CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	January 14, 2019, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Classify the fixed point of the two-dimensional dynamical system:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\,, \qquad \text{where } \mathbb{A} = \begin{pmatrix} -3 & 4 \\ -4 & 2 \end{pmatrix}.$$

A. It is a saddle point.

B. It is a stable spiral.

C. It is an unstable spiral.

D. It is a stable node.

E. It is an unstable node.

b) The normal forms of typical bifurcations for dynamical systems of dimensionality one are the following:

Type	saddle-node	transcritical	supercrit. pitchfork	subcrit. pitchfork
Normal form	$\dot{x} = r + x^2$	$\dot{x} = rx - x^2$	$\dot{x} = rx - x^3$	$\dot{x} = rx + x^3$

How does the stability time of the fixed points close to a subcritical pitchfork bifurcation scale with the bifurcation parameter r?

A. $\sim \frac{1}{r}$ B. $\sim \frac{1}{\sqrt{r}}$ C. ~ 1 D. $\sim \sqrt{r}$ E. $\sim r$

c) Which type(s) of bifurcation(s) does the system $\dot{x} = 5 - re^{-x^2}$ have?

A. Saddle-node bifurcation

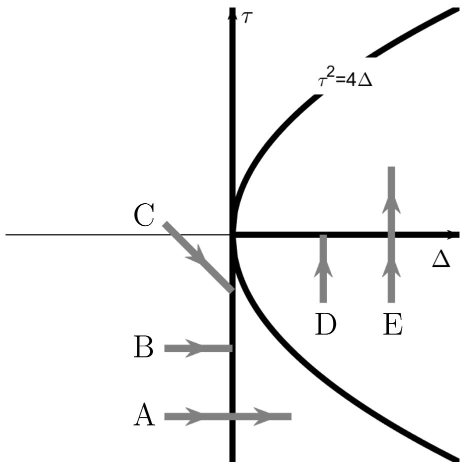
B. Transcritical bifurcation

C. Supercritical pitchfork bifurcation

D. Subcritical pitchfork bifurcation

E. No bifurcation occurs

d) Each path A–E in the Δ-τ diagram below (Δ and τ are the determinant and trace of the stability matrix of dimensionality two) is obtained from τ and Δ of a fixed point upon increasing a parameter.

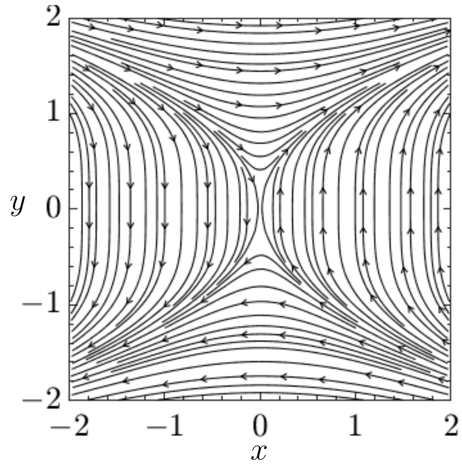


Which of the paths corresponds to one of the fixed points in a normal form of the saddle-node bifurcation in dimensionality two?

A. B. C. D. E.

e) The figure below shows the phase portrait of the system:

$$\dot{x} = y^3$$
$$\dot{y} = x$$



What is the index of the fixed point at the origin?

A. -2 B. -1 C. 0 D. 1 E. 2

f) Which properties hold in general for the Lyapunov exponents in a continuous dynamical system of dimensionality three with a strange attractor that is globally attracting?

A. One Lyapunov exponent is negative

B. One Lyapunov exponent is zero

C. One Lyapunov exponent is positive

D. The sum of all Lyapunov exponents is negative

E. The sum of all Lyapunov exponents is zero

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Write down the equations for a Hamiltonian dynamical system of your choice.

b) What is meant by a catastrophe in the context of bifurcation theory?

c) Explain what the difference between a global and a local bifurcation is. Give two examples of global bifurcations.

d) Explain what is meant by a secular term in perturbation theory.

e) What value does the maximal Lyapunov exponent of a stable limit cycle take? Explain why.

f) Explain why the transition from regular dynamics to chaos is typically very different in dissipative and in Hamiltonian dynamical systems.

3. Imperfect bifurcations [2.5 points] Consider the system

$$\dot{x} = 2(4 + a + 3r) + (12 + a + 5r)x + (6 + r)x^2 + x^3 \tag{1}$$

where r and a are real parameters.

a) The system (1) has one fixed point x^* which is independent of the parameters a and r. Find the value of this fixed point.

b) Find and classify all bifurcations that occur in the system (1) when $a = 0$. Sketch the bifurcation diagram.
Hint: It may be helpful to know that the shifted coordinate $\xi = x - x^*$, where x^* is the parameter-independent fixed point in subtask a), has the following dynamics

$$\dot{\xi} = (a + r)\xi + r\xi^2 + \xi^3 \,.$$

c) Find and classify all bifurcations that occur in the system (1) when $a = -1$. Sketch the bifurcation diagram.

4. Linear stability analysis and phase portrait [1.5 points] Consider the following dynamical system

$$\begin{aligned} \dot{x} &= ax^2 - xy \\ \dot{y} &= -y + x^2 \end{aligned} \tag{2}$$

where $0 \le a \le 1$ is a real parameter.

a) Identify all fixed points of the system (2) and classify them according to linear stability analysis for the parameter range $0 \le a \le 1$.

b) Using the nullclines as a guide, sketch the phase portrait of the system (2) for the case $a = 0$. Describe in words how trajectories behave close to the fixed point.

5. Hopf bifurcation [2 points] Consider the following dynamical system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \mu y - x^2 y - 2y^3 \end{aligned} \tag{3}$$

where μ is a real parameter.

a) Show that the system (3) undergoes a Hopf bifurcation as μ passes zero.

b) Consider the case $\mu = 0$ in the system (3) and classify the fixed point at the origin.

c) Consider the case $\mu = 1$. Using the Poincaré-Bendixon theorem, show that the system (3) has at least one closed orbit.

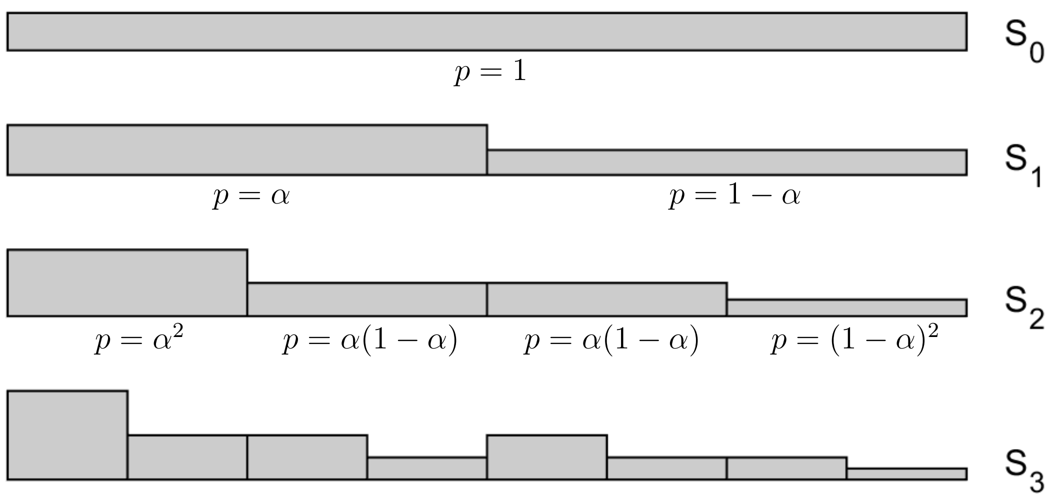
6. Fractal dimension of a weighted Cantor set [2 points] The generalized fractal dimension D_q is defined by

$$D_q \equiv \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)}$$

with

$$I(q,\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon)\,.$$

Here p_k is the probability to be in the k:th occupied box (box with $p_k \neq 0$) and N_{box} is the total number of occupied boxes.
Consider a set S_n where n labels the generation. Start with S_0 being the unit interval. S_n is obtained by dividing each interval in the set S_{n-1} into two subintervals. Upon each division, allocate a fraction α (assume $0 \le \alpha \le 1$) of the probability to be in the original interval to the left subinterval, and a fraction $1-\alpha$ to the right subinterval. The figure below illustrates the first few generations S_0, S_1, S_2 and S_3:



The probability to be in different intervals is displayed in the text below the intervals. The height of an interval illustrates the relative probability to be in that interval for the case $\alpha = 2/3$. In what follows, consider the fractal dimension of the set S_∞ obtained by iterating $n \to \infty$.

a) Evaluate the generalized fractal dimension D_q of S_∞ for the case $\alpha = 1$.

b) Evaluate the generalized fractal dimension D_q of S_∞ for the case $\alpha = \frac{1}{2}$.

c) Evaluate the box-counting dimension D_0 of S_∞ for $0 < \alpha < 1$.

d) Evaluate the generalized fractal dimension D_q of S_∞ for general values of α.
Hint: To verify your result, you can check that the results in subtasks a), b) and c) come out correctly and that your result is symmetric upon replacing $\alpha \to 1 - \alpha$.