Central Limit Theorem:

Let $X_1, X_2, \ldots, X_n$ denote the items of a random ~~to~~ sample from a distribution that has mean μ and variance σ^2. Then the random variable

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$

$$Y_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}.$$

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has limiting standard normal distribution. $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$

proof: Consider $S_n = X_1 + X_2 + \cdots + X_n$, $n = 1, 2, \ldots$

Since $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, we have

$$E(S_n) = E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i) = n\mu.$$

(Since $X_1, X_2, \ldots, X_n$ are independent)

$$V(S_n) = V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i) = n\sigma^2.$$

Thus we have the standardised r.v.

$$Y_n = \frac{\sum X_i - n\mu}{\sigma\sqrt{n}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \quad n = 1, 2, \ldots$$

Corresponding to S_n.

Consider the mgf of Y_n.

$$M_{Y_n}(t) = E\left(e^{tY_n}\right) = \cancel{E\left(e^{t\left(\frac{S_n - n\mu}{\sigma\sqrt{n}}\right)}\right)}$$

$$= E\left(e^{t\left(\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}\right)}\right)$$

$$= E\left[e^{t\left(\frac{X_1-\mu}{\sigma\sqrt{n}}\right)} e^{t\left(\frac{X_2-\mu}{\sigma\sqrt{n}}\right)} \cdots e^{t\left(\frac{X_n-\mu}{\sigma\sqrt{n}}\right)}\right]$$

Since the random variables are independent,

$$M_{Y_n}(t) = \prod_{i=1}^{n} E\left(e^{\frac{t(X_i-\mu)}{\sigma\sqrt{n}}}\right).$$

Since the random variable are identically distributed with $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, we have

$$M_{Y_n}(t) = \left\{E\left[e^{\frac{t(X_i-\mu)}{\sigma\sqrt{n}}}\right]\right\}^n.$$

$\left(n^{-1/2}\right)^3 = o(n^{-3/2})$

$\left(\frac{1}{\sqrt{n}}\right)^3$

$$= \left\{E\left\{1 + \frac{t(X_i-\mu)}{\sigma\sqrt{n}} + \frac{t^2(X_i-\mu)^2}{2\sigma^2 n} + o\left(n^{-3/2}\right)\right\}\right]^n.$$

$$= \left\{1 + \frac{t}{\sigma\sqrt{n}} E(X_i-\mu) + \frac{t^2}{2\sigma^2 n} E(X_i-\mu)^2 + o\left(n^{-3/2}\right)\right\}^n.$$

$$= \left\{1 + \frac{t}{\sigma\sqrt{n}} \cdot 0 + \frac{t^2}{2\sigma^2 n} \cdot \sigma^2 + o\left(n^{-3/2}\right)\right]^n.$$

(Since $E(X_i) = \mu$ and $E(X_i-\mu)^2 = \sigma^2$)

$$= \left[1 + \frac{t^2}{2n} + o\left(n^{-3/2}\right)\right]^n.$$

For every fixed 't', the terms of $o(n^{-3/2}) \to 0$ as $n \to \infty$

Therefore as $n \to \infty$, we get.

$$\lim_{n\to\infty} M_{Y_n}(t) = \lim_{n\to\infty}\left\{1 + \frac{t^2}{2n} + \underbrace{o\left(n^{-3/2}\right)}_{=0}\right]^n.$$

$$= e^{t^2/2}$$

Since $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$

This ~~the~~ is the mgf of a random variable with distribution $N(0,1)$. Thus by uniqueness theorem of mgf, we get $\quad Y_n = \dfrac{\sum_i x_i - n\mu}{\sigma\sqrt{n}}$ has limiting

standarised normal distribution.

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Problems on Central Limit Theorem:

① Suppose that X_j, $j = 1, 2, ---, 50$ are independent random variables each having a poisson Distribution $\lambda = 0.03$. Let $S = X_1 + X_2 + \cdots + X_{50}$. Using the central limit theorem, evaluate $pr(S \geq 3)$.

Sol: we have $E(S) = E(X_1) + \cdots + E(X_{50})$

$$= 50(0.03) = 1.5 = \mu$$

$$V(S) = V(X_1) + \cdots + V(X_{50}) = 1.5 = \sigma^2.$$

Let $$y = \frac{S - E(S)}{\sqrt{V(S)}} = \frac{S - 1.5}{\sqrt{1.5}}$$

Then by central limit theorem, $y \sim N(0,1)$

Therefore $$pr(S \geq 3) = pr\left\{ Y \geq \frac{3 - 1.5}{1.225} \right\}.$$

$$pr\left\{ \frac{S - E(S)}{\sqrt{V(S)}} \right\} = pr\{ Y \geq 1.225 \}$$

$$= 1 - pr\{ Y < 1.225 \}$$

$$= 1 - \Phi(1.225) \; \cancel{0.111}$$

$$= 1 - 0.8888 = 0.1112$$

② Compute an approximate probability that the mean of a random sample of size 15 from a distribution having pdf $f(x) = \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases}$

is between $\frac{3}{5}$ and $\frac{4}{5}$.

Sol.

Let $\frac{\bar{X} - E(\bar{X})}{V(\bar{X})} \sim N(0,1)$

~~Now~~ To find $pr\left\{\frac{3}{5} < \bar{X} < \frac{4}{5}\right\}$

$E(\bar{x}) = \mu$

$V(\bar{x}) = \frac{\sigma^2}{n}$, where $\bar{x}$ is the mean of the random sample.

Then by central limit theorem

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

But $\mu = \int_0^1 x\, 3x^2\, dx = \frac{3}{4}$ $(E(X))$

$E(x^2) = 3/5$

Therefore $V(x) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0.0375.$ $\left(E(x^2) - (E(x))^2\right)$

Therefore $\sigma = \sqrt{0.0375} = 0.1936.$

So $pr\left\{\frac{3}{5} < \bar{x} < \frac{4}{5}\right\} = pr\left\{\frac{\frac{3}{5} - \mu}{\sigma/\sqrt{n}} < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < \frac{\frac{4}{5} - \mu}{\sigma/\sqrt{n}}\right\}.$

$= pr\left\{\frac{\frac{3}{5} - \mu}{\sigma/\sqrt{n}} < z < \frac{\frac{4}{5} - \mu}{\sigma/\sqrt{n}}\right\}$, $n = 15.$

$= pr\{-3.06 < z < 1.02\}$

$= \Phi(1.0) - \Phi(-3.6)$

$= \Phi(1.02) + \Phi(3.6) - 1$

$= 0.8461 + 0.9998 - 1 = 0.8459$

(3) Let $\bar{x}$ denote the mean of a random sample of size 100 from a distribution which is $\chi^2(50)$. Compute the approximate value of $pr\{49 < \bar{x} < 51\}$.

<u>Sol:</u> If $X \sim \chi^2(k)$, then $\mu = E(X) = k = 50$

$V(X) = 2k = 100$,

where k is the degrees of freedom.

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_{100}}{100}$$

$$E(\bar{x}) = \frac{1}{100}\left[E(x_1) + \dots + E(x_{100})\right]$$

$$= \frac{1}{100} \cdot 100 \cdot 50 = \underline{\underline{50}}$$

$$V(\bar{x}) = \frac{1}{100}(100 \times 100) = 100 = \sigma^2.$$

Therefore $\mu = 50$, $\sigma = 10$

By central limit theorem,

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{x} - 50}{10/\sqrt{100}} = \frac{\bar{x} - 50}{1} \sim N(0,1)$$

Therefore $pr\{49 < \bar{x} < 51\}$

$$= pr\{49 - 50 < z < 51 - 50\}$$

$$= pr\{-1 < z < 1\} = \Phi(1) - \Phi(-1)$$

$$= 2\Phi(1) - 1$$

$$= \underline{0.6826}.$$

(4) Let $\bar{x}$ be the mean of a random sample of size 5 from a normal distribution with $\mu = 0$ and $\sigma^2 = 144$. Determine c so that $pr\{\bar{x} < c\} = 0.9$

Sol: By central limit theorem,

$$E(\bar{X}) = \mu = 0 \qquad V(\bar{X}) = \frac{\sigma^2}{n} = \frac{144}{5}$$

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-0}{12/\sqrt{5}} = \frac{\bar{X}}{12/\sqrt{5}} \sim N(0,1)$$

Therefore $pr\{\bar{X} < c\} = pr\left\{Z < \frac{\sqrt{5}\,c}{12}\right\} = 0.9$ (given)

$$\Rightarrow \Phi\left(\frac{\sqrt{5}\,c}{12}\right) = 0.9 = \Phi(1.29)$$

$$\Rightarrow \frac{c\sqrt{5}}{12} = 1.29$$

$$\Rightarrow c = \frac{12 \times 1.29}{\sqrt{5}} = 6.922.$$

(5) A random sample of size 64 is taken from an infinite population having $\mu = 112$ and $\sigma^2 = 144$. Find the probability of getting $\bar{X} > 114.5$

Sol: By central limit theorem,

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-112}{12/\sqrt{64}} = \frac{8(\bar{x}-112)}{12} \sim N(0,1)$$

Therefore $pr\{\bar{x} > 114.5\} = pr\left\{z > \frac{8(114.5-112)}{12}\right\}$

$$= pr\{z > 1.67\}$$

$$= 1 - pr\{z \leq 1.67\} = 1 - \Phi(1.67)$$

$$= 1 - 0.9525 = \underline{0.0475}$$

(6) A random sample of size 100 is taken from an infinite population with $\mu = 53$ and $\sigma^2 = 400$. Find $pr\{50 < \bar{x} < 56\}$.

Sol: By central limit theorem,

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{X}-53}{20/10} = \frac{\bar{X}-53}{2} \sim N(0,1)$$

Therefore $pr\{50 < \bar{X} < 56\} = pr\{\frac{50-53}{2} < Z < \frac{56-53}{2}\}$

$= pr\{\frac{-3}{2} < Z < \frac{3}{2}\}$

$= 2\,\Phi(1.5)-1 = 2\times 0.9332 - 1 = 0.8664.$

(7) Let $\bar{X}$ denote the mean of a random sample of size 128 from a gamma distribution with $\alpha = 2$ and $\beta = 4$. Approximate $pr\{7 < \bar{X} < 9\}$.

Sol: for gamma distribution,

$\mu = \alpha\beta = 2\times 4 = 8$, and $\sigma^2 = \alpha\beta^2 = 2\times 4^2 = 32.$

By central limit th:

$$Z = \frac{\bar{X}-8}{\sqrt{32}/\sqrt{128}} \sim N(0,1)$$

i.e., $Z = \frac{\bar{X}-8}{0.5} \sim N(0,1)$

Therefore $pr\{7 < \bar{X} < 9\} = pr\{\frac{7-8}{0.5} < Z < \frac{9-8}{0.5}\}$

$= pr\{-2 < Z < 2\} = 2\,\Phi(2) - 1 = 0.9544.$

(8) If $\bar{X}$ is the mean of a random sample of size n from a normal distribution with mean μ and variance $\sigma^2 = 100$, find n so that

$pr\{\mu - 5 < \bar{X} < \mu + 5\} = 0.954$

Sol. By Central limit theorem,

$$Z = \frac{\sqrt{n}\,(\bar{x} - \mu)}{\sigma} = \frac{\sqrt{n}\,(\bar{x} - \mu)}{10} \sim N(0,1)$$

Therefore $pr\{\mu - 5 < \bar{x} < \mu + 5\} = 0.954$

$$\Rightarrow pr\left\{\frac{\sqrt{n}\,(\mu-5-\mu)}{10} < Z < \frac{\sqrt{n}\,(\mu+5-\mu)}{10}\right\} = 0.954$$

$$\Rightarrow pr\left\{\frac{-5\sqrt{n}}{10} < Z < \frac{5\sqrt{n}}{10}\right\} = 0.954.$$

$$\Rightarrow 2\,\Phi\left(\frac{5\sqrt{n}}{10}\right) - 1 = 0.954$$

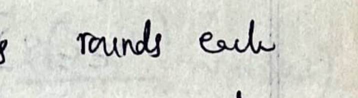
$$\Rightarrow \Phi\left(\frac{5\sqrt{n}}{10}\right) = \frac{1.954}{2} = 0.977.$$

$$\Rightarrow \frac{5\sqrt{n}}{10} = 2.0$$

$$\Rightarrow \sqrt{n} = 4 \quad \Rightarrow n = 16.$$

(9) A computer in adding numbers rounds each number off to the nearest integer. Suppose that all rounding errors are independent and uniformly distributed over $(-0.5, 0.5)$. If 1500 numbers are added what is the probability that the magnitude of the total error exceeds 15. How many numbers may be added together inorder that the magnitude of the total error is less than 10 with prob. 0.9?

Ans: Let X = rounding error.



Then $X \sim U(-0.5, 0.5)$

$$E(X) = \frac{0.5 - 0.5}{2} = 0.$$

$$V(X) = \frac{[0.5-(-0.5)]^2}{12} = \frac{1}{12}$$

Let $Y = X_1 + X_2 + \cdots + X_{1500}$

Total error $= Y$, $E(Y) = 0$.

$$V(Y) = 1500 \times \frac{1}{12} = 1500 \times \frac{1}{12} = 125$$

Then by central limit theorem,

$$Z = \frac{Y - E(Y)}{\sqrt{V(Y)}} \sim N(0,1)$$

Now to find $pr\{|Y| > 15\}$, Consider $pr\{|Y| \leq 15\}$

$pr\{-15 \leq Y \leq 15\}$.

$$= pr\left\{\frac{-15}{11.1803} \leq Z \leq \frac{15}{11.1803}\right\}.$$

$$= pr\{-1.34 \leq Z \leq 1.34\}$$

$$= 2\Phi(1.34) - 1$$

$$= 2(0.9099) - 1 = 0.8198$$

Therefore $pr\{|Y| > 15\} = 1 - 0.8198 = 0.1802$

(b) $pr\{|\bar{x}| < 10\} = 0.9$

$$\Rightarrow pr\{-10 < \bar{x} < 10\} = 0.9$$

$$\Rightarrow pr\left\{\frac{-10}{\sqrt{n}/\sqrt{12}} < Z < \frac{10}{\sqrt{n}/\sqrt{12}}\right\} = 0.9$$

$$\Rightarrow pr\left\{\frac{-34.64}{\sqrt{n}} < Z < \frac{34.64}{\sqrt{n}}\right\} = 0.9$$

$$\Rightarrow \Phi\left(\frac{34.64}{\sqrt{n}}\right) - \Phi\left(\frac{-34.64}{\sqrt{n}}\right) = 0.9$$

$$\Rightarrow 2\Phi\left(\frac{34.64}{\sqrt{n}}\right) - 1 = 0.9$$

$$\Rightarrow \Phi\left(\frac{34.64}{\sqrt{n}}\right) = \frac{1.9}{2} = 0.95,$$

~~Therefore $\sqrt{n}$ =~~

Therefore $\dfrac{34.94}{\sqrt{n}} = 1.65$

$$\Rightarrow \sqrt{n} = \frac{34.64}{1.65} = 20.99$$

So $\underline{n \sim 440}$