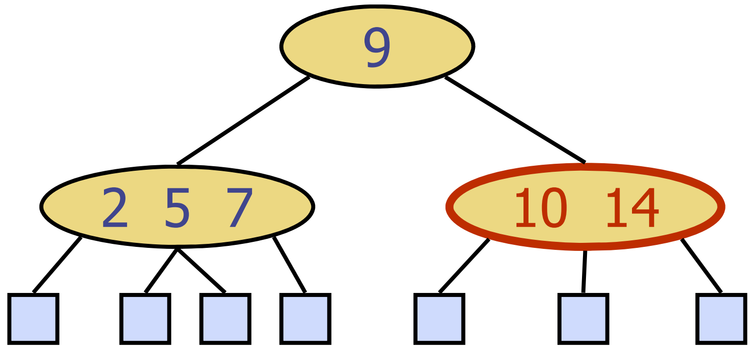
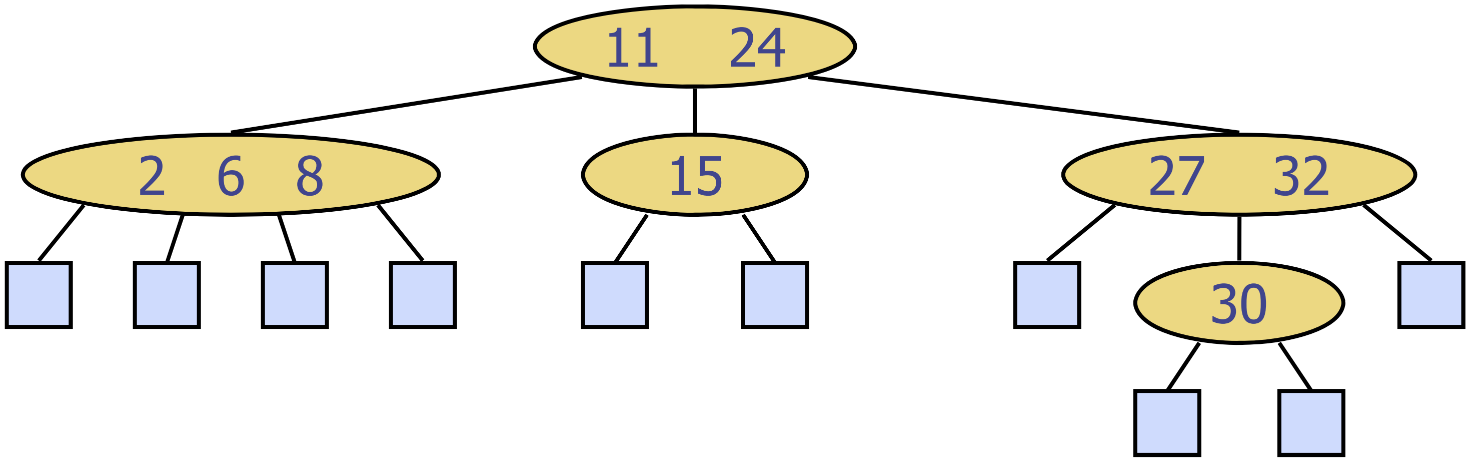
Presentation for use with the textbook Data Structures and Algorithms in Java, 6th edition, by M. T. Goodrich, R. Tamassia, and M. H. Goldwasser, Wiley, 2014

(2,4) Trees



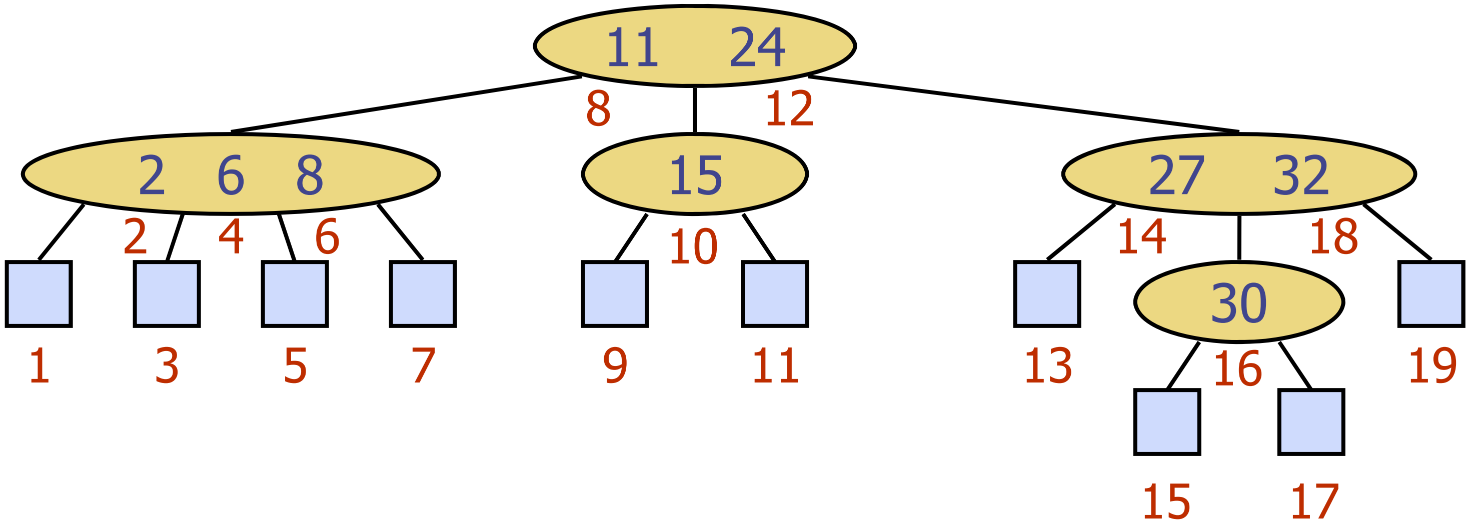
Multi-Way Search Tree

- A multi-way search tree is an ordered tree such that
  - Each internal node has at least two children and stores $d-1$ key-element items (k_i, o_i), where d is the number of children
  - For a node with children $v_1\, v_2\, \ldots\, v_d$ storing keys $k_1\, k_2\, \ldots\, k_{d-1}$
    - keys in the subtree of v_1 are less than k_1
    - keys in the subtree of v_i are between k_{i-1} and k_i $(i = 2, \ldots, d-1)$
    - keys in the subtree of v_d are greater than k_{d-1}
  - The leaves store no items and serve as placeholders



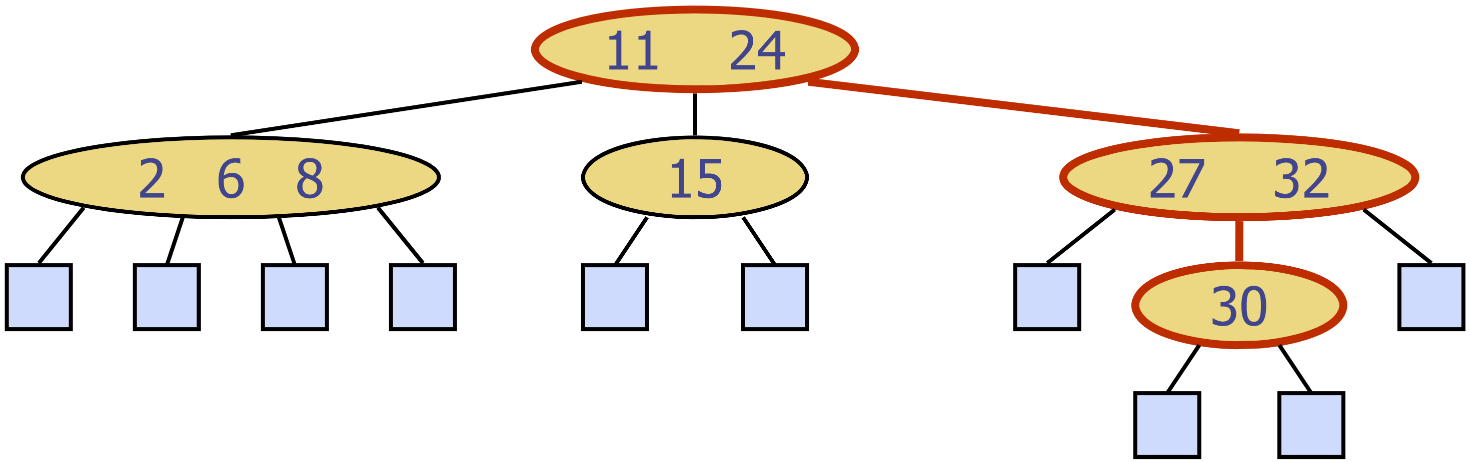
Multi-Way Inorder Traversal

- extend traversal from binary trees
- visit entry (k_i, o_i) of node v
  - between the recursive traversals of the subtrees of v rooted at children v_i ("left") and v_{i+1} ("right")
- Consequently, visit the keys in increasing order



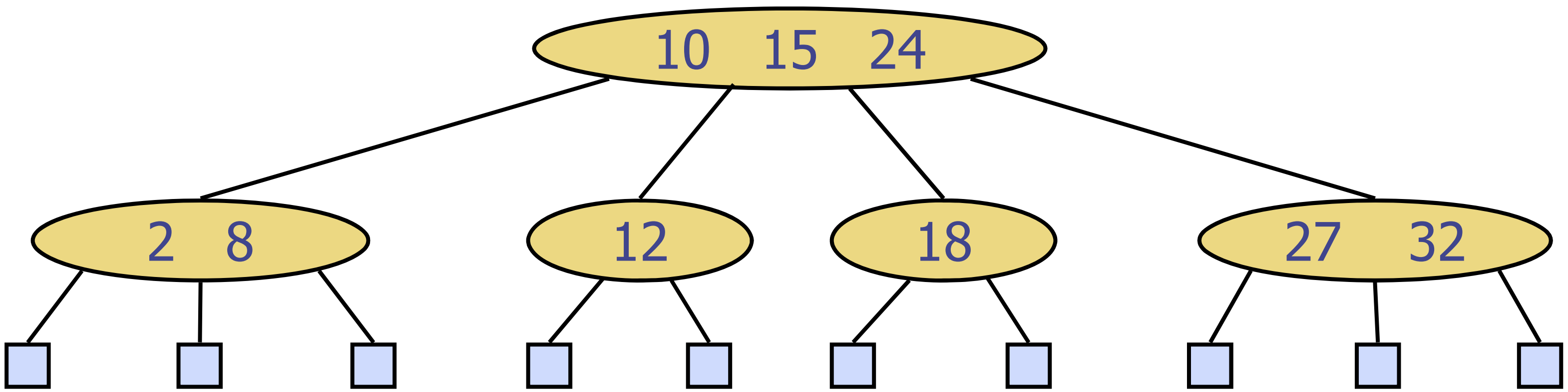
Multi-Way Searching

- Similar to search in a binary search tree
- A each internal node with children $v_1\ v_2\ \ldots\ v_d$ and keys $k_1\ k_2\ \ldots\ k_{d-1}$
  - $k = k_i$ $(i = 1, \ldots, d-1)$: the search terminates successfully
  - $k < k_1$: we continue the search in child v_1
  - $k_{i-1} < k < k_i$ $(i = 2, \ldots, d-1)$: we continue the search in child v_i
  - $k > k_{d-1}$: we continue the search in child v_d
- Reaching an external node terminates the search unsuccessfully
- Example: search for 30



(2,4) Trees

- A (2,4) tree (also called 2-4 tree or 2-3-4 tree) is a multi-way search with the following properties
  - Node-Size Property: every internal node has at most four children
  - Depth Property: all the external nodes have the same depth
- Depending on the number of children
  - an internal node of a (2,4) tree is called a 2-node, 3-node or 4-node

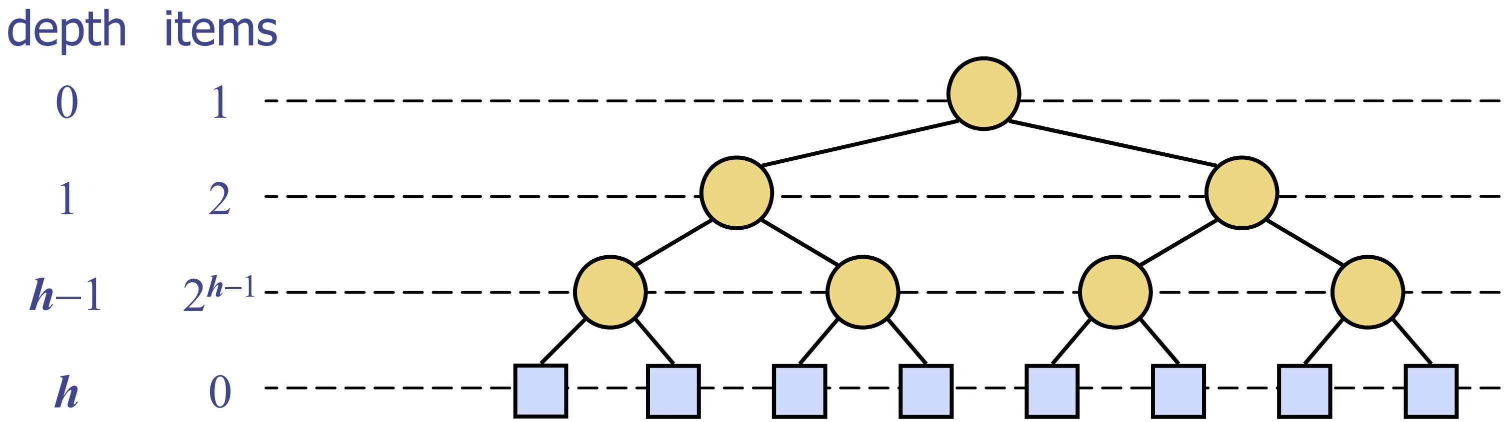


Height of a (2,4) Tree

- **Theorem:** A (2,4) tree storing n items has height $O(\log n)$

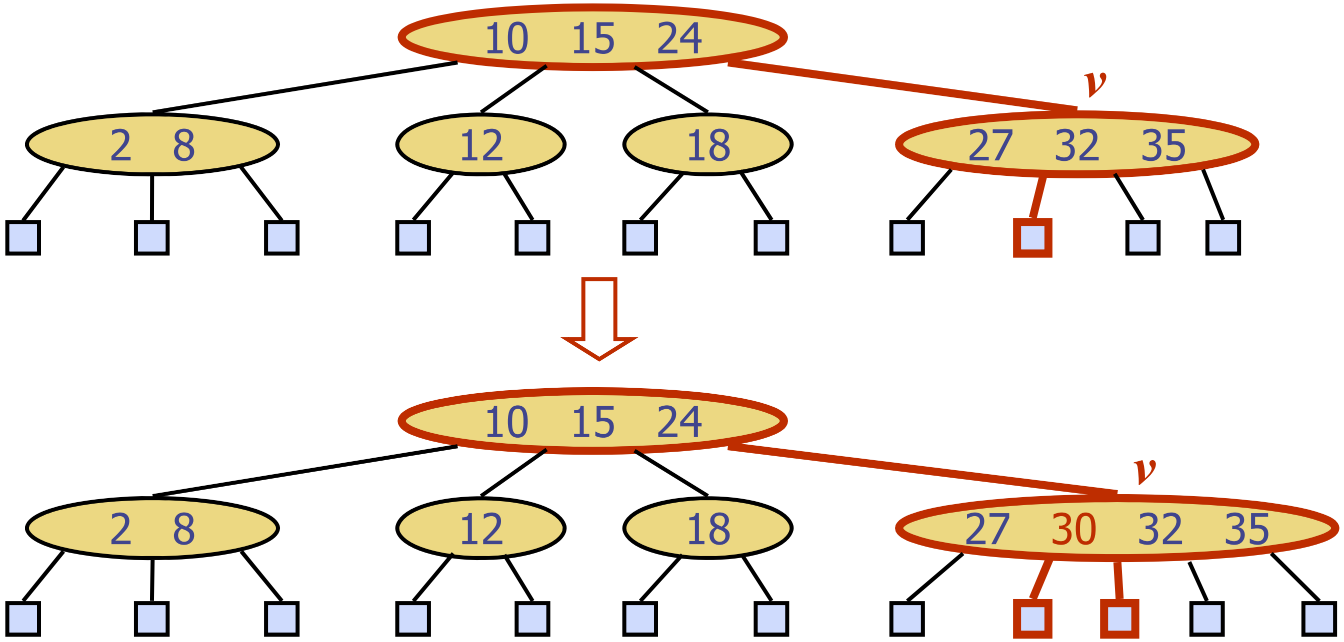
  Proof (worst case is complete binary tree):
  - Let h be the height of a (2,4) tree with n items
  - Since there are at least 2^i items at depth $i = 0, \ldots, h-1$ and no items at depth h, we have

    $$n \geq 1 + 2 + 4 + \ldots + 2^{h-1} = 2^h - 1$$
  - Thus, $h \leq \log(n + 1)$
- Searching in a (2,4) tree with n items takes $O(\log n)$ time



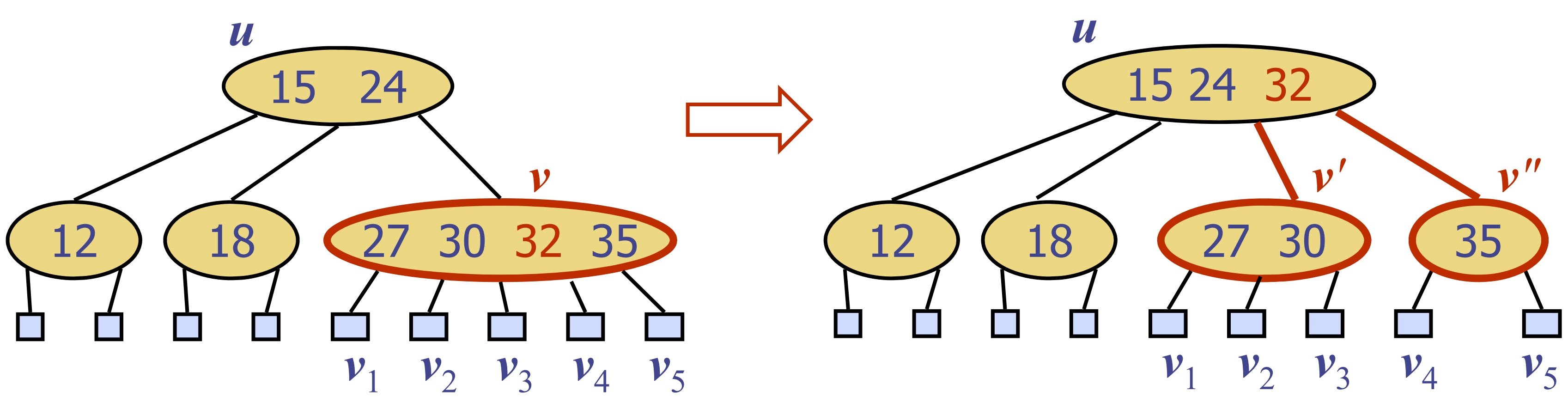
Insertion

- We insert a new item (k, o) at the parent v of the leaf reached by searching for k
  - We preserve the depth property but
  - We may cause an overflow (i.e., node v may become a 5-node)
- Example: inserting key 30 causes an overflow

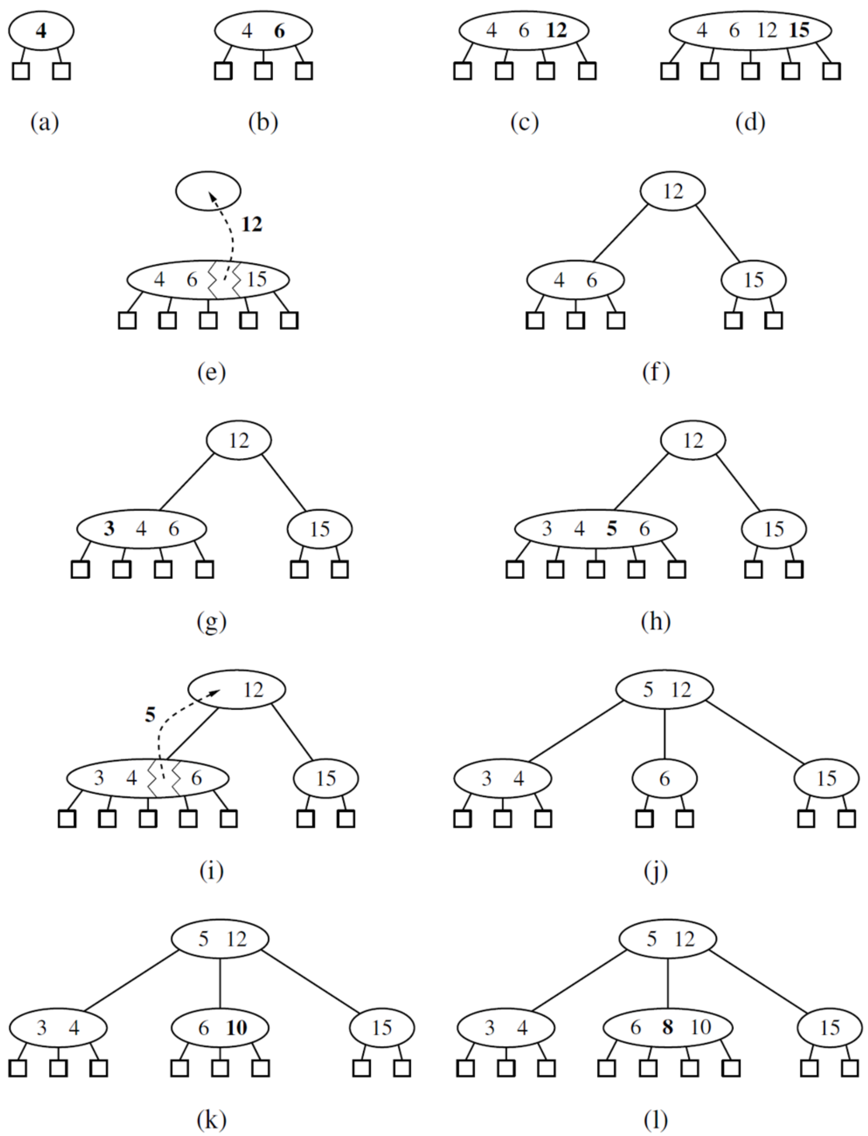


Overflow and Split

- We handle an overflow at a 5-node v with a split operation:
  - let $v_1 \dots v_5$ be the children of v and $k_1 \dots k_4$ be the keys of v
  - node v is replaced nodes v' and v''
    - v' is a 3-node with keys k_1 k_2 and children v_1 v_2 v_3
    - v'' is a 2-node with key k_4 and children v_4 v_5
  - key k_3 is inserted into the parent u of v (a new root may be created)
- The overflow may propagate to the parent node u



Example: Insert 4,6,12,15,3,5,10,8



Pseudocode of Insertion

Algorithm *put*(*k*, *o*)

1. We search for key ***k*** to locate the insertion node ***v***
2. We add the new entry (***k***, ***o***) at node ***v***
3. **while** ***overflow***(***v***)

    if ***isRoot***(***v***)

        create a new empty root above ***v***

    v $\leftarrow$ ***split***(***v***) # split returns a parent node if any

Analysis of Insertion

Algorithm *put*(k, o)

1. We search for key k to locate the insertion node v
2. We add the new entry (k, o) at node v
3. **while** *overflow*(v)

    if *isRoot*(v)

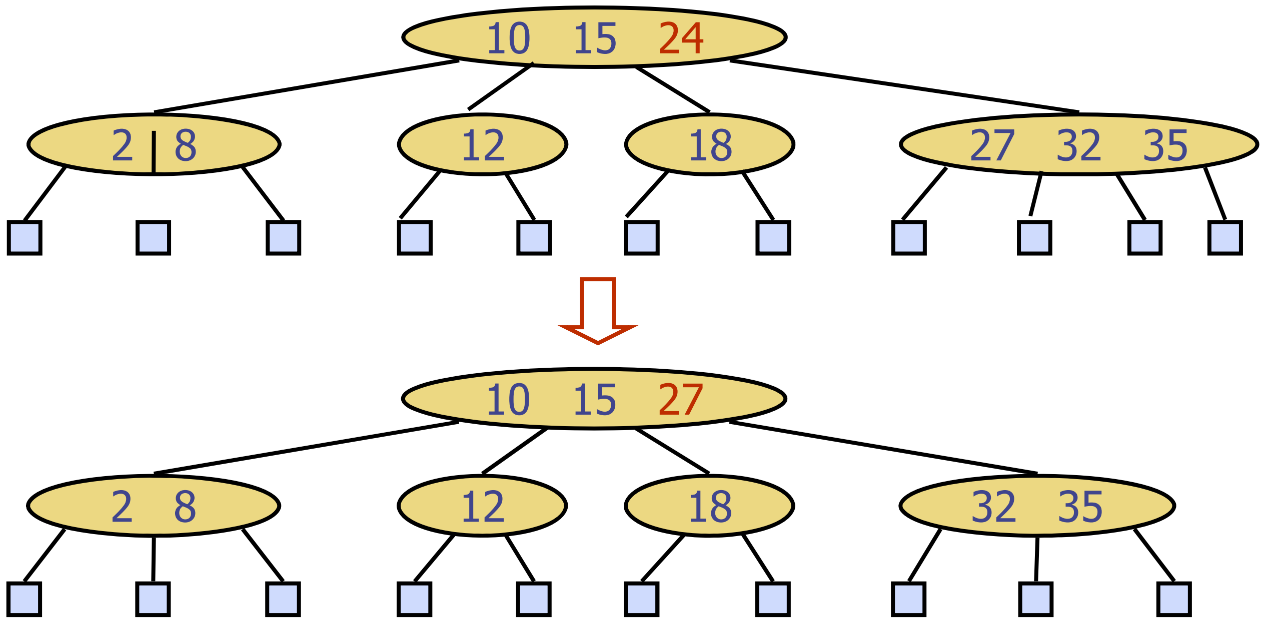
        create a new empty root above v

    $v \leftarrow$ *split*(v)

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
  - Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
  - Step 2 takes $O(1)$ time
  - Step 3 takes $O(\log n)$ time because each split takes $O(1)$ time and we perform $O(\log n)$ splits
- Thus, an insertion in a (2,4) tree takes $O(\log n)$ time

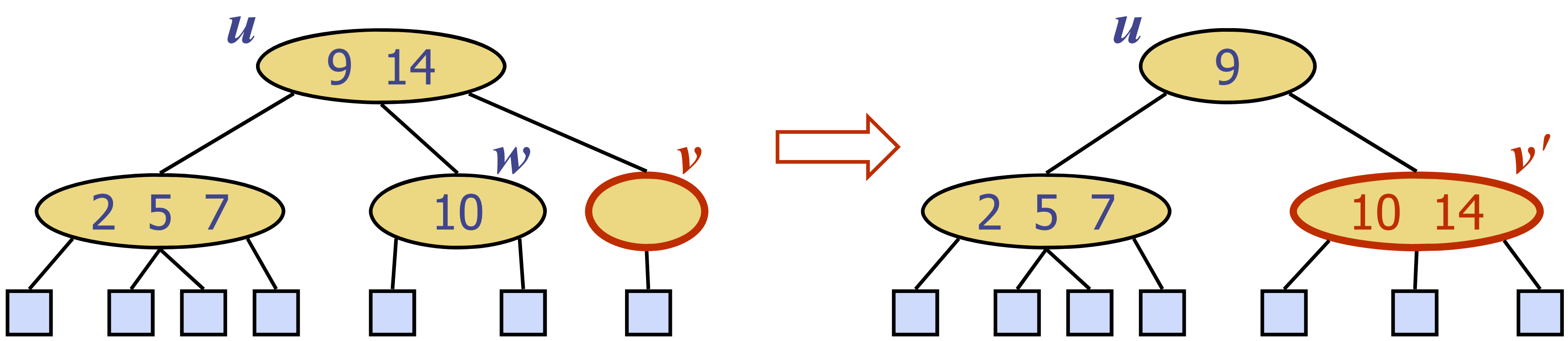
Deletion

- We reduce deletion of an entry to the case where the item is at the node with leaf children
- Otherwise, we replace the entry with its inorder successor (or, equivalently, with its inorder predecessor) and delete the latter entry
- Example: to delete key 24, we replace it with 27 (inorder successor)



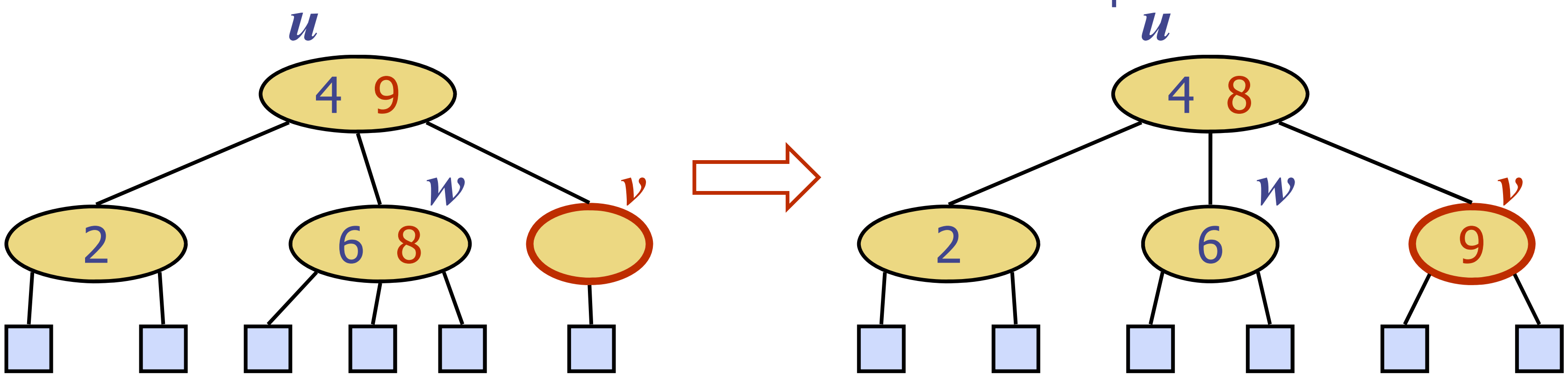
Underflow and Fusion

- Deleting an entry from a node v may cause an underflow
  - node v becomes a 1-node with one child and no keys
- To handle an underflow at node v with parent u, we consider two cases
- Case 1: the adjacent siblings of v are 2-nodes
  - Fusion operation: we merge v with an adjacent sibling w and move an entry from u to the merged node v'
  - After a fusion, the underflow may propagate to the parent u

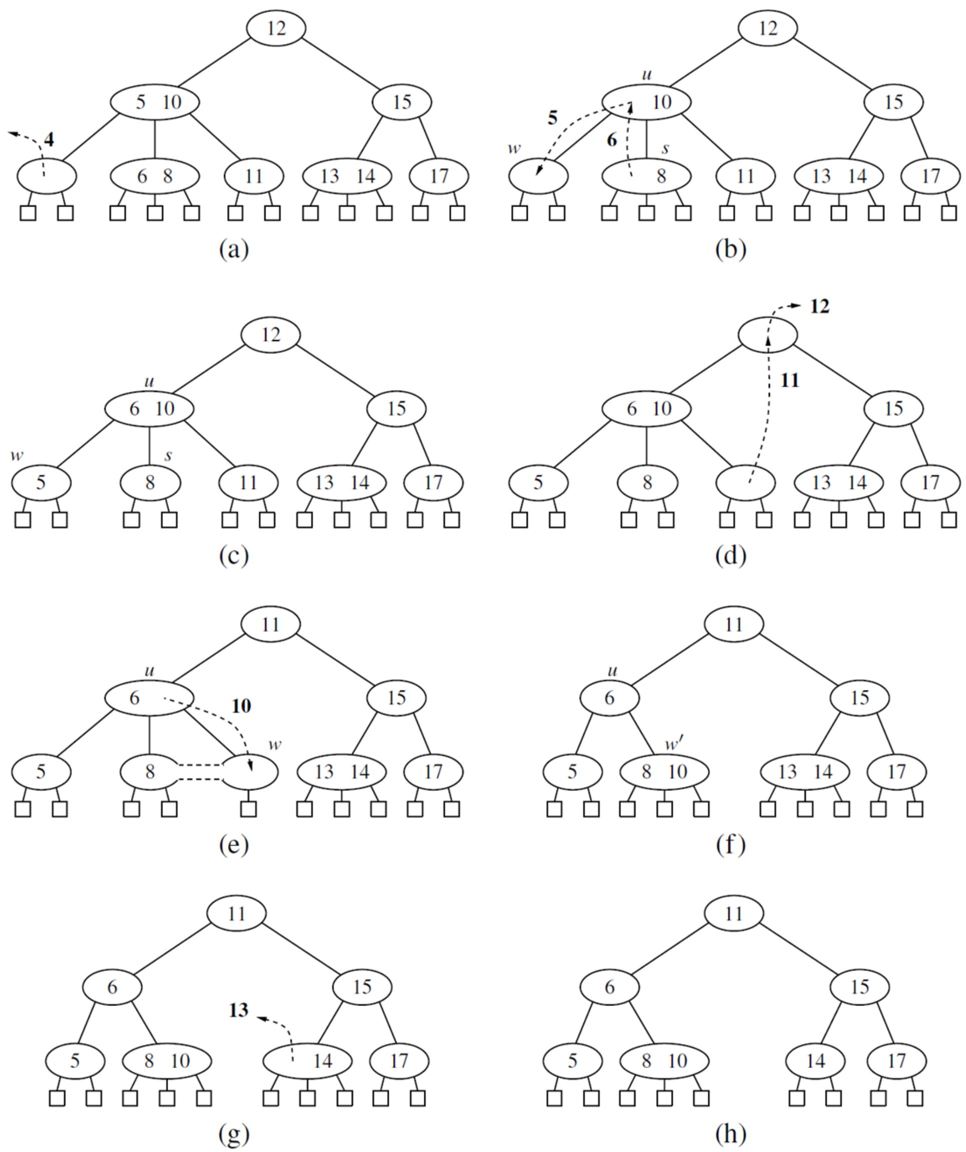


Underflow and Transfer

- To handle an underflow at node v with parent u, we consider two cases
- Case 2: an adjacent sibling w of v is a 3-node or a 4-node
  - Transfer operation:
    1. we move a child of w to v
    2. we move an item from u to v
    3. we move an item from w to u
  - After a transfer, no underflow occurs
  - Default: Do transfer first if case 1 & 2 are both possible



Example: Delete 4,12,13



Analysis of Deletion

- Let T be a (2,4) tree with n items
  - Tree T has $O(\log n)$ height
- In a deletion operation
  - We visit $O(\log n)$ nodes to locate the node from which to delete the entry
  - We handle an underflow with
    - a (0 or more) series of $O(\log n)$ fusions (one fewer key/child in parent),
    - followed by at most one transfer (no changes in children)
  - Each fusion and transfer takes $O(1)$ time
- Thus, deleting an item from a (2,4) tree takes $O(\log n)$ time

Comparison of Map Implementations

	Search	Insert	Delete	Notes
Sorted Array	log n	n	n	○ ordered map methods ○ simple to implement
AVL and (2,4) Tree	log n worst-case	log n worst-case	log n worst-case	○ ordered map methods ○ complex to implement

In-Class Exercise

- Implement the search & other utility functions
- Download source code from Brightspace
- Upload your source code to Gradescope