CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	January 10, 2022, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 50 points (need 20 points to pass).
Maximum score for homework problems: 50 points (need 20 points to pass).
CTH $\geqslant$40 passed; $\geqslant$60 grade 4; $\geqslant$80 grade 5,
GU $\geqslant$40 grade G; $\geqslant$70 grade VG.

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1. Multiple choice questions [**8 points**] For each of the following questions identify **all** the correct alternatives labelled by A,B,.... Answer with letters among the alternatives. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters.

a) Consider the two-dimensional linear dynamical system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbb{A}\begin{pmatrix}x\\ y\end{pmatrix}, \qquad \text{where } \mathbb{A} = \begin{pmatrix}4 & 4\\ -1 & 8\end{pmatrix}.$$

What type is the fixed point of the dynamics above?

A. Node

B. Spiral

C. Saddle point

D. Center

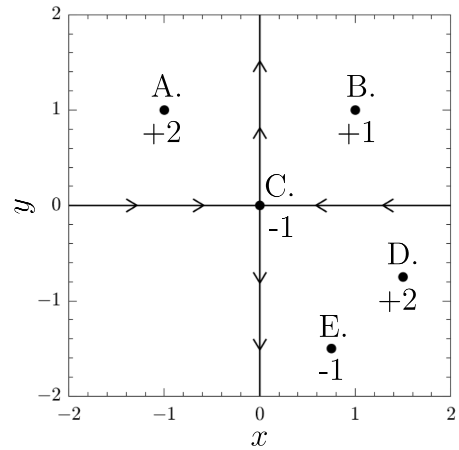
E. Degenerate node

F. Star

b) Which of the following alternatives are eigenvectors to the fixed point in subtask a)?

A. $\begin{pmatrix}2\\ 1\end{pmatrix}$ B. $\begin{pmatrix}1\\ 2\end{pmatrix}$ C. $\begin{pmatrix}0\\ 1\end{pmatrix}$ D. $\begin{pmatrix}1\\ -2\end{pmatrix}$ E. $\begin{pmatrix}2\\ -1\end{pmatrix}$

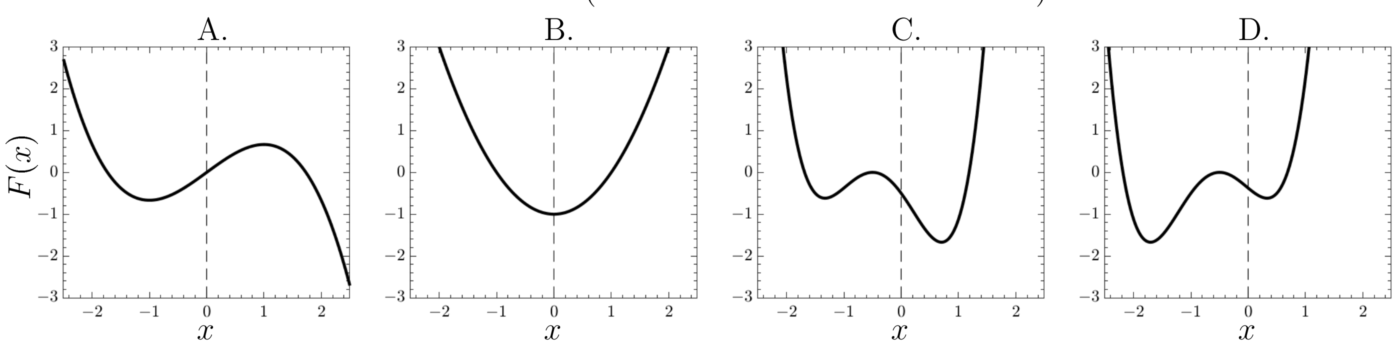
c) The figure below shows the phase plane of a two-dimensional system with 5 fixed points and invariant manifolds along the x- and y-axes. Circles mark the locations of the fixed points, each labeled with A–E and its index. Which fixed points can be located inside a closed orbit?



d) Consider systems on the form

$$\epsilon\dot{x} = z - F(x)$$
$$\dot{z} = -\epsilon x$$

with $0 < \epsilon \ll 1$. Which of the choices A–D of $F(x)$ below may give rise to relaxation oscillations (vertical dashed lines show $x = 0$)?



e) Hamiltonian systems in one spatial dimension has centers at potential minima. What type(s) of fixed points can occur at minima of $V(x)$ in the system $\dot{x} = p$ and $\dot{p} = +V'(x)$ (note the sign)?

A. Nodes

B. Spirals

C. Saddle points

D. Centers

E. Degenerate nodes

F. Stars

f) Which of the following bifurcations can be involved in an intermittency transition to chaos?

A. Saddle-node bifurcation

B. Transcritical bifurcation

C. Subcritical pitchfork bifurcation

D. Supercritical pitchfork bifurcation

E. Hopf bifurcation

2. Short questions [8 points] For each of the following questions give a concise answer within a few lines per question.

a) Sketch the bifurcation diagram of the system $\dot{x} = r/2 - re^{-x^2}$ with real parameter r. Comment on the type of bifurcation(s) that occurs.

b) Explain why the index of a test curve not encircling a fixed point is zero, while it can be non-zero for a test curve encircling a fixed point.

c) Explain what the difference between a global and a local bifurcation is. Give two examples of global bifurcations.

d) Explain why the dynamics in billiard systems of rectangular shape is equivalent to the dynamics of uncoupled oscillators on a torus. What kinds of long-term behaviours are possible?

e) Explain why at least one Lyapunov exponent is zero for a trajectory starting on a closed orbit.

f) Explain what an attractor is.

3. Biochemical switch [10 points] A biochemical switch consists of a gene that is normally inactive, but can be activated to produce a gene product (for example pigment) if a signal substance exceeds a threshold. A simple model for a biochemical switch is given by

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2} \,. \tag{1}$$

Here $g(t)$ denotes the concentration of the gene product, s_0 is a constant concentration of a signal substance, and all k_1, k_2, k_3 and k_4 are positive parameters.

a) Show that by introducing suitable dimensionless units, the system (1) can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = s - rx + \frac{x^2}{1 + x^2} \,, \tag{2}$$

where $s \geqslant 0$ and $r > 0$ are dimensionless parameters. Answer by expressing x, τ, s and r in terms of the original variables and parameters.

b) Consider the case $s = 0$ and $r > 0$ in the system (2). Perform the following tasks

   i) find all fixed points;

   ii) sketch the bifurcation diagram;

   iii) mark up location and type of all bifurcations with $s = 0$ and $r > 0$.

c) Use a geometric approach to qualitatively sketch the bifurcation diagram against r for the two cases of small positive s and large s.
Hint Qualitative locations of the fixed points can for example be obtained by sketching $x^2/(1 + x^2)$ and the linear part of Eq. (2).

d) Assume that there initially is no gene product concentration, $x(0) = 0$, and that s is slowly increased from zero, i.e. the activating signal is slowly turned on. What happens to $x(t)$? What happens if s then goes back to zero? Does the gene turn off again?

4. Hopf bifurcations [8 points] One system with a Hopf bifurcation in Cartesian coordinates is given by the following dynamics in polar coordinates

$$\begin{aligned} \dot{r} &= r^3 - \mu r \\ \dot{\theta} &= \omega \end{aligned} \tag{3}$$

Here r is the radius, θ the polar angle, and μ is a real parameter.

a) Consider the one-dimensional system for the radius r. Sketch the bifurcation diagram against μ and label all bifurcations with their type.

b) Consider the dynamics corresponding to the two-dimensional system (3), but in Cartesian coordinates x and y. Explain the dynamics for $\mu < 0$ and $\mu > 0$. Are there stable or unstable limit cycles?

c) Repeat subtasks a) and b) for the following system

$$\begin{aligned} \dot{r} &= r^2 - \mu r \\ \dot{\theta} &= \omega \end{aligned} \tag{4}$$

d) Make a variable substitution to write down the dynamics (3) and (4) in Cartesian coordinates. Discuss differences in the two cases. Why would you prefer one system over the other?

5. Hamiltonian systems [8 points] Consider a Hamiltonian system in two spatial dimensions

$$\begin{aligned} \dot{\boldsymbol{x}} &= \frac{\partial H}{\partial \boldsymbol{p}} \\ \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{x}} \end{aligned} \tag{5}$$

with coordinates $\boldsymbol{x} = (x, y)$ and $\boldsymbol{p} = (p_x, p_y)$, and Hamiltonian function

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m}|\boldsymbol{p}|^2 + \frac{m\omega_0^2}{2}|\boldsymbol{x}|^2 - \frac{b}{4}|\boldsymbol{x}|^4 \,. \tag{6}$$

a) Show that the dynamical system (5) for the Hamiltonian (6) in suitable dimensionless units, t', $\boldsymbol{x}'$ and $\boldsymbol{p}'$, can be written as

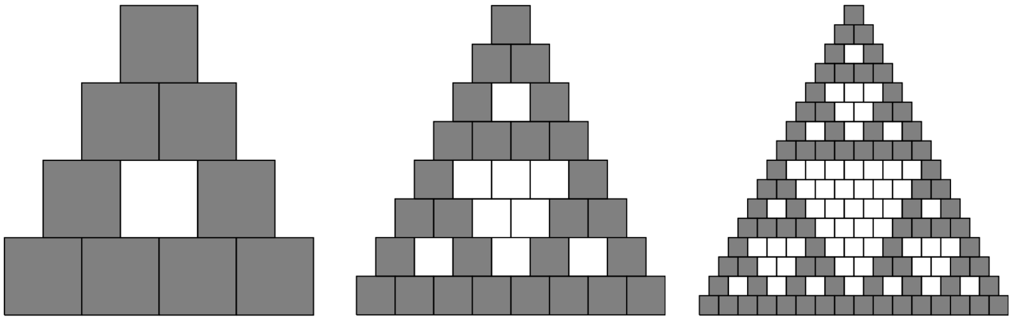
$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}t'} &= \boldsymbol{p}' \\ \frac{\mathrm{d}\boldsymbol{p}'}{\mathrm{d}t'} &= \boldsymbol{x}'(-1 + |\boldsymbol{x}'|^2) \end{aligned} \tag{7}$$

b) Consider trajectories starting with $y = p_y = 0$. Analyze the dynamics (7) and sketch the phase portrait for such trajectories.

c) Find two integrals of motion to the system (7).

d) Discuss whether there are initial conditions (no longer constrained to $y = p_y = 0$) that can give rise to chaotic dynamics in the system (7).

6. Fractal dimension of Pascal's triangle [8 points] Pascal's triangle is a triangular array of binomial coefficients. The first $N = 2^k$ rows are shown in the figure below for $k = 2, 3, 4$. Odd entries ar colored grey and even entries are white.

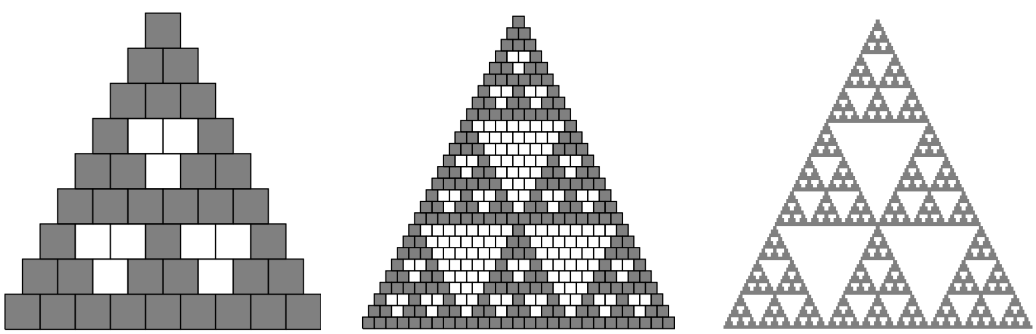


$N = 2^2$ $\qquad$ $N = 2^3$ $\qquad$ $N = 2^4$

The pattern of odd numbers approximates a fractal, becoming more accurate the more rows are taken into account. Consider the triangle for the $N = 2^k$ first rows. The triangle of $2N = 2^{k+1}$ rows is obtained by making two copies of the pattern in the original N-row triangle, and attaching them below the two lower corners of the original triangle, see the figure above for the first few values of k. In the limit of $N \to \infty$, the pattern becomes fractal.

a) What is the box-counting dimension of odd entries in Pascal's triangle as $N \to \infty$?

If instead entries not divisible by 3 are colored grey, the pattern for the first $N = 3^k$ rows becomes



$N = 3^2$ $\qquad$ $N = 3^3$ $\qquad$ $N = 3^4$

Starting from the triangle with $N = 3^k$ rows, the triangle of $3N = 3^{k+1}$ rows is obtained by making five copies of the pattern in the original N-row triangle, attaching two of these copies below the lower corners of the original triangle, and then attaching the three remaining copies below these two. See the figure above for the first few values of k. In the limit of $N \to \infty$, the pattern becomes fractal.

b) What is the box-counting dimension of entries not divisible by 3 in Pascal's triangle as $N \to \infty$?

c) What is the box-counting dimension of even entries in Pascal's triangle as $N \to \infty$ [white entries in subtask a)]?