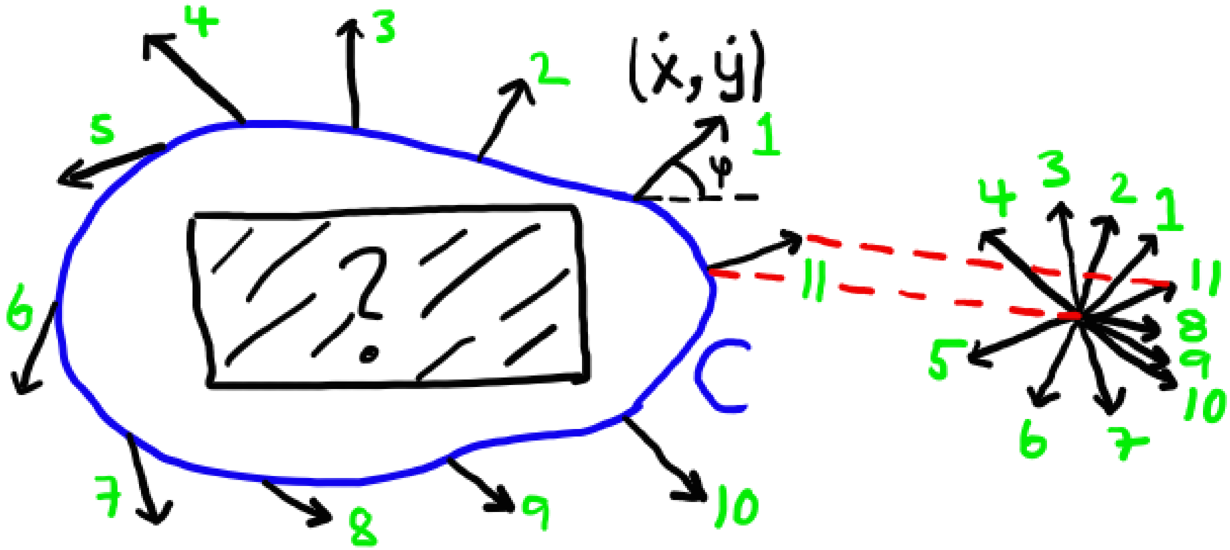
6 Index theory (Strogatz 6.8)

A general vector field $\boldsymbol{f}(\boldsymbol{x})$ is characterized by a magnitude and a direction for most $\boldsymbol{x}$. However, in singular points $\boldsymbol{x}^*$ where the magnitude of the vector field is zero (fixed points in dynamical systems), the direction is undefined. The index of a two-dimensional vector field (in the plane) is an integer that describes global information about the phase portrait around isolated zeroes (the value of the index is determined by how the vector field orients around the zeroes).

6.1 Illustration

The index I_C of any closed, non-intersecting curve C not passing through any fixed point is defined as the winding of a vector field as we traverse one counter-clockwise lap on C:



Here the vector field is $\boldsymbol{f} = (\dot{x}, \dot{y})$. At each point along the curve it forms an angle $\varphi = \text{atan}(\dot{y}/\dot{x})$ (chosen in the correct quadrant) to the x-axis.
Let $\Delta\varphi = \varphi_{\text{start}} - \varphi_{stop}$ be the angular change as one lap is traversed. For the Fig. above, the angular change can be visualized by labelling of all vectors in counter-clockwise order and then parallel transport them to a common origin (right panel). Traversing the vectors through the labels in order we see that the angle φ has increased by 2π when we come back to the starting vector.

For a general closed curve, since the start and end positions are the same, $\Delta\varphi = I_{\mathrm{C}} 2\pi$, where the index of curve C, I_{C}, is an integer.
For the example above $I_{\mathrm{C}} = \Delta\varphi/(2\pi) = +1$.

Remark 1 We only use information of the vector field on the curve that encloses a black box we know nothing about.
Using the index of the curve, I_{C}, it is possible to deduce information on the possible content of the black box.

Remark 2 φ is undefined at fixed points (where $\dot{x} = \dot{y} = 0$). This is why C is not allowed to pass fixed points.

6.1.1 Mathematical formulation

When the dynamical system is given in terms of equations

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

rather than a graphical vector field one can either

- Plot the flow as a vector field
- Use an analytical evaluation:

$$\begin{aligned} \varphi &= \operatorname{atan}\left(\frac{\dot{y}}{\dot{x}}\right) = \operatorname{atan}\left(\frac{g(x,y)}{f(x,y)}\right) \\ \mathrm{d}\varphi &= \frac{\partial\varphi}{\partial f}\mathrm{d}f + \frac{\partial\varphi}{\partial g}\mathrm{d}g = -\frac{g}{f^2+g^2}\mathrm{d}f + \frac{f}{f^2+g^2}\mathrm{d}g \\ \Rightarrow I_{\mathrm{C}} &\equiv \frac{\Delta\varphi}{2\pi} = \frac{1}{2\pi}\oint_{\mathrm{C}} \mathrm{d}\varphi = \frac{1}{2\pi}\oint_{\mathrm{C}} \frac{f\mathrm{d}g - g\mathrm{d}f}{f^2+g^2} \end{aligned}$$

6.2 Properties of indices

1. A continuous transformation of curve C to new curve C' without passing a fixed point $\Rightarrow I_C = I_{C'}$.
   Proof: I_C must vary continuously, but since I_C can only take discrete values it must remain constant. (Crossing a fixed point could change the index because the orientation is undefined at the fixed point.)

2. A curve C that does not enclose any fixed points has $I_C = 0$.
   Proof: Shrink C to a very small curve C' in which vectors are close to parallel due to smoothness of the flow $\Rightarrow I_C = I_{C'} = 0$.

3. Reversing all arrows in vector field ($t \to -t$) leaves index unchanged.
   Proof: All angles change from φ to $\varphi + \pi \Rightarrow$ Index unchanged.

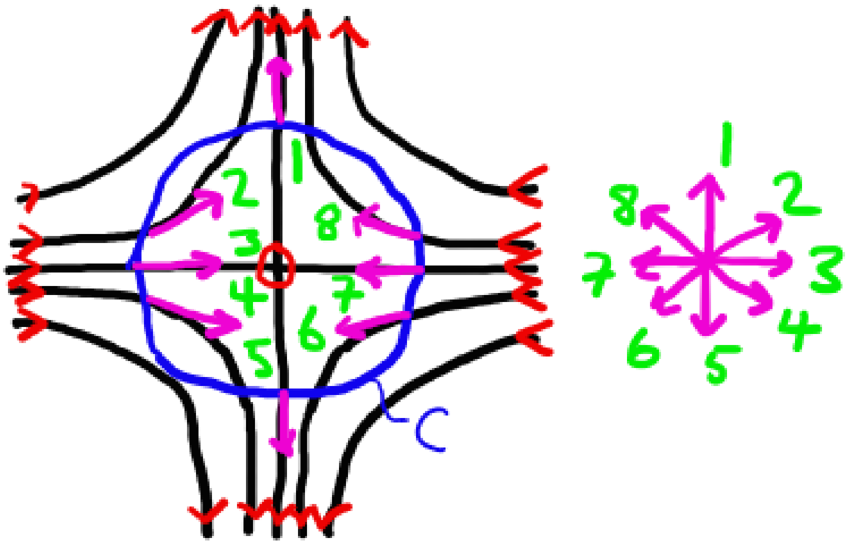
4. The index of a closed orbit (i.e. the curve C is the closed orbit), has $I_C = +1$.
   Proof: Flow is tangential to the curve at all times $\Rightarrow$ vector field rotates once as curve is traversed.

5. Let x^* be an isolated fixed point. The index of a fixed point I is equal to $I_{\rm C}$ for any curve that encloses x^* but does not enclose any other fixed points.

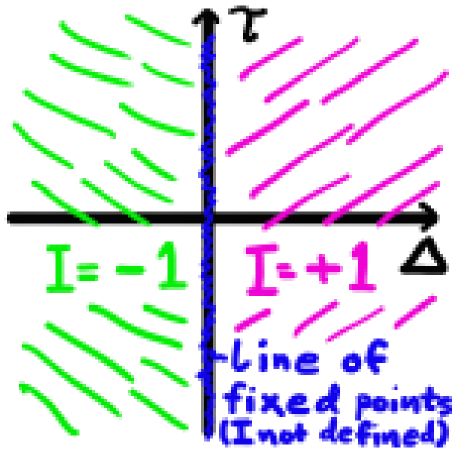
6. A curve C surrounding n isolated fixed points has index $I_C = I_1 + I_2 + \cdots + I_n$.

6.2.1 Indices of fixed points

The index of a saddle point can be obtained using the method above:



The change in φ is -2π, so the index of a saddle point is $I = -1$. In this way it is possible to show that all other isolated fixed points of the types found in linear systems (nodes, spirals, centers, degenerate nodes, stars) have $I = +1$. Fixed points with -1 and $+1$ each cover one half-plane in the classification diagram from Lecture 3:

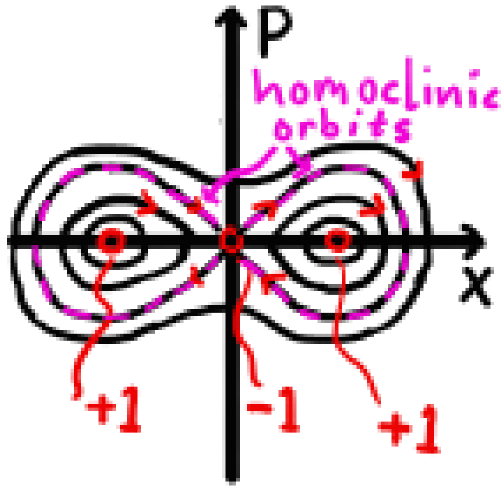


Along the line $\Delta = 0$ the linear system has a line of fixed points and the index cannot be defined (allowing the discontinuity in I as Δ passes zero).

Fixed points corresponding to higher-order roots of $\boldsymbol{f} = 0$ may give other integer values on I.

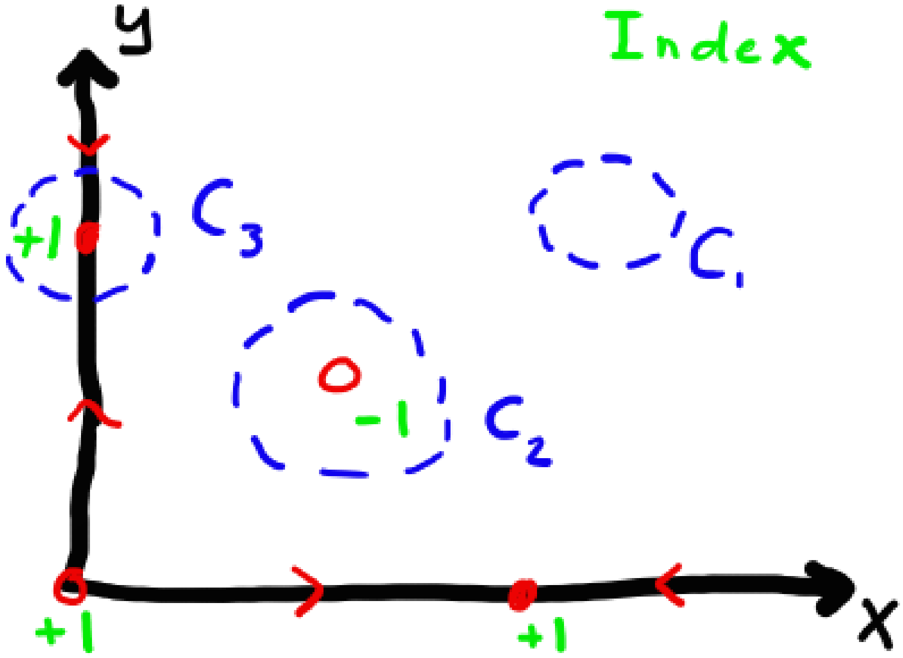
6.2.2 Consequences (in the plane)

- Closed orbits must encircle fixed points whose indices sum to $+1 \Rightarrow$ at least one fixed point (not a single saddle) inside any closed orbit in the phase plane.
  Example: Double-well potential revisited:



  Closed orbits may surround the centers (+1 each), or surround both the centers and the saddle ($I_C = 2 \cdot (+1) - 1 = +1$). Trajectories surrounding one center and one saddle can not happen. The separating homoclinic orbits do not have well defined indices (the curves cross the fixed point).

- Sometimes index theory can be used to rule out closed orbits in parts of the phase plane.
  Example: Since any closed orbit must encircle fixed points whose indices sum to +1, it is sometimes possible to rule out closed orbits, c.f. Strogatz Example 6.8.5 (Rabbits vs. sheep):



Curves C_1 and C_2 can be ruled out because the indices of the encircled fixed points does not sum to $+1$.
The trajectories connecting the origin to $(1, 0)$ and the origin to $(0, 1)$ are heteroclinic trajectories, preventing any trajectories to encircle the fixed points on the axes, ruling out curve C_3 (the flow move vertically along the y-axis in contradiction with the blue trajectory that cut the y-axis at an angle).

- Index theory provides information about trajectories around higher-order fixed points (not achievable with linearisation).
- Index theory is important for bifurcation theory to answer which types of bifurcations are allowed (next lecture).