SESSIONAL ADDITIONAL SHEET

Semester: .... Branch: .... Section: ....

Subject: .... Roll No.: ....

Semantics of FL

An interpretation is a pair $I = (D, \phi)$, where D is a nonempty set, called the domain or universe of I, and ϕ is a function associating function symbols with partial functions on D and predicates with relations on D. Further, ϕ preserves arity, that is,

(a) If P is a 0-ary predicate, then $\phi(P)$ is a sentence in D, which is either true or false.

(b) If P is $\approx$, then $\phi(P)$ is the equality relation on D, expressing "same as", that is $\phi(P) = \{(d, d) : d \in D\}$.

(c) If P is an n-ary predicate for $n \geq 1$, other than $\approx$, then $\phi(P)$ is an n-ary relation on D, a subset of D^n.

(d) If f is a 0-ary function symbol (a constant, a name), then $\phi(f)$ is an object in D; that is $\phi(f) \in D$.

(e) If f is an n-ary function symbol, $n \geq 1$, then $\phi(f) : D^n \to D$ is a partial function of n arguments on D.

A valuation under the interpretation $I = (D, \phi)$ is a function l that assigns terms to the elements of D, which is first defined for variables, with $l(x) \in D$ for any variable x, and then extended to terms satisfying:

(i) If f is a 0-ary function symbol, then $\ell(f) = \phi(f)$.

(ii) If f is an n-ary function symbol, $n \geq 1$, and $t_1, \ldots, t_n$ are terms, then
$$\ell(f(t_1, \ldots, t_n)) = \phi(f)(\ell(t_1), \ldots \ell(t_n)).$$

For a valuation ℓ, a variable x, and an object $a \in D$, we write $\ell[x \to a]$ as a new valuation obtained from ℓ which fixes x to a, but assigns every other variable to what ℓ assigns.

That is,
$$\ell[x \to a](x) = a, \quad \ell[x \to a](y) = \ell(y), \; y \neq x.$$

A <u>state</u> I_ℓ is a triple (D, ϕ, ℓ), where $I = (D, \phi)$ is an interpretation and ℓ is a valuation under I.

Let X, Y, Z be formulas and let $I_\ell = (D, \phi, \ell)$ be a state. Satisfaction of a formula in the state I_ℓ is defined inductively by the following. We read $I_\ell \vDash X$ as I_ℓ <u>satisfies</u> X or as I_ℓ <u>verifies</u> X or as I_ℓ is a <u>state-model</u> of X.

1. $I_\ell \vDash \top$
2. $I_\ell \nvDash \bot$.
3. For a 0-ary predicate P, $I_\ell \vDash P$ iff $\phi(P)$ is a true sentence in D.
4. For terms s, t, $I_\ell \vDash (s \approx t)$ iff $\ell(s) = \ell(t)$ as elements of D.
5. For an n-ary predicate P other than $\approx$, $n \geq 1$, and

terms $t_1, \ldots, t_n$,

$I_l \models p(t_1, \ldots, t_n)$ iff $(l(t_1) \ldots, l(t_n)) \in \Phi(P)$.

6. $I_l \models \neg X$ iff $I_l \not\models X$.
7. $I_l \models X \wedge Y$ iff $I_l \models X$ and $I_l \models Y$ hold.
8. $I_l \models X \vee Y$ iff at least one of $I_l \models X$ or $I_l \models Y$ holds.
9. $I_l \models X \rightarrow Y$ iff at least one of $I_l \not\models X$ or $I_l \models Y$ holds.
10. $I_l \models X \leftrightarrow Y$ iff either $I_l \models X$ and $I_l \models Y$ hold, or $I_l \not\models X$ and $I_l \not\models Y$ hold.
11. $I_l \models \forall x Y$ iff for each $d \in D$, $I_{l[x \to d]} \models Y$.
12. $I_l \models \exists x Y$ iff for at least one (some) $d \in D$, $I_{l[x \to d]} \models Y$.

<u>Example</u>: Consider the formulas Pxy and $Pcf(c)$ along with the following interpretations and valuations l, m:

I, J and valuations l, m, where
$I = (\mathbb{N}, \Phi)$, where
$\Phi(P) =$ "is less than"
$\Phi(c) = 0$,
$\Phi(f) =$ 'successor function',
$l(x) = 0$, $l(y) = 1$.

$I_l \models Pcf(c)$ iff $(l(c), l(f(c))) \in \Phi(P)$.
iff $(l(c), \Phi(f)(l(c))) \in \Phi(P)$
iff $(\Phi(c), \Phi(f)(\Phi(c))) \in \Phi(P)$
iff $(\Phi(c), \Phi(f)(\Phi(c))) \in \Phi(P)$.
iff $(0, 0+1) \in \Phi(P)$.
iff $0 < 0+1$
which is true in $\mathbb{N}$.

Hence $I_l \models Pcf(c)$

Example: $\forall x\, P_{x f(x)}$.

$I = (\mathbb{N}, P', f')$

$f' : \mathbb{N} \longrightarrow \mathbb{N}$

$f'(n) = n+4$.

$P' \subseteq \mathbb{N} \times \mathbb{N}$, $(m, n) \in P'$ iff $m \leq n$.

$l(x) = 3$.

$I_l \models \forall x\, P_{x f(x)}$

iff for each $n \in \mathbb{N}$, $I_{[x \mapsto n]} \models P_{x f(x)}$

iff for each $n \in \mathbb{N}$, $(n, f'(n)) \in P'$

iff for each $n \in \mathbb{N}$, $n \leq n+4$

iff for each $n \in \mathbb{N}$,

which true for $n \in \mathbb{N}$.

So, $I_l \models \forall x\, P_{x f(x)}$

Example: $\forall x\, P_{x f(y)}$ $l(y) = 101$.

$I_l \models \forall x\, P_{x f(y)}$

iff for each $n \in \mathbb{N}$, $I_{l[x \mapsto n]} \models P_{x f(y)}$

iff for each $n \in \mathbb{N}$,

H.W: Determine whether the state $I_l = (D, \phi, l)$ satisfies the following formulus, where $D = \{1, 2\}$, $\phi(c) = 1$; $\phi(P) = \{1\}$, $\phi(Q) = \{(1,1), (2,2)\}$, and l is some valuation.

a) $\forall x(Pc \vee Qxx) \to Pc \vee \forall x\, Qxx$

b) $\forall x(Pc \wedge Qxx) \to Pc \vee \forall x\, Qxx$

c) $\forall x(Pc \vee Qxx) \to Pc \wedge \forall x\, Qxx$

d) $\forall x(Pc \wedge Qxx) \to Pc \wedge \forall x\, Qxx$.

SESSIONAL ADDITIONAL SHEET

Semester: .... Branch: .... Section: ....

Subject: .... Roll No.: ....

Lemma (Substitution lemma): Let t be a term free for a variable x in a formula Y. Let s be any term. Let $I_l = (D, \Phi, \ell)$ be a state. Then

1) $\ell[x \to \ell(t)](s) = \ell(s[x/t])$;

2) $I_{\ell[x \to \ell(t)]} \vDash X$ iff $I_\ell \vDash X[x/t]$.

Roughly, it says that substitution of a free variable by a term can be done either at the formula, or at the valuation; so that when interpreted, they have same effect.

Satisfiability and validity

Let A be a formula: we say A is satisfiable iff some state satisfies it; A is unsatisfiable iff each state falsifies it; A is valid iff each state satisfies it; and A is invalid iff some state falsifies it.

For a predicate P, its corresponding relation $\Phi(P)$ is informally written as P'. For example, to interpret the formula $P_x f(yz)$ we write the interpretation $I = (N, \Phi)$, with $\Phi(P) =$ '<', $\Phi(f) =$ 'sum of so and so'; as $I = (N, P', f')$, wher P' is '<' and f' is 'sum of so and so'.

Example: Is the formula $A = \forall x P_x f(a)$ satisfiable? Is it valid?

Take an interpretation $I = (N, P', f')$, where P' is '<' relation and f' is

the function 'plus 5', i.e., $f'(n) = n+5$.
Let ℓ be a valuation with $\ell(x) = 2$.
The state $I_\ell \models \forall x\, P x f(x)$ iff for each $n \in \mathbb{N}$,
$I_{\ell[x \mapsto n]} \models P x f(x)$. Suppose we want to verify for $n = 3$. Now $\ell[x \mapsto n] = \ell[x \mapsto 3]$ maps x to 3, and every other variable y to $\ell(y)$. However, it does not matter what this $\ell(y)$ is since y does not occur in the formula. We see that $I_{\ell[x \mapsto 3]} \models Pxf(x)$ iff $3 < f'(3)$ iff $3 < 3+5$. Since $3 < 3+5$ is true in $\mathbb{N}$, $I_{\ell[x \mapsto 3]} \models Px f(x)$.
It can be show that $I_{[x \mapsto n]} \models Pxf(x)$ for each $n \in \mathbb{N}$.
Thus the formula $\forall x\, Pxf(x)$ is satisfiable as it has a state-model such as I_ℓ.

Consider the interpretation $J = (\mathbb{N}, \bar{P}, f')$, where $\bar{P}$ is the 'greater than' relation.
Take f' as the same as 'plus 5' as above.
Now $J_\ell \models Pxf(x)$ iff for each $n \in \mathbb{N}$, $\ell[x \mapsto n] \models Px f(x)$. For $n = 3$, we see that $\ell[x \mapsto 3] \models Pxf(x)$ iff $3 > 3+5$. As $3 \not> 3+5$, $J_\ell \not\models Pxf(x)$. That is, the formula $\forall x\, Pxf(x)$ is <u>invalid</u>

We define the notions of consequence and equivalence in FL as follows.
Let Σ be a set of formulas, I an interpretation, and ℓ a valuation under I. The state I_ℓ is a <u>state-model</u> of (or satisfies, or verifies) the set Σ, written $I_\ell \models \Sigma$ iff for each $X \in \Sigma$, $I_\ell \models X$.
The set Σ is called <u>satisfiable</u> iff Σ has a state-model.

For a formula B, Σ semantically entails B, written as $\Sigma \models B$, iff each state-model of Σ is a state-model of B. For $\Sigma = \{X_1, \ldots, X_n\}$, we also write $\Sigma \models B$, as $X_1, \ldots, X_n \models B$. We read $\Sigma \models B$ as "Σ entails B", "B is a semantic consequence of Σ", and also as "the consequence $\Sigma \,!\vdash B$ is valid".

Two formulas A, B are called equivalent, written $A \equiv B$, iff each state-model of A is a state-model of B, and also each state-model of B is a state-model of A. That is, $A \equiv B$ iff "for each interpretation I and for each valuation l under I, we have $I_l \models A$ iff $I_l \models B$."

Example: Show that $\exists y \forall x P_{xy} \models \forall x \exists y P_{xy}$.

Soln: Let $I = (D, \phi)$ be an interpretation and l a valuation under I. Assume that $I_l \models \exists y \forall x P_{xy}$. This means $\phi(P)$ is some subset of $D \times D$, and for some $d \in D$, for each $d' \in D$, we have $(d', d) \in \phi(P)$.

It demands that $(d_1', d) \in \phi(P)$, $(d_2', d) \in \phi(P), \ldots$

Then for each $d' \in D$ we have a corresponding element of D, here is $d \in D$, such that $(d', d) \in \phi(P)$. Thus $I_l \models \forall x \exists y P_{xy}$.

Example: Show that $\forall x \exists y P_{xy} \models \exists y \forall x P_{xy}$

We try with an interpretation. Let $I = (D, P')$, where $D = \{2, 3\}$ and $P' = \{(2,3), (3,2)\}$. Suppose $l(x) = 2$, $l(y) = 3$. To check $I_l \models \forall x \exists y P_{xy}$, we must check whether both of $I_{[x \to 2]} \models \exists y P_{xy}$ and $I_{l[x \to 3]} \models \exists y P_{xy}$.

Now, $I_l[x \mapsto 2] \models \exists y\, Pxy$ holds when at least one of

$I_l[x \mapsto 2][y \mapsto 2] \models Pxy$ or

$I_l[x \mapsto 2][y \mapsto 3] \models Pxy$

holds. Clearly $I_l[x \mapsto 2][y \mapsto 2] \not\models Pxy$ since $(2,2) \notin P'$! But $I_l[x \mapsto 2][y \mapsto 3] \models Pxy$ holds since $(2,3) \in P'$.

Therefore $I_l[x \mapsto 2] \models \exists y\, Pxy$ holds. Similarly, one can check that $I_l[x \mapsto 3] \models \exists y\, Pxy$ holds.

For $I_l \models \exists y\, \forall x\, Pxy$, we must have $I_l[y \mapsto 2] \models \forall x\, Pxy$ or $I_l[y \mapsto 3] \models \forall x\, Pxy$.

Now, $I_l[y \mapsto 2] \models \forall x\, Pxy$ holds iff $I_l[y \mapsto 2][x \mapsto 2] \models Pxy$ and $I_l[y \mapsto 2][x \mapsto 3] \models Pxy$.

Since $(2,2) \notin P'$, $I_l[y \mapsto 2][x \mapsto 2] \models Pxy$ does not hold. Consequently, $I_l[y \mapsto 2] \models \forall x\, Pxy$ does not hold.

Similarly, one can check that $I_l[y \mapsto 3] \models \forall x\, Pxy$ does not hold.

Thus. $\forall x\, \exists y\, Pxy \not\models \exists y\, \forall x\, Pxy$.

Semester: .... Branch: .... Section: ....

Subject: .... Roll No.: ....

Example: Let t be a term free for a variable x in a formula X. we show that $\forall x X \rightarrow X[x/t]$ is valid.

Solution: Let $I_\ell = (D, \phi, \ell)$ be a state.

If $I_\ell \nvDash \forall x X$, then $I_\ell \vDash \forall x X \rightarrow X[x/t]$.

So, suppose that $I_\ell \vDash \forall x X$. Then for each $d \in D$, $I_\ell[x \mapsto d] \vDash X$. In particular $I_\ell[x \mapsto \ell(t)] \vDash X$. However, this is same as $I_\ell \vDash X[x/t]$. Therefore, $I_\ell \vDash \forall x X \rightarrow X[x/t]$.

Example: We show that $\exists x \neg X \equiv \neg \forall x X$.

Solution: Let $I_\ell = (D, \phi, \ell)$ be a state.

Now, $I_\ell \vDash \exists x \neg X$ iff for some $d \in D$, $I_\ell[x \mapsto d] \vDash \neg X$ iff for some $d \in D$, $I_\ell[x \mapsto d] \nvDash X$ iff it is not the case that for each $d \in D$, $I_\ell[x \mapsto d] \vDash X$ iff $I_\ell \vDash \neg \forall x X$.

Some metatheorems:

Thm (Relevance lemma). Let X be a formula, and let $I_\ell = (D, \phi, \ell)$ and $J_m = (D, \psi, m)$ be states. Assume that for each predicate P occurring in X, $\phi(P) = \psi(P)$; for each function symbol occurring in X, $\phi(f) = \psi(f)$; and for each variable x occurring free in X, $\ell(x) = m(x)$. Then $I_\ell \vDash X$ iff $J_m \vDash X$.

Informally speaking, the Relevance lemma asserts that if two states agree on all free variables, predicates, and function symbols occurring in a formula, then either both satisfy the formula, or both falsify the formula.

A sentence A does not have any free variables. Let l and m be two valuations under an interpretation I. Vacuously, l and m agree on all free variables of A. By the Relevance lemma, $I_l \vDash A$ iff $I_m \vDash A$. Thus satisfiability of sentences is independent of the valuations; an interpretation interprets a sentence.

An interpretation I satisfies a formula A, written as $I \vDash A$, iff each valuation l under I, the state $I_l \vDash A$. We also read $I \vDash A$ as I verifies A or as I is a model of A. Similarly, we can define $I \nvDash A$, $I \vDash \Sigma$ and so on.

Example: Let $A = \forall x \forall y (Pxa \wedge Pyx \to \neg Pya)$. Take $I = (D, P', a')$, where $D = \{a', b', c'\}$ and $P' = \{(a', a'), (b', a'), (c', a')\}$. Here, we are implicitly writing $\phi(P) = P'$ and $\phi(a) = a'$. Now, does the interpretation I satisfy A?

Theorem (Relevance Lemma for sentences). Let A be a sentence, and let I be an interpretation.

(1) $I \vDash A$ iff $I_l \vDash A$ for some valuation l under I.

(2) Either $I \vDash A$ or $I \vDash \neg A$.

Proof: Since sentences have no free variables, Relevance lemma implies that if some state satisfies a sentence, then every state under the same interpretation satisfies the sentence. This proves (1). Since any state is either a state-model of A or of $\neg A$, (2) follows from (1).

Example: Let $I = (\mathbb{N}, p')$ be an interpretation of the formula px, where $p' \subseteq \mathbb{N}$ is the set of all prime numbers. Let l and m be valuations under I such that $l(x) = 4$ and $m(x) = 5$. Now $I_l \models px$ iff 4 is a prime number; and $I_m \models \neg px$ iff 5 is not a prime number. We see that $I_l \not\models px$ and $I_m \not\models \neg px$. Therefore, $I \not\models px$ and $I \models \neg px$.

Exercise: Let X and Y be formulas. Show that

(i) $X \equiv Y$ iff $\models X \leftrightarrow Y$ iff ($X \models Y$ and $Y \models X$)

(ii) $\models X$ iff $X \equiv \top$ iff $\top \models X$ iff $\emptyset \models X$ iff $\neg X \equiv \bot$ iff $\neg X \models \bot$.

(iii) X is unsatisfiable iff $X \equiv \bot$ iff $\neg X \equiv \top$ iff $X \models \bot$ iff $\models \neg X$.

Thm (Paradox of material implication). A set of formulas Σ is unsatisfiable iff $\Sigma \models X$ for each formula X.

Thm (M: monotonicity). Let $\Sigma \subseteq \Gamma$ be sets of formulas, and let X be a formula.

(1) If Γ is satisfiable, then Σ is satisfiable.

(2) If $\Sigma \models X$, then $\Gamma \models X$.

Thm (RA: Reductio ad absurdum). Let Σ be a set of formulas, and let X be a formula.

(1) $\Sigma \models X$ iff $\Sigma \cup \{\neg X\}$ is unsatisfiable

(2) $\Sigma \models \neg X$ iff $\Sigma \cup \{X\}$ is unsatisfiable.

Thm (DT: Deduction Theorem) Let Σ be a set of formulus. Let X and Y be formulus. $\Sigma \models X \to Y$ iff $\Sigma \cup \{X\} \models Y$.

~~Eq~~ Equality sentences

Let X be a given formula. Let E be a binary predicate, which does not occure in X. We replace each occurence of $\approx$ with E in X, and call the resulting formula $\overline{X}$. Let D be a nonempty set. Let Φ be a function that associates all function symbols to partial functions on D, and all predicates to relations on D, preserving arrity. We take $\Phi(E) = \overline{E}$, which is a binary relation on D satisfying the following properties:

a) $\overline{E}$ is an equivalence relation.

b) If $(s,t) \in \overline{E}$, then for all $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n \in D$ and for each n-ary function $g: D^n \to D$,
$$(g(d_1, \ldots, d_{i-1}, s, d_{i+1}, \ldots, d_n),\ g(d_1, \ldots, d_{i-1}, t, d_{i+1}, \ldots, d_n)) \in \overline{E}.$$

c) If $(s,t) \in \overline{E}$, then for all $d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n \in D$, for each n-ary relation $R \subseteq D^n$, $(d_1, \ldots, d_{i-1}, s, d_{i+1}, \ldots, d_n) \in R$ implies $(d_1, \ldots, d_{i-1}, t, d_{i+1}, \ldots, d_n) \in R$.

Due to the relevance, we need only to describe the properties that $\overline{E}$ must satisfy with regard to the function symbols and the predicates occurring in X.

SESSIONAL ADDITIONAL SHEET

Semester: .... Branch: .... Section: ....

Subject: .... Roll No.: ....

our aim is to characterize those special properties of $\bar{E}$ by some first order formulus. we thus formulate the eaulity axioms as follows.

E1. $\forall x\, Exx$

E2. $\forall x \forall y\, (Exy \to Eyx)$

E3. $\forall x \forall \forall z\, (Exy \wedge Eyz \to Exz)$.

E4. For any n-ary function symbol f,

$\forall x_1 \ldots, \forall x_n \forall y_1 \ldots \forall y_n$
$(Ex_1y_1 \wedge \ldots \wedge Ex_ny_n \to Ef(x_1 \ldots, x_n) f(y_1 \ldots, y_n))$

E5. For any n-ary predicate P,

$\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n$
$(Ex_1y_1 \wedge \ldots \wedge Ex_ny_n \to (Px_1 \ldots x_n \to Py_1 \ldots y_n))$.

<u>Example</u>: consider the set of formulus $A = \{\forall x\, (Pxa \to Qx),\ f(a) \approx b\}$. Following our proposal, we form the set

$A' = \{\forall x\, (Pxa \to Qx),\ Ef(a)b,\ \forall x\, Exx,$
$\forall x \forall y\, (Exy \to Eyx), \forall x \forall y \forall z\, (Exy \wedge Eyz \to Exz), \forall x \forall y\, (Exy \to Ef(x)f(y)),$
$\forall x \forall y \forall u \forall v\, (Exu \wedge Eyv \to (Pxy \to Puv)),$
$\forall x \forall u\, (Exu \to (Qx \to Qu))\}$.

If A has a model, extend this model (since E is new to A) by interpreting the predicate E as $=$. The extended model is a model of A'. Similarly, if A is valid, so is the conjunction of all sentences in A'. The reason is that the equality axioms are valid when E is replaced by $\approx$.

Conversely, suppose A' has a model. Here, E is interpreted as any binary predicate and not as the equality relation $=$. Thus, the elements of the domain related by E are not necessarily equal. Since E is an equivalence relation, we take its equivalence classes and then all elements related by this E become a single element, so to say. In the new domain of equivalence classes, E behave as '$=$'.

Let Σ be a set of formulas in at least one of which $\approx$ occurs. Assume that E is never used in any formula. Construct the sets Σ' and Σ_E as follows:

$\Sigma' = \{ Y : Y$ is obtained by replacing each occurrence of $(s \approx t)$ by Est in X, for $X \in \Sigma$ and for terms s and $t\}$.

$\Sigma_E = \{ \forall x\, Exx,\ \forall x \forall y (Exy \to Eyx),$
$\forall x \forall y \forall z (Exy \wedge Eyz \to Exz)\}$
$\cup \{ \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n (Ex_1y_1 \wedge \cdots \wedge Ex_ny_n \to Ef(x_1, \ldots, x_n)f(y_1, \ldots, y_n)) : f$ is an n-ary function symbol occurring in $\Sigma\}$
$\cup \{ \forall x_1 \cdots \forall x_n \forall y_1 \cdots \forall y_n (Ex_1y_1 \wedge \cdots \wedge Ex_ny_n \to (Px_1 \ldots x_n \to Py_1 \ldots y_n)) :$ P is an n-ary predicate occurring in $\Sigma\}$.

When $\Sigma = \{X\}$, we write Σ' as X' and Σ_E as X_E. We call the sentences in Σ_E as the equality sentences appropriate to Σ.

Let ℓ be a valuation under an interpretation $I = (D, \phi)$ of the set $\Sigma' \cup \Sigma_E$. Notice that

(10)

E is interpreted as $\phi(E)$. Suppose $I_\ell \models \Sigma' \cup \Sigma_E$. Since the first three sentences in Σ_E are true in I_ℓ, $\phi(E)$ is an equivalence relation on D. For each $d \in D$, write the equivalence class to which d belongs, as $[d]$. Let $[D]$ be the set of all equivalence classes of $\phi(E)$. That is,

$$[d] = \{s \in D : (d, s) \in \phi(E)\}, \quad [D] = \{[d] : d \in D\}$$

Define the interpretation $J = ([D], \psi)$, where the map ψ assigns function symbols and predicates to functions and relations over $[D]$ as in the following:

1. For any n-ary function symbol f, $\psi(f)$ is a partial function from $[D]^n$ to $[D]$ with
$$\psi(f)([d_1], \ldots, [d_n]) = [\phi(f)(d_1, \ldots, d_n)]$$

2. For any n-ary predicate P, $\psi(P) \subseteq [D]^n$ with $([d_1], \ldots, [d_n]) \in \psi(P)$ iff $(d_1, \ldots, d_n) \in \phi(P)$.

For any 0-ary predicates, i.e., propositions P, we declare it as a convention that $I \models P$ iff $J \models P$ (why?).

We define a valuation ℓ' under J corresponding to the valuation ℓ under I as in the following:

a) $\ell'(x) = [\ell(x)]$, for any variable x;

b) $\ell'(c) = [\ell(c)]$, for any constant c; and

c) $\ell'(f(t_1, \ldots, t_n)) = \psi(f)(\ell'(t_1), \ldots, \ell'(t_n))$

for any n-ary function symbol f and terms $t_1, \ldots, t_n$.

Lemma: Let X be a formula. Let $d \in D$, x a variable, and let t be a term. Construct X' and X_E corresponding to the formula X, as before.... Then

1) $\ell'[x \to [d]] = (\ell[x \to d])'$;

2) $\ell'(t) = [\ell(t)]$;

3) if $I_\ell \vDash \{X'\} \cup X_E$, then $J_{\ell'} \vDash X$.

Theorem: (Equality Theorem) Let Σ be a set of formulus. Let Σ' be the set of all formulus obtained from those of Σ by replaceing $\approx$ with E. Let Σ_E be the set of equality sentences appropriate to Σ. Then Σ is satisfiable iff $\Sigma' \cup \Sigma_E$ is satisfiable.

Proof: Let $I_\ell \vDash \Sigma$. The predicate E is new to Σ. For interpreting Σ', we extend I_ℓ by interpreting E as equality relation $=$. The equality sentences in Σ_E are satisfied by the extended I_ℓ. Moreover, extended I_ℓ also satisfies all the formulus in Σ'. Hence the extended I_ℓ is a model of $\Sigma' \cup \Sigma_E$.

Conversly, suppose $I_\ell \vDash \Sigma' \cup \Sigma_E$. Construct the equivalence classes and the interpretation $J_{\ell'}$ as is done for the above lemma (see before the lemma) by adding the equility sentences appropriate to Σ. ~~Since $I_\ell \vDash X$ for each $X \in \Sigma$, by the above lemma, $J_{\ell'} \vDash \{X'\} \cup X_E$ for each $X \in \Sigma$. Since $\Sigma' \cup \Sigma_E = \{X' : X \in \Sigma\} \cup \{X_E : X \in \Sigma\}$~~, we have $J_{\ell'} \vDash \Sigma' \cup \Sigma_E$. For each $X \in \Sigma$, since $I_\ell \vDash \Sigma' \cup \Sigma_E$, $I_\ell \vDash \{X'\} \cup X_E$ by monotonicity. Then $J_{\ell'} \vDash X$ by the above lemma. Thus $J_{\ell'} \vDash \Sigma$.

ADDITIONAL SHEET

Registration No.: ....   Roll No.: ....

Corollary: Let Σ be a set of formulas, and
let X be a formula. Let Σ' be the set
of all formulas obtained from those
of Σ by replacing $\approx$ with E. Let
$\Delta = (\Sigma \cup \{X\})_E$ be the set of all
equality sentences appropriate to
$\Sigma \cup \{X\}$. Then, $\Sigma \models X$ iff $\Sigma' \cup \Delta \models X'$.

Proof: $\Sigma \models X$ iff $\Sigma \cup \{\neg X\}$ is unsatisfiable
(by RA) iff $(\Sigma \cup \{\neg X\})' \cup \Delta$ is
unsatisfiable, by the above theorem.
However, $(\Sigma \cup \{\neg X\})' \cup \Delta = \Sigma' \cup \Delta \cup \{\neg X'\}$.
Thus by RA, $\Sigma' \cup \Delta \models X'$.

Example: Does there exist a sentence all
of whose models have m elements?
For $k=1$, the sentence $A_1 = \forall x \forall y (x \approx y)$
For $k=2$, the sentence A_2 is $\forall x_1 \forall x_2 \forall x_3$
$(((x_3 \approx x_1) \vee (x_3 \approx x_2)) \wedge \neg(x_1 \approx x_2))$.
In general, take A_m as follows:

$\forall x_1 \forall x_2 \ldots \forall x_{m+1} (((x_{m+1} \approx x_1) \vee \ldots (x_{m+1} \approx x_m))$
$\wedge \neg(x_1 \approx x_2) \wedge \ldots \wedge \neg(x_1 \approx x_m)$
$\wedge \neg(x_2 \approx x_3) \wedge \ldots \wedge \neg(x_2 \approx x_m)$
$\wedge \ldots \wedge \neg(x_{m-1} \approx x_m))$

Example: Let $\Sigma = \{\forall x \forall y (x \approx y)\}$. All
models of Σ are singleton sets. Now
$\Sigma' \cup \Sigma_E = \{\forall x Exx, \forall x \forall y Exy, \forall x \forall y (Exy \to Eyx),$
$\forall x \forall y \forall z (Exy \wedge Eyz \to Exz)\}$.
Consider any nonempty set D and the
interpretation $I = (D, \phi)$, with $\phi(E) = D \times D$.
We see that I is a model of $\Sigma' \cup \Sigma_E$.
This shows that $\Sigma' \cup \Sigma_E$ has models of
every cardinality.

Remark: From the above example, we can say that Equality Theorem and its corollary provide a way to eliminate the equality predicate without derailing satisfiability and validity. However, the predicate E along with the appropriate equality sentences (in Σ_E) do not quite account for the equality relation $=$. The reason is that the equivalence classes identify a bunch of elements as the same element but do not make them same element. The semantic equality is thus never captured in its entirety by a set of first order sentences without equality.

H.W: Consider the sentence $\forall x P x$. Show that it has models of all cardinalities.

H.W: Prove that in the absence of $\approx$, a satisfiable sentence can have models of arbitrary cardinalities.

Remark: Elimination of the equality predicate will come to help at many places later. It has two note worthy applications; one in proving completeness of an axiomatic system for FL, and two, in constructing a model in an abstract ~~fashion~~ fashion

Ex: Let X, Y and Z be three sentences. Answer the following:

a) If $X \models Y$, does it follow that $\neg X \not\models Y$?

b) If $X \wedge Y \models Z$, then does it follow that $X \models Y$ and $X \models Z$?

c) If $X \wedge Y \models Z$, then does it follow that $X \models Z$ and $X \models Z$?

d) If ~~$X \vee$~~ $X \models Y \vee Z$, then does it follow that $X \models Y$ and $X \models Z$?

e) If one of $X \models Y$ or $Z \models Y$ holds, then does $X \vee Z \models Y$ holds?

f) If $X \models (Y \to Z)$, then do $X \models Y$ and/or $X \models Z$ hold?

2. Show that the sentence $\forall x \exists y\, Pxy \wedge \forall x \neg Pxx \wedge \forall x \forall y \forall z\, (Pxy \wedge Pyz \to Pxz)$ is true in some infinite domain and is false in some finite domain.