$$T : \underset{x}{\mathbb{R}^n} \longrightarrow \underset{Ax}{\mathbb{R}^m} \quad \text{by}$$

$$T(X) = \underset{m \times n}{A} \; \underset{n \times 1}{X}.$$

$$\operatorname{Ker} T = \{ x \in \mathbb{R}^n \mid T(X) = 0 \}$$

$$= \{ x \in \mathbb{R}^n \mid AX = 0 \}$$

$$= N(A).$$ (Null space of A)

$$T^* : \mathbb{R}^m \to \mathbb{R}^n$$

$$T^*(X) = \underset{n \times m}{A^T} \; \underset{m \times 1}{X}.$$

$$\operatorname{Ker} T^* = \{x \in \mathbb{R}^m \mid T^*(x) = 0\}$$

$$= \{x \in \mathbb{R}^m \mid A^T x = 0\}$$

$$= N(A^T) \quad \text{(null space of } A^T\text{)}$$

(*)

$C(A)$ = span of columns of A.

$\rightarrow$ is a subspace of $\mathbb{R}^m$.

Any vector space V is always a direct sum of a subspace and its orthogonal complement.

In particular, $\mathbb{R}^m = C(A) \oplus C(A)^\perp$.

Result : $C(A)^\perp = \,?$ (Answer: $N(A^T)$).

Proof : Take $Y \in C(A)^\perp$.

$$\Leftrightarrow \langle Y, Ax \rangle = 0 \quad \forall\, x \in \mathbb{R}^n \text{ (domain)}$$

(Defn. of $\perp$)

$$\Leftrightarrow Y^T(Ax) = 0 \quad \forall\, x \in \mathbb{R}^n$$

(vectorization)

$$\Leftrightarrow (Y^T A) X = 0 \quad \forall \; X \in \mathbb{R}^n$$

(Associativity of matrix multiplication)

$$\Leftrightarrow (A^T Y)^T X = 0 \quad \forall \; X \in \mathbb{R}^n$$

$$\Leftrightarrow \langle A^T Y, X \rangle = 0 \quad \forall \; X \in \mathbb{R}^n$$

$\Leftrightarrow$ $A^T Y$ is perpendicular to every vector X in $\mathbb{R}^n$.

$$\Leftrightarrow A^T Y = 0$$

$$\Leftrightarrow X \in \ker T^* = N(A^T)$$

(by (*)).

$$\therefore \boxed{C(A)^\perp = N(A^T)}$$

Eg: Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\mathbb{R}^3 = \underset{\text{plane}}{C(A)} \oplus \underset{\text{line}}{C(A)^\perp}$

$\underset{m}{3} \times 2 = n$

$$C(A) = \left\{ \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; \alpha_i \in \mathbb{R} \right\}$$

$\hookrightarrow$ plane

$$C(A)^\perp = \left\{ x \in \mathbb{R}^3 \;\middle|\; \langle x, y \rangle = 0 \;\; \forall \; y \in C(A) \right\}$$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$N(A^T) = \{ x \in \mathbb{R}^3 \mid A^T x = 0 \}$$

$$= \left\{ \underset{(x,y,z)}{x} \in \mathbb{R}^3 \;\middle|\; \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \right\}$$

$$= \left\{ (x, y, z) \;\middle|\; \begin{array}{l} x + z = 0 \\ y + z = 0 \end{array} \right\}$$

$$= \{ (-z, -z, z) \mid z \in \mathbb{R} \}$$

$$= \{ z(-1, -1, 1) \mid z \in \mathbb{R} \}$$ ↪ a line.