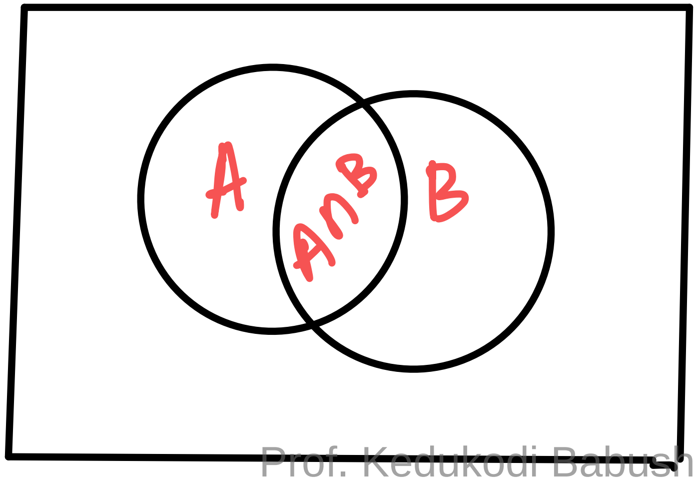
For every vector space V,

$$V = W \oplus W^{\perp}.$$

W → any subspace of V.

Analogous notion → Principle of Inclusion - Exclusion.



$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

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Result : Let U, W be subspaces of a vector space V. Then

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Proof : Define $T : U \times W \longrightarrow V$ by

cartesian product of U & W.

$$T(u, w) = u + w.$$

Claim : T is a linear transformation.

Take $x = (u_1, w_1)$, $y = (u_2, w_2) \in U \times W$.

i) $T(x+y) = T(u_1 + u_2, w_1 + w_2)$

$= (u_1 + u_2) + (w_1 + w_2)$

$= (u_1 + w_1) + (u_2 + w_2)$

$= T(u_1, w_1) + T(u_2, w_2)$

$= T(x) + T(y)$.

Take a scalar α,

(ii). $T(\alpha x) = T(\alpha(u_1, w_1))$

$= T(\alpha u_1, \alpha w_1)$

$= \alpha u_1 + \alpha w_1$

$= \alpha(u_1 + w_1)$

$= \alpha T(u_1, w_1)$

$= \alpha T(x)$.

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From i). & ii). , T is a linear transformation.

Now, $\ker T = \{(u,w) \in U \times W \mid T(u,w) = 0\}$

$= \{(u,w) \in U \times W \mid u + w = 0\}$

$= \{(u,w) \in U \times W \mid u = -w\}$

$= \{(u,-u) \mid u \in U \cap W\}$. (as

(as $u \in U$ & $u = -w \in W$

$\Rightarrow u \in U \cap W$)

Claim: $U \cap W \cong \ker T.$

Isomorphic

Define $\phi : U \cap W \longrightarrow \ker T$ by

$$\phi(x) = (x, -x).$$

1). To show: ϕ is onto.

Take $y \in \ker T \Rightarrow y = (u, -u)$ for some $u \in U \cap W$

$$\Rightarrow \phi(u) = (u, -u) = y$$

$\Rightarrow \phi$ is onto.

2). <u>To Show</u>: ϕ is one-one.

Take $\phi(x) = \phi(y)$

$\Rightarrow (x, -x) = (y, -y)$

$\Rightarrow x = y$

$\Rightarrow \phi$ is <u>one-one</u>.

3). <u>To Show</u>: ϕ is a L.T.

Take $x, y \in U \cap W$.

$$\phi(x+y) = (x+y, -(x+y))$$

$$= (x+y, -x-y)$$

$$= (x,-x) + (y,-y)$$

$$= \phi(x) + \phi(y)$$

Take a scalar α.

$$\phi(\alpha x) = (\alpha x, -\alpha x)$$

$$= \alpha(x, -x) = \alpha\phi(x)$$

$\therefore$ ϕ is a L.T.

from 1), 2) & 3),

$$U \cap W \cong \operatorname{Ker} T.$$

$$\Rightarrow \dim(U \cap W) = \dim(\operatorname{Ker} T) \quad \text{--- (1)}$$

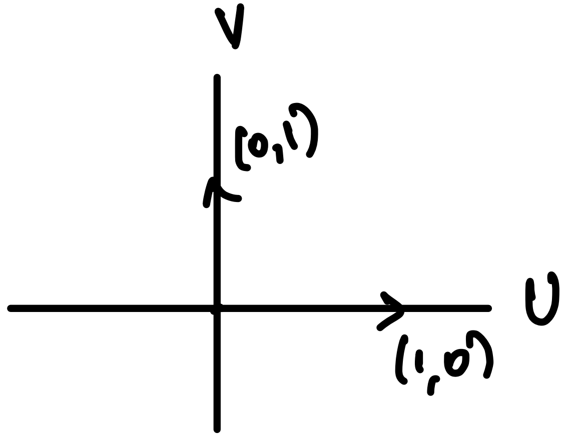
By Rank-Nullity theorem for T,

$$\dim(\operatorname{Ker} T) + \dim(\operatorname{Im} T) = \dim(U \times W)$$

$$\Rightarrow \dim(U \cap W) + \dim(U + W) = \dim(U \times W). \quad \text{--- (2)}$$

(by (1))

Eg:



$\dim U = 1$

$\dim V = 1$ $\Rightarrow$ $\dim U \times V = \dim \mathbb{R}^2 = 2.$

Take $B_1 = \{u_1, u_2, \ldots, u_m\}$ be a basis for U.

$B_2 = \{w_1, w_2, \ldots, w_n\}$ be a basis for W.

Claim: $B = \{(u_1, 0), (u_2, 0), \dots, (u_m, 0), (0, w_1), (0, w_2), \dots, (0, w_n)\}$ is a basis for $U \times W$.

spanning set

L.I.

Take $\alpha_1(u_1, 0) + \alpha_2(u_2, 0) + \dots + \alpha_m(u_m, 0) + \beta_1(0, w_1)$

$+ \beta_2(0, w_2) + \dots + \beta_n(0, w_n) = (0, 0)$

$\Rightarrow (\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_m u_m, \beta_1 w_1 + \beta_2 w_2 + \dots + \beta_n w_n)$

$= (0, 0).$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m = 0$$

$$\beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n = 0$$

$$\Rightarrow \alpha_1 = 0, \ \alpha_2 = 0, \ \ldots, \ \alpha_m = 0$$ (because B_1 is L.I)

$$\beta_1 = 0, \ \beta_2 = 0, \ \ldots, \ \beta_n = 0.$$

(because B_2 is L.I)

$\Rightarrow$ B is L.I.

Take $x \in U \times W$.

$\Rightarrow x = (u, w)$ for some $u \in U$, $w \in W$.

$$\Rightarrow u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$$ (as B_1 is a basis for U)

and

$$w = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n$$ (because B_2 is a basis for W)

$$\Rightarrow x = (u, w)$$

$$= (\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m , \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_n w_n)$$

$$= \alpha_1 (u_1, 0) + \alpha_2 (u_2, 0) + \cdots + \alpha_m (u_m, 0) + \beta_1 (0, w_1) + \beta_2 (0, w_2) + \cdots + \beta_n (0, w_n)$$

$\Rightarrow$ B spans $U \times W$.

$\therefore$ B is a basis for $U \times W$.

$$\Rightarrow \dim(U \times W) = n(B)$$

$$= m + n$$

$$= \dim U + \dim W \quad \text{——} \quad (3)$$

Put (3) in (2),

$$\dim(U \cap W) + \dim(U + W) = \dim U + \dim W.$$

$$\Rightarrow \dim(U+W) = \dim U + \dim W - \dim(U \cap W).$$

Corollary 1:

$$\dim(U \oplus W) = \dim U + \dim W - \dim(\{0\})$$

(as $U \cap W = \{0\}$)

$$= \dim U + \dim W.$$

Corollary 2:

If $V = W \oplus W^{\perp}$, $\dim V = \dim W + \dim W^{\perp}$.

subspace of V.

(because $W \cap W^{\perp} = \{0\}$)

$$A_{m\times n}$$

$$\Big\updownarrow \text{L.T}$$

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(x) = A_{m\times n}\, x_{n\times 1}$$

$$Col(A) = C(A)$$

$$= \{ Ax \mid x \in \mathbb{R}^n \}$$

$$= \{ x_1 c_1 + x_2 c_2 + \dots + x_n c_n \mid x_i \in \mathbb{R} \} \quad ; A = [c_1 \dots c_n]$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$C(A) = \{ \text{LCs of columns of } A \}$

↳ is a subspace of $\mathbb{R}^m$.

Take $W = C(A)$.

$$\boxed{\mathbb{R}^m = C(A) \oplus C(A)^{\perp}}$$