CSCI-SHU 220: Algorithms
Dynamic Programming II

NYU Shanghai
Spring 2025

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- In all these problems, when we consider a subproblem $P \in \mathcal{S}$, each choice in $C(P)$ reduces P to one (smaller) subproblem in $\mathcal{S}$.
- In some more complicated problems, it can happen that after taking a choice in $C(P)$, the problem is reduced to a combination of multiple subproblems in $\mathcal{S}$ that are independent.

Cut the cake!

Problem (cut the cake!)

We have a cake which is a rectangle consisting of $n \times m$ square cells. Each cell might be empty or contain a cherry. Now we want to cut the cake into pieces where each piece contains at most one cherry. The cutting procedure is done as follows.

- In each step, we pick one piece of cake and cut it into two.
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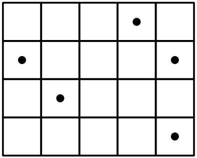

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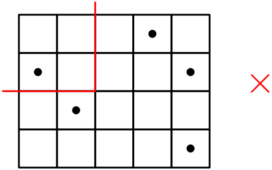

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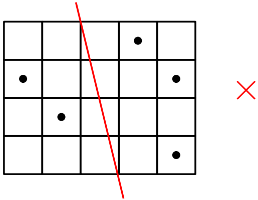

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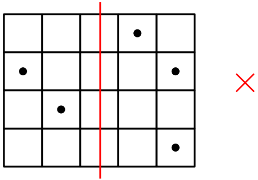

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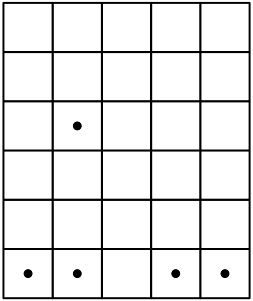

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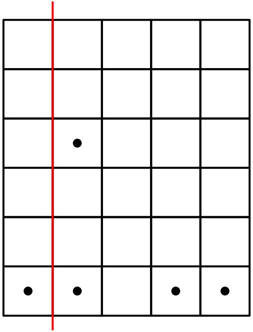

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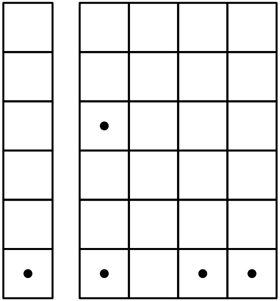

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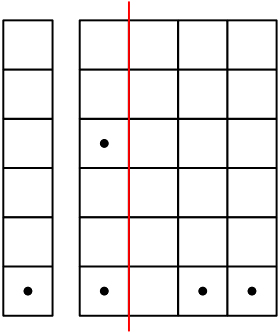

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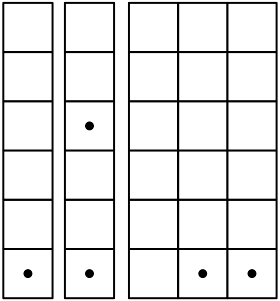

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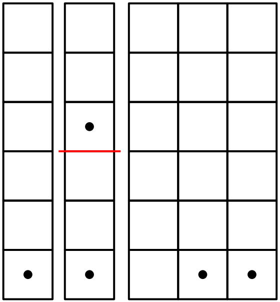

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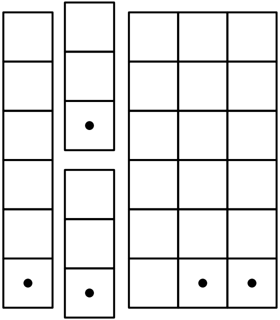

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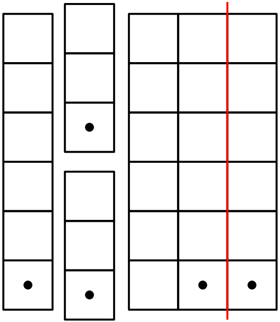

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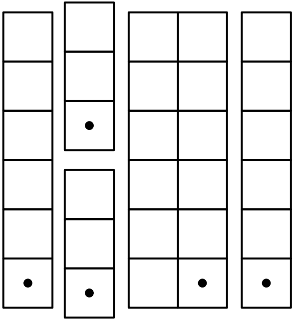

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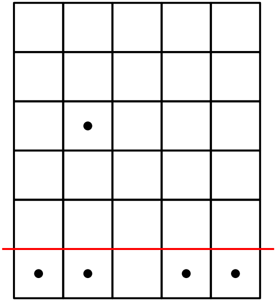

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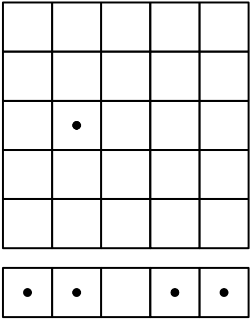

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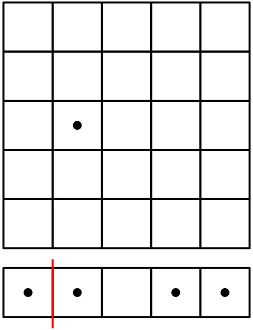

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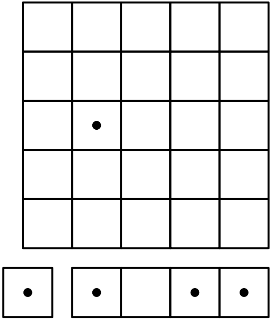

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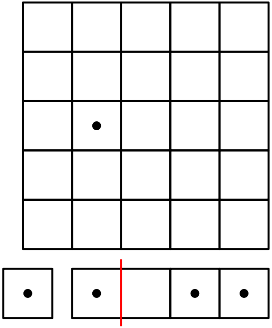

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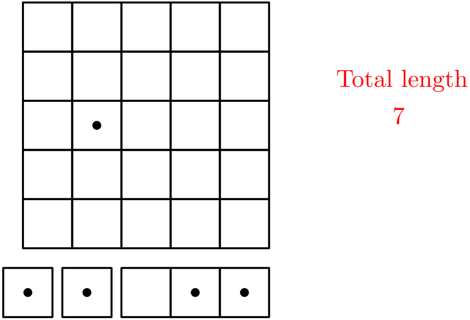

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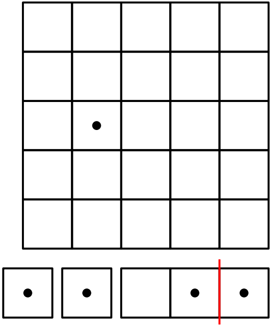

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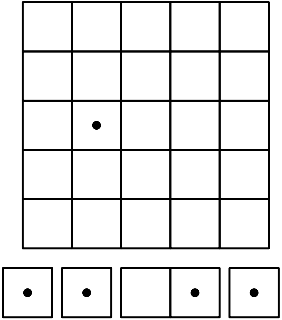

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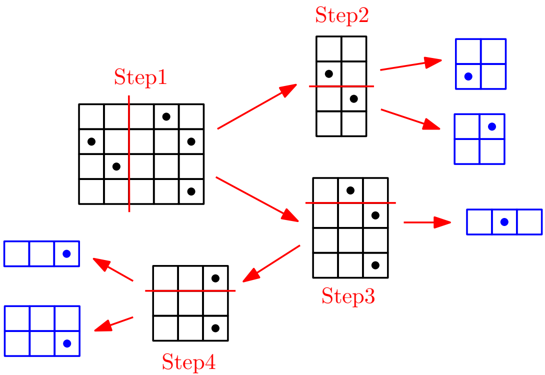

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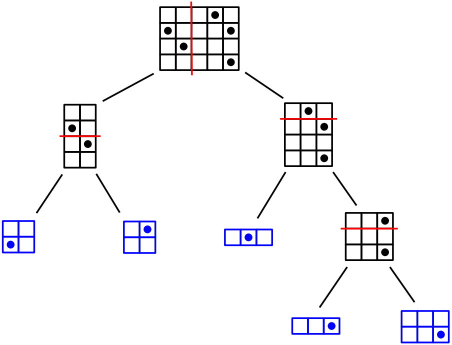

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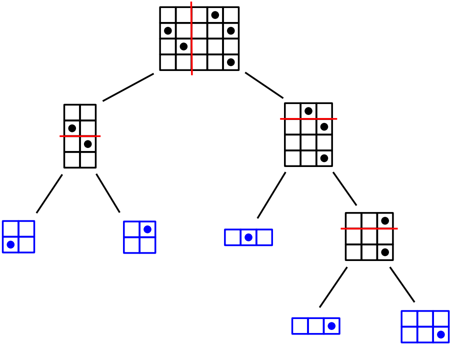

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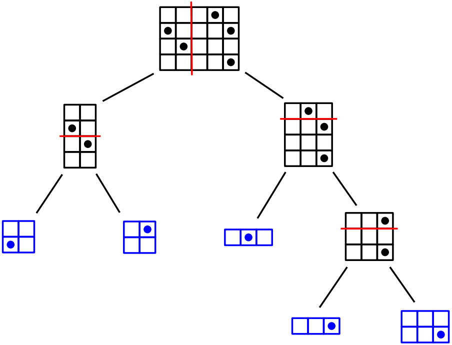

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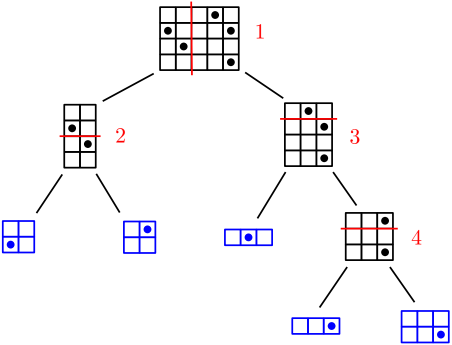

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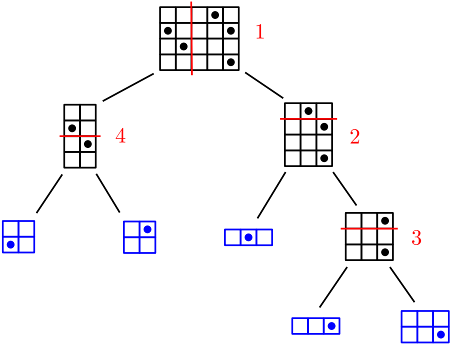

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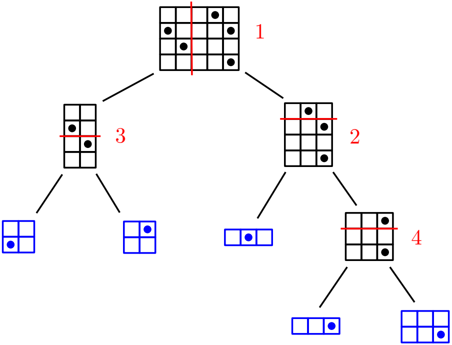

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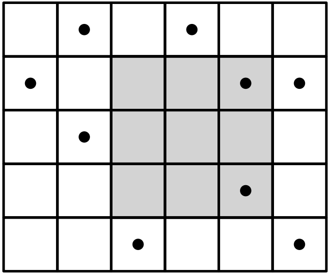

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- **Boundary case**
  $\mathrm{opt}_{i^-,i^+,j^-,j^+} = 0$ if (i^-, i^+, j^-, j^+) contains at most one cherry.

Expression evaluation

Problem (expression evaluation)

We have an expression with n numbers and $n-1$ "$+$" operations. In order to evaluate the expression, we can perform these $n-1$ "$+$" operations in any order and, the outcome is always the sum of the n numbers. Suppose computing the sum $x+y$ of two numbers x and y has a cost, which can be known from a given oracle $\textsc{cost}(x, y)$. Our goal is to find an optimal order to perform the "$+$" operations which minimizes the total cost.

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- **Example:** $\textsc{cost}(x,y) = xy$

$$7 + 3 + 8 + 11 + 4$$

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We have an expression with n numbers and $n-1$ "$+$" operations. In order to evaluate the expression, we can perform these $n-1$ "$+$" operations in any order and, the outcome is always the sum of the n numbers. Suppose computing the sum $x+y$ of two numbers x and y has a cost, which can be known from a given oracle $\textsc{cost}(x,y)$. Our goal is to find an optimal order to perform the "$+$" operations which minimizes the total cost.

- **Example:** $\textsc{cost}(x,y) = xy$

$$\begin{array}{ccccccccc} 7 & + & 3 & + & 8 & + & 11 & + & 4 \\ & 2 & & 4 & & 3 & & 1 & \end{array}$$

$$\text{Total cost} = 0$$

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- **Example:** $\text{COST}(x, y) = xy$

$$\underset{}{7} \underset{2}{+} 3 \underset{4}{+} 8 \underset{3}{+} \quad 15$$

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- **Example:** $\textsc{cost}(x,y) = xy$

$$\begin{array}{ccccc} 10 & \underset{4}{+} & 8 & \underset{3}{+} & 15 \end{array}$$

$$\text{Total cost} = 65$$

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$$\underset{4}{10 \quad + \quad 23}$$

$$\text{Total cost} = 185$$

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- **Example:** $\textsc{cost}(x, y) = xy$

$$33$$

$$\text{Total cost } = 415$$

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- The cost of the last "+" is $\textsc{Cost}(\sum_{j=1}^{i} x_j, \sum_{j=i+1}^{n} x_j)$.
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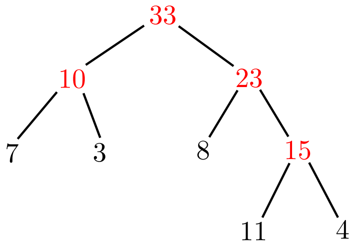

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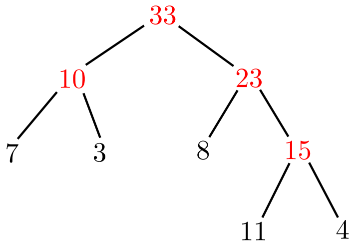

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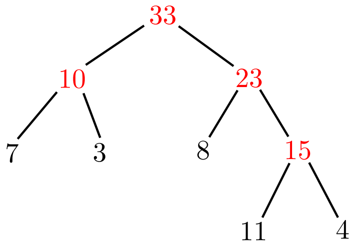

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- The tree determines the total cost. So we can always evaluate the left subtree first, and then the right subtree.

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- **A typical way to do this**
  Start from the most trivial definition of subproblems, and try to work out the recursive dependency. If you fail, you will find a more proper way to define subproblems. Repeat this procedure until success.

Generalized (weighted) activity selection

Problem (generalized activity selection)

In NYUSH, there are n proposed activities $a_1, \ldots, a_n$, which wish to use the same room. The room is big enough and can hold c activities at the same time for a constant c. Each activity a_i has a start time s_i, a finish time f_i, and a weight $w_i > 0$. We want to select a subset of c-compatible activities with maximum total weight.

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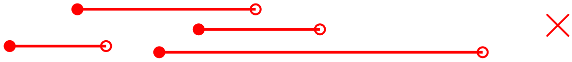

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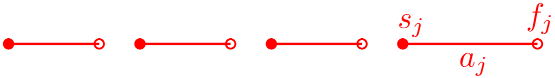

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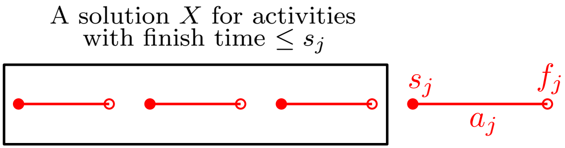

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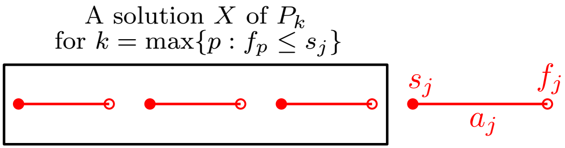

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- However, not every feasible solution of P_{j-1} remains feasible after adding the activity a_j.

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- **New problem**

  Given an activity set A and a number x, compute the max-weighted 2-compatible subset of A that is 1-compatible in $[x, \infty)$.

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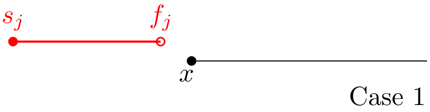

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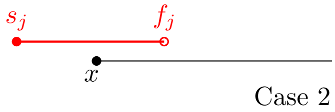

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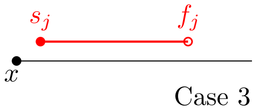

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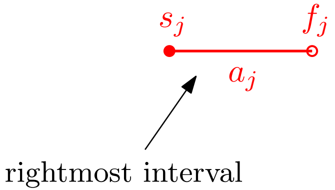

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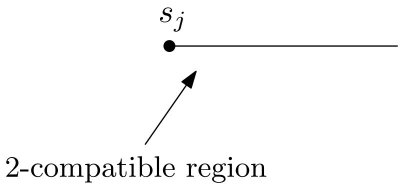

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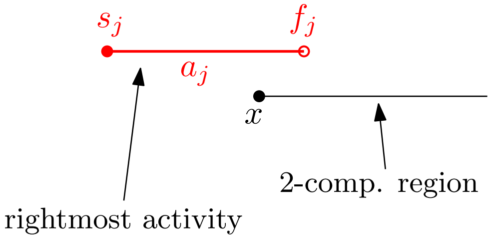

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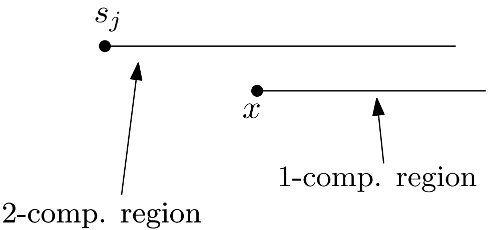

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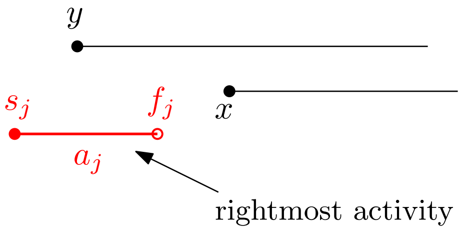

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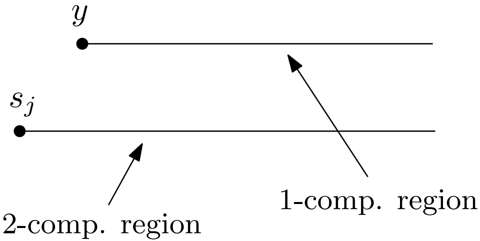

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- Time complexity $= O(n^c)$
  (We don't need to enumerate the rightmost activity a_j. Instead, we only consider $P_{i-1,x_1,\ldots,x_{c-1}}$ and the case a_i is the rightmost activity.)

Picking non-adjacent entries

Problem (picking non-adjacent entries)

Suppose we have a $4 \times n$ matrix M where each entry contains a number. Our goal is to pick some non-adjacent entries in M such that the sum of these entries is maximized.

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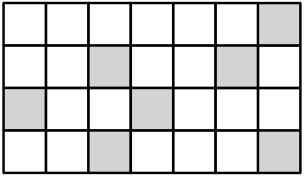

M

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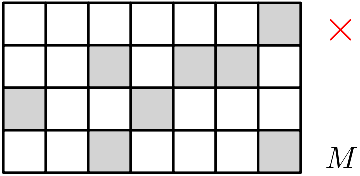

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- When M is $4 \times n$, we can define P_i as the subproblem of picking non-adjacent entries in the first i column of M.

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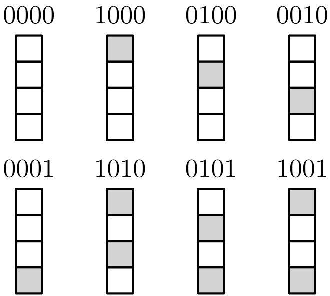

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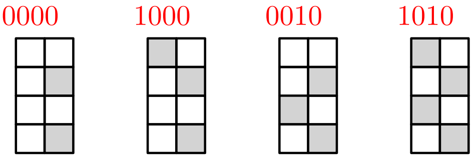

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