$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$B = AA^T = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}$$

$$333 x_1^2 + 162 x_1 x_2 + 117 x_2^2 = Q(x_1, x_2)$$ (Quadratic form)

$$= Q(x)$$

Subject to $x_1^2 + x_2^2 = 1$.

$\lambda_1 = 360$

$\lambda_2 = 90$

$u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \longleftrightarrow x_1 - 3x_2 = 0 : u_1$

$u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \longleftrightarrow 3x_1 + x_2 = 0 \; : u_2 \Big\}$ principal axes

Also, $B = PDP^T$; $P = [u_1 \;\; u_2]$

$$D = \begin{bmatrix} 360 & 0 \\ 0 & 90 \end{bmatrix}.$$

Max^m lies on the principal axis $x = u_1$.

Min. lies on the principal axis $x = u_2$.

$X = PY = Pe_1 = u_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \longleftrightarrow \boxed{x - 3y = 0}$ line

$X = PY = Pe_2 = u_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix} \longleftrightarrow \boxed{3x + y = 0}$ line

} Principal Axes.

$Q(X) = 360\, y_1^2 + 90 y_2^2 = 360$ (max^m value)

$$\frac{y_1^2}{1^2} + \frac{y_2^2}{4} = 1 \longrightarrow \text{ellipse}$$

vertices: $(\pm 1, 0)$

$(0, \pm 2)$

$X = PY = P\begin{bmatrix} \pm 1 \\ 0 \end{bmatrix} = [u_1\ u_2]\begin{bmatrix} \pm 1 \\ 0 \end{bmatrix} = \pm u_1$

$$X = PY = P\begin{bmatrix} 0 \\ \pm 2 \end{bmatrix} = [u_1 \; u_2]\begin{bmatrix} 0 \\ \pm 2 \end{bmatrix} = \pm 2u_2$$

$$Q(X) = 360\, y_1^2 + 90\, y_2^2 = 90 \quad \text{(min. value)}$$

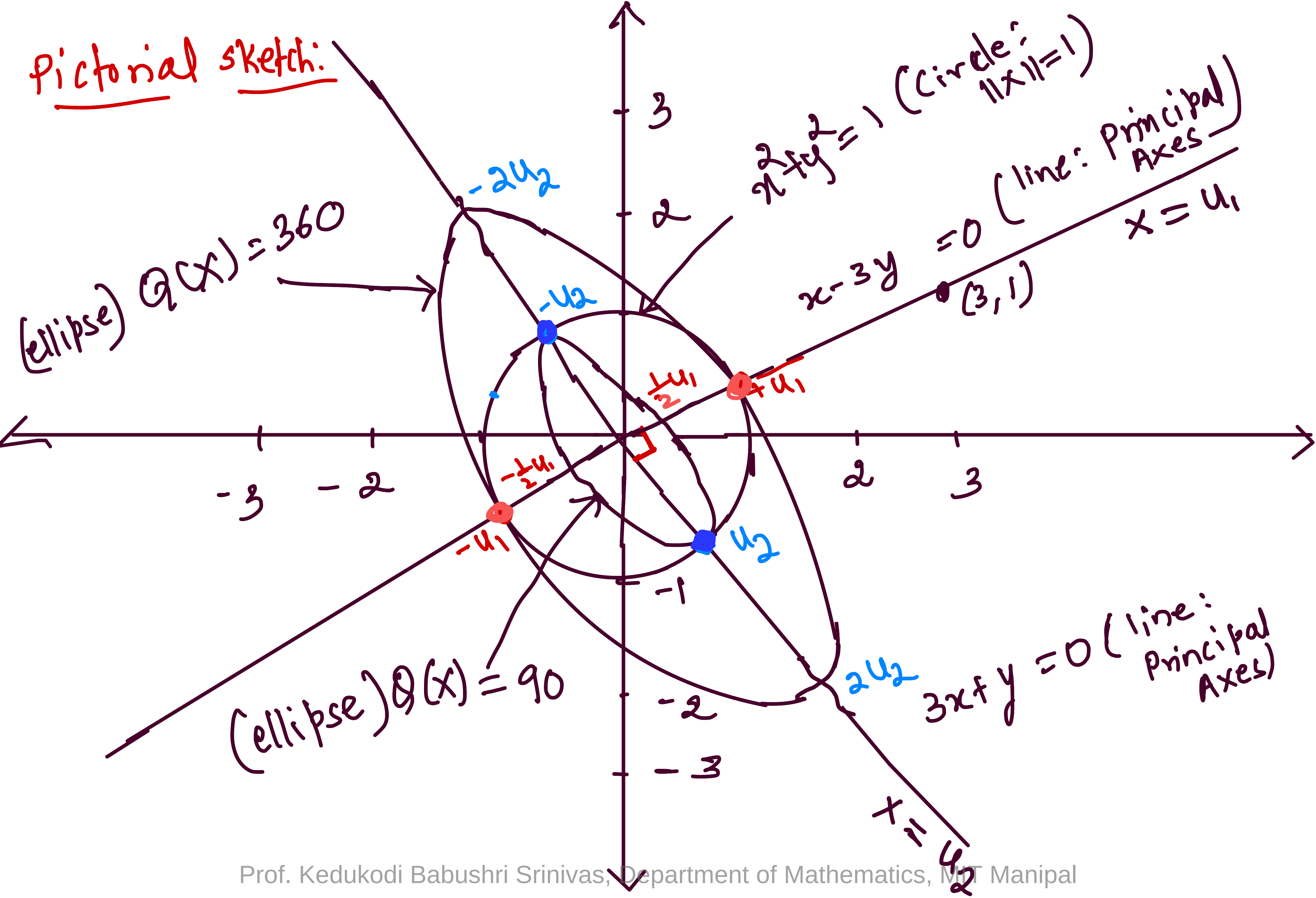
$$\frac{y_1^2}{\left(\frac{1}{4}\right)^2} + \frac{y_2^2}{1^2} = 1 \longrightarrow \text{ellipse}$$

vertices: $\left(\pm\frac{1}{2}, 0\right), \; (0, \pm 1)$

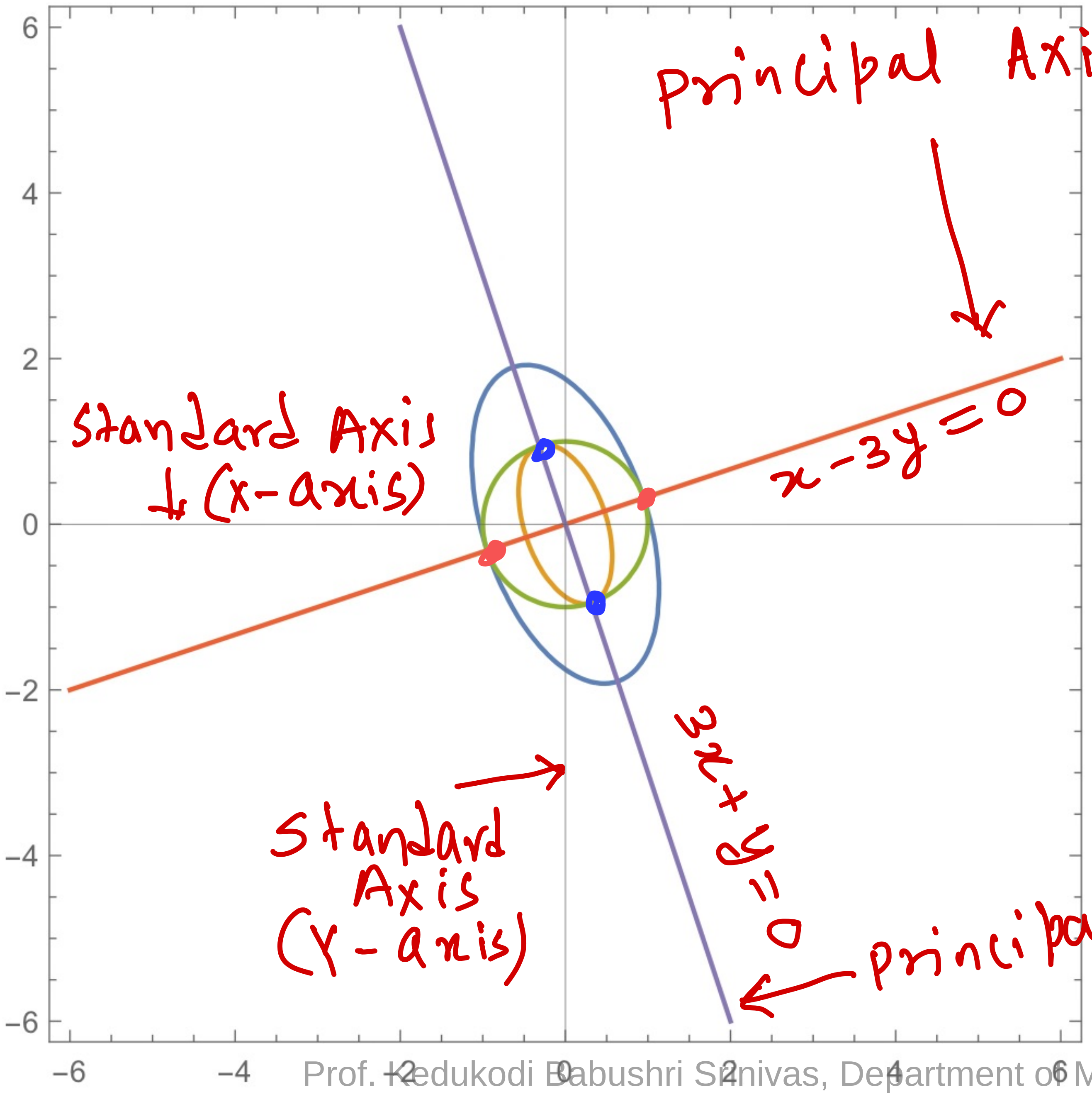
$$X = PY = P\begin{bmatrix} \pm\frac{1}{2} \\ 0 \end{bmatrix} = [u_1 \; u_2]\begin{bmatrix} \pm\frac{1}{2} \\ 0 \end{bmatrix} = \pm\frac{1}{2}u_1$$

$$X = PY = P\begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} = [u_1 \; u_2]\begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} = \pm u_2$$

Pictorial sketch:



Sketch Using a Computer:



(*) Ellipse Q(x) = 360

(*) Ellipse: Q(x) = 90.

$162\,x\,y + 333\,x^2 + 117\,y^2 = 360$ — (*)

$162\,x\,y + 333\,x^2 + 117\,y^2 = 90$ — (*)

$x^2 + y^2 = 1$ (Unit circle).

$x - 3\,y = 0$ } principal Axes.

$3\,x + y = 0$

← principal Axis.

(Refer lab files).

- points of minima.
- → points of maxima

Principal Axes theorem

Let $A_{n \times n}$ be a symmetric matrix. Then there exists an orthogonal change of variable $X = PY$ that transforms the quadratic form $X^T A X$ into quadratic form $Y^T D Y$ with no crossproduct term.

Proof: $Q(X) = X^T A X$

$$= \underbrace{X^T P}_{} D \underbrace{P^T X}_{Y}$$ (By spectral decomposition of symmetric matrix A)

$$= Y^T D Y \; ; \; Y = P^T X$$

$$PP^T = I = P^T P \qquad \text{— (1)}$$

$$y = P^T x$$

$$\Rightarrow x = (P^T)^{-1} y = Py \qquad (\text{by (1)})$$

$$Q(x) = Q(y)$$

$$= y^T D y$$

$$= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n \underline{y_n^2}.$$ (no crossproduct terms)

Q. Find eigen values / eigenvectors of $A^T A$.

for $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

Sol:

$$\underset{3\times 2}{A^T}\;\underset{2\times 3}{A} = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}.$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}_{3\times 3}.$$

Theorem: The non-zero eigen values of A^TA and AA^T coincide; $A_{m\times n}$ (wide matrix)

Let X be the eigenvector corresponding to nonzero eigenvalue λ. Then A^TX is the eigenvector of A^TA corresponding to eigenvalue λ.

Proof: Let λ be a nonzero eigen value of AA^T.

Then $AA^TX = \lambda X$ where $\lambda \neq 0$, $X \neq 0$

$$\underbrace{\qquad\qquad}_{(1).}$$

$$\Rightarrow A^T(AA^TX) = A^T(\lambda X)$$

$$\Rightarrow (A^TA)\underbrace{(A^TX)}_{\text{nonzero vector}} = \lambda A^TX \quad \text{--- (2).}$$

Can $A^TX = 0$?

If $A^TX = 0$ then by (1), $\lambda X = 0$

$\Rightarrow \lambda = 0$ or $X = 0$; not possible.

Take $C = A^TA$, $v = A^TX$.

Then (2) becomes,

$Cv = \lambda v$; where $\lambda \neq 0$, $v \neq 0$

$\Rightarrow$ Eigenvalue of $C = A^T A$ is λ & corresponding eigenvector of C is $v = A^T X$.

Eg : $C = \begin{bmatrix} 80 & 100 & 140 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} = A^T A$.

nonzero eigen values of C = 360, 90.

$\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(C)$

$360 + 90 + \lambda_3 = 450 \Rightarrow \lambda_3 = 0$

Eigen values of C are $360, 90, 0$.

Eigen vector of C : $v = A^T x$.

$$v_1 = A^T u_1$$

$$= \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = ?$$

$$v_2 = A^T u_2 = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = ?$$