Continuous Random Variable: A random variable X is said to be continuous if it can take all possible values between certain limits, here the range space of X is infinite. Therefore the probability distribution function named for such random variable is Probability density function (PDF), which is defined as the pdf of X is a function $f(x)$ satisfying the following properties i) $f(x) \geq 0$

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

iii) $\Pr\{a \leq X \leq b\} = \int_a^b f(x)dx$ for any a, b such that $-\infty < a < b < \infty$.

Note: 1. If X is a continuous random variable with pdf f(x), then

$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b) = \int_a^b f(x)dx$.

2. $P(X = a) = 0$, if X is a continuous random variable.

Cumulative distribution function: Let X be random variable (discrete or continuous), we define F to be the cumulative distribution function of a random variable X given by $F(x) = \Pr\{X \leq x\}$.

Case i) If X is discrete random variable then

$$F(t) = \Pr\{X \leq t\} = P(x_1) + P(x_2) + \cdots + P(t)$$

Case ii) If x is a continuous random variable then $F(x) = \Pr\{X \leq x\} = \int_{-\infty}^{x} f(x)dx$.

Two dimensional random variable: Let E be an experiment and S be a sample space associated with E. Let X=X(s) and Y=Y(s) be two functions each assigning a real number to each outcome s of S. We call (X, Y) to be two dimensional random variable.

Discrete 2D: If the possible values of (X, Y) are finite or countably infinite then (X, Y) is called discrete and it is defined as $P(x_i, y_j)$ satisfying the following condition,

i) $P(x_i, y_j) \geq 0$ and

ii) $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} P(x_i, y_j) = 1$. The function $P(x_i, y_j)$ defined is called as Joint probability distribution function (Jpdf).

Continuous 2D: If (X, Y) is a continuous random variable assuming all values in some region R of the Euclidean plane, then the Joint probability density function $f(x, y)$ is a function satisfying the following conditions

i) $f(x, y) \geq 0$ for all (x, y)ϵR

ii) $\iint f(x, y)dx\,dy = 1$ over the region R.

Marginal Probability distribution: The marginal probability distribution is defined as

Case i) In the discrete (X, Y), it is defined as $p(x_i) = P\{X = x_i\} = \sum_{j=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of X. Similarly $q(y_j) = P\{Y = y_j\} = \sum_{i=1}^{\infty} P(x_i, y_j)$ is the marginal probability distribution of Y.

Case ii) In the continuous (X, Y), it is defined as the marginal probability function of X is defined as $g(x) = \int_{-\infty}^{\infty} f(x, y)dy$ and the marginal probability function of Y is defined as $h(y) = \int_{-\infty}^{\infty} f(x, y)dx$.

To calculate the conditional probability:

Case i) Discrete: Probability of x_i given y_j is defined as $= \frac{P(x_i,y_j)}{q(y_j)}$, $q(y_j) > 0$

Probability of y_j given x_i is defined as $= \frac{P(x_i,y_j)}{p(x_i)}$, $p(x_i) > 0$

Case ii) Continuous: The pdf of X for given Y=y is $= \frac{f(x,y)}{h(y)}, h(y) > 0$

The pdf off Y for given X=x is $= \frac{f(x,y)}{g(x)}, g(x) > 0$.

Independent Random variable: If X and Y are independent random variable then two dimensional random variable in case of discrete is defined as $P(x_i, y_j) = p(x_i).q(y_j)$ for all the values of i and j. In case of Continuous it is defined as $f(x,y) = g(x).h(y)$.

Mathematical Expectation: If X is a discrete random variable with pmf p(x), then the expectation of X is given by $E(X) = \sum_x xp(x)$, provided the series is absolutely convergent.
If X is continuous with pdf f(x), then the expectation of X is given by $E(X) = \int xf(x)dx$, provided $\int |x|f(x)dx < \infty$.
Variance of X is given by $V(X) = E(X - E(X))^2 = E(X^2) - \big(E(X)\big)^2$.
Chebyshev's inequality: Let x be random variable with mean μ and variance σ^2 then for any positive real number k(k>0)

$$p\{|x-\mu| \geq k\} \leq \frac{\sigma^2}{k^2} \text{ (Upper bound)}$$

$$p\{|x-\mu| < k\} > 1 - \frac{\sigma^2}{k^2} \text{ (Lower bound)}$$

Note: some other forms

1. $p\{|x-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$ and $p\{|x-\mu| < k\sigma\} \geq 1$ (Upper bound)
2. $p\{|x-\mu| \geq \in\} \leq \frac{1}{\in^2} E(x-c)^2$ and $p\{|x-\mu| < \in\} \geq 1$

DISTRIBUTIONS:

Distribution	PMF/PDF	Mean	Variance
Binomial distribution $X \sim B(n,p)$	$P(x) = {}^nC_k p^k (1-p)^{n-k} \quad ,k = 0,1,2,...,n$	$E(x) = np$	$V(x) = np(1-p)$
Poisson's Distribution $X \sim P(\alpha)$	$P(x) = \frac{e^{-\alpha}\alpha^k}{k!}, \; k = 0,1,2,..., \alpha > 0$	$E(x) = \alpha = np$	$V(x) = \alpha = np$
Uniform Distribution $X \sim U(a,b)$	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$	$E(x) = \frac{b+a}{2}$	$V(x) = \frac{(b-a)^2}{12}$
Normal Distribution $X \sim N(\mu, \sigma^2)$	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad -\infty < x, \mu < \infty, \; \sigma > 0$	$E(x) = \mu$	$V(x) = \sigma^2$
Exponential Distribution $X \sim E(\lambda)$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & otherwise \end{cases}$	$E(x) = \frac{1}{\lambda}$	$V(X) = \frac{1}{\lambda^2}$
Gamma Distribution $X \sim G(r,\alpha)$	$f(x) = \begin{cases} \frac{x^{r-1}e^{-\alpha x}\alpha^r}{\Gamma(r)}, & x > 0, \; \alpha, r > 0 \\ 0, & elsewhere \end{cases}$	$E(x) = \frac{r}{\alpha}$	$V(x) = \frac{r}{\alpha^2}$

Chi-square Distribution $X \sim \chi^2(n)$	$f(x)=\begin{cases}\dfrac{x^{\frac{n}{2}-1}e^{-\frac{x}{2}}}{\Gamma(n/2)2^{\frac{n}{2}}}, & x>0\\ 0, & elsewhere\end{cases}$	$E(x)=n$	$V(x)=2n$

Uniform distribution on a two dimensional set: If R is a set in the two-dimensional plane, and R has a finite area, then we may consider the density function equal to the reciprocal of the area of R inside R, and equal to 0 otherwise:

$$f(x,y)=\begin{cases}\frac{1}{area\ R}; & if\ (x,y)\in R\\ 0 & Otherwise\end{cases}.$$

Covariance: $Cov(x,y)=E(xy)-E(x)E(y)$

Correlation coefficient: $\rho_{xy}=\rho=\dfrac{E(xy)-E(x)E(y)}{\sqrt{V(x)V(y)}}$

Properties:

1. $E(c)=$ c, where c is a constant.
2. $V(c)=0$, where c is a constant.
3. If $E(xy)=0$ then x and y are orthogonal.
4. $V(Ax+b)=A^2V(x)$ when Ax+B is linear function of x.
5. If $\rho=0$ then x and y are un correlated.
6. $V(Ax+by)=A^2V(x)+B^2V(y)+2ABCOV(x,y)$

FUNCTIONS OF ONE DIMENSIONAL RANDOM VARIABLES

Let S be a sample space associated with a random experiment E, then it is known that a random variable X on S is a real valued function, i.e., $X: S\to R$, for each element s $\in$ S, there is a real number associated.

Let X be a random variable defined on S. Let $y=H(x)$ is a real valued function of x. Then $Y=H(X)$ is a random variable on S. i.e., for each element s $\in$ S, there is a real number associated, say $y=H(X(s))$. Here Y is called a function of the random variable X.

Notations:

1. R_X – the set of all possible values of the function X, called the **range space** of the random variable X.
2. R_Y – the set of all possible values of the function $Y=H(X)$, called the **range space** of the random variable Y.

Equivalent Events: Let C be an event associated with the range space R_Y. Let $B\subset R_X$ defined by $B=\{x\in R_X; H(x)\in C\}$, then B and C are called equivalent events.

Distribution function of functions of random variables:

Case 1: Let X be a discrete random variable with p.m.f. $p(x_i)=P(X=x_i)$ for $i=1,2,3,\ldots$ Let $Y=H(X)$ then Y is also a discrete random variable. If $Y=H(X)$ is a one to one function then the probability distribution of Y is as follows:

For the possible values of $y_i=H(x_i)$ for $i=1,2,3,\ldots$. The p.m.f. of $Y=H(X)$ is $q(y_i)=P(Y=y_i)=P(X=x_i)=p(x_i)$ for $i=1,2,3,\ldots$.

Case 2: Let X be a discrete random variable with p.m.f. $p(x_i)=P(X=x_i)$ for $i=1,2,3,\ldots$ Let $Y=H(X)$ then Y is also a discrete random variable. Suppose tat for one value of $Y=y_i$ there corresponds several values of X say $x_{i_1},x_{i_2},\ldots,x_{i_j},\ldots$ then the p.m.f. of of $Y=H(X)$ is

$$q(y_i)=P(Y=y_i)=p(x_{i_1})+p(x_{i_2})+\cdots+p\left(x_{i_j}\right)+\cdots$$

Sampling Theory:

Note: Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2)$. Then

1. $\frac{(X-\mu)\sqrt{n}}{\sigma}\sim N(0,1)$
2. $\frac{nS^2}{\sigma^2}\sim\chi^2(n-1)$.

Estimation: (1- α) 100% Confidence interval for mean when σ^2 is known is $\left(\bar{X}-\frac{a\sigma}{\sqrt{n},},\bar{X}+\frac{a\sigma}{\sqrt{n},}\right)$, and $\left(\bar{X}-\frac{aS}{\sqrt{(n-1)},},\bar{X}+\frac{aS}{\sqrt{(n-1)},}\right)$.