Testing of Hypothesis

Suppose that the life length X of an item say bulb is normally distributed with mean 100 and Variance 9. Suppose that a new manufacturing scheme has been introduced whose purpose is to extend the life length of bulb. Suppose that the life length of the newly manufactured bulb is $N(\mu, 9)$ [Assuming that the variance is same).

The problem is to decide whether $\mu = 100$ or $\mu > 100$ for the new scheme.

Statements like $\mu = 100$, $\mu > 100$ are called **statistical hypo.**

Def: A statistical hypo. is an assertion about the distribution of one or more r.v.s.

If the statistical hypo. completely specifies the distribution, then it is called simple hypo. Otherwise it is called composite hypo.

$\mu = 100$ simple hypo, $\mu > 100$ composite hypo.

Def. procedures which enable us to decide whether to accept or reject a given hypothesis is called test of hypo.

Errors: In a test process there can be four possible situations of which two of the situations leads to the two types of errors.

	Accepting the hypo.	Rejecting the hypo
Hypo. is true	Correct decision	Wrong decision (Type I error)
Hypo. is false	Wrong decision (Type II error)	Correct decision.

Def: The hypo. formulated for the purpose of its rejection under the assumption that it is true is called as the null hypo., denoted by H_0.

Any hypo. which is complimentary to the null hypo. is called alternative hypo., denoted by H_1.

Def: (i) Let C be subset of the sample space which leads to the rejection of the null hypo. Then C is called the critical region of the test.

(ii) The power function is that which yields the prob. that the sample point falls in the critical region.

Power of the text = Pr{critical region}.

Def: Let H_0 and H_1 be null and alternative hypo. respectively. The significance level of the test is the maximum value of the power function of the test when H_0 is true.

Def: (Significance level) The probability level, below which leads to the rejection of the hypo. is known as the significance level.

The prob. is conventionally fixed at 0.05 or 0.01 ie, 5% or 1% These are called sig. levels.

Let $Z = \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

From the tables 95% of the area lies b/w $Z=-1.96$ and $Z=1.96$

Further 5% ~~of the~~ level of significance is denoted by $Z_{0.05}$

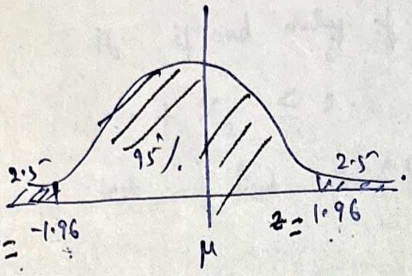
Thus, $-1.96 \leq \dfrac{\bar{x}-\mu}{\sigma/\sqrt{n}} \leq 1.96$

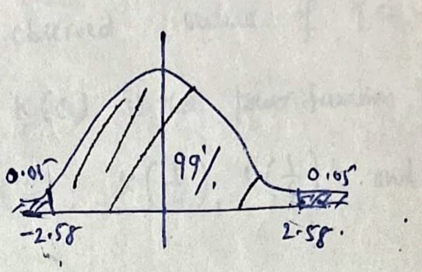
From this we can arrive at the confidence level 95%.

is $\bar{x} - (1.96)\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu \le \bar{x} + 1.96\left(\frac{\sigma}{\sqrt{n}}\right)$

Similarly 99% of the area lies b/w. -2.58 and 2.58, the equivalent form is

$$\bar{x} - 2.58\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu \le \bar{x} + 2.58\left(\frac{\sigma}{\sqrt{n}}\right)$$





Test of significance for large samples:

(i) find mean proportion of successes

(ii) find S.D.

(v). probable limits: $p \pm 2.58\sqrt{pq/n}$

Consider $z = \frac{x-\mu}{\sigma} = \frac{x-np}{\sqrt{npq}}$

(iii) If $|z| > 2.58$, then reject the hypo.

(iv) critical values of z: -1.96 to 1.96 (5% level), -2.58 to 2.58 (1% level)

Ex: A coin is tossed 1000 times and turnup head 540 times. Decide on the hypo. that the coin is unbiased.

Sol: Let us suppose that the coin is unbiased, p = prob. getting a head in one toss $= \frac{1}{2}$.

Since $p+q=1$, $q = \frac{1}{2}$.

Expected no. of heads in 1000 tosses $= 1000 \times \frac{1}{2} = 500$

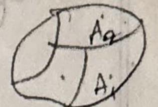
Actual number of heads $= 540$

The diff: $\underbrace{x}_{\text{actual}} - \underbrace{np}_{\text{expected}} = 540 - 500 = 40$

$$z = \frac{x-np}{\sqrt{npq}} = \frac{40}{\sqrt{1000 \times \frac{1}{2} \times \frac{1}{2}}} = 2.53 < 2.58$$

Therefore we can say that the coin is unbiased.

$E \rightarrow$ repeated n times



Definition: (Multinomial distribution)

Let E be an experiment is to be repeated independently n times. Let $A_1, A_2, \ldots, A_k$ be k mutually exclusive exhaustive events.

$P(A_i) = p_i$ which remains the same for all repetitions.

Let X_i = no. of repetitions favourable for A_i, $i = 1, 2, \ldots, n-1$.

$$X_k = n - X_1 - X_2 - \cdots - X_{k-1}$$

Let $x_1, x_2, \ldots, x_{k-1}$ are all positive integers with

$$x_1 + x_2 + \cdots + x_{k-1} \leq n \quad \text{and let}$$

$$x_k = n - (x_1 + x_2 + \cdots + x_{k-1})$$

Therefore $\Pr\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\}$

$$= \frac{n!}{x_1!\, x_2! \cdots x_k!} \, p_1^{x_1} \cdots p_k^{x_k}.$$

The n-dim r.v. $(X_1, \ldots, X_k)$ is said to have a multinomial distribution with parameters $n, p_1, p_2, \ldots, p_k$.

~~Chi-qu~~

Chi-Square Test:

Let $(X_1, X_2, \ldots, X_k)$ have a multinomial distribution with parameters $n, p_1, p_2, \ldots, p_k$.

To test a simple hypo.

$H_0: p_1 = p_{10}, \; p_2 = p_{20}, \ldots, p_k = p_{k0}$ against all alternatives

Consider

$$\chi^2_{(k-1)} = Q_{k-1} = \sum_{i=1}^{k} \frac{(X_i - n p_i)^2}{n p_i}$$

If H_0 is true then Q^*_{k-1} has limiting distribution $\chi^2(k-1)$.

Let α be the significance level of the test

Let C be a constant such that

$$Pr\{\chi^2_{(n-1)} \geq c\} = \alpha$$

If the value of $Q_{n-1} < c$, Accept the hypo.

(Calculated) < (tabulated)

If $Q_{n-1} \geq c$: reject the hypo.

$$\boxed{Q_{n-1} = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}}$$

O_i = observed values
E_i = Expected values
$i = 1, 2, \ldots, n$

Example: The following figures show the distribution of digits in numbers at random from a telephone directory. Test whether the digits occur equally frequently at 5% significance level.

Digit	0	1	2	3	4	5	6	7	8	9
Frequency (O_i)	1026	1107	977	966	1075	913	1107	972	964	853

E_i: 1000 1000 . . . .

Sol: Suppose that all the digits are equally probable.

$n = 10,000$

$E = np = 10,000 \times \frac{1}{10} = 1000$

χ^2 Variate has $(10-1) = 9$ degrees of freedom

$$\chi^2_9 = \sum_{i=0}^{9} \frac{(O_i - E_i)^2}{E_i} = \frac{1}{1000}\left[(26)^2 + (107)^2 + \cdots + (147)^2\right] = 61.622$$

If c is the Critical value, then

$$P(\chi^2 > c) = 0.05$$

ie, $pr(\chi^2 < c) = 0.95$

$\Rightarrow c = 16.9$

Now $\underset{(\text{cal})}{61.622} > \underset{(\text{tab})}{16.9}$

So we reject the hypo. that the digits are equally probable.

problem: A survey of 320 families with 5 childeren each revealed the following distribution. Is the result consistent with the hypo. that male and female births are equally probable at 5% significance level.

No. of boys:	5	4	3	2	1	0	
No. of girls:	0	1	2	3	4	5	
No. of families:	14	56	110	88	40	12	$(O_i's)$

Solution: Suppose that the male and female births are equally probable (Say H_0:)

Ie; $p = \frac{1}{2}$, $q = \frac{1}{2}$.

let X = no. of boys in the family. (Success)

$P(5 \text{ boys}, 0 \text{ girls}) = 5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = \frac{1}{32}$

$P(4b, 1g) = 5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} = \frac{5}{32}$

$P(3b, 2g) = \frac{10}{32}$, $P(2b, 3g) = \frac{10}{32}$, $P(1b, 4g) = \frac{5}{32}$,

$P(0b, 5g) = \frac{1}{32}$.

In 320 families, expected number of families with

(i) no boys $E_1 = np = 320 \times \frac{1}{32} = 10$

1 boy $E_2 = \ldots 320 \times \frac{5}{32} = 50$

2 boys $= - - -$

3 boys $= - -$

$\vdots$

$$Q_{6-1} = Q_5 = \sum_{E_0}^{5} \frac{(O_i - E_i)^2}{E_i} = \underline{7.16}$$

At $\alpha = 0.05$ significance level:

$$P(\chi_5^2 \geq c) = 0.05$$

$$\Rightarrow \Pr\{\chi_5^2 < c\} = 0.95$$

$$\Rightarrow c = \underline{11.07}$$

Therefore the calculated value < tabulated value.

Therefore accept H_0.

ie: we accept the hypo. that male and female births are equally possible.

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problem: A die was thrown $n = 120$ independent times and the following data resulted.

Spots up	1	2	3	4	5	6
frequency	b	20	20	20	20	40-b

If we use chi-square test for what values of b was the die is unbiased be rejected at 0.025 significance.

Sol: H_0: the die is unbiased.

$n = 120, \quad p = \frac{1}{6}$ (equal probability)

$E_i = 120 \times \frac{1}{6} = 20, \quad i = 1, 2, \cdots 6$

$$\chi^2_5 = \sum_{i=1}^{6} \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{(b-20)^2}{20} + \frac{(40-b-20)^2}{20}$$

$$= \frac{b^2 - 40b + 400}{10}.$$

At $\alpha = 0.025$ significance level

$P\{\chi^2_5 < c\} = 0.975$

$\Rightarrow c = 12.8.$

To reject H_0, we must have

Calculated value $\geq$ observed value.

ie, $\frac{b^2 + 400 - 40b}{10} \geq 12.8$

ie, $b^2 - 40b + 272 \geq 0$

$$\Rightarrow b = \frac{+40 \pm \sqrt{1600 - 1088}}{2}.$$

$$= \frac{40 \pm 22.62}{2} \qquad \cancel{31.31, \; 8.69}.$$

$\Rightarrow 8.68 \leq b \leq 31.31.$

Therefore we reject the hypothesis that the die is unbiased if b lies between 8.68 and 31.31

Problem: The following table gives number of aircraft accidents that occur during the various days of a week. Test whether the accidents are uniformly distributed over a week at 10% significance level.

Days:	Mon	Tue	Wed	Thur	Fri	Sat	Total
No. of accidents:	14	18	12	11	15	14	84

Sol: Let the hypothesis be accidents are uniformly distributed over the week.

Expected frequency $= np = 84 \times \frac{1}{6} = 14$.

$$\chi^2_5 = 2.143$$

Assuming 0.01 significan level

$$P(\chi^2_5 \geq c) = \cancel{0.01}\ 0.1\ 2$$

ie, $P(\chi^2_5 < c) = 0.99$

$\Rightarrow c = \cancel{15.1}\ 9.24$

Now $2.143 < \cancel{18.1}\ 9.24$, we accept the hypo. that accidents are uniformly distributed.