Chapter 3

Functions of Random Variables

In the analysis of electrical systems, we will be often interested in finding the properties of a signal after it has been subjected to certain processing operations by the system, such as integration, weighted averaging, etc. These signal processing operations may be viewed as transformations of a set of input variables to a set of output variables. If the input is a set of random variables (RVs), then the output will also be a set of RVs. In this chapter, we deal with techniques for obtaining the probability law (distribution) for the set of output RVs when the probability law for the set of input RVs and the nature of transformation are known.

Function of One Random Variable

Let X be an RV with the associated sample space S_x and a known probability distribution. Let g be a scalar function that maps each $x \in S_x$ into $y = g(x)$. The expression $Y = g(X)$ defines a new RV Y. For a given outcome, $X(s)$ is a number x and $g[X(s)]$ is another number specified by $g(x)$. This number is the value of the RV Y, i.e., $Y(s) = y = g(x)$. The sample space S_y of Y is the set

$$S_y = \{ y = g(x) : x \in S_x \}$$

How to find $f_Y(y)$, when $f_X(x)$ is known

Let us now derive a procedure to find $f_Y(y)$, the pdf of Y, when $Y = g(X)$, where X is a continuous RV with pdf $f_X(x)$ and $g(x)$ is a strictly monotic function of x.

Case (i): $g(x)$ is a strictly increasing function of x.

$$F_Y(y) = P(Y \le y), \text{ where } F_Y(y) \text{ is the cdf of } Y$$
$$= P[g(X) \le y]$$

$$= P[X \le g^{-1}(y)]$$
$$= F_X(g^{-1}(y))]$$

Differentiating both sides with respect to y,

$$f_Y(y) = f_X(x)\frac{dx}{dy}, \text{ where } x = g^{-1}(y) \quad (1)$$

Case (ii): $g(x)$ is a strictly decreasing function of x.

$$F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y]$$
$$= P[X \ge g^{-1}(y)]$$
$$= 1 - P[X \le g^{-1}(y)]$$
$$= 1 - F_X[g^{-1}(y)]$$

$$\therefore \quad f_Y(y) = -f_X(x)\frac{dx}{dy} \quad (2)$$

Combining (1) and (2), we get

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

i.e.,
$$f_Y(y) = \frac{f_X(x)}{|g'(x)|}$$

Note *The above formula for $f_Y(y)$ can be used only when $x = g^{-1}(y)$ is single valued.*

When $x = g^{-1}(y)$ takes finitely many values $x_1, x_2, \cdots x_n$, i.e., when the roots of the equation $y = g(x)$ are $x_1, x_2, \cdots, x_n$, the following extended formula should be used for finding $f_Y(y)$:

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|} \text{ or}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| + \ldots + f_X(x_n)\left|\frac{dx_n}{dy}\right|$$

One Function of Two Random Variables

If a RV Z is defined as $Z = g(X, Y)$ where X and Y are RVs, we proceed to find $f_Z(z)$ in the following way.

If z is a given number, we can find a region D_Z in the xy-plane such that all points in D_Z satisfy the condition $g(x, y) \le z$.

i.e.,
$$(Z \le z) = [g(X, Y) \le z] = [(X, Y) \in D_Z)]$$

Now
$$F_Z(z) = P(Z \le z) = P[(X, Y) \in D_Z] = \iint_{D_Z} f(x, y)\, dx\, dy$$

where $f(x, y)$ is the joint pdf of (X, Y). Thus, to find $F_Z(z)$ it is sufficient to find the region D_Z for every Z and to evaluate the above integral. $f_Z(z)$ is then found out as usual.

Theorem 1

If two RVs are independent, then the density function of their sum is given by the convolution of their density functions.

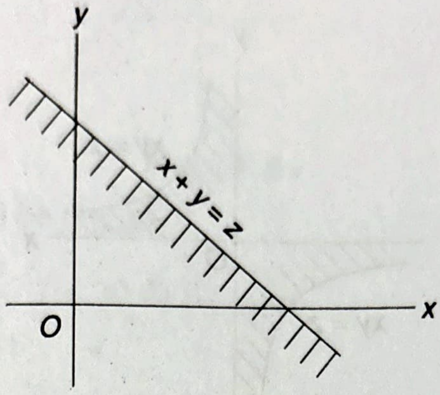

Fig. 3.1

Proof

Let the joint pdf of (X, Y) be $f(x, y)$

Let $$Z = X + Y$$

$$F_Z(z) = P(X + Y \leq z)$$

$$= \iint_{(x+y\leq z)} f(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f(x, y)\, dx\, dy$$

Differentiating both sides with respect to z (Note that the upper limit for the inner integral is a function of z),

$$f_Z(z) = \int_{-\infty}^{\infty} f(z - y, y)\, dy \tag{1}$$

Since X, Y are independent RVs

$$f(x, y) = f_X(x)\, f_Y(y)$$

$$\therefore \quad f(z - y, y) = f_X(z - y)\, f_Y(y) \tag{2}$$

Using (2) in (1), we get

$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y)\, f_y(y)\, dy$, which is the convolution of $f_X(x)$ and $f_Y(y)$.

Corollary

If $f_X(x) = 0$, for $x < 0$, and $f_Y(y) = 0$, for $y < 0$, then $f_X(z - y) f_Y(y) \neq 0$, only when $0 < y < z$.

$$\therefore \quad f_Z(z) = \int_0^z f_X(z-y)\, f_Y(y)\, dy,\ z > 0$$

Theorem 2

If two RVs X and Y are independent, find the pdf of $Z = XY$ in terms of the density functions of X and Y.

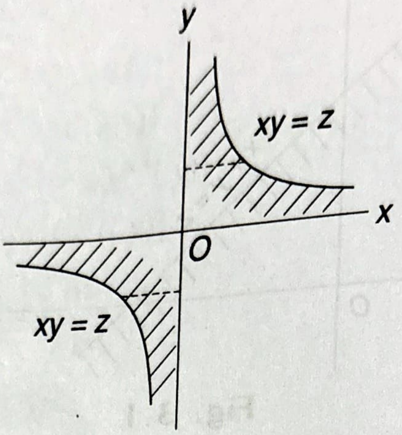

Fig. 3.2

Let the joint pdf of (X, Y) be $f(x, y)$

$$F_Z(z) = \iint_{xy \le z} f(x, y)\, dx\, dy$$

Note $xy = z$ *is a rectangular hyperbola as shown in the figure.)*

$$F_Z(z) = \int_{-\infty}^{0} \int_{z/y}^{0} f(x, y)\, dx\, dy + \int_{0}^{\infty} \int_{0}^{z/y} f(x, y)\, dx\, dy$$

Differentiating both sides with respect to z,

$$f_Z(z) = \int_{-\infty}^{0} -\frac{1}{y} f\left(\frac{z}{y}, y\right) dy + \int_{0}^{\infty} \frac{1}{y} f\left(\frac{z}{y}, y\right) dy$$

$$= \int_{-\infty}^{\infty} \left|\frac{1}{y}\right| f\left(\frac{z}{y}, y\right) dy$$

$$= \int_{-\infty}^{\infty} \left|\frac{1}{y}\right| f_X\left(\frac{z}{y}\right) f_Y(y)\, dy \quad \text{(since } X \text{ and } Y \text{ are independent)}$$

Theorem 3

If two RVs X and Y are independent, find the pdf of $Z = \dfrac{X}{Y}$ in terms of the density functions of X and Y.

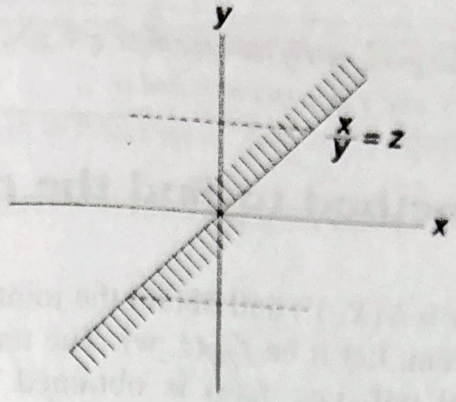

Fig. 3.3

Let the joint pdf of (X, Y) be $f(x, y)$.

$$F_Z(z) = \iint\limits_{\frac{x}{y} \le z} f(x, y)\, dx\, dy$$

i.e.,
$$F_Z(z) = \int_{-\infty}^{0} \int_{yz}^{\infty} f(x, y)\, dx\, dy + \int_{0}^{\infty} \int_{-\infty}^{yz} f(x, y)\, dx\, dy$$

Differentiating both sides with respect to z,

$$f_Z(z) = \int_{-\infty}^{0} - yf(yz, y)\, dy + \int_{0}^{\infty} yf(yz, y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| f(yz, y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| f_X(yz)\, f_Y(y)\, dy \quad \text{(since } X \text{ and } Y \text{ are independent)}$$

Two Functions of Two Random Variables

Theorem

If (X, Y) is a two-dimensional RV with joint pdf $f_{XY}(x, y)$ and if $Z = g(X, Y)$ and $W = h(X, Y)$ are two other RVs, then the joint pdf of (Z, W) is given by

$$f_{ZW}(z, w) = |J|\, f_{XY}(x, y), \text{ where } J = \frac{\partial(x, y)}{\partial(z, w)}$$

is called the Jacobian of the transformation and is given by

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix}$$

Note *This theorem holds good, only if the equations $z = g(x, y)$ and $w = h(x, y)$ when solved, give unique values of x and y in terms of z and w.*

An alternative method to find the pdf of $Z = g(X, Y)$

Introduce a second RV $W = h(X, Y)$ and obtain the joint pdf of (Z, W), as suggested in the above theorem. Let it be $f_{ZW}(z, w)$. The required pdf of Z is then obtained as the marginal pdf, i.e., $f_Z(z)$ is obtained by simply integrating $f_{ZW}(z, w)$ with respect to w.

i.e., $$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w)\, dw$$

Worked Example 3

Example 1

Find the distribution function of the RV $Y = g(X)$, in terms of the distribution function of X, if it is given that

$$g(x) = \begin{Bmatrix} x - c & \text{for } x > c \\ 0 & \text{for } |x| \le c \\ x + c & x < -c \end{Bmatrix}$$

(MSU — Apr. 96)

If $y < 0$, $\quad F_Y(y) = P(Y \le y)$

$$= P(X + c \le y)$$

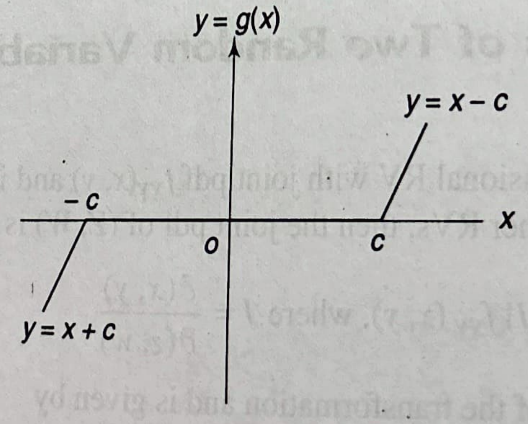

Fig. 3.4

$$= P(X \le y - c)$$

$$= F_X(y - c)$$

If $y \ge 0$, $\quad F_Y(y) = P(X - c \le y)$

$$= F_X(y + c)$$

Example 2

The random variable Y is defined by $Y = \frac{1}{2}(X + |X|)$, where X is another RV. Determine the density and distribution function of Y in terms of those of X.

(MSU — Nov. 96)

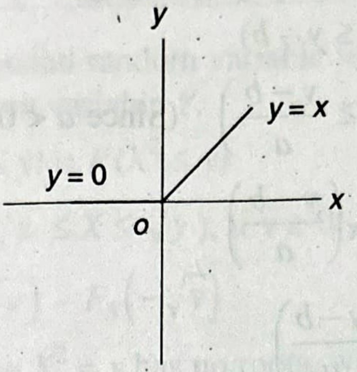

Fig. 3.5

When $X \geq 0$, $Y = X$

When $X < 0$, $Y = 0$

If $y < 0$, $F_Y(y) = P(Y \leq y) = 0$ (since there is no X, for which $Y \leq y$)

If $y \geq 0$, $F_Y(y) = P(Y \leq y)$

$$= P(X \leq y / X \geq 0)$$

$$= P(0 \leq X \leq y)/P(X \geq 0)$$

$$= \frac{F_X(y) - F_X(0)}{1 - F_X(0)}$$

$\therefore$ When $y < 0$, $f_Y(y) = 0$

and

when $y \geq 0$, $f_Y(y) = f_X(y)/[1 - F_X(0)]$

Example 3

(a) Find the density function of $Y = aX + b$ in terms of the density function of X.

(b) Let X be a continuous RV with pdf

$$f(x) = \frac{x}{12}, \text{ in } 1 < x < 5$$

$$= 0, \text{ elsewhere}$$

find the probability density function of $Y = 2X - 3$

(MU — Apr. 96 and Nov. 96)

(a) (i) Let $a > 0$

$$F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$$

$$= P\left(X \le \frac{y-b}{a}\right) \quad \text{(since } a > 0\text{)}$$

$$= F_X\left(\frac{y-b}{a}\right) \qquad (1)$$

(ii) Let $a < 0$

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$

$$= P(aX \le y - b)$$

$$= P\left(X \ge \frac{y-b}{a}\right) \quad \text{(Since } a < 0\text{)}$$

$$= 1 - F_X\left(\frac{y-b}{a}\right)$$

From (1), $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$ (3)

From (2), $f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right)$ (4)

Combining (3) and (4),

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

(b) $y = 2x - 3$, since $Y = 2X - 3$

$\therefore$ $x = \frac{1}{2}(y+3)$, i.e., x is a single valued function of y

$$\therefore \quad f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{x}{12} \times \frac{1}{2}$$

$$= \frac{1}{48}(y+3), \text{ in } -1 < y < 7$$

Note *The range of y is obtained from that of x (given in the problem) using the relation between x and y.*

Example 4

If X is a continuous RV with some distribution defined over (0, 1) such that $P(X \le 0.29) = 0.75$, determine k so that $P(Y \le k) = 0.25$, where $Y = 1 - X$

$P(Y \le k) = 0.25$

i.e., $P(1 - X \le k) = 0.25$

i.e., $P(X \ge 1 - k) = 0.25$

$\therefore$ $P(X \le 1 - k) = 0.75$ (1)

But it is given that $P(X \le 0.29) = 0.75$ (2)

Comparing (1) and (2), $k = 0.71$

Example 5

If $Y = X^2$, where X is a Gaussian random variable with zero mean and variance σ^2, find the pdf of the random variable Y. (BU — Nov. 96)

$$F_Y(y) = P(Y \le y) = P(X^2 \le y)$$

$$= P(-\sqrt{y} \le X \le \sqrt{y}), \text{ if } y \ge 0$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad (1)$$

and $F_Y(y) = 0$, if $y < 0$ [since $X^2 = y$ has no roots, when $y < 0$]

Differentiating (1) with respect to y,

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left\{f_X(\sqrt{y}) + f_X(-\sqrt{y})\right\}, \text{ if } y \ge 0 \quad (2)$$

$$= 0, \text{ if } y < 0$$

It is given that X follows $N(0, \sigma)$.

$$\therefore \quad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \quad -\infty < x < \infty$$

Using this value in (2), we get

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi y}} e^{-y/2\sigma^2} \quad y > 0$$

Example 6

If the continuous RV X has pdf $f_X(x) = \frac{2}{9}(x + 1)$, in $-1 < x < 2$ and $= 0$, elsewhere, find the pdf of $Y = X^2$.

The transformation function $y = x^2$ is not monotonic in $(-1, 2)$. So we divide the interval into two parts.

i.e., $(-1, 1)$ and $(1, 2)$

Since $(-1, 1)$ is a symmetric interval, $f_Y(y)$ is found out by using the formula (2) of the previous problem.

$\therefore$ When $-1 < x < 1$, i.e., $0 < y < 1$

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left\{\frac{2}{9}(1 + \sqrt{y}) + \frac{2}{9}(1 - \sqrt{y})\right\}$$

$$= \frac{2}{9\sqrt{y}}$$

When $1 < x < 2$, i.e., $1 < y < 4$, $y = x^2$ is strictly increasing

$$\therefore \quad f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$= \frac{2}{9}(x + 1) \times \frac{1}{2x}$$

$$= \frac{1}{9}\left(1 + \frac{1}{\sqrt{y}}\right)$$

Example 7

According to the Maxwell-Boltzmann law of theoretical physics, the pdf of V, the velocity of a gas molecule is given by

$$f_V(v) = \begin{cases} kv^2 e^{-av^2}, & v > 0 \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant depending on its mass and the absolute temperature and k is an appropriate constant. Show that the kinetic energy $Y = \frac{1}{2} mV^2$ is a random variable having Gamma distribution.

By the property of pdf,

$$k \int_0^\infty v^2 e^{-av^2} dv = 1$$

i.e., $$\frac{k}{2a\sqrt{a}} \int_0^\infty t^{3/2 - 1} e^{-t} dt = 1, \text{ by putting } t = av^2$$

i.e., $$\frac{k}{2a\sqrt{a}} \Gamma\left(\frac{3}{2}\right) = 1$$

i.e., $$\frac{k}{2a\sqrt{a}} \times \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = 1$$

$$\therefore \quad k = \frac{4a\sqrt{a}}{\sqrt{\pi}}, \text{ since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$Y = \frac{m}{2} V^2$$

$$\therefore \quad v = \pm\sqrt{\frac{2y}{m}}$$

since $v > 0$, $v = \sqrt{\frac{2y}{m}}$ is the only admissible value.

Now $$f_Y(y) = f_V(v)\left|\frac{dv}{dy}\right|$$

$$= \frac{4a\sqrt{a}}{\sqrt{\pi}} \times \frac{2y}{m} e^{-2ay/m} \frac{1}{\sqrt{2my}}$$

$$= \frac{4\sqrt{2}\, a\sqrt{a}}{m\sqrt{m}\,\sqrt{\pi}} \times y^{1/2}\, e^{-2ay/m}$$

$$= \frac{(2a/m)^{3/2}}{\Gamma(3/2)} \times y^{3/2-1}\, e^{-(2a/m)y}, \quad y > 0$$

which is a 2-parameter Gamma distribution or Erlang distribution.

Example 8

Given the RV X with density function

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

find the pdf of $Y = 8X^3$. (MU — Nov. 96)

Since $y = 8x^3$ is a strictly increasing function in (0, 1),

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|, \text{ where } x = \frac{1}{2}\, y^{1/3}$$

$$= y^{1/3} \times \frac{1}{6}\, y^{-2/3}$$

$$= \frac{1}{6}\, y^{-1/3} \quad 0 < y < 8$$

Example 9

If X is a Gaussian random variable with mean zero and variance σ^2, find the pdf of $Y = |X|$.

$$F_Y(y) = P(Y \le y) = P(|X| \le y)$$
$$= P\{-y \le X \le y\}$$
$$= F_X(y) - F_X(-y)$$

Differentiating both sides with respect to y,

$$f_Y(y) = f_X(y) + f_X(-y) \quad y > 0 \tag{1}$$

Now X follows $N(0, \sigma)$

$$\therefore \quad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}, \quad -\infty < x < \infty \tag{2}$$

$$\therefore \quad \frac{dx_i}{dr} = \frac{1}{\sqrt{a^2 - r^2}}$$

Now
$$f_R(r) = f_X(x_1)\left|\frac{dx_1}{dr}\right| + f_X(x_2)\left|\frac{dx_2}{dr}\right|$$

$$= \frac{2}{\pi\sqrt{a^2 - r^2}}, 0 < r < a$$

$$= 0, r > a$$

Example 13

(i) If X is uniformly distributed in $(-\pi/2, \pi/2)$ find the pdf of $Y = \tan X$.
(ii) If X has the Cauchy's distribution with parameter 1, find the pdf of $Y = \tan^{-1} X$.

(i) X is $U(-\pi/2, \pi/2)$

$$\therefore \quad f_X(x) = \frac{1}{\pi}$$

$$y = \tan x$$

Therefore, $x = \tan^{-1} y$, which is single valued in $(-\pi/2, \pi/2)$ i.e., for a given value of y, there exists only one value of $\tan^{-1} y$ in $(-\pi/2, \pi/2)$.

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{\frac{1}{\pi}}{1 + y^2} \quad -\infty < y < \infty$$

which is the pdf of a Cauchy's distribution.

(ii) $y = \tan^{-1} x$ is a monotonic increasing function

$$\therefore \quad f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{1}{\pi(1 + x^2)} \sec^2 y \quad (\text{since } x = \tan y)$$

$$= \frac{1}{\pi}, \quad \frac{-\pi}{2} < y < \frac{\pi}{2}$$

i.e., Y is uniformly distributed in $(-\pi/2, \pi/2)$.

Example 14

(i) If X has an arbitrary distribution function $F_X(x)$, find $g(x)$ so that the random variable $Y = g(X)$ may be uniformly distributed in $(0, 1)$.

(ii) If X is uniformly distributed in (0, 1), find $g(x)$, so that the random variable $Y = g(X)$ may have an arbitrary distribution with cdf $F_Y(y)$.

(iii) If X is uniformly distributed in (−1, 1), find $g(x)$, so that the random variable $Y = g(X)$ may have the density function $f_Y(y) = 2e^{-2y}$, $y > 0$.

(i) Y is to be uniform in (0, 1)

$$\therefore \quad f_Y(y) = 1 \quad \text{and} \quad F_Y(y) = \int_0^y f_Y(y)\,dy = y$$

$$\therefore \quad F_Y\{g(x)\} = g(x) \tag{1}$$

Now $\quad F_Y(y) = P(Y \le y) = P[g(X) \le y]$

$$= P[X \le g^{-1}(y)]$$

$$= P(X \le x) \qquad \text{[since } y = g(x) \text{ and hence } x = g^{-1}(y)]$$

$$= F_X(x) \tag{2}$$

i.e., $\quad F_Y\{g(x)\} = F_X(x)$

i.e., $\quad g(x) = F_X(x)$, from (1)

(ii) X is uniform in (0, 1)

$\therefore \quad F_X(x) = x$

By (2), $F_Y(y) = F_X(x)$

$\therefore \quad F_Y[g(x)] = x$

$$\therefore \quad g(x) = F_Y^{-1}(x)$$

(iii) X is uniform in (−1, 1)

$$\therefore \quad f_X(x) = \frac{1}{2} \text{ and } F_X(x) = \frac{1}{2}(x+1)$$

$$f_Y(y) = 2e^{-2y},\ y > 0$$

$$\therefore \quad F_Y(y) = \int_0^y 2e^{-2y}\,dy = 1 - e^{-2y}$$

By (2), $\quad 1 - e^{-2y} = \dfrac{1}{2}(x+1)$

i.e., $\quad 1 - e^{-2g(x)} = \dfrac{1}{2}(x+1)$

$$\therefore \quad g(x) = \frac{1}{2}\log\left(\frac{2}{1-x}\right)$$

Example 15

If X and Y are independent RVs having density functions.

$$f_1(x) = \begin{cases} 2e^{-2x}, & x \ge 0 \\ 0, & x < 0 \end{cases} \quad \text{and}$$

$$f_R(r) = \frac{r^2}{9}, \text{ if } 0 \le r \le 3$$

Hence the limits for r are taken as e and 3.

$$\therefore \quad f_E(e) = \frac{2e}{9}(3 - e), \quad 0 \le e \le 3$$

Example 19

If X and Y are independent RVs each following $N(0, 2)$ prove that $Z = \frac{X}{Y}$ follows a Cauchy's distribution.

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} e^{-x^2/8} \text{ and } f_Y(y) = \frac{1}{2\sqrt{2\pi}} e^{-y^2/8} \quad -\infty < x, \quad y < \infty$$

By Theorem 3

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_X(yz) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} |y| \frac{1}{2\sqrt{2\pi}} e^{-y^2 z^2/8} \times \frac{1}{2\sqrt{2\pi}} e^{-y^2/8}\, dy$$

$$= \frac{1}{4\pi} \int_{0}^{\infty} y \; e^{-(1+z^2)y^2/8}\, dy$$

(since the integrand is an even function)

$$= \frac{1}{\pi} \times \frac{1}{1+z^2} \quad -\infty < z < \infty$$

which is the pdf of Cauchy's distribution.

Example 20

If X and Y are independent RVs each following $N(0, 2)$, find the pdf of $Z = 2X + 3Y$.

Introduce the auxiliary RV $W = Y$.

$\therefore z = 2x + 3y$ and $w = y$

solving, $x = \frac{1}{2}(z - 3w)$ and $y = w$

$$J = \frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{-3}{2} \\ 0 & 1 \end{vmatrix} = \frac{1}{2}$$

Since X and Y are independent normal RVs $f_{XY}(x, y) = \dfrac{1}{8\pi} e^{-(x^2 + y^2)/8}$, $-\infty < x, y < \infty$. The joint pdf of (Z, W) is given by

$$f_{ZW}(z, w) = |J|\, f_{XY}(x, y)$$

$$= \frac{1}{2} \times \frac{1}{8\pi} e^{-\{(z-3w)^2 + 4w^2\}/32} \qquad -\infty < z \quad w < \infty$$

The pdf of Z is the marginal pdf, obtained by integrating $f_{ZW}(z, w)$ with respect to w over the range of w.

$$\therefore \quad f_Z(z) = \frac{1}{16\pi} \int_{-\infty}^{\infty} e^{-(13w^2 - 6zw + z^2)/32}\, dw$$

$$= \frac{1}{16\pi} e^{-z^2/8 \times 13} \int_{-\infty}^{\infty} e^{-13\left(w - \frac{3z}{13}\right)^2/32}\, dw$$

$$= \frac{1}{(2\sqrt{13})\sqrt{2\pi}} e^{-z^2/2(2\sqrt{13})^2} \qquad -\infty < z < \infty$$

which is $N(0, 2\sqrt{13})$.

Example 21

If X and Y each follow an exponential distribution with parameter 1 and are independent, find the pdf of $U = X - Y$.

$$f_X(x) = e^{-x},\ x > 0, \text{ and } f_Y(y) = e^{-y},\ y > 0$$

Since X and Y are independent.

$$f_{XY}(x, y) = e^{-(x+y)};\quad x, y > 0$$

Consider the auxiliary RV $V = Y$ along with

$$U = X - Y.$$

$$\therefore \quad x = u + v \quad \text{and} \quad y = v$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

The joint pdf of (U, V) is given by

$$f_{UV}(u, v) = |J|\, f_{XY}(x, y)$$

$$= e^{-(x+y)} = e^{-(u+2v)}$$

The range space of (U, V) is found out from the map of the range space of (X, Y) under the transformations $x = u + v$ and $y = v$.

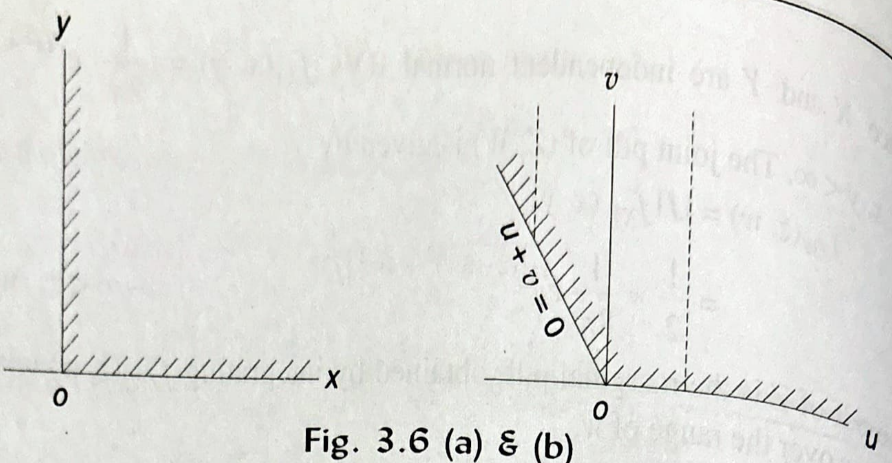

Fig. 3.6 (a) & (b)

Therefore, the range space of (U, V) is given by $v > -u$, when $u < 0$ and $v > 0$, when $u > 0$.

Now the pdf of U is given by

$$f_U(u) = \int_{-u}^{\infty} e^{-(u+2v)}\, dv, \quad \text{when } u < 0$$

and
$$= \int_{0}^{\infty} e^{-(u+2v)}\, dv, \quad \text{when } u > 0$$

$\therefore$
$$f_U(u) = \frac{1}{2}\, e^{u}, \quad \text{when } u < 0.$$

and
$$= \frac{1}{2}\, e^{-u}, \quad \text{when } u > 0.$$

Example 22

If the joint pdf of (X, Y) is given by $f_{XY}(x, y) = x + y;\ 0 \le x, y \le 1$, find the pdf of $U = XY$.

Introduce the auxiliary RV $V = Y$.

$\therefore$
$$x = \frac{u}{v} \quad \text{and} \quad y = v$$

$$J = \begin{vmatrix} \dfrac{1}{v} & -\dfrac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

The joint pdf of (U, V) is given by

$$f_{UV}(u, v) = \frac{1}{|v|}\, f_{XY}(x, y) = \frac{1}{|v|}\left(\frac{u}{v} + v\right)$$

Range space of (X, Y) is given by $0 \le x \le 1$ and $0 \le y \le 1$

$\therefore$ Range space of (U, V) is given by

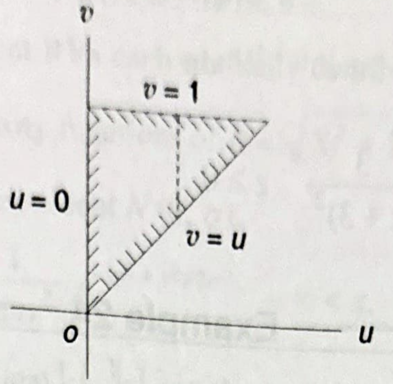

Fig. 3.7

$$0 \le \frac{u}{v} \le 1 \text{ and } 0 \le v \le 1$$

i.e., $\quad 0 \le u \le v \quad$ and $0 \le v \le 1$

The pdf of U is given by

$$f_U(u) = \int_u^1 f_{UV}(u, v)\, dv$$

$$= \int_u^1 \frac{1}{v}\left(\frac{u}{v} + v\right) dv$$

$$= 2(1 - u), \quad 0 < u < 1$$

Example 23

If X and Y are independent RVs with $f_X(x) = e^{-x}\, U(x)$ and $f_Y(y) = 3e^{-3y}\, U(y)$, find $f_Z(z)$, if $Z = \dfrac{X}{Y}$.

Since X and Y are independent, $f_{XY}(x, y) = 3e^{-(x + 3y)}$, $x, y \ge 0$

Introduce the auxiliary RV $W = Y$.

$$\therefore \qquad x = zw \text{ and } y = w$$

$$J = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$$

The joint pdf of (Z, W) is given by

$$f_{ZW}(z, w) = |J|\, f_{XY}(x, y)$$

$$= |w| \times 3e^{-(z + 3)\, w}; \quad z, w \ge 0$$

The range space is obtained as follows:

Since $y \ge 0$, $w \ge 0$. Since $x \ge 0$, $zw \ge 0$.

As $w \ge 0$, $z \ge 0$.