

Singular Value Decomposition (SVD).

Let $A_{m \times n}$ be a wide matrix ($m < n$). Then there exists orthogonal matrices $U_{m \times m}$, $V_{n \times n}$ and rectangular matrix Σ with diagonal entries as singular values of A s.t

$$A = U \sum V^T$$

↑ ↗ ↗
rotation scaling rotation/reflection
or reflection

$$AA^T \xrightarrow[m \times n \times m]{\rightarrow m \times m} \downarrow U$$

smaller

$$A^T A \xrightarrow[n \times m]{\rightarrow n \times n} \downarrow V$$

larger

Proof: The nonzero eigen values of AA^T and A^TA coincide and as the matrices are symmetric, the eigen values are always real.

Take $\lambda_1 \neq 0$ s.t $AA^T x = \lambda_1 x$ (nonzero eigen value of AA^T)

$$AA^T u_1 = \lambda_1 u_1 \quad \text{--- (1)}$$

$$\begin{aligned} A^T AA^T \underbrace{u_1}_v &= A^T \lambda_1 u_1 \\ &= \lambda_1 \underbrace{A^T u_1}_v \end{aligned}$$

$\Rightarrow A^T u_1$ is an eigen vector of $A^T A$.

Take $v_1 = \frac{A^T u_1}{\|A^T u_1\|}$ — (2)

Form $U = [u_1, u_2, \dots, u_m]_{m \times m}$

$$V = [v_1, v_2, \dots, v_n]_{n \times n}.$$

Consider $\|A^T u_1\|^2 = \langle A^T u_1, A^T u_1 \rangle$

$$= (A^T u_1)^T A^T u_1$$

$$= u_1^T (A^T)^T A^T u_1$$

$$= u_1^T A A^T u_1$$

$$= u_r^T \lambda_1 u_r, \quad (\text{by (i)})$$

$$= \lambda_1 u_r^T u_r$$

$$= \lambda_1 \|u_r\|^2$$

$$= \lambda_1 \cdot 1 \quad (\text{as } u_r \text{ is unit vector})$$

$$= \lambda_1.$$

Put in (2).

$$v_1 = \frac{A^T u_1}{\sqrt{\lambda_1}}.$$

$$\Rightarrow A v_1 = \frac{A A^T u_1}{\sqrt{\lambda_1}}$$

$$= \frac{\lambda_1 u_1}{\sqrt{\lambda_1}}$$

$$= \sqrt{\lambda_1} u_1$$

$$= \sigma_1 u_1 \quad ; \quad \text{where } \sigma_1 = \sqrt{\lambda_1}$$

↑
singular value

$$\therefore A v_1 = \sigma_1 u_1$$

$$A v_2 = \sigma_2 u_2$$

$$A v_3 = \sigma_3 u_3$$

$$A v_m = \sigma_m u_m$$

$$A v_{m+1} = 0.0$$

:

:

$$A v_n = 0.0$$

vectorizing this leads to decomposition of A.

where $\sigma_i = \sqrt{\lambda_i}$, $i = 1, 2, \dots, m$ and

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0.$$

$$A_{m \times n} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}_{n \times n} = \begin{bmatrix} u_1 & u_2 & \dots & u_m \end{bmatrix}_{m \times m} \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \sigma_m \\ & & & & \end{bmatrix}_{m \times n}$$

(vectorization of above n equations)

$$\Rightarrow Av = U\Sigma$$

$$\Rightarrow A = U\Sigma V^{-1}$$

$$= U\Sigma V^T \quad (\text{as } V \text{ is orthogonal, } V^{-1} = V^T)$$

$$\therefore \boxed{A = U\Sigma V^T.}$$

Suppose B be a tall matrix.

Then $B^T = A$ is a wide matrix.

$$\Rightarrow A = U\Sigma V^T$$

$$\Rightarrow B^T = A^T = (U\Sigma V^T)^T$$

$$B = V \Sigma^T U^T.$$

↳ gives SVD of B.

$$AX = \lambda X$$

$$A u_i = \lambda_i u_i$$

↓
square
matrix

eigen
value

generalized to $A u_i = \sigma_i u_i$

↓
rectangular
matrix

singular
value

Q. Find SVD of $A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}_{2 \times 3}$.

S1: $A^T A \rightarrow 3 \times 3$

$A A^T \rightarrow 2 \times 2$

$$A A^T = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 24 \\ 24 & 40 \end{bmatrix}.$$

$$\lambda_1 + \lambda_2 = 80$$

$$\lambda_1 \lambda_2 = \det(AA^T)$$

$$= 1024$$

$$\Rightarrow \lambda_1 = 64 ; \lambda_2 = 16$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{64} = 8$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{16} = 4.$$

For $\lambda_1 = 64$:

$$\begin{bmatrix} 40 - \lambda_1 & 24 \\ 24 & 40 - \lambda_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-24x + 24y = 0$$

$$24x - 24y = 0$$

$$\Rightarrow x = y.$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 16$:

$$24x + 24y = 0$$

$$24x - 24y = 0$$

$$\Rightarrow x = -y$$

$$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$U = [u_1 \ u_2]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{where } \theta = 45^\circ$$

(rotation in anti clockwise)

Nm. zero eigen values of AAT and ATA coincide.

$$V = A^T X$$

$$v_1 = A^T u_1 = \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3×2

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 16 \\ 0 \\ 16 \end{bmatrix}$$

$$= \beta \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = A^T u_2 = \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$$= P \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v \perp v_2$$

$$A^T A = \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 & 32 \\ 0 & 16 & 0 \\ 32 & 0 & 32 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A^T A)$$

$$\begin{matrix} \downarrow & \downarrow \\ 64 & 16 \end{matrix} = 80$$

$$\Rightarrow \lambda_3 = 0.$$

$$\begin{bmatrix} 32 - 0 & 0 & 32 \\ 0 & 16 - 0 & 0 \\ 32 & 0 & 32 - 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 32x + 0y + 32z = 0 \\ 16y = 0$$

$$32x + 0y + 32z = 0$$

$$\Rightarrow y = 0 \\ x + z = 0 \quad \Rightarrow -x = z$$

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$V = [v_1 \ v_2 \ v_3]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array}$$

$$A_{3 \times 2} = U \Sigma V^T$$

scaling

$$: \begin{bmatrix} u_1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}_{3 \times 2} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2 \times 3} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}_{3 \times 3} \begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array}$$

rotation in
xy-plane (45° , anticlockwise)

↳ rotation (?)