CSCI-SHU 220: Algorithms
Dynamic Programming I

NYU Shanghai
Spring 2025

Weakness of greedy algorithms

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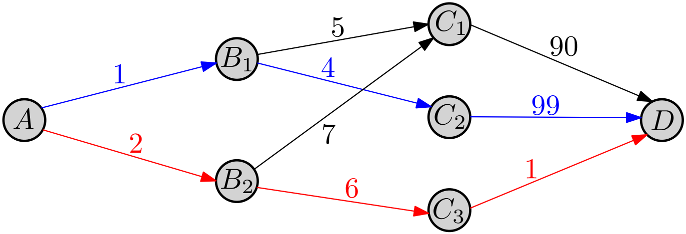
  - If locally optimal $=$ globally optimal, the greedy algorithm works.
  - If locally optimal $\neq$ globally optimal, the greedy algorithm fails.
- So for some problems, we have to consider optimality globally.

Weakness of greedy algorithms

- **Finding the shortest path from left to right**



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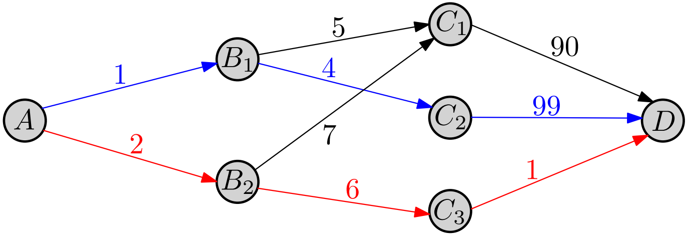

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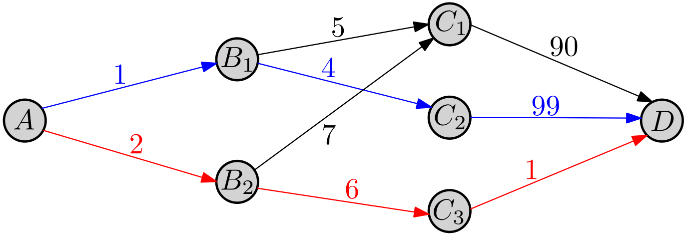

- Moving $A \to B_1$ only costs 1 but brings us to a "bad" position B_1. Moving $A \to B_2$ costs 2 but brings us to a "good" position B_2.
- Should consider the cost and the position simultaneously.

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Dynamic programming

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- $A \to B_1$ or $A \to B_2$? Depend on which of the above two is smaller.
- Knowing the shortest path from B_1 to D and the shortest path from B_2 to D, we can make a globally optimal choice at A, thus directly compute the shortest path from A to D.

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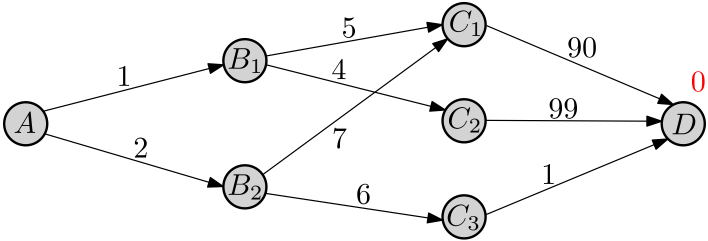

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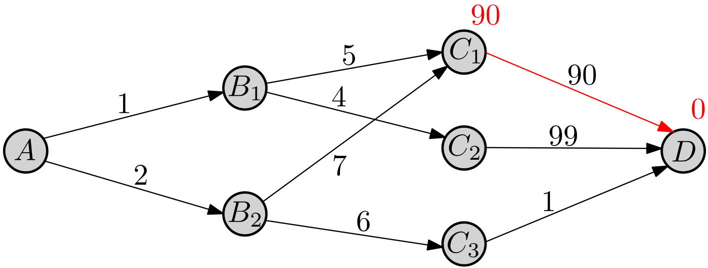

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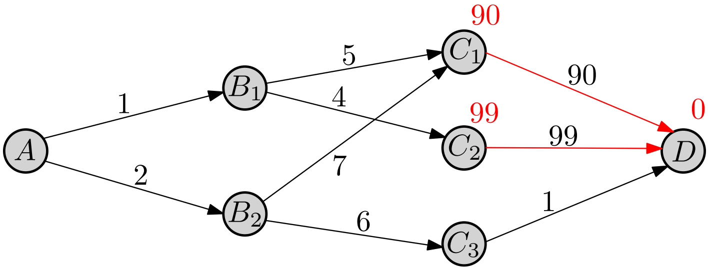

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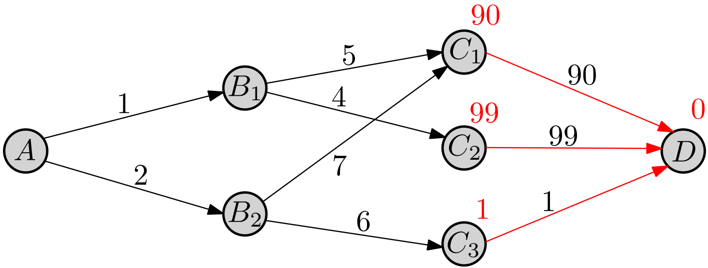

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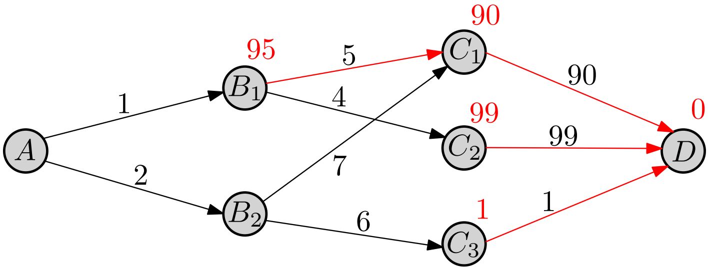

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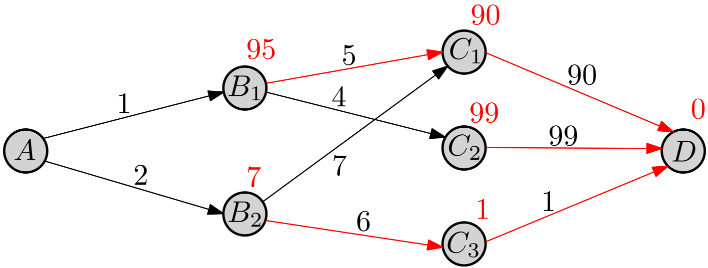

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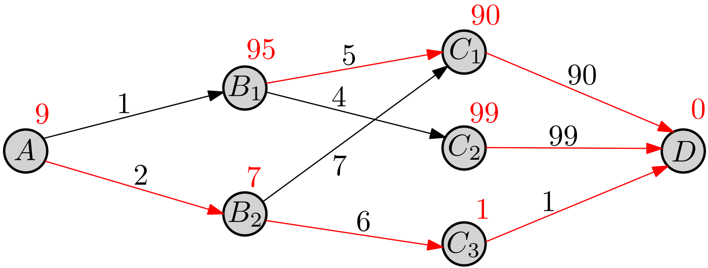

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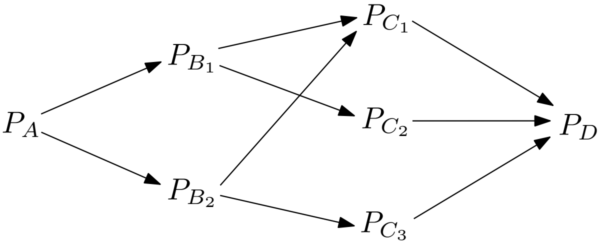

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- Step 1 is the most technical and important step.

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- $C(P)$ = the set of choices of the first (or last) step for P
- $\text{opt}(P) = \min_{c \in C(P)} f_c(\text{opt}(Q_1), \ldots, \text{opt}(Q_r))$ for minimization.
  $\text{opt}(P) = \max_{c \in C(P)} f_c(\text{opt}(Q_1), \ldots, \text{opt}(Q_r))$ for maximization.

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Problem (shortest path)

Suppose there are n cities $A_1, \ldots, A_n$ and a set E of m edges connecting these cities. Each path directs from a city A_i to another city A_j for $j > i$, and has a length. Our goal is to find a shortest path from A_1 to A_n.

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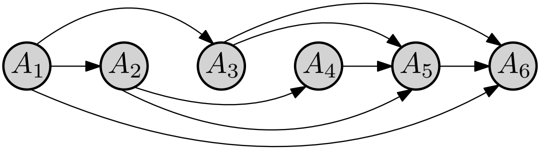

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- $\textsc{ShortestPath}(n, A_1, \ldots, A_n, E)$

  opt[1] $\leftarrow 0$
  for $i = 2, \ldots, n$ **do**
  $\quad E_i \leftarrow \{e \in E : \text{target}(e) = i\}$
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  return $\text{opt}[n]$
- Time complexity $= O(n^2)$ or $O(n + |E|)$ if implemented carefully

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- If we have edges between all $\binom{n}{2}$ pairs of cities...
  $T(n) \geq \sum_{i=1}^{n-1} T(i) \implies T(n) = \Omega(2^n)$

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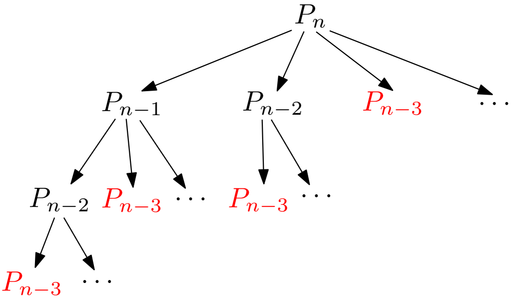

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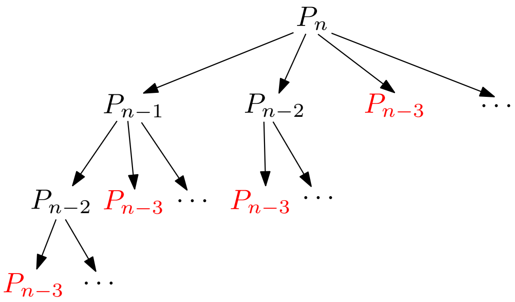

- When we have $\binom{n}{2}$ edges, P_{n-3} is solved 4 times.

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  $\quad\quad \text{last}[i] \leftarrow \arg\min_{e \in E_i}(\text{length}(e) + \text{opt}[\text{source}(e)])$
  $\text{path} \leftarrow [\,]$ and $i \leftarrow n$
  while $i > 1$ **do**
  $\quad \text{path} \leftarrow [\text{last}[i]] + \text{path}$
  $\quad i \leftarrow \text{source}(\text{last}[i])$
  return $(\text{opt}[n], \text{path})$

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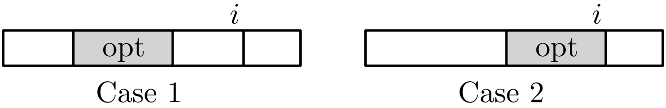

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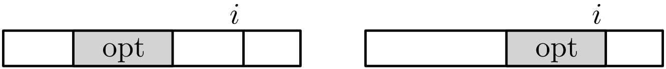

Case 1 Case 2

- Seems impossible... An optimal solution of P_i may end at $A[i]$, and in this case the optimal solutions of $P_1, \ldots, P_{i-1}$ are not helpful.

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- Consider a subproblem P_i. Construct a solution of P_i step by step?

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- Dependency among $P_1, \ldots, P_n$?

- Consider a subproblem P_i. Construct a solution of P_i step by step?
  Start at $A[i]$, choose "go left" or "stop" in each step.

- **Second attempt**

  P_i = computing the maximum subarray of A ending at $A[i]$
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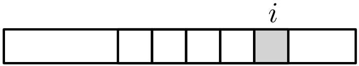

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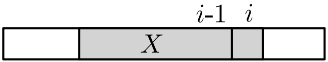

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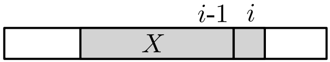

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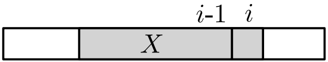

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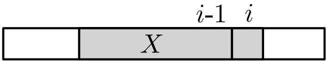

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- $\textsc{MaxSubarray}(n, A)$

  $\text{opt}[1] \leftarrow A[1]$ and $C[1] \leftarrow$ "stop"
  for $i = 2, \ldots, n$ **do**
  $\quad \text{opt}[i] \leftarrow \max\{0, \text{opt}[i-1]\} + A[i]$
  $\quad$ **if** $0 < \text{opt}[i-1]$ **then** $C[i] \leftarrow$ "go left"
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  $i^* \leftarrow \arg\max_{i \in \{1,\ldots,n\}} \text{opt}[i]$
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Problem (weighted activity selection)

In NYUSH, there are n proposed activities $a_1, \ldots, a_n$, which wish to use the same room. Each activity a_i has a start time s_i, a finish time f_i, and a weight $w_i > 0$. We want to select a subset of compatible activities with maximum total weight.

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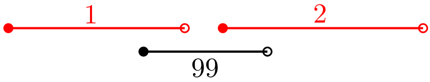

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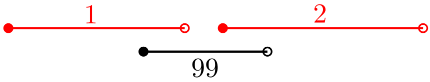

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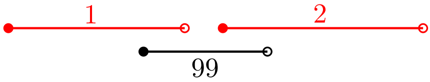

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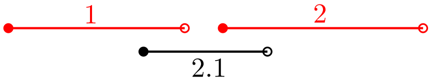

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- Time complexity? At least $\Omega(2^n)$.

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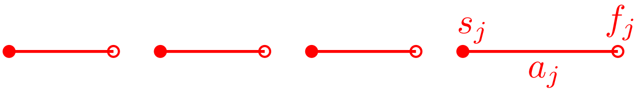

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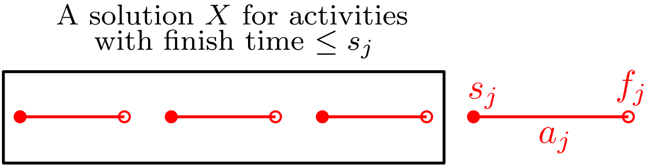

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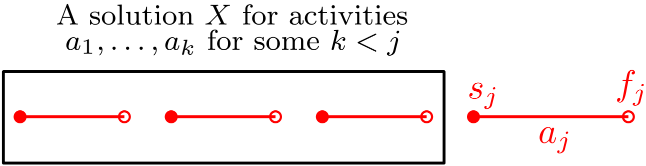

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  Determine the rightmost activity in the solution.
- $C(P_i) = \{1, \ldots, i\}$
  Choosing $j \in C(P_i)$ means a_j is the rightmost activity in the solution.

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- Can we do this more efficiently?

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- Recall that $\text{pre}(j) = \max\{k : f_k \leq s_j\}$. Since $f_1 < \cdots < f_n$, we can use binary search to compute $\text{pre}(j)$ in $O(\log n)$ time.

Weighted activity selection

- $\textsc{WeightedActSelect}(n, S, F, W)$

  Sort the activities such that $F[1] \leq \cdots \leq F[n]$

  $F[0] \leftarrow -\infty$

  for $j = 1, \ldots, n$ **do**

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- Time complexity $= O(n \log n)$

Longest increasing subsequence

- A subsequence of $A[1 \ldots n]$ is an array $B[1 \ldots m]$ for $m \leq n$ such that there exists $i_1 < \cdots < i_m$ satisfying $B[j] = A[i_j]$ for $j \in \{1, \ldots, m\}$.

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- **Example**

6	3	7	5	6	1	9	18	14	4	15

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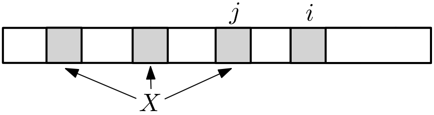

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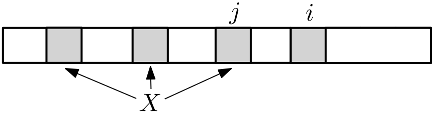

- The solution consists of $A[i]$ and an IS X of A ending at $A[j]$. Score $=$ lengt$(X) + 1$, maximized when lengt$(X) =$ opt(P_j).

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- $\mathrm{opt}(P_i) = \max_{j \in C(P_i)} (\mathrm{opt}(P_j) + 1) = \max_{j \in C(P_i)} \mathrm{opt}(P_j) + 1$

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- $\text{LIS}(n, A)$

for $i = 1, \ldots, n$ **do**
  $m \leftarrow 0$ and $k \leftarrow 0$
  for $j = 1, \ldots, i-1$ **do**
    if $A[j] < A[i]$ and $\text{opt}[j] > m$ **then**
      $m \leftarrow \text{opt}[j]$ and $k \leftarrow j$
  $\text{opt}[i] \leftarrow m + 1$ and $\text{pos}[i] \leftarrow j$
$i^* = \arg\max_{i \in \{1,\ldots,n\}} \text{opt}[i]$
$i \leftarrow i^*$ and $B \leftarrow [A[i]]$
while $\text{pos}[i] > 0$ **do**
  $i \leftarrow \text{pos}[i]$ and $B \leftarrow [A[i]] + B$
return B

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C	K	A	T	U	B	C	X	P	A	Z

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     Consider a common subsequence $A[i_1, \ldots, i_m] = B[j_1, \ldots, j_m]$ of A, B and observe the subsequence $A'[i_1, \ldots, i_m]$ of A'. What do you find?

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  3. Compute an LIS of A', which corresponds to an LCS of A and B.

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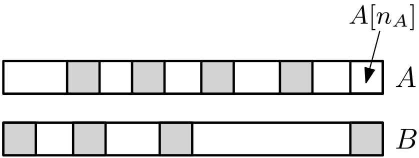
- **Case 1.** C doesn't contain $A[n_A]$.



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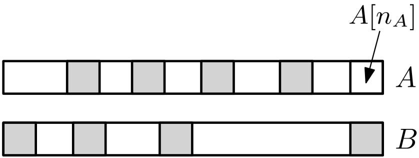

In this case, C is an LCS of $A[1 \ldots n_A - 1]$ and B.

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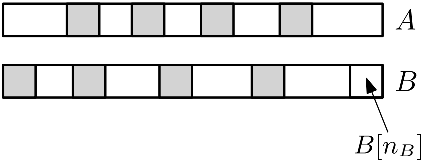

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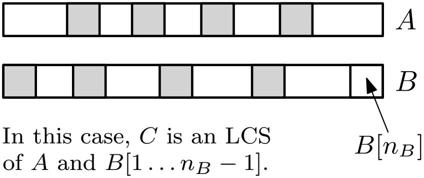

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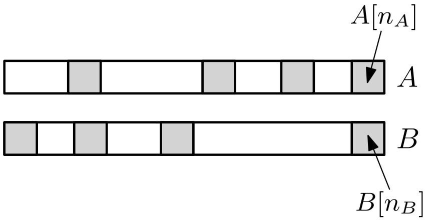

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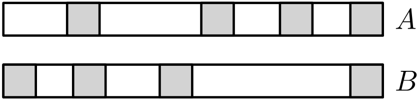

In this case, C consists of $A[n_A] = B[n_B]$
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- **Boundary case**
  $\text{opt}(P_{i,0}) = \text{opt}(P_{0,j}) = 0$ for all i and all j

Longest common subsequence

- $\mathrm{LCS}(n_A, A, n_B, B)$

  opt$[i, 0] \leftarrow 0$ and opt$[0, j] \leftarrow 0$ for $i, j \in \{0, 1, \ldots, n\}$
  for $i = 1, \ldots, n_A$ **do**
      for $j = 1, \ldots, n_B$ **do**
          opt$[i, j] \leftarrow$ max$\{$opt$[i-1, j]$, opt$[i, j-1]\}$
          if $A[i] = B[j]$ and opt$[i, j] <$ opt$[i-1, j-1] + 1$ **then**
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