Consider $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Then

$$AA^T = \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}$$

$$A^TA = \underbrace{\begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}}_{2\times 3}$$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

The non-zero eigenvalues (and corresponding eigenvectors) of AA^T and A^TA are related. This relation gives rise to a decomposition of A called the Singular Value Decomposition (SVD).

Hint: Given $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$, test whether the non-zero eigen values of AA^T and A^TA coincide (that is, they are equal).

(Two eigen values of A^TA are 360, 90.

$$\lambda_1 + \lambda_2 + \lambda_3 = 80 + 170 + 200 = 450$$

$$360 + 90 + \lambda_3 = 450$$

Third eigenvalue of A^TA should be $\lambda_3 = 0$)

Let $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3}$

$$AA^T = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 333 & 81 \\ 81 & 117 \end{bmatrix}_{2\times 2}$$

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

AA^T

Eigenvalues

$\lambda_1 = 360$

$\lambda_2 = 90$ (done)

unit eigen vectors

$$u_1 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \quad \text{(done)}$$

$$u_2 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Eigenvalues

$\lambda_1 = 360$ $\quad$ $\lambda_2 = 90$ $\quad$ $\lambda_3 = 0$

$$\lambda_1 + \lambda_2 + \lambda_3 = 450 \Rightarrow \lambda_3 = 0$$

Prof. Kedukodi Babushri Srinivas, Department of Mathematics, MIT Manipal

$$A^T x = A^T u_1$$

$$= \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$A^T x = \frac{1}{\sqrt{10}} \begin{bmatrix} 20 \\ 40 \\ 40 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

↑ Make it unit vector

unit eigen vector $= v_1 = \frac{1}{\sqrt{9}}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$

$$A^T u_2 = \begin{bmatrix}4 & 8\\11 & 7\\14 & -2\end{bmatrix}\begin{bmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{bmatrix}$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix}-20\\-10\\20\end{bmatrix}$$

← Make it unit vector

$$= \beta \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

unit eigen vector, $v_2 = \dfrac{1}{\sqrt{9}} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$\lambda_3 = 0$

$\therefore (A^T A) X = 0 \cdot X$

$$\begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$80x + 100y + 40z = 0$$

$$4x + 5y + 2z = 0$$

$$V_3 = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}_{3\times 1}$$

$$= \sqrt{360} \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} = \sqrt{360}\, u_1$$

(verify it)

$$Av_2 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3} \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} = \sqrt{90} \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} = \sqrt{90}\, u_2$$

(verify it)

$$\therefore \quad Av_1 = \sqrt{\lambda_1}\, u_1$$

$$\Rightarrow \quad Av_1 = \sigma_1 u_1 \quad \text{where} \quad \sigma_1 = \sqrt{\lambda_1}$$

(σ_1 is called singular value).

$$Av_1 = \sqrt{\lambda_1}\, u_1 = \sqrt{360}\, u_1$$

$$Av_2 = \sqrt{\lambda_2}\, u_2 = \sqrt{90}\, u_2$$

$$Av_3 = 0 = 0$$

$$\underset{2\times 3}{A} \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}_{3\times 3}}_{V} = \underbrace{\begin{bmatrix} u_1 & u_2 \end{bmatrix}_{2\times 2}}_{U} \underbrace{\begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}_{2\times 3}}_{\Sigma}$$

$$\Rightarrow \quad AV = U\Sigma$$

$$\Rightarrow \quad A = U\Sigma V^{-1}$$

$$A = U \Sigma V^T$$

$V^T \to$ rotation/reflection

$\Sigma \to$ scaling

$U \to$ rotation or reflection

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}_{2\times 3} = \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix}_{2\times 2} \end{array} \begin{array}{c} \text{scaling matrix} \\ \uparrow \\ \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}_{2\times 3} \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}^{T}_{3\times 3} \end{array}$$

$$\underset{2\times 3}{A} = U \cdot \Sigma \cdot V^T$$

This decomposition is called the singular value decomposition (SVD) of the matrix A.

$$\underset{\sigma_1}{\underset{\downarrow}{\sqrt{360}}},\ \underset{\sigma_2}{\underset{\downarrow}{\sqrt{90}}} \longrightarrow \text{are called singular values of } A.$$

Q: Find SVD of $B = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix}_{3\times 2}$ (Tall Matrix)

S/: $B_{3\times 2} = A^T$

$= (U \Sigma V^T)^T$

$= V \Sigma^T U^T$

$$= \begin{bmatrix} \frac{1}{3} & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}_{3\times 3} \begin{bmatrix} \sqrt{360} & 0 \\ 0 & \sqrt{90} \\ 0 & 0 \end{bmatrix}_{3\times 2} \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \end{bmatrix}_{2\times 2}$$

H.W.: Find SVD of

$$A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}$$

Ans:

$$A = \begin{bmatrix} 4 & -2\sqrt{2} & 4 \\ 4 & 2\sqrt{2} & 4 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{array}{c} \begin{matrix} u_1 & u_2 \end{matrix} \\ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \overset{\sigma_1}{8} & 0 & 0 \\ 0 & \underset{\sigma_2}{4} & 0 \end{bmatrix} \begin{array}{c} \begin{matrix} v_1 & \quad v_2 & \quad v_3 \end{matrix} \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T \end{array}$$

A has two non-zero singular values σ_1 and σ_2. This implies that A has rank 2 (i.e one redundant column; see 1st and last column of A).