CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	August 21, 2019, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geqslant$18 passed; $\geqslant$26 grade 4; $\geqslant$31 grade 5,
GU $\geqslant$18 grade G; $\geqslant$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) The fixed point of the following two-dimensional dynamical system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbb{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where } \mathbb{A} = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$$

has complex eigenvalues, select the correct statements.

A. The fixed point is a stable spiral.

B. The fixed point is a center.

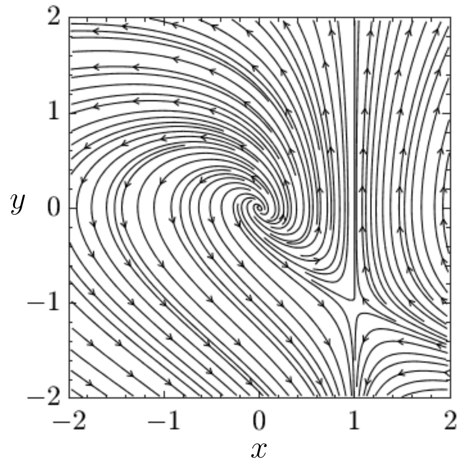
C. The fixed point is an unstable spiral.

D. Trajectories run clockwise ($\circlearrowright$) in (x,y) space.

E. Trajectories run counterclockwise ($\circlearrowleft$) in (x,y) space.

b) The figure below shows the phase portrait of the system:

$$\dot{x} = -y + xy$$
$$\dot{y} = x + y$$



What is the sum of the indices of all fixed points of this system?

A. -2 B. -1 C. 0 D. 1 E. 2

c) Which of the following statements about the indices of smooth flows of dimensionality two are correct?

A. The index of a fixed point must either be -1 or 1.

B. The index of a closed curve intersecting a fixed point is 0.

C. The index of a closed curve not intersecting a fixed point is 1.

D. The index of a closed orbit cannot be 0.

E. Pitchfork bifurcations conserve the sum of fixed-point indices.

d) Which of the following statements about bifurcations in smooth flows are correct?

A. A Hopf bifurcation is a local bifurcation.

B. A heteroclinic bifurcation is a local bifurcation.

C. The period time of a limit cycle usually approaches infinity close to a saddle-node bifurcation of limit cycles.

D. It is possible to have bifurcations between two unstable fixed points, for example a saddle point and an unstable node.

E. A homoclinic bifurcation usually occurs when a limit cycle collides with a spiral.

e) Which of the following statements about the generalized dimension spectrum D_q are true?

A. When $q > 0$ low-density regions of the attractor give the dominant contribution to D_q.

B. When $q > 0$ high-density regions of the attractor give the dominant contribution to D_q.

C. For a monofractal, D_q is independent of q.

D. For a multifractal, D_q decreases with increasing q.

E. The Kaplan-Yorke conjecture states that, in most cases, the Lyapunov dimension is equal to D_1.

f) The figure below shows the first few generations in the construction of a fractal. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

S_1 S_2 S_3

Which of the following alternatives describe the box-counting dimension of the fractal above?

A. $\dfrac{\log(2)}{\log(3)}$ B. $\dfrac{\log(3)}{\log(4)}$ C. $\dfrac{\log(4)}{\log(3)}$ D. $\dfrac{3}{2}$ E. $\dfrac{\log(3)}{\log(2)}$

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) A simple harmonic oscillator $m\ddot{x} = -kx$ is a system that oscillates along the x-axis in one dimension. Explain how this is consistent with the statement that dynamical systems of dimensionality one cannot show oscillations.

b) What is meant by a catastrophe in the context of bifurcation theory?

c) What is meant by a reversible dynamical system? Give an example of a time reversible dynamical system of dimensionality two.

d) What does the Poincaré-Bendixon theorem state?

e) Construct a fractal set (for example a variation of the Cantor set) with fractal dimension $D_0 = \frac{\ln 3}{\ln 5}$.

f) Explain the difference in the typical solutions of an integrable Hamiltonian system and a non-integrable Hamiltonian system.

3. Bifurcation [2 points] In this problem, consider different one-dimensional systems with a variable x and a parameter r.

a) Consider the system

$$\dot{x} = x^2 - r\,.$$

Explain what is meant by a slow passage in this system.

b) Using for example a geometric approach or a normal form, determine whether the following system has any bifurcation(s) and, if so, determine the type of bifurcation(s):

$$\dot{x} = x(r - e^x)$$

c) Using for example a geometric approach or a normal form, determine whether the following system has any bifurcation(s) and, if so, determine the type of bifurcation(s):

$$\dot{x} = r - e^{-x^2}$$

d) Sketch the bifurcation diagrams for the system

$$\dot{x} = r^a - x^2$$

for $a = 1$, $a = 2$ and $a = 4$, clearly marking the stability of fixed points.

4. Phase portrait [2 points] Consider the system

$$\begin{aligned} \dot{x} &= 2(3 - y)x - 3x^2 + 2y + y^2 \\ \dot{y} &= 2(1 - y)x + x^2 + 6y - 3y^2 \end{aligned} \tag{1}$$

a) This system has a fixed point at the origin $(x^*, y^*) = (0, 0)$. Classify this fixed point.

b) Find the eigendirections $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ of the fixed point at the origin.

c) Write the system (1) in the eigenbasis of the fixed point at the origin by introducing transformed coordinates $\boldsymbol{\xi} = \mathbb{P}\boldsymbol{x}$, where $\boldsymbol{x} = (x, y)^{\mathrm{T}}$ and $\mathbb{P}$ has $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (from subtask b)) as columns, $\mathbb{P} = [\boldsymbol{v}_1 \boldsymbol{v}_2]$. Show that your result is equivalent (up to a scaling $\xi_i \to a\xi_i$ or an interchange $\xi_1 \leftrightarrow \xi_2$) to the system

$$\begin{aligned} \dot{\xi}_1 &= 4\xi_1(2 - \xi_1) \\ \dot{\xi}_2 &= 4\xi_2(1 - 2\xi_1) \end{aligned} \tag{2}$$

d) Evaluate the nullclines of the transformed system (2). Use the nullclines to sketch the phase portrait in (ξ_1, ξ_2) space as well as the (x, y) space, clearly marking all fixed points and their types.

5. Lorenz equations [2 points] Consider the Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - z \end{aligned} \tag{3}$$

where σ and r are positive parameters (usually there is an additional parameter b which is set to unity here).

a) The Lorenz system (3) has one fixed point at the origin, $(x_1^*, y_1^*, z_1^*) = (0, 0, 0)$. Find a condition on the system parameters such that this fixed point is stable.

b) Two additional fixed points $C_- = (x_2^*, y_2^*, z_2^*)$ and $C_+ = (x_3^*, y_3^*, z_3^*)$ of the Lorenz system (3) are formed in a bifurcation as the parameter r changes. Find the fixed points C_- and C_+ and determine the bifurcation point r_c where C_- and C_+ are created.

c) A straightforward calculation shows that the characteristic equation for the eigenvalues λ of the stability matrix $\mathbb{J}$ evaluated at C_- and C_+ is

$$0 = \det(\lambda\mathbb{I} - \mathbb{J}) = \lambda^3 + (2 + \sigma)\lambda^2 + (r + \sigma)\lambda + 2(r - 1)\sigma\,. \tag{4}$$

Use this relation to determine whether C_- and C_+ are stable close to the bifurcation point r_c. What type of bifurcation occurs at r_c?
Hint: Eq. (4) is too messy to be solved in general. Therefore, consider the stability when $r = r_\mathrm{c} + \delta r$ with small δr.

d) Each of the fixed points C_- and C_+ undergoes a Hopf bifurcation at the same bifurcation point r_H. Use Eq. (4) to determine r_H.
Hint: To avoid solving Eq. (4) directly, make an ansatz for the form of one root λ at the Hopf bifurcation.

6. Damped driven pendulum [2 points] Consider the damped driven pendulum

$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = I\sin(\omega_{\mathrm{F}}t) \tag{5}$$

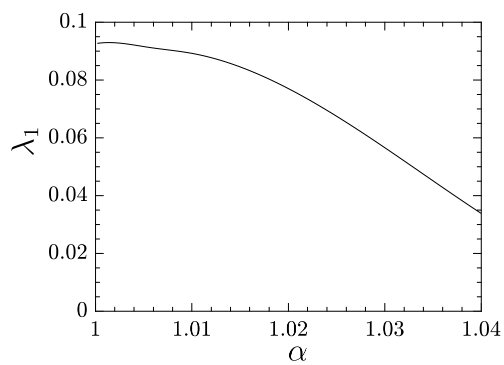
where α, I and ω_{F} are positive parameters.

a) Give possible physical interpretations of the different terms and parameters in Eq. (5).

b) Write Eq. (5) as a dynamical system of dimensionality 3 for the coordinates $(\theta,\ y,\ \tau)$ where $y = \dot{\theta}$ and $\tau = \omega_{\mathrm{F}}t$.

c) Determine all fixed points of the system for $(\theta,\ y,\ \tau)$ and determine their stability.

d) Consider $\omega_{\mathrm{F}} = 2/3$, $F = 2.048$ and $1 \leqslant \alpha \leqslant 1.04$. In this range of α the system for $(\theta,\ y,\ \tau)$ is chaotic with maximal Lyapunov exponent λ_1 approximated by the curve:



Use this data to sketch the remaining Lyapunov exponents λ_2 and λ_3 of the system for $(\theta,\ y,\ \tau)$ as functions of α (in the range $1 \leqslant \alpha \leqslant 1.04$).