# CSCI-SHU 220: Algorithms
# Subset Sum and Knapsack

NYU Shanghai
Spring 2025

## Subset sum problems

**Problem (subset sum)**

Given a set $A = \{a_1, \ldots, a_n\}$ of integers and an integer $s$, we want to check whether there exists a subset $A' \subseteq A$ such that $s = \sum_{a \in A'} a$.

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- **Some variants**
  - Finding a smallest subset $A'$ satisfying $s = \sum_{a \in A'} a$.
  - Finding a largest subset $A'$ satisfying $s = \sum_{a \in A'} a$.
  - The variants where $A'$ can contain multiple copies of each $a_i$.
    (call it infinite subset sum for convenience)

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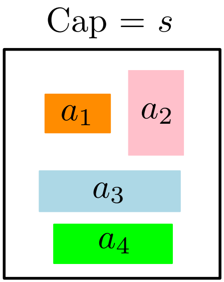

Can we fill the knapsack?

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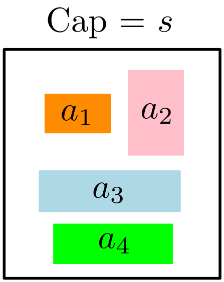

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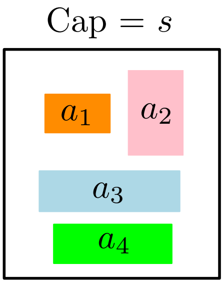

Fill the knapsack using most items?

## Knapsack problems

**Problem (0-1 knapsack)**

Given a knapsack of weight capacity $W$ and $n$ items where the $i$-th item has weight $w_i \in \mathbb{N}$ and value $v_i \in \mathbb{R}$, include in the knapsack items of total weight at most $W$ with maximum total value.

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- **Example**

  $W = 9$

  Item 1: $w_1 = 4, v_1 = 6$

  Item 2: $w_2 = 3, v_2 = 4$

  Item 3: $w_3 = 5, v_3 = 5$

  The optimal solution chooses Item 1 and Item 3.

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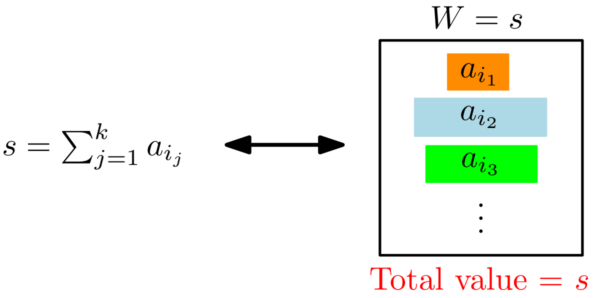

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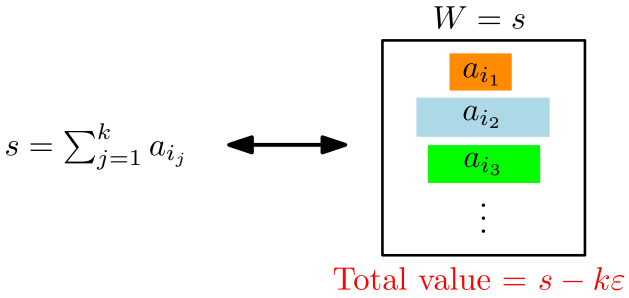

$W = s$
$a_{i_1}$
$a_{i_2}$
$a_{i_3}$
$\vdots$
$s = \sum_{j=1}^{k} a_{i_j}$
Total value $= s - k\varepsilon$

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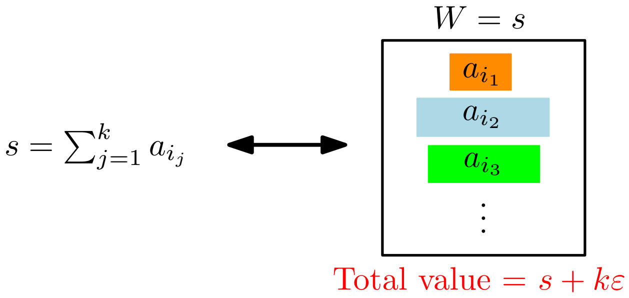
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Given a knapsack of weight capacity $W$ and $n$ types of (infinite) items where the $i$-th type has weight $w_i \in \mathbb{N}$ and value $v_i \in \mathbb{R}$, include in the knapsack items of total weight at most $W$ with maximum total value.

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- **Example**

  $W = 9$
  Item 1: $w_1 = 4, v_1 = 6$
  Item 2: $w_2 = 3, v_2 = 4.1$

  The optimal solution chooses 3 copies of Item 2.

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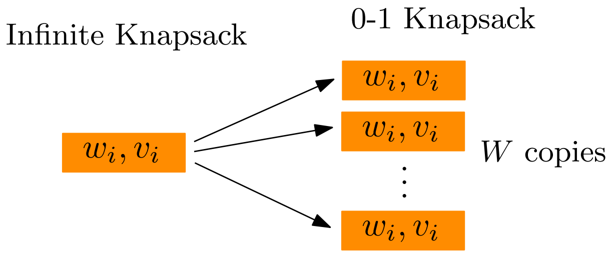

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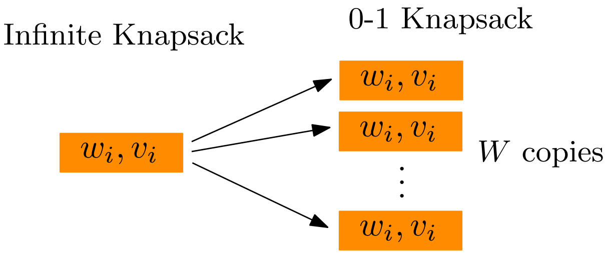

- We usually don't use this reduction, since it increases the number of items significantly from $n$ to $O(nW)$.

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- **Exercise**

  Improve the time cost to $O(nw^*)$, where $w^* = \max_{i \in \{1,\ldots,n\}} w_i^2$.

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- Suppose we want to solve the subproblem $P_{w,\{1,\ldots,n\}}$, i.e., achieve the maximum value with total weight $w$ when all items are available.
- What if we choose the $j$-th item as the one in our solution with the largest index? Then the remaining problem is $P_{w-w_j,\{1,\ldots,j-1\}}$.

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- **Standard implementation**
- $\text{KNAPSACK}(W, w_1, \ldots, w_n, v_1, \ldots, v_n)$

```
(assume $\text{opt}[w,k] = -\infty$ for all $w$ and $k$ initially)
$\text{opt}[0,k] \leftarrow 0$ for all $k \in \{1, \ldots, n\}$
**for** $k = 1, \ldots, n$ **do**
    **for** $w = 1, \ldots, W$ **do**
        $\text{opt}[w,k] \leftarrow \max\{\text{opt}[w,k-1], \text{opt}[w-w_k, k-1] + v_k\}$
**return** $\max_{w \in \{1,\ldots,W\}} \text{opt}[w,n]$
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- **A better implementation**
- $\textsc{Knapsack}(W, w_1, \ldots, w_n, v_1, \ldots, v_n)$

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opt[0] $\leftarrow 0$
for $k = 1, \ldots, n$ do
    for $w = W, \ldots, 1$ do
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- **Example**

  $W = 6$
  Item 1: $w_1 = 4, v_1 = 5$
  Item 2: $w_2 = 3, v_2 = 4$
  Item 3: $w_3 = 5, v_3 = 5$

  The optimal solution is $\rho_1 = 75\%$, $\rho_2 = 100\%$, $\rho_3 = 0\%$.

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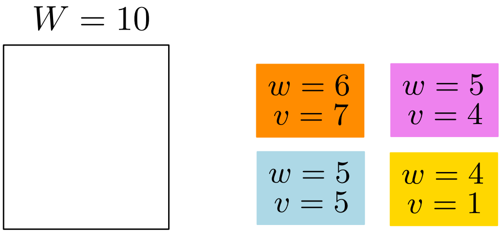

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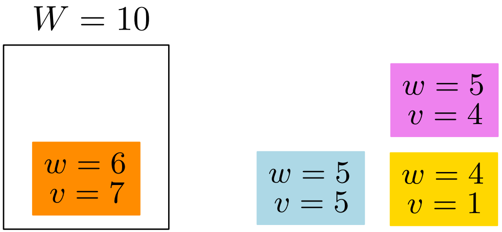

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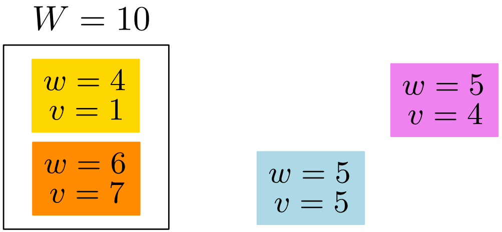

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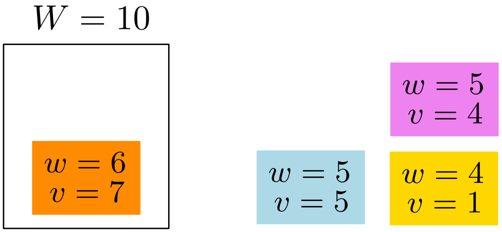

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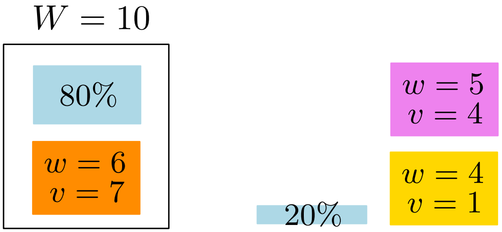

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- Without loss of generality, suppose $\frac{v_1}{w_1} \geq \cdots \geq \frac{v_n}{w_n}$.
- Find the maximum index $k$ satisfying $\sum_{i=1}^{k} w_i \leq W$.
- $\rho_i = 1$ for all $i \in \{1, \ldots, k\}$
  $\rho_{k+1} = (W - \sum_{i=1}^{k} w_i)/w_{k+1}$
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## Fractional knapsack

- Next let's formally describe the greedy strategy.
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  $\rho_i = 0$ for all $i \in \{k+2, \ldots, n\}$
- **Proof of correctness**
  Assume there exists an optimal solution whose $\rho_i$ is the same as the greedy solution for all $i < t$, and show the existence of an optimal solution whose $\rho_i$ is the same as the greedy solution for all $i \leq t$.

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**Problem (Fibonacci sum)**

Given a positive integer $s$...

1. Check whether $s$ is the sum of several distinct Fibonacci numbers.
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- **Example**

  $s = 31$

  1. Yes
  2. $s = 2 + 8 + 21$
  3. $s = 5 + 13 + 13$

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    $i =$ the largest index such that $F_i \leq s$
    $\implies s = F_i + s'$ where $s' < F_i$
    By our hypothesis, $s' = F_{j_1} + \cdots + F_{j_r}$ where $j_1, \ldots, j_r$ are distinct.
    $\implies s = F_i + F_{j_1} + \cdots + F_{j_r}$ and $i \notin \{j_1, \ldots, j_r\}$ as $s' < F_i$

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  1. Reduce to the minimization version of subset sum.
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- The first observation can be proved by contradiction, and the second one can be proved by induction on $i$.

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  - Observation 2 further shows that if $j < i$, then $s > F_i \geq \text{sum}(S)$.
- Thus, we have $i = j$. Now try to further prove the optimality of the greedy solution by induction.

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  - $68 = 34 + 34$, $165 = 55 + 55 + 55$, $84 = 21 + 21 + 21 + 21$
  - $68 = 55 + 13$, $165 = 144 + 21$, $84 = 55 + 21 + 8$
- Now we should suspect that the greedy algorithm might be correct.

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  We can keep modifying $S$ using the equality $2F_i = F_{i+1} + F_{i-2}$.

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- It's not clear whether the modification of $S$ will terminate or not.

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- Now we see there's always an optimal solution with distinct numbers.