Result: Let $A_{n\times n}$ be a symmetric matrix.

Then i). All eigen values of A are real.

ii). Eigenvectors corresponding to distinct eigen values are orthogonal (that is, they form axes)

Proof: $\langle x, y \rangle = x^T y$ —— (1)

$\langle Ax, Ax \rangle \geq 0$ (Axiom)

$$\Rightarrow (AX)^T AX \geq 0$$ (by (1))

$$\Rightarrow (X^T A^T) AX \geq 0$$

$$\Rightarrow (X^T A^T A) X \geq 0$$

$$\Rightarrow (A^T A X)^T X \geq 0$$ (as $(AB)^T = B^T A^T$)

$$\Rightarrow \langle A^T A X , X \rangle \geq 0$$

$$\Rightarrow \langle A^2 X , X \rangle \geq 0 \quad (2)$$ (because $A = A^T$)

given A is symmetric.

i). $Ax = \lambda x$.

eigen value of A

$$\Rightarrow A^2x = A(Ax)$$

$$= A(\lambda x)$$

$$= \lambda Ax$$

$$= \lambda^2 x$$

eigen value of A^2.

By (2), $\langle \lambda^2 x, x \rangle \geq 0$

$$\Rightarrow \lambda^2 \underbrace{\langle x, x \rangle}_{\geqslant 0} \geqslant 0$$

$$\Rightarrow \lambda^2 \geqslant 0$$

If possible; $\lambda = u + iv$, where u, v are real.

we have $\lambda^2 \geqslant 0$

$$\Rightarrow (u+iv)^2 \geqslant 0$$

$$\Rightarrow u^2 - v^2 + 2iuv \geqslant 0$$

$$\Rightarrow uv = 0$$

$$\Rightarrow u = 0 \text{ or } v = 0$$

If $v = 0$; $\lambda = u$.

$\hookrightarrow$ real

If $u = 0$; $\lambda = iv$

Then $\lambda^2 \geq 0$

$\Rightarrow i^2 v^2 \geq 0$

$\Rightarrow -v^2 \geq 0$

$\Rightarrow v^2 \leq 0$

$\Rightarrow v = 0$

$\Rightarrow \lambda = 0$; is real.

$\therefore\ \lambda$ is real.

Symmetric matrix has all its eigenvalues as real numbers.

(ii) Take $\lambda_1 \neq \lambda_2$

$$AX = \lambda_1 X \quad ; \quad AY = \lambda_2 Y$$

To <u>prove</u> : $\langle X, Y \rangle = 0$

ie, $X^T Y = 0$.

$$\langle AX, Y \rangle = \langle \lambda_1 X, Y \rangle$$

$$= \lambda_1 \langle x, y \rangle$$

$$= \lambda_1 x^T y \quad \text{--- (1)}$$

Given A is symmetric $\Rightarrow A = A^T$.

$$\langle Ax, y \rangle = \langle A^T x, y \rangle \qquad (A = A^T)$$

$$= \langle x, Ay \rangle \qquad (\text{Associativity})$$

$$= \langle x, \lambda_2 y \rangle$$

$$= x^T (\lambda_2 y) = \lambda_2 (x^T y) \quad \text{--- (2)}$$

(1) − (2) ;

$$\lambda_1 x^T y - \lambda_2 x^T y = 0$$

$$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} x^T y = 0$$

$$\Rightarrow x^T y = 0 \quad (\text{as } \lambda_1 \neq \lambda_2)$$

$$\Rightarrow \underline{\underline{x \perp y.}}$$

Spectral theorem (Spectral decomposition of a symmetric matrix)

Take a symmetric matrix $A_{n\times n}$. Then there exists an orthogonal matrix P such that $A = PDP^T$ (A is orthogonally diagonizable).

Proof: Let $A_{n\times n}$ be a symmetric matrix.

$\Rightarrow$ All eigen values of A are real & corresponding eigenvectors are orthogonal.

If eigenvalues of A are not distinct, then we use Gram-Schmidt process to get orthonormal basis.

Let $\{u_1, u_2, \dots, u_n\}$ be unit eigen vectors of A or by the Gram-Schmidt process (in case of repeated eigenvalues).

Then $\langle u_i, u_j \rangle = 0$ for $i \neq j$ and

$$\langle u_i, u_i \rangle = 1.$$

Choose $P = [u_1, u_2, \dots, u_n] \longrightarrow$ orthogonal matrix

Consider $AP = A[u_1, u_2, \dots, u_n]$

$= [Au_1, Au_2, \dots, Au_n]$

$= [\lambda_1 u_1, \lambda_2 u_2, \dots, \lambda_n u_n]$

$$= [u_1, u_2, \dots, u_n]_{n\times n} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}_{n\times n}$$

Diagonal matrix D.

$= PD$

$$\Rightarrow AP = PD$$

$$\Rightarrow A = PDP^{-1}$$

$$= PDP^{T} \quad \text{(as } P \text{ is orthogonal)}$$

Explicit spectral thm:

$$A = PDP^{T}$$

$$= [u_1, u_2, \ldots, u_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

projectⁿ matrices of rank 1

projection matrix of rank 1

$$\Rightarrow \underset{\hookrightarrow \text{ symmetric}}{A_{n\times n}} = \underbrace{\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T}_{\text{linear combination of outer products}}$$

Take $H = u_i u_i^T$

Then $H^T = (u_i u_i^T)^T$

$= u_i u_i^T = H$

and $H^2 = u_i u_i^T (u_i u_i^T)$

$= u_i (u_i^T u_i) u_i^T$

$= u_i I u_i^T = u_i u_i^T = H$

$\Rightarrow$ H is a projection matrix

Note that rank(H) is 1.

$$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$v_{2\times 1} v^T_{1\times 2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$$

$\Rightarrow$ rank $(v_{2\times 1} v^T_{1\times 2}) = 1$ (each column is a multiple of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$)

Q. Find spectral decomposition of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Soln: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \quad ; \quad \begin{aligned} \lambda_1 + \lambda_2 &= 5 \\ \lambda_1 \lambda_2 &= 0 \end{aligned}$$

$$\Rightarrow (1-\lambda)(4-\lambda) - 4 = 0$$

$$\Rightarrow \lambda^2 - \underbrace{5}_{\text{trace}}\lambda + \underbrace{0}_{\text{det}} = 0$$

$$\Rightarrow \lambda(\lambda-5) = 0$$

$$\Rightarrow \lambda_1 = 5$$
$$\lambda_2 = 0$$

(distinct eigenvalues)

$$AX = \lambda X$$

$$\Rightarrow (A-\lambda I)X = 0$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For $\underline{\lambda_1 = 5}$: $\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

$$\Rightarrow \quad -4x + 2y = 0$$

$$2x - y = 0$$

Take $x = 1 \Rightarrow y = 2$.

$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $\lambda_2 = 0$: $\quad x + 2y = 0$

$$2x + 4y = 0$$

Take $y = 1 \Rightarrow x = -2$

$$u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$P = [u_1 \quad u_2]$$

$$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}.$$

$$A_{2\times 2} = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$

$$= 5 u_1 u_1^T + 0 \, u_2 u_2^T$$

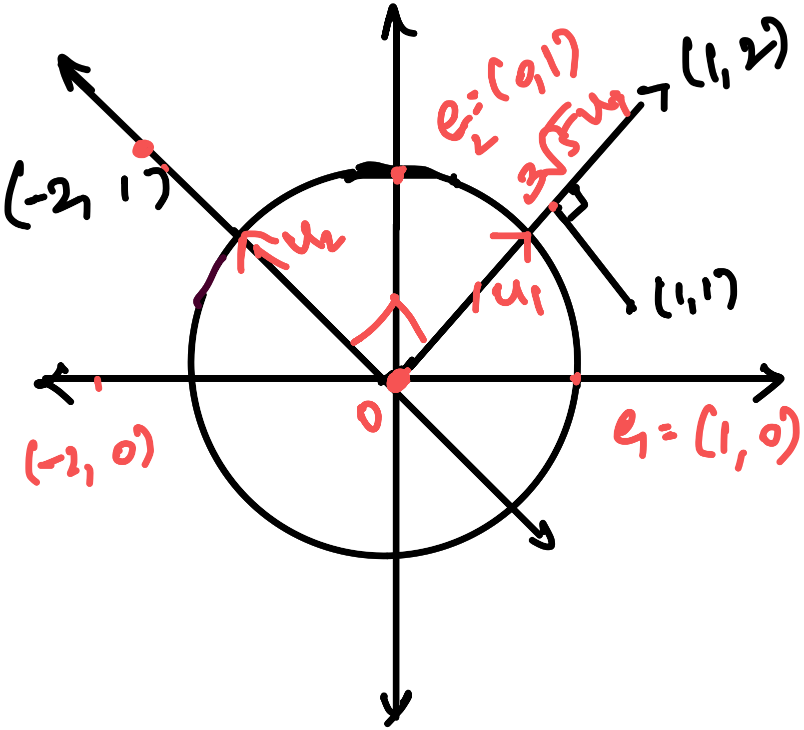
$$= 5 u_1 u_1^T.$$

$$u_1 u_1^T = \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}}_{2\times 1} \underbrace{\frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2\end{bmatrix}}_{1\times 2}$$

$$= \underbrace{\frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}}_{2\times 2}$$

$$\Rightarrow 5u_1u_1^T = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix} = A.$$

$$AX = 5\underbrace{(u_1u_1^T)}_{\text{projection matrix which projects on } u_1.}X$$



$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 5(u_1 u_1^T)\, x$$

$$= 5 \cdot \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 5 \cdot \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}$$

$$= (3\sqrt{5}) \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 3\sqrt{5}\, u_1 .$$