Q: Solve the system of simultaneous differential equations given by

$$\frac{dy_1}{dt} = y_1 + y_2$$

$$\frac{dy_2}{dt} = y_1$$

i.e $y_1(t) = ?$

$y_2(t) = ?$

Sol: Given system can be written as

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

↑ Fibonacci Matrix

$$\Rightarrow \quad Y' = AY \quad \text{———} \quad (1)$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

Consider the simpler case of

$$y_1' = \alpha y_1$$

$$\Rightarrow \frac{y_1'}{y_1} = \alpha$$

$$\Rightarrow \log y_1 = \alpha t + k$$

$$\Rightarrow y_1 = e^{\alpha t + k} = e^k \cdot e^{\alpha t}$$

$$\Rightarrow y_1 = a e^{\alpha t}$$ where $a = e^k$

Similarly, $y_2 = b e^{\alpha t}$

Now, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} e^{\alpha t} = e^{\alpha t} x$

where $x = \begin{bmatrix} a \\ b \end{bmatrix}$.

Put in $Y = e^{\alpha t} x$ in (1)

i.e in $Y' = AY$

$$\Rightarrow \frac{d}{dt}\left(e^{\alpha t} x\right) = A\left(e^{\alpha t} x\right)$$

$$\Rightarrow \alpha e^{\alpha t} x = A e^{\alpha t} x$$

$$\Rightarrow \quad \alpha X = AX \qquad (\text{As } e^{\alpha t} \neq 0).$$

$\alpha = ?$

$X = ?$

This is the standard eigenvalue problem

eigenvalue

eigenvector

We have $\alpha = \lambda_1 = \dfrac{1+\sqrt{5}}{2}$

$$X_1 = \begin{bmatrix} 1 \\ -\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\left(\frac{1-\sqrt{5}}{2}\right) \end{bmatrix}$$

One particular solution of the system is

$$e^{\alpha t} X = e^{\lambda_1 t} \begin{bmatrix} 1 \\ -\lambda_2 \end{bmatrix}$$

We have $\alpha = \lambda_2 = \dfrac{1-\sqrt{5}}{2}$

$$X_2 = \begin{bmatrix} 1 \\ -\lambda_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\left(\dfrac{1+\sqrt{5}}{2}\right) \end{bmatrix}$$

Another particular solution of the system is

$$e^{\alpha t} X = e^{\lambda_1 t} \begin{bmatrix} 1 \\ -\lambda_1 \end{bmatrix}.$$

The general solution of the given system is a linear combination of particular solutions i.e $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ -\lambda_2 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ -\lambda_1 \end{bmatrix}$

i.e
$$\left.\begin{aligned} y_1(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ y_2(t) &= -c_1 \lambda_2 e^{\lambda_1 t} - c_2 \lambda_1 e^{\lambda_2 t} \end{aligned}\right\} (\mathrm{I})$$

where c_1, c_2 are constants

$$\lambda_1 = \frac{1+\sqrt{5}}{2} ;$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} .$$

Note that as $|\lambda_2| < 0$; as $t \to \infty$

$$e^{\lambda_2 t} \to 0.$$

As $t \to \infty$; the solution (I) mainly depends on the term $e^{\lambda_1 t}$ i.e on $\lambda_1 t$ i.e on the golden ratio multiple of time t.

Take $t \to \infty$; $|\lambda_2| < 1$

$$e^{\lambda_2 t} \to 0$$

$$\therefore \quad y_1(t) = c_1 e^{\lambda_1 t}$$

$$y_2(t) = -c_1 \lambda_2 e^{\lambda_1 t}$$

$$= -c_1 \left(\frac{-1}{\lambda_1} e^{\lambda_1 t} \right) = \frac{c_1}{\lambda_1} e^{\lambda_1 t}$$

entirely in terms of golden ratio and time t

Theorem : Let $A_{m \times n}$ be a matrix of rank r. Then $A = U \Sigma V^T$ (SVD) and

(i). $u_1, u_2, \ldots, u_r$ is a basis for $C(A) = R(A^T)$.

(ii). $u_{r+1}, u_{r+2}, \ldots, u_m$ is a basis for $N(A^T)$.

(iii). $v_1, v_2, \ldots, v_r$ is a basis for $C(A^T) = R(A)$.

(iv). $v_{r+1}, \ldots, v_n$ is a basis for $N(A)$.

(Actually, SVD is computing basis for the Four Fundamental Subspaces of a Matrix).

Proof : $A = U\Sigma V^T$ (done already).

$$A v_i = \sigma_i u_i$$

$$A^T(A v_i) = \sigma_i A^T u_i$$

$$\lambda_i v_i = \sigma_i A^T u_i$$

$$v_i = \frac{\sigma_i}{\lambda_i} A^T u_i \quad \text{for } i = 1, 2, \ldots, r.$$

↳ a linear combination of columns of A^T.

$$\Rightarrow v_i \in C(A^T) = R(A) \quad ; \quad i = 1, 2, \ldots, r.$$

that is, we have proved (iii).

Now,

$$A v_{r+1} = 0 \; ; \; A v_{r+2} = 0 \; ; \; \cdots \; ; \; A v_n = 0 .$$

$$\Rightarrow v_{r+1}, v_{r+2}, \ldots, v_n \in N(A).$$

Given $\mathbb{R}^n = R(A) \oplus N(A)$.

Replace A by A^T to prove (i) and (ii).

Example :

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}_{\substack{2\times 3 \\ m \times n}} .$$

$T : \mathbb{R}^{3=n} \to \mathbb{R}^{2=m}$

$T(X) = A_{2\times 3} \cdot X_{3\times 1}$

$\mathbb{R}^3 = R(A) \oplus N(A)$

$\mathbb{R}^2 = R(A^T) \oplus N(A^T)$

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}_{2\times 3}.$$

Then
$$A = \frac{1}{\sqrt{2}} \begin{array}{c} \begin{matrix} u_1 & u_2 \end{matrix} \\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \sqrt{18} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1/3 & 2/\sqrt{5} & 2/\sqrt{45} \\ -1/3 & 0 & 5/\sqrt{45} \\ 2/3 & -1/\sqrt{5} & 4/\sqrt{45} \end{bmatrix}^T \end{array}.$$

Find basis for $R(A)$, $N(A)$, $R(A^T)$, $N(A^T)$.

$\text{rank}(A) = 1 \quad ; \quad m = 2 \quad ; \quad n = 3.$

$= r.$

$T : \mathbb{R}^3 \to \mathbb{R}^2$

$T(X) = AX$

$\mathbb{R}^3 = R(A) \oplus N(A).$

Basis for $R(A^T)$: $\left\{ \overset{u_1}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \right\}$

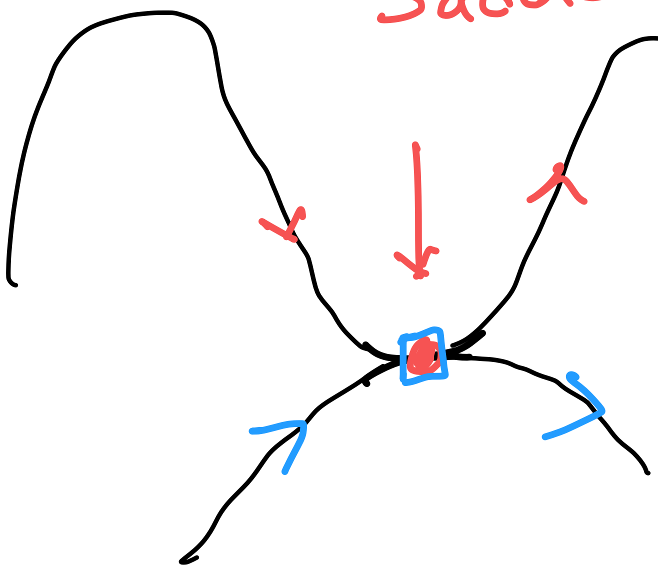
Basis for $N(A^T)$: $\left\{ \overset{u_2}{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right\}$

Basis for $C(A^T) = R(A)$: $\left\{ \overset{v_1}{\frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}} \right\}$

Basis for $N(A)$: $\left\{ \underset{v_2}{\dfrac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}}, \underset{v_3}{\dfrac{1}{\sqrt{45}}\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}} \right\}$

Game Theory:

Saddle point (Acts like maximum and minimum at the same time depending on direction)



→ maxm □

→ min ●

Q1: Solve the game

Player II

Player I
$$\begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & -10 \\ 1 & 2 \end{bmatrix} \end{array}$$

Payoff Matrix

S/:

Player II

Player I

$$
\begin{array}{c|c}
 & \begin{array}{cc} 1 & 2 \end{array} \\
\hline
\begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} 0 & -10 \\ \boxed{1} & \boxed{2} \end{bmatrix}
\end{array}
\qquad
\begin{array}{c} \text{row min} \\ -10 \\ 1 \end{array}
$$

col. max $\square$ 1   2

saddle point (column max = row min)

Solution: player I → option 2.

player II → option 1.

(saddle point)

Q2: Saddle point:

$$
\begin{array}{cc|cc|c}
 & & \text{II} & & \\
 & & 1 & 2 & \text{row min} \\
\hline
\text{I} & 1 & \boxed{0} & 1 & 0 \\
 & 2 & -10 & 2 & -10 \\
\hline
 & \text{col. max.} & \boxed{0} & \boxed{2} &
\end{array}
$$

Strategy Player I : option 1. (saddle point)

Strategy Player II : option 1.

Q3:

Prisoner's dilemma.

$$\begin{array}{ccc} & & \text{Prisoner II} \\ & & \begin{array}{cc} \text{confess} & \text{deny} \end{array} \\ \text{Prisoner I} & \begin{array}{c} \text{confess} \\ \text{deny} \end{array} & \begin{bmatrix} -6,-6 & 0,-10 \\ -10,0 & -1,-1 \end{bmatrix} \end{array}$$

↑ payoff matrix.

→ solution?

5/ Find saddle points of payoff matrix.

solution ; if they exist.

$\square$: column max.

$\bigcirc$: Row min.

} along 1^{st} component.

$\square$: column min

$\bigcirc$: Row max

} along 2^{nd} component.

Prisoners can't believe each other (Non-cooperative game).

$\square \rightarrow$ col. max.
$\bigcirc \rightarrow$ row min.

$\square \rightarrow$ col. min
$\bigcirc \rightarrow$ row max.

Nash Equilibrium

$$
\begin{array}{c|cc}
 & \text{Confess} & \text{Don't Confess} \\
\text{Confess} & \boxed{-6}, \boxed{-6} & \boxed{0}, \boxed{-10} \\
\text{Don't Confess} & -10, 0 & -1, -1
\end{array}
$$

$\rightarrow$ max

Ist component saddle point at $\boxed{-6}$

IInd component saddle point at $\boxed{-6}$

Solution is Confess, Confess.

In games without saddle points, we find probabilities of playing various options (called Mixed Strategies).

Mathematician Prof. John Nash

↓

Nobel Prize Winner 1994

↓

Movie on him "A Beautiful Mind" in 2001 with Box Office \$316 Million.

QED □