Registration No.: ....

First order Calculus

Axiomatic System F C

The inherent circularity present in PL extends to FL also. For proving an argument, we translate it to ~~FL also~~. in FL. Next, for justifying a consequence in FL. we consider all possible interpretations, and require that in all these interpretations, the consequence must be true. Notice that one of the interpretations is the argument we begin with!

The Axiom Schemes of FC:

For formulas X, Y, Z, variable x, and terms s and t:

A1> $X \rightarrow (Y \rightarrow X)$

A2> $(X \rightarrow (Y \rightarrow Z)) \rightarrow ((X \rightarrow Y) \rightarrow (X \rightarrow Z))$

A3> $(\neg X \rightarrow \neg Y) \rightarrow ((\neg X \rightarrow Y) \rightarrow X)$

A4> $\forall x Y \rightarrow Y[x/t]$, provided t is free for x in Y.

A5> $\forall x (Y \rightarrow Z) \rightarrow (Y \rightarrow \forall x Z)$, provided x does not occur free in Y.

A6> $(t \approx t)$

A7> $(s \approx t) \rightarrow (X[x/s] \rightarrow X[x/t])$, provided s, t are free for x in X.

The Rules of Inference of FC:

(MP) $\dfrac{X \quad X \to Y}{Y}$

(UG) $\dfrac{X}{\forall x\, X}$ provided x is not free in any premise used thus far.

The rule ~~of~~ UG is the rule of universal generalization. The phrase "premise used thus far" means that in order to apply ~~S~~ UG on a formula, say $X(x)$ in Line n, we must make sure that the variable x is not free in any premise used in the proof up to and including Line n.

To compensate for the missing connectives and quantifiers, we include the definitions (D1) → (D6) and the rule (RD) as in the following

(D1) $X \wedge Y \overset{\circ}{=} \neg(X \to \neg Y)$

(D2) $X \vee Y \overset{\circ}{=} \neg X \to Y$

(D3) $X \leftrightarrow Y \overset{\circ}{=} \neg((X \to Y) \to \neg(Y \to X))$

(D4) $\top \overset{\circ}{=} X \to X$

(D5) $\bot \overset{\circ}{=} \neg(X \to X)$

(D6) $\exists x\, X \overset{\circ}{=} \neg \forall x \neg X$.

(RD) $\dfrac{X \overset{\circ}{=} Y \quad Z}{Z[X := Y]}$ $\qquad \dfrac{X \overset{\circ}{=} Y \quad Z}{Z[Y := X]}$

As in PC, a proof is a finite sequence of formulas, each of which is either an axiom (an instance of an axiom scheme), or is obtained (derived) by an application of some inference rule on earlier formulas.

The last formula of a proof is a <u>theorem</u>. The fact that a formula X is a theorem of FC is written as $\vdash_{FC} X$; in that case, we also say that the formula X is <u>provable</u>.

For a set of formulas Σ and a formula Y, a <u>proof of the consequence</u> $\Sigma ! \vdash Y$ is again a finite sequence of formulas, each of which is an axiom, or a premise (a formula) in Σ, or is derived from earlier formulas by an application of an inference rule; the last formula of the sequence is Y. The fact that there is a proof of the consequence $\Sigma ! \vdash Y$ is written simply as $\Sigma \vdash_{FC} Y$; $\Sigma \vdash Y$. In this case, we also say that the consequence $\Sigma ! \vdash Y$ is <u>provable</u>; or FC- provable.

We also write $\{X_1, \ldots, X_n\} \vdash Y$ as $X_1, \ldots, X_n \vdash Y$. Notice that $\phi \vdash Y$ expresses the same fact as $\vdash Y$.

<u>Example 1</u>: If x does not occur in Z, then show that $\forall y Z \vdash \forall x Z[y/x]$.

$\forall y Z$	P
$\forall y Z \to Z[y/x]$	A4, since x is free for y in Z.
$Z[y/x]$	MP
$\forall x Z[y/x]$	UG, since x is not free in $\forall y Z$.

<u>Example 2</u>: The following is a proof of $\forall x \forall y Z \vdash \forall y \forall x Z$

$\forall x \forall y Z$	P
$\forall x \forall y Z \to \forall y Z$	A4 x is free for x in Z.

$\forall y Z$ MP

$\forall y Z \to Z$ A4, y is free for y in Z

Z MP

$\forall x Z$ UG

$\forall y \forall x Z$ UG.

Example 3: In proving $\forall x Y \to \forall x Y$, you can imitate the proof of $p \to p$ in PC. However, the same can be proved using the quantifier axioms.

$\forall x Y \to Y$ A4

$\forall x (\forall x Y \to Y)$ UG

$\forall x (\forall x Y \to Y) \to (\forall x Y \to \forall x Y)$ A5

$\forall x (\cancel{\forall x Y} \to Y)$ MP.

$\forall x Y \to \forall x Y$

Since $\forall x Y$ has no free occurence of the variable x, the third line is indeed A5.

Example 4: The following is a proof of $\forall x \neg Y \to \neg \forall x Y$

$\forall x \neg Y$ P

$\forall x \neg Y \to \neg Y$ A4

1. $\neg Y$ MP

$\forall x Y \to Y$ A4

$(\forall x Y \to Y) \to (\neg Y \to \neg \forall x Y)$ Th

$\neg Y \to \neg \forall x Y$ MP

$\neg \forall x Y$ 1, MP.

(In this example, we use the PC-theorem $\vdash (p \to q) \to (\neg q \to \neg p)$ after a suitable uniform replacement of the propositional variables by formulas of FC. In fact, we do not formulate any such principle of replacement; it is implicitly assumed due to the

~~natura~~

nature of axiom schemes, rule schemes, theorem schemes, and the fact that PC is a subsystem of FC. It means that a proof of $\vdash (p \to q) \to (\neg q \to \neg p)$ can simply be duplicated with p, q substituted with suitable formulas from FC.

Example: The following proof shows that $(s \approx t) \vdash (t \approx s)$.

$(s \approx t)$	P
$(s \approx t) \to ((s \approx s) \to (t \approx s))$	A7, $X = (x \approx s)$
$(s \approx s) \to (t \approx s)$	MP
$(s \approx s)$	A6
$(t \approx s)$	MP

Exercise: Try proving $\vdash (s \approx t) \to (t \approx s)$ in F.C.

Exercise: Show that $\neg Y \to Y \vdash Y$

Some Theorems About FC

A set of formulas Σ is said to be inconsistent (in FC) iff there exists a formula Y such that $\Sigma \vdash Y$ and $\Sigma \vdash \neg Y$, else Σ is said to be consistent (in FC). We also say that a formula X is inconsistent or consistent according as the set $\{X\}$ is inconsistent or consistent.

Theorem (M: monotonicity) Let $\Sigma \subseteq \Gamma$ be sets of formulas, and let X be any formula. Then

(1) If $\Sigma \vdash X$, then $\Gamma \vdash X$

(2) If Σ is inconsistent, then so is Γ.

Theorem (DT: Deduction Theorem): Let Σ be a set of formulus. Let X and Y be any formulus. Then

$\Sigma \vdash X \to Y$ iff $\Sigma \cup \{X\} \vdash Y$.

Theorem (RA: Reductio ad Absurdum) Let Σ be a set of formulus, and let Y be any formula

1) $\Sigma \vdash Y$ iff $\Sigma \cup \{\neg Y\}$ is inconsistent.

2) $\Sigma \vdash \neg Y$ iff $\Sigma \cup \{Y\}$ is inconsistent.

Theorem (Finiteness) Let Σ be a set of formulus, and let X be any formula.

1) If $\Sigma \vdash X$, then there exists a finite subset Γ of Σ such that $\Gamma \vdash X$.

2) If Σ is inconsistent, then there exists a finite inconsistent subset of Σ.

Theorem (Paradox of material Implication) Let Σ be an inconsistent set of formulus, and let X be any formula. Then $\Sigma \vdash X$.

Example: Show that $\vdash \forall x(X \to Y) \to (\forall x \neg Y \to \forall x \neg X)$

Due to DT, we only show that $\{\forall x(X \to Y), \forall x \neg Y\} \vdash \forall x \neg X$.

$\forall x(X \to Y)$ P

$\forall x(X \to Y) \to (X \to Y)$ A4

$X \to Y$ MP

$(X \to Y) \to (\neg Y \to \neg X)$ Th

MP

1. $\neg Y \to \neg X$ P

$\forall x \neg Y$ A4

$\forall x \neg Y \to \neg Y$ MP

$\neg Y$ 1, MP

$\neg X$ UG.

$\forall x \neg X$

Example: Show that $\forall x (X \to Y) \to (\neg \forall x \neg X \to \neg \forall x \neg Y)$

Due to DT, we only show that $\{\forall x (X \to Y), \neg \forall x \neg X\} \vdash \neg \forall x \neg Y$.

Due to RA, it is enough to show $\{\forall x (X \to Y), \neg \forall x \neg X, \forall x \neg Y\}$ is inconsistent.

Again due to RA, $\{\forall x (X \to Y), \forall x \neg Y\} \vdash \forall x \neg X$

The proof of the last consequence is in the preceding example.

Example: If x is not free in Y, then $\vdash \neg(\forall x X \to Y) \to \forall x \neg (X \to Y)$.

~~Using DT~~

$\vdash \neg(\forall x X \to Y) \to \forall x \neg (X \to Y)$

iff (DT) $\{\neg(\forall x X \to Y), \neg \forall x \neg (X \to Y)\}$ is inconsistent

iff (RA) $\{\neg (\forall x X \to Y)\}$ ...

iff (RA) $\{\neg \forall x \neg (X \to Y)\} \vdash \forall x X \to Y$

iff (DT) $\neg \forall x \neg (X \to Y), \forall x X \vdash Y$

iff (RA) $\{\neg \forall x \neg (X \to Y), \forall x X, \neg Y\}$ is inconsistent

i.e. $\forall x X, \neg Y \vdash \forall x \neg (X \to Y)$.
(RA)

The last consequence has the following proof:

$$
\begin{array}{ll}
\forall x X & P \\
\forall x X \to X & A4 \\
 & MP \\
X & \\
X \to (\neg Y \to \neg (X \to Y)) & Th \\
\neg Y \to \neg (X \to Y) & MP \\
 & P \\
\neg Y & \\
 & MP \\
\neg (X \to Y) & \\
 & UG. \\
\forall x \neg (X \to Y) &
\end{array}
$$

__Example__ Show that $\vdash \forall x ((x \approx f(y)) \to Qx) \to Qf(y)$.

Due to DT, we give a proof of $\forall x ((x \approx f(y)) \to Qx) \vdash Qf(y)$

$$
\begin{array}{ll}
\forall x ((x \approx f(y)) \to Qx) & P. \\
\forall x ((x \approx f(y)) \to Qx) \to ((f(y) \approx f(y)) \to Qf(y)) & A4. \\
 & MP \\
(f(y) \approx f(y)) \to Qf(y) & \\
 & A6 \\
f(y) \approx f(y) & \\
 & MP. \\
Qf(y) &
\end{array}
$$

__H.W:__ 1. Show that $\{Pa, \forall x (Px \to Qx), \forall x (Rx \to \neg Qx), Rb\} \vdash \neg (a \approx b)$

2. Show that $\forall x \forall y (f(x,y) \approx f(y,x)), \forall x \forall y (f(x,y) \approx y) \vdash \forall x \neg \forall y (x \approx y)$

Theorem (Strong Generalization). Let Σ be a set of formulus, X a formula, x a variable, and let c be a constant. suppose, in any formula of $\Sigma \cup \{X\}$, x does not occur free, and c does not occur at all. If $\Sigma \vdash X[y/c]$, then $\Sigma \vdash \forall x\, X[y/x]$. moreover, there exists a proof of $\Sigma \vdash \forall x\, X[y/x]$ in which c does not occur at all.

ADEQUACY OF FC TO FL

Recall that a proof system is sound with respect to a logic if each provable formula in the system is also valid in the logic. The proof system is called complete ~~iff~~ with respect to the logic if each valid formula in the logic is provable in the proof system. And, the adjective strong is used for similar ~~+~~ notions with regards to conseauences.

Theorem (Strong Soundness of FC). Let Σ be a set of formulas, and let A be a formula.

1> If $\Sigma \vdash A$, then $\Sigma \vDash A$.

2> If Σ is satisfiable, then Σ is consistent.

Proof: We check that the axioms are valid; the rules of inference are valid consequences; and then apply induction on the lengths of proofs.

Let P be a proof of $\Sigma \vdash A$. In the proof P, all the premises in Σ might not have been used. Let Σ_P be the set of premises that have been actually used in P. Then Σ_P is a finite subset of Σ and $\Sigma_P \vdash A$.

We use induction on n, the number of (occurences of) formulas in P ~~proves~~ and then use monotonicity in FL.

In the basis step, n = 1; A is either an axiom or a premise in Σ_P. ~~Then by monotonicity~~ Clearly, $\Sigma_P \vDash A$.

~~$\Sigma \vDash A$~~

Lay out the induction hypothesis that if $\Sigma_p \vdash X$ having m formulas, then $\Sigma_p \models X$ for any formula X. Let $P1$ be a proof of $\Sigma_p \vdash A$ having m formulas. If A is again an axiom or a premise in Σ_p, then clearly $\Sigma_p \models A$ holds. Otherwise, A has been obtained in $P1$ by an application of (a) mp or (b) UG.

(a) There are formulas B and $B \to A$ occurring earlier to A in $P1$. By the induction hypothesis, $\Sigma_p \models B$ and $\Sigma_p \models B \to A$. Since $\{B, B \to A\} \models A$, we have $\Sigma_p \models A$.

(b) There is a formula C occurring ~~prior~~ prior to A in $P1$ such that $A = \forall x C$, for some variable x. Further, let Σ_C be the subset of Σ_p containing exactly those formulas in $P1$ in which have been used deriving C. Then the variable x does not occur free in any formula of Σ_C due to the restriction on applicability of UG. By induction hypothesis, $\Sigma_C \models C$. Then, $\Sigma_C \models \forall x C$ (why?), i.e., $\Sigma_C \models A$. Since $\Sigma_C \subseteq \Sigma$, by monotonicity, $\Sigma \models A$.

... left as an exercise.

Semester: .... Branch: .... Section: ....

Subject: .... Roll No.: ....

Compactness of FL

Thm (compactness of FL): Let Σ be any nonempty set of formulas, and let X be any formula.

1) If $\Sigma \vDash X$, then Σ has a finite subset Γ such that $\Gamma \vDash X$

2) If Σ is unsatisfiable, then Σ has a finite subset Γ which is unsatisfiable.

3) If all finite nonempty subsets of Σ are satisfiable, then Σ is satisfiable.

We say that a formula X has arbitrarily large models iff for each $n \in \mathbb{N}$, there exists a model of X having at least n elements in its domain.

A typical application of compactness is to extend a property from arbitrary large to infinite.

Theorem (Skolem): If a set of first order sentences has arbitrarily large models, then it has an infinite model.

Proof: Let Σ be a set of sentences. Assume that Σ has arbitrarily large models. For each positive number n, let $\psi_n = \exists x_0 \cdots \exists x_n \left(\bigwedge_{0 \leq i < j \leq n} \neg (x_i \approx x_j) \right)$

(for instance $\psi_2 = \exists x_0 \exists x_1 \exists x_2 (\neg(x_0 \approx x_1) \wedge \neg (x_0 \approx x_2) \wedge \neg (x_1 \approx x_2))$. Next, let $S = \{\psi_n : n \text{ is a positive integer}\}$

If A is any finite subset of $\Sigma \cup S$, then it contains a finite number of ψ_n's.

If m is the maximum of all those n, then A has a model whose domain has at least $m+1$ elements (why?). The fact that A has a model is enough for us. By the compactness theorem, $\Sigma \cup S$ has a model, say, $I=(D,\phi)$. If D is a finite set, say, with k elements, then it does not satisfy Y_{k+1}. As $Y_{k+1} \in S$, it contradicts the fact that I is a model of $\Sigma \cup S$. Therefore, I must be an infinite model of $\Sigma \cup S$.

Recall that a first order language starts with a set of symbols that include constants, function symbols, and predicates, along with all connectives and quantifiers. A first order language is like a consequence, where we require only some function symbols and some predicates out of the infinite supply of them.

Example: Consider a first order languge L which has two constants a and b, two function symbols f and g, and a binary predicate P. Interpret the language L in the system of natural numbers $\mathbb{N}$ with a as 0, b as 1, f as addition, g as muliplication, and p as the relation of 'less than'. Let c be a new constant. Let Σ be the set of all true sentences of $\mathbb{N}$. Define a set of formula Γ by
$\Gamma = \{P(n,c) : n \in \mathbb{N}\} = \{P(0,c), P(1,c), \ldots\}$
If $S \subseteq \Sigma \cup \Gamma$ is a finite set, then it contains a finite number of sentences from Γ. Clearly, the same interpretation with domain $\mathbb{N}$ is a model of S.

That is, every finite subset of $\Sigma \cup \Gamma$ is satisfiable. By the compactness theorem, the set $\Sigma \cup \Gamma$ is also satisfiable. In such a model all true sentences of $\mathbb{N}$ are true, and also there exists an element which is bigger than all natural numbers. We may think of such an element in this model as an infinity.

Example: Consider a first order language with constants a, b, binary function symbols f, g; and a binary predicate P. Interpret the language in the system of real number $\mathbb{R}$ with a as 0, b as 1, f as addition, g as multiplication, and P as the relation 'less than'. Let c be a new constant. Let Σ be the set of all true sentences of $\mathbb{R}$. Define

$$\Gamma = \{ P(0, c) \wedge P(g(n, c), 1) : n \in \mathbb{N} \}.$$

Now each finite subset of $\Sigma \cup \Gamma$ has the same ~~model~~ interpretation with domain as $\mathbb{R}$ is a model of it. By the compactness theorem, the set $\Sigma \cup \Gamma$ is satisfiable. In this new model of axioms of $\mathbb{R}$ there exists an element (that corresponds to c) which is smaller than all positive real numbers and is also greater than 0. We do not say that such a number is positive, for, 'positive' is applicable only to real numbers, and this number is not a real number. Such an element in this nonstandard model of the axioms of $\mathbb{R}$ is called an infinitesimal.

This trick of extending a model gives rise to the Skolem - Löwenheim upward theorem which states that if a set of sentences having cardinality α has a model with domain of an infinite cardinality β, and $\gamma \geq \max\{\alpha, \beta\}$, then the set of sentences also has a model with domain of cardinality γ.

proof: The proof of this remarkable theorem requires first order languages allowing the set of constants to have cardinality γ.

Suppose Σ is a set of cardinality α which has a model with a domain of cardinality β. We start with a set S of cardinality γ, and introduce γ number of constants c_a to our language for each $a \in S$. Next, we define the set of sentences

$$\Gamma = \Sigma \cup \{\neg(c_a \leftrightarrow c_b) : a, b \in S, a \neq b\}$$

Then using compactness we show that Γ has a model which has a domain with cardinality γ.

H.W 1: Write a formula X involving a binary predicate P so that if $I = (D, \phi)$ is any interpretation, then $I \models X$ iff D is an infinite set.

2. Show that a first order sentence X cannot be found for any interpretation $I = (D, \phi)$, we have $I \models X$ iff D is a finite set.

3. Find a sentence which is true in a denumerable domain, but false in each finite domain.