

Chi-Square distribution:

Let $Z \sim N(0, 1)$. Then the pdf of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Let $X = Z^2$. Then we find the pdf of X .

$$X = Z^2 \Rightarrow Z = \pm \sqrt{x}$$

$$f_X(x) = f_Z(\sqrt{x}) \left| \frac{dz}{dx} \right| + f_Z(-\sqrt{x}) \left| \frac{dz}{dx} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{2\pi}} \cdot e^{-x/2} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \quad x > 0.$$

Therefore $\boxed{f_X(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \quad x > 0}$ Chi-square
pdf 1 df.
 $X \sim \chi^2(1)$.

If Z_1, Z_2, \dots, Z_k are independent $N(0, 1)$ random variables, then

$X = Z_1^2 + \dots + Z_k^2$ follows Chi-square distribution with k degrees of freedom, written as $X \sim \chi^2(k)$. Then $X \sim \text{Gamma}\left(\alpha = \frac{k}{2}, \beta = 2\right)$

From gamma distribution:

$$f(x) = \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} e^{-x/\beta}.$$

Take $\alpha = \frac{k}{2}$, $\beta = 2$, we get

$\boxed{f(x) = \frac{1}{\Gamma\left(\frac{k}{2}\right) 2^{k/2}} x^{\frac{k}{2}-1} e^{-x/2}, \quad x > 0}$ pdf of Chi-square dis.

let $X = \sum_{i=1}^k Z_i^2$ where $Z_i \sim N(0,1)$
 $i=1, 2, \dots, k.$

for $k=1$, $Z \sim N(0,1)$

$$M_{Z^2}(t) = E[e^{tZ^2}]$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} e^{t z^2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(\frac{1}{2}-t)z^2} \end{aligned}$$

The integral converges only if $\frac{1}{2}-t > 0$ i.e; $t < \frac{1}{2}$.
 (Verify!)

$$M_{Z^2}(t) = \frac{1}{\sqrt{1-2t}} = (1-2t)^{-\frac{1}{2}}, \quad t < \frac{1}{2}.$$

Now mgf of the sum

$$Z_1^2 + \dots + Z_k^2 = \text{(independent...)} \quad \text{product of mgf.}$$

Therefore

$$M_X(t) = (M_{Z^2}(t))^k$$

$$= ((1-2t)^{-\frac{1}{2}})^k$$

$$= (1-2t)^{-\frac{k}{2}}, \quad t < \frac{1}{2}.$$

Therefore $\boxed{M_X(t) = (1-2t)^{-\frac{k}{2}}, \quad t < \frac{1}{2}.}$

let $a = \frac{1}{2}-t$
 $M_X(t) = \int_{\sqrt{2a}-\infty}^{\infty} e^{-az^2} dz$
 $= \frac{1}{\sqrt{\pi}} \sqrt{\frac{\pi}{a}}$
 $M_X(t) = \frac{1}{\sqrt{2a}} \cdot \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}}$
 $2a = 1-2t$
 $\frac{1}{\sqrt{2a}} = \frac{1}{\sqrt{1-2t}}$.

$$M_x(t) = (1-2t)^{-k_2}$$

$$M'_x(t) = -\frac{k}{2}(1-2t)^{-k_2-1} \cdot (-2) = k(1-2t)^{-k_2-1}$$

$$E(x) = \left[M'_x(t) \right]_{t=0} = M'_x(0) = k.$$

$$M''_x(t) = k(k+2)(1-2t)^{-k_2-2} \quad (\text{verifiy!})$$

$$E(x^2) = \left[M''_x(t) \right]_{t=0} = k(k+2).$$

$$V(x) = E(x^2) - [E(x)]^2 = k(k+2) - k^2 = 2k.$$
