QR decomposition :

$$A = QR$$

orthogonal matrix $(QQ^T = Q^TQ = I)$

Product of two orthogonal matrices A, B is orthogonal.

Proof : $(AB)(AB)^T = (AB)(B^TA^T)$

$$= A(BB^T)A^T$$

$$= AIA^T = AA^T = I.$$

$$(AB)^T(AB) = B^T A^T (AB)$$

$$= B^T (A^T A) B$$

$$= B^T I B = B^T B = I.$$

Q. Find QR decomposition of $A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 11 & 5 \\ 0 & 12 & 2 \end{bmatrix}$.

$\begin{array}{c} x \\ y \end{array}$

Soln:

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{3^2 + 4^2}$$

$$= 5.$$

Givens formula: $c = \frac{x}{r} = \frac{3}{5}$

$$s = -\frac{y}{r} = -\frac{4}{5}.$$

$$G_1 A = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 4 & 11 & 5 \\ 0 & 12 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 4 \\ 0 & 5 & 3 \\ 0 & 12 & 2 \end{bmatrix}.$$

$5 \to x$, $12 \to y$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} 5 \\ 12 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \quad ; \quad r = \sqrt{x^2+y^2} = \sqrt{5^2+(12)^2}$$

$$= 13$$

$$c = \frac{x}{r} = \frac{5}{13}$$

$$s = \frac{-y}{r} = -\frac{12}{13}.$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5/13 & 12/13 \\ 0 & -12/13 & 5/13 \end{bmatrix}}_{G_2} \begin{bmatrix} 5 & 10 & 4 \\ 0 & 5 & 3 \\ 0 & 12 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 4 \\ 0 & 13 & 3 \\ 0 & 0 & -2 \end{bmatrix} = R$$

$\hookrightarrow$ upper triangular

ie $G_2 G_1 A = R$.

$$\Rightarrow A = \underbrace{G_1^T G_2^T}_{Q} R \quad ; \quad \text{where}$$

(orthogonal)

$$Q = G_1^T \, G_2^T$$

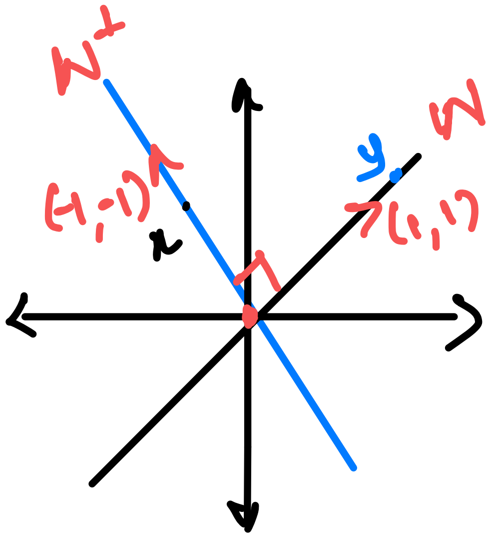
$$= \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5/13 & -12/13 \\ 0 & 12/13 & 5/13 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 & -4/13 & 48/65 \\ 4/5 & 3/13 & -36/65 \\ 0 & 12/13 & \underbrace{25/65}_{5/13} \end{bmatrix}.$$

Then $QR = A$.

Take W as a subspace of V.

$W \subseteq V$



$$W^{\perp} = \{x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in W\}$$

↳ orthogonal complement of a subspace W.

Eg: $W = \{\alpha(1,1) \mid \alpha \in \mathbb{R}\}$

$$W^{\perp} = \{x \in V = \mathbb{R}^2 \mid \langle x, y \rangle = 0 \text{ for all } y \in W\}$$

$$= \{(x_1, x_2) = x \in \mathbb{R}^2 \mid \langle (x_1, x_2), \alpha(1,1) \rangle = 0\}$$

$$= \{(x_1, x_2) = x \in \mathbb{R}^2 \mid \alpha x_1 + \alpha x_2 = 0\}$$

$$= \{(x_1, x_2) \mid x_i \in \mathbb{R}\}$$

$$= \{(x_1, -x_1) \mid x_1 \in \mathbb{R}\}$$

(as $\alpha x_1 + \alpha x_2 = 0$
$\Rightarrow \alpha(x_1 + x_2) = 0$

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$= \{x_1(1,-1) \mid x_1 \in \mathbb{R}\}.$$

$\Rightarrow x_1 + x_2 = 0$

$\Rightarrow x_2 = -x_1)$

Result 1: $W \cap W^{\perp} = \{0\}$.

Proof: Take $t \in W \cap W^{\perp}$

$\Rightarrow t \in W$ and $t \in W^{\perp}$.

$\Rightarrow \langle t, t \rangle = 0$.

$\Rightarrow t = 0$

$\Rightarrow W \cap W^{\perp} \subseteq \{0\}$ —— (1)

To prove $\{0\} \subseteq W \cap W^\perp$; take $y \in \{0\}$

$\Rightarrow$ $y = 0 \in W$ (as W is a subspace of V and every subspace contains 0)

Now, $\langle 0, x \rangle = 0$ for all $x \in W$

$\Rightarrow 0 \in W^\perp$.

$\Rightarrow 0 \in W \cap W^\perp$ (as $0 \in W$ and $0 \in W^\perp$).

$\Rightarrow \{0\} \subseteq W \cap W^\perp$ —— (2)

From (1) & (2), $W \cap W^\perp = \{0\}$.

Result 2 : $W^{\perp}$ is a subspace of V.

Proof : (i). Take $x, y \in W^{\perp}$.

$$\left.\begin{array}{l} \Rightarrow \langle x, t\rangle = 0 \quad \text{and} \\ \langle y, t\rangle = 0 \quad ; \text{ for all } t \in W. \end{array}\right\} \text{(1)}$$

Consider $\langle x+y, t\rangle = \langle x, t\rangle + \langle y, t\rangle$

$= 0 + 0$

$= 0 \quad ; \text{ for all } t \in W.$

$\Rightarrow x+y \in W^{\perp}.$

ie, $W^\perp$ is closed under $+$.

(ii). Take a scalar 'α'.

Consider $\langle \alpha x, t \rangle = \alpha \langle x, t \rangle$

$= \alpha \cdot 0$ (by ①)

$= 0$; for all $t \in W$.

$\Rightarrow \alpha x \in W^\perp$.

ie, $W^\perp$ is closed under scalar multiplication.

$\therefore W^\perp$ is a subspace of V (by (i) & (ii))

Direct sum of subspaces.

If $V = V_1 + V_2$, V_1 and V_2 are subspaces of V and $V_1 \cap V_2 = \{0\}$, then V is called the direct sum of V_1 and V_2.

This is denoted by

$$V = V_1 \oplus V_2 .$$

Result 3 : Take any subspace W of V. Then
$V = W \oplus W^{\perp}$.

Proof : Let P be the projection matrix corresponding to the projection on the column space of W.

Take $x \in V$.

Then $x = \underbrace{Px}_{\in W?} + \underbrace{(x - Px)}_{\in W^{\perp}?}$

Px : projection of x on the column space of W.

$$\Rightarrow Px \in W.$$

<u>To prove</u> : $x - Px \in W^{\perp}$.

Take $t \in W$.

Consider $\langle t, x - Px \rangle = \langle Pt, x - Px \rangle$ (as $t \in W$, $Pt = t$)

$$= (Pt)^T (x - Px) \quad \text{(vectorization)}$$

$$= t^T P^T (x - Px)$$

$$= t^T P (x - Px) \qquad (\text{as } P = P^T)$$

$$= t^T (Px - P^2 x)$$

$$= t^T (Px - Px) \qquad (\text{as } P^2 = P)$$

$$= t^T (0) = 0.$$

$$\Rightarrow x - Px \in W^\perp.$$

Also, $W \cap W^\perp = \{0\}$ (By Result 1)

$$\therefore V = W \oplus W^\perp.$$