

## Horizontal slices

$$X = \left\{ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3 \times 4}, \begin{bmatrix} 13 & 14 & 15 & 16 \\ 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 \end{bmatrix}_{3 \times 4} \right\}$$

$$= X(i, :, :, :)$$

<sup>i=1</sup>

→ matrices kept on top of each other : stack of matrices  
 $i=1 \rightarrow \text{top}; i=2 \rightarrow \text{next below}$

$i \in \{1, 2\} \rightarrow$  how many matrices

$j \in \{1, 2, 3\}$       }      order of each matrix  
 $k \in \{1, 2, 3, 4\}$

vector space under consideration :  $\mathbb{R}^{2 \times 3 \times 4}$

## Lateral Slices

$$x(:, j, :) = \left\{ \begin{array}{l} \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{array} \right], \quad j=1 \\ \left[ \begin{array}{cccc} 5 & 6 & 7 & 8 \\ 17 & 18 & 19 & 20 \end{array} \right], \quad j=2 \\ \left[ \begin{array}{cccc} 9 & 10 & 11 & 12 \\ 21 & 22 & 23 & 24 \end{array} \right] \end{array} \right\}, \quad j=3$$

$$x(:, :, 1) = \left[ \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 13 & 14 & 15 & 16 \end{array} \right]$$

## Frontal Slices.

$$x(:,:,k) = \left\{ \begin{array}{l} \left[ \begin{array}{ccc} 1 & 5 & 9 \\ 13 & 17 & 21 \end{array} \right], \quad k=1 \\ \left[ \begin{array}{ccc} 2 & 6 & 10 \\ 14 & 18 & 22 \end{array} \right], \\ \left[ \begin{array}{ccc} 3 & 7 & 11 \\ 15 & 19 & 23 \end{array} \right], \quad k=3 \\ \left[ \begin{array}{ccc} 4 & 8 & 12 \\ 16 & 20 & 24 \end{array} \right] \end{array} \right\}, \quad k=2$$

$x_{ijk}$  : Tensors.

↳ generalization of arrays.

a. Compute a basis for four fundamental subspaces  $X(:, 1, :)$ ; given

$$X = \left\{ \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

$i=1$        $2 \times 4$        $i=2$        $2 \times 4$        $i=3$        $2 \times 4$

Soln:

$$\underbrace{X(:, i, :)}_{\text{Lateral slicing}} = \left\{ \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

$i=1$        $j=2$

Lateral  
slicing

Now,  $x(:, 1, :) := \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} = A$ .

*$3 \times 4$*

We have  $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$

$$T(x) = Ax$$

*$3 \times 4 \times 1$*

$$\mathbb{R}^4 = R(A) \oplus N(A)$$

$$\mathbb{R}^3 = R(A^T) \oplus N(A^T)$$

$$R_2 \rightarrow R_2 - R_1 \quad ; \quad R_3 \rightarrow R_3 - R_1 .$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_1.$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_{\text{red}}$$

Echelon form of A.

$$\text{rank}(A) = r = 2$$

Basis for  $R(A) =$

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \dim R(A) = 2.$$

$$R(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$N(A) = \{x \in \mathbb{R}^4 \mid Ax = 0\}$$

$$= \{x \in \mathbb{R}^4 \mid A \text{red } x = 0\}$$

$$= \{x = (x_1, x_2, x_3, x_4) \mid \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$$

$x_2, x_4$  are non-pivot variables (free variables)

$$N(A) = \left\{ (x_1, x_2, x_3, x_4) \mid \begin{array}{l} x_1 + 2x_2 + x_3 + 2x_4 = 0 \\ x_3 + x_4 = 0 \end{array} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$\hookrightarrow$  2-dim. (as  $R^4 = R(A) \oplus N(A)$ )

$\downarrow$  4-dim       $\downarrow$  2-dim

Put  $x_2 = 1, x_4 = 0$  in above eqns.

$$x_1 + 2 \cdot 1 + x_3 + 2 \cdot 0 = 0$$

$$\Rightarrow x_1 + x_3 = -2$$

$$\underline{\underline{x_3 = 0}}$$

$$\Rightarrow \underline{\underline{x_1 = -2}}$$

Put  $x_2 = 0$ ,  $x_4 = 1$ .

$$x_1 + 0 + x_3 + 2 \cdot 1 = 0$$

$$x_3 + 1 = 0$$

$$x_1 + x_3 = -2$$

$\Rightarrow$

$$\underline{\underline{x_3 = -1}}$$

$$\Rightarrow \underline{\underline{x_1 = -1}}.$$

$$\left\langle \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 0, \text{ etc.}$$

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 0 \end{bmatrix} = B.$$

$$R_2 \rightarrow R_2 - 2R_1 \quad ; \quad R_3 \rightarrow R_3 - R_1 \quad ; \quad R_4 \rightarrow R_4 - 2R_1$$

$$B \sim \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$B \sim \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] = B_{\text{red}}$$

*Echelon form of B.*

$$\text{rank}(B) = r = 2$$

$$\text{Row}(B) = R(B) = R(A^T)$$

$$= \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$\Rightarrow \dim(R(A^T)) = 2.$$

$$N(A^T) = N(B)$$

$$= \{x \in \mathbb{R}^3 \mid Bx = 0\}$$

$$= \{x \in \mathbb{R}^3 \mid \text{Bred } x = 0\}$$

$$= \left\{ \mathbf{x} = (x_1, x_2, x_3) \mid \begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ (x_1, x_2, x_3) \mid \begin{array}{l} x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \end{array} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$x_3$  : non-pivotal variable.

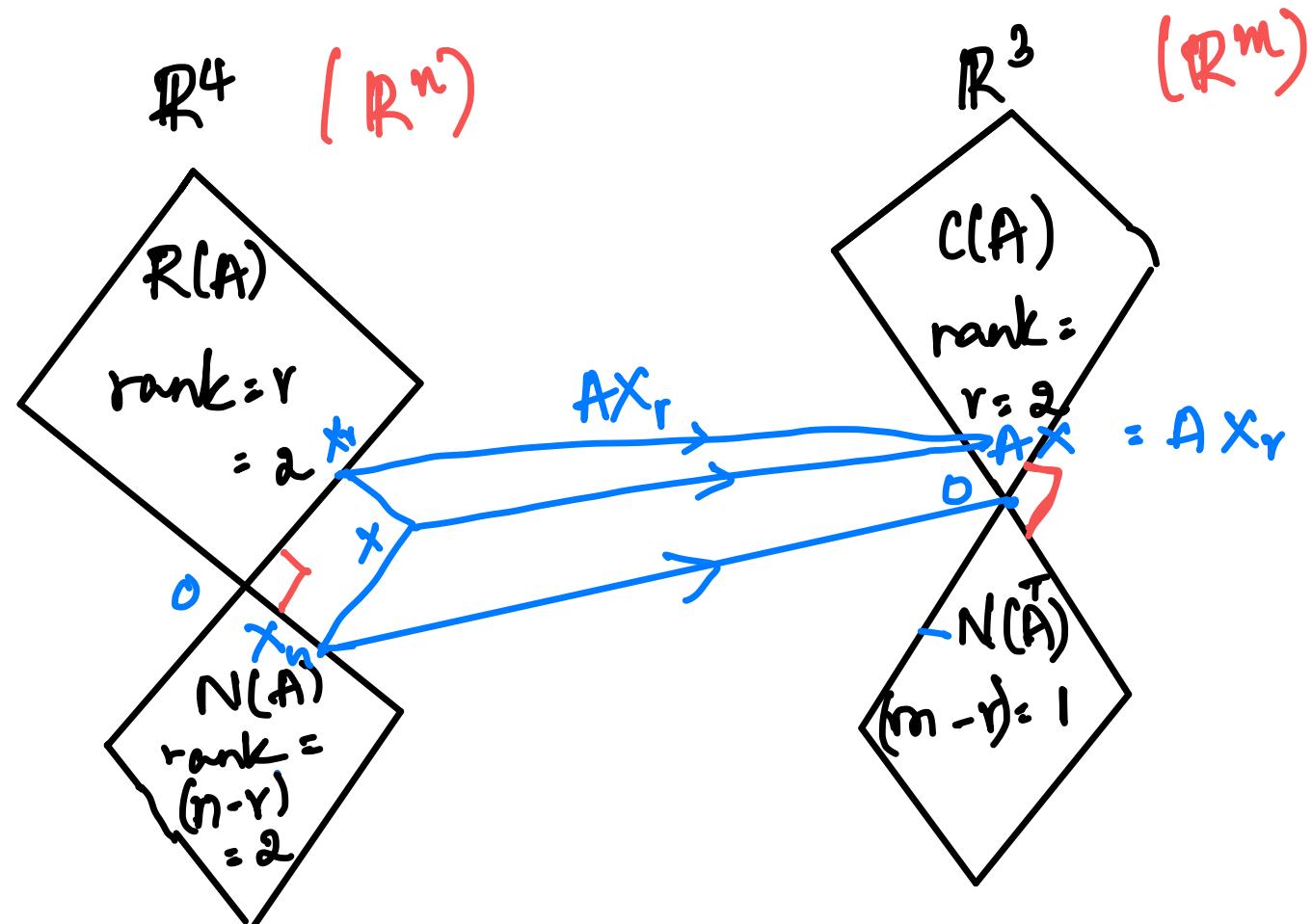
$$\Rightarrow \dim N(A^T) = 1 ; \quad \left\langle \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \underline{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right\rangle = 0.$$

etc.

$$T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

$$T(x) = Ax$$

$3 \times 4$



Any

$x \in \mathbb{R}^4$ ;  $x = x_r + x_n$ ;  $x_r \in R(A)$   
 $x_n \in N(A)$ .

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$$\begin{aligned} \text{Then } AX &= A(x_r + x_n) \\ &= Ax_r + Ax_n \\ &= Ax_r \quad (\text{as } x_n \in N(A)) \end{aligned}$$

