CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	April 13, 2022, at $14^{00} - 18^{00}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 15^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 50 points (need 20 points to pass).
Maximum score for homework problems: 50 points (need 20 points to pass).
CTH $\geqslant$40 passed; $\geqslant$60 grade 4; $\geqslant$80 grade 5,
GU $\geqslant$40 grade G; $\geqslant$70 grade VG.

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1. Multiple choice questions [8 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Consider the linear dynamical system of dimensionality two

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbb{A} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where } \mathbb{A} = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix}.$$

What type is the fixed point of the dynamics above?

A. Stable node

B. Stable spiral

C. Saddle point

D. Unstable node

E. Unstable spiral

b) Consider the dynamics $\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$ with real parameter γ. Which kind of bifurcation occurs at $\gamma = 0$?

A. Saddle-node bifurcation

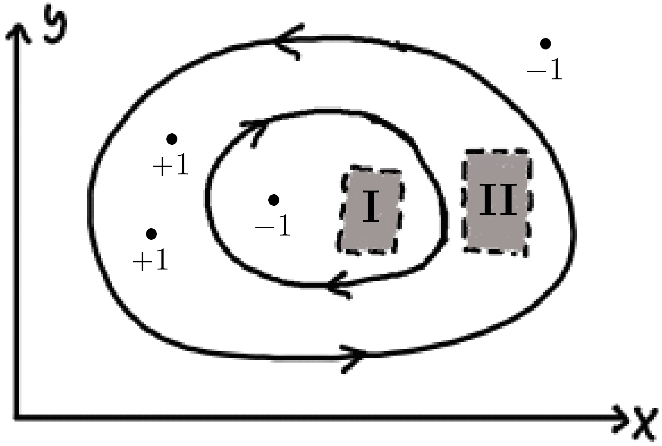
B. Transcritical bifurcation

C. Pitchfork bifurcation

D. Hopf bifurcation

E. Homoclinic bifurcation

c) The figure below shows the phase plane of a two-dimensional system with exactly two closed orbits. Filled circles mark the locations of four fixed points, each labeled with its index. Two regions **I** and **II** are missing in the plot and may contain any number of fixed points.



What is the index of a test curve surrounding the fixed points in region **II** and no other fixed points?

A. -2 B. -1 C. 0 D. $+1$ E. $+2$

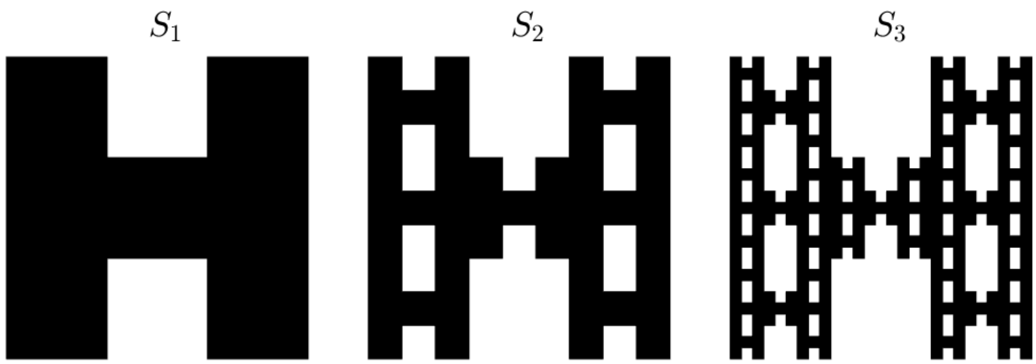
d) Which of the following structures are attractors?

A. Stable node
B. Saddle point
C. Stable limit cycle
D. Semi-stable limit cycle
E. Stable strange attractor

e) Which of the following statements about the generalized dimension spectrum D_q are true?

A. When evaluating D_q, the total number of point-containing boxes is weighted with a power q.
B. When evaluating D_q, the fraction of points in each point-containing box is weighted with a power q.
C. The Kaplan-Yorke conjecture states that, in most cases, the information dimension is equal to D_1.
D. The information sum can be used to efficiently evaluate D_1 for experimental data.
E. The correlation sum can be used to efficiently evaluate D_2 for experimental data.

f) The figure below shows the first few generations in the construction of a fractal. Split the unit square into nine equal-sized squares and remove the center squares in the upper and lower rows to obtain S_1. Repeat for each of the seven remaining squares to obtain S_2. Repeat again to obtain S_3. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.



Which of the following alternatives describe the box-counting dimension of the fractal above?

A. $\dfrac{\log(7)}{\log(2)}$ B. $\dfrac{\log(9)}{\log(2)}$ C. $\dfrac{\log(7)}{\log(3)}$ D. $\dfrac{\log(9)}{\log(3)}$ E. $\dfrac{\log(9)}{\log(7)}$

2. Short questions [8 points] For each of the following questions give a concise answer within a few lines per question.

a) Explain what is meant by critical slowing down after a saddle-node bifurcation. Give a real-world example.

b) Explain what the difference between a stable direction and a stable manifold is.

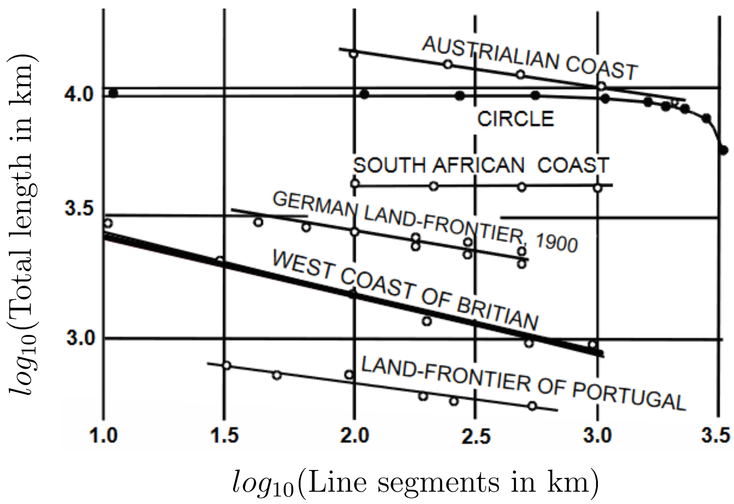
c) Explain what a relaxation oscillator is and give a real-world example.

d) Show that the following system of dimensionality two does not have any closed orbits.

$$\dot{x} = xe^{-x}$$
$$\dot{y} = 1 + x + y^2$$

e) Show that if a non-autonomous system, $\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}, t)$ (explicit time-dependence), is converted to an autonomous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ (no explicit time-dependence), then at least one Lyapunov exponent is zero in the autonomous system.

f) The figure below shows the total border length of different countries measured using line segments of different lengths [Richardson 1961]. Give an estimate of the fractal dimension of the west coast of Britain based on the scales in this data.



3. Bifurcations in model for fishery [10 points] Consider the following dynamical system for the dynamics of a fish population of size x

$$\dot{x} = x(1-x) - b\frac{x}{a+x}\,. \tag{1}$$

Here a and b are parameters. Assume $a \geqslant 0$, $b \geqslant 0$ and $x \geqslant 0$.

a) Consider the case $a = 0$ in the system (1). Perform the following tasks

i) find all fixed points;

ii) sketch the bifurcation diagram against the parameter b;

iii) mark up location and type of all bifurcations with $a = 0$.

b) Consider $a \neq 0$ and find conditions on the parameters a and b for which the system (1) has a saddle-node bifurcation with $x \neq 0$. One approach is to evaluate the fixed points of Eq. (1), but any method is acceptable. Verify that your result agrees with your findings in subtask a).

c) Analyze the dynamics close to $x = 0$ and show that there is a bifurcation at $x = 0$ when $b = a$. What type of bifurcation is it?

d) Sketch regions in the (a,b)-plane that are separated by the bifurcations obtained in subtasks a)–c). Label each region with the number of positive fixed points in that region.

4. Classification of fixed points in the plane [8 points] Consider the system

$$\begin{aligned} \dot{x} &= -2x - x^2 + \mu \\ \dot{y} &= -2y - xy \end{aligned} \tag{2}$$

where μ is a real parameter.

a) Find all fixed points of the system (2) and give conditions on μ for when the fixed points exist (be careful with the case $\mu = 0$).

b) For allowed values of μ, classify all fixed points in the system (2) according to linear stability analysis.
Hint You may find it easiest to directly evaluate the stability exponents rather than the invariants τ and Δ.

c) Find an explicit solution y as a function of x when $\mu = 0$. Use this to sketch the dynamics when $\mu = 0$. Does the prediction from linear stability come out correct, or does non-linear terms ruin the prediction?

5. Pendulum driven by a constant torque [8 points] Consider a pendulum whose angular coordinate θ ($-\pi < \theta \leqslant \pi$) is governed by the dynamics

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{\tau}{I_0}\,, \tag{3}$$

where g is gravitational acceleration, τ is an external torque, l is the length of the pendulum, and I_0 its moment of inertia. Assume g, τ, l and I_0 are positive constants.

a) Rewrite Eq. (3) as a dynamical system for θ and $y = \frac{\mathrm{d}\theta}{\mathrm{d}t}$. Introduce suitable dimensionless units to write the system in terms of a single parameter: a dimensionless torque I.

b) Find all fixed points of the system and classify them using, for example, that the system is Hamiltonian. You can neglect classification of higher-order (degenerate) fixed points.

c) Describe the bifurcation that happens in the system. What is the bifurcation point I_{c}?

d) Using the nullclines and the fixed points in subtask b) as guide, sketch the phase portrait for two values of the dimensionless torque I: $I = 0$ and $I = I_{\mathrm{c}}/2$.

6. Lyapunov exponents [8 points] Consider an experiment on a chaotic mechanical system that is not driven (such as the double pendulum). The experiment is repeated many times with initial conditions within a distance of accuracy $|\delta \boldsymbol{x}_0|$ from an initial point $\boldsymbol{x}_0$ in phase space. On average, the different experimental realisations agree approximately for 10 seconds, whereafter most trajectories diverge (their phase-space separations become larger than $100|\delta \boldsymbol{x}_0|$).

a) Give an estimate of the maximal Lyapunov exponent of this system.

b) How much must you improve the accuracy of the initial condition to allow to predict the system for twice as long, i.e. the time scale until the trajectories diverge is twice as large?

c) Discuss whether you expect the chaotic dynamics to be intermittent or transient.

d) Discuss whether you expect the results in subtasks a) and b) to be the qualitatively the same for any initial condition with the same initial accuracy $|\delta \boldsymbol{x}|$.