8 Example 1: The van der Pol oscillator (Strogatz Chapter 7)

So far we have seen some different possibilities of what can happen in two-dimensional systems (local and global attractors and bifurcations) using mainly constructed examples. In this and the next lectures we consider two examples with real-world applications.

8.1 Self-sustained oscillations

Self-sustained oscillations are frequent in nature and in technology, some examples being stick-slip oscillations, unwanted mechanical vibrations, and oscillators in biology. Usually a system with self-sustained oscillations has a feedback mechanism such that small-amplitude oscillations grow in size. The small oscillations are often modeled using an oscillator with negative damping, $\gamma = -\mu < 0$

$$\ddot{x} - \mu\dot{x} + \omega_0^2 x = 0\,.$$

The system for x and $y = \dot{x}$ has an unstable spiral at the origin, blowing up small-scale oscillations. As oscillations grow, non-linear terms may form a stable limit cycle and the system shows self-sustained oscillations. One simple and common model in mathematics, physics, engineering, biology and economic theory, is the van der Pol oscillator:

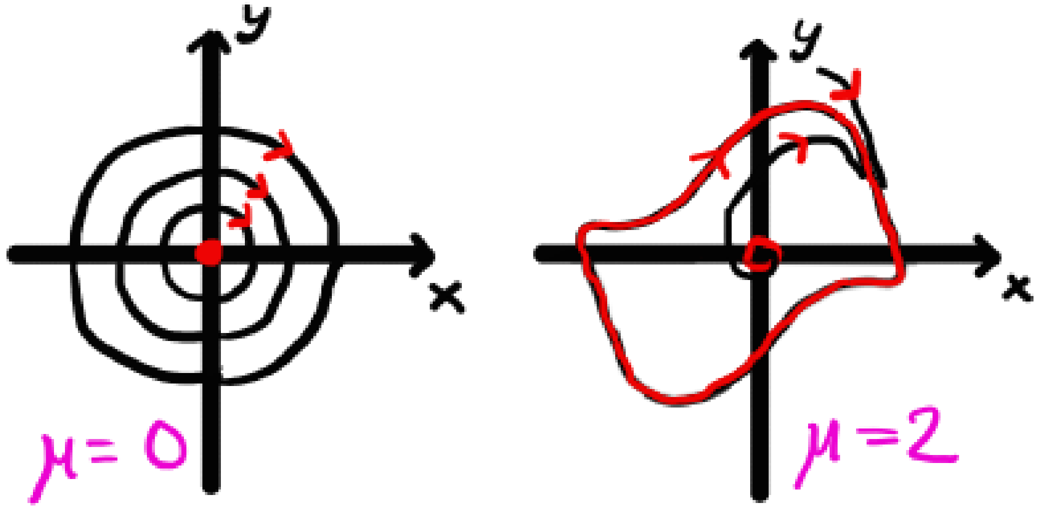
$$\ddot{x} + \underbrace{\mu(x^2-1)}_{f(x)}\dot{x} + x = 0 \tag{1}$$

Nonlinear damping coefficient $f(x)$ damps large oscillations (friction when $|x| > 1$) and amplifies small oscillations (forcing when $|x| < 1$) $\Rightarrow$ We expect self-sustained oscillations to be possible.

Indeed the corresponding dynamical system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x - \mu(x^2-1)y\end{aligned}$$

shows a stable limit cycle if $\mu > 0$ (after a rescaling $x = x'/\sqrt{\mu}$ and $y = y'/\sqrt{\mu}$, the system has a supercritical Hopf bifurcation):



Period time and shape of the cycle depends on μ.

The van der Pol oscillator is an example of a Liénard system

$$\ddot{x} + \underbrace{f(x)}_{\text{even}}\,\dot{x} + \underbrace{g(x)}_{\text{odd}} = 0$$

For such systems a unique stable limit cycle surrounds the origin if certain criteria (Strogatz 7.4) are fulfilled that ensures that:

- the nonlinear damping f is negative for small $|x|$ and positive for large x
- displacements are reduced by the nonlinear restoring force g

The van der Pol oscillator, Eq. (1), can not be solved analytically for general values of μ. In certain limits however, we can find approximate solutions, as seen by the following sections.

8.2 Relaxation oscillations: Case of large μ

Now consider the van der Pol oscillator (1)

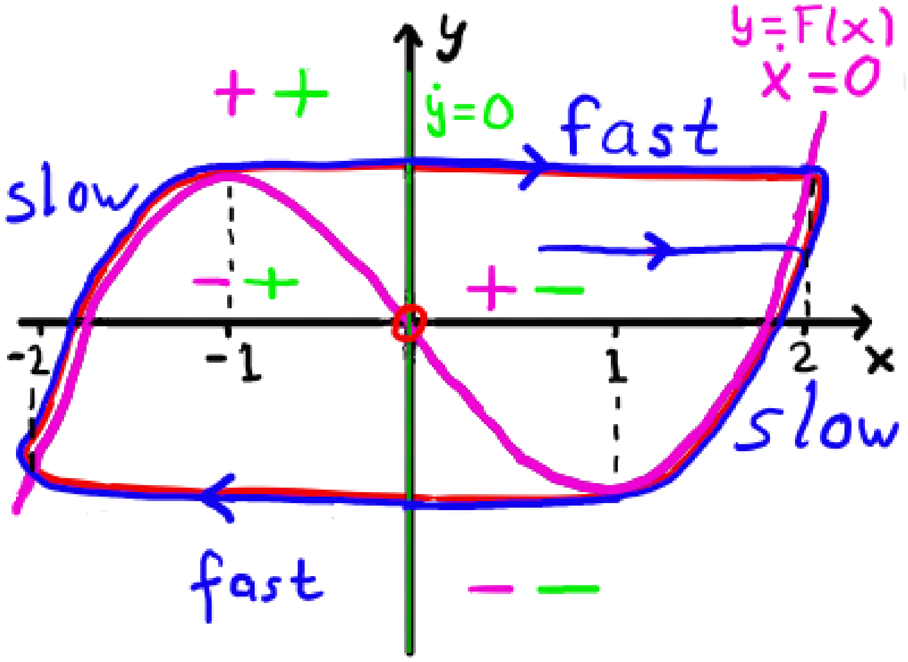
$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

with $\mu \gg 1$. Let $\epsilon = 1/\mu \ll 1$ be small. Let $y = \epsilon\dot{x} + F(x)$ with $F(x) = x^3/3 - x$:

$$\begin{aligned}\epsilon\dot{x} &= y - F(x)\\ \dot{y} &= \epsilon\ddot{x} + F'(x)\dot{x} = -\epsilon x\,.\end{aligned}$$

Two kinds of dynamics emerge:
Fast: If $|y - F(x)| \sim 1$: $|\dot{x}| \gg 1$ and $|\dot{y}| \sim \epsilon \ll 1$
Slow: If $|y - F(x)| \sim \epsilon^2$: $|\dot{x}| \sim \epsilon$ and $|\dot{y}| \sim \epsilon$

Dynamics can be understood by plotting the nullclines $\dot{x} = 0$ ($y = F(x)$) and $\dot{y} = 0$



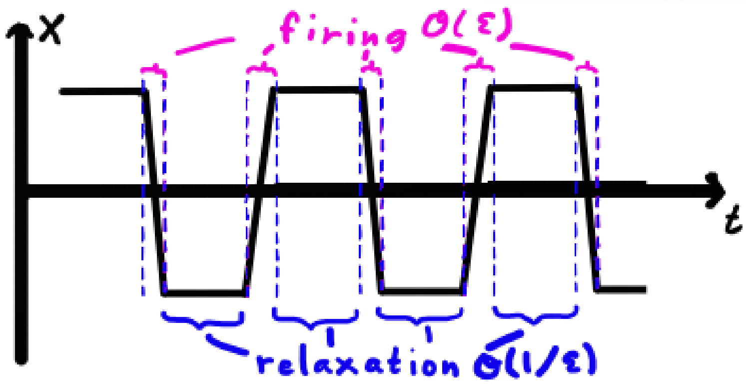
Starting from any point (except fixed point in origin) the trajectory moves quickly horizontally onto the cubic nullcline $y = F(x)$, then it moves slowly along nullcline until the 'jump-off points' (max and min of $F(x)$) where the direction of the flow and the nullcline starts to deviate. After the jump-off point the trajectory quickly moves over to the branch on the opposite side, and so it continues (the red underlying limit cycle is approached).

The period time can be approximated by the travel time along the two slow branches. On the slow branches $y \approx F(x) \Rightarrow \dot{y} \approx F'(x)\dot{x}$. But we also have $\dot{y} = -\epsilon x$

$$\Rightarrow F'(x)\frac{\mathrm{d}x}{\mathrm{d}t} \approx -\epsilon x\,, \Rightarrow \mathrm{d}t \approx -\frac{1}{\epsilon x}F'(x)\mathrm{d}x$$
$$\Rightarrow T_{\text{slow}} \approx -\int_{x_1}^{x_2} \frac{1}{\epsilon x}F'(x)\mathrm{d}x = -\frac{1}{\epsilon}\left[\frac{x^2}{2} - \ln x\right]_{x_1}^{x_2}$$
$$= [\text{Take slow branch from } x_1 = 2 \text{ to } x_2 = 1] = \frac{1}{2\epsilon}[3 - 2\ln 2]$$

Slow compared to $T_{fast} \sim \epsilon \Rightarrow$ period time is $\approx 2T_{slow}$.

This is an example of a relaxation oscillator:



Relaxation oscillations: very slow build-up and sudden discharge (for example periodic firing of nerve cells, geysers, stick-slip oscillations, e.g. squeaking of door hinges, of chalk on a blackboard, of violin bow setting strings in vibration),...

Relaxation oscillators have two widely separated time scales acting sequentially. In the opposite limit $\mu \ll 1$ the situation is more complicated: two time scales act at the same time.

8.3 van der Pol oscillator with small μ

Consider the van der Pol oscillator Eq. (1) with $0 < \mu \ll 1$ and some initial condition $x(0) = 0$, $\dot{x}(0) = 1$ (arbitrary condition that starts off the limit cycle).

8.3.1 Regular perturbation theory

Search for a solution for small values of μ by a series expansion

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \dots$$

and insert this expansion into Eq. (1) and collect terms to order μ

$$\begin{aligned} 0 &= \ddot{x} + \mu(x^2 - 1)\dot{x} + x \\ &= [\ddot{x}_0 + \mu\ddot{x}_1] + \mu([x_0 + \mu x_1]^2 - 1)[\dot{x}_0 + \mu\dot{x}_1] + [x_0 + \mu x_1] + O(\mu^2) \\ &= \ddot{x}_0 + x_0 + \mu[\ddot{x}_1 + (x_0^2 - 1)\dot{x}_0 + x_1] + O(\mu^2)\,. \end{aligned}$$

With initial conditions

$$\begin{aligned} 0 = x(0) &= \underbrace{x_0(0)}_{=0} + \mu \underbrace{x_1(0)}_{=0} \\ 1 = \dot{x}(0) &= \underbrace{\dot{x}_0(0)}_{=1} + \mu \underbrace{\dot{x}_1(0)}_{=0}\,. \end{aligned}$$

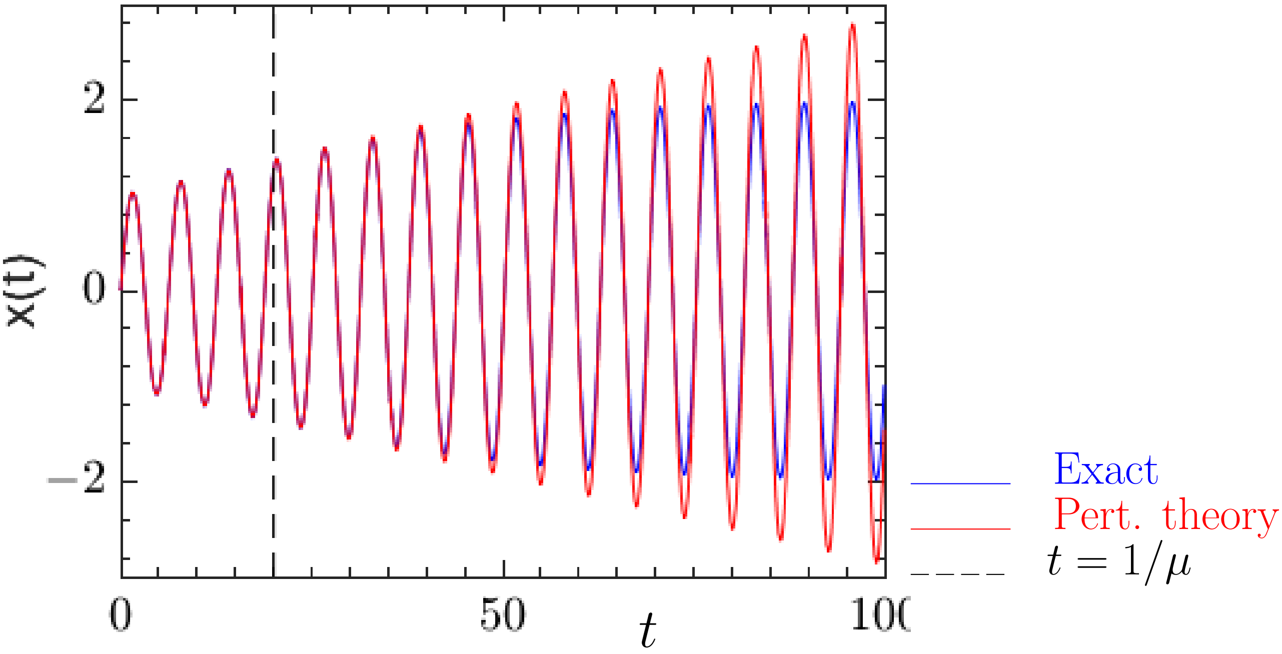
To order μ^0 we have

$$\ddot{x}_0 + x_0 = 0 \;\Rightarrow\; x_0(t) = \sin t\,.$$

To order μ^1 we have

$$\ddot{x}_1 + (\sin^2 t - 1)\cos t + x_1 = 0 \;\Rightarrow\; x_1(t) = (6t + \sin(2t))\sin t/16\,.$$

Problem: x_1 contains <u>secular terms</u> (terms that approaches infinity as $t \to \infty$):



Thus the perturbation theory fails to describe the formation of the limit cycle (the amplitude of oscillations grow to infinity). Even though the perturbation theory is identical to the series expansion of the actual solution, it fails for times of order $t \sim 1/\mu$.

In order to obtain a perturbation theory valid for large values of t, we need to make a high-order expansion. Alternatively, we can notice that there are (at least) two time scales in the problem: one for the oscillations (fast, $O(1)$) and one for the peak amplitude (slow,$O(1/\mu)$). Separating these time scales in the perturbation expansion, so called two-timing, gives more accurate results for large t even though we only consider the lowest order in μ.

8.3.2 Two-timing

Let $\tau = t$ denote the fast time scale and $T = \mu t$ denote the slow time scale (T is of order unity when $t \sim 1/\mu \gg 1$) and treat these as independent variables: $x = x(\tau, T)$. The reason this works is that when we have a large time separation $T \gg \tau$, x is roughly constant w.r.t. T during the time scale τ , and x fluctuates so rapidly that the variable τ is effectively averaged during the time scale T. Evaluate time derivatives:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \underbrace{\frac{\partial x}{\partial \tau}}_{\partial_\tau x}\underbrace{\frac{\mathrm{d}\tau}{\mathrm{d}t}}_{1} + \underbrace{\frac{\partial x}{\partial T}}_{\partial_T x}\underbrace{\frac{\mathrm{d}T}{\mathrm{d}t}}_{\mu} = \partial_\tau x + \mu\partial_T x$$
$$\ddot{x} = \partial_\tau \dot{x} + \mu\partial_T \dot{x} = \partial_\tau^2 x + 2\mu\partial_\tau\partial_T x + O(\mu^2)$$

Make expansion in terms of small μ, $x(\tau, T) = x_0(\tau, T) + \mu x_1(\tau, T) + \ldots$, and insert this into the van der Pol equation (1)

$$\begin{aligned}
0 &= \ddot{x} + \mu(x^2-1)\dot{x} + x\\
&= \partial_\tau^2 x + 2\mu\partial_\tau\partial_T x + \mu(x^2-1)[\partial_\tau x + \mu\partial_T x] + x + O(\mu^2)\\
&= \partial_\tau^2 x + x + \mu[2\partial_\tau\partial_T x + (x^2-1)\partial_\tau x] + O(\mu^2)\\
&= \partial_\tau^2[x_0 + \mu x_1] + [x_0 + \mu x_1] + \mu[2\partial_\tau\partial_T x_0 + ([x_0]^2-1)\partial_\tau[x_0]] + O(\mu^2)\\
&= \partial_\tau^2 x_0 + x_0 + \mu[\partial_\tau^2 x_1 + x_1 + 2\partial_\tau\partial_T x_0 + ([x_0]^2-1)\partial_\tau[x_0]] + O(\mu^2)
\end{aligned}$$

To order μ^0 we have

$$\ddot{x}_0 + x_0 = 0 \;\Rightarrow\; x_0 = A(T)\sin\tau + B(T)\cos\tau\,.$$

Where $A(T)$ and $B(T)$ are T-dependent coefficients. The initial condition for x_0

$$\begin{aligned}
0 = x(0) &= \underbrace{x_0(0,0)}_{=0} + \mu\underbrace{x_1(0,0)}_{=0} + \ldots\\
1 = \dot{x}(0) &= \underbrace{\partial_\tau x_0(0,0)}_{=1} + \mu\underbrace{[\partial_T x_0(0,0) + \partial_\tau x_1(0,0)]}_{=0} + \ldots\,.
\end{aligned}$$

gives $A(T)$ and $B(T)$ are any functions satisfying $A(0) = 1$, $B(0) = 0$.

To order μ^1 we have

$$\begin{aligned}
0 &= \partial_\tau^2 x_1 + x_1 + 2\partial_\tau\partial_T[A(T)\sin\tau + B(T)\cos\tau]\\
&\quad + ([A(T)\sin\tau + B(T)\cos\tau]^2 - 1)\partial_\tau[A(T)\sin\tau + B(T)\cos\tau]]\\
&= \partial_\tau^2 x_1 + x_1 + 2(A'(T)\cos\tau - B'(T)\sin\tau)\\
&\quad + ([A(T)\sin\tau + B(T)\cos\tau] - 1)[A(T)\cos\tau - B(T)\sin\tau]\,.
\end{aligned}$$

This equation can be solved for x_1 (preferably Mathematica). The solution contains a secular term on the form $f_1(T)\tau\sin\tau + f_2(T)\tau\cos\tau$ with

$$f_1(T) = -A'(T) + A(T)(4 - A(T)^2 - B(T)^2)/8$$
$$f_2(T) = -B'(T) + B(T)(4 - A(T)^2 - B(T)^2)/8\,.$$

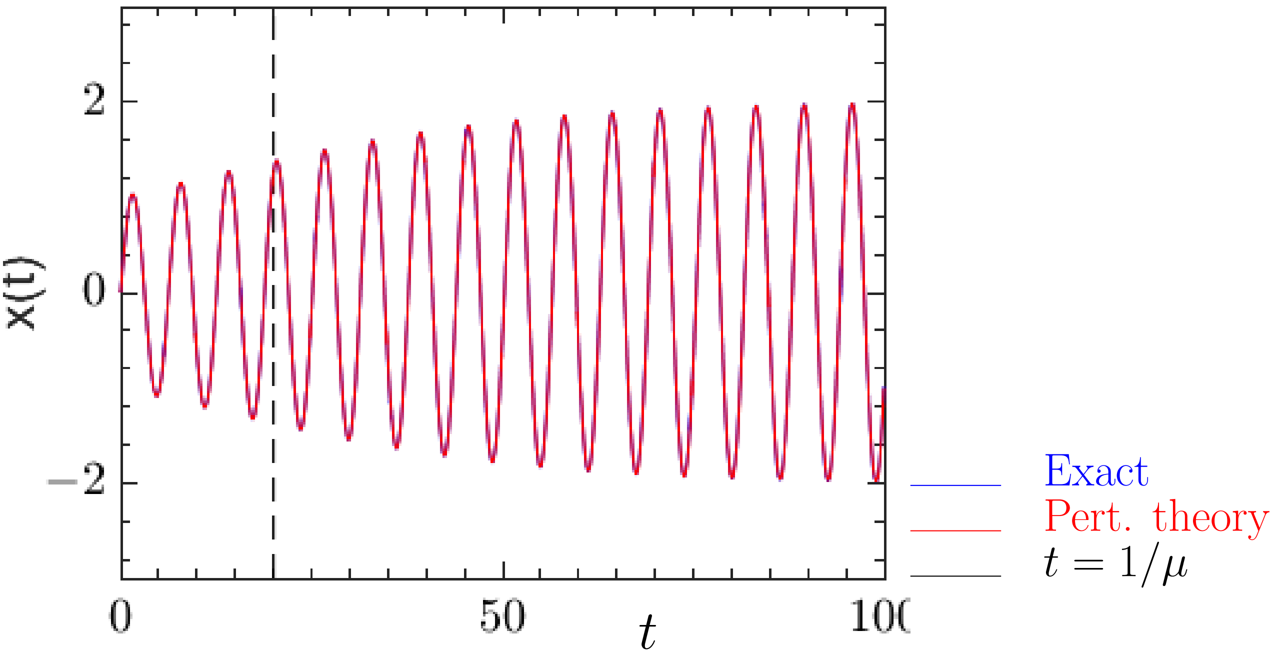
We choose the coefficients A and B in a self-consistent manner as to remove the secular divergence from the terms proportional to $\tau\sin\tau$ and $\tau\cos\tau$, i.e. we solve $f_1 = f_2 = 0$ for $A(T)$ and $B(T)$ with $A(0) = 1$ and $B(0) = 0$ [Mathematica]:

$$A(T) = \frac{2}{\sqrt{1+3e^{-T}}}\,, \quad B(T) = 0\,. \tag{2}$$

In conclusion, the two-timing gives the solution

$$x(t) = A(T)\sin\tau = \frac{2}{\sqrt{1+3e^{-\mu t}}}\sin t$$

To lowest order in μ the van der Pol oscillator approaches a circular limit cycle with amplitude $\lim_{T\to\infty} A(T) = 2$. The two-timing result for $x(t)$ agrees very well with the numerical solution, also for large times:



8.3.3 Average over fast variable

If we are not interested in the fast dynamics, an easier method than two-timing, frequently applied in mechanics, is to average over the fast variable. Consider van der Pol's equation (1) as a dynamical system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x-\mu(x^2-1)y\,.\end{aligned}$$

Change to polar coordinates

$$r=\sqrt{x^2+y^2}\,,\qquad\qquad \phi=\arctan(y/x)$$

to obtain (these expressions for $\dot{r}$ and $\dot{\theta}$ were derived in Lecture 4)

$$\begin{aligned}\dot{r} &= \frac{x\dot{x}+y\dot{y}}{r} = -\mu r\sin^2\phi(r^2\cos^2\phi-1)\\ \dot{\phi} &= \frac{x\dot{y}-y\dot{x}}{r^2} = -1-\mu(r^2\cos^2\phi-1)\cos\phi\sin\phi\,.\end{aligned}$$

When $|\mu|\ll 1$, r changes slowly (time scale $\sim 1/\mu$) compared to ϕ (time scale ~ 1). Introduce a slow, smoothed variable R obtained by filtering out the fast oscillations in ϕ. Its dynamics is given by averaging the $\dot{r}$ equation over the fast variable ϕ

$$\begin{aligned}\dot{R} &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi\,\dot{r}|_{r\to R} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\phi[-\mu R\sin^2\phi(R^2\cos^2\phi-1)]\\ &= -\frac{\mu R}{8}(R^2-4)\,.\end{aligned}\tag{3}$$

When $\mu>0$ we have unstable fixed point at $R=0$ and stable fixed point at $R=2$. Thus, to lowest order in μ the system has a stable limit cycle of radius 2. An exact solution of Eq. (3) gives

$$R=\frac{2}{\sqrt{1+(4R_0^{-2}-1)e^{-\mu t}}}\,.$$

This is the same result that was obtained for the amplitude Eq. (2) with $R_0=A(0)=1$.