

Chi-Square distribution:

Let $Z \sim N(0, 1)$. Then the pdf of Z is

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Let $X = Z^2$. Then we find the pdf of X .

$$X = Z^2 \Rightarrow Z = \pm \sqrt{X}$$

$$f_X(x) = f_Z(\sqrt{x}) \left| \frac{dz}{dx} \right| + f_Z(-\sqrt{x}) \left| \frac{dz}{dx} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{2\pi}} \cdot e^{-x/2} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{1}{\sqrt{x}}, \quad x > 0.$$

$$\text{Therefore } \boxed{f_X(x) = \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \quad x > 0.}$$

Chi-square
pdf 1 df.

$$X \sim \chi^2(1).$$

If Z_1, Z_2, \dots, Z_k are independent $N(0, 1)$ random variables, then

$X = Z_1^2 + \dots + Z_k^2$ follows Chi-square distribution with k degrees of freedom, written as $X \sim \chi^2(k)$. Then $X \sim \text{Gamma}(\alpha = \frac{k}{2}, \beta = 2)$

From gamma distribution:

$$f(x) = \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} e^{-x/\beta}.$$

Take $\alpha = \frac{k}{2}$, $\beta = 2$, we get

$$\boxed{f(x) = \frac{1}{\Gamma(\frac{k}{2}) 2^{k/2}} x^{k/2-1} e^{-x/2}, \quad x > 0.}$$

pdf of
Chi-square
dis.

$$\text{let } X = \sum_{i=1}^k Z_i^2 \quad \text{where } Z_i \sim N(0,1) \\ i=1, 2, \dots, k.$$

for $k=1$, $Z \sim N(0,1)$

$$M_{Z^2}(t) = E[e^{tZ^2}]$$

$$= \int_{-\infty}^{\infty} e^{tZ^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{Z^2}{2}} dZ \\ = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{1}{2}-t\right)Z^2} dZ$$

The integral converges only if $\frac{1}{2}-t > 0$ i.e. $t < \frac{1}{2}$.
(verify!)

$$M_{Z^2}(t) = \frac{1}{\sqrt{1-2t}} = (1-2t)^{-1/2}, \quad t < \frac{1}{2}.$$

Now mgf of the sum

$$Z_1^2 + \dots + Z_k^2 = (\text{independent..}) \\ \text{product of mgf.}$$

Thus

$$M_X(t) = \left(M_{Z^2}(t)\right)^k$$

$$= \left((1-2t)^{-1/2}\right)^k$$

$$= (1-2t)^{-k/2}, \quad t < \frac{1}{2}.$$

$$\text{Thus } \boxed{M_X(t) = (1-2t)^{-k/2}, \quad t < \frac{1}{2}.}$$

$$\text{let } a = \frac{1}{2}-t$$

$$M_X(t) = \int_{-\infty}^{\infty} e^{-aZ^2} dZ \\ = \sqrt{\frac{\pi}{a}}.$$

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{a}} = \frac{1}{\sqrt{2a}}$$

$$2a = 1-2t$$

$$\frac{1}{\sqrt{2a}} = \frac{1}{\sqrt{1-2t}}.$$

$$M_X(t) = (1-2t)^{-k/2}$$

$$M_X'(t) = -\frac{k}{2}(1-2t)^{-k/2-1} \cdot (-2) = k(1-2t)^{-k/2-1}$$

$$E(X) = \left[M_X'(t) \right]_{t=0} = M_X'(0) = k.$$

$$M_X''(t) = k(k+2)(1-2t)^{-k/2-2} \quad (\text{watch!})$$

$$E(X^2) = \left[M_X''(t) \right]_{t=0} = k(k+2).$$

$$V(X) = E(X^2) - [E(X)]^2 = k(k+2) - k^2 = 2k.$$
