Q. Solve
$$x_1 + x_2 + x_3 = 1$$
$$x_1 + x_2 + x_3 = 2.$$

Sl:.

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{2\times3} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

$$Ax = b$$

$$x = A^{+}b$$

↓ pseudo inverse of A.

SVD of A :

$$A \underset{2\times3}{A^T} \underset{3\times2}{=} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}_{2\times2}$$

$$\lambda_1 + \lambda_2 = 6$$

$$\lambda_1 \lambda_2 = 0$$

$$\Rightarrow \quad \lambda_1 = 6 \quad ; \quad \lambda_2 = 0$$

$$\sigma_1 = \sqrt{6} \quad ; \quad \sigma_2 = 0$$

$$\begin{bmatrix} 3-6 & 3 \\ 3 & 3-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3x + 3y = 0$$

$$3x - 3y = 0$$

$$\Rightarrow x = y$$

$$\therefore u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

$$v_1' = A^T u_1$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_1' = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$= \sqrt{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Reduced SVD of A is given by

$$A = u_1 \sigma_1 v_1^T$$

$$= \frac{1}{\sqrt{2}} \sqrt{6} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$A = U \Sigma V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [\sqrt{6}] \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T$$

$$2\times 1$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{1}{\sqrt{2}} & * \end{bmatrix}_{2\times 2} \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1/\sqrt{3} & * & * \\ 1/\sqrt{3} & * & * \\ 1/\sqrt{3} & * & * \end{bmatrix}_{3\times 3}$$

$$A^+ = V \Sigma^{-1} U^T$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & * & * \\ 1/\sqrt{3} & * & * \\ 1/\sqrt{3} & * & * \end{bmatrix}_{3\times 1} \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ * & * \end{bmatrix}_{2\times 2}$$

$$A^+ = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 1/6 \\ 1/6 & 1/6 \end{bmatrix}_{3 \times 2}$$

$$x = A^+ b$$

$$= \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & 1/6 \\ 1/6 & 1/6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$\hookrightarrow$ solution x^*

(using SVD)

$$\Rightarrow \quad x_1 = \frac{1}{2} \quad ; \quad x_2 = \frac{1}{2} \quad ; \quad x_3 = \frac{1}{2}.$$

Meaning:

$$AX = b$$

$$A^T(AX) = A^T b$$

$$(A^TA)X = A^T b \qquad \text{(Least Square Solution)}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $2x_1 + 2x_2 + 2x_3 = 3$ (listed thrice)

$\Rightarrow$ $x_1 + x_2 + x_3 = 1.5$

$\longrightarrow$ Infinitely many solutions.

Finding $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ s.t $x_1 + x_2 + x_3 = 1.5$ with

least norm:

Lagrange's method :

$$F = x_1^2 + x_2^2 + x_3^2 + \lambda(x_1 + x_2 + x_3 - 1.5)$$

↑
minimize the norm.

$$\frac{\partial F}{\partial x_1} = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial x_1} = 2x_1 + \lambda = 0$$

$$\frac{\partial F}{\partial x_2} = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial x_2} = 2x_2 + \lambda = 0$$

$$\frac{\partial F}{\partial x_3} = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial x_3} = 2x_3 + \lambda = 0$$

$$x_1 = \frac{\lambda}{2} = x_2 = x_3 .$$

(All x_i's are equal)

$$X^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} .$$

Q. Solve :

(I). $\begin{bmatrix} 80 & -41 \\ 40 & -21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix} .$

(II).
$$\begin{bmatrix} 80 & -41 \\ 41 & -21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

S: (I). $x = -10$

$y = -20$

(II). $x = 400$

$y = 780$

* Small changes in input leads to large change in output. called perturbations.

Such system is ill-conditioned or matrix is ill-conditioned matrix.

Euclidean norm :

Given $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$; $\|X\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

(Norm)

$= \|X\|_E$

Take $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$\|A\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2}$

Column norm (1-norm) :

Maximum absolute column sum norm.

$$\|A\|_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$$

Eg: $A = \begin{matrix} 1 & -2 \\ -3 & 4 \end{matrix}$

$$\|A\|_1 = \max \left\{ |1| + |-3|, \quad |-2| + |4| \right\}$$

$$= \max \{ 4, 6 \} = \underline{6}.$$

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Row norm (∞-norm):

Maximum absolute row sum norm.

$$\|A\|_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

Eg: $A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$.

$$\|A\|_\infty = \max \left\{ |1| + |-2|, \ |-3| + |4| \right\}$$

$$= \max \left\{ 3, 7 \right\}$$

$$= 7$$

2- Norm (Spectral norm):

$$\|A\|_2 = \sqrt{\lambda_{max}(A^TA)}$$

$$= \sigma_{max}(A)$$

$$= \sigma_1(A)$$

$$= \sigma_1.$$

Eg: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ with $\sigma_1 = \sqrt{6}.$

$\|A\|_2 = \sigma_1 = \sqrt{6}.$

$$\|A\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2}$$

$$= \sqrt{6}.$$

$$\|A\|_{\infty} = \max \left\{ 1+1+1, \ 1+1+1 \right\}$$

$$= 3.$$

$$\|A\|_{1} = \max \left\{ 1+1, \ 1+1, \ 1+1 \right\}$$

$$= 2.$$

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