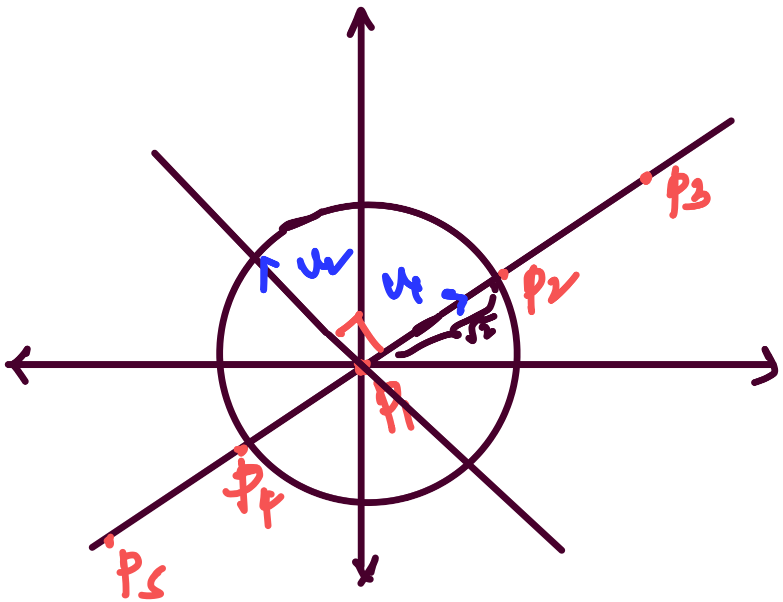
PCA :

$$X : \begin{array}{ccccc} p_1 & p_2 & p_3 & p_4 & p_5 \\ \end{array}$$
$$\begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix}_{2\times 5}.$$



Step 1 (change of origin) :

$$X_c = X - \bar{X}$$
$$= X - 0$$
$$= X$$

centered data

(done)

<u>Step 2</u> : SVD of X_c. (done).

<u>Step 3</u> : Interpretation using projections.

projection of p_2 on u_1 $= \dfrac{\langle p_2, u_1 \rangle}{\|u_1\|^2}$

$$= \frac{\langle p_2, u_1 \rangle}{1^2}$$

$$= \langle p_2, u_1 \rangle$$

$$= \langle u_1, p_2 \rangle$$

$$= u_1^T p_2$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} (1+1)$$

$$= \sqrt{2}.$$

↳ 1D data.

$$\frac{\langle p_2, u_1 \rangle}{\| u_1 \|^2} u_1 = \sqrt{2}\, u_1$$

$$= \sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= p_2 .$$

Vectorize the above calculation as $U^T X$.

$$U^T X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix}$$

$$= \begin{array}{ccccc} p_1 & p_2 & p_3 & p_4 & p_5 \\ \left[\begin{array}{ccccc} 0 & \sqrt{2} & 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

→ projection coefficent on PC1 (u_1)

projection coeffient on PC2 (u_2)

called PC scores.

Move $2\sqrt{2}$ along PC1 (u_1) and 0 along PC2 (u_2) to get p_3

that is, $2\sqrt{2}\, u_1 = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} = p_3 .$$

Move $-\sqrt{2}$ along PC1 (u_1) and 0 along PC2 (u_2) to get p_4.

$$-\sqrt{2}\, u_1 = -\sqrt{2} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = p_4$$

$$X_c = U \Sigma V^T$$

$$\Rightarrow \underbrace{U^T X_c}_{\text{PC scores}} = U^T (U \Sigma V^T)$$

$$= \underbrace{(U^T U)}_{I} \Sigma V^T$$

$$= \Sigma V^T \quad \text{(as } U \text{ is orthogonal)}$$

$$= \begin{bmatrix} \sqrt{20} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{0} & 0 & 0 & 0 \end{bmatrix}_{2\times 5} \frac{1}{\sqrt{10}} \begin{bmatrix} 0 & * & * & * & * \\ 1 & * & * & * & * \\ 2 & * & * & * & * \\ -1 & * & * & * & * \\ -2 & * & * & * & * \end{bmatrix}^T_{5\times 5} .$$

$$= \begin{bmatrix} 0 & \sqrt{2} & 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{2\times 5} .$$

$\leftarrow$ 1D Data.

$$U.(\text{PC scores}) = U(\Sigma V^T)$$

$$= X_c .$$

$$U \cdot (\text{PC scores}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2} & 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 & -1 & -2 \\ 0 & 1 & 2 & -1 & -2 \end{bmatrix}$$

$\uparrow$ original centred 2D data.

$$= X_c$$

Solving Difference Equations :

Fibonacci sequence : $\underset{F_0}{0}, \underset{F_1}{1}, \underset{F_2}{1}, \ldots, 2, 3, 5, 8, 13, 21, \ldots$

$$F_{n+2} = F_{n+1} + F_n \quad ; \quad n = 0, 1, 2, \ldots$$

(called recurrence relation or difference equation)

$$F_{n+1} = F_{n+1}$$

2D form of Fibonacci sequence.

$$\Rightarrow \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

We will go to 1D again at last.

$$\bar{F}_{n+1} = A\bar{F}_n \quad ; \quad \text{where} \quad \bar{F}_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix},$$

$$\bar{F}_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

<u>Finding</u> F_n :

$$\bar{F}_{n+1} = A\bar{F}_n$$

$$= A\,(A\bar{F}_{n-1})$$

$$= A^2\bar{F}_{n-1}$$

$$= A^2 (A\bar{F}_{n-2})$$

$$= A^3 \bar{F}_{n-2}$$

$$\vdots$$

$$= A^{n+1} \bar{F}_{n-n}$$

$$= A^{n+1} \bar{F}_0$$

$$= A^{n+1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$$= A^{n+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 = \text{trace}(A) = 1$$

$$\lambda_1\lambda_2 = \det(A) = -1$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(1-\lambda) - 1 = 0$$

$$\Rightarrow -\lambda + \lambda^2 - 1 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$= \underbrace{\frac{1+\sqrt{5}}{2}}_{\lambda_1} \text{ or } \underbrace{\frac{1-\sqrt{5}}{2}}_{\lambda_2}$$

golden ratio.

Eigen vector corresponding to λ_1 :

$$\begin{bmatrix} 1-\lambda_1 & 1 \\ 1 & -\lambda_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad (1-\lambda_1)x + y = 0$$

$$x - \lambda_1 y = 0$$

$$\Rightarrow \quad \lambda_2 x + y = 0$$

$$x - \lambda_1 y = 0$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -\lambda_2 \end{bmatrix}.$$

Consider λ_2 :

$$(1-\lambda_2)x + y = 0$$

$$x - \lambda_2 y = 0$$

$$\Rightarrow \quad \lambda_1 x + y = 0$$

$$x - \lambda_2 y = 0$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -\lambda_1 \end{bmatrix}$$ ← eigenvector corresponding to λ_1

$$P = \begin{bmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{bmatrix} \quad ; \quad P^{-1} = \frac{1}{-\sqrt{5}} \begin{bmatrix} -\lambda_1 & -1 \\ \lambda_2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A = PDP^{-1} \quad \text{and} \quad \det P = -\lambda_1 + \lambda_2 = -\sqrt{5}.$$

$$A^{n+1} = P D^{n+1} P^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1^{n+1} & 0 \\ 0 & \lambda_2^{n+1} \end{bmatrix} \frac{1}{-\sqrt{5}} \begin{bmatrix} -\lambda_1 & -1 \\ \lambda_2 & 1 \end{bmatrix}$$

$$= -\frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+1} & \lambda_2^{n+1} \\ -\lambda_2\lambda_1^{n+1} & -\lambda_1\lambda_2^{n+1} \end{bmatrix} \begin{bmatrix} -\lambda_1 & -1 \\ \lambda_2 & 1 \end{bmatrix}$$

$$= -\frac{1}{\sqrt{5}} \begin{bmatrix} -\lambda_1^{n+2} + \lambda_2^{n+2} & -\lambda_1^{n+1} + \lambda_2^{n+1} \\ \lambda_2\lambda_1^{n+2} - \lambda_1\lambda_2^{n+2} & \lambda_2\lambda_1^{n+1} - \lambda_1\lambda_2^{n+1} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+2} - \lambda_2^{n+2} & \lambda_1^{n+1} - \lambda_2^{n+1} \\ \lambda_2\lambda_1^{n+2} - \lambda_1\lambda_2^{n+2} & \lambda_2\lambda_1^{n+1} - \lambda_1\lambda_2^{n+1} \end{bmatrix}$$

$$A^{n+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+2} - \lambda_2^{n+2} \\ \lambda_2\lambda_1^{n+2} - \lambda_1\lambda_2^{n+2} \end{bmatrix}$$

that is, $\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^{n+2} - \lambda_2^{n+2} \\ \lambda_2 \lambda_1^{n+2} - \lambda_1 \lambda_2^{n+2} \end{bmatrix}$

$F_{n+2} = ?$

$F_n = ?$

(1D form)

$$\Rightarrow F_{n+2} = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\sqrt{5}} \quad \text{—— } (*) \text{ ; } n = 0, 1, 2, \ldots$$

(Infact, F_{n+2} is the number of binary strings of length n without consecutive 1s. $\rightarrow$

eg: $n = 2$
00, 01, 10
$F_{n+2} = F_4 = 3$.)

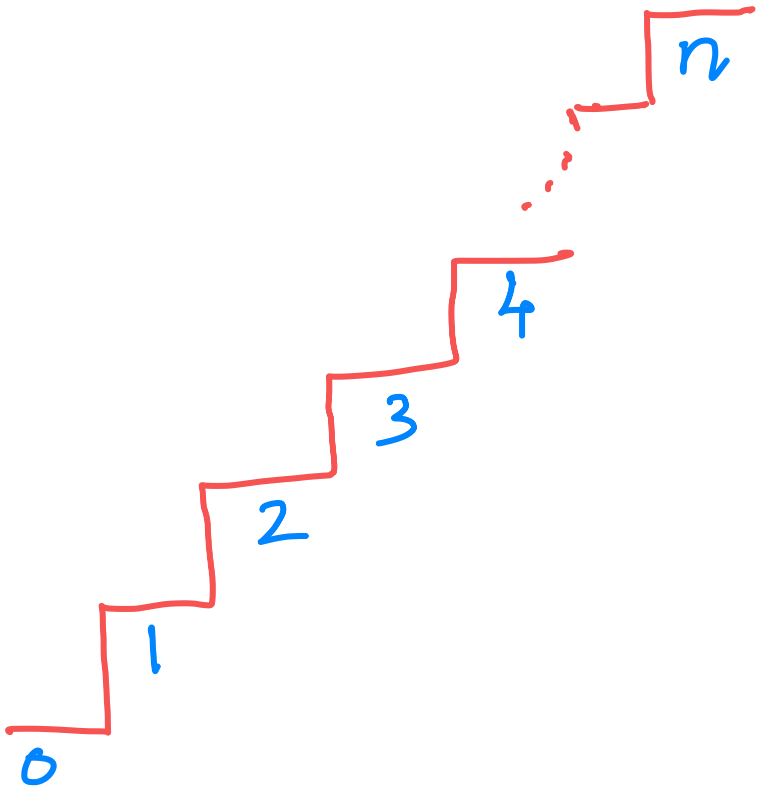
$$\Rightarrow F_n = \frac{\lambda_1^{n} - \lambda_2^{n}}{\sqrt{5}}$$

(Replace n by $n-2$ in $(*)$)

$$\Rightarrow F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

This formula is called Binet's Formula for F_n.

H.W:



Given n stairs, in how many ways you can reach nth stair if you are allowed to take either 1 step (stair) or 2 steps (stairs) at a time?

(Hint: Answer is F_n).