

# Singular Value Decomposition (SVD).

Let  $A_{m \times n}$  be a wide matrix ( $m < n$ ). Then there exists orthogonal matrices  $U_{m \times m}$ ,  $V_{n \times n}$  and rectangular matrix  $\Sigma$  with diagonal entries as singular values of  $A$  s.t

$$A = U \Sigma V^T$$

↑ rotation  
↑ scaling  
or reflection

→ rotation/reflection

$$A A^T \xrightarrow{m \times n \times m} m \times m \rightarrow U \downarrow \text{smaller}$$

$$A^T A \xrightarrow{n \times m \times n} n \times n \rightarrow V \downarrow \text{larger}$$

**Proof:** The nonzero eigen values of  $AA^T$  and  $A^T A$  coincide and as the matrices are symmetric, the eigen values are always real.

Take  $\lambda_1 \neq 0$  s.t.  $AA^T X = \lambda_1 X$  (nonzero eigen value of  $AA^T$ )

$$AA^T u_1 = \lambda_1 u_1 \quad \text{--- (1)}$$

$$\begin{aligned} A^T \underbrace{AA^T}_{v} u_1 &= A^T \lambda_1 u_1 \\ &= \lambda_1 \underbrace{A^T u_1}_v \end{aligned}$$

$\Rightarrow A^T u_1$  is an eigen vector of  $A^T A$ .

Take  $v_1 = \frac{A^T u_1}{\|A^T u_1\|}$  — (2)

Form  $U = [u_1, u_2, \dots, u_m]_{m \times m}$

$$V = [v_1, v_2, \dots, v_n]_{n \times n}.$$

Consider  $\|A^T u_1\|^2 = \langle A^T u_1, A^T u_1 \rangle$

$$= (A^T u_1)^T A^T u_1$$

$$= u_1^T (A^T)^T A^T u_1$$

$$= u_1^T A A^T u_1$$

$$= u_i^T \lambda_i u_i \quad (\text{by (i)})$$

$$= \lambda_i u_i^T u_i$$

$$= \lambda_i \|u_i\|^2$$

$$= \lambda_i \cdot 1 \quad (\text{as } u_i \text{ is unit vector})$$

$$= \lambda_i.$$

Put in (2).

$$v_i = \frac{A^T u_i}{\sqrt{\lambda_i}}.$$

$$\Rightarrow Av_1 = \frac{AA^T u_1}{\sqrt{\lambda_1}}$$

$$= \frac{\lambda_1 u_1}{\sqrt{\lambda_1}}$$

$$= \sqrt{\lambda_1} u_1$$

$$= \sigma_1 u_1 \quad ; \quad \text{where } \sigma_1 = \sqrt{\lambda_1}$$

↑  
singular value

$$\therefore Av_1 = \sigma_1 u_1$$

$$Av_2 = \sigma_2 u_2$$

$$Av_3 = \sigma_3 u_3$$

$$A v_m = \sigma_m u_m$$

$$A v_{m+1} = 0 \cdot 0$$

$$\vdots$$

$$A v_n = 0 \cdot 0$$

vectorizing this leads to  
decomposition of  $A$ .

where  $\sigma_i = \sqrt{\lambda_i}$ ,  $i = 1, 2, \dots, m$  and

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0.$$

$$A_{m \times n} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}_{n \times n} = \begin{bmatrix} u_1 & u_2 & \dots & u_m \end{bmatrix}_{m \times m} \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ 0 & & \ddots & \\ & & & \sigma_m \\ & & & & 0 \end{bmatrix}_{m \times n}$$

(vectorization of above  $n$   
equations)

$$\Rightarrow Av = U\Sigma$$

$$\Rightarrow A = U\Sigma V^{-1}$$

$$= U\Sigma V^T \quad (\text{as } V \text{ is orthogonal, } V^{-1} = V^T)$$

$$\therefore \boxed{A = U\Sigma V^T.}$$

Suppose  $B$  is a tall matrix.

Then  $B^T$  is a wide matrix.

$$\Rightarrow A = U\Sigma V^T$$

$$\Rightarrow B = A^T = (U\Sigma V^T)^T$$

$$B = V \Sigma^T U^T.$$

↳ gives SVD of  $B$ .

$$AX = \lambda X$$

$$AU_i = \lambda_i U_i$$

↓  
square  
matrix

↳ eigen  
value

generalized

to

$$AU_i = \sigma_i U_i$$

↙  
rectangular  
matrix

↳ singular  
value

Q. Find SVD of  $A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}_{2 \times 3}$ .

S1:  $A^T A \rightarrow 3 \times 3$   
 $A A^T \rightarrow 2 \times 2$

$$A A^T = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 24 \\ 24 & 40 \end{bmatrix}.$$

$$\lambda_1 + \lambda_2 = 80$$

$$\lambda_1 \lambda_2 = \det(AA^T)$$

$$= 1024$$

$$\Rightarrow \lambda_1 = 64 \quad ; \quad \lambda_2 = 16$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{64} = 8$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{16} = 4.$$

For  $\lambda_1 = 64$  :

$$\begin{bmatrix} 40 - \lambda_1 & 24 \\ 24 & 40 - \lambda_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-24x + 24y = 0$$

$$24x - 24y = 0$$

$$\Rightarrow x = y.$$

$$u_1 = \frac{1}{\sqrt{2}} \underline{\underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}}.$$

For  $\lambda_2 = 16$  :

$$24x + 24y = 0$$

$$24x + 24y = 0$$

$$\Rightarrow x = -y$$

$$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$U = [u_1 \quad u_2]$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{where } \theta = 45^\circ$$

(rotation in anticlockwise)

Non-zero eigen values of  $AA^T$  and  $A^T A$  coincide.

$$V = A^T X$$

$$V_1 = A^T u_1 = \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3x2 2x1

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 16 \\ 0 \\ 16 \end{bmatrix}$$

$$= \beta \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$v_2 = A^T u_2 = \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

$$= \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_1 \perp v_2$$

$$A^T A = \begin{bmatrix} 4 & 4 \\ -2\sqrt{2} & 2\sqrt{2} \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 0 & 32 \\ 0 & 16 & 0 \\ 32 & 0 & 32 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A^T A)$$

$\downarrow$        $\downarrow$   
 64      16      = 80

$$\Rightarrow \lambda_3 = 0.$$

$$\begin{bmatrix} 32-0 & 0 & 32 \\ 0 & 16-0 & 0 \\ 32 & 0 & 32-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 32x + 0y + 32z &= 0 \\ 16y &= 0 \end{aligned}$$

$$32x + 0y + 32z = 0$$

$$\Rightarrow y = 0$$

$$x + z = 0 \quad \Rightarrow -x = z$$

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$$V = [v_1 \quad v_2 \quad v_3]$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix}$$

$$A_{3 \times 2} = U \Sigma V^T$$

$$= \begin{bmatrix} \overset{u_1}{1/\sqrt{2}} & \overset{u_2}{-1/\sqrt{2}} \\ \underset{2 \times 2}{1/\sqrt{2}} & \underset{2 \times 2}{1/\sqrt{2}} \end{bmatrix} \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \begin{matrix} \text{scaling} \\ \underset{2 \times 3}{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ \underset{3 \times 3}{\end{matrix} \end{matrix}$$

rotation in  
xy-plane (45°, anticlockwise)

rotation (?)