Given matrices $B$, $C$ :

$$BC = \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_{C}$$

columns of $B$ span $C(A)$.

rows of $C$ span $R(A)$.

$$= \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{21} & b_{21}c_{12} + b_{22}c_{22} \end{bmatrix} = A$$

$$\underbrace{1^{st} \text{ column of}}_{\text{denoted by } A[:1]} A = \begin{bmatrix} b_{11}c_{11} + b_{12}c_{21} \\ b_{21}c_{11} + b_{22}c_{21} \end{bmatrix}$$

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$$= c_{11}\begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} + c_{21}\begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix}$$

linear combinations of columns of B with coefficients given by C.

$A[:2]$ = 2nd column of A.

$$= \begin{bmatrix} b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{12} + b_{22}c_{22} \end{bmatrix} = c_{12}\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} + c_{22}\begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix}$$

$$\underbrace{\alpha_1 A[:1] + \alpha_2 A[:2]}_{\text{columns of } B} \quad \text{span} \quad C(A).$$

$$A[1:] = \underbrace{b_{11}[c_{11} \quad c_{12}] + b_{12}[c_{21} \quad c_{22}]}_{\text{linear combinations of rows of } C\text{, with coefficients given by } B}$$

$$A[2:] = b_{21}[c_{11} \quad c_{12}] + b_{22}[c_{21} \quad c_{22}].$$

$$\alpha_1 A[1:] + \alpha_2 A[2:] \rightsquigarrow \text{row space } R(A)$$

Eg:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= BC = B \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

echelon form.

C: should not have redundant rows.

$R_3 \to R_3 - R_1$

$$A = IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ ;

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Echelon form

Now,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Echelon form of A.

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{C}$$

$$A[1:] = 1\,[1 \quad 0] + 0\,[0 \quad 1] + 0\,\underbrace{[0 \quad 0]}_{\text{redundant row.}}$$

$$A[2:] = 0\,[1 \;\; 0] + 1\,[0 \;\; 1] + 0\,[0 \;\; 0]$$

$$A[3:] = 1\,[1 \;\; 0] + 1\,[0 \;\; 1] + 1\,[0 \;\; 0].$$

linear combination of $A[1:]$, $A[2:]$, $A[3:]$ gives $R(A)$

$\underbrace{\text{rank of row space}}_{\text{minimum number of rows which span } R(A).}$ $R(A)$ of $A$ = row rank of $A$

$= 2.$

$$A[:1] = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$A[:2] = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

redundant column.

linear combinations of $A[:1]$, $A[:2]$ gives $C(A)$.

Rank of column space $C(A)$ of $A$

$=$ min. no. of columns which span $C(A)$

$=$ column rank of $A$

$= 2$.

**Theorem:** Let $A$ be a $m \times n$ matrix. Then row rank of $A$ = column rank of $A$.

**Proof:** If $A = 0$, then

row rank of $A$ = column rank of $A$ = 0.

Let $A \neq 0$.

Take the smallest natural number '$r$' s.t

$$\boxed{A_{m \times n} = B_{m \times r} \, C_{r \times n}} \quad \text{——— (1)}$$

; $C$ is in Echelon form.

$\hookrightarrow$ BC-decomposition of $A$

Row rank of A = min. number of rows in C that are non-redundant.

$= r$ —— (2)

Column rank of A = min. number of columns in B that are non-redundant

$= r$ —— (3)

By (2) & (3),

row rank of A = column rank of A

$= r$

$= \underline{\text{rank}(A)}.$

Q. Find the basis for 4 fundamental subspaces of $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix}_{3\times 4}.$

Soln: $T : \underset{\text{domain}}{\mathbb{R}^4} \to \underset{\text{codomain}}{\mathbb{R}^3}$

$T(X) = \underset{3\times 4}{A}\ \underset{4\times 1}{X}$

domain $= \mathbb{R}^4$

$$= R(A^T) \oplus C(A^T).$$

codomain $= \mathbb{R}^3$

$$= R(A) \oplus C(A).$$