Given $A_{m\times n}$. $T: \mathbb{R}^n \to \mathbb{R}^m$

$$T(X) = \underset{m\times n}{A} X_{n\times 1}$$

$$C(A) \oplus C(A)^{\perp} = \mathbb{R}^m.$$

codomain.

Revisit to the Question: $C(A)^{\perp} = ?$

Result : $C(A)^{\perp} = N(A^T).$

Proof:

Take $y \in C(A)^{\perp}$.

inner product in $\mathbb{R}^n$.

for all

$$\Leftrightarrow \langle y, Ax \rangle = 0 \quad \forall \; x \in \mathbb{R}^n \quad \text{(by definition)}$$

$$\Leftrightarrow y^T(Ax) = 0 \quad \forall \; x \in \mathbb{R}^n \quad \text{(vectorization)}$$

$$\Leftrightarrow (y^T A)x = 0 \quad \forall \; x \in \mathbb{R}^n \quad \text{(associativity)}$$

$$\Leftrightarrow (A^T y)^T x = 0 \quad \forall \; x \in \mathbb{R}^n$$

$$\Leftrightarrow \langle A^T y, x \rangle = 0 \quad \forall \; x \in \mathbb{R}^n$$

Inner product in $\mathbb{R}^n$.

$\Leftrightarrow$ $A^T y$ is perpendicular to every vector x in $\mathbb{R}^n$.

$$\Leftrightarrow A^T y = 0$$

$$\Leftrightarrow y \in N(A^T).$$

$$\therefore \; C(A)^{\perp} = N(A^T)$$

So, $C(A) \oplus N(A^T) = \mathbb{R}^m$ —— (1)

$T^* : \mathbb{R}^m \to \mathbb{R}^n$

$T^*(y) = \underset{n\times m}{A^T} \overset{m\times 1}{y}.$

$C(A^T)^\perp = N((A^T)^T)$

$= N(A).$

Replace A by A^T in (1).

Then $C(A^T) \oplus N(A) = \mathbb{R}^n.$

$C(A) \oplus N(A^T) = \mathbb{R}^m$

$C(A^T) \oplus N(A) = \mathbb{R}^n$

Four Fundamental subspaces, given a matrix A:

are: $C(A)$, $C(A^T)$, $N(A)$, $N(A^T)$.

Eg:

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{b}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$AX = b$

$A\hat{X} = \hat{b}$ where $\hat{X} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$.

$$\hat{e} = b - \hat{b}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ -1/3 \\ 1/3 \end{bmatrix}.$$

Now,

$$\hat{b} = A\hat{x} \in C(A)$$

$$b = \underbrace{\hat{b} + \hat{e}}_{\text{representation is unique.}}.$$

We have $N(A^T) = \{z(-1, -1, 1) \mid z \in \mathbb{R}\}$

$$= \text{span}\left\{\begin{bmatrix}-1\\-1\\1\end{bmatrix}\right\}$$

$\Rightarrow \hat{e} \in N(A^T).$

Given $V = W \oplus W^{\perp}$. Take $x \in V$

Suppose $x = x_1 + y_1 = x_2 + y_2$; $x_1, x_2 \in W$ &

$y_1, y_2 \in W^{\perp}$.

To show : $x_1 = x_2$

$y_1 = y_2$

$$x_1 + y_1 = x_2 + y_2$$

$$\Rightarrow -x_2 + x_1 = y_2 - y_1 \in W^{\perp}$$ (as $W^{\perp}$ is a subspace of V)

Also, $-x_2 + x_1 \in W$ (as W is a subspace of V)

$$\Rightarrow -x_2 + x_1 \in W \cap W^{\perp} = \{0\}$$

$$\Rightarrow x_1 = x_2$$

$$\Rightarrow \underline{y_1 = y_2}.$$

$C(A)$: 2-dim.

$N(A^T)$: 1-dim.

$$\hat{e} = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \text{line } (1\text{-dim. error vector})$$

Now, $R(A) = C(A^T)$.

$\uparrow$ row space of A

$C(A^T) \oplus N(A) = \mathbb{R}^n \quad \leftrightarrow \quad R(A) \oplus N(A) = \mathbb{R}^n.$

$C(A) \oplus N(A^T) = \mathbb{R}^m \quad \leftrightarrow \quad R(A^T) \oplus N(A^T) = \mathbb{R}^m.$

Q. Find the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}_{\underset{m}{\underset{n}{3}} \times 2 = n}.$$

Soln: $m = 3$

$n = 2$

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$T(X) = \underset{3\times 2}{A} X_{2\times 1}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$.

i). $R(A^T)$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \text{Echelon form.}$$

$r = \text{rank}(A^T) = 2$

$\Rightarrow R(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow r = \dim R(A^T) = 2.$

ii) $N(A^T) = \{ x \in \mathbb{R}^3 \mid A^T x = 0 \}$

$$= \left\{ X = (x, y, z) \in \mathbb{R}^3 \;\middle|\; \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \right\}$$

$$= \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x + z = 0 \\ y + z = 0 \end{array} \right\}$$

$$= \{ (-z, -z, z) \mid z \in \mathbb{R} \}$$

$$= \{ z(-1, -1, 1) \mid z \in \mathbb{R} \}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \dim(N(A^T)) = 1.$$

$$\langle (-1,-1,1), (1,0,1) \rangle = -1\cdot 1 + (-1)\cdot 0 + 1\cdot$$

$$= -1+1 = 0$$

$$\langle (-1,-1,1), (0,1,1) \rangle = (-1)\cdot 0 + (-1)\cdot 1 + 1\cdot 1$$

$$= 0.$$

$$\underset{\dim 2}{R(A^T)} \oplus \underset{\dim 1}{N(A^T)} = \underset{\dim 3}{\mathbb{R}^m} \quad (m=3)$$

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iii) $R(A)$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_1$.

$$A \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$. $\quad A \sim \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_{red}}$. $\longrightarrow$ Echelon form

$$r = \text{rank } A = 2$$

$$R(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

$$\Rightarrow \dim R(A) = 2.$$

iv). $$N(A) = \{ x \in \mathbb{R}^2 \mid \underset{3\times 2}{A} \underset{2\times 1}{x} = 0 \}$$

$$= \{ x = (x, y) \in \mathbb{R}^2 \mid A_{red}\, x = 0 \}$$

$$= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$N(A) = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{matrix} x = 0 \\ y = 0 \end{matrix} \right\}$$

$$= \{(0,0)\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

$$\Rightarrow \dim N(A) = 0.$$

$$\langle (0,0), (1,0) \rangle = 0 = \langle (0,0), (0,1) \rangle.$$

$$N(A) \cap R(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

$$\underbrace{R(A)}_{\dim 2} \oplus \underbrace{N(A)}_{\dim 0} = \underbrace{\mathbb{R}^n}_{\dim 2} \quad (n = 2).$$