CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	April 26, 2019, at $14^{00} - 18^{00}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 15^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$18 passed; $\geq$26 grade 4; $\geq$31 grade 5,
GU $\geq$18 grade G; $\geq$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Classify the fixed point of the two-dimensional dynamical system:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\,, \qquad \text{where } \mathbb{A} = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}.$$

A. It is a saddle point.

B. It is a stable spiral.

C. It is an unstable spiral.

D. It is a stable node.

E. It is an unstable node.

b) The following system has a single fixed point:

$$\dot{x} = y(1 + x^2)$$
$$\dot{y} = 3x - 2y$$

Which of the following are stable directions of the fixed point?
A. (0,1) B. (1,1) C. (-1,1) D. (-1,3) E. (2,3)

c) The normal forms of typical bifurcations for dynamical systems of dimensionality one are the following:

Type	saddle-node	transcritical	supercrit. pitchfork	subcrit. pitchfork
Normal form	$\dot{x} = r + x^2$	$\dot{x} = rx - x^2$	$\dot{x} = rx - x^3$	$\dot{x} = rx + x^3$

How does the stability time of the fixed points close to a transcritical bifurcation scale with the bifurcation parameter r?

A. $\sim \frac{1}{r}$ B. $\sim \frac{1}{\sqrt{r}}$ C. ~ 1 D. $\sim \sqrt{r}$ E. $\sim r$

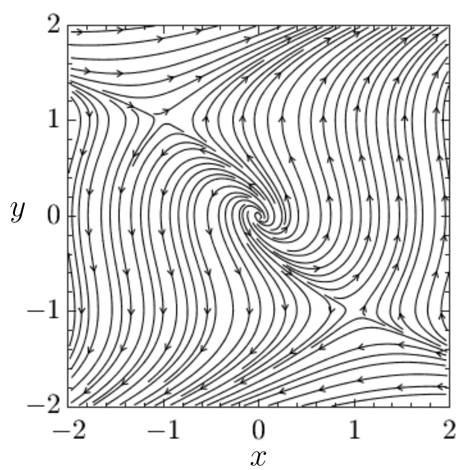
d) Consider the dynamical system:

$$\dot{x} = y$$
$$\dot{y} = g(x, y)$$

Which of the following choices of $g(x, y)$ make the system conservative?
A. x^2 B. e^{-x} C. y D. y^2 E. $x^2 y$

e) The figure below shows the phase portrait of the system:

$$\dot{x} = -y + y^3$$
$$\dot{y} = x + y$$



What is the sum of the indices of all fixed points of this system?
A. -2 B. -1 C. 0 D. 1 E. 2

f) Which of the following properties are satisfied in a continuous dynamical system with a globally attracting strange attractor?

A. Close-by trajectories remain close-by at all times
B. At least one Lyapunov exponent must be negative
C. If the dimensionality is two, the attractor formed in a stretch and fold process
D. The Lyapunov dimension D_L satisfies $D_\mathrm{L} \leq$ dimensionality n
E. The strange attractor forms a fractal structure

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Give a real-world example of a dynamical system with a transcritical bifurcation.

b) Explain why the index I_C of a curve C that is continuously transformed, without crossing any fixed points, into a new curve C' must remain constant, $I_C = I_{C'}$. Why may $I_C \neq I_{C'}$ if the curve passes a fixed point?

c) Evaluate the nullclines and sketch the phase portrait in the upper right quadrant for the following dynamical system

$$\dot{x} = x(2 - x - y)$$
$$\dot{y} = y(x - 1)$$

d) What is meant by a heteroclinic bifurcation? Let r_c be the bifurcation point and sketch example phase portraits for $r < r_\mathrm{c}$, $r = r_\mathrm{c}$ and $r > r_\mathrm{c}$.

e) Explain what a relaxation oscillator is and give a real-world example.

f) Explain the difference between Lyapunov exponents and stability exponents of separations.

3. Bifurcations [2 points] Consider the system

$$\dot{x} = a - 2x + \frac{x^n}{1 + x^n} \,. \tag{1}$$

Assume that x is a positive function of time, that a is a non-negative parameter, $a \geq 0$, and that the parameter n is large, $n \gg 1$.

a) Sketch the function

$$g(x) = \frac{x^n}{1 + x^n}$$

for positive x and with $n \gg 1$. What simple shape does $g(x)$ approach as $n \to \infty$?

b) Let $h(x) = 2x - a$ and note that the system (1) can be written as $\dot{x} = g(x) - h(x)$, where $g(x)$ is given in subtask a). Using a geometrical approach, sketch the phase portrait (dynamics along the line) for the following three cases:

i) $a < 1$

ii) $a = 3/2$

iii) $a > 2$

c) Sketch the bifurcation diagram for the system (1) with $a \geq 0$. Label all branches with stable and unstable fixed points and all bifurcations that occur in the system. You do not need to find analytical expressions for the fixed points nor bifurcation points.

d) Discuss how x behaves if a is very slowly increased from $a = 0$ to $a > 2$, and then very slowly decreased back to $a = 0$.

4. Construction of star nodes [2 points] Star nodes are fixed points to linear systems whose stability matrix (Jacobian) has two equal eigenvalues and more than one eigenvector.

a) Construct an example of a **linear** dynamical system of dimensionality two with a stable star node at $x = y = 0$.

b) Write the system in subtask a) in polar coordinates. Explain the form of the system in polar coordinates.

c) Construct an example of a **non-linear** dynamical system of dimensionality two with a stable star node that remains a star node (unchanged structure of the phase portrait close to the fixed point) also in the presence of non-linear terms.

d) Construct a dynamical system (could be a linear system) with a parameter a such that the system is an unstable star node when $a = 0$, unstable node when $a > 0$ and unstable spiral when $a < 0$.

5. Driven and self-sustained oscillations [2 points]

a) Give an example of a physical system exhibiting self-sustained (limit cycle) oscillations.

b) The equation below shows the dynamics of an undamped oscillator driven by an external periodic torque

$$\ddot{\theta} + \omega_0^2 \theta = I \cos(\omega_{\mathrm{F}} t) \,. \tag{2}$$

Show that by a suitable rescaling of t and θ this system can be written in terms of a single dimensionless parameter. What parameter do you obtain?

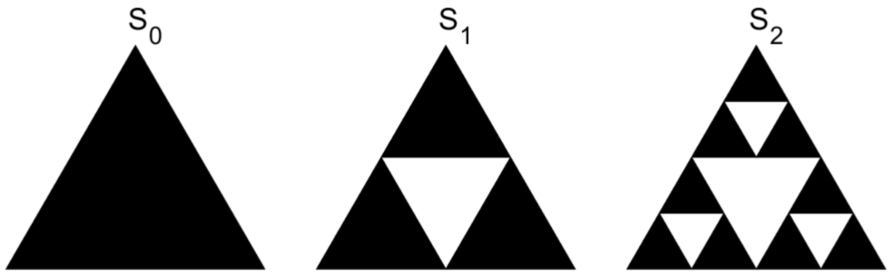
c) For suitable initial conditions (or adding a small degree of damping), the solution of Eq. (2) becomes

$$\theta(t) = \frac{I}{\omega_0^2 - \omega_{\mathrm{F}}^2} \cos(\omega_{\mathrm{F}} t) \,.$$

Discuss how these driven oscillations differ from the typical oscillations in a system with self-sustained oscillations.

d) Consider the case without external forcing, $I = 0$. Discuss how the system (2) can be modified to show self-sustained oscillations.

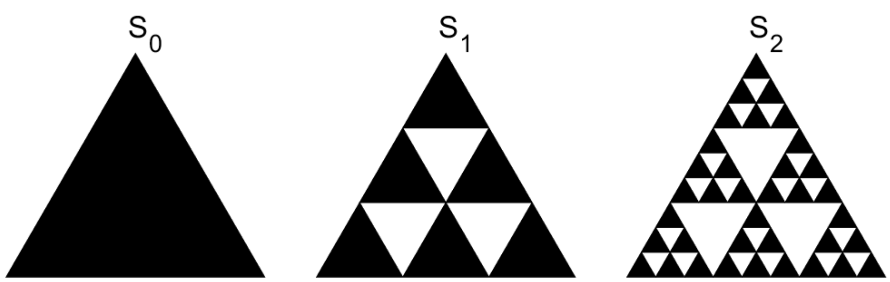
6. Fractal dimension of the Sierpinski gasket [2 points] The figure below shows the first generations S_0, S_1, S_2 in the construction of the level 2 Sierpinski gasket constructed by a set of equilateral triangles. It is obtained by, at each generation n, dividing each remaining triangle of side length L_{n-1} into four equilateral triangles of side length $L_{n-1}/2$ and removing the central one. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.



a) Analytically find the box-counting dimension D_0 for the level 2 Sierpinski gasket.

b) Analytically find the box-counting dimension D_0 for the **inverted** level 2 Sierpinski gasket, i.e. for the set of white regions inside the borders of the original equilateral triangle S_0 in the figure above.

The level k Sierpinski gasket is obtained similar to the level 2 case above, but by dividing remaining triangles of side length L_{n-1} into a grid of equilateral triangles of side length L_{n-1}/k ($k = 2$ gives the example above) and removing all triangles pointing downwards. An example for $k = 3$ is given below:



c) Analytically find the box-counting dimension D_0 for the level 3 Sierpinski gasket.

d) Analytically find the box-counting dimension D_0 for the level k Sierpinski gasket. Explicitly make sure that your formula agrees with what you expect for the cases $k = 2$ and $k = 3$.