4 Phase plane (Strogatz 6)

The previous lecture dealt with linear two-dimensional flows. This lecture considers non-linear two-dimensional flows living in a phase space of dimensionality two: the phase plane.

4.1 Geometrical approach: Phase portraits

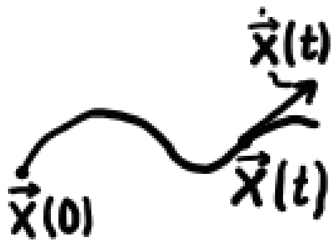
Consider a general dynamical systems of dimensionality two:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}$$

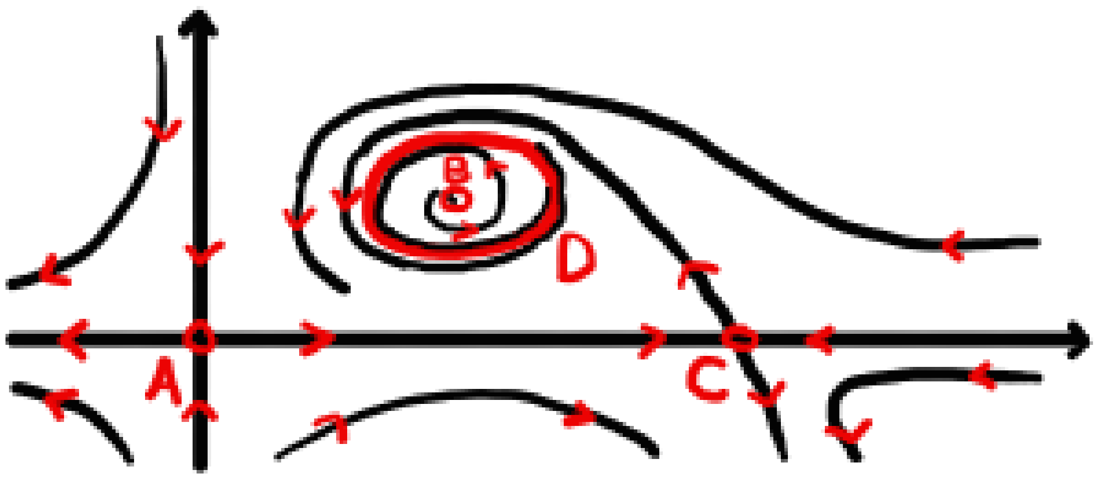
with $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$. To have a cleaner notation without indices, we often use

$$x = x_1\,,\quad y = x_2\,,\quad f = f_1\,,\quad g = g_2\,,\quad \Rightarrow\quad \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

The trajectory $\boldsymbol{x}(t)$ depends on the initial condition $\boldsymbol{x}(0)$:



In non-linear systems it is usually not possible to find $\boldsymbol{x}(t)$ analytically. Phase portraits are typically much more complicated compared to the linear flows considered so far. One example: (Strogatz Fig. 6.1.2)



- Fixed points (A,B,C)

$$\boldsymbol{f}(\boldsymbol{x}^*) = 0$$

- Closed orbits (D) [periodic solution $\boldsymbol{x}(t) = \boldsymbol{x}(t+T)$].
- Arrangement of trajectories near different fixed points and different closed orbits may differ:
  - A, C saddle
  - B spiral
- Stability
  - A, B, C unstable
  - D stable

As for the one-dimensional case: if the flow is smooth (both $\boldsymbol{f}$ and its partial derivatives $\partial_i f_j$ are continuous) in some open connected domain $D \in R^n$, then for $\boldsymbol{x}_0 \in D$ the initial-value problem has a unique solution in some time interval around $t = 0$.

As a consequence different trajectories cannot intersect. If they did, there would be two solutions starting from the point of intersection, i.e. breaks the uniqueness condition.

4.1.1 Numerical computation of phase portraits

Using a low-level language such as C++ without suitable external libraries, one may use a Runge-Kutta integration scheme (Strogatz 6.1) to compute trajectories from a set of initial conditions. Using Matlab or Mathematica, it is more convenient to use the built-in functions, e.g. StreamPlot[] in Mathematica to plot the flow, or NDSolve[] to find the trajectories.

4.1.2 Sketching the phase portrait by hand

To draw a phase portrait by pen and paper, it is often instructive to first determine the nullclines. These are the curves defined by

$$\dot{x} = 0 \qquad \text{or} \qquad \dot{y} = 0\,.$$

Along the nullclines the flow is either vertical ($\dot{x} = 0$) or horizontal ($\dot{y} = 0$). They divide the phase plane into regions where direction of flow is known or approximately known:

	$\dot{x} < 0$	$\dot{x} = 0$	$\dot{x} > 0$
$\dot{y} < 0$	$\swarrow$	$\downarrow$	$\searrow$
$\dot{y} = 0$	$\leftarrow$	$\cdot$	$\rightarrow$
$\dot{y} > 0$	$\nwarrow$	$\uparrow$	$\nearrow$

Intersection points between a nullcline with $\dot{x} = 0$ and one with $\dot{y} = 0$ give the fixed points of the flow.

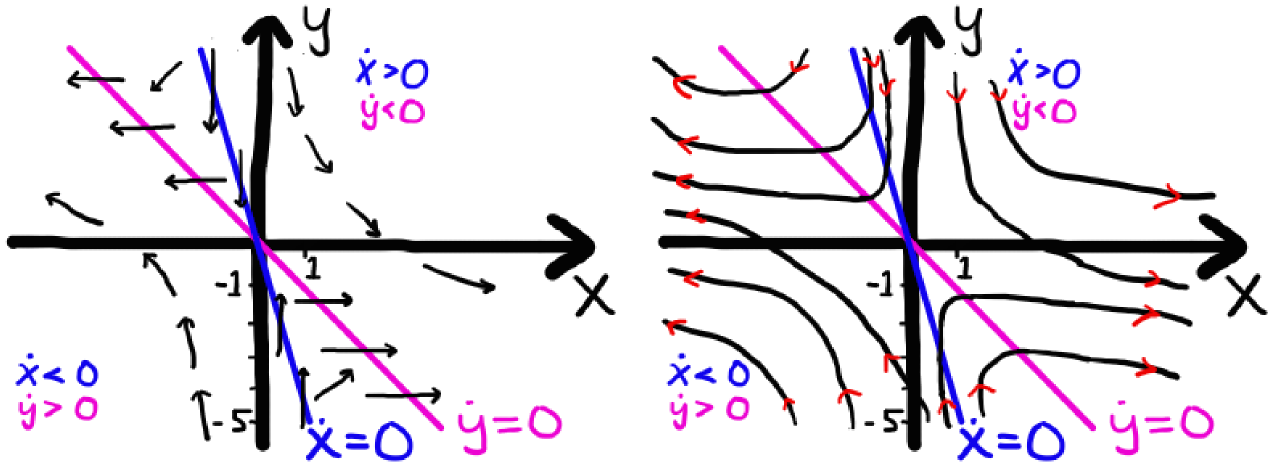
Since trajectories are not allowed to cross, the information given by the nullclines often allows to make a qualitative plot of the dynamics.

Linear example

$$\begin{aligned}\dot{x} &= 5x + y\\ \dot{y} &= -x - y\end{aligned}$$

Nullclines:

$$\begin{aligned}\dot{x} = 0: &\quad y = -5x\\ \dot{y} = 0: &\quad y = -x\end{aligned}$$



From the plotted trajectories we see that the fixed point at the intersection of the nullclines is a saddle point.

Consistency check:

$$\mathbb{A} = \begin{pmatrix} 5 & 1 \\ -1 & -1 \end{pmatrix} \quad \Rightarrow \quad \Delta = \det \mathbb{A} = -4 \quad \Rightarrow \quad \text{Saddle point}$$

4.2 Analytical approach: Linear stability analysis

A dynamical system of dimensionality two

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

has fixed points (x^*, y^*) where $f(x^*, y^*) = g(x^*, y^*) = 0$. Linearize around the fixed point (c.f. Lecture 2):

$$\eta = x - x^*, \qquad \mu = y - y^*$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \eta \\ \mu \end{pmatrix} = \mathbb{J}(x^*, y^*)\begin{pmatrix} \eta \\ \mu \end{pmatrix} + \dots, \qquad \text{with } \mathbb{J}(x^*, y^*) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \quad (1)$$

where $\mathbb{J}$ is the stability matrix (Jacobian matrix in Strogatz) and the derivatives are evaluated at the fixed point (x^*, y^*).

In linear stability analysis, we neglect the higher-order terms and the deviation (η, μ) satisfies a linear system that can be analyzed and classified as in Lecture 3.

4.2.1 Example on phase-plane analysis

Analyze the dynamical system:

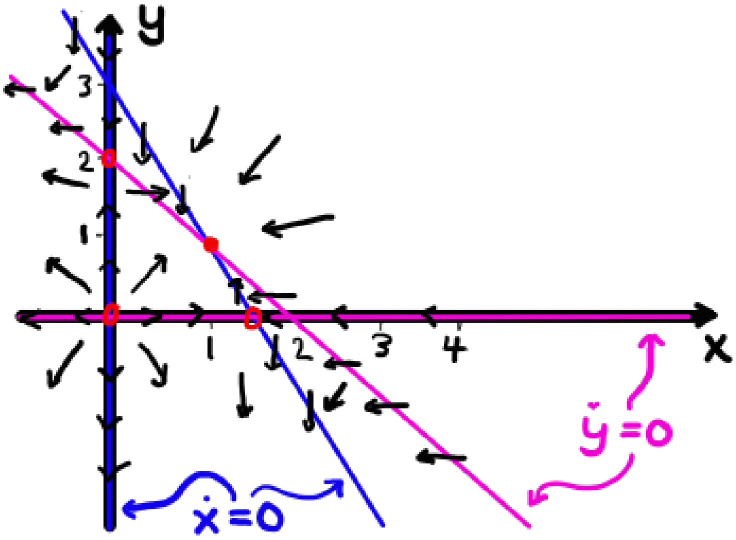
$$\dot{x} = x(3 - 2x - y)$$
$$\dot{y} = y(2 - x - y)\,.$$

The nullclines are

$$\dot{x} = 0: \quad x = 0 \quad \text{or} \quad x = (3 - y)/2$$
$$\dot{y} = 0: \quad y = 0 \quad \text{or} \quad y = 2 - x\,.$$

On the nullclines the flow is one-dimensional and therefore straightforward to analyze:

Nullcline	$x = 0$	$x = (3 - y)/2$	$y = 0$	$y = 2 - x$
Flow	$\dot{y} = y(2 - y)$	$\dot{y} = y(1 - y)/2$	$\dot{x} = x(3 - 2x)$	$\dot{x} = x(1 - x)$



The system has 4 fixed points at the intersections of the two types of nullclines:

$$(x_1^*, y_1^*) = (0, 0)\,, \quad (x_2^*, y_2^*) = (0, 2)\,, \quad (x_3^*, y_3^*) = (3/2, 0)\,, \quad (x_4^*, y_4^*) = (1, 1)\,.$$

The nullclines give a rough picture of the flow, but it is complicated to figure out what happens close to the fixed points using nullclines only. Therefore, use linear stability analysis for the fixed points:

$$\mathbb{J} = \begin{pmatrix} (3-2x-y)+x(-2) & x(-1) \\ y(-1) & (2-x-y)+y(-1) \end{pmatrix}$$
$$= \begin{pmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{pmatrix}$$

Fixed point (x^*, y^*)	$(0,0)$	$(0,2)$	$(3/2,0)$	$(1,1)$
$\tau \equiv \mathrm{tr}\mathbb{J}(\mathrm{x}^*, \mathrm{y}^*)$	5	-1	-5/2	-3
$\Delta \equiv \det \mathbb{J}(x^*, y^*)$	6	-2	-3/2	1
$\lambda_{1,2} = (\tau \pm \sqrt{\tau^2 - 4\Delta})/2$	$(2,3)$	$(-2,1)$	$(-3,1/2)$	$(-3 \pm \sqrt{5})/2$
Type	Unstable node	Saddle	Saddle	Stable node
$\boldsymbol{v}_1$	$(0,1)$	$(0,1)$	$(1,0)$	$(1-\sqrt{5},2)$
$\boldsymbol{v}_2$	$(1,0)$	$(-3,2)$	$(-3/7,1)$	$(1+\sqrt{5},2)$

Stable|unstable directions The real part of the eigenvalues determine the stability of a fixed point. Small deviations from a fixed point along an eigenvector $\boldsymbol{v}_\mathrm{u}$ corresponding to a real positive eigenvalue, $\lambda_\mathrm{u} > 0$, remain in the direction of $\boldsymbol{v}_\mathrm{u}$ and grow exponentially fast. This follows from Eq. (1) using $(\eta, \mu) = \epsilon(t)\boldsymbol{v}_\mathrm{u}$ with $\epsilon \ll 1$:

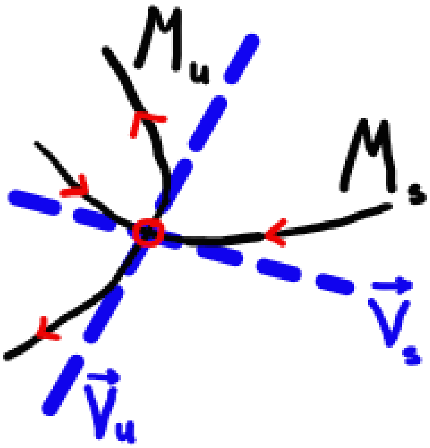
$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t}\boldsymbol{v}_\mathrm{u} = \mathbb{J}(x^*, y^*)\epsilon\boldsymbol{v}_\mathrm{u} = \lambda_\mathrm{u}\epsilon\boldsymbol{v}_\mathrm{u} \quad \Rightarrow \quad \epsilon(t) = \epsilon(0)e^{\lambda_\mathrm{u} t}\,.$$

The (normed) eigenvector $\boldsymbol{v}_\mathrm{u}$ is an unstable direction of the fixed point. Similarly, a normed eigenvector $\boldsymbol{v}_\mathrm{s}$ corresponding to $\lambda_\mathrm{s} < 0$ is a stable direction of the fixed point: small deviations in this direction shrink exponentially fast. For complex eigenvalues, all directions are either unstable ($\mathrm{Re}\,\lambda > 0$) or stable ($\mathrm{Re}\,\lambda < 0$) directions of the fixed point (for dimensionality 2).

Stable|unstable manifolds The stable manifold M_s of a fixed point is either a point, curve, or surface in the phase-plane. It is defined as the set of points (including the fixed point) that approach the fixed point in the limit $t \to \infty$. Similarly, the unstable manifold M_u consists of the set of points that approach the fixed point in the limit $t \to -\infty$, i.e. if the flow is reversed, then M_s and M_u switch stability.

- In a **linear system** the stable|unstable manifold is given by the subspace spanned by the set of stable|unstable directions. For example, a saddle point has one negative and one positive eigenvalue, it attracts along the stable direction, but repels along the unstable direction. Its stable and unstable manifolds are lines in these directions. Attractors|repellers are stable|unstable in all directions and the stable|unstable manifold is a surface (the entire phase plane).

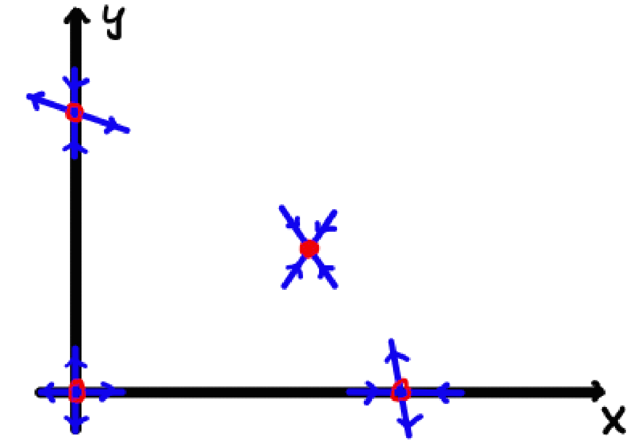
- For a **non-linear system**, M_s and M_u approach the manifolds of the linearized fixed point close to it, but may deviate further away due to non-linear effects. Example for a saddle point:



  The stable|unstable manifold approaches the stable|unstable direction $v_\mathrm{s}|v_\mathrm{u}$ of the fixed point close to the fixed point.
  The two-dimensional stable|unstable manifold of an attractor|repeller may become bounded.

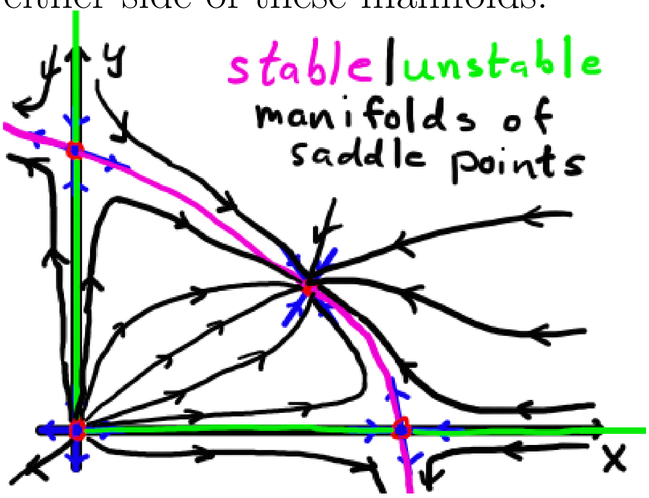
To numerically evaluate the stable|unstable manifold: start close to the fixed point in the stable|unstable direction and integrate the system backwards|forward in time.

Coming back to our example, from the table, the stable|unstable directions close to the fixed points are:



As shown by the nullclines, the flow aligns with the coordinate axes. Therefore, since trajectories cannot cross, the four quadrants are isolated from each other.

Consider first the upper-right quadrant. Since the flow is negative for large values of x or y, trajectories do not escape to infinity, and must therefore be attracted by the stable node at $(1,1)$ (its stable manifold or basin of attraction is the upper right quadrant). In particular, the unstable manifolds of the saddle points must connect with the stable node. As a consequence, trajectories become trapped on either side of these manifolds:



This is a generic behaviour, the manifolds of saddle points often divide the phase space into regions of qualitatively different long-term dynamics (C.f. Strogatz 6.4).

One example is the stable manifolds of the saddle points along the coordinate axes: these separate dynamics that are attracted to the stable node, from the rest of the phase plane where trajectories run off to infinity. The stable manifolds of the saddle points are examples of separatrices (singular: separatrix): they divide the phase space into regions of different long-term behaviour.

Outside the upper-right quadrant, the unstable manifolds of the saddle points must run away to infinity (no attractor can attract them).

Note: The stable|unstable manifolds and the nullclines can sometimes coincide (the coordinate axes in the example above), but in general they are different curves, also close to the fixed point.

4.2.2 Effect of small non-linear terms

When is it safe to neglect quadratic terms in the stability analysis?

Linear stability analysis gives a qualitatively correct picture if the fixed point is a node, spiral, or saddle (as in the Example in Section 4.2.1).

For the border-line cases (degenerate node, star, center, or not isolated) non-linear terms may (or may not) change the dynamics qualitatively from the border-line case to one of the neighbouring cases in the Δ-τ diagram. To find what type the fixed point is, one must analyze the non-linear dynamics close to the fixed point, either analytically, or using a geometric approach.

Example Consider the system

$$\begin{aligned}\dot{x} &= -y + ax(x^2+y^2) \equiv f(x,y)\\ \dot{y} &= x + ay(x^2+y^2) \equiv g(x,y)\end{aligned}$$

We have one fixed point $(x^*, y^*) = (0,0)$ and stability matrix

$$\mathbb{J}(x,y) = \begin{pmatrix} 3ax^2 & -1+2axy \\ 1+2axy & 3ay^2 \end{pmatrix} .$$

At the fixed point

$$\mathbb{J}(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

with eighenvalues $\lambda_{1,2} = \pm\mathrm{i}$. The fixed point is a center according to linear stability analysis.
Analyse the non-linear behaviour of the system. In polar coordinates

$$
\begin{aligned}
&r = \sqrt{x^2+y^2}\,, \qquad \theta = \mathrm{atan}(y/x) \\
&\Rightarrow \dot r = \frac{1}{2\sqrt{x^2+y^2}}(2x\dot x + 2y\dot y) \\
&= \frac{1}{r}(x[-y+ax(x^2+y^2)] + y[x+ay(x^2+y^2)]) \\
&= \frac{1}{r}(ax^2r^2 + ay^2r^2) = ar^3\,. \\
&\dot\theta = \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{atan(x)} = \frac{1}{1+x^2}\right] = \frac{1}{1+(y/x)^2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{y}{x} \\
&= \frac{1}{1+(y/x)^2}\left(\frac{\dot y}{x} - \frac{y}{x^2}\dot x\right) = \frac{1}{x^2+y^2}\left(x\dot y - y\dot x\right) \\
&= \frac{1}{r^2}\left(x[x+ay(x^2+y^2)] - y[-y+ax(x^2+y^2)]\right) = \frac{1}{r^2}\left(x^2+y^2\right) = 1
\end{aligned}
$$

The dynamics $\dot r = ar^3$ and $\dot\theta = 1$ is simpler to analyze. If $a = 0$ the system is linear and we have a center. Otherwise, we have a spiral (unstable if $a > 0$ or stable if $a < 0$).

In conclusion, the non-linear terms change the stability in this case. Similarly, stars and degenerate nodes can be changed into spirals or nodes by small non-linearities, but their stability does not change.

More generally:

- If both $\mathrm{Re}\,\lambda_1$ and $\mathrm{Re}\,\lambda_2$ are non-zero (attractors, repellers, saddle points), the qualitative dynamics is robust to small perturbations. The fixed point is <u>hyperbolic</u> and the dynamics is <u>structurally stable</u>.

- If $\mathrm{Re}\,\lambda_1$ and/or $\mathrm{Re}\,\lambda_2$ are zero (centers or non-isolated fixed points), the dynamics is marginal and not structurally stable: a small perturbation may change closed orbits and the number of fixed points.