CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	April 06, 2018, at $14^{00} - 18^{00}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 15^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Multiple choice questions [**2 points**] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Classify the fixed point of the three-dimensional dynamical system:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\,, \qquad \text{where } \mathbb{A} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ 2 & 0 & -3 \end{pmatrix}.$$

A. It is a saddle point (all eigenvalues are real but of different signs).

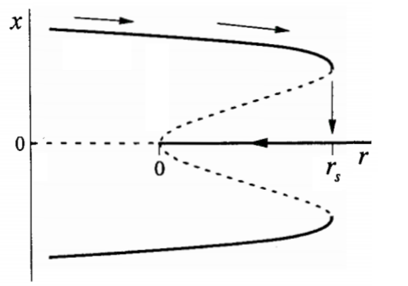
B. It is a stable spiral (eigenvalues form a complex pair and the real parts of all eigenvalues are negative).

C. It is an unstable spiral (eigenvalues form a complex pair and the real parts of all eigenvalues are positive).

D. It is a stable node (all eigenvalues are negative).

E. It is an unstable node (all eigenvalues are positive).

b) The figure below shows a bifurcation diagram for a dynamics in x with a parameter r. Solid lines show stable fixed points, dashed lines show unstable fixed points.



Consider the path (arrows) obtained by increasing r above r_s and then decreasing r again. Where does the system end up when r becomes smaller than zero?

A. The system ends up at either the lower or upper branches of fixed points with 50% probability for each

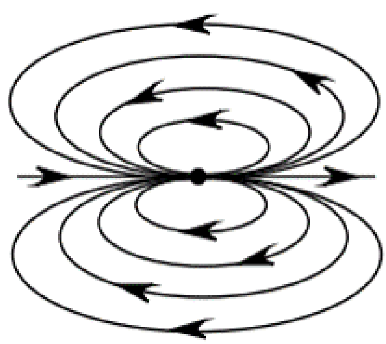
B. The system ends up either over or under the upper branch of fixed points with 50% probability for each

C. The system ends up over the upper branch of fixed points

D. The system ends up under the upper branch of fixed points

E. The system ends up at the middle branch of fixed points

c) The figure below shows a flow with a single fixed point at its center:



What is the index of this fixed point if the trajectories are traversed backwards in time?

A. -2 B. -1 C. 0 D. 1 E. 2

d) Consider the two-dimensional system

$$\dot{x} = y\,, \qquad \dot{y} = x + x^4\,.$$

Which of the following is a conserved quantity of this system?

A. $y + x + x^4$

B. $y - x - x^4$

C. $y^2/2 + x + x^4$

D. $y^2/2 + 1 + 4x$

E. $y^2/2 - x^2/2 - x^5/5$

e) A three-dimensional dynamical system has the following Lyapunov exponents: $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -2$. Which of the following statements are true?

A. The system may be chaotic.

B. The system may be volume preserving.

C. The system may be Hamiltonian.

D. The system may have a globally attracting limit cycle.

E. The Lyapunov spectrum is unchanged if time changes sign.

f) Which of the following statements about the generalized dimension spectrum D_q are true?

A. For a multifractal, D_q increases with increasing q

B. For a monofractal, D_q is independent of q

C. When $q > 0$ low-density regions of the attractor gives the dominant contribution to D_q

D. When $q > 0$ high-density regions of the attractor gives the dominant contribution to D_q

E. The Kaplan-Yorke conjecture states that, in most cases, the correlation dimension is equal to D_1.

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Explain a method that can be used to decrease the dimensionality of a dynamical system. What assumptions does your method rely on?

Solution

b) Explain what it means that a dynamical system shows hysteresis. Give an example of a system with hysteresis.

Solution

c) What are the stable manifolds of a fixed point?

Solution

d) Give two examples of different types of global bifurcations. Explain how one can experimentally distinguish the different bifurcations.

Solution

Examples: Bifurcation between limit cycles, infinite period bifurcation, homoclinic bifurcation. Close to the bifurcation, all of these have characteristic dependence of the period time on the bifurcation parameter. Measuring how the period time changes with the bifurcation parameter one can distinguish the three types of bifurcations.

e) Explain what a fractal (strange) attractor is and how it may form in a dynamical system.

Solution

f) In a dynamical system of dimensionality larger than two, what does the **second** largest Lyapunov exponent characterize?

Solution

3. Bifurcations [2 points] Consider the system

$$\dot{r} = \mu r - g r^3 - r^5 \,. \tag{1}$$

Assume that $r \geq 0$, and that μ and g are real parameters.

a) Sketch the bifurcation diagram for the system (1) for the two cases $g = +1$ and $g = -1$. Identify the types of all bifurcations that occur in your bifurcation diagrams.

Solution
The bifurcation diagram is obtained by plotting the solutions of $\dot{r} = 0$, i.e. $r^* = 0$ for all μ and plotting $\mu = g(r^*)^2 + (r^*)^4$ for positive r. The fixed point at $r^* = 0$ is stable for $\mu < 0$ and unstable otherwise. The stability of the other fixed points is determined by investigation of the sign of the flow around them. When $g = +1$, the system has a supercritical pitchfork bifurcation at $\mu = 0$, where negative r is omitted. When $g = -1$, the system has a subcritical pitchfork bifurcation at $\mu = 0$ and a saddle point bifurcation (at $\mu = -g^2/4$), where negative r is omitted.

b) When $g = 0$ a special kind of bifurcation occurs. For the two cases of $g = 0$ and a finite value $g > 0$, analytically determine how the locations of the fixed points depend on μ close to the bifurcation at $\mu = 0$. Sketch the bifurcation diagrams for the cases of $g = 0$ and a finite value $g > 0$ in the same plot.

Solution
The condition $\dot{r} = 0$ has at most three positive solutions: $r_1^* = 0$ and $r_{2,3}^* = \frac{1}{\sqrt{2}}\sqrt{-g \pm \sqrt{g^2 + 4\mu}}$. When $g \geq 0$, only two solutions are valid: $r_1^* = 0$ and $r_2^* = \frac{1}{\sqrt{2}}\sqrt{-g + \sqrt{g^2 + 4\mu}}$. For the case of $g = 0$, the second fixed point becomes $r_2^* = \mu^{1/4}$. For any finite $g > 0$, the bifurcation occurs at $\mu \ll g$. Series expansion of r_2^* in terms of small μ gives $r_2^* \sim \sqrt{\mu/g}$.
The sketch should show that the bifurcation for $g = 0$ is much steeper ($\sim \mu^{1/4}$) than the regular pitchfork bifurcation ($\sim \mu^{1/2}$).

c) Discuss what would happen if a small imperfection h is added to the system (1) for the case $g = 0$:

$$\dot{r} = h + \mu r - r^5 \,.$$

Make qualitative sketches for the cases $h > 0$ and $h < 0$.

Solution
The curve of fixed points becomes $\mu = ((r^*)^5 - h)/r^*$. When $h > 0$, the curve is monotonically increasing: $\mathrm{d}\mu/\mathrm{d}r^* > 0$. The imperfection joins $r_1^* = 0$ for negative μ smoothly with r_2^* for positive μ. When $h < 0$, the curve has one minimum (a saddle point). The imperfection joins $r_1^* = 0$ for positive μ smoothly with r_2^* for positive μ.

4. Model of a national economy [2.5 points] A simple model for a national economy is provided by

$$\begin{aligned} \dot{I} &= I - \alpha C \\ \dot{C} &= \beta(I - C - G(I)), \end{aligned} \tag{2}$$

where $I \geq 0$ is the national income, $C \geq 0$ is the rate of consumer spending, and $G(I) \geq 0$ is the rate of government spending. Assume that $\alpha > 1$ and $\beta \geq 1$.

a) Show that if $G(I) = G_0 = \text{const.}$, the system (2) has a single fixed point. Classify this fixed point in terms of α and β. Make a diagram in α-β-space that shows the regions of the different types of fixed points you find.

Solution

The system (2) has fixed points where $C^* = I^*/\alpha$ and where $I^* - I^*/\alpha - G(I^*) = 0$.

The Jacobian becomes

$$\begin{aligned} \mathbb{J} &= \begin{pmatrix} 1 & -\alpha \\ \beta(1 - G'(I)) & -\beta \end{pmatrix} \\ \operatorname{tr} \mathbb{J} &= 1 - \beta \\ \det \mathbb{J} &= \beta(\alpha - 1 - \alpha G'(I)) \end{aligned}$$

When $G = G_0$, there is a single fixed point at

$$(I^*, C^*) = G_0(\alpha, 1)/(\alpha - 1).$$

If $\beta = 1$, then $\operatorname{tr} \mathbb{J} = 0$ and $\det \mathbb{J} = \alpha - 1 > 0$, i.e. we have a center and an oscillating economy. For $\beta > 1$, $\operatorname{tr} \mathbb{J}$ is negative and the fixed point is attracting and the economy is stable. When $(\operatorname{tr} \mathbb{J})^2 < 4 \det \mathbb{J}$, i.e. when $\alpha > (1+\beta)^2/(4\beta)$, the fixed point is a stable spiral. When $\alpha < (1+\beta)^2/(4\beta)$, the fixed point is a stable node. At the boundary $\alpha = (1+\beta)^2/(4\beta)$, the Jacobian is not a multiple of the unit matrix and the fixed point is a stable degenerate node.

b) Now consider the case of a government spending that increases linearly with the national income: $G(I) = G_0 + kI$, where $k > 0$. Show that there exists a value k_c such that for $k > k_c$ there exists no positive fixed points in the system. Does the system have any attractors when $k > k_c$?

Solution

The fixed point moves to:

$$(I^*, C^*) = G_0(\alpha, 1)/(\alpha - 1 - \alpha k).$$

If $k > k_c \equiv (\alpha - 1)/\alpha$ the fixed point becomes negative. Since $\det \mathbb{J} = \beta(\alpha - 1 - \alpha k) < 0$ the fixed point becomes a saddle point when $k > k_c$. The system therefore does not have any regular attractors, most trajectories will run off to infinity.

c) Solve the dynamics in subtask b) for $I(t)$ and $C(t)$ as functions of time in terms of a matrix exponential. Explicitly write out the components of the solution for the case $k = 1$, $\alpha = 2$ and $\beta = 1$. What is the long-term fate of the dynamics according to this solution? Is the result consistent with your result in subtask b)?

Solution

Use $\boldsymbol{x} = (I - I^*, C - C^*)$ to write the system (2) on matrix form

$$\dot{\boldsymbol{x}} = \mathbb{M}\boldsymbol{x}\,, \text{ with } \mathbb{M} = \begin{pmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{pmatrix}\,,$$

with solution

$$\boldsymbol{x}(t) = e^{\mathbb{M}t}\boldsymbol{x}_0\,.$$

In conclusion

$$\begin{pmatrix} I(t) \\ C(t) \end{pmatrix} = \begin{pmatrix} I^* \\ C^* \end{pmatrix} + e^{\mathbb{M}t}\begin{pmatrix} I_0 - I^* \\ C_0 - C^* \end{pmatrix}$$

When $k = 1$, $\alpha = 2$ and $\beta = 1$ we have

$$\mathbb{M} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}\,,\ \mathbb{M}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow e^{\mathbb{M}t} = \sum_{i=0}^{\infty}\frac{1}{2i!}\mathbb{M}^{2i}t^{2i} + \sum_{i=0}^{\infty}\frac{1}{(2i+1)!}\mathbb{M}^{2i+1}t^{2i+1} = \mathbb{I}\sum_{i=0}^{\infty}\frac{1}{2i!}t^{2i} + \mathbb{M}\sum_{i=0}^{\infty}\frac{1}{(2i+1)!}t^{2i+1}$$

$$= \mathbb{I}\cosh t + \mathbb{M}\sinh t = \begin{pmatrix} e^t & -2\sinh t \\ 0 & e^{-t} \end{pmatrix}$$

In the long run $I(t)$ will grow exponentially fast, while $C(t)$ will decay exponentially fast. Since the system does not have any attractor for $k > k_c$, the dynamics must escape towards infinity in some way. For this particular case, along the $C = 0$-axis.

5. Damped driven pendulum [1.5 points] The equation of motion for the angle θ of a driven pendulum may be approximated by

$$\ddot{\theta} = -\frac{a}{m}\dot{\theta} - \frac{g}{b}\sin\theta + \frac{c}{I_0}\,. \tag{3}$$

Assume that m is the mass of the pendulum, g is the gravitational acceleration and I_0 is the moment of inertia of the pendulum.

a) Briefly explain the origin of the three terms on the right hand side. Determine the dimensions of the parameters a, b, and c and give plausible physical interpretations of these three parameters.

Solution

The first term comes from friction with a viscous medium, the second term comes from the gravitational force acting on the pendulum, and the third term comes from a constant external torque that drives the pendulum. In units of mass M, time T and length L we have $[\dot{\theta}] = T^{-1}$, $[\ddot{\theta}] = T^{-2}$, $[m] = M$, $[g] = LT^{-2}$, $[I_0] = ML^2$ (from Beta if one does not remember). Consequently, the dimensions of the remaining parameters are $[a] = MT^{-1}$, $[b] = L$ and $[c] = ML^2T^{-2}$. These can be interpreted as the damping coefficient (a), the distance to the center of mass of the pendulum (b), and a constant torque acting on the pendulum (c).

b) Show that Eq. (3) can be reduced to a dimensionless two-dimensional flow

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t'} &= y\\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= -\sin\theta - \alpha y + I\end{aligned}$$

by a proper rescaling $t = t_0 t'$ and by suitable choices of the time-dependent function $y(t')$ and of the dimensionless parameters α and I. Explicitly check that α and I are dimensionless.

Solution

Start from Eq. (3). Using $y(t') = \frac{\mathrm{d}x}{\mathrm{d}t'} = t_0\dot{\theta}$, we have

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t'} &= y\\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= t_0^2\ddot{\theta} = -t_0^2\frac{a}{m}\dot{\theta} - t_0^2\frac{g}{b}\sin\theta + t_0^2\frac{c}{I_0}\,.\end{aligned}$$

Choosing $t_0 = \sqrt{b/g}$ gives

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t'} &= y\\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= t_0^2\ddot{\theta} = -\sqrt{\frac{b}{g}}\frac{a}{m}y - \sin\theta + \frac{b}{g}\frac{c}{I_0}\,.\end{aligned}$$

which is on the desired form if

$$\begin{aligned}\alpha &= \sqrt{\frac{b}{g}}\frac{a}{m}\\ I &= \frac{b}{g}\frac{c}{I_0}\,.\end{aligned}$$

The dimensions of these parameters become

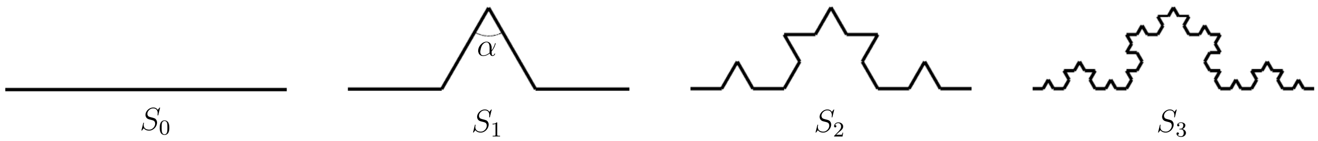
$$\begin{aligned}[][\alpha] &= \sqrt{\frac{L}{LT^{-2}}}\frac{MT^{-1}}{M} = 1\\ [I] &= \frac{L}{LT^{-2}}\frac{ML^2T^{-2}}{ML^2} = 1\,.\end{aligned}$$

c) Explain how this system may undergo a homoclinic bifurcation if the pendulum is weakly driven (consider fixed points separated by 2π in the θ-direction to be the same fixed point). Explain a method that can be used to find the corresponding bifurcation point. An explanation is enough, you do not need to explicitly evaluate the bifurcation point.

Solution

Melnikov's method, Lecture notes 9.1

6. Koch curve [2 points] The figure below shows the first few generations in the construction of the Koch curve. It is obtained by, at each generation, replacing the central 1/3 of each line with two new lines joined at a bend angle $\alpha = \pi/3$. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.



a) Analytically find the box-counting dimension D_0 for the Koch curve.

Solution

Using boxes (sticks) of side length $\epsilon_n = 3^{-n}$ it takes $N_n = 4^n$ boxes to cover the fractal. Thus

$$D_0 = -\lim_{n\to\infty} \frac{\ln(4^n)}{\ln(3^{-n})} = \frac{\ln 4}{\ln 3}\,.$$

b) Now consider a general bend angle α. Assume that all lines in a given generation have the same lengths. Analytically, find the box-counting dimension as a function of the bend angle.

Solution

Changing the bend angle does not affect the number of boxes needed to cover the fractal, but it changes how the natural box length ϵ_n changes with generation n. A line of length ϵ_n at generation n (for example S_0 in the figure above) is replaced by four lines of length ϵ_{n+1} in generation $n+1$ (S_1 in figure above). Since all lines have the same length, the new line length is chosen by dividing the baseline of length ϵ_n into two lines of length ϵ_{n+1} and one gap of length $2\epsilon_{n+1}\sin(\alpha/2)$, i.e. $\epsilon_n = 2\epsilon_{n+1} + 2\epsilon_{n+1}\sin(\alpha/2)$. Solving this recursion relation gives the line length $\epsilon_n = (2 + 2\sin(\alpha/2))^{-n}$. The box-counting dimension becomes

$$D_0 = -\lim_{n\to\infty} \frac{\ln(4^n)}{\ln((2+2\sin(\alpha/2))^{-n})} = \frac{\ln 4}{\ln(2+2\sin(\alpha/2))}\,.$$

c) Evaluate the box-counting dimension you obtained in subtask b) for the special cases $\alpha = 0$, $\alpha = \pi/3$ and $\alpha = \pi$. Discuss the results.

Solution

When $\alpha = 0$, the box-counting dimension becomes $D_0 = 2$. This is expected because at each generation the lines form a rectangular grid (enclosed by a triangle). As the generation n goes to infinity, this grid uniformly fills space. For $\alpha = \pi/3$ we have $\sin(\alpha/2) = 1/2$ and the Koch result is reobtained. When $\alpha = \pi$, the box-counting dimension becomes $D_0 = 1$. This is expected since if the bending angle is π the initial line maps to itself at each generation.

d) Analytically find the area between the Koch curve and the baseline (the line in S_0) as a function of the bend angle α. You can assume that the length of the baseline is $L_0 = 1$.

Solution

From subtask b) we have that $N_n = 4^n$ and $\epsilon_n = (2 + 2\sin(\alpha/2))^{-n}$. Each generation the area is increased by adding one triangular surface for each line segment that is replaced. At generation n, 4^{n-1} triangles are added, each with area $\epsilon\cos(\alpha/2) \cdot \epsilon\sin(\alpha/2) = \epsilon^2(\sin\alpha)/2$. The total area becomes

$$A = \sum_{n=1}^{\infty} 4^{n-1}(2 + 2\sin(\alpha/2))^{-2n}(\sin\alpha)/2 = \frac{\sin\alpha}{8}\sum_{n=1}^{\infty}\left[\frac{1}{(1+\sin(\alpha/2))^2}\right]^n$$

$$= \frac{\sin\alpha}{8}\left[-1 + \frac{1}{1 - \frac{1}{(1+\sin(\alpha/2))^2}}\right] = \frac{\cos(\alpha/2)}{4(2+\sin(\alpha/2))}.$$