$$A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}$$

$$= \begin{array}{cc} u_1 & u_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}} \end{array} \begin{bmatrix} 8 = \sigma_1 & 0 & 0 \\ 0 & 4 = \sigma_2 & 0 \end{bmatrix} \begin{array}{ccc} v_1 & v_2 & v_3 \\ \multicolumn{3}{c}{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}^T} \end{array}$$

U (rotation/reflection) $\qquad \Sigma$ (scaling) $\qquad V^T$ (rotation/reflection)

(Full) SVD

$$= [u_1, u_2] \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} [v_1 \quad v_2 \quad v_3]^T$$

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$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T \quad \text{(outer product)}$$

$$A_1 = \sigma_1 u_1 v_1^T$$

$$= 8 \cdot \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{2\times 1} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}}_{1\times 3}$$

$$= \underbrace{\frac{8}{4} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{2\times 3}$$

$$A \cong A_1 = \begin{bmatrix} 4 & 0 & 4 \\ 4 & 0 & 4 \end{bmatrix}$$ (called rank 1 approximation of A)

$$\|A - A_1\|_2 = \sqrt{(-2\sqrt{2})^2 + (2\sqrt{2})^2}$$

$$= \sqrt{8+8} = 4.$$

error

For $m \times n$ matrix A;

$$\|A - A_k\|_2 = \sqrt{\sum_{i=k+1}^{n} \sigma_i^2}$$

So, $\|A - A_1\|_2 = \sqrt{\sum_{i=2}^{3} \sigma_i^2}$

$$= \sqrt{\sigma_2^2 + \sigma_3^2} = \sqrt{4^2 + 0^2} = \underline{\underline{4.}}$$

$$\sigma_2 u_2 v_2^T = 4 \cdot \frac{1}{\sqrt{2}} \underset{2\times 1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}$$

$$= 2\sqrt{2} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} & 0 \end{bmatrix}.$$

$$\Rightarrow \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T = A_2.$$

(Rank approximation of A)

$$= A.$$

$$\|A - A_2\|_2 = \sqrt{\sum_{i=3}^{3} \sigma_i^2}$$

$$= \sqrt{\sigma_3^2}$$

$$= \sigma_3 = 0.$$

$\hookrightarrow$ error

Called $\underline{\text{Reduced SVD}}$ (or rank 2 approximation of A).

$$A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{2\times 2} \begin{bmatrix} \overset{\sigma_1}{8} & 0 \\ 0 & \underset{\sigma_2}{4} \end{bmatrix}_{2\times 2} \begin{array}{c} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} \end{array} \begin{array}{l} \to v_1 \\ \to v_2 \end{array}$$

Practically, a coloured image (in RGB format) is stored in 3 matrices — one for each colour Red, Green, and Blue.

Each matrix represents the intensity values for the colour Red, Green, Blue.

$Red_{m\times n}$, $Green_{m\times n}$, $Blue_{m\times n}$ $\longrightarrow$ Matrices of order $m\times n$

$$\begin{array}{r} 1273 \\ +\ 07 \\ \hline 1280 \end{array}$$

For the original image in the next slide $m = 1280$, $n = 1280$ for each colour.

$\text{rank}(Red_{m\times n}) = \text{rank}(Green_{m\times n}) = \text{rank}(Blue_{m\times n}) = 1273.$

We can have approximation using top rank singular values from $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{1273} \geq \underbrace{0, 0, 0, 0, 0, 0, 0}_{\text{7 Zero singular values.}}$

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(σ_1 used)

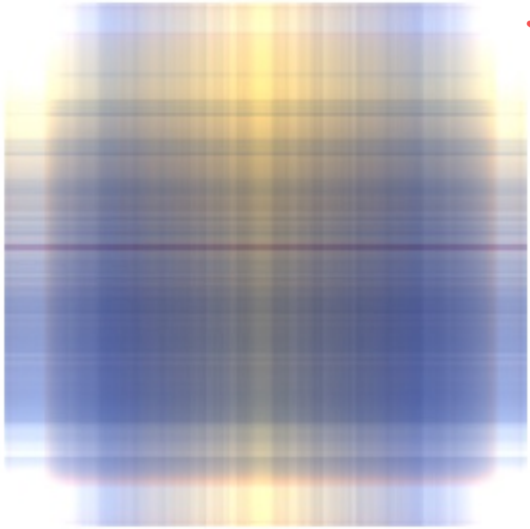
Rank 1 approximation

($\sigma_1, \ldots \sigma_5$ used)

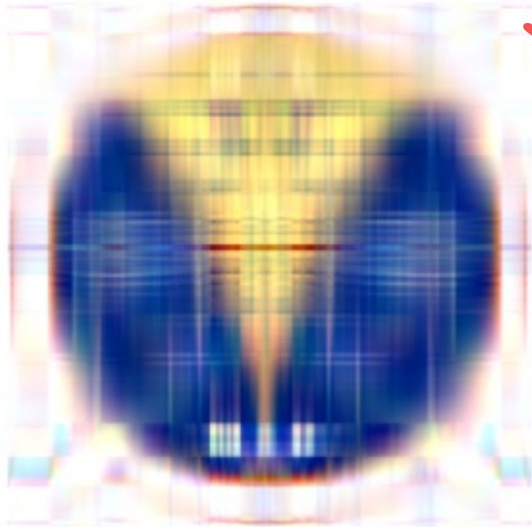
Rank 5 approximation

Original Image

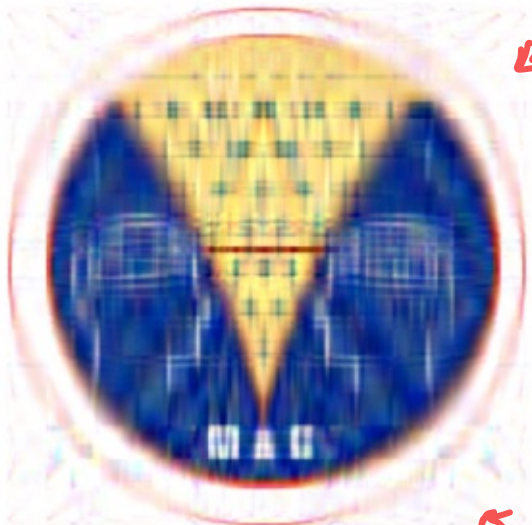
Compression percentage 99.84%



Compression percentage 99.22%



Compression percentage 98.44%



Rank 10 approximation

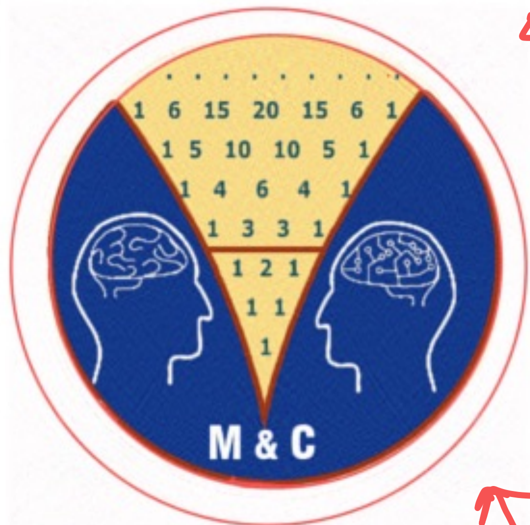
($\sigma_1, \ldots, \sigma_{10}$ used)

Compression percentage 92.18%

Rank 50 approximation

($\sigma_1, \ldots \sigma_{50}$ used).

Compression percentage 84.37%



Rank 100 approximation

As good as original image with 84% compression achieved.

($\sigma_1, \ldots, \sigma_{100}$ used).

Revisit to Reduced SVD:

$$A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}_{2\times 3} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}_{2\times 2} \begin{bmatrix} 8 = \sigma_1 & 0 \\ 0 & \sigma_2 = 4 \end{bmatrix}_{2\times 2} \begin{array}{c} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3} \end{array} \begin{array}{l} \to v_1 \\ \to v_2 \end{array}$$

Q. Solve

$$4x_1 - 2\sqrt{2}\, x_2 + 4x_3 = 1 \quad \text{— (1)}$$

$$4x_1 + 2\sqrt{2}\, x_2 + 4x_3 = 2 \quad \text{— (2)}$$

Sl: (1) + (2) ; $8x_1 + 8x_3 = 3$

(2) − (1) ; $4\sqrt{2}\, x_2 = 1$

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$$\Rightarrow \quad x_1 = 3/8 - x_3$$

$$x_2 = \frac{1}{4\sqrt{2}}$$

Put $x_3 = t$; $t \in \mathbb{R}$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/8 - t \\ \frac{1}{4\sqrt{2}} \\ t \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 \\ 1/4\sqrt{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(Infinitely many solutions)

Find a solution with minimum norm.

Let $g(t) = \left(\frac{3}{8} - t\right)^2 + \left(\frac{1}{4\sqrt{2}}\right)^2 + t^2$.

norm of x.

$$= \frac{9}{64} + t^2 - \frac{3}{4}t + \frac{1}{32} + t^2.$$

$$g'(t) = 2t - \frac{3}{4} + 2t = 0$$

$$\Rightarrow 4t - \frac{3}{4} = 0$$

$$\Rightarrow t = \frac{3}{16}.$$

$g''(t) = 4 > 0$ (g has min at $t = 3/16$)

$$x^* = \begin{bmatrix} \frac{3}{8} - \frac{3}{16} \\ \frac{1}{4\sqrt{2}} \\ \frac{3}{16} \end{bmatrix} = \begin{bmatrix} \frac{3}{16} \\ \frac{1}{4\sqrt{2}} \\ \frac{3}{16} \end{bmatrix}.$$

Min. norm solution

Now, $Ax = b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$(U \Sigma V^T) x = b$ (Full SVD of A)

$$\Rightarrow x^* = (U \Sigma V^T)^{-1} b$$

$$= (V^T)^{-1} \Sigma^{-1} U^{-1} b$$

$$= \underbrace{V \Sigma^{-1} U^T}_{A^+} b \quad \text{(as } U, V \text{ are orthogonal)}$$

$$= \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{8} & 0 \\ 0 & 1/4 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{A^+ b}$$

$$= \begin{bmatrix} 3/16 \\ 1/4\sqrt{2} \\ 3/16 \end{bmatrix}.$$

$$\underset{m\times n}{A}\underset{n\times m}{A^{+}} = A(V\Sigma^{-1}U^{T})$$

$$= (U\Sigma V^{T})(V\Sigma^{-1}U^{T})$$

$$= U\Sigma(V^{T}V)\Sigma^{-1}U^{T}$$

$$= U\Sigma I\Sigma^{-1}U^{T}$$

$$= UIU^{T}$$

$$= UU^{T}$$

$$= I_{m\times m}$$

$A^{+} \rightarrow$ called pseudo inverse or Moore-Penrose inverse.

Eg: For $A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}$,

$$A^{+} = U\Sigma^{-1}V^{T}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{8} & 0 \\ 0 & 1/4 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/16 & 1/16 \\ 1/4\sqrt{2} & 1/4\sqrt{2} \\ 1/16 & 1/16 \end{bmatrix}.$$

$$AA^{+} = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix} \begin{bmatrix} 1/16 & 1/16 \\ 1/4\sqrt{2} & 1/4\sqrt{2} \\ 1/16 & 1/16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Q. Solve

$$x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 + x_3 = 2.$$

Sl : :

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b}$$

$$AX = b$$

$$X^* = A^+ b$$

$$= A^+ \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

find this

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$$AA^{+} = I$$

$$(U\Sigma V^{T})A^{+} = I$$

$$\Rightarrow A^{+} = (U\Sigma V^{T})^{-1}$$

$$= V\Sigma^{-1}U^{T}.$$

$$\underset{2\times3\ \ 3\times2}{AA^{T}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \underset{2\times2}{\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}}.$$

$$\lambda_1 + \lambda_2 = 6$$

$$\lambda_1 \lambda_2 = 0$$

$$\Rightarrow \quad \lambda_1 = 6$$

$$\lambda_2 = 0$$

$$\Rightarrow \quad \sigma_1 = \sqrt{\lambda_1} = \sqrt{6}$$

$$\sigma_2 = 0.$$

$$\therefore \quad A = \sigma_1 u_1 v_1^T + 0$$

<u>Finding u_1</u> :

$$\begin{bmatrix} 3-6 & 3 \\ 3 & 3-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3x + 3y = 0$$

$$3x - 3y = 0$$

$$\Rightarrow x = y$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Nonzero eigen values of AA^T & A^TA coincide.