14 Order to chaos in Hamiltonian system

In Lecture 13 transitions from order to chaos in dissipative systems were discussed. The transition typically occurs in a sequence of one or more bifurcations of regular attractors into a low-dimensional strange attractor. Hamiltonian systems (or other volume conserving systems) cannot have attractors, implying that the transition to chaos must be of a different nature. In regular integrable Hamiltonian systems, trajectories form closed orbits or quasi-periodic orbits around centers and saddles. The transition to chaos typically occurs via a break-up process of closed orbits.

Hamiltonian systems were introduced in Lecture 5:

$$\dot{\boldsymbol{x}} = \frac{\partial H}{\partial \boldsymbol{p}}\,, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{x}}\,. \tag{1}$$

An important example is Newtons law $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$ without friction written on dynamical-system form with position $\boldsymbol{x}$, momentum $\boldsymbol{p} = m\dot{\boldsymbol{x}}$ and energy function

$$H(\boldsymbol{x},\boldsymbol{p}) = \underbrace{\frac{|\boldsymbol{p}|^2}{2m}}_{\text{kinetic energy}} + \underbrace{V(\boldsymbol{x})}_{\text{potential energy}}\,.$$

More compactly, we let $\boldsymbol{\xi} \equiv (\boldsymbol{x},\boldsymbol{p})$, $H = H(\boldsymbol{\xi})$ and write Eq. (1) as

$$\dot{\boldsymbol{\xi}} = \mathbb{S}\frac{\partial H}{\partial \boldsymbol{\xi}} \equiv \boldsymbol{f}\,, \qquad \text{with} \begin{pmatrix} \mathbb{O}_{N\times N} & \mathbb{I}_{N\times N} \\ -\mathbb{I}_{N\times N} & \mathbb{O}_{N\times N} \end{pmatrix}\,, \tag{2}$$

where $\mathbb{S}$ is a 'symplectic matrix'.

Energy $E = H(\boldsymbol{x},\boldsymbol{p})$ is an integral of motion, as we showed in one spatial dimension in Lecture 5. Hamiltonian systems are also volume conserving everywhere because the divergence of $\boldsymbol{f}$ vanishes:

$$\sum_{i=1}^{n}\frac{\partial}{\partial \xi_i}f_i = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial \xi_i}S_{ij}\frac{\partial}{\partial \xi_j}H = \sum_{i=1}^{n}\sum_{j=1}^{n}\underbrace{S_{ij}}_{\text{antisymmetric}}\underbrace{\frac{\partial^2}{\partial \xi_i\partial \xi_j}}_{\text{symmetric}}H = 0$$

The symplectic structure in Eq. (2) not only preserves the magnitude of phase-space volume but also their topology (symplectic symmetry).

14.1 Integrable systems

Trajectories in Hamiltonian systems are either periodic|quasiperiodic (integrable systems) or aperiodic (non-integrable, chaotic systems). The periodic|quasiperiodic motion in integrable systems becomes apparent in 'action-angle coordinates'.

14.1.1 Action-angle coordinates

One spatial dimension Consider the case of one spatial dimension $H(x,p)$. Bounded trajectories are periodic orbits (isolines of constant energy). Make a canonical (preserves the form in Eq. (2), and thus area conservation) change of coordinates to 'action variable' I

$$I = \frac{1}{2\pi}\oint \mathrm{d}x\, p$$

and a so far unspecified 'angle variable' ϕ. The integral runs over a periodic orbit. Energy conservation $H(x,p) = E \Rightarrow p$ can be rewritten in terms of x and $E \Rightarrow I$ depends on E only. Consequently $I(t) = I(0) = \text{const}$. Moreover, since $I = f(E)$ we have $H = E = f^{-1}(I)$, i.e. H is a function of I only (not ϕ) in the new coordinates.

To preserve the form of Eq. (2) we have (these equations define ϕ)

$$\begin{aligned}\dot{\phi} &= \frac{\partial H(I)}{\partial I} \equiv \omega(I)\\ \dot{I} &= -\frac{\partial H(I)}{\partial \phi} = 0\end{aligned}$$

with solution $I(t) = I(0)$ and $\phi(t) = \phi(0) + \omega(I(0))t$.

I selects a periodic trajectory with angular frequency ω. Angle variables ϕ are useful for finding frequencies of rotations or librations without solving the equations of motion.

Example: Harmonic oscillator Hamiltonian $H(x,p) = p^2/(2m) + m\omega_0^2 x^2/2 = E$. This is an equation for elliptic orbits with axes of magnitude $\sqrt{2mE}$ and $\sqrt{2E/(m\omega_0^2)}$. The action coordinate becomes

$$
\begin{aligned}
I &= \frac{1}{2\pi}\oint_C \mathrm{d}x\, p \quad \Big[\text{Green's theorem: } \oint_C (L\mathrm{d}x + M\mathrm{d}p) = \iint_S (\tfrac{\partial M}{\partial x} - \tfrac{\partial L}{\partial p})\mathrm{d}x\mathrm{d}p\Big] \\
&= -\frac{1}{2\pi}\iint_S \mathrm{d}x\mathrm{d}p \quad [\text{Harmonic oscillator has clockwise } C \Rightarrow \text{sign(S)} = -1] \\
&= \frac{\text{Area of ellipse}}{2\pi} = \frac{1}{2}\sqrt{2mE}\sqrt{\frac{2E}{m\omega_0^2}} = \frac{E}{\omega_0}\,.
\end{aligned}
$$

I is a function of E only as expected. It follows that $H = E = \omega_0 I$ and $\dot\phi = H'(I) = \omega_0$ is the angular frequency.

Higher dimension Hamiltonian systems in one spatial dimension are examples of integrable systems. The dynamics can be solved formally for all times in terms of I and ϕ (while the integral defining I is not always possible to evaluate explicitly).

The notion of integrability is generalised to general spatial dimension d if the system has d integrals of motion $C(\boldsymbol{x},\boldsymbol{p})$ such that

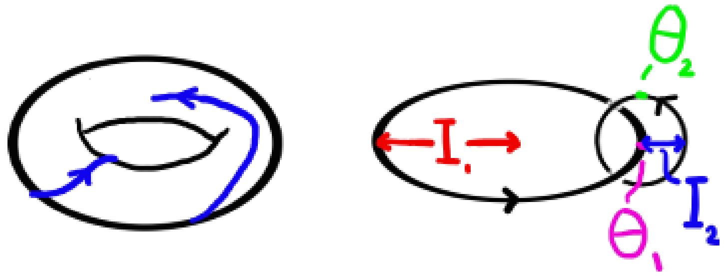
$$
\begin{aligned}
\dot C &= \dot{\boldsymbol{x}}\frac{\partial C}{\partial \boldsymbol{x}} + \dot{\boldsymbol{p}}\frac{\partial C}{\partial \boldsymbol{p}} \\
&= \frac{\partial H}{\partial \boldsymbol{p}}\frac{\partial C}{\partial \boldsymbol{x}} - \frac{\partial H}{\partial \boldsymbol{x}}\frac{\partial C}{\partial \boldsymbol{p}} \equiv \{C, H\} = 0\,.
\end{aligned}
$$

If the Hamiltonian system has d independent integrals of motion (such as energy, momentum in each coordinate, angular momentum etc.) it is integrable. Solutions are constrained to a d-dimensional hypersurface defined by $C_i(\boldsymbol{x},\boldsymbol{p}) = \text{const.}$, $i = 1,\dots,d$. The solutions are topologically equivalent to d-dimensional non-intersecting tori .

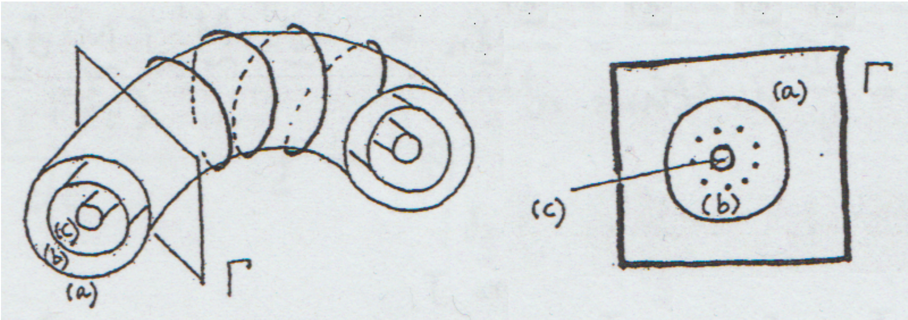
In terms of action-angle coordinates, the equations of motion are

$$
\begin{aligned}
\dot{\boldsymbol{\phi}} &= \frac{\partial H(\boldsymbol{I})}{\partial \boldsymbol{I}} \equiv \boldsymbol{\omega}(\boldsymbol{I}) = \text{constant vector of frequencies} \\
\dot{\boldsymbol{I}} &= \boldsymbol{0}\,.
\end{aligned}
$$

When $d = 2$ the dynamics of the angle variables is identical to that of uncoupled oscillators on a torus (discussed in Lecture 9):



Using the constant action variables I_i as the torus radii, solutions of the four-dimensional dynamics can be visualized:



The motion show periodicity or quasiperiodicity depending on whether the ratio $\omega_1(\boldsymbol{I})/\omega_2(\boldsymbol{I})$ is rational or not, i.e. if there exists integers k and l such that $k\omega_1 + l\omega_2 = 0$, then trajectories close into periodic orbits. In the illustration above, trajectory (b) is periodic (rational torus) and trajectories (a) and (c) are quasiperiodic (irrational torus).

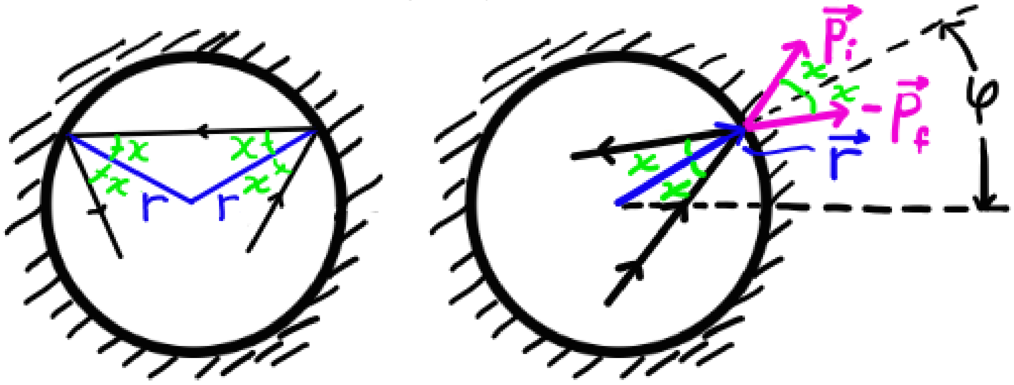
Action-angle coordinates are suited for investigating perturbations, leading to near-integrable systems which we consider next.

14.2 Perturbation of system remaining integrable

In few exceptional cases the dynamics of an integrable Hamiltonian system remains integrable after a small perturbation.

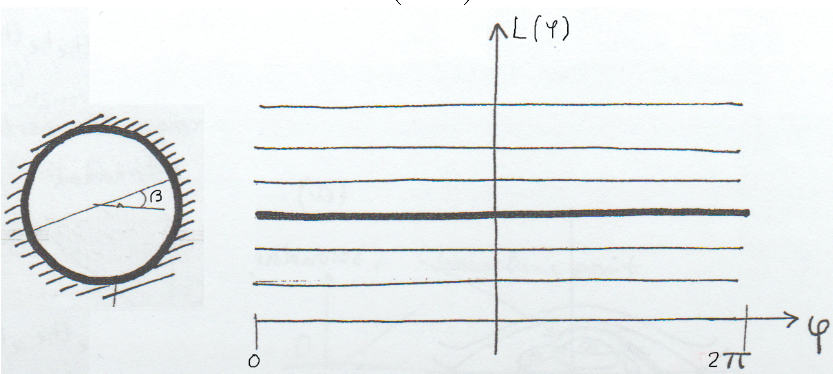
Example: Circular billiards Consider a circular billiard (dimensionality $n = 4$, i.e. two spatial dimensions). As we discussed in Lecture 9, this is an example of an integrable system because it has two integrals of motion: energy $E = p^2/(2m)$ and angular momentum with respect to the circle center.

Conservation of angular momentum For the circular billiard, the reflection angle χ is the same for any collision (left panel).



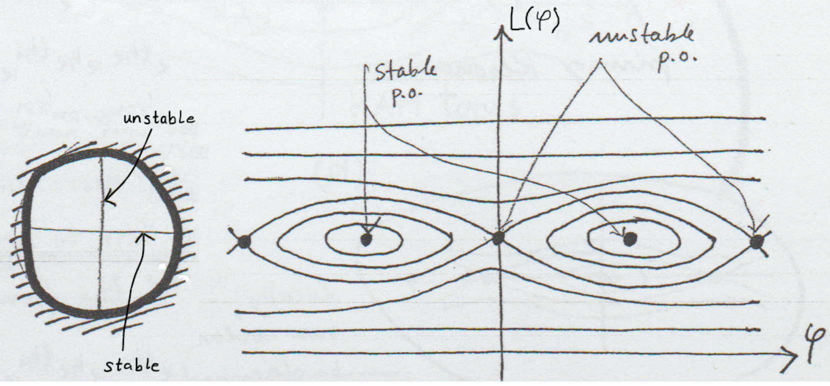
Right panel: Angular momentum w.r.t. the center is: $L = |\boldsymbol{r}||\boldsymbol{p}|\sin\alpha$, where α is the angle between $\boldsymbol{r}$ and $\boldsymbol{p}$. Before a collision with the wall, the angle between $\boldsymbol{r}$ and $\boldsymbol{p}_{\rm i}$ equals the reflection angle, $\alpha = \chi$, and after the collision the angle between $\boldsymbol{r}$ and $\boldsymbol{p}_{\rm f}$ are related by $\alpha = \pi - \chi$. Since $\sin\chi = \sin(\pi - \chi)$ the angular momentum is conserved at collisions. In addition, uniform motion has constant angular momentum and angular momentum is therefore a conserved quantity.

Visualisation One example of a periodic orbit in the circular billiard is shown below (left):



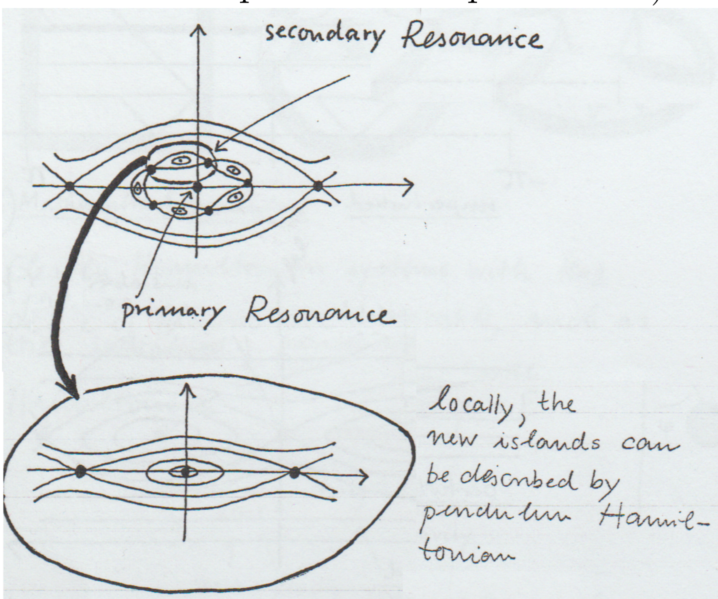
To visualize the dynamics, it is convenient to sample the angle φ and angular momentum $L(\varphi)$ at collisions, neglecting the intermediate motion. To the right, the dynamics is illustrated by plotting φ and $L(\varphi)$ upon collisions with the boundary. Thin lines show irrational tori and the thick line shows the rational tori corresponding to the periodic motion in the left panel (for each initial condition β we obtain a discrete set of two points, averaging over initial conditions we obtain the continuous line). Due to circular symmetry, there is a continuous family of periodic orbits, determined by parameter β (left).

Small perturbation After a small perturbation as shown below (left), only two isolated periodic orbits remain:



Thus, a small perturbation breaks up the rational torus into one (marginally) stable (small perturbations do not grow) and one unstable (small perturbations grow) periodic orbit. The system in the vicinity of the new fixed points can be described by the Hamiltonian of an undamped pendulum. The system is integrable because there exists two sets of integrals of motion, separated by the heteroclinic orbits between saddle points. This is a consequence of the stable and unstable manifolds of the saddle points joining smoothly.

If the system is perturbed slightly more, other rational tori break up, possibly one of those created around the new stable periodic orbit. In the latter case a chain of integrable islands (local phase portraits resemble the portrait of a pendulum) is created (secondary resonance):



Stronger perturbations lead to higher-order resonances, creating a hierarchy of island chains (nested tori).

Two important observations

1. **KAM theorem** Adding a small perturbation to an integrable system, tori with rational ω_1/ω_2 breaks up first. Tori with strongly irrational ω_1/ω_2 break up last (KAM tori), as the perturbation becomes larger. Any real quotient ω_1/ω_2 can be approximated by a continued fraction

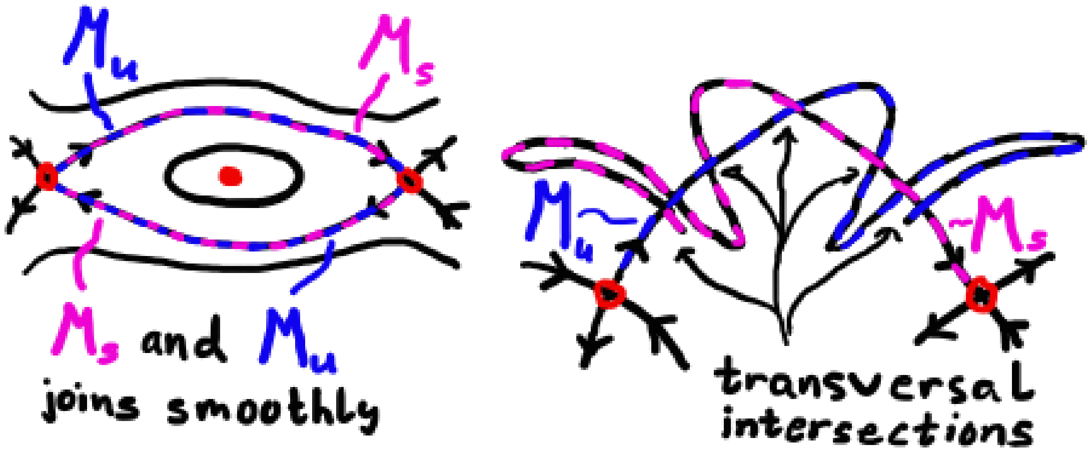
$$\frac{\omega_1}{\omega_2} = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}$$

   with a sequence where a_0 is an integer and a_n with $n = 1, 2, \ldots$ are positive integers. The continued fraction converge quicker the faster the coefficients in the sequence a_n grow. Strongly irrational ω_1/ω_2 have sequences a_n that grow slowly. The limiting case of a non-growing (constant) sequence a_n with $a_n = 1$ for all n gives the golden ratio $G = (\sqrt{5} + 1)/2$. Tori with $\omega_1/\omega_2 = G \pm k$ with k an integer are the last ones to break up (KAM tori).

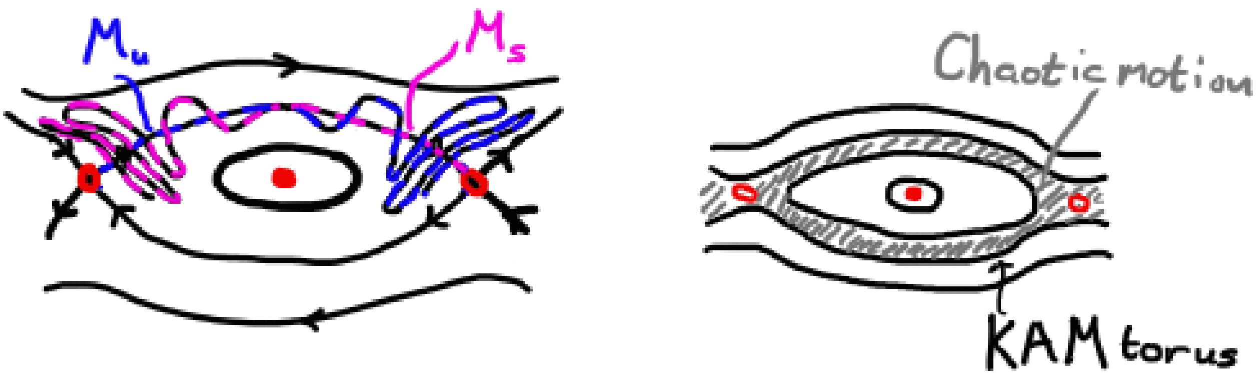
2. **Poincaré-Birchoff theorem** When a torus of rational ω_1/ω_2 breaks up, an equal number of (marginally) stable and unstable fixed points appear. Around the (marginally) stable fixed points, new resonant tori are formed that generates a new sequence of (marginally) stable and unstable fixed points appear. Iterating gives a self-similar structure of fixed points around each (marginally) stable fixed point (regular islands).

14.3 Transition to chaos

In the example for the circular billiard above, chains of integrable islands are generated when rational tori break up. Locally the dynamics can be described by the Hamiltonian of a pendulum. This is a special case, more generally integrability is destroyed by small perturbations of Hamiltonian systems. Typically the manifolds of the newly created unstable fixed points do not join smoothly in heteroclinic trajectories as for the circular billiard (left), but rather intersect transversally (right):



Note that the plotted dynamics is projected (e.g. by plotting one component of position and momentum at regular time intervals), allowing the manifolds to cross. It is possible to show that, under quite general conditions, the manifolds of the projected dynamics will intersect an infinite number of times if they intersect once, similar to the right panel above. This means that he manifolds become more and more intricate closer to the saddle and a thin band of chaotic motion is formed:



The band is bounded by KAM tori that are more resistent to perturbations. If the perturbation is increased, an infinite hierarchy of bands of chaotic motion is created (similar to the hierarchy of islands for the circular billiard). When perturbation is strong enough also KAM tori break up and global chaotic motion is obtained (close inspection on small scales often reveals tiny integrable islands).

The transition to chaos outlined above is valid in two spatial dimensions. In higher dimensions, chaotic regions are no longer sandwiched between remaining KAM tori, allowing chaotic trajectories of slightly perturbed systems to wander off in phase space (Arnold diffusion).