Condition Number of a Matrix :

$$K(A) = \|A\| \, \|A^{-1}\| .$$

challenging to compute.

$$\|AB\| \le \|A\| \, \|B\|$$ (sub-multiplicative property)

Take $B = A^{-1}$. Then

$$\|AA^{-1}\| \le \|A\| \, \|A^{-1}\|$$

$$\Rightarrow K(A) \ge \|AA^{-1}\| = \|I\| = 1.$$

$$\Rightarrow K(A) \ge 1$$ (condition number is atleast 1)

If $AX = \lambda X$; $\lambda \neq 0$

$\Rightarrow A^{-1}AX = \lambda A^{-1}X$

$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$

If $\lambda \neq 0$ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Also, $Av_i = \sigma_i u_i$ where σ_i is a singular value of A

$A^{-1}Av_i = \sigma_i A^{-1}u_i$

$v_i = \sigma_i A^{-1}u_i$

$$A^{-1}u_i = \frac{1}{\sigma_i} v_i$$

If $0 \neq \sigma_i$ is a singular value of A, then $\frac{1}{\sigma_i}$ is a singular value of A^+.

$$\kappa(A) := \|A\|_2 \|A^+\|_2$$

$$= \sigma_{max}(A) \cdot \frac{1}{\sigma_{min}(A)}$$

$$= \frac{\sigma_{max}(A)}{\sigma_{min}(A)}.$$

Eg : $A = \begin{bmatrix} 80 & -41 \\ 40 & -21 \end{bmatrix}$.

(I). <u>Column norm</u> :

$$K(A) = \|A\|_1 \|A^{-1}\|_1$$

where $\|A\|_1 = \max\{|80| + |40|, |-41| + |-21|\}$

$= \max\{120, 62\} = 120.$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} -21 & 41 \\ -40 & 80 \end{bmatrix}.$$

$$\|A^{-1}\|_1 = \max\left\{ \left|\frac{-21}{40}\right| + |-1| \; , \; \left|\frac{41}{40}\right| + |2| \right\}$$

$$= \frac{41}{40} + 2 = \frac{121}{40}$$

$$K(A) = \|A\|_1 \, \|A^{-1}\|_1 = 120 \times \frac{121}{40} = 3 \times 121$$

$$= 363 \gg 1.$$

$\therefore$ A is ill-conditioned.

(II). <u>Row norm</u> :

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty \quad ; \quad \text{where}$$

$$\|A\|_\infty = \max\{|80| + |-41|,\ |40| + |-21|\}$$

$$= \max\{121,\ 61\}$$

$$= 121$$

$$\|A^{-1}\|_\infty = \max\left\{\left|-\frac{21}{40}\right| + \left|\frac{41}{40}\right|,\ |-1| + |2|\right\}$$

$$= 3$$

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = 121 \times 3 = 363 >> 1.$$

$\therefore$ A is ill-conditioned.

(III). <u>2-Norm</u>:

$$K(A) = \|A\|_2 \|A^{+}\|_2$$

$$= \|A\|_2 \|A^{-1}\|_2$$

$$= \frac{\sigma_{max}(A)}{\sigma_{min}(A)}.$$

$$AA^T = \begin{bmatrix} 80 & -41 \\ 40 & -21 \end{bmatrix} \begin{bmatrix} 80 & 40 \\ -41 & -21 \end{bmatrix}$$

$$= \begin{bmatrix} 8081 & 4061 \\ 4061 & 2041 \end{bmatrix}$$

$$\sigma_{max} = \sigma_1 = \sqrt{\lambda_1} = 100.607 \qquad \sigma_{min} = \sigma_2 = \sqrt{\lambda_2} = 0.3976$$

$$K(A) = \frac{100.607}{0.3976} = 253.0357 >> 1.$$

$\therefore$ A is ill-conditioned.

Data Analysis by PCA and Dimension Reduction:

Principal Component Analysis (PCA)

↓

Orthogonal linear transformation

↓

Transforms the data into new co-ordinate system

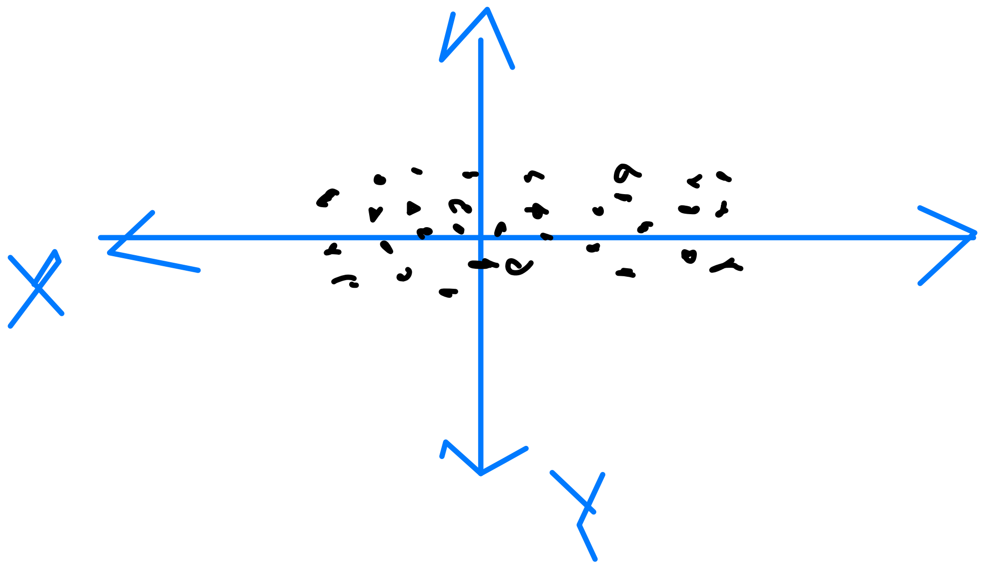
↓ such that

the greatest variance of some scalar projection of data comes to lie on the first co-ordinate (called principal component PC1), the 2nd greatest on the 2nd co-ordinate (called PC2),

. . .

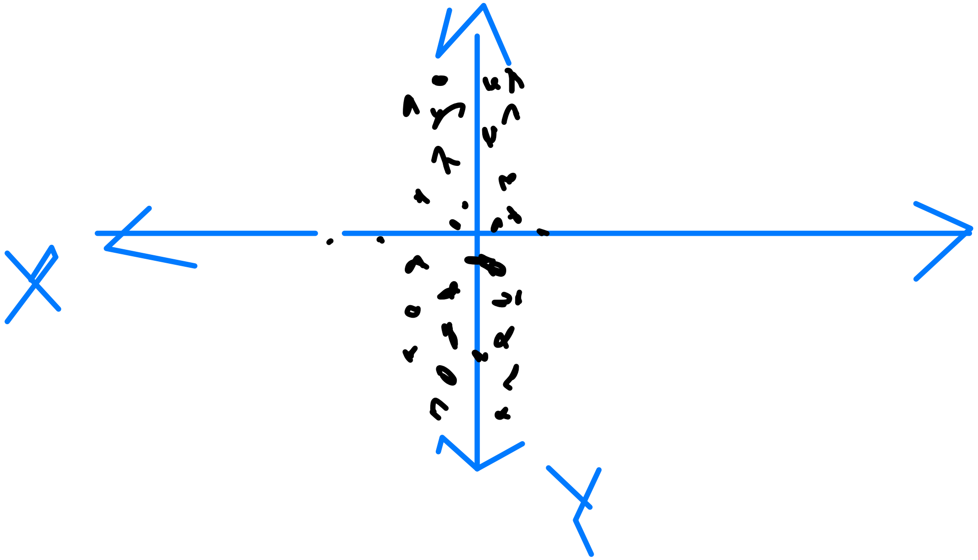
so on.

Recall that



① var X > var Y

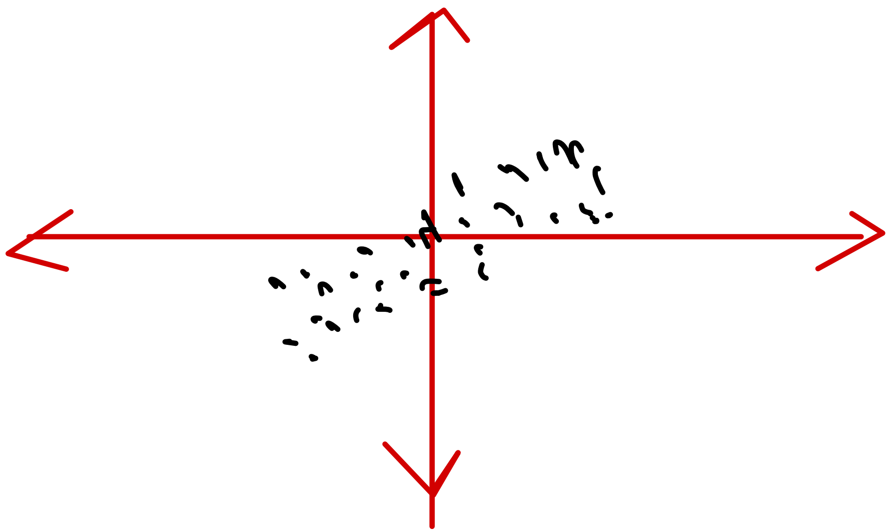
② var Y > var X.



① $\mathrm{var}\ X > \mathrm{var}\ Y$

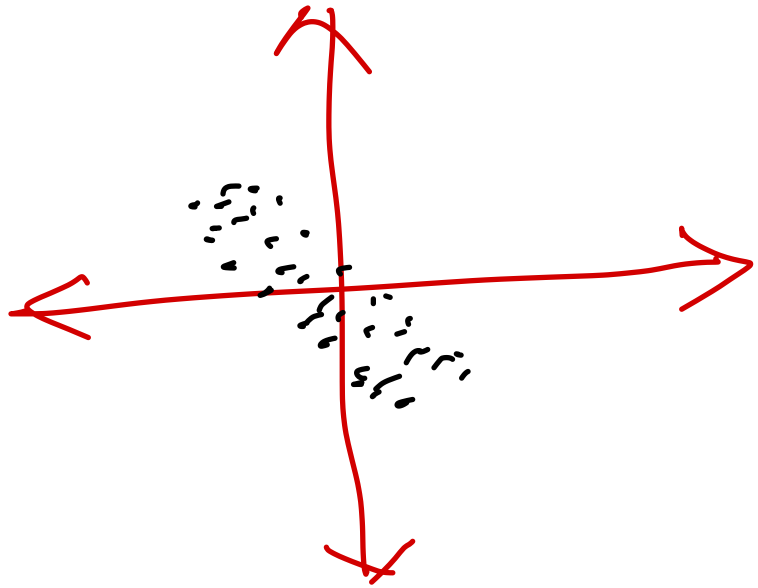
② $\mathrm{var}\ Y > \mathrm{var}\ X$.

$$\rho \text{ or } r = \frac{\mathrm{Cov}(X,Y)}{\sqrt{V(X)\,V(Y)}}$$ (correlation coefficient)



1

r or $\mathrm{Cov}(X,Y) > 0$.

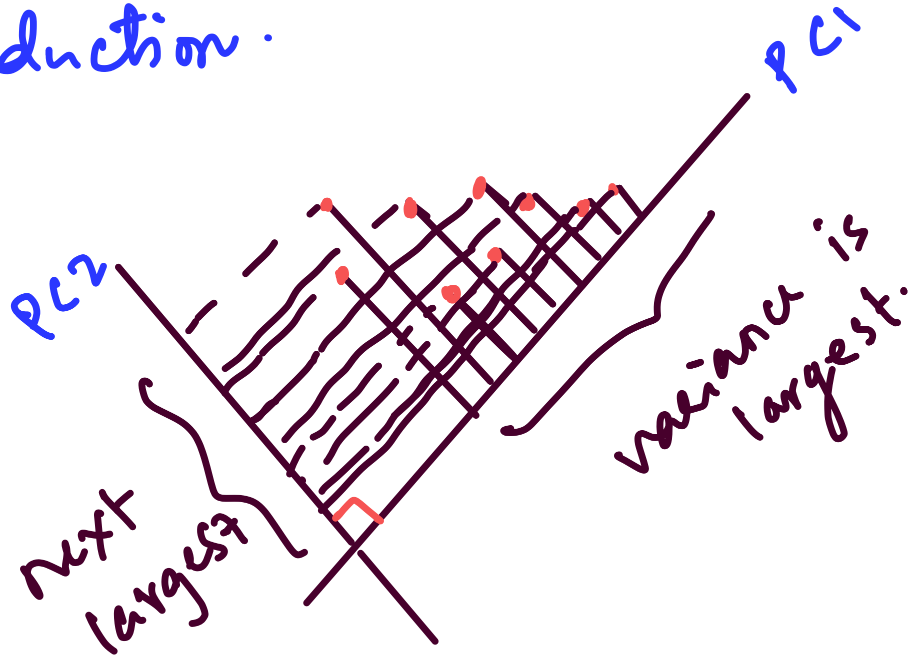


2

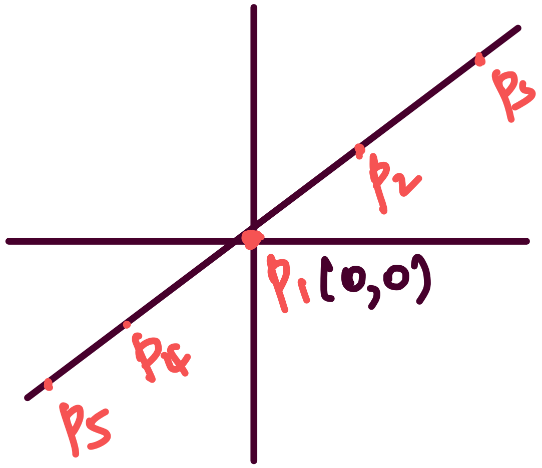
r or $\mathrm{Cov}(X,Y) < 0$.

Principal component Analysis (PCA):

↳ Dimension reduction.



Given $X = \begin{array}{c|c} & \begin{array}{ccccc} p_1 & p_2 & p_3 & p_4 & p_5 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix}_{2\times 5} \end{array}$. Reduce dim. of X to 1D.



step 1 :

$$\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{1i}$$ (row average) ; n = no. of data points

$$= \frac{1}{5}(0) = 0$$

$$\bar{x}_2 = \frac{1}{n}\sum_{i=1}^{n} x_{2i} = \frac{1}{5}(0) = 0$$

Centered data $X_c = X - \bar{X}$

$$= \begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix} - \begin{bmatrix} \bar{x}_1 & \bar{x}_1 & \bar{x}_1 & \bar{x}_1 & \bar{x}_1 \\ \bar{x}_2 & \bar{x}_2 & \bar{x}_2 & \bar{x}_2 & \bar{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix}$$

$= X$.

(change of origin)

Step 2: Do SVD of X_c.

$X_c = X$.

$$\underset{2\times 5}{X_c}\underset{5\times 2}{X_c^T} = \begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ -1 & -1 \\ -2 & -2 \end{bmatrix}$$

called covariance matrix.

$$= \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}.$$ ← Covariance matrix of the centred data.

$$\lambda_1 + \lambda_2 = 20$$
$$\lambda_1 \lambda_2 = 0$$

$$\Rightarrow \lambda_1 = 20; \quad \lambda_2 = 0.$$

$$\Rightarrow \sigma_1 = \sqrt{20} \quad ; \quad \sigma_2 = 0.$$

Eigen vector for $\lambda_1 = 20$:

$$\begin{bmatrix} 10 - 20 & 10 \\ 10 & 10 - 20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$-10x + 10y = 0$$

$$10x - 10y = 0$$

$$\Rightarrow \quad x = y.$$

$$\therefore \quad u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Eigen vector for $\lambda_2 = 0$:

$$\begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$10x + 10y = 0 \quad \Rightarrow \quad x = -y.$$

$$\therefore \; u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\Sigma = \begin{bmatrix} \sqrt{20} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{0} & 0 & 0 & 0 \end{bmatrix}_{2\times 5} \; ; \quad U = \begin{array}{c} \begin{array}{cc} u_1 & \quad u_2 \end{array} \\ \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{2\times 2} \end{array}.$$

$$v_1' = X_c^T u_1$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ -1 & -1 \\ -2 & -2 \end{bmatrix}_{5\times 2} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 2 \\ 4 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ 4/\sqrt{2} \\ -2/\sqrt{2} \\ -4/\sqrt{2} \end{bmatrix}.$$

$$v_1' = \begin{bmatrix} 0 \\ \sqrt{2} \\ 2\sqrt{2} \\ -\sqrt{2} \\ -2\sqrt{2} \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$$

$$v_1 = \frac{1}{\sqrt{0^2+1^2+2^2+(-1)^2+(-2)^2}} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}.$$

Reduced SVD of X_c :

$$X_c = \underset{2\times 2}{U}\;\underset{2\times 5}{\Sigma}\;\begin{array}{c} v_1 \\ \begin{bmatrix} 0 & * & * & * & * \\ 1/\sqrt{10} & * & * & * & * \\ 2/\sqrt{10} & * & * & * & * \\ -1/\sqrt{10} & * & * & * & * \\ -2/\sqrt{10} & * & * & * & * \end{bmatrix}_{5\times 5} \end{array}$$

Step 3: Interpretation of step 2 using projections.