Robotic kinematics and dynamics

Autonomous robots, TME290

Ola Benderius

ola.benderius@chalmers.se

Applied artificial intelligence
Vehicle engineering and autonomous systems
Mechanics and maritime sciences
Chalmers

Introduction

Kinematics and dynamics

- **Kinematics** is the study of motion (of rigid bodies) without regard to the forces that cause the motion
- **Dynamics** concerns the motion of bodies under the action of forces

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- To make simulations of robots

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Today we will look at differentially steered robots!

Real-world robots

Figure 1: Lawn mower (left), vacuum cleaner (right)

A differential steered robot has two independently driven wheels for combined propulsion and steering

Kinematics

The differential drive: Kinematics

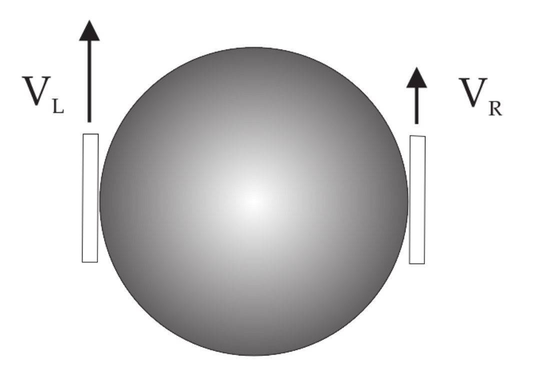


Figure 2: A schematic representation

The differential drive: Kinematics

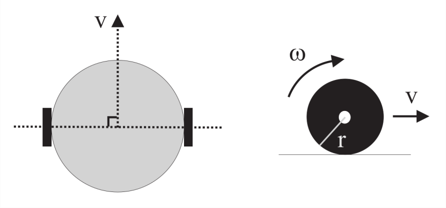


Figure 3: The combined speed (left), the rotation of the wheel (right)

- The robot's frame is a rigid body of radius R

The differential drive: Kinematics

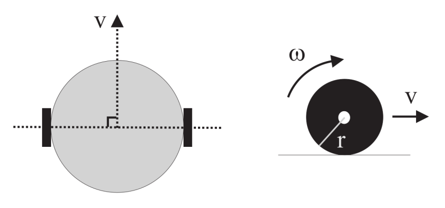
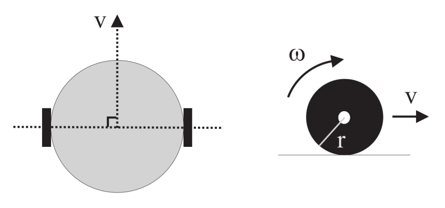


Figure 3: The combined speed (left), the rotation of the wheel (right)

- The robot's frame is a rigid body of radius R
- The wheels, radius r, roll without slipping

The differential drive: Kinematics



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- The wheels can only move in the direction perpendicular to the wheel axis

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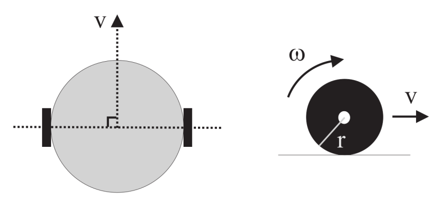

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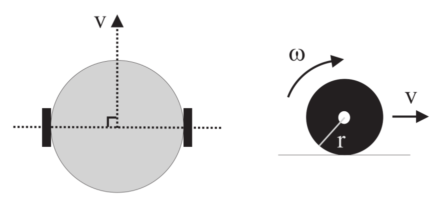
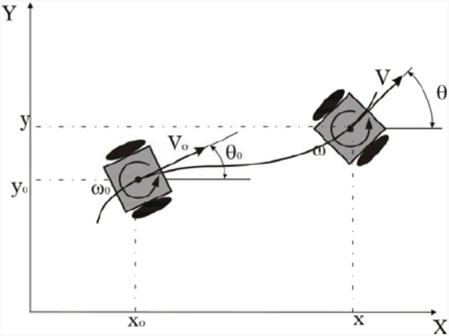


Figure 3: The combined speed (left), the rotation of the wheel (right)

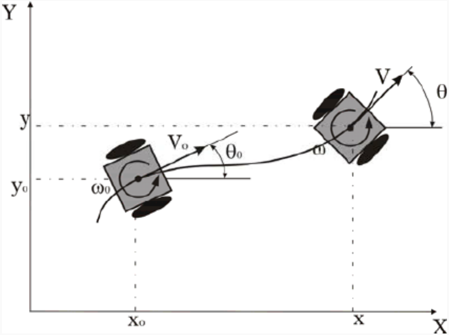
- The robot's frame is a rigid body of radius R
- The wheels, radius r, roll without slipping
- The wheels can only move in the direction perpendicular to the wheel axis
- The wheels can rotate independently of each other
- The forward speed of the wheel is $v = \omega r$

The differential drive: Kinematics



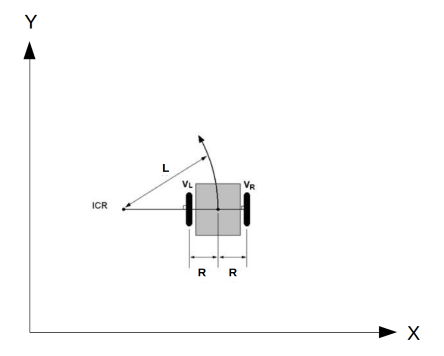
- The yaw angle (heading) φ (θ in the figure) and position x and y form the robot's pose (state), relative an global coordinate system.

The differential drive: Kinematics



- The yaw angle (heading) φ (θ in the figure) and position x and y form the robot's pose (state), relative an global coordinate system.
- The **forward kinematics**, i.e. $x(t)$, $y(t)$, and $\varphi(t)$ are determined by the speeds of the wheels, v_L, v_R

The differential drive: Kinematics



Instantaneous centre of rotation (ICR)

For any values of v_L, v_R, the motion of the robot can be seen as a pure rotation around the ICR, with angular velocity $\dot{\varphi}$

The differential drive: Kinematics

Note that:

- The ICR is located at a distance L from the centre of the robot

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- $v_L = \dot{\varphi}(L - R)$ and $v_R = \dot{\varphi}(L + R)$
- The speed of the centre of the robot: $V = \dot{\varphi}L \Leftrightarrow L = \frac{V}{\dot{\varphi}}$

Thus, L can be eliminated:

- $v_L = \dot{\varphi}(\frac{V}{\dot{\varphi}} - R) = V - \dot{\varphi}R$
- $v_R = \dot{\varphi}(\frac{V}{\dot{\varphi}} + R) = V + \dot{\varphi}R$

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- $v_R = \dot{\varphi}(\frac{V}{\dot{\varphi}} + R) = V + \dot{\varphi}R$

Next step is to solve for V and $\dot{\varphi}$!

The differential drive: Kinematics

Solve for V and $\dot{\varphi}$:

1. $v_L = V - \dot{\varphi}R \Leftrightarrow \dot{\varphi}R = V - v_L \Leftrightarrow \dot{\varphi} = \frac{V - v_L}{R}$

The differential drive: Kinematics

Solve for V and $\dot{\varphi}$:

1. $v_L = V - \dot{\varphi}R \Leftrightarrow \dot{\varphi}R = V - v_L \Leftrightarrow \dot{\varphi} = \frac{V - v_L}{R}$
2. $v_R = V + \dot{\varphi}R = [Ins.1] = V + \frac{V - v_L}{R}R = 2V - v_L \Leftrightarrow \boxed{V = \dfrac{v_R + v_L}{2}}$

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3. $v_L = V - \dot{\varphi}R \Leftrightarrow V = v_L + \dot{\varphi}R$

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3. $v_L = V - \dot{\varphi}R \Leftrightarrow V = v_L + \dot{\varphi}R$
4. $v_R = V + \dot{\varphi}R = [Ins.3] = v_L + \dot{\varphi}R + \dot{\varphi}R = v_L + 2\dot{\varphi}R \Leftrightarrow \boxed{\dot{\varphi} = \frac{v_R - v_L}{2R}}$

The differential drive: Kinematics

When simulating we normally want each velocity component separately:

- $\cos\varphi = \frac{V_x}{V} \Leftrightarrow \boxed{V_x = V\cos\varphi}$
- $\sin\varphi = \frac{V_y}{V} \Leftrightarrow \boxed{V_y = V\sin\varphi}$

The differential drive: Kinematics

It is then possible to obtain the pose (x_1, y_1, φ_1) of the robot at time t_1, from a known state (x_0, y_0, φ_0) at time t_0, by means of *integration*.

The differential drive: Kinematics

It is then possible to obtain the pose (x_1, y_1, φ_1) of the robot at time t_1, from a known state (x_0, y_0, φ_0) at time t_0, by means of *integration*.

- $x_1 = x_0 + \int_{t_0}^{t_1} V_x(t)\mathrm{d}t = x_0 + \int_{t_0}^{t_1} \frac{v_R(t)+v_L(t)}{2} \cos\varphi(t)\mathrm{d}t$
- $y_1 = y_0 + \int_{t_0}^{t_1} V_y(t)\mathrm{d}t = y_0 + \int_{t_0}^{t_1} \frac{v_R(t)+v_L(t)}{2} \sin\varphi(t)\mathrm{d}t$
- $\varphi_1 = \varphi_0 + \int_{t_0}^{t_1} \dot{\varphi}(t)\mathrm{d}t = \varphi_0 + \int_{t_0}^{t_1} \frac{v_R(t)-v_L(t)}{2R}\mathrm{d}t$

The differential drive: Kinematics

The vehicle model of differential drive can be used for:

- Simulation environments
  - In each time step, translate actuation into movement in the global frame
  - Ground truth

The differential drive: Kinematics

The vehicle model of differential drive can be used for:

- Simulation environments
  - In each time step, translate actuation into movement in the global frame
  - Ground truth
- Online movement estimation (internal model)
  - To model the behaviour of the robot
  - Wheel speeds (v_L, v_R) are usually not given *a priori*. Instead, some estimates $\hat{v}_L$, $\hat{v}_R$ must be used
  - Odometry is prone to errors!

The differential drive: Kinematics

- Given the constraints and the wheel speeds, the new state (pose: position, heading) can be determined using the kinematic model
- However, the kinematics does not say how the the motion is achieved.
- **Dynamics** by contrast considers the motion of the robot under the action of the forces and torques acting on it!

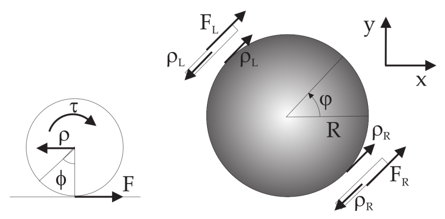
Dynamics

The differential drive: Dynamics

- Note that we assume a horizontal, flat surface ($-F_N = mg$ can be neglected)!
- No rolling resistance
- The motors generate torque, which propel the wheels
- When the wheels turn they apply a force to the ground, in the horizontal plane
- Due to friction an opposite force F will drag the robot forward, according to Newton's third law
- Using Newton's second law, $F = ma$, the equations of motion of the robot can be derived

The differential drive: Dynamics

Consider a free-body diagram of the robot in order to determine the forces in action:



- τ is the torque from the motor
- F is the reaction force from the ground
- ρ is the reaction force from the robot body acting on the wheel, due to the moment of inertia I of the robot body
- $\dot{\phi}$ is the angular velocity of the wheel
- φ is the yaw angle of the robot

The differential drive: Dynamics

Note that *angular momentum* is defined as:

$$\boxed{L = I\omega} \tag{1}$$

and *torque* is defined as:

$$\boxed{\tau = \frac{\mathrm{d}L}{\mathrm{d}t}} \tag{2}$$

Therefore,

$$\tau = I\frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{3}$$

where $\frac{\mathrm{d}\omega}{\mathrm{d}t}$ is the *angular acceleration*.

In the case of *circular motion* $\tau = I\frac{\mathrm{d}\omega}{\mathrm{d}t}$ can be interpreted as analogous to $F = ma$ in the case of linear motion.

The differential drive: Dynamics

Using Newton's second law, the forces and torques acting on the wheels

Translation:

- $m\dot{v}_L = F_L - \rho_L$
- $m\dot{v}_R = F_R - \rho_R$

Rotation:

- $I_w\ddot{\phi}_L = \tau_L - F_L r$
- $I_w\ddot{\phi}_R = \tau_R - F_R r$

where, for the wheel, m is the mass, I_w is the moment of inertia, and r is the radius.

The differential drive: Dynamics

Correspondingly, for the robot body

Translation:

- $M\dot{V} = \rho_L + \rho_R$

Rotation:

- $I\ddot{\varphi} = (\rho_R - \rho_L)R$

where, for the robot body, M is the mass, I is the moment of inertia, and R is the radius.

The differential drive: Dynamics

Now, we have 6 equations:

- $m\dot{v}_L = F_L - \rho_L$
- $m\dot{v}_R = F_R - \rho_R$
- $I_w\ddot{\phi}_L = \tau_L - F_L r$
- $I_w\ddot{\phi}_R = \tau_R - F_R r$
- $M\dot{V} = \rho_L + \rho_R$
- $I\ddot{\varphi} = (\rho_R - \rho_L)R$

and 10 unknown variables: v_L, v_R, F_L, F_R, ρ_L, ρ_R, ϕ_L, ϕ_R, V, and φ

The differential drive: Dynamics

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Four more equations are needed!

The differential drive: Dynamics

The kinematics constraints:

- Wheels should roll without slipping:
  - $v_L = \dot{\phi}_L r$
  - $v_R = \dot{\phi}_R r$
- The two kinematic equations:
  - $V = \frac{v_L + v_R}{2}$
  - $\dot{\varphi} = \frac{v_R - v_L}{2R}$

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Now we have what we need!

The differential drive: Dynamics

Eliminate F_L and F_R:

$$\left.\begin{aligned} m\dot{v}_L &= F_L - \rho_L \\ I_w\ddot{\phi}_L &= \tau_L - F_L r \end{aligned}\right\} \Rightarrow \boxed{m\dot{v}_L = \frac{\tau_L - I_w\ddot{\phi}_L}{r} - \rho_L}$$

$$\left.\begin{aligned} m\dot{v}_R &= F_R - \rho_R \\ I_w\ddot{\phi}_R &= \tau_R - F_R r \end{aligned}\right\} \Rightarrow \boxed{m\dot{v}_R = \frac{\tau_R - I_w\ddot{\phi}_R}{r} - \rho_R}$$

Two variables less!

The differential drive: Dynamics

Note that:

$$v_L = \dot{\phi}_L r \Leftrightarrow \dot{\phi}_L = \frac{v_L}{r} \Leftrightarrow [Differentiate] \Leftrightarrow \ddot{\phi}_L = \frac{\dot{v}_L}{r}$$

$$v_R = \dot{\phi}_R r \Leftrightarrow \dot{\phi}_R = \frac{v_R}{r} \Leftrightarrow [Differentiate] \Leftrightarrow \ddot{\phi}_R = \frac{\dot{v}_R}{r}$$

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Eliminate $\ddot{\phi}_L$ and $\ddot{\phi}_R$ and solve for ρ_L and ρ_R:

$$m\dot{v}_L = \frac{\tau_L - I_w \ddot{\phi}_L}{r} - \rho_L \Leftrightarrow \boxed{\rho_L = \frac{\tau_L}{r} - \left(\frac{I_w}{r^2} + m\right) \dot{v}_L}$$

$$m\dot{v}_R = \frac{\tau_R - I_w \ddot{\phi}_R}{r} - \rho_R \Leftrightarrow \boxed{\rho_R = \frac{\tau_R}{r} - \left(\frac{I_w}{r^2} + m\right) \dot{v}_R}$$

The differential drive: Dynamics

Recall that the translation of the centre of mass of the robot is determined by (Newton II):

$$M\dot{V} = \rho_L + \rho_R$$

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Eliminate ρ_L and ρ_R using the previous results:

$$M\dot{V} = \frac{\tau_L}{r} - \left(\frac{I_w}{r^2} + m\right)\dot{v}_L + \frac{\tau_R}{r} - \left(\frac{I_w}{r^2} + m\right)\dot{v}_R$$

$$= \boxed{\frac{\tau_L + \tau_R}{r} - \left(\frac{I_w}{r^2} + m\right)(\dot{v}_L + \dot{v}_R)}$$

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$$= \boxed{\frac{\tau_L + \tau_R}{r} - \left(\frac{I_w}{r^2} + m\right)(\dot{v}_L + \dot{v}_R)}$$

Note that:

$$V = \frac{v_L + v_R}{2} \Leftrightarrow v_L + v_R = 2V \Rightarrow [Differentiate] \Rightarrow \boxed{\dot{v}_L + \dot{v}_R = 2\dot{V}} \Rightarrow$$

The differential drive: Dynamics

$$\Rightarrow M\dot{V} = \frac{\tau_R + \tau_L}{r} - 2\left(\frac{I_w}{r^2} + m\right)\dot{V}$$

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Rearrange the terms:

$$M\dot{V}\left(1 + 2\left(\frac{I_w}{Mr^2} + \frac{m}{M}\right)\right) = \frac{1}{r}(\tau_R + \tau_L)$$

The differential drive: Dynamics

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or

$$M\dot{V} = A(\tau_R + \tau_L)$$

The differential drive: Dynamics

Similarly for the rotational motion (Newton II):

$$I\ddot{\varphi} = (\rho_R - \rho_L)R$$

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Recall the kinematic equation:

$$\dot{\varphi} = \frac{v_R - v_L}{2R} \Rightarrow [\textit{Differentiate}] \Rightarrow \dot{v}_L - \dot{v}_R = -2R\ddot{\varphi}$$

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(1) Eliminate ρ_L and ρ_R, and (2) insert the result for $(\dot{v}_L - \dot{v}_R)$

$$\Rightarrow I\ddot{\varphi} = \left(\frac{\tau_R}{r} - \left(\frac{I_w}{r^2} + m\right)\dot{v}_R - \frac{\tau_L}{r} + \left(\frac{I_w}{r^2} + m\right)\dot{v}_L\right)R$$

$$= (\tau_R - \tau_L)\frac{R}{r} + R\left(\frac{I_w}{r^2} + m\right)(\dot{v}_L - \dot{v}_R)$$

$$= (\tau_R - \tau_L)\frac{R}{r} - 2R^2\left(\frac{I_w}{r^2} + m\right)\ddot{\varphi} \Rightarrow$$

The differential drive: Dynamics

Rearrange the terms and solve for $I\ddot{\varphi}$:

$$\Rightarrow I\ddot{\varphi}\left(\frac{r}{R}+2\left(\frac{I_w R}{Ir}+\frac{mRr}{I}\right)\right)=(\tau_R-\tau_L)$$

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or

$$I\ddot{\varphi}=B(\tau_R-\tau_L)$$

The differential drive: Dynamics

Due to friction and limited strength of the motors, linear damping terms are introduced:

$$M\dot{V} + \alpha V = \frac{1}{r}\left(1 + 2\left(\frac{I_w}{Mr^2} + \frac{m}{M}\right)\right)^{-1}(\tau_R + \tau_L)$$

$$I\ddot{\varphi} + \beta\dot{\varphi} = \left(\frac{r}{R} + 2\left(\frac{I_w R}{Ir} + \frac{mRr}{I}\right)\right)^{-1}(\tau_R - \tau_L)$$

where α and β are empirical constants.

The motor torques τ_L and τ_R can be controlled. Therefore, the motion of the robot can be determined by integration of the above equations.

The differential drive: Dynamics

However, assuming that $m \ll M$ and $I_w \ll I$:

$$M\dot{V} + \alpha V = \frac{1}{r}\left(1 + 2\left(\frac{I_w}{Mr^2} + \frac{m}{M}\right)\right)^{-1}(\tau_R + \tau_L)$$

$$I\ddot{\varphi} + \beta\dot{\varphi} = \left(\frac{r}{R} + 2\left(\frac{I_w R}{Ir} + \frac{mRr}{I}\right)\right)^{-1}(\tau_R - \tau_L)$$

can be simplified to:

$$\boxed{M\dot{V} + \alpha V = \frac{1}{r}(\tau_R + \tau_L)}$$

$$\boxed{I\ddot{\varphi} + \beta\dot{\varphi} = \frac{R}{r}(\tau_R - \tau_L)}$$

which can be used as our final dynamic model of motion.

Integration

The differential drive: Euler integration

The robot motion can be obtained numerically obtained by a *solver*, for example the **Euler method**.

The differential drive: Euler integration

For each time step t_0, t_1, t_2, ... of length Δt:

1. Compute $\dot{V}$ and $\ddot{\varphi}$ using

$$\boxed{M\dot{V}_{n+1} + \alpha V_n = \frac{1}{r}(\tau_R + \tau_L)} \qquad \boxed{I\ddot{\varphi}_{n+1} + \beta\dot{\varphi}_n = \frac{R}{r}(\tau_R - \tau_L)}$$

2. Then update the velocities

$$\boxed{V_{n+1} = V_n + \dot{V}_{n+1}\Delta t} \qquad \boxed{\dot{\varphi}_{n+1} = \dot{\varphi}_n + \ddot{\varphi}_{n+1}\Delta t}$$

3. Then update the yaw angle

$$\boxed{\varphi_{n+1} = \varphi_n + \dot{\varphi}_{n+1}\Delta t}$$

4. Finally update the position

$$\boxed{X_{n+1} = X_n + V_{n+1}\cos\varphi_{n+1}\Delta t} \qquad \boxed{Y_{n+1} = Y_n + V_{n+1}\sin\varphi_{n+1}\Delta t}$$

Questions

Please post all questions on the Canvas discussion pages, in that way we can all benefit from the answers, and I can highlight important outcomes.