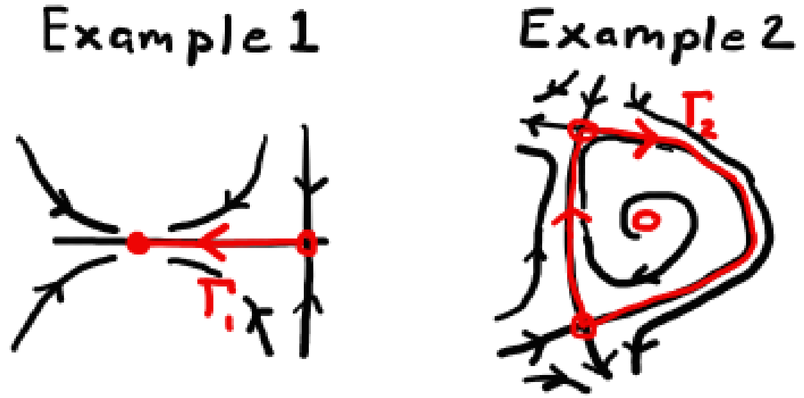
11 Strange attractors and Lyapunov dim.

11.1 Attractors (Strogatz 9.3)

The attractors we have encountered so far are stable fixed points and stable limit cycles. More generally, an attractor is defined as

1. an invariant set: trajectories starting on the attractor can not leave it.
2. attracting: there exists an open set of initial conditions that contains the attractor A and whose trajectories reach the attractor as $t \to \infty$. The basin of attraction is the largest such set.
3. minimal: no subset on the attractor satisfies conditions 1 and 2.

Example 1 Heteroclinic trajectory Γ_1 between (and including) a saddle and a stable node:



Γ_1 satisfies condition 1 (trajectories starting on Γ_1 remains on Γ_1) and Γ_1 is attractive. But it is not minimal, there is a subset (the node) that satisfies 1 and 2 $\Rightarrow$ the node is the attractor.

Example 2 Cycle of heteroclinic trajectories between two saddle points, Γ_2, surrounding an unstable spiral. The candidate invariant sets are Γ_2 or the two saddles. Which saddle a trajectory starting inside the cycle ends up at as $t \to \infty$ can not be determined (for any large t we can make t even larger in order to closely follow a heteroclinic trajectory to the opposite saddle). This implies that both saddles and the interconnecting heteroclinic trajectories would constitute the minimal invariant set. However, since the region outside of the cycle is not attracted, we cannot create an open set of initial conditions containing Γ_2 and the saddle points. The cycle is therefore not an attractor, it is instead a limit set of the trajectories inside the cycle, i.e. the state of the system as t approaches infinity. In the same way, a half-stable limit cycle or a half-stable fixed point count as limit sets, not attractors.

11.2 Lyapunov exponents of attractors

For the previously encountered attractors, stable fixed points and limit cycles, we expect that all Lyapunov exponents are non-positive.

11.2.1 Attracting fixed point

For trajectories in the basin of attraction of an attracting fixed point, $\mathrm{Re}\,\sigma_i < 0$, separations must in the long run shrink in all directions because, as shown in Lecture 10, for this case the stability exponents of separations $\tilde{\sigma}_i = \sigma_i$, and hence all Lyapunov exponents $\lambda_i = \mathrm{Re}\,\tilde{\sigma}_i$ are negative, $\lambda_i < 0$, in a system with globally attracting fixed points.

11.2.2 Attracting limit cycle

For any bounded, autonomous (time-independent) flow $\boldsymbol{f}(\boldsymbol{x}(t))$ without attracting fixed points, one Lyapunov exponent is zero. This follows from (f_i are components of $\boldsymbol{f}$)

$$\dot{f}_i = \underbrace{\partial_t f_i}_{=0} + \sum_j \underbrace{\dot{x}_j}_{f_j} \underbrace{\partial_j f_i}_{J_{ij}} = \sum_j J_{ij} f_j \,,$$

i.e. the phase-space velocity $\dot{\boldsymbol{x}} = \boldsymbol{f}$ satisfies the same time evolution as $\mathbb{M}$ and $\boldsymbol{\delta}$ (tangent equations). For an initial separation $\boldsymbol{\delta}(0) = \dot{\boldsymbol{x}}(0)$, the separation grows to $\boldsymbol{\delta}(t) = \dot{\boldsymbol{x}}(t)$ at a later time t and the corresponding stretching rate is

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\dot{\boldsymbol{x}}(t)|}{|\dot{\boldsymbol{x}}(0)|} \,.$$

This is zero unless $\dot{\boldsymbol{x}}$ depends exponentially on t for large t, which would happen close to a stable fixed point or if infinity is approached exponentially fast. But, in the bounded system considered here, regular trajectories do not diverge exponentially with time and λ must vanish. The vanishing $\lambda = 0$ must be equal to one of the Lyapunov exponents (by rewriting $\dot{\boldsymbol{x}}(t) = \mathbb{M}\dot{\boldsymbol{x}}(0) = \mathbb{Q}\mathbb{R}\dot{\boldsymbol{x}}(0)$ and decomposing $\dot{\boldsymbol{x}}(0) = \sum_j a_j \boldsymbol{v}_j$ in terms of eigenvectors $\boldsymbol{v}_i$ of $\mathbb{R}$, λ approaches the largest Lyapunov exponent λ_k corresponding to non-zero a_k).

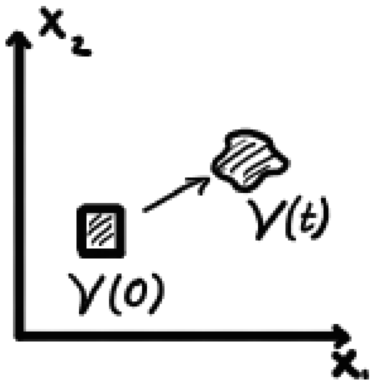
As a consequence, two trajectories starting close-by on a closed orbit (i.e. their separation points along the direction of velocity) does not on average separate or contract. $\Rightarrow$ attracting limit cycles have $\lambda_1 = 0$ and the remaining Lyapunov exponents are non-positive.
Similarly: Attracting m-frequency quasiperiodic orbit (motion on m-torus) has $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ and the remaining Lyapunov exponents are negative.

11.2.3 Volume conserving system (no attractors), Strog. 9.2

A dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is volume conserving in phase-space if

$$\boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x}) = 0 \tag{1}$$

everywhere. Consider a phase-space volume $\mathcal{V}(t)$ with shape $D(t)$:



In a volume-conserving system a phase-space element changes shape

$D(t)$, but the volume $\mathcal{V}(t)$ is constant. To show this, take a small time step δt. Positions evolve according to

$$\boldsymbol{x}(\delta t) = \boldsymbol{x}(0) + \delta t \boldsymbol{f}(\boldsymbol{x}(0)) .$$

The coordinate transformation $\boldsymbol{y} \equiv \boldsymbol{x}(\delta t)$ transforms all coordinates $\boldsymbol{x}_0 \in D(0)$ into the coordinates $\boldsymbol{y} \in D(\delta t)$

$$\begin{aligned}
\mathcal{V}(\delta t) &= \int_{D(\delta t)} \mathrm{d}^n y = \int_{D(0)} \mathrm{d}^n x_0 \left|\det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}_0}\right)\right| \\
&= \int_{D(0)} \mathrm{d}^n x_0 \underbrace{\left|\det\left(1 + \delta t \mathbb{J}(\boldsymbol{x}_0)\right)\right|}_{\approx 1 + \delta t\, \mathrm{tr}\mathbb{J}(\boldsymbol{x}_0)} = \mathcal{V}(0) + \delta t \int_{D(0)} \mathrm{d}^n x_0 \underbrace{\mathrm{tr}\mathbb{J}(\boldsymbol{x}_0)}_{\boldsymbol{\nabla}\cdot\boldsymbol{f}(\boldsymbol{x}_0)} \\
\Rightarrow \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} &= \int_{D(t)} \mathrm{d}^n x \, \boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x}) \qquad (2)
\end{aligned}$$

If $\boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x}) = 0$ everywhere, $\mathcal{V}(t)$ is constant and the system is volume conserving. A system that has $\boldsymbol{\nabla} \cdot \boldsymbol{f}(\boldsymbol{x}) < 0$ somewhere is called dissipative. Dissipative systems have attractors, while volume-conserving systems cannot have attractors nor repellers.

Example Consider again the last example from Lecture 4

$$\begin{aligned}
\dot{x} &= -y + ax(x^2 + y^2) \equiv f(x, y) \\
\dot{y} &= x + ay(x^2 + y^2) \equiv g(x, y)
\end{aligned}$$

with a single fixed point at $(0, 0)$. Evaluate the phase-space contraction

$$\boldsymbol{\nabla} \cdot \boldsymbol{f} = \frac{\partial}{\partial x}[-y + ax(x^2 + y^2)] + \frac{\partial}{\partial y}[x + ay(x^2 + y^2)] = 3a(x^2 + y^2)$$

If $a < 0$, the system is dissipative and has an attractor, i.e. the fixed point is a stable spiral. Similarly, if $a > 0$, the fixed point must be an unstable spiral. When $a = 0$, $\boldsymbol{\nabla} \cdot \boldsymbol{f} = 0$ everywhere $\Rightarrow$ the system cannot have attractors or repellers and the fixed point is a center.

Constant phase-space contraction If $\boldsymbol{\nabla} \cdot \boldsymbol{f} = \mathrm{tr}\mathbb{J}$ is constant, Eq. (2) simplifies to $\dot{\mathcal{V}} = \mathcal{V}\mathrm{tr}\mathbb{J}$, giving

$$\mathcal{V} = \mathcal{V}_0 e^{\mathrm{tr}\mathbb{J}\mathrm{t}} ,$$

Comparison to the definition of the Lyapunov exponents in Lecture 10

$$\mathcal{V} \sim \mathcal{V}_0 e^{(\lambda_1+\lambda_2+\dots\lambda_n)t}$$

the sum of Lyapunov exponents are equal to tr$\mathbb{J}$. It follows that the sum $\lambda_1 + \dots \lambda_n$ is zero in volume conserving systems. Note that λ_i are averaged quantities, i.e. $\lambda_1 + \dots \lambda_n = 0$ does not imply a volume conserving system (where tr$\mathbb{J}$ is zero everywhere).

11.3 Strange attractors

Summarizing the previous section for dimensionality $n = 3$:

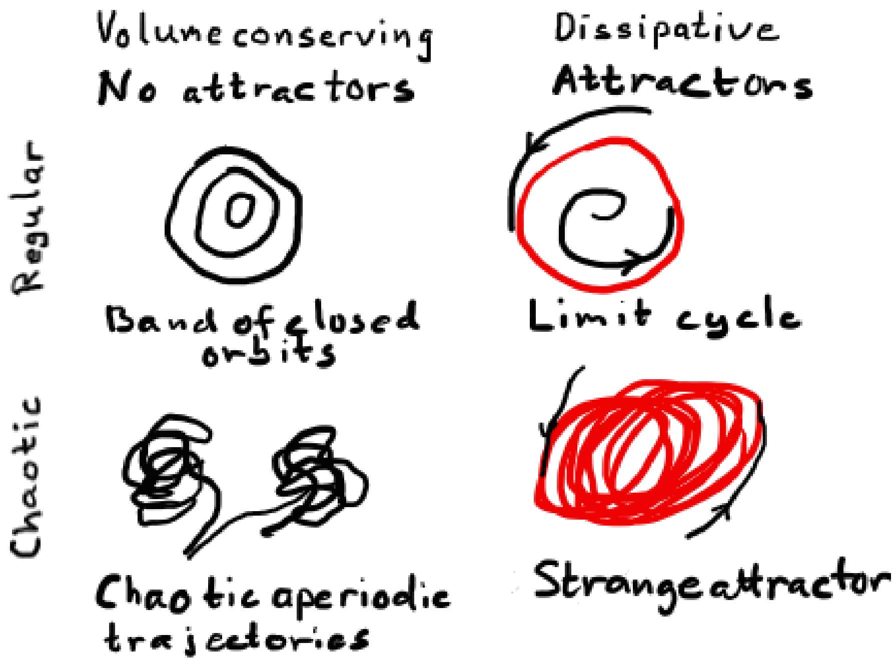
Attractor	λ_1	λ_2	λ_3	Dimension
Fixed point	< 0	< 0	< 0	0
Limit cycle	0	< 0	< 0	1
Limit torus	0	0	< 0	2
Volume conserving with $\lambda_1 \leq 0$ (no attractor)	0	0	0	3

Now consider the case of a chaotic system ($\lambda_1 > 0$) with trajectories bounded in a finite region:

- In volume conserving chaotic system ($\lambda_1 + \lambda_2 + \lambda_3 = 0$) then aperiodic chaotic trajectories (corresponding to different initial conditions) fill out space uniformly. Examples are Hamiltonian systems (billiards, double pendulum, ...) and chaotic advection.

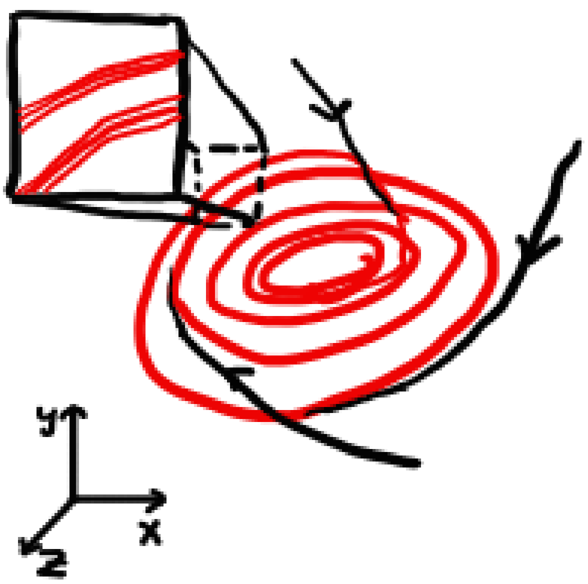
- In dissipative chaotic system ($\lambda_1+\lambda_2+\lambda_3 < 0$) then phase-space volumes shrink. One Lyapunov exponent must be zero, $\lambda_2 = 0$ (see Section 11.2.2), and consequently $\lambda_3 < 0$. Now, since $\lambda_1 > 0$ and therefore $\lambda_1 + \lambda_2 > 0$, we expect small areas to grow and small volumes to shrink. What kind of attractor can we have? It cannot be a limit cycle|torus (because these have $\lambda_1 = 0$). The resolution is a new kind of attractor, strange attractor (fractal attractor), that takes on a fractional dimensionality, somewhere between 2 and 3.

In summary, typical non-local limit sets are:



11.3.1 Properties of strange attractors

Illustration of a strange attractor:



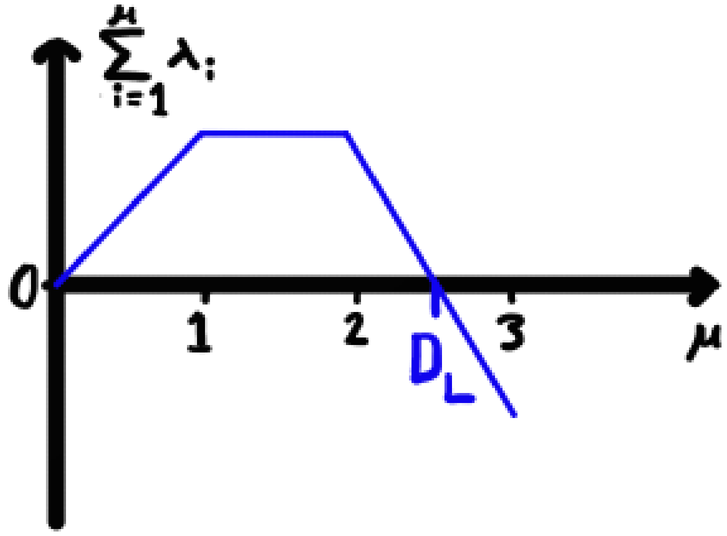
Some properties of a strange attractor:

- Sensitive dependence of initial conditions. Close-by trajectories end up at different places on the attractor.

- It is bounded but aperiodic (periodic would imply limit cycle).
- It requires a phase-space dimensionality of $n \geq 3$. For $n < 3$, trajectories cannot pass and aperiodic motion is ruled out by the Poincaré-Bendixson theorem (Lecture 5).
- It has structure at all scales (since it outlines an infinitely long aperiodic trajectory in a confined region).
- It cannot be plotted (there is always more structure if you zoom). Curves lying arbitrarily close to the attractor are obtained by choosing an initial point in the basin of attraction, solving the flow equations for some time to get close to the attractor, and then plot the aperiodic dynamics for a long time.
- A system may have one or several regular|strange attractors with different basins of attraction (the basin of attraction can itself be a fractal). .

11.3.2 Lyapunov dimension

There are several ways to define the fractal dimension of a strange attractor (next lecture). One estimate of the dimensionality of the strange attractor is the Lyapunov dimension D_{L}. It is defined as the number of ordered Lyapunov exponents that sum to zero. For the attractors listed in the table above, D_{L} becomes 0 for the fixed point, 1 for the limit cycle, 2 for the limit torus and 3 for the volume conserving system. For the strange attractor illustrated above, $\lambda_1 + \lambda_2 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 < 0$ sum to non-zero numbers:



$D_{\rm L}$ is given by a linear interpolation $D_{\rm L}(\lambda_1+\lambda_2) = A + B(\lambda_1+\lambda_2)$ where A and B are determined by $D_{\rm L}(0) = 2$ and $D_{\rm L}(-\lambda_3) = 3$, i.e.

$$D_{\rm L} = 2 - \frac{\lambda_1+\lambda_2}{\lambda_3}\,.$$

This is a number between 2 and 3 as desired (seen from the constraints $\lambda_1+\lambda_2 > 0$ and $\lambda_1+\lambda_2+\lambda_3 < 0 \Rightarrow \lambda_3 < -(\lambda_1+\lambda_2)$).

Similarly, the Lyapunov dimension can be generalized to other dimensionalities of phase space.

11.3.3 Example of strange attractor: Lorenz attractor

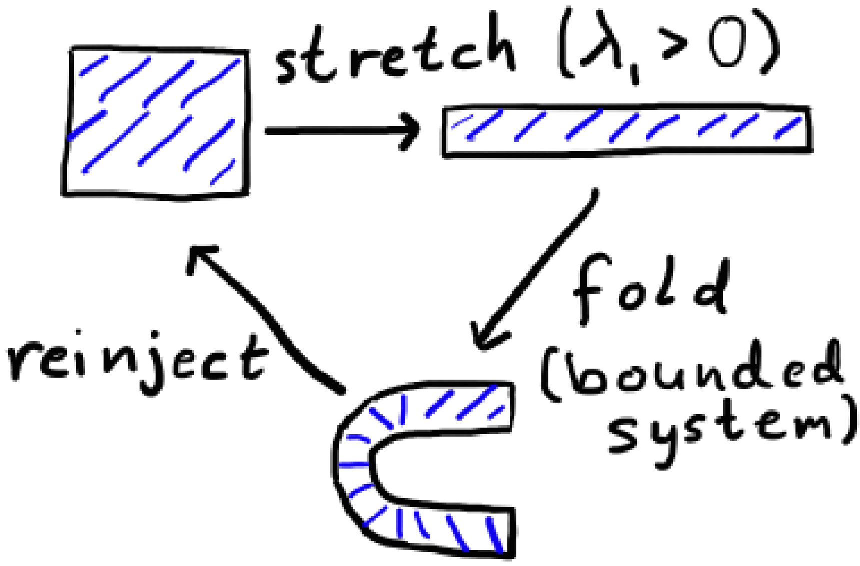
The standard example of system with a strange attractor

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

This is a toy model for convection rolls in the atmosphere It also describes the motion of a particular water wheel (Strogatz 9.1). Attracting fixed points exist for small r, but system jumps to a strange attractor after a subcritical Hopf Bifurcation at $r_{\rm H}$ (the value of $r_{\rm H}$ depends on σ and b).

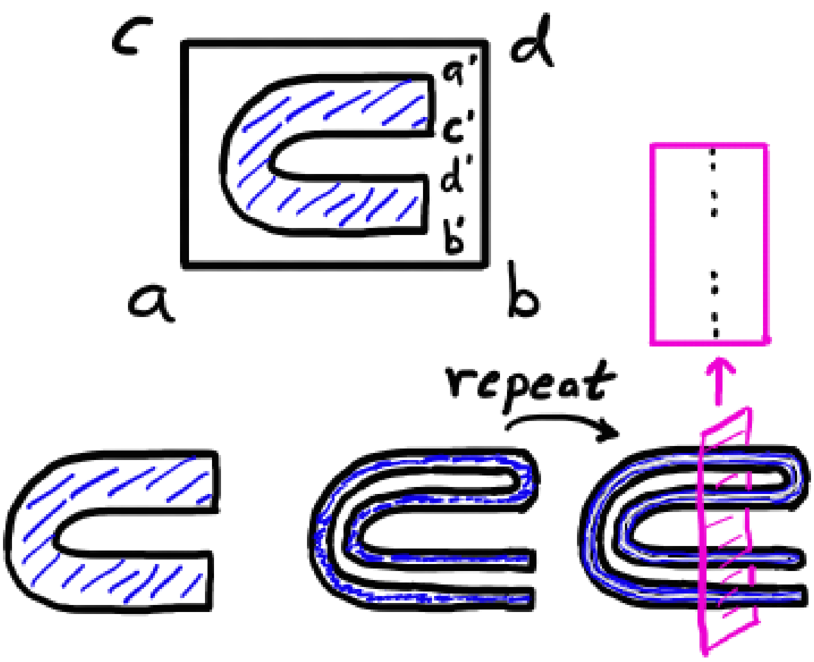
11.4 Formation of strange attractors

Typical behaviour in chaotic systems:



We typically have a strange attractor if the system is also dissipative. **Geometric illustration: Simple horseshoe map** Repeated mapping of a rectangle into itself.

- Map rectangle $abcd$ into a horseshoe $a'b'c'd'$ by stretching and folding as above. Area of horseshoe smaller than original image $\Rightarrow$ dissipation.
- Apply the map again by stretching and folding the horse shoe.
- Repeat this to get thinner and thinner filaments



After infinite iterations, a vertical cut through the middle resembles a fractal (a topological deformation of the Cantor set, next lecture).

This is a typical situation: locally the strange attractor consists of a bundle of a large (infinite) number of close-to parallel filaments. By putting a $n-1$-dimensional cross-section through the bundle a fractal pattern emerges.