Stochastic process & Markov chains

Def.: A probability space is a triple $(\Omega, \mathcal{F}, P)$

Ω : Sample space, $\mathcal{F}$: all subsets of Ω

σ- algebra:

1) $\Omega \in \mathcal{F}$
2) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
3) If $A_1, A_2, \ldots \in \mathcal{F}$ $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

P: probability measure:

$P: \mathcal{F} \to [0,1] \ni P(\Omega) = 1$

If $A_1, A_2, \ldots \in \mathcal{F}$, then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Ex.: probability space for "rolling a fair die".

$\Omega = \{1, 2, 3, 4, 5, 6\}$.

$\mathcal{F} = 2^{\Omega}$ (all subsets of Ω)

If $A = \{\text{all even outcomes}\} = \{2, 4, 6\} \subseteq \Omega$

$P(A) = \frac{3}{6} = \frac{1}{2}$.

Def.: Let $(\Omega, \mathcal{F}, P)$ be a probability space, T: index set (usually, time), S: state space (usually subset of $\mathbb{R}^n$).

A stochastic process is a collection of random variables $\{X_t : t \in T\}$ such that for each fixed $t \in T$,

$X_t : \Omega \to S$ is a measurable function. (prob. exists)

(i.e., for each fixed $t \in T$, X_t is a random variable)

Ex.: Toss a fair coin several times $\left(\begin{matrix} H: 1 \\ T: 0 \end{matrix}\right)$

Sample space $\Omega = \{\text{all infinite sequences of H and T}\}$

$$X_n = \begin{cases} 1 & \text{if } n^{th} \text{ toss is head} \\ 0 & \text{if } n^{th} \text{ toss is tail} \end{cases}$$

for $n = 1, 2, 3, \ldots$

Then $\{X_n : n = 1, 2, \cdots\}$ is a stochastic process.

One particular outcome say $\omega = \{H, T, H, H, T, \cdots\}$.

Then $X_1(\omega) = 1$, $X_2(\omega) = 0$, $X_3(\omega) = 1$, $\cdots$

Here: Index set T: $\{1, 2, 3, \cdots\}$ time steps - coin toss number

State space S: $\{0, 1\}$ $(0 : T, 1 : H)$

Sample space : All finite sequences of coin tosses

Stochastic process: $X_n = \begin{cases} 1 & \text{if Head} \\ 0 & \text{if Tail.} \end{cases}$

Examples:

1) T: discrete, S: discrete (Discrete random sequence)

X_n: outcome of the n^{th} toss of a fair dice,

$= \{X_n \mid n \geq 1\}$. discrete.

$S = \{1, 2, 3, \cdots, 6\}$.

$T = \{1, 2, 3, \cdots\}$

2) T: is discrete, S: Continuous. (Continuous random process)

X_n: temperature at the end of n^{th} hour of a day.

$= \{X_n : 1 \leq n \leq 24\}$

Temp. $\rightarrow$ interval. $(\underbrace{\quad}_{\text{hour.}})$

3). T: Continuous, S: discrete (discrete random process)

X_t: number of telephone calls received in $\underbrace{(0, t)}_{\text{interval.}}$

$S = \{0, 1, 2, 3, \cdots\}$.

4). T: Continuous, S: Continuous. (Continuous random process)

X_t: $\underbrace{\text{max. temperature}}_{\text{~~temp~~ interval.}}$ at a place in $\underbrace{(0, t)}_{\text{interval.}}$.

Stochastic Process - Problems

1. Classification of Random Processes

Index set

Depending on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified into four types:

- If both T and S are discrete, the random process is called a discrete random sequence.
  For example, if X_n represents the outcome of the n^{th} toss of a fair dice, then $\{X_n; n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, ...\}$ and $S = \{1, 2, 3, ...\}$.

- If T is discrete and S is continuous, the random process is called a continuous random sequence.
  Example: If X_n represents the temperature at the end of the n^{th} hour of a day, then $\{X_n; 1 \leq n \leq 24\}$ is a continuous random sequence, since temperature can take any value in an interval and hence continuous.

- If T is continuous and S is discrete, the random process is called a discrete random process.
  For example, if X_t represents the number of telephone calls received in the interval $(0, t)$ then $\{X_t\}$ is a discrete random process, since $S = \{0, 1, 2, 3, ...\}$.

- If both T and S are continuous, the random process is called a continuous random process. For example, if X_t represents the maximum temperature at a place in the interval $(0, t)$ then $\{X_t\}$ is a continuous random process.

In the names given above, the word 'discrete' or 'continuous' is used to refer to the nature of S and the word 'sequence' or 'process' is used to refer to the nature of T.

2. Markov Process / Markov Chain

Consider a stochastic process

$\{X(t) \mid t = 0, 1, 2, 3 \cdots\}$

$"X_t"$

Suppose $X_0, X_1, X_2, \cdots$ are discrete r.v.

with

$x_{01}, x_{02}, x_{03} \cdots$ are the values of X_0

$x_{11}, x_{12}, x_{13} \cdots$ are the values of X_1

$x_{21}, x_{22}, x_{23} \cdots$ are the values of X_2

⋮

and so on

$\therefore$ The set $S = \{x_{ij} \mid i = 0, 1, 2 \cdots, j = 1, 2, \cdots\}$ is the state space of the stochastic process.

If S is finite: then $A = \{a_1, a_2, \cdots a_m\}$

then x_{ij} is equal to some a_i in A.

Here Index set is $T = \{0, 1, 2, 3 \cdots\}$

States are $a_1, a_2 \cdots a_m$

& $A = \{a_1, a_2 \cdots a_n\}$ is the state space

∴ This process is discrete state discrete parameter process.

Suppose $X_0, X_1, X_2, \ldots$ are such that

(i) the values taken by X_1 depends on the values taken by X_0.

(ii) the values taken by X_2 depends on the values taken by X_1 but not depends upon the values taken by X_0.

(iii) the values taken by X_3 depends on the values taken by X_2 but not depending on the values taken by X_1 & X_0

and so on

further, Suppose that for each pair of states (a_i, a_j) and for any successive random variables say, X_r & X_{r+1} in the process the conditional probability that the event $X_{r+1} = a_j$ occurs given that $X_r = a_i$ has already occurred, is specified.

$$P\left(X_{r+1} = a_j \,\middle|\, \underbrace{X_r = a_i}_{}\right) = ?$$

← random Vari X is at a_i state r^{th} time.

~~State~~ r.v. X is at state a_i at r^{th} time

Def: (Markov chain).

→ The Stochastic process $\{X(t) \mid t = 0,1,2,3 \ldots\}$ is called a Markov chain if for $j, k, j_1, j_2, \ldots j_{n-1} \in \mathbb{N}$ then

$$P\{X_n = k \mid X_{n-1} = j, X_{n-2} = j_1, X_{n-3} = j_2 \ldots X_0 = j_{n-1}\}$$

$$= P\{X_n = k \mid X_{n-1} = j\} = p_{jk}$$

→ Transition probabilities

Let $A = \{a_1, a_2, \ldots a_m\}$ be the state space of a Markov chain & for each pair of states a_i, a_j

Let $p_{ij} = P\{X_{r+1} = a_j \mid X_r = a_i\}$ where X_r & X_{r+1} are any two Succesive r.v's in the Stochastic process

Since i, j varies from $\{1, 2, \ldots m\}$ then we've, m^2 probabilities p_{ij}, are called the transition probabilities of the Markov Chain

→ Matrix

t.p.m.

$$\begin{array}{c|c} & m \text{ col'ns} \downarrow \text{ TO.} \\ \begin{array}{c} a_1 \text{ From} \\ m \text{ Rows} \rightarrow \end{array} & \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1m} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2m} \\ \vdots & & & & \\ p_{m1} & p_{m2} & & \cdots & p_{mm} \end{pmatrix}_{m \times m} \end{array}$$

Transition matrix or Stochastic matrix

A Student of

Example:

A Student study habits are as follows:
If he studies one night, he is 30% sure to study the next night. On the other hand if he doesn't study one night, he is 40% sure to study the next night. Find the transition matrix of the chain of his study? And draw the Stochastic graph?

Ans: There are two possible states

They are a_1 : studying
a_2 : not studying } States.

$\therefore$ Transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} a_1 & a_2 \end{array} \text{ Next night} \\ \begin{array}{c} a_1 \\ \text{One night (previous) } a_2 \end{array} & \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \end{array}$$

$p_{11} = P(\text{Studying one night (next)} \mid \text{he has studied previous night})$
$= 30\% = 0.3$

$p_{12} = P(X_{r+1} = a_2 \mid X_r = a_1) = 70\% = 0.7$

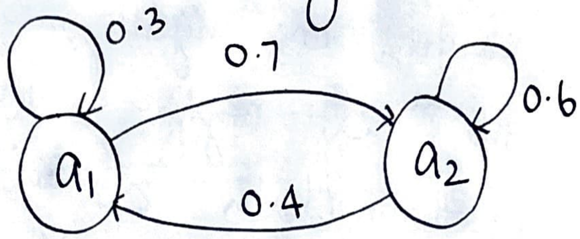
$p_{21} = P(X_{r+1} = a_1 \mid X_r = a_2) = 40\% = 0.4$

$p_{22} = P\{X_{r+1} = a_2 \mid X_r = a_2) = 60\% = 0.6$

$\therefore$ Transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} 0.3 = p_{11} & 0.7\ p_{12} \\ 0.4\ p_{21} & 0.6\ p_{22} \end{pmatrix} \end{array}$$

∴ Stochastic graph is



But we are interested on discrete state Markov process, known as Markov Chains where the system can occupy only a finite or countable number of states.

Definition 1.4. *Let $X(t)$ be a Markov process with states $X(t_r) = X_r = a_r$ for $t_0 < t_1 < t_2 < ... < t_n$.*

If for all n,

$$P[X_n = a_n | X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, ... X_1 = a_1, X_0 = a_0]$$
$$= P[X_n = a_n | X_{n-1} == a_{n-1}]$$

Then the sequence of random variables $\{X_n; n = 0, 1, , 3, ..\}$ is called a Markov chain with $a_1, a_2, ..., a_n$ are the states of the Markov chain.

States

Definition 1.5. ***Transition probabilities and Transition Matrix:*** *Let $A = \{a_1, a_2, ..., a_n\}$ be the state space of a Markov chain, and for any two states a_i, a_j, let p_{ij} denote the conditional probability that $X_{r+1} = a_j$ given that $X_r = a_i$.*

$$p_{ij} = P\left[X_{r+1} = a_j | X_r = a_i\right]$$

where X_{r+1} & X_r are any two successive random variables present in the process. The probabilities p_{ij} are called ***transition probabilities.***

The square matrix P whose elements are the transition probabilities p_{ij} is called a ***transition probability matrix*** *or* ***transition matrix (t.p.m.)***.

X_n

States of X_{r+1} (present)

(next)

X_{n-1} States of X_r (previous) (present)

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_n \end{array} \\
\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_n \end{array} &
\begin{pmatrix}
p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\
p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\
\cdots & & & & \\
\cdots & & & & \\
p_{m1} & p_{m2} & p_{m3} & \cdots & p_{mn}
\end{pmatrix},
\end{array}
$$

Remark 1.6. In P it is clear that, $p_{ij} \geq 0$ for all $i, j = 1, 2, 3, .., n$ and $\Sigma_{j=1}^{n} p_{ij} = 1$ for $i = 1, 2, 3.., n$.

Definition 1.7. ***Stochastic Matrix*** *A transition probability matrix $P = [p_{ij}]$ with the above two properties is called a* ***stochastic matrix.***

Problem 1.8. *A travelling salesman S sells in three cities namely A, B and C. He never sells in the same city on successive days. If he sells in the city A on a day, then the next day he sells in the city B. However, if he sells in either B or C then the next day he is twice as likely to sell in city A as in the other city. Find the transition matrix of the Markov chain of selling.*

Solution

Prob. of selling.

Here, a_1: Selling in city A
a_2: Selling in city B
a_3: Selling in city C

equally likely $p(\text{selling}) = \frac{1}{3}$.

t.p.m.

$$P = \begin{array}{cc} & \text{States of } X_n \\ & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \text{States of } X_{n-1} \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} \end{array}$$

sum 1.

$P_{11} = P(\text{Selling city A} \mid \text{he had sold in city A on the previous day})$
$= 0$

$P_{12} = P\{X_n = a_2 \mid X_{n-1} = a_1) = 1$ (since he sells in A then next day he sells in B).

$P_{13} = P(X_n = a_3 \mid X_{n-1} = a_1) = 0$ (since prob. already reached is 1 ~~a_1 and a_3 are not successive~~),

$P_{21} = P\{X_n = a_1 \mid X_{n-1} = a_2) = \frac{2}{3}$ (twice).

$P_{22} = 0$, $P_{23} = P(X_n = a_3 \mid X_{n-1} = a_2) = 1/3$

$P_{31} = 2/3$, $P_{32} = 1/3$, $P_{33} = 0$

Contd.

$\therefore$ The req'd t.p.m is

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix}$$

The joint pdf of $\{X_n\}_{n\geq 0}$ $(x_0, x_1, \dots, x_n)$ given that $P(X_0)$ is known.

Collection of values taken by X_t's.

2. Probability distribution

Consider the Markov chain $\{X(t) = X_t\}$ with state space $S = \{a, b, ...i, j, k\}$. Then the probability distribution of $X_r, X_{r+1}, X_{r+2}, ..., X_{r+n}$ can be determined in terms of the transition probabilities p_{jk} and the initial distribution of X_r.

$P(A \cap B) = P(A|B) \cdot P(B)$

Joint prob. of $(x_0, x_1, \dots x_n) =$

For our convenience take $r = 0$ then

$$P\{X_0 = a, X_1 = b, X_2 = c, \dots X_{n-2} = i, X_{n-1} = j, X_n = k)$$

$$= P\{\underbrace{X_n = k}_{A}, \underbrace{X_{n-1} = j, X_{n-2} = i, \dots X_2 = c, X_1 = b, X_0 = a}_{B}\}$$

$$= P(X_n = k \mid X_{n-1} = j, X_{n-2} = i, \dots X_2 = c, X_1 = b, X_0 = a\}$$

Since $\{X_n\}$ is Markov chain

$$P\{X_{n-1} = j, X_{n-2} = i, \dots X_2 = c, X_1 = b, X_0 = a\}$$

$$= P(X_n = k \mid X_{n-1} = j) P\{\underbrace{X_{n-1} = j}_{A}, \underbrace{X_{n-2} = i \dots X_2 = c, X_1 = b, X_0 = a}_{B}\}$$

$P(X_{n-1} = j \mid X_{n-2} = i, X_{n-3} = h \dots X_0 = a)$

$$= p_{jk} P\{X_{n-1} = j \mid X_{n-2} = i\} P\{X_{n-2} = i, X_{n-3} = h, \dots X_2 = c, X_1 = b, X_0 = a\}$$

$$= p_{jk} \cdot p_{ij} P\{\underbrace{X_{n-2} = i}_{A}, \underbrace{X_{n-3} = h, \dots X_2 = c, X_1 = b, X_0 = a}_{B}\}$$

$$= p_{jk} p_{ij} p_{hi} \dots p_{ab} P\{X_0 = a\}$$

Therefore

$$\therefore P\{X_0 = a, X_1 = b, X_2 = c, \dots X_{n-1} = j, X_n = k\}$$

$$= P\{X_0 = a\} p_{ab} p_{bc} \dots p_{hi} p_{ij} p_{jk}$$

$$\boxed{P\{X_r = a, X_{r+1} = b \dots X_{r+n-1} = j, X_{r+n} = k\} = P\{X_r = a\} p_{ab} p_{bc} \dots p_{hi} p_{ij} p_{jk}}$$

In general for any r^{th} random variable.

Problem 2.1. *Let $\{X_n; n \geq 0\}$ be a Markov chain with three states 0, 1, 2 and transition matrix is given by* $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$. *Find the following probabilities by assuming that the initial distribution is equally likely for the* X_n *three states i.e.,* $P\{X_0 = i\} = \frac{1}{3}$ *for* $i = 0, 1, 2$.

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \hline \begin{matrix} 0 \\ 1 \\ X_{n-1}\ 2 \end{matrix} & \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \end{array}$$

1. $P\{X_1 = 1 | X_0 = 2\}$
2. $P\{X_2 = 2 | X_1 = 1\}$
3. $P\{X_2 = 2, X_1 = 1, X_0 = 2\}$
4. $\frac{3}{64} =$ $P\{X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2\}$

⑤ Draw the Stochastic Graph of $\{X_n\}_{n\geq 0}$

Solution $X_1 = i \quad X_0 = j \quad p_{ji}$

Ans:. ① $P\{X_1 = 1 | X_0 = 2\} = p_{21}^{"} = \frac{3}{4}$

what is the prob of moving from state 2 to state 1 in 1 step?

② $P\{X_2 = 2 | X_1 = 1\} = P_{12} = 1/4$

③ $P\{\underbrace{X_2 = 2}_{A}, \underbrace{X_1 = 1, X_0 = 2}_{B}\} = P\{X_2 = 2 | X_1 = 1\} \cdot P\{X_1 = 1, X_0 = 2\}$

$= P\{X_2 = 2 | X_1 = 1\} P\{X_1 = 1 | X_0 = 2\} P\{X_0 = 2\}$

$= p_{12}\, p_{21}\, P\{X_0 = 2\} = \frac{1}{4} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{16}$

⑤ $P\{\underbrace{X_2 = 2}_{A}, \underbrace{X_1 = 1}_{B} | \underbrace{X_0 = 2}_{C}\} = P\{X_2 = 2 | X_1 = 1\} P\{X_1 = 1 | X_0 = 2\}$

$= p_{12} \times p_{21}$

$= \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$

$P(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(C)}$

$= \frac{P(A|B) P(B|C) P(C)}{P(C)}$

$= P(A|B) P(B|C)$

$P(X_2 = 2, X_1 = 1 | X_0 = 2)$

$= \frac{P(X_2 = 2, X_1 = 1, X_0 = 2)}{P(X_0 = 2)}$

$= \frac{p_{12}\, p_{21} \times \cancel{P(X_0 = 2)}}{\cancel{P(X_0 = 2)}}$

$P(A|B) P(B \cap C)$

$P(A \cap B \cap C) = \cancel{P(A|B)} \cancel{P(B|C)} \cancel{P(C)}$

$= P(A|B) P(B|C) P(C)$

Contd.

(6) Draw the stochastic graph