CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	August 17, 2022, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 50 points (need 20 points to pass).
Maximum score for homework problems: 50 points (need 20 points to pass).
CTH $\geqslant$40 passed; $\geqslant$60 grade 4; $\geqslant$80 grade 5,
GU $\geqslant$40 grade G; $\geqslant$70 grade VG.

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1. Multiple choice questions [**8 points**] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Consider the differential equation $\ddot{x} = 3x + 2\dot{x}$. If $y = \dot{x}$, this can be written as a dynamical system of dimensionality two on the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbb{A} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Which of the following matrices gives the correct form of $\mathbb{A}$?

A. $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$ C. $\begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$ D. $\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$ E. $\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$

b) Which type(s) of bifurcation(s) does the system $\dot{x} = rx - 4x^3$ have?

A. Saddle-node bifurcation

B. Transcritical bifurcation

C. Supercritical pitchfork bifurcation

D. Subcritical pitchfork bifurcation

E. No bifurcation occurs

c) Which of the following statements hold for a homoclinic bifurcation?

A. It is a local bifurcation.

B. It may occur as an isolated stable spiral becomes unstable.

C. It may occur as a saddle point collides with a limit cycle.

D. The amplitude of an involved limit cycle approaches zero at the bifurcation point.

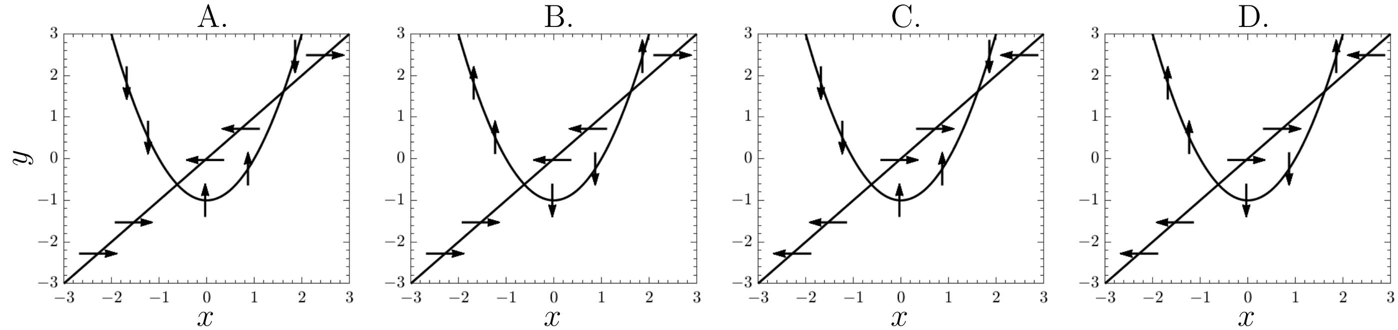
E. The period time of an involved limit cycle approaches infinity at the bifurcation point.

d) Consider the dynamical system

$$\dot{x} = x^2 - y - 1$$
$$\dot{y} = x - y$$

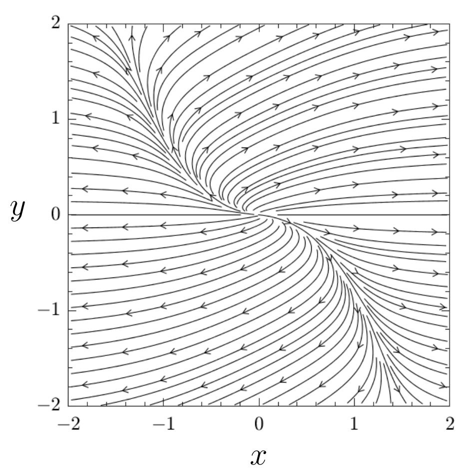
The figures below show the nullclines of this system with arrows indicating possible flow direction. Which of the figures A–D is correct?

A. B. C. D.



e) The figure below shows the phase portrait of the system:

$$\dot{x} = x + 2y - x^3$$
$$\dot{y} = y$$

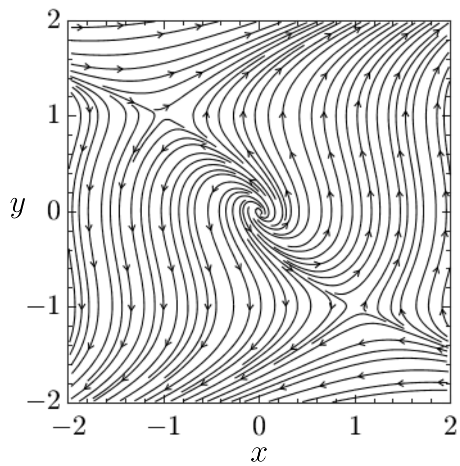


What is the index of the fixed point in this system?

A. -2 B. -1 C. 0 D. 1 E. 2

f) The figure below shows the phase portrait of the system:

$$\dot{x} = -y + y^3$$
$$\dot{y} = x + y$$



The system has three fixed points. What is the index of a test curve encircling the two rightmost fixed points?

A. -2 B. -1 C. 0 D. 1 E. 2

2. Short questions [8 points] For each of the following questions give a concise answer within a few lines per question.

a) Give a definition for what a dynamical system is.

b) Explain how one can numerically calculate the unstable directions and unstable manifolds for a fixed point in a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$.

c) Explain why saddle-node, transcritical and pitchfork bifurcations in systems of dimensionality two or higher are very similar to the corresponding bifurcations in dimensionality one systems.

d) Explain what a limit cycle is. Give an example of a system with a limit cycle.

e) Explain what the difference between a limit set and an attractor is.

f) Explain what transient chaos is. Give an example of a system with transient chaos.

3. Critical slowing down in model for crypto currency [10 points] The system below is simple model for the price x ($x \geqslant 0$) of a crypto currency

$$\dot{x} = -m + rx - x^3 \,. \tag{1}$$

Here $m \geqslant 0$ and $r \geqslant 0$ are parameters. A stochastic version of Eq. (1) has been used to study early warning signals for critical transitions in x.

a) Introduce dimensionless units to rewrite Eq. (1) such that all parameter dependence lies in a constant term α. What is the expression for α?

b) Find all realistic (non-negative) fixed points when $\alpha = 0$ and determine their stability. If you did not solve subtask a), you can set $m = \alpha$ and $r = 1$ in Eq. (1).

c) Analytically identify all realistic bifurcations in the system. Sketch the bifurcation diagram against α and label the types of all bifurcations.

d) For one realistic fixed point that is stable and that undergoes a bifurcation, evaluate the stability exponent close to the bifurcation.

4. Construction of fixed points [8 points]

a) Construct a **linear** dynamical system of dimensionality two with a fixed point that is a center.

b) Construct a **non-linear** dynamical system of dimensionality two with a non-linear center, i.e. a fixed point whose linear stability indicates a center and which is surrounded by closed orbits.

c) Introduce a parameter μ to your system in subtask a) such that, as the parameter passes a critical value μ_c, the center becomes a saddle point. *Hint* The dynamics does not need to depend smoothly on μ.

5. Lorenz system for large r [8 points] In a certain limit for large values of the parameter r, the Lorenz system can be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -xz \\ \dot{z} &= xy \end{aligned} \tag{2}$$

The dynamics (2) turns out to be bounded at all times. i.e. it does not run off to infinity in any direction.

a) Identify all fixed points of the system (2) and describe their linear stability according to their stability exponents.

b) Show that the system (2) conserves phase space volume. Discuss whether phase space volume is an integral of motion.

c) Find two integrals of motion for the system (2).

d) Using the results of subtasks a)–c) discuss what values the Lyapunov exponents can take. Can the dynamics be chaotic?

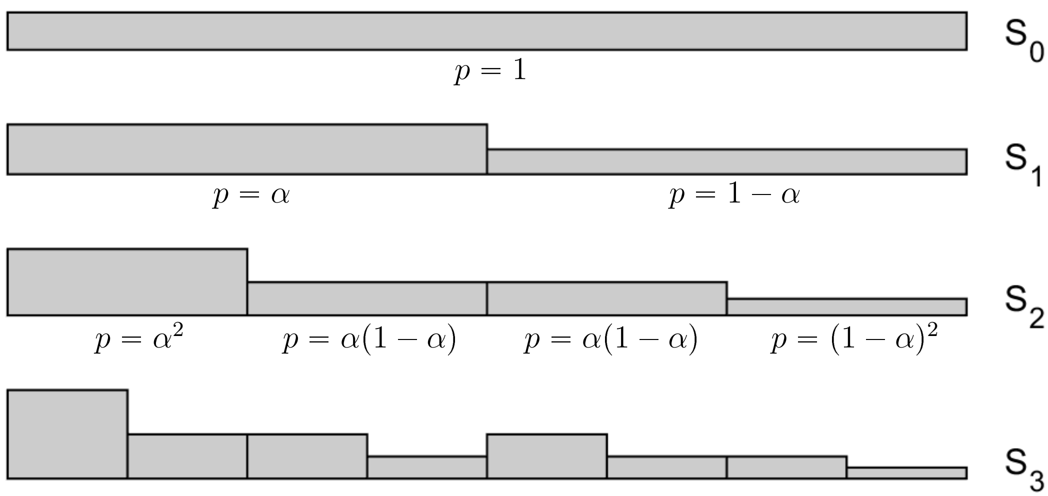
6. Fractal dimension of a weighted Cantor set [8 points] The generalized fractal dimension D_q is defined by

$$D_q \equiv \frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)}$$

with

$$I(q,\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon)\,.$$

Here p_k is the probability to be in the k:th occupied box (box with $p_k \neq 0$) and N_{box} is the total number of occupied boxes.
Consider a set S_n where n labels the generation. Start with S_0 being the unit interval. S_n is obtained by dividing each interval in the set S_{n-1} into two subintervals. Upon each division, allocate a fraction α (assume $0 \leqslant \alpha \leqslant 1$) of the probability to be in the original interval to the left subinterval, and a fraction $1-\alpha$ to the right subinterval. The figure below illustrates the first few generations S_0, S_1, S_2 and S_3:



The probability to be in different intervals is displayed in the text below the intervals. The height of an interval illustrates the relative probability to be in that interval for the case $\alpha = 2/3$. In what follows, consider the fractal dimension of the set S_∞ obtained by iterating $n \to \infty$.

a) Evaluate the generalized fractal dimension D_q of S_∞ for the case $\alpha = 1$.

b) Evaluate the generalized fractal dimension D_q of S_∞ for the case $\alpha = \frac{1}{2}$.

c) Evaluate the box-counting dimension D_0 of S_∞ for $0 < \alpha < 1$.

d) Evaluate the generalized fractal dimension D_q of S_∞ for general values of α.
Hint: To verify your result, you can check that the results in subtasks a), b) and c) come out correctly and that your result is symmetric upon replacing $\alpha \to 1-\alpha$.