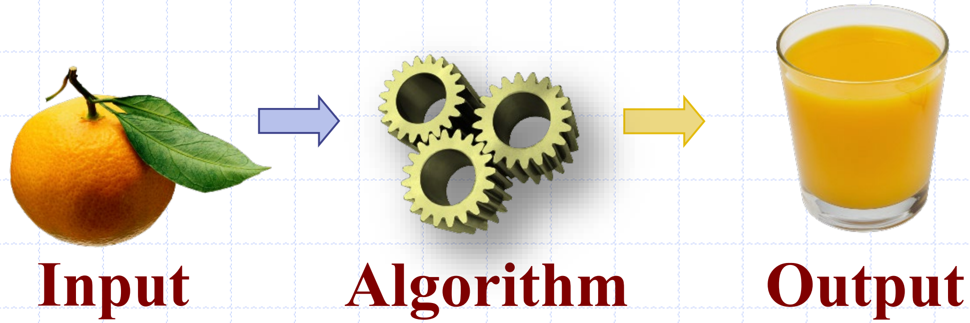
Analysis of Algorithms

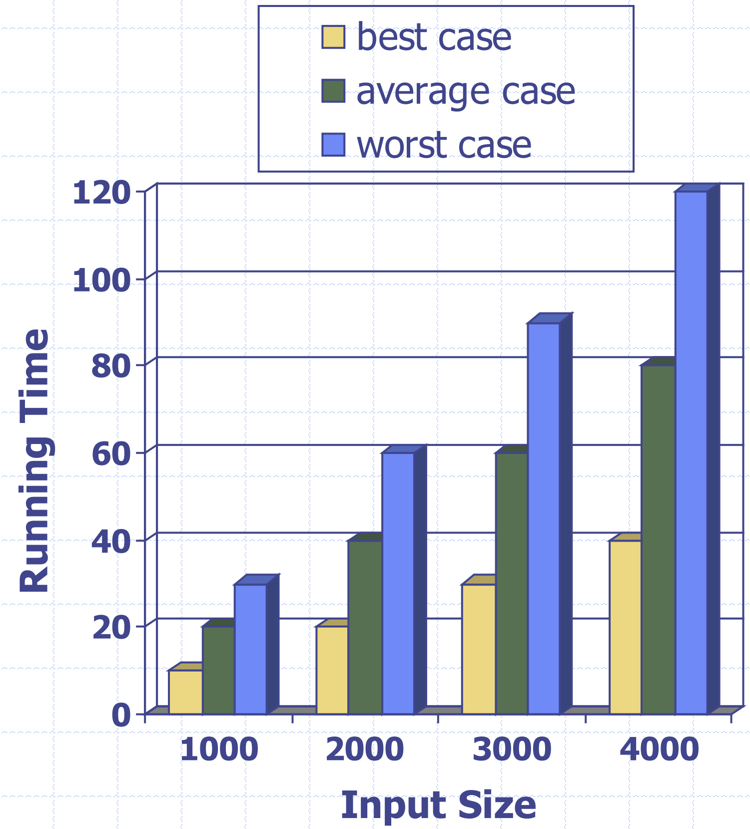


How fast is your algorithm?

- Low memory usage?
- Small amount of time measured on a stopwatch?
- Low power consumption?

Running Time

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics

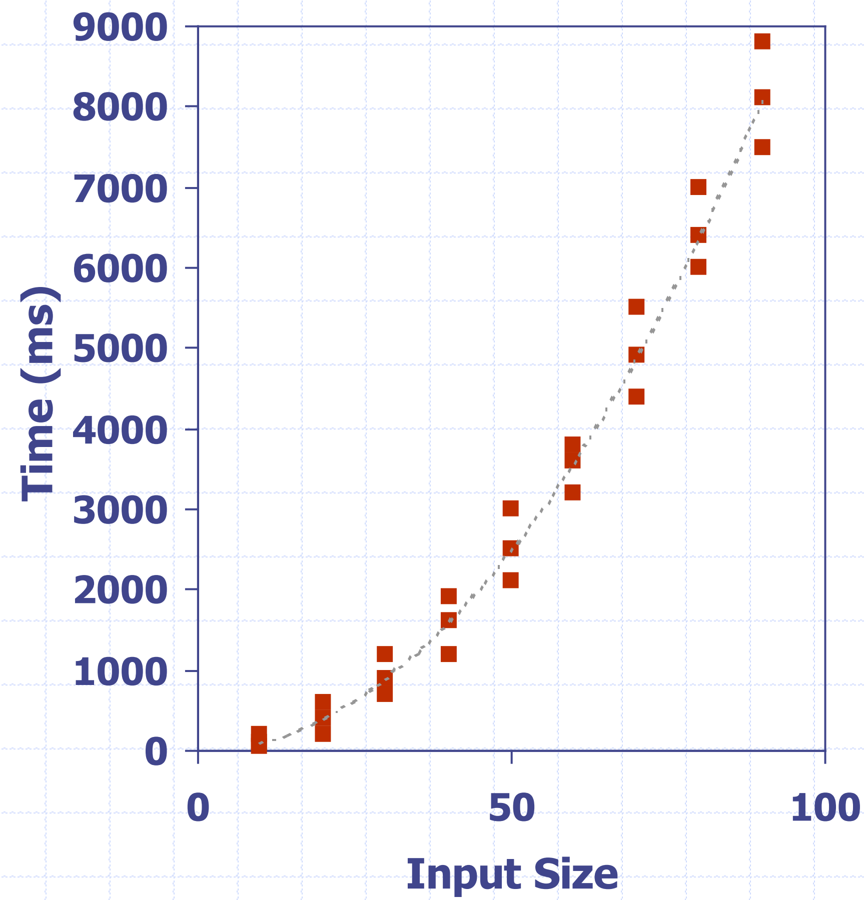


Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition, noting the time needed:

```
from time import time
start_time = time( )
run algorithm
end_time = time( )
elapsed = end_time − start_time
```

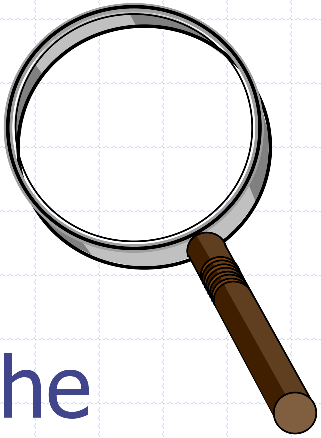
- Plot the results



Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used

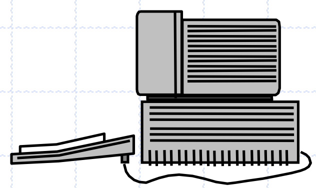
Theoretical Analysis



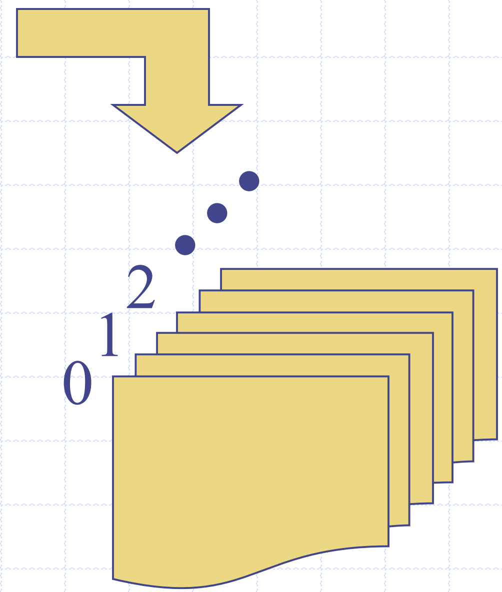
- Uses a high-level description of the algorithm instead of an implementation
- Characterizes running time as a function of the input size, n.
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment

The Random Access Machine (RAM) Model

- A **CPU**



- An potentially unbounded bank of **memory** cells, each of which can hold an arbitrary number or character



- Memory cells are numbered and accessing any cell in memory takes unit time.

Elementary Operations

- Algorithmic "time" is measured in elementary operations
  - Math (+, -, *, /)
  - Comparisons (==, >, <=, ...)
  - Function calls and value returns
  - Variable assignment
  - Variable increment or decrement
  - Array allocation
  - Creating a new object (may have elementary ops too!)
- In practice, all of these operations take different amounts of time
- For the purpose of algorithm analysis, we assume each of these operations takes the same time: "1 operation"

Elementary Operations

- Basic computations performed by an algorithm
- Largely independent from the programming language
- Exact definition not important (we will see why later)
- Assumed to take a constant amount of time in the RAM model

- Examples:
  - Evaluating an expression
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

Example: Constant Running Time

```
function first(array):
    // Input: an array
    // Output: the first element
    return array[0] // index 0 and return, 2 ops
```

- How many operations are performed in this function if the list has ten elements? If it has 100,000 elements?

Example: Constant Running Time

```
function first(array):
    // Input: an array
    // Output: the first element
    return array[0] // index 0 and return, 2 ops
```

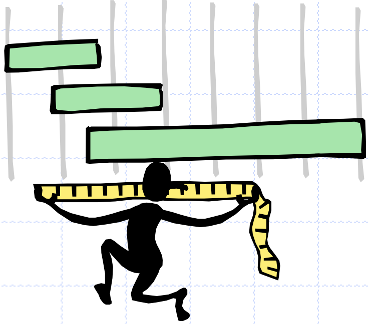
- How many operations are performed in this function if the list has ten elements? If it has 100,000 elements?
  - Always 2 operations performed
  - Does not depend on the input size

Example: Linear Running Time

```
def argmax(array):
   // Input: an array
   // Output: the index of the maximum value
   index = 0 // assignment, 1 op
   for i in range(len(array))://1 + 1 op per loop
      if array[i] > array[index]://3 ops per loop
         index = i // 1 op per loop, sometimes
   return index // 1 op
```

- How many operations if the list has 10 elements? 100,000 elements?
  - Varies proportionally to the size of the input list: $6n + 2$
  - We'll be in the `for` loop longer and longer as the input list grows
  - If we were to plot, the runtime would increase linearly

Estimating Running Time



- Algorithm **argmax** executes $6n + 2$ primitive operations in the worst case, $5n + 2$ in the best case. Define:

  a = Time taken by the fastest primitive operation

  b = Time taken by the slowest primitive operation

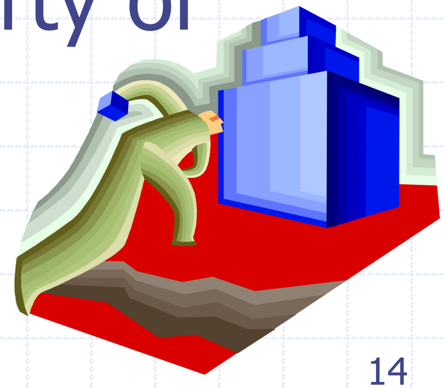
- Let $T(n)$ be worst-case time of **argmax.** Then

$$a\,(5n + 2) \leq T(n) \leq b(6n + 2)$$

- Hence, the running time $T(n)$ is bounded by two linear functions.

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects $T(n)$ by a constant factor, but
  - Does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm **argmax**



Example: Quadratic Running Time

```
def possible_products(array):
    // Input: an array
    // Output: a list of all possible products
    //    between any two elements in the list
    products = [] // make an empty list, 1 op
    for i in range(len(array)): // 1+1=2 op per loop
        for j in range(len(array)): // 1+1=2 op per loop per loop
            products.append(array[i] * array[j]) // 4 ops per loop per loop
    return products // 1 op
```

- Requires about $6n^2 + 2n + 2$ operations (okay to approximate!)
  - If we were to plot this, the number of operations executed grows quadratically!
- Consider adding one element to the list: the added element must be multiplied with every other element in the list
- Notice that the linear algorithm on the slide #14 had only one `for` loop, while this quadratic one has `two` for loops, nested. What would be the highest-degree term (in number of operations) if there were three nested loops?

Some Common Computing Times

$\log_2 n$	n	$n \log_2 n$	n^2	2^n
1	2	2	4	4
2	4	8	16	16
3	8	24	64	256
4	16	64	256	65,536
5	32	160	1,024	4,294,967,296
6	64	384	4,096	1.84×10^{19}
7	128	896	16,384	3.40×10^{38}
8	256	2,048	65,536	1.16×10^{77}
9	512	4,608	262,144	1.34×10^{154}
10	1,024	10,240	1,048,576	1.80×10^{308}

Slide by Matt Stallmann included with permission.

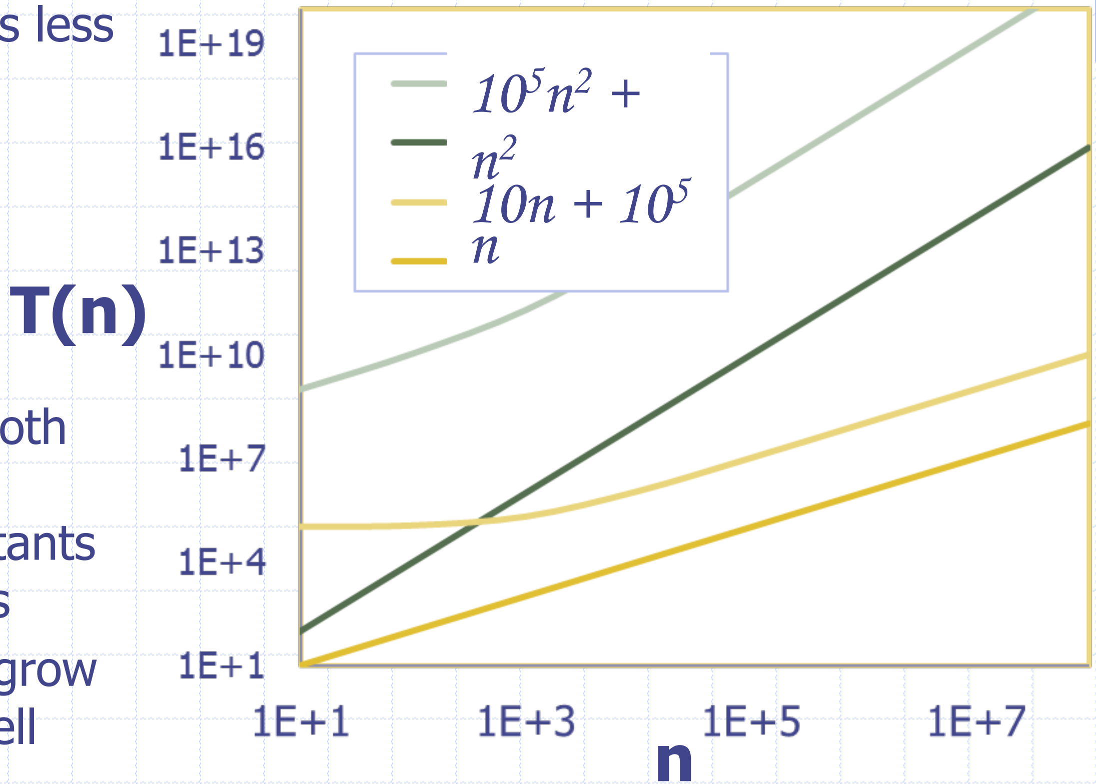
Why Growth Rate Matters

if runtime is...	time for n + 1	time for 2 n	time for 4 n
$c \lg n$	$c \lg (n + 1)$	$c (\lg n + 1)$	$c(\lg n + 2)$
$c n$	$c (n + 1)$	$2c n$	$4c n$
$c n \lg n$	$\sim c n \lg n + c n$	$2c n \lg n + 2cn$	$4c n \lg n + 4cn$
$c n^2$	$\sim c n^2 + 2c n$	$\mathbf{4c n^2}$	$16c n^2$
$c n^3$	$\sim c n^3 + 3c n^2$	$8c n^3$	$64c n^3$
$c 2^n$	$c 2^{n+1}$	$c 2^{2n}$	$c 2^{4n}$

runtime quadruples when problem size doubles

Summarizing Function Growth

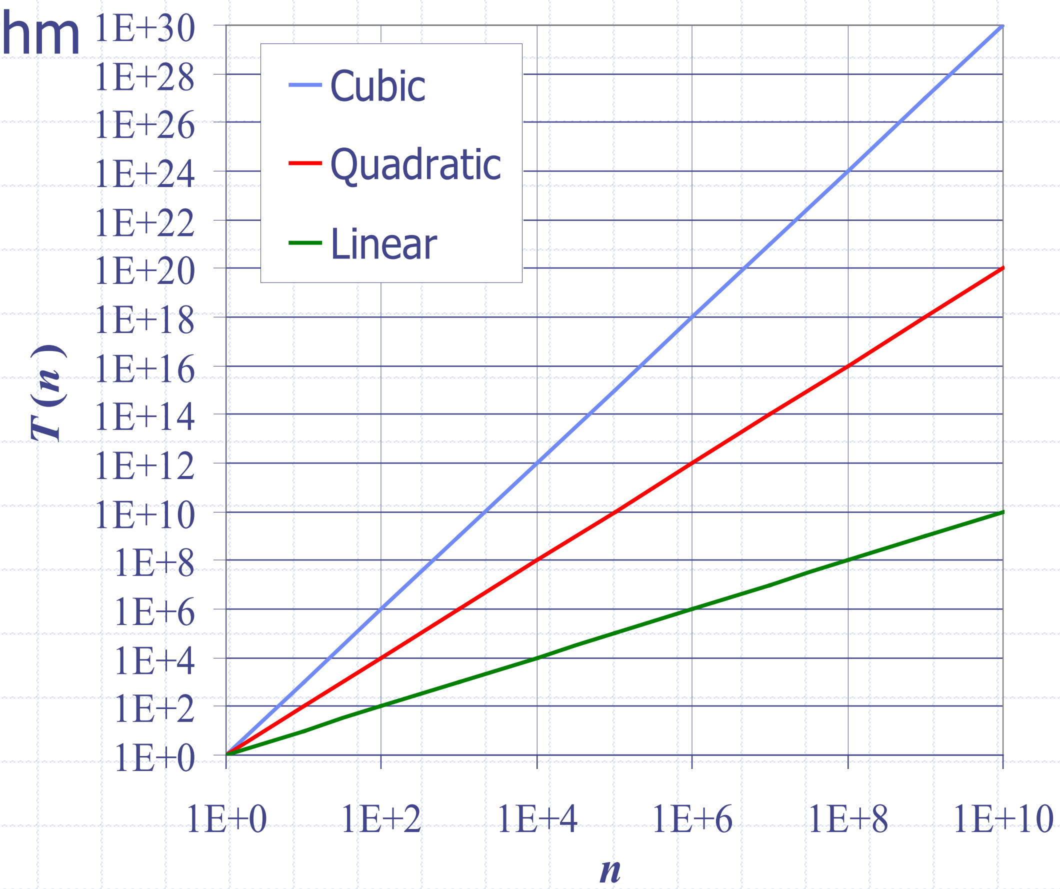
- For very large inputs, the growth rate of a function becomes less affected by:
  - constant factors or
  - lower-order terms
- Examples
  - $10^5n^2 + 10^8n$ and n^2 both grow with same slope despite differing constants and lower-order terms
  - $10n + 10^5$ and n both grow with same slope as well



In this graph (log scale on both axes), the slope of a line corresponds to the growth rate of its respective function

Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ 1
  - Logarithmic ≈ $\log n$
  - Linear ≈ n
  - N-Log-N ≈ $n \log n$
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
  - Exponential ≈ 2^n
- In a log-log chart, the slope of the line corresponds to the growth rate



Functions Graphed Using “Normal” Scale

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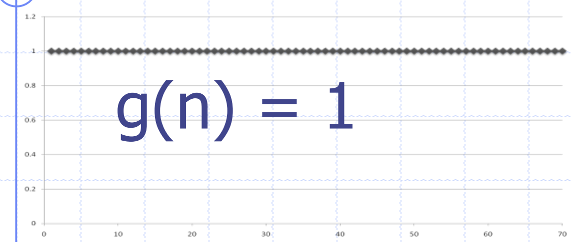
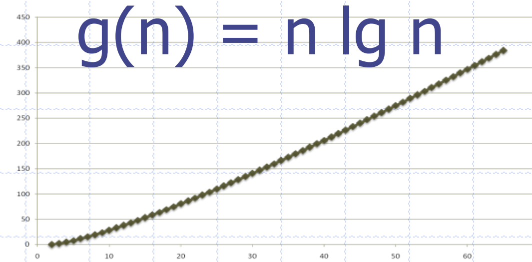
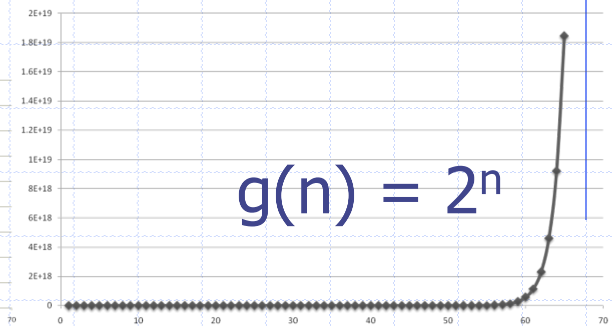
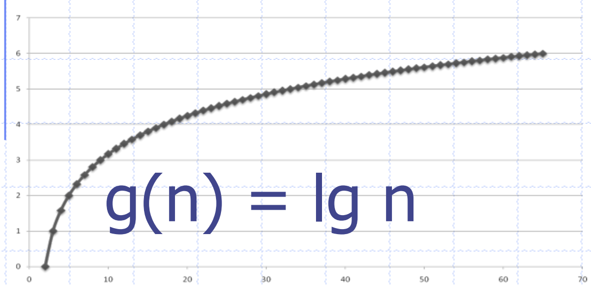
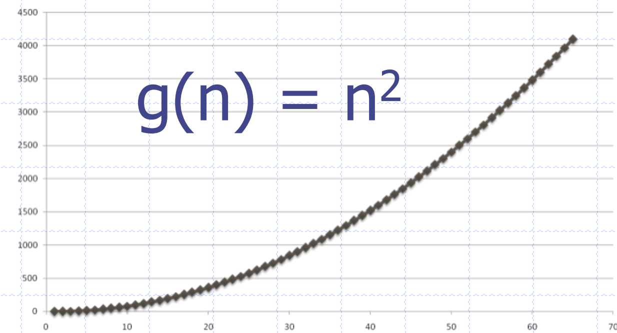
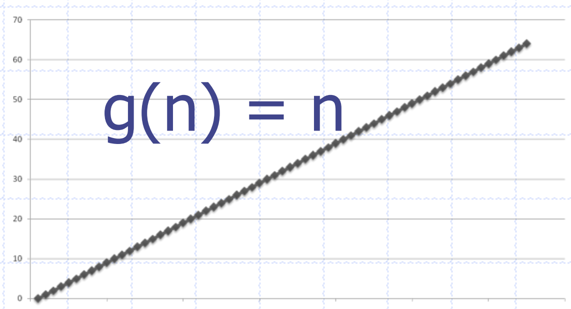


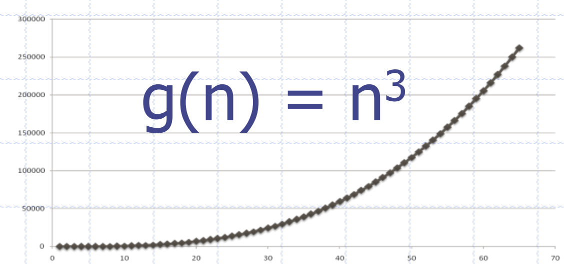




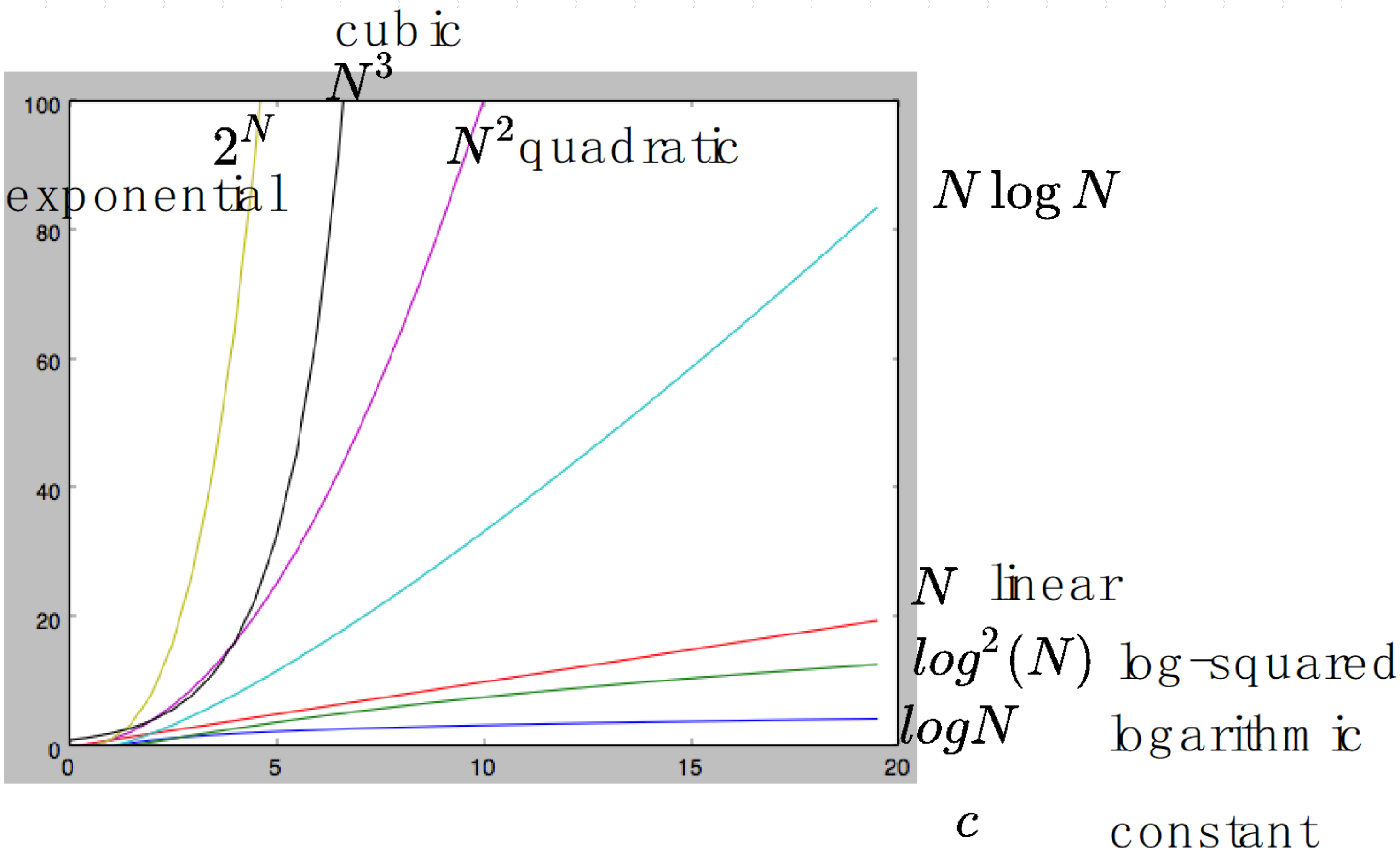








Typical Growth Rates



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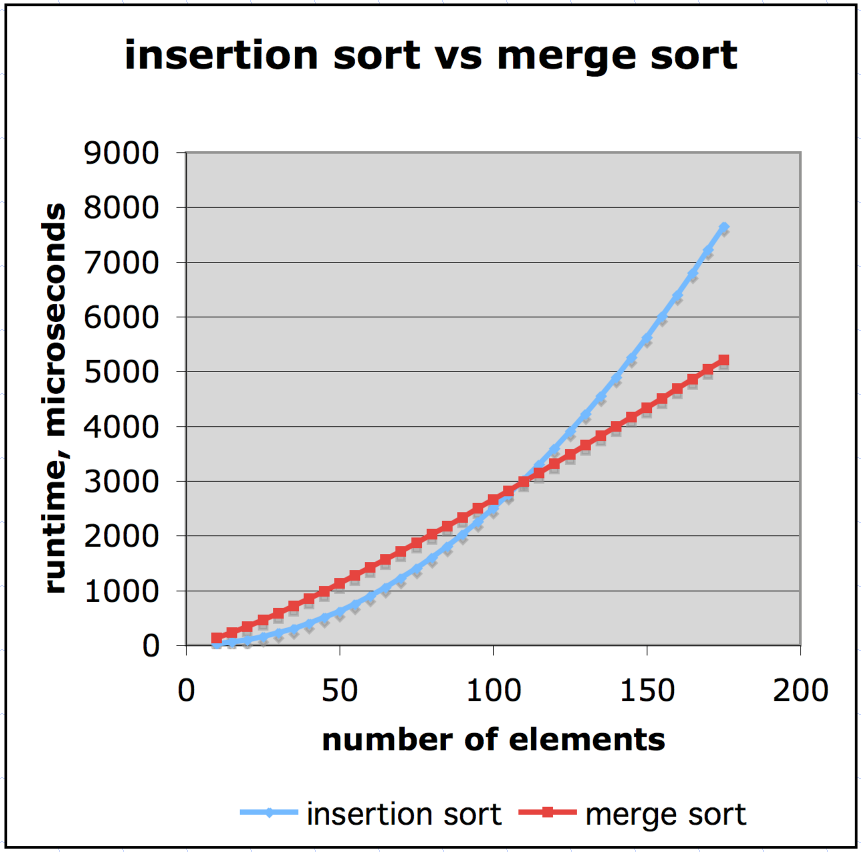
Comparison of Two Algorithms



insertion sort is

$n^2 / 4$

merge sort is

2 n lg n

sort a million items?

insertion sort takes roughly 70 hours

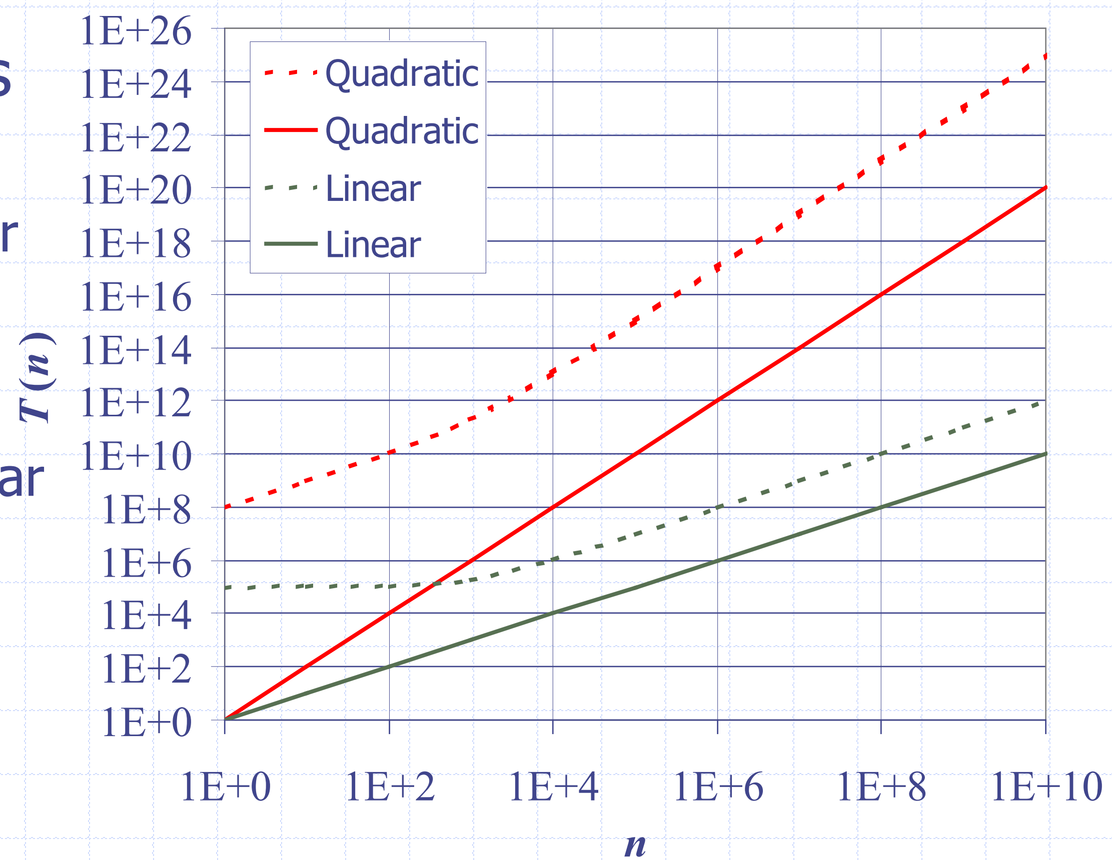
while

merge sort takes roughly 40 seconds

This is a slow machine, but if 100 x as fast then it's 40 minutes versus less than 0.5 seconds

Constant Factors

- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Comparison of Insertion Sort and Python Built In Sort Function

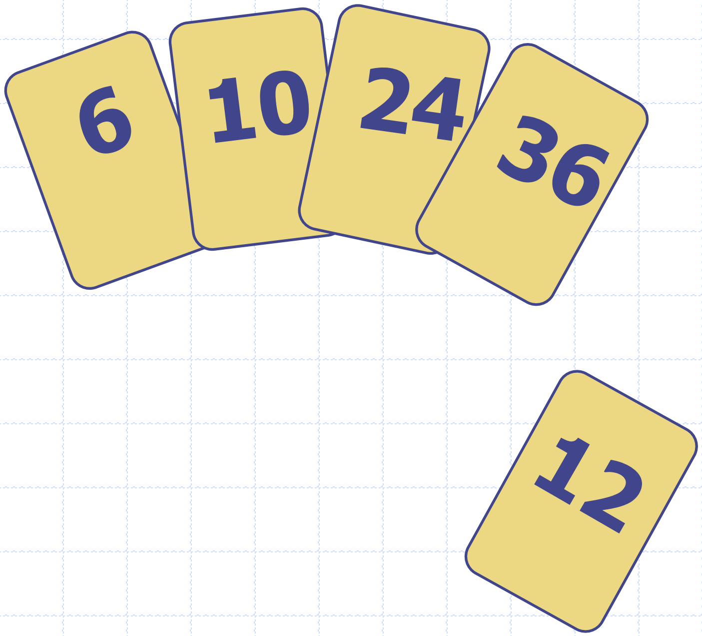
- Please go to Brightspace to open the InsertionVSbuiltinClassVersion.py file
- Implement Insertion sort in that code.
- Use the Python built in sort from list class
- Compare the runtime. Which one is better???
- Submit code to Gradescope

Insertion Sort

- Idea: like sorting a hand of playing cards
  - Start with an empty left hand and the cards facing down on the table.
  - Remove one card at a time from the table, and insert it into the correct position in the left hand
    - compare it with each of the cards already in the hand, from right to left
  - The cards held in the left hand are sorted
    - these cards were originally the top cards of the pile on the table

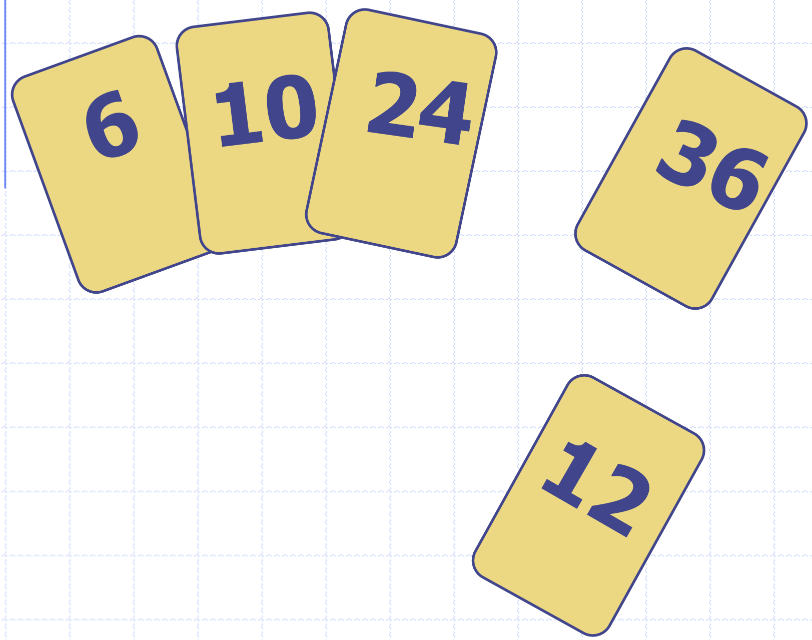
Visualization: https://visualgo.net/en/sorting

Insertion Sort

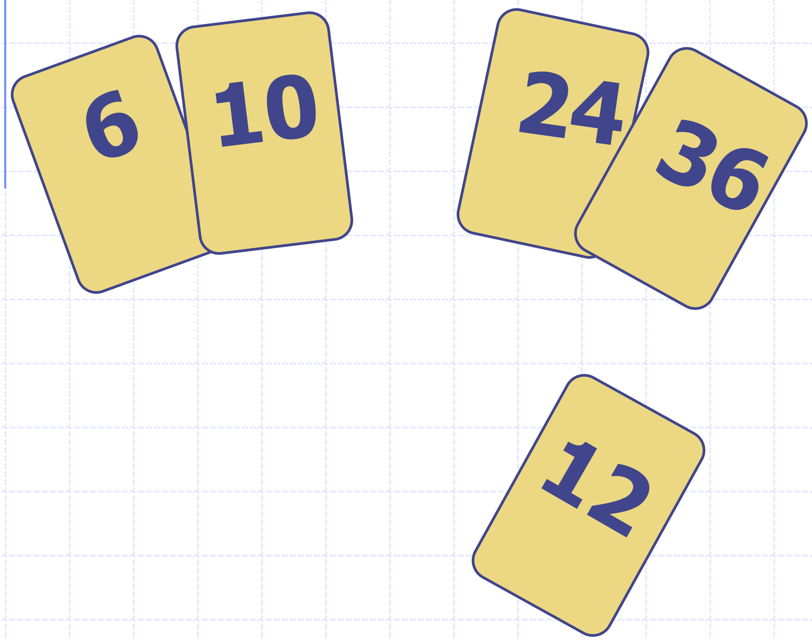


To insert 12, we need to make room for it by moving first 36 and then 24.

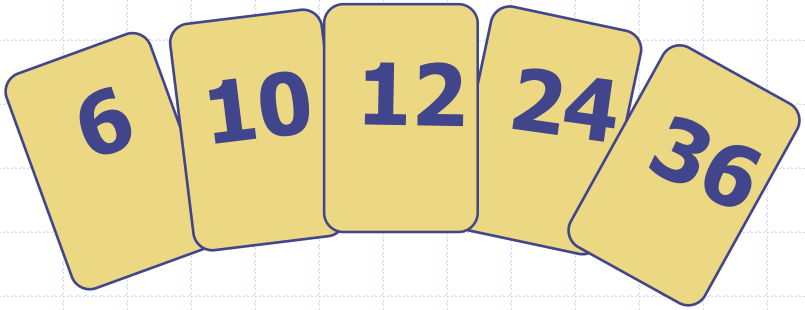
Insertion Sort



Insertion Sort



Insertion Sort



Insertion Sort

input array

5 2 4 6 1 3

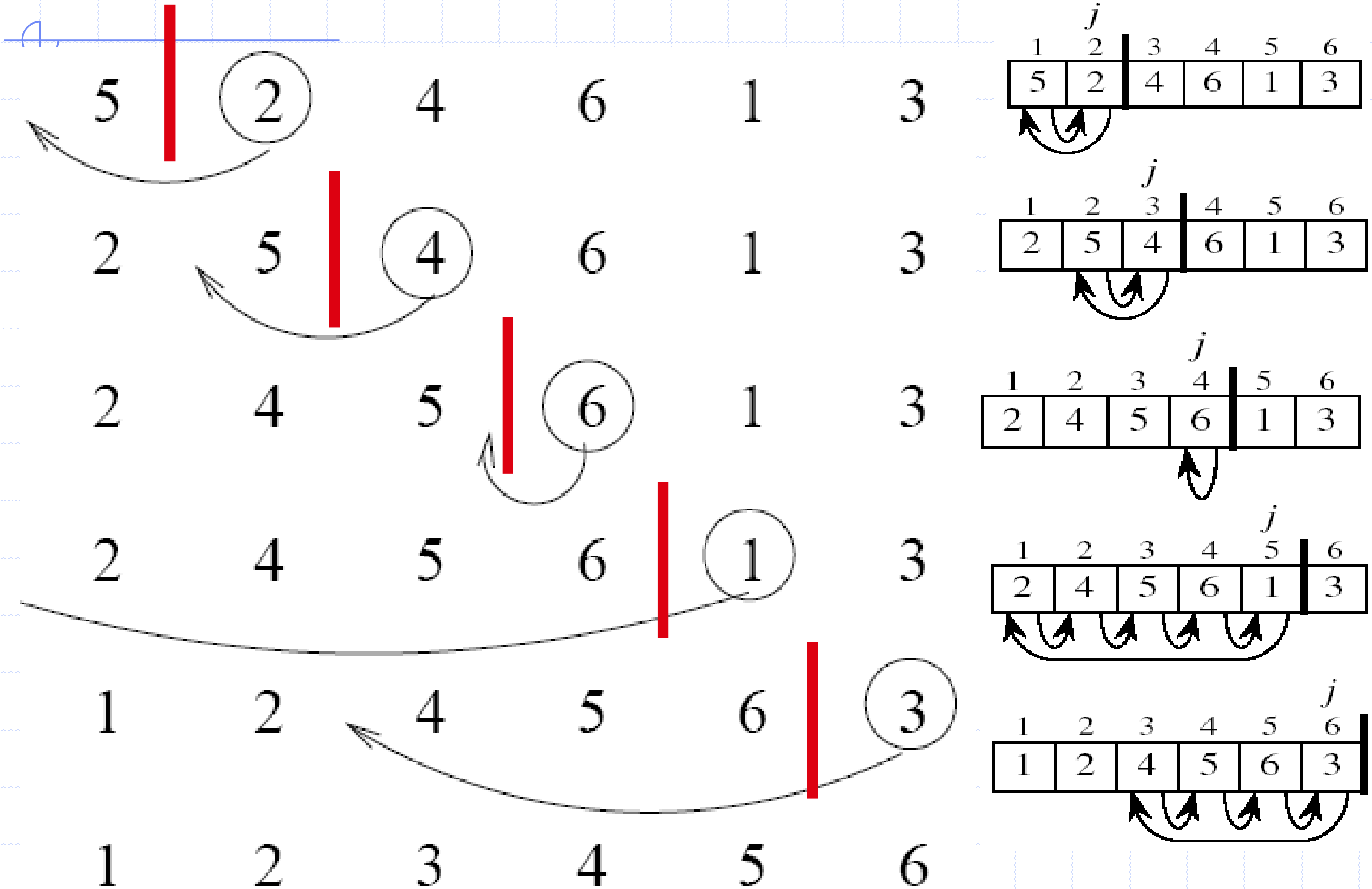
at each iteration, the array is divided in two sub-arrays:

left sub-array right sub-array

2 5 4 6 1 3

sorted unsorted

Insertion Sort



INSERTION-SORT

0	1	2	3	4	5	6	7
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7

key

Alg.: INSERTION-SORT*(A)*

```
for j ← 1 to n-1
      do key ← A[ j ]
▷         # Insert key into the sorted sequence A[0 . . j-1]
          i ← j - 1
          while i >= 0 and A[i] > key
                do A[i + 1] ← A[i]
                   i ← i - 1
          A[i + 1] ← key
```

- Insertion sort – sorts the elements in place

Big-O Notation

- Given functions $f(n)$ and $g(n)$, we say that

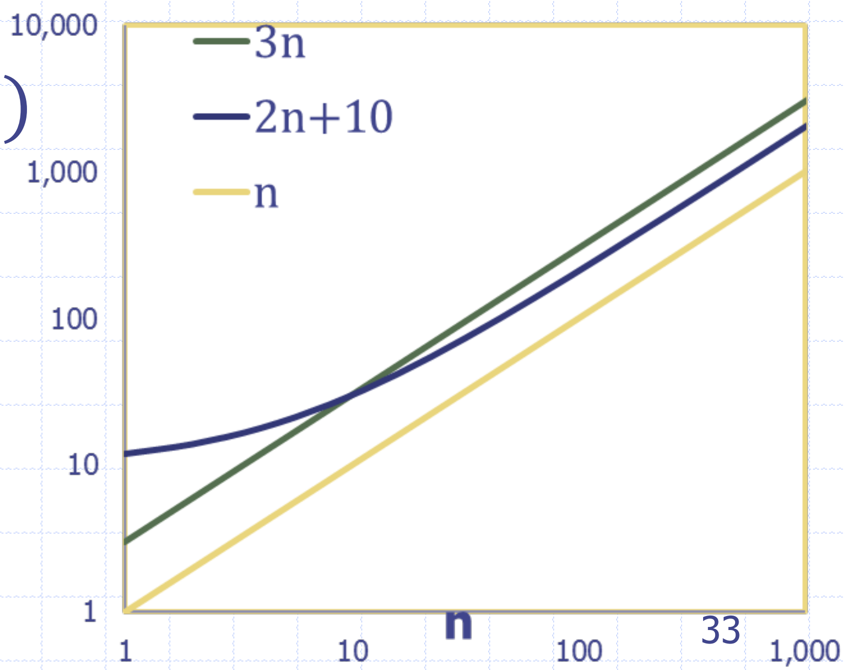
$$f(n) \text{ is } O(g(n))$$

if there exist positive constants c and n_0 such that

$$f(n) \leq c\, g(n) \text{ for all } n \geq n_0$$

- Example: $2n + 10$ is $O(n)$
  - Pick $c = 3$ and $n_0 = 10$

$$2n + 10 \leq 3n$$
$$2(10) + 10 \leq 3(10)$$
$$30 \leq 30$$



Big-O Notation (continued)

Example: n^2 is not $O(n)$

$$n^2 \leq cn$$

$$n \leq c$$

The above inequality cannot be satisfied because c must be a constant, therefore for any $n > c$ the inequality is false

Big-O and Growth Rate

- Big-O notation gives an upper bound on the growth rate of a function
- We say "an algorithm is $O(g(n))$" if the growth rate of the algorithm is no more than the growth rate of $g(n)$
- We saw on the previous slide that n^2 is not $O(n)$
  - But n is $O(n^2)$
  - And n^2 is $O(n^3)$
  - Why? Because Big-O is an upper bound!

Summary of Big-O Rules

- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $O(n^d)$. In other words:
  - forget about lower-order terms
  - forget about constant factors
- Use the smallest possible degree
  - It's true that $2n$ is $O(n^{50})$, but that's not a helpful upper bound
  - Instead, say it's $O(n)$, discarding the constant factor and using the smallest possible degree

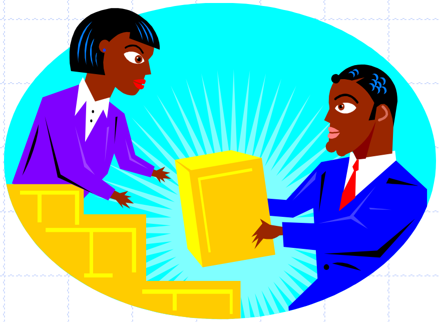
Constants in Algorithm Analysis

- Find the number of primitive operations executed as a function (T) of the input size
  - `first`: $T(n) = 2$
  - `argmax`: $T(n) = 6n + 2$
  - `possible_products`: $T(n) = 6n^2 + 2n + 2$
- In the future we can skip counting operations and replace any constants with c since they become irrelevant as n grows
  - `first`: $T(n) = c$
  - `argmax`: $T(n) = c_0 n + c_1$
  - `possible_products`: $T(n) = c_0 n^2 + n + c_1$

Big-O in Algorithm Analysis

- Easy to express T in big-O by dropping constants and lower-order terms
- In big-O notation
  - `first` is $O(1)$
  - `argmax` is $O(n)$
  - `possible_products` is $O(n^2)$
- The convention for representing $T(n) = c$ in big-O is $O(1)$.

More Big-Oh Examples



- $7n-2$

  $7n-2$ is $O(n)$

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \bullet n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \bullet n^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- $3 \log n + 5$

  $3 \log n + 5$ is $O(\log n)$

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \bullet \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Asymptotic Algorithm Analysis

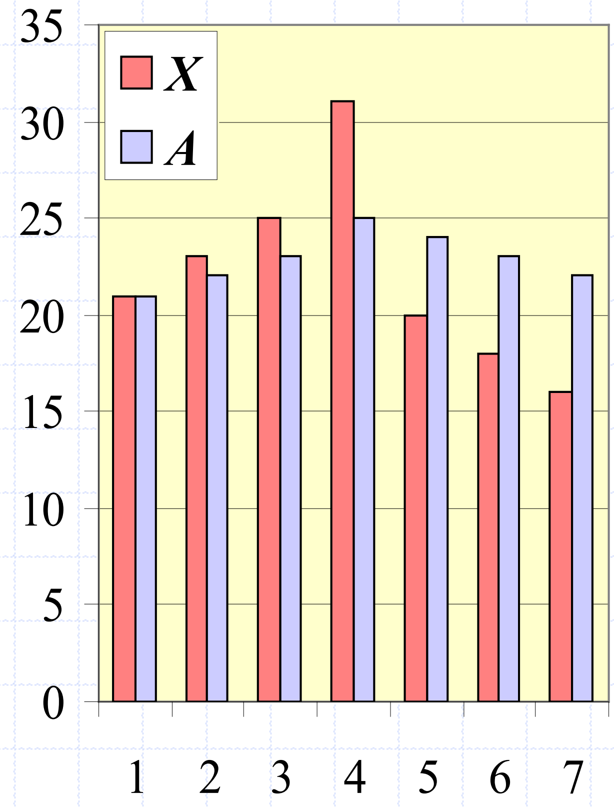
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We say that algorithm **argmax** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages

- We further illustrate asymptotic analysis with two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i + 1)$ elements of X:

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Computing the array A of prefix averages of another array X has applications to financial analysis



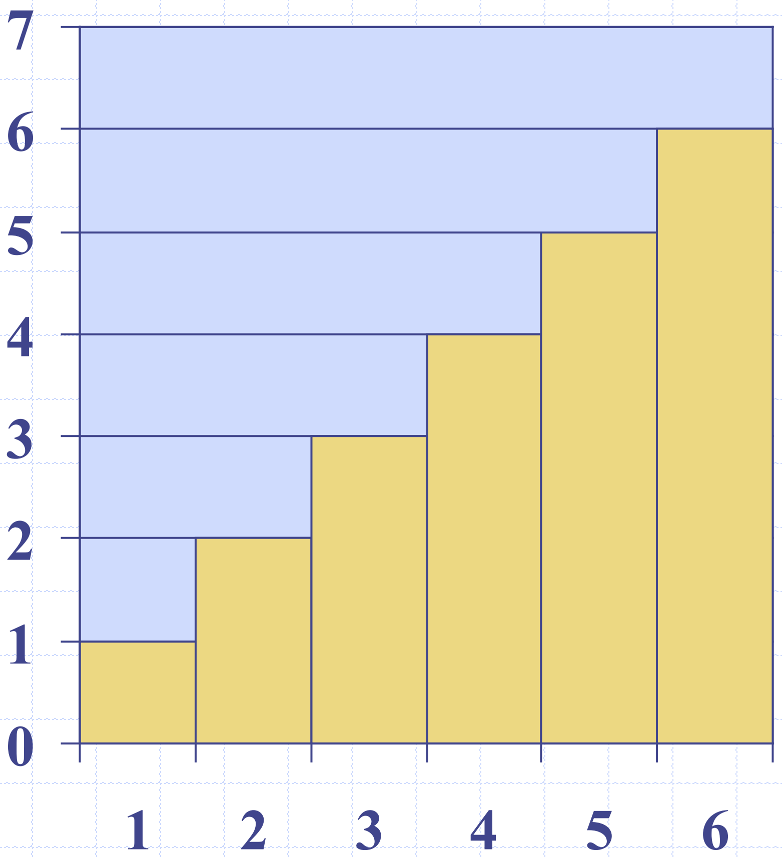
Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

```
def prefix_average1(S):
  """Return list such that, for all j, A[j] equals average of S[0], ..., S[j]."""
  n = len(S)
  A = [0] * n                    # create new list of n zeros
  for j in range(n):
    total = 0                    # begin computing S[0] + ... + S[j]
    for i in range(j + 1):
      total += S[i]
    A[j] = total / (j+1)         # record the average
  return A
```

Arithmetic Progression

- The running time of ***prefixAverage1*** is $O(1 + 2 + \ldots + n)$
- The sum of the first n integers is $n(n + 1) / 2$
  - There is a simple visual proof of this fact
- Thus, algorithm ***prefixAverage1*** runs in $O(n^2)$ time



Prefix Averages 2 (Looks Better)

- The following algorithm uses an internal Python function to simplify the code

```
def prefix_average2(S):
  """Return list such that, for all j, A[j] equals average of S[0], ..., S[j]."""
  n = len(S)
  A = [0] * n                          # create new list of n zeros
  for j in range(n):
    A[j] = sum(S[0:j+1]) / (j+1)       # record the average
  return A
```

- Algorithm *prefixAverage2* **still** runs in $O(n^2)$ time!

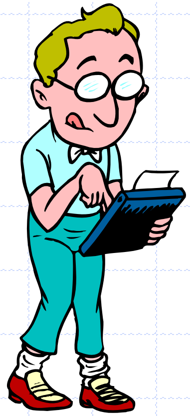
Prefix Averages 3 (Linear Time)

- The following algorithm computes prefix averages in linear time by keeping a running sum

```
def prefix_average3(S):
  """Return list such that, for all j, A[j] equals average of S[0], ..., S[j]."""
  n = len(S)
  A = [0] * n                      # create new list of n zeros
  total = 0                        # compute prefix sum as S[0] + S[1] + ...
  for j in range(n):
    total += S[j]                  # update prefix sum to include S[j]
    A[j] = total / (j+1)           # compute average based on current sum
  return A
```

- Algorithm ***prefixAverage3*** runs in $O(n)$ time

Math you need to Review



- Summations
- Logarithms and Exponents
  - **properties of logarithms:**
    - $\log_b(xy) = \log_b x + \log_b y$
    - $\log_b (x/y) = \log_b x - \log_b y$
    - $\log_b x^a = a\log_b x$
    - $\log_b a = \log_x a/\log_x b$
  - **properties of exponentials:**
    - $a^{(b+c)} = a^b a^c$
    - $a^{bc} = (a^b)^c$
    - $a^b / a^c = a^{(b-c)}$
    - $b = a^{\log_a b}$
    - $b^c = a^{c*\log_a b}$
- Proof techniques
- Basic probability

Composition Rules for Big-O

If $T_1(N) = O(f(N))$ and $T_2(N) = O(g(N))$

$T_1(N) + T_2(N) \quad = \quad O(f(N)) + O(g(N))$

$O(max(f(N), g(N))$

$T_1(N) * T_2(N) \quad = \quad O(f(N)) * O(g(N))$

General Rules – Basic for-loops

Compute $\sum_{i=1}^{N} i^3$

```java
public static int sum(int n){
  int partialSum = 0;

  for (int i = 1; i <= n; i++)
        partialSum += i * i * i;
  return partialSum;
}
```

1 step (initialization) +1 step for last test

`int partialSum = 0;`: 1 step

N iterations

`for (int i = 1; i <= n; i++)`: 2 steps each

`partialSum += i * i * i;`: 4 steps each

`return partialSum;`: 1 step

```python
def sum(n):
   partialSum = 0
   for i in range(1,n+1):
      partialSum += i*i*i
   return partialSum
```

T(N) = 6N + 2 = O(N)

(running time of statements in the loop) X (iterations)

If loop runs a constant number of times: *O(c)*

General Rules – Nested Loops

```
for (i=0; i < n; i++)
  for (j=0; j < n; j++)
    k++;
```

N iterations $O(N) * O(N) = O(N^2)$

N iterations $O(N)$

2 steps each $O(c)$

```
for i in range(n):
    for j in range(n):
        k += 1
```

General Rules – Consecutive Blocks

```
for (i = 0; i < n; i++)
  a[i] = 0;
for (i=0; i < n; i++)
  for (j = 0; j < n; j++)
      a[i] += a[j] + i + j;
```

$O(N)$

$O(N^2)$

```
for i in range(n):
  a[i] = 0
for i in range(n):
  for j in range(n):
    a[i] += a[j] + i + j
```

$$O(N) + O(N^2) = O(N^2)$$

General Rules - Conditionals

```
if (condition)
    S1
else
    S2
```

$T_{\text{test}}(N)$

$T_{S_1}(N)$

$T_{S_2}(N)$

```
if condition:
    S1
else:
    S2
```

$$T(N) = O(\max(T_{S_1}(N), T_{S_2}(N)) + T_{\text{test}}(N))$$

Logarithms in the Runtime

```java
public static int binarySearch(int[] a, int x) {
  int low = 0;
  int  high = a.length - 1;

  while ( low <= high) {
    int mid = (low + high) / 2;
     if (a[mid] < x)
        low = mid + 1;
      else if(a[mid] > x)
        high = mid - 1;
      else
        return mid; // found
  }
  return -1; // Not found.
}
```

```python
def binarySearch(a, x):
    low = 0
    high = len(a) – 1

    while low <= high:
        mid = (low+high) // 2
        if a[mid] < x:
            low = mid+1
        elif a[mid] > x:
            high = mid-1
        else:
            return mid #found
    return -1 #not found
```

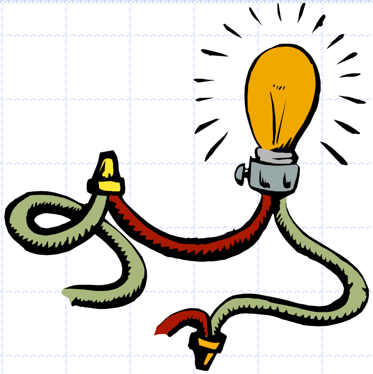
Reduces the search space by half at every step

k steps until $N+1 \geq 2^k \geq N$

$$\mathrm{Log}_2(N+1) \geq k \geq \mathrm{Log}_2 N$$

$T(N) = O(\mathrm{Log}(N))$

In-class exercise



- Please complete the in-class Big-O exercises on Gradescope.

Big-Omega (Ω)

- Recall that $f(n)$ is $O(g(n))$ if $f(n) \leq cg(n)$ for some constant c as n grows
  - Big-O expresses the idea that $f(n)$ grows no faster than $g(n)$
  - $g(n)$ acts as an upper bound to $f(n)$'s growth rate

- What if we want to express a lower bound?

Big-Omega

- We say $f(n)$ is $\Omega(g(n))$ if $f(n) \geq cg(n)$
  - $f(n)$ grows no **slower** than $g(n)$

Big-Theta (Θ)

- What about an upper *and* lower bound?

Big-Theta

- We say f(n) is $\Theta(g(n))$ if

  f(n) is $O(g(n))$ **and** $\Omega(g(n))$

  - f(n) grows the same as g(n) (tight-bound)

Some More Examples

Function, f(n)	Big-Θ
$an + b$	$\Theta(n)$
$an^2 + bn + c$	$\Theta(n^2)$
a	$\Theta(1)$
$3^n + an^{40}$	$\Theta(3^n)$
$an + b \log n$	$\Theta(n)$

Common Time Complexities

Name	Running Time
Constant	$O(1)$
Log-logarithmic	$O(\log \log N)$
Logarithmic	$O(\log N)$
Polylogarithmic	$O((\log N)^2)$
Fractional power	$O(N^c)$ where $0 < c < 1$
Linear	$O(N)$
Linearithmic	$O(N \log N)$
Quadratic	$O(N^2)$
Cubic	$O(N^3)$
Polynomial	$O(N^c)$ where $c > 3$
Exponential	$O(c^N)$ where $c \geq 2$
Factorial	$O(N!)$

source: https://en.wikipedia.org/wiki/Time_complexity#Table_of_common_time_complexities

Relatives of Big-Oh

- **big-Omega**
  - f(n) is $\Omega(g(n))$ if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$
- **big-Theta**
  - f(n) is $\Theta(g(n))$ if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$