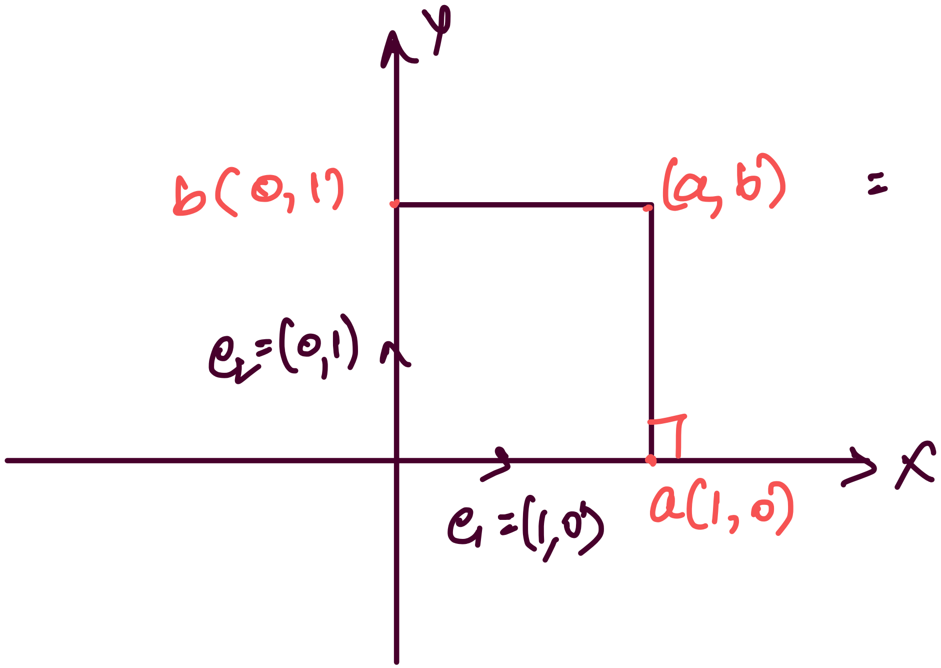

$$= a(1,0) + b(0,1)$$

projections

$$A = \begin{array}{cc} e_1 & e_2 \\ \hline \end{array}$$
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AX = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{X}$$

$$AX = \underset{\uparrow \text{ eigenvalue}}{1} \cdot \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{e_1 e_1^T} \begin{bmatrix} a \\ b \end{bmatrix} + \underset{\uparrow \text{ eigenvalue}}{1} \cdot \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{e_2 e_2^T} \begin{bmatrix} a \\ b \end{bmatrix}$$

(projection matrices)

Eigenvalues of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\lambda_1 = 1$; $\lambda_2 = 1$.

$$AX = 1 \cdot e_1 e_1^T X + 1 \cdot e_2 e_2^T X \quad ; \quad \text{where}$$

$$e_1 e_1^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e_2 e_2^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Spectral thm:

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$ (Outer product expansion)

↳ a linear combination of projection matrices

$$AX = \underbrace{\lambda_1}_{\text{eigen value}} \underbrace{u_1 u_1^T X}_{\text{projection of } X \text{ on } u_1\text{-axis}} + \underbrace{\lambda_2}_{\text{eigen value}} \underbrace{u_2 u_2^T X}_{\text{projection of } X \text{ on } u_2\text{-axis.}}$$

$\underbrace{\text{new axes}}_{\text{called principal axes}}$: u_1-axis, u_2-axis

(change of basis)

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$AX = 1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Axes : standard x-axis , y-axis.

$$\underbrace{A}_{\text{symmetric matrix}} = P \underbrace{D}_{\text{diagonal}} P^T \quad ; \quad P = \text{orthogonal matrix.}$$

(rotations)

Hence,

symmetric matrix

$$A = P\,D\,P^T$$

rotation or reflection

rotation or reflections

scaling along reflection axis

For $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$,

$$B = \underset{2\times 3}{A}\,\underset{3\times 2}{A^T} = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}_{2\times 2}$$

Q: Find the maximum & minimum value of $333x_1^2 + 162x_1x_2 + 117x_2^2$ subject to

$x_1^2 + x_2^2 = 1.$ $(\Leftrightarrow \|x\|^2 = 1)$

S/: $$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\|X\|^2 = \langle x, x \rangle = X^T X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2$$

$$X^T X = X^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X$$

↓ (generalize identity matrix to symmetric matrix B).

$$333\,x_1^2 + 162\,\underbrace{x_1 x_2}_{\text{cross product term.}} + 117\,x_2^2 = Q\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q(X)$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}}_{B\text{ (symmetric)}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= X^T B X$$ (called Quadratic form)

Q: Find spectral decomposition of

$$B = AA^T = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}.$$

$$\lambda_1 + \lambda_2 = 333 + 117 = \text{trace}(B) = 450$$

$$\lambda_1 \lambda_2 = \det(B)$$

$$\Rightarrow \lambda_1 \lambda_2 = 32400$$

$$\Rightarrow \lambda_1 (450 - \lambda_1) = 32400$$

$$\Rightarrow 450 \lambda_1 - \lambda_1^2 = 32400$$

$$\Rightarrow \quad \lambda_1^2 - 450\lambda_1 + 32400 = 0$$

$$\Rightarrow \quad \lambda_1 = 360 \; ; \; \underline{\underline{\lambda_2 = 90.}}$$

Eigen vector for $\lambda_1 = 360$:

$$BX = \lambda_1 X$$

$$(B - \lambda_1 I)\, X = 0$$

$$\begin{bmatrix} 333-360 & 81 \\ 81 & 117-360 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} -27 & 81 \\ 81 & 243 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Prof. Redukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$\Rightarrow \quad -27x_1 + 81x_2 = 0$$

$$81x_1 + 243x_2 = 0$$

$$\Rightarrow \quad -x_1 + 3x_2 = 0$$

$$\Rightarrow 3x_2 = x_1.$$

Take $x_2 = 1$. Then $x_1 = 3$.

$$X = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = u_1.$$

Eigen <u>vector for $\lambda_2 = 90$</u>:

$$(B - \lambda_2 I)X = 0$$

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$$\begin{bmatrix} 333-90 & 81 \\ 81 & 117-90 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \quad 243\,x_1 + 81\,x_2 = 0$$

$$81\,x_1 + 27\,x_2 = 0$$

$$\Rightarrow \quad 3x_1 + x_2 = 0$$

$$\Rightarrow \quad 3x_1 = -x_2$$

Take $x_1 = 1$. Then $x_2 = -3$.

$$X = u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$P = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix} ; \qquad D = \begin{bmatrix} 360 & 0 \\ 0 & 90 \end{bmatrix}$$

$$Q(X) = X^T B X$$

$$= X^T (P D P^T) X$$

$$= (X^T P) D (P^T X)$$

$$= Y^T D Y \quad ; \quad \text{where } \boxed{Y = P^T X}$$

$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 360 & 0 \\ 0 & 90 \end{bmatrix} \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{Y}$$

$$= 360\, y_1^2 + 90\, y_2^2$$ (without cross-product terms)

$$= Q(Y)$$

Now, $Y = P^T X$

$$\Rightarrow (P^T)^{-1} Y = X$$

$$\Rightarrow PY = X$$ (as P is orthogonal)

and $\|x\|^2 = 1$

$$\Leftrightarrow \langle x, x \rangle = 1$$

$$\Leftrightarrow \langle Py, Py \rangle = 1$$

$$\Leftrightarrow \langle y, P^T P y \rangle = 1 \quad \text{(as } P \text{ is orthogonal)}$$

$$\Leftrightarrow \langle y, I y \rangle = 1$$

$$\Leftrightarrow \langle y, y \rangle = 1$$

$$\Leftrightarrow \|y\|^2 = 1$$

$$\Leftrightarrow y_1^2 + y_2^2 = 1.$$

$$Q(Y) = 360\, y_1^2 + 90\, y_2^2$$

$$\leq 360\, y_1^2 + 360\, y_2^2$$

$$= 360\, (y_1^2 + y_2^2)$$

$$\leq 360(1) = 360$$

$$Q(X) = Q(Y) \leq 360.$$

(Found an upper bound for $Q(X)$ as 360)

Find X s.t Q actually takes value 360:-

(we will find a point in domain that actually takes the upper bound value).

$$Y = Pe_1$$

$$= [u_1 \;\; u_2] \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= u_1$$

$$= \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = X.$$

Take $x_1 = 3/\sqrt{10}$

$x_2 = 1/\sqrt{10}$

$$Q\Big(\underbrace{3/\sqrt{10},\ 1/\sqrt{10}}_{u_1}\Big) = 333\left(3/\sqrt{10}\right)^2 + 162 \cdot \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} + 117\left(1/\sqrt{10}\right)^2$$

eigen vector

$$= \frac{1}{10}\left[333 \times 9 + 162 \times 3 + 117\right)$$

$$= \frac{1}{10}(3600) = \underline{\underline{360}}.$$

$\Rightarrow$ Maximum value of $Q(X)$ is actually 360.

$$Q(X) = Q(Y)$$
$$= 360\, y_1^2 + 90\, y_2^2$$
$$\geqslant 90\, y_1^2 + 90\, y_2^2$$
$$= 90\,(y_1^2 + y_2^2)$$
$$\geqslant 90\,(1) = 90.$$

(Lower bound for $Q(X)$).

$$y = P e_2 = [u_1 \;\; u_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

$$Q\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right) = 333\left(\frac{1}{\sqrt{10}}\right)^2 + 162 \cdot \frac{1}{\sqrt{10}} \cdot \frac{-3}{\sqrt{10}} + 117\left(\frac{-3}{\sqrt{10}}\right)^2$$

$$\underbrace{\left(\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\right)}_{u_2}$$

u_2 ↑ eigen vector

$$= \frac{1}{10}\left(333 - 162 \times 3 + 117 \times 9\right)$$

$$= 90.$$

∴ Minimum value of $Q(x)$ is actually 90.

(We found a point in domain that actually took the lower bound value).

$$\Rightarrow \max_{\|x\|=1} Q(x) = 360 \quad (\text{given by } x = u_1)$$

and

$$\min_{\|x\|=1} Q(x) = 90. \quad \left(\text{given by } x = u_2\right)$$

(We have solved the constraint optimization problem using Linear Algebra).