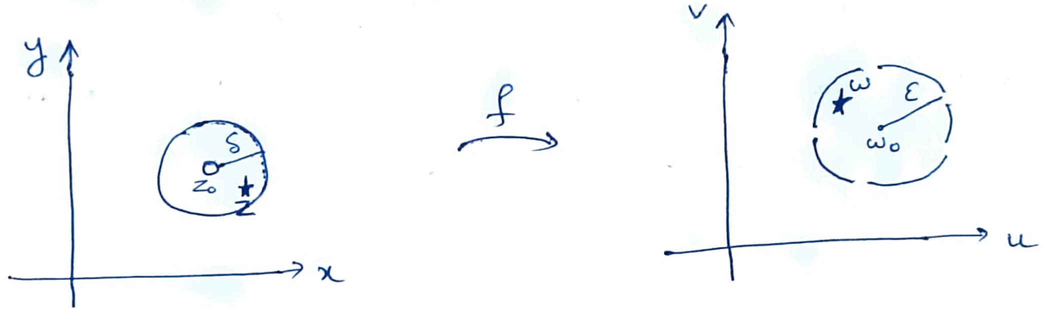
(1)

$\lim_{z \to z_0} f(z) = w_0$ means that the point $w = f(z)$ can be made arbitrarily close to w_0 if we choose the point z close enough to z_0 but distinct from it.

Equivalently, for each +ve no. ε, $\exists\ \delta > 0$ s.t
$|f(z) - w_0| < \varepsilon$ whenever $0 < |z - z_0| < \delta$. } (★)

(2)

Geometrically, this says that for each ε nbhd $|w - w_0| < \varepsilon$ of w_0, there is a deleted δ neighbourhood $0 < |z - z_0| < \delta$ of z_0 such that every point z in it has an image w lying in the ε nbhd of w_0.



$u = \text{Re } f(z)$
$v = \text{Im } f(z)$

PS: i) It is not necessary that every pt of $B(w_0, \varepsilon)$ has a pre-image in $B^{del}(z_0, \delta)$.

ii) For given $\varepsilon > 0$, if some $\delta > 0$ satisfies ★, so does $\frac{\delta}{2}$, etc....

~~Lemma: When a li~~ If f satisfies (★), w₀ is said to be the **Limit** of f(z) as z approaches z_0.

Lemma:- When a limit of a function f(z) exists at a point z_0, it is unique.

Proof :- Let $\lim_{z \to z_0} f(z) = w_1$ & $\lim_{z \to z_0} f(z) = w_2$. Then,

$|w_1 - w_2| \le |w_1 - f(z) + f(z) - w_2|$ $\forall$ z in domain of defn of f

Note that by defn, for any given $\varepsilon > 0$, $\exists\ \delta_1 > 0$ st

$|f(z) - w_1| < \varepsilon$ whenever $0 < |z - z_0| < \varepsilon_0$

& $\exists\ \delta_2 > 0$ st

$|f(z) - w_2| < \varepsilon$ whenever $0 < |z - z_0| < \varepsilon_0$

$\therefore\ |w_1 - w_2| \le 2\varepsilon$ holds for any $\varepsilon > 0$

Hence, $w_1 = w_2$

FACT: A limit point of a set is either in Int(S) or ∂S.
~~Defn (†) or (‡) requires that~~.

If z_0 is an interior point of S, then such a deleted nbhd of z_0 always exists inside S.

If z_0 is a boundary pt of S, then we modify (‡) or (†) as follows

Let z be a boundary pt z_0 of S ~~is said~~. ~~f is said to have limit w_0 lead to~~

~~The limit of f(z) as z approaches z_0~~ is w_0

$\lim_{z \to z_0} f(z) = w_0$ means $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ st

$|f(z) - w_0| < \varepsilon$ whenever z lies in S & satisfies $0 < |z - z_0| < \delta$.

Let $f : S \to \mathbb{C}$ & let z_0 be a limit pt of S. We say $\lim_{z \to z_0} f(z) = L$ provided that for all $\varepsilon > 0$, $\exists\ \delta > 0$ such that whenever $0 < |z - z_0| < \delta$ ($\& z \in S$), $|f(z) - L| < \varepsilon$

Eg 1) Let $f(z) = \frac{i\bar{z}}{2}$, $|z| < 1$. Then, $\lim_{z \to 1} f(z) = \frac{i}{2}$.

Proof:- Let $\varepsilon > 0$ be any arbitrary +ve no. We need to show that $\exists\ \delta > 0$ st

$0 < |z - 1| < \delta \Rightarrow |f(z) - \frac{i}{2}| < \varepsilon$.

Sps $|f(z) - \frac{i}{2}| < \varepsilon$, ie, $\left|\frac{i\bar{z}}{2} - \frac{i}{2}\right| = \frac{|\bar{z} - 1|}{2} = \cancel{\left|\frac{x - i(y+1)}{2}\right|}\ \left|\frac{z-1}{2}\right|$

Eg 2) Let $f(z) = \frac{z}{\bar{z}}$. Show that $\lim_{z \to 0} f(z)$ doesn't exist.

For any ε

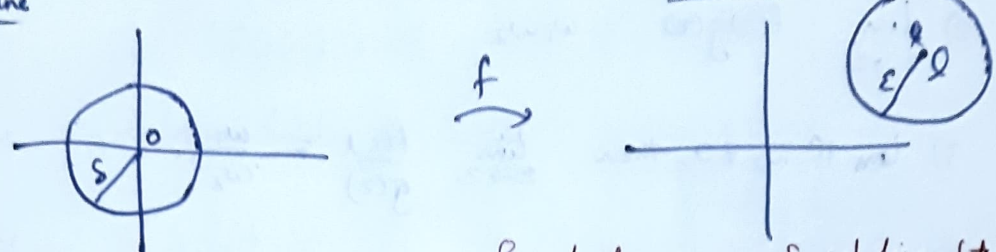
Clearly if $z \in \mathbb{R}$, $\lim_{z \to 0} f(z) = 1$ & if $z = ix, x \in \mathbb{R}$, $\lim_{z \to 0} f(z) = \frac{ix}{-ix} = -1$

Thus, we need to prove by contradiction.

Suppose $\underset{z\to 0}{Lt}\ f(z) = l$. Then, for any given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t

$$0 < |z| < \delta \Rightarrow |f(z) - l| < \varepsilon. \quad — (*)$$

ie, $z \in B_\delta^{del}(0) \Rightarrow f(z) \in B(l, \varepsilon)$



To contradict,
We need to show that there is an $\varepsilon > 0$ for which even no δ satisfies $(*)$

Clearly any deleted neighbourhood of 0 contains ~~pts~~ pts on the real axis & ~~another~~ points ~~&~~ on the imaginary pts.

Let $\varepsilon < 1$. Let $z_1 = \frac{\delta}{2}$ & $z_2 = \frac{i\delta}{2}$

$$|f(z_1) - f(z_2)| = |f(z_1) - l + l - f(z_2)| \le |f(z_1) - l| + |f(z_2) - l| < 2\varepsilon$$

However, as $f(z_1) = 1$ & $f(z_2) = -1$, $|f(z_1) - f(z_2)| = 2$. $\rightarrow\leftarrow$

~~Hence,~~ Hence, ~~our assumption was~~ ~~no such δ can exist~~ given any $\varepsilon > 0$, ~~no such~~ no such δ can exist.

$\therefore\ \underset{z\to 0}{Lt}\ f(z)$ doesn't exist.

Lec 18

Theorem 1) Suppose that $f(z) = u(x,y) + iv(x,y)$ where $z = x+iy$, $z_0 = x_0 + iy_0$ & $w_0 = u_0 + iv_0$. Then,

$$\underset{z\to z_0}{Lt}\ f(z) = w_0 \quad \text{iff}$$

$$\underset{(x,y)\to(x_0,y_0)}{Lt} u(x,y) = u_0 \quad \text{and} \quad \underset{(x,y)\to(x_0,y_0)}{Lt} v(x,y) = v_0.$$

Recall: Meaning : $\underset{\|(x,y)-(x_0,y_0)\|\to 0}{Lt} \|u(x,y) - u_0\| = 0.$

Theorem 2 (Algebra on limits) Suppose

$\lim_{z \to z_0} f(z) = w_1$ & $\lim_{z \to z_0} g(z) = w_2$. Then,

1) $\lim_{z \to z_0} \left(f(z) + g(z) \right) = w_1 + w_2$

2) $\lim_{z \to z_0} f(z) g(z) = w_1 w_2$

3) If $w_2 \neq 0$, then $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{w_1}{w_2}$.

Continuity

A function $f : S \to \mathbb{C}$ is continuous at $z_0 \in S$ if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Note that it is implicit in this defn that

1) $\lim_{z \to z_0} f(z)$ exists

2) $f(z_0)$ exists.

In other words, for each +ve ε, $\exists\ \delta > 0$ s.t

$|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$.

Further $\lim_{z \to z_0} f(z) = f(z_0) = f(\lim_{z \to z_0} z)$ ie, limit & f can be interchanged if f is cts at z_0.

f is continuous in a region R if it is continuous at each point in R.

FACT 1) If f & g are continuous at $z_0 \in S$, ($f, g : S \to \mathbb{C}$), then $f + g$, fg, $\frac{f}{g}$ (provided $g(z_0) \neq 0$) are also continuous at z_0.

2) A composition of two continuous functions is also continuous.

3) If a function $f(z)$ is continuous and non-zero at a point z_0, then $f(z) \neq 0$ throughout some neighbourhood of z_0.

4) The function $f(z) = u(x,y) + iv(x,y)$ is continuous at a point $z = \cancel{(x_0, y_0)}\ x_0 + iy_0$ iff $u(x,y)$ & $v(x,y)$ are continuous at (x_0, y_0).

Theorem 3) If a function $f(z)$ is continuous throughout a region R that is both closed & bounded, then $\exists\ M \geq 0$ such that

$$|f(z)| \leq M \qquad \forall\ z \in R$$

where equality holds for atleast one such z.

Exercise 1: ~~What is Lt~~ ~~Show that the limit~~ Discuss the limiting behaviour of the function $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ as $z \to 0$ ~~doesn't exist~~. (Hint: Line $y = x$)

Derivatives

Let $z_0 \in \mathbb{C}$.

Let f be a function whose domain of defn contains a neighbourhood $|z - z_0| < \varepsilon$ of the pt z_0. The derivative of f at z_0 is the limit

$$\underset{z \to z_0}{Lt} \frac{f(z) - f(z_0)}{z - z_0}.$$

if it exists. If the above limit exists, we denote it by $f'(z_0)$ & say that "f is differentiable at z_0".

If $\Delta z = z - z_0$, then equivalently, we may also write

$$f'(z_0) \text{ as } \underset{\Delta z \to 0}{Lt} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Since f is defined on some $B(z_0, \varepsilon)$, ($f(z_0 + \Delta z)$ is defined on pts $z_0 + \Delta z \in B(z_0, \varepsilon)$ &) hence, ~~limit can be~~ ~~is well-defined~~ the expression $\underset{\Delta z \to 0}{Lt} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ makes sense.

Eg. 1) $f(z) = \bar{z}$ is NOT differentiable anywhere on $\mathbb{C}$.

$$\underset{\Delta z \to 0}{Lt} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \underset{\Delta z \to 0}{Lt} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$ doesn't exist as seen in Eg 2) Limits class.

2) $f(z) = |z|^2 = z\bar{z}$

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{z\bar{h}+\bar{z}h+h\bar{h}}{h}$$

Along real axis, $(h \in \mathbb{R})$, this limit exists & equals $z+\bar{z}$

Along ~~the~~ imaginary axis, $(h = ik, k \in \mathbb{R})$, this limit exists &

equals $\lim_{k\to 0}\left(z\frac{-ik}{ik} + \bar{z}\frac{ik}{ik} + -ik\right) = \bar{z}-z$

Thus, ~~$\lim_{k\to 0}$~~ f has no ~~limit~~ derivative anywhere on $\mathbb{C}$. except ~~unless~~

at $z=0$. (Note: $z+\bar{z} = \bar{z}-z \iff 2z=0 \iff z=0$)

This is an example of a function $(f(z)=|z|^2)$ ~~$f(x,y)=x^2+y^2$~~ which is complex differentiable at a single point & nowhere else in any nbhd of that point.

<u>Usual identities involving derivatives of real valued functions work here also:-</u>

1) $\frac{d}{dz}(c) = 0$, $c \in \mathbb{C}$

2) $\frac{d}{dz}(z^n) = nz^{n-1}$, $n \in \mathbb{N}$

3) $\frac{d}{dz}(z^n) = nz^{n-1}$, $n \in \mathbb{Z}_{<0}$ if $z \neq 0$.

4) Sum rule

5) Product rule

6) Quotient rule

7) <u>Chain rule</u>: Sps f has deriv. at z_0 & g has deriv. at $f(z_0)$. Then the function $F(z) = g[f(z)]$ has derivative at z_0 &

$$F'(z_0) = g'[f(z_0)]\, f'(z_0).$$

Cauchy – Riemann Equations

AIM : To obtain alternate ways to talk about complex differentiability.

Let $f(z) = u(x,y) + iv(x,y)$ & $z_0 = x_0 + iy_0$.

Let $\Delta z = \Delta x + i\Delta y = h + ik$

$$\Delta w := f(z_0+\Delta z) - f(z_0)$$

$$= u(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) + i\left[v(x_0+\Delta x, y_0+\Delta y) - v(x_0,y_0)\right]$$

Recall

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \cancel{\lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z}} = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z}$$

$= \cancel{\lim_{(\Delta x,\Delta y)\to(0,0)}}$ ~~Re~~ Recall that ~~if~~ $\lim_{z\to z_0} f(z)$ ~~can be~~ is equivalent to ~~saying~~ $\lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$

$$\therefore f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \mathrm{Re}\left[\frac{\Delta w}{\Delta z}\right] + i \lim_{(\Delta x,\Delta y)\to(0,0)} \mathrm{Im}\left[\frac{\Delta w}{\Delta z}\right].$$

Theorem 4) Suppose that $f(z) = u(x,y) + iv(x,y)$ & that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then, ~~the first order pa~~

1) the first order partial derivatives of u & v exist at (x_0,y_0).

2) first order partial derivatives must satisfy the CR eqns

$$u_x = v_y \quad \& \quad u_y = -v_x.$$

3) Further,

$$f'(z_0) = u_x + iv_x \Big|_{(x_0,y_0)}.$$

Eg 1) $f(z) = z^2 \Rightarrow u(x,y) = x^2 - y^2$ & $v(x,y) = 2xy$

$= (x+iy)^2$ $\qquad u_x = 2x \qquad v_y = 2x$

$= x^2 - y^2 + i2xy$ $\qquad u_y = -2y \qquad v_x = 2y$

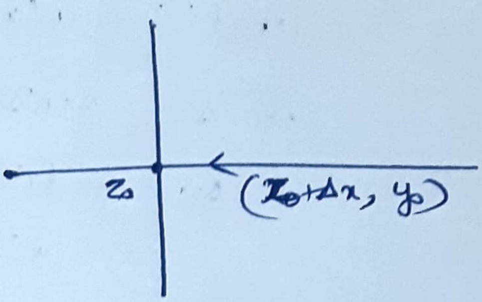
& $f'(z) = 2z = u_x + iv_x$

Proof: Since $f'(z_0)$ exists, the limit

$$\cancel{\underset{\Delta z \to 0}{Lt} \frac{f(z)-f(z_0)}{\Delta z}} \quad \underset{\Delta z \to 0}{Lt} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$ exists & hence, is

same along all directions.

Consider along X axis :-



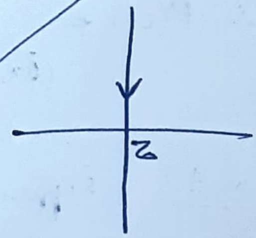
$$\underset{\Delta x \to 0}{Lt} \frac{f(x_0+\Delta x+iy_0) - f(x_0+iy_0)}{\Delta x}$$

$$= \underset{\Delta x \to 0}{Lt} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\left[\frac{v(x_0+\Delta x, y_0) - v(x_0, y_0)}{\Delta x}\right]$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x, y_0)$$ both these exist from Thm 1

ie, $\boxed{f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)}$ Part 2 is proved

Consider along Y axis :-



$$\underset{\Delta y \to 0}{Lt} \frac{f(x_0+i(y_0+\Delta y)) - f(x_0+iy_0)}{i\Delta y}$$

$$= \underset{\Delta y \to 0}{Lt} \frac{u(x_0, y_0+\Delta y) - u(x_0, y_0)}{i\Delta y} + i\left[\frac{v(x_0, y_0+\Delta y) - v(x_0, y_0)}{i\Delta y}\right]$$

$$= -i\, u_y(x_0, y_0) + v_y(x_0, y_0)$$ Thus part (1) is proved.

ie, $\boxed{f'(z_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)}$

In particular, $u_x(x_0, y_0) = v_y(x_0, y_0)$ &

$u_y(x_0, y_0) = -v_x(x_0, y_0)$ part 2 is proved, On the other hand, if u_x, u_y, v_x, v_y exist at (x_0, y_0) & satisfy CR eqns, it is not necessary that $f(z)$ is differentiable at $z_0 = x_0 + iy_0$.

Eg. $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & \text{if } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$

$$f(z) = \frac{(x-iy)^2}{x+iy} = \frac{(x-iy)^3}{x^2+y^2},$$

$$\Rightarrow f(z) = \frac{1}{|z|^2}\left[x^3 - 3xy^2 + i\left(y^3 - 3x^2y\right)\right]$$

Let $u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ & \text{o/w} \end{cases}$ & $v(x,y) = \begin{cases} \dfrac{y^3 - 3x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{o/w} \end{cases}$

a) Verify that CR eqns are satisfied at $(0,0)$.
However, $f'(0)$ doesn't exist.

$$u_x(0,0) = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

$$u_y(0,0) = \lim_{k \to 0} \frac{u(0,k) - 0}{k} = \lim_{k \to 0} 0 = 0$$

$$v_x(0,0) = \lim_{h \to 0} \frac{v(h,0)}{h} = \lim_{h \to 0} 0 = 0$$

$$v_y(0,0) = \lim_{k \to 0} \frac{v(0,k)}{k} = \lim_{k \to 0} \frac{k^3}{k^2 \times k} = 1$$

ie, $u_x = v_y$ & $u_y = -v_x$ at $(0,0)$.

b) Show that ~~f(z) at z=0~~ $f'(z_0)$ doesn't exist. (Exercise)

~~By taking~~ partial derivatives, ~~we can see that~~ exist for u & v, for eg.

$$u_x(x,y) = \begin{cases} \dfrac{x^4 + 6x^2y^2 - 3y^4}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

However,

$$\lim_{h\to 0} u_x(h,0) = \lim_{h\to 0} \frac{h^4}{h^4} = 1 \text{ but}$$

$$\lim_{k\to 0} u_x(0,k) = \lim_{k\to 0} \frac{-3k^4}{k^4} = -3$$

That is, $\therefore u_x$ is not continuous at $(0,0)$.

~~Since~~ ~~Hence~~ Hence, sufficient conditions for differentiability are not met. To prove that f is not differentiable, (See 20 Q9) notice that

$$\lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z} \text{ doesn't exist as}$$

along $y = x$ line, say $(\Delta x, \Delta x)$

$$\cancel{\lim_{(\Delta x,\Delta x)\to(0,0)} \frac{(\Delta x)^2/\Delta x}{\Delta x} \Rightarrow \lim_{\Delta x\to 0}} \qquad \lim_{\Delta z\to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z\to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2}$$

Along $Y = X$ line: $$\lim_{\Delta x\to 0} \frac{(\overline{\Delta x + i\Delta x})^2}{(\Delta x + i\Delta x)^2} = \lim_{\Delta x\to 0} \frac{(1-i)^2}{(1+i)^2} = -1$$

$$\text{but } \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - 0}{\Delta x} = \lim_{\Delta x\to 0} \frac{(\Delta x)^2}{(\Delta x)^2} = 1$$

(along X axis)

Pr 2) Show that the complex fn $f(z) = e^z$ is differentiable everywhere on $\mathbb{C}$.

$$f(z) = e^x \cos y + i e^x \sin y = u(x,y) + i v(x,y)$$

$u_x = e^x \cos y \qquad v_y = e^x \cos y$

$u_y = -e^x \sin y \qquad v_x = e^x \sin y$

Clearly, u_x, u_y, v_x, v_y exists everywhere on $\mathbb{R}^2$ & are continuous on $\mathbb{R}^2$. Since they satisfy CR eqns, $f(z)$ is differentiable everywhere on $\mathbb{C}$.

Sufficient condn. for differentiability

Theorem : Let $f(z)$ be defined throughout some ϵ-nbhd of a point $z_0 = x_0 + iy_0$. Suppose that

a) u_x, u_y, v_x, v_y exist everywhere on the neighbourhood,

b) u_x, u_y, v_x, v_y are continuous at (x_0, y_0)

c) u & v satisfy the CR equations at (x_0, y_0)

$$u_x = v_y \;,\quad u_y = -v_x \;,$$

then $f'(z_0)$ exists &

$$f'(z_0) = u_x + iv_x \Big|_{(x_0, y_0)} .$$

Verify that $u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{o/w} \end{cases}$

& $v(x,y) = \begin{cases} \dfrac{y^3 - 3x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{o/w} \end{cases}$

doesn't satisfy CR eqns at $(x,y) \neq (0,0)$.

$$u_x(x,y) = \frac{(x^2+y^2)(3x^2-3y^2) - (x^3-3xy^2)(2x)}{(x^2+y^2)^2} = \frac{3(x^4-y^4) - 2x^4 + 6x^2y^2}{(x^2+y^2)^2}$$

$$= (x^4 + 6x^2y^2 - 3y^4)/(x^2+y^2)^2$$

$$v_y(x,y) = \frac{(x^2+y^2)(3y^2-3x^2) - (y^3-3x^2y)(2y)}{(x^2+y^2)^2} = \frac{3(y^4-x^4) - 2y^4 + 6x^2y^2}{(\quad)^2}$$

$$= (y^4 + 6x^2y^2 - 3x^4)/(\quad)^2$$

Thus, ~~$u_x \neq v_y$ if~~ $u_x(x,y) \neq v_y(x,y)$ unless $x = \pm y$.

Show that a) $f(z) = 3x+y+i(3y-x)$ & b) $f(z) = \sin x \cosh y$ (22)

$+ i \cos x \sinh y$ are differentiable everywhere on $\mathbb{C}$.

Let $f: D \to \mathbb{C}$ where D is an open connected set (domain)

1) A function f of the complex variable z is <u>analytic at a point z_0</u> if it has a derivative at each point in some neighbourhood of z_0.

$B(z_0, \varepsilon) = \{ z \in \mathbb{C} \mid |z - z_0| < \varepsilon \}$.

By defn, f is ~~holo~~ analytic at $z_0 \Rightarrow f$ is analytic at each pt $z \in B(z_0, \varepsilon)$.

2) A function f is analytic in an open set if it has a derivative everywhere in that set.

<u>Note</u>: If $S \subseteq \mathbb{C}$ is not open, we say f is analytic on S if f is analytic in an open set containing S.

a) $f(z) = \frac{1}{z}$ is analytic ~~on $\mathbb{C} \setminus \{0\}$~~ at every point of $\mathbb{C}$ except 0.

b) $f(z) = |z|^2$ is <u>NOT</u> analytic at any point.

Recall it is differentiable at $z_0 = 0$, but not on any ~~cont~~ neighbourhood containing 0.

3) ~~An entire~~ function is said to be <u>entire</u> if it is analytic at every point of $\mathbb{C}$.

Eg a) Polynomials $p(z)$ as derivative exists for them everywhere on $\mathbb{C}$.

~~4) Singular point or singularity of f~~

~~If a function fails to be analytic at a point z but is analytic at some~~

4) A necessary condition for analyticity of f in a domain D is satisfaction of CR eqns. Sufficient conditions for analyticity ~~in D~~ of f in D is given in prev. thm.

5) If two functions are analytic in D, then their sum, product, quotient are respectively analytic in D provided $g(z) \neq 0 \ \forall z \in D$

6) Composition of two analytic fns is analytic.

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	function	domain	holo.	derivative
1)	$f(z) = z$	$\mathbb{C}$	✓	1
2)	$\frac{1}{z}$	~~$\mathbb{C}\setminus\{0\}$~~ Any S open subset of $\mathbb{C}$, $0 \notin S$	✓	$-\frac{1}{z^2}$
3)	$\bar{z}$	$\mathbb{C}$	X	

Pr 1) Show that $f(z) = \dfrac{z^3+4}{(z^2-3)(z^2+1)}$ is analytic throughout the z plane except at $\pm\sqrt{3}, \pm i$.

Derivative of f exists everywhere on $\mathbb{C}\setminus\{\pm\sqrt{3}, \pm i\}$ & hence is analytic there.

Pr 2) Show that $f(z) = \cosh x \cos y + i \sinh x \sin y$ is analytic on $\mathbb{C}$, i.e. entire

$u(x,y) = \cosh x \cos y$ Recall $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $x \in \mathbb{R}$

$v(x,y) = \sinh x \sin y$ $\sinh x = \dfrac{e^y - e^{-y}}{2}$, $x \in \mathbb{R}$

Verify $\dfrac{d}{dx} \cosh x = \sinh y$

$\dfrac{d}{dx} \sinh x = \cosh y$

u_x, u_y, v_x, v_y clearly exist everywhere on ~~$\mathbb{C}$~~ $\mathbb{R}^2$ & are continuous.

$u_x = \sinh x \cos y$ $v_y = \sinh x \cos y$

$u_y = -\cosh x \sin y$ $v_x = \cosh x \sin y$

$\Rightarrow u_x = v_y$ & $u_y = -v_x$.

f diff $\Rightarrow f$ cts

Lec 20: Harmonic functions

Pr 1) Suppose that a function $f(z)$ & its conjugate $\overline{f(z)}$ are both analytic on a given domain D, then $f(z)$ must be constant throughout D.

Ans: $f(z) = u(x,y) + iv(x,y) \Rightarrow u_x(x,y) = v_y(x,y)$ on D

$\quad\quad \& \; u_y = -v_x$

Now, $\overline{f}(z) = u(x,y) - iv(x,y) \Rightarrow u_x(x,y) = -v_y(x,y)$

$\quad\quad \& \; u_y(x,y) = v_x(x,y)$ on D

ie, $u_x = v_y = -v_y$ on D & $u_y = -v_x = v_x$ on D

$\Rightarrow v_y = v_x = 0$ on D

$\Rightarrow v = f_1(x)$ on D (as $v_y = 0$)

& $v = f_2(y)$ on D

$\therefore v = c_2$ on D

III^{ly}, $u = c_1$ on D $\Rightarrow f(z) = c_1 + ic_2 = c \quad \forall z \in D$

~~Ex 1) S.T if $f(z)$ is diff at z_0, then $f(z)$ is cts at z_0.~~

Pr 2) If f is analytic on a domain D, and $|f(z)|$ is constant throughout D, ~~also~~ then show that $f(z)$ must be a constant on D

Ans:- let $|f(z)| = c$.

$\Rightarrow |f(z)|^2 = u^2 + v^2 = c^2$. Either $c = 0$ or $c \neq 0$.

Case 1 $c \neq 0 \Rightarrow \overline{f}(z) = \dfrac{c^2}{f(z)}$. Since $|f(z)| \neq 0$ throughout D, $\overline{f}(z)$ is also analytic on D. Thus, by prev. problem, $f(z) = c'$ on D.

Case 2: $c = 0 \Rightarrow |f(z)| = 0$ on D $\Rightarrow f(z) = 0$ on D

Pr 3) Show Pr 2) directly using CR eqns without using Pr 1)

Let $|f(z)|^2 = u^2+v^2 = c^2$ on D

$\Rightarrow \quad 2uu_x + 2vv_x = 0 \quad \& \quad 2uu_y + 2vv_y = 0$

$\Rightarrow \quad uv_y + vv_x = 0 \rightarrow (1) \quad \& \quad -uv_x + vv_y = 0 \rightarrow (2)$

$(1) \times v + (2) \times u = uvv_y + v^2v_x + u^2v_x - uvv_y = 0$

$\Rightarrow (u^2+v^2)v_x = 0$ on D.

But $u^2+v^2 = c^2$. If $c=0$, then $u(x,y) = 0$ & $v(x,y) = 0$ on D

$\Rightarrow f(z) = 0$ on D.

If $u^2+v^2 = c^2 \neq 0$ on D, then $v_x(x,y) = 0$ on D

$\Rightarrow v(x,y) = f_1(y)$ on D

~~lllly, (1) × u + (2) × v = $u^2v_y + v^2v_y = 0$~~

~~$\Rightarrow (u^2+v^2)v_y = 0$~~

~~$\Rightarrow v(x,y) = f_2(x)$ on D~~

If so, by CR $u_y(x,y)$ on D $\Rightarrow u(x,y) = f_1(x)$ on D &

ie, $v(x,y) = c'$ on D

lllly, $(1) \times u + (2) \times v = u^2v_y + v^2v_y = 0$

$\Rightarrow (u^2+v^2)v_y = 0$

$\Rightarrow v(x,y) = f_2(x)$ on D

ie, $v(x,y) = f_1(y) = f_2(x)$ on D

$\Rightarrow v(x,y) = c_2$ on D

lllly, ~~it can b~~ it can be ~~shos~~ shown that

$u(x,y) = c_1$ on D

$\therefore f(z) = c_1 + ic_2$ on D.

(4) B. If $f(z) = z^2$, then what is the value of $u_{xx}+u_{yy}$ & $v_{xx}+v_{yy}$

$u(x,y) = x^2-y^2 \qquad v(x,y) = 2xy$

$u_y = -2y \qquad u_x = 2x \qquad v_x = 2y \qquad v_y = 2x$

$u_{yy} = -2 \qquad u_{xx} = 2 \qquad v_{xx} = 0 \qquad v_{yy} = 0$

$\Rightarrow u_{xx} + u_{yy} = 0 \qquad \Rightarrow v_{xx} + v_{yy} = 0$

Harmonic Functions

A real valued function $h : \mathbb{R}^2 \to \mathbb{R}$ is said to be harmonic in a given domain D of the XY plane if,

i) it has continuous partial derivatives of 1st & 2nd order &

2) it satisfies the Laplace eqn $h_{xx} + h_{yy} = 0$ on D

$$\nabla^2 h(x,y) = 0 \text{ on } D.$$

Lec 21 :-

Pr 3) If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component functions u & v are harmonic in D.

For this, we assume the following facts:

If $f(z)$ is analytic at z_0, then all its n^{th} derivatives, $f^{(n)}(z)$ exists at z_0.

This will be further explored ~~Its proof will be taken~~ while developing Cauchy's integral formula.

Ans :- Recall that if $f(z) = u(x,y) + iv(x,y)$, then

$$f'(z) = u_x + iv_x = v_y - iu_y$$

Since $f''(z_0)$ exists at z_0, $f'(z)$ is cts at z_0.

$\therefore$ u_x & v_y (also v_y & u_y) are cts at (x_0, y_0)

Further, $f''(z_0) = \frac{\partial}{\partial x}(u_x) + i\frac{\partial}{\partial x}(v_x)$ $\Big|$ $\frac{\partial}{\partial x}(v_y) + i\frac{\partial}{\partial x}(-u_y)$

$= u_{xx} + iv_{xx}$ $\Big|$ $= v_{yx} - i u_{yx}$

Also, $f''(z_0) = \frac{\partial}{\partial y}(-u_y) - i\frac{\partial}{\partial y}(v_y)$ $\Big|$ $\frac{\partial}{\partial y}$

$= -u_{yy} - iv_{yy}$

ie, $u_{xx} + u_{yy} = 0$ on D

& $v_{xx} + v_{yy} = 0$ on D

Further, since $f^{(3)}(z_0)$ exists, $f''(z_0) = u_{xx} + iv_{xx}$ is ~~$f^{(3)}(z_0) =$~~ continuous at z_0 & hence u_{xx} & v_{xx} are cts at z_0 & hence, ~~$u_{xy} = u_{yx}$ & $v_{xy} = v_{yx}$ also.~~ we get $(u_x)_x = (v_y)_x = v_{yx}$ $= (-u_y)_y = (v_x)_y = v_{xy}$

(lly, $u_{xy} = u_{yx}$ too)

__Corollary__ If $f(z) = u(x,y) + iv(x,y)$ is analytic at $z_0 = (x_0, y_0)$, then u & v have continuous partial derivatives of all orders at that point.

__Theorem__ A function $f(z) = u + iv$ is analytic in a domain D iff v is a harmonic conjugate of u.

__Defn__:- If two given functions u & v are harmonic in a domain D & their first order partial derivatives satisfy CR eqns throughout D then v is said to be a __harmonic conjugate__ of u.

__Proof__:- $\Rightarrow$ $f(z)$ is analytic in D

$\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$ on D

& by pr 3), u & v satisfy $u_{xx} + u_{yy} = 0$ on D

$\Rightarrow$ v is a harmonic conjugate of u

$\Leftarrow$ By defn of harmonic fns, $u_x, u_y, v_x, v_y, u_{xx}, u_{yy}$ (& hence v_{yx} & v_{xy}), v_{xx} & v_{yy} (& hence u_{yx} & u_{xy}) exist & are cont ~~exist~~ on D. Further $u_x = v_y$ & $u_y = -v_x$ on D. ~~If all first order partial derivatives are continuous in D, we have that f(z) is analytic on D.~~ However, since $f'(z)$ is continuous on D, its real part u_x & img. part v_x are also cont. on D & hence, $f(z)$ is analytic on D.

__Finding harmonic conjugate v of u__

Let $u(x,y) = y^3 - 3x^2y$

Check that it is harmonic on $\mathbb{C}$.

Let v be a harmonic function conjugate of u. Then, automatically v is harmonic on D & ~~$v \in C^1(\mathbb{C})$~~. satisfies CR eqns.

$\therefore \quad v_x = -u_y = -3y^2 + 3x^2 \qquad -①$

$\& \quad v_y = u_x = -6xy \qquad -②$

Then, $\quad v = -3y^2x + x^3 + c_1(y) \qquad$ from ①

$\qquad = -3xy^2 \quad + c_2(x) \qquad$ from ②

$\Rightarrow \quad v(x,y) = x^3 - 3xy^2 + C$

& hence, $\quad f(z) = u + iv$ is analytic on $\mathbb{C}$, ie an entire function.

$\qquad = y^3 - 3x^2y + i\,(x^3 - 3xy^2 + C)$

This can be seen to be equal to $f(z) = i(z^3 + C)$ by verification.

To guess: put $y=0$. Then $f(x) = i(x^3 + C)$

FACT :- Any harmonic function on an open ball $B(a,r)$ has a harmonic conjugate in $B(a,r)$. (More generally, in simply connected domains)

2) Harmonic ~~funcl~~ conjugate of a ~~harmonic~~ function $u(x,y)$, when exists, is unique except for an additive constant. (Ex 2 Sec 26)

Elementary functions

1) Exponential function

Recall $e^z = e^x e^{iy}$ where $e^{iy} = \cos y + i \sin y$ (y radian) — ③

~~Note that $e^{1/n}$ was defined as all (n many) distinct nth roots of e. This was a set. However, when $z = \frac{1}{n}$ in ③ we write e^x where $x = \frac{1}{n}$, we consider $\sqrt[n]{e}$, the the nth root of e.~~

~~This expone~~ The complex exponential function is an extension of the real exponential function e^x.