CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	January 13, 2020, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once around 10^{00}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geqslant$18 passed; $\geqslant$26 grade 4; $\geqslant$31 grade 5,
GU $\geqslant$18 grade G; $\geqslant$ 28 grade VG.

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1. Multiple choice questions [2 points] For each of the following questions identify **all** the correct alternatives A–E. Answer with letters among A–E. Some questions may have **more than one correct alternative**. In these cases answer with all appropriate letters among A–E.

a) Consider the two-dimensional linear dynamical system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \mathbb{A}\begin{pmatrix}x\\ y\end{pmatrix}, \qquad \text{where } \mathbb{A} = \begin{pmatrix}1 & -2\\ 2 & 6\end{pmatrix}.$$

What type is the fixed point of the dynamics above?

A. Stable node

B. Stable spiral

C. Saddle point

D. Unstable node

E. Unstable spiral

b) Which of the following alternatives are eigenvectors to the fixed point in subtask a)?

A. $\begin{pmatrix}-2\\ 1\end{pmatrix}$ B. $\begin{pmatrix}-1\\ 2\end{pmatrix}$ C. $\begin{pmatrix}0\\ 1\end{pmatrix}$ D. $\begin{pmatrix}1\\ -2\end{pmatrix}$ E. $\begin{pmatrix}2\\ -1\end{pmatrix}$

c) One property of indices is that the index of a counter-clockwise closed orbit is +1. What is the index of a clockwise closed orbit?

A. -2 B. -1 C. 0 D. $+1$ E. $+2$

d) Which of the following statements hold for a homoclinic bifurcation?

A. It may occur as an isolated stable spiral becomes unstable.

B. It may occur as a saddle point collides with a limit cycle.

C. It is a local bifurcation.

D. The period time of an involved limit cycle approaches infinity at the bifurcation point.

E. The amplitude of an involved limit cycle approaches zero at the bifurcation point.

e) Consider a dynamical system with four Lyapunov exponents

$$\lambda_1 = 1\,, \quad \lambda_2 = 0\,, \quad \lambda_3 = -1\,, \quad \lambda_4 = -2\,.$$

Which of the following statements are correct in the limit of large times?

A. Small phase-space distances grow exponentially fast.

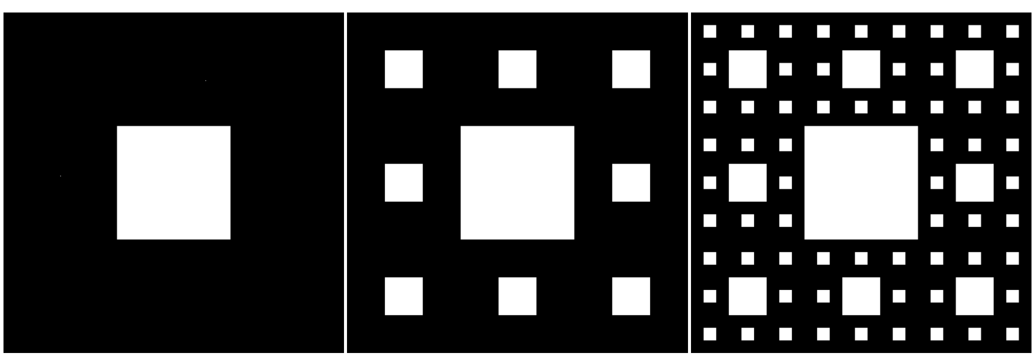
B. Small phase-space areas grow exponentially fast.

C. Small phase-space volumes grow exponentially fast.

D. The Lyapunov dimension is $D_{\rm L} = 3$.

E. The Lyapunov dimension lies between 3 and 4: $3 < D_{\rm L} < 4$.

f) The figure below shows the first few generations in the construction of a fractal. Split the unit square into nine equal-sized squares and remove the center square to obtain S_1 below. Then repeat for each of the eight remaining squares to obtain S_2. Repeat again to obtain S_3. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.



S_1 S_2 S_3

Which of the following alternatives describe the box-counting dimension of the fractal above?

A. $\dfrac{\log(3)}{\log(8)}$ B. $\dfrac{\log(3)}{\log(9)}$ C. $\dfrac{\log(3)}{\log(1)}$ D. $\dfrac{\log(8)}{\log(3)}$ E. $\dfrac{\log(9)}{\log(3)}$

2. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Explain a mechanism for obtaining self-sustained oscillations.

Solution
The typical example is an oscillator with negative damping in the linearized regime, resulting in an unstable spiral blowing up small oscillations. This blow up is counteracted by non-linear terms that regularize the oscillations at a finite amplitude. This is the mechanism underlying the van der Pol oscillator and many other limit cycles.

b) Explain why regular perturbation theory on the form $x(t) = x_0(t) + x_1(t)\mu + x_2(t)\mu^2 + \dots$ fails for the system

$$\ddot{x} = -2\mu\dot{x} - (1 + \mu^2)x$$

which has the exact solution $x(t) = e^{-\mu t}\sin t$.

Solution
The solution has two time scales ~ 1 and $\sim 1/\mu$. Series expansion of the solution in small μ

$$x(t) \approx \left(1 - t\mu + \frac{t^2}{2}\mu^2\right)\sin t$$

shows that the perturbative solutions

$$x_0(t) = \sin t\,, \quad x_1(t) = -t\sin t\,, \quad x_2(t) = \frac{t^2}{2}\sin t$$

have diverging amplitudes for x_1, x_2, ...for large t. Thus, the series expansion of the exponential give rise to the failure, and since the solution has two time scales, it is not possible to regularize the expansion by rescaling time with μ.

The generalized fractal dimension D_q is defined by

$$D_q \equiv \frac{1}{1-q}\lim_{\epsilon\to 0}\frac{\ln I(q,\epsilon)}{\ln(1/\epsilon)}$$

with

$$I(q,\epsilon) = \sum_{k=1}^{N_{\text{box}}} p_k^q(\epsilon)\,.$$

Here p_k is the probability to be in the k:th occupied box and N_{box} is the total number of occupied boxes.

c) Explain why the limit $q \to 1$ in D_q is not infinite.

Solution
Because p_k is a probability it sums to unity, $\sum_{k=1}^{N_{\text{box}}} p_k = 1$, giving $I(q,\epsilon) \to 1$ and $\ln I(q,\epsilon) \to 0$ as $q \to 1$. This cancels the divergence from $q - 1$ in the denominator.

d) Give an interpretation of the parameter q in D_q.

Solution
Introduction of the parameter q weighs different boxes differently on the fractal set. When $q = 0$, p_k is taken to the power 0 in $I(q, \epsilon)$ and all boxes are weighed equally. When $q > 0$, large p_k are weighed higher, meaning that the fractal dimension weigh dense regions of the fractal set higher. When $q < 0$, sparse regions of the fractal set are weighed higher.

e) Explain what is meant by the Lyapunov time. Give examples of the approximate Lyapunov time for two systems.

Solution
The Lyapunov time is obtained by the inverse maximal Lyapunov exponent when it is positive (chaotic system). It determines the time scale under which the system is predictable. Two examples are the the solar system with Lyapunov time $\sim 10^7$ years, or weather systems with Lyapunov time $\sim$ days.

f) Explain how the intermittency transition from regular dynamics to chaos (Pomeau-Manneville) works.

Solution
A system with a stable regular attractor exhibit a bifurcation to a distant strange attractor (catastrophe). The ghost of the former fixed point affect the dynamics by introducing a bottleneck (slow region) where the dynamics is close to regular. After the trajectory escapes the bottleneck it moves around irregularly in phase space (intermittent chaotic outburst) until it reach the bottleneck again.

3. Imperfect transcritical bifurcation [2 points] Consider the system

$$\dot{x} = x(r - x) - a^2 \tag{1}$$

where r and a are real parameters.

a) Sketch the bifurcation diagram against r when $a = 0$, what kind of bifurcation do you get?

Solution
When $a = 0$, the system is on the normal form for transcritical bifurcation at $r = 0$. See lecture notes for the dynamics and the bifurcation diagram.

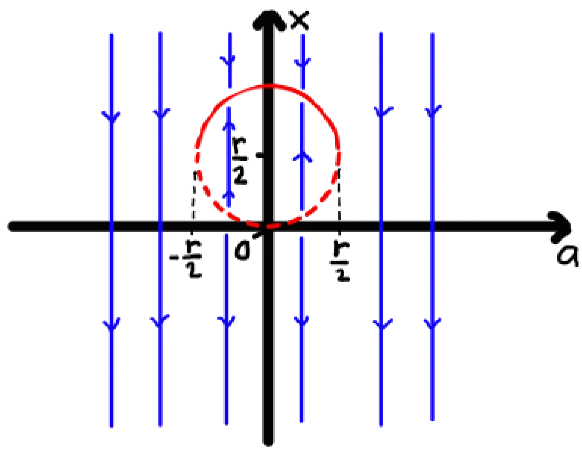
b) Now consider a fixed value of r. Make a descriptive sketch of the bifurcation diagram against a. Describe how the bifurcation diagram changes for different values of r. Which kinds of bifurcations occur?

Solution
The fixed points x^* of Eq. (1) are solutions to $x(r - x) - a^2 = 0$.

Rewriting this condition

$$0 = x^2 - xr + a^2 = \left(x - \frac{r}{2}\right)^2 + a^2 - \left(\frac{r}{4}\right)^2$$

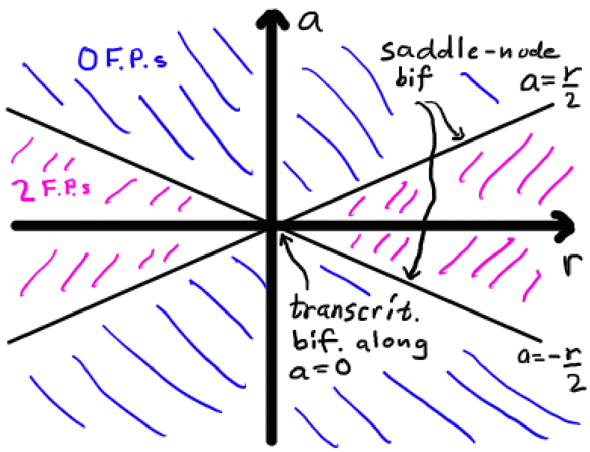
shows that fixed points in the a-x plane lie on circles centered at $x = r/2$ and $a = 0$ with radius $|r|/2$. The system therefore has saddle-node bifurcations at $x^* = r/2$ and $a_c = \pm r/2$. Stability is obtained by evaluating the direction of the flow at, for example, $x = r/2$ and $a = 0 \Rightarrow \dot{x} = r^2/4 > 0$ and at $x \to \pm\infty \Rightarrow \dot{x} \sim -x^2 < 0$. Thus, the lower half-circle repels trajectories in both directions and is unstable, the upper half-circle attracts trajectories in both directions and is stable. The case of $r > 0$ is sketched below ($r < 0$ shifts the circle to negative x).



When $r = 0$, the radius goes to zero and the system only has a single, marginally stable, fixed point $x^* = 0$ when $a = 0$. I would not consider this a bifurcation since the solutions do not change structure as a passes 0.

c) Sketch all curves in the (r,a)-plane for which a bifurcation occur in the dynamics (1). Label each curve with the type of bifurcation. Label regions separated by the bifurcation curves by the number of fixed points.

Solution
As found in subtask b) we have saddle-node bifurcations at the curves $a = \pm r/2$, and from subtask a) we have a transcritical bifurcation at $r = 0$ along the line $a = 0$.



4. Classification of fixed points [2 points] Consider the system

$$
\begin{aligned}
\dot{x} &= y + ax \\
\dot{y} &= y(y^2 - 2) - 3ax
\end{aligned}
\tag{2}
$$

where a is a real parameter.

a) Find all fixed points of the system (2) for general values of a (be sure to treat the case $a = 0$ separately).

Solution

Solving $\dot{x} = 0$ gives $y = -ax$. Inserting this in $\dot{y} = 0$ gives $0 = -ax((-ax)^2 - 2) - 3ax = -ax((ax)^2 + 1)$. When $a \neq 0$ we have a single fixed point at the origin. When $a = 0$ we have a line of fixed points (or 'fixed line' as suggested in class) at $y^* = -ax = 0$.

b) For arbitrary values of a, use linear stability analysis to identify all possible major-type fixed points of the system (2) (you can neglect the boundary types). List the parameter ranges of a for the different cases.

Solution

First, calculate the stability matrix of (2) along the line $y = 0$ where all fixed points from subtask a) lie

$$
\begin{aligned}
\mathbb{J}|_{y=0} &= \begin{pmatrix} a & 1 \\ -3a & 3y^2 - 2 \end{pmatrix}\Bigg|_{y=0} = \begin{pmatrix} a & 1 \\ -3a & -2 \end{pmatrix} \\
\operatorname{tr}\mathbb{J}|_{y=0} &= a - 2 \\
\det\mathbb{J}|_{y=0} &= a
\end{aligned}
$$

All fixed points can be classified along the line $\tau = \Delta - 2$ in the Δ-τ diagram. The condition $\tau^2 = 4\Delta$ is satisfied when $a^2 - 8a + 4 = 0$, i.e. when $a = 4 \pm \sqrt{12}$. Following the line $\tau = \Delta - 2$ in the Δ-τ diagram, we move through all regions (neglecting boundary types):

$a < 0$	saddle point
$0 < a < 4 - \sqrt{2}$	stable node
$4 - \sqrt{2} < a < 2$	stable spiral
$2 < a < 4 + \sqrt{2}$	unstable spiral
$4 + \sqrt{2} < a$	unstable node

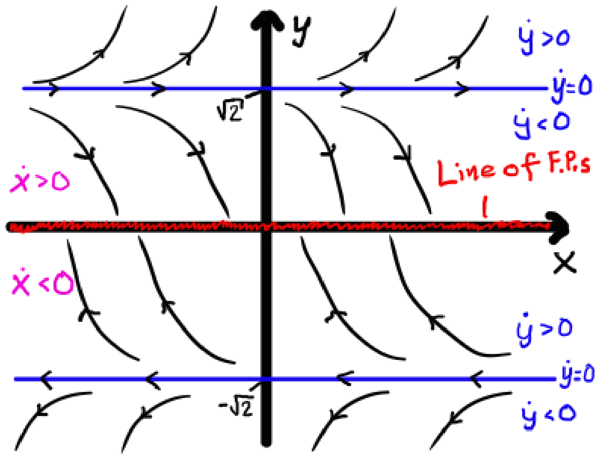
c) For the case $a = 0$, sketch the phase portrait of the system (2). Use the nullclines of the system to guide your sketch.

Solution

When $a = 0$ we have

$$\dot{x} = y$$
$$\dot{y} = y(y^2 - 2)$$

The nullcline for $\dot{x} = 0$ is $y = 0$ and the nullclines for $\dot{y} = 0$ are $y = 0$ and $y = \pm\sqrt{2}$. We remark that these curves form invariant sets, i.e. if we start on these curves we do not leave them (blue lines and line of fixed points below). From the structure of signs of $\dot{x}$ and $\dot{y}$, separated by the nullclines, it is straightforward to sketch the dynamics.



5. Damped double-well oscillator [**2 points**] Consider the system

$$\dot{x} = y$$
$$\dot{y} = -\alpha y + x - x^3 \tag{3}$$

where α is a real parameter.

a) Explain what an integral of motion is. Confirm that

$$E = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} \tag{4}$$

is an integral of motion for the system (3) when $\alpha = 0$.

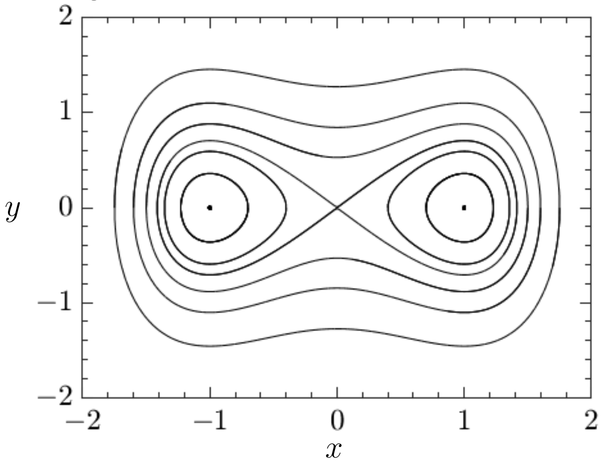
Solution

An integral of motion is a quantity that is constant along a trajectory, but non-constant on open sets in phase space. Evaluating the time derivative of E in Eq. (4) along trajectories, we have

$$\dot{E} = y\dot{y} - x\dot{x} + x^3\dot{x} = y(x - x^3) - xy + x^3 y = 0\,,$$

i.e. E is constant along trajectories. On the other hand, E in Eq. (4), being a function of both x and y, is non-constant on any open set, as illustrated in the figure in subtask b). In conclusion, E is an integral of motion.

b) The figure below shows curves for different values of E in Eq. (4):



Using this information, sketch the phase portrait for the system (3) with $\alpha = 0$ and

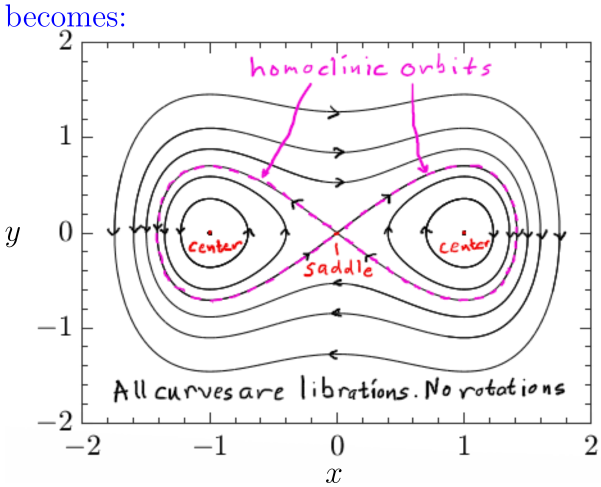
i) label all fixed points with their type;

ii) indicate any homoclinic orbits or heteroclinic trajectories;

iii) indicate in which regions the system shows librations or rotations.

Solution

Since E is an integral of motion, trajectories follow the curves shown in the figure above. The system has two nonlinear centers at $y = 0$ and $x = \pm 1$, and one saddle point at the origin. By considering the dynamics with $y = 0$ in Eq. (3), we have $\dot{x} = 0$ and $\dot{y} > 0$ if $|x| < 1$ and $\dot{y} < 0$ if $|x| > 1$. As a consequence, trajectories form clockwise closed orbits, encircling one or three fixed points. The phase portrait becomes:



c) Explain why the system (3) cannot have an integral of motion when $\alpha \neq 0$.

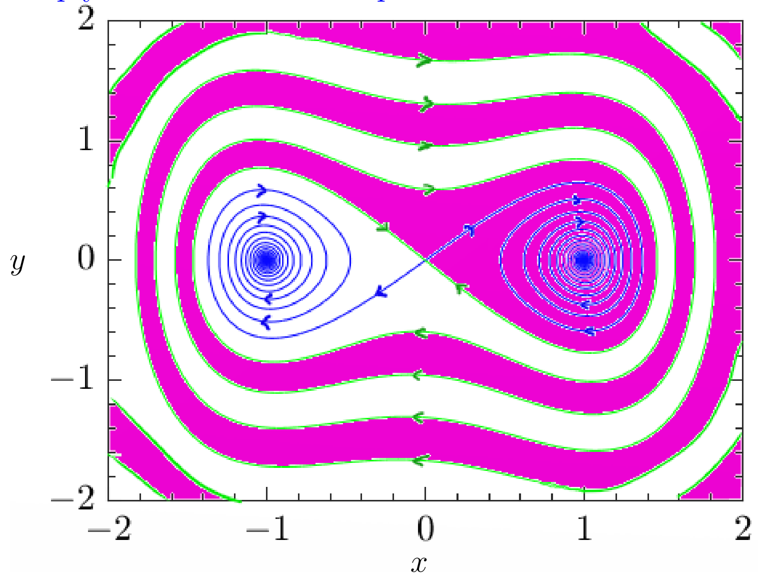
Solution

When $\alpha > 0$, the system is dissipative and all trajectories end up at attractors (the fixed points at $x = \pm 1$). Since all trajectories approach fixed points, a hypothetical integral of motion for all trajectories in the vicinity of an attracting fixed point must take the same value in an open set, contradicting the assumption that an integral of motion exist. When $\alpha < 0$, the same argument applies: all trajectories originate at repelling fixed points and can therefore not have any integral of motion.

d) Consider the case of a small, positive value of α, $0 < \alpha \ll 1$. Make a sketch of the phase portrait for the dynamics (3). Highlight the stable and unstable manifolds of all the different fixed points of the system.

Solution

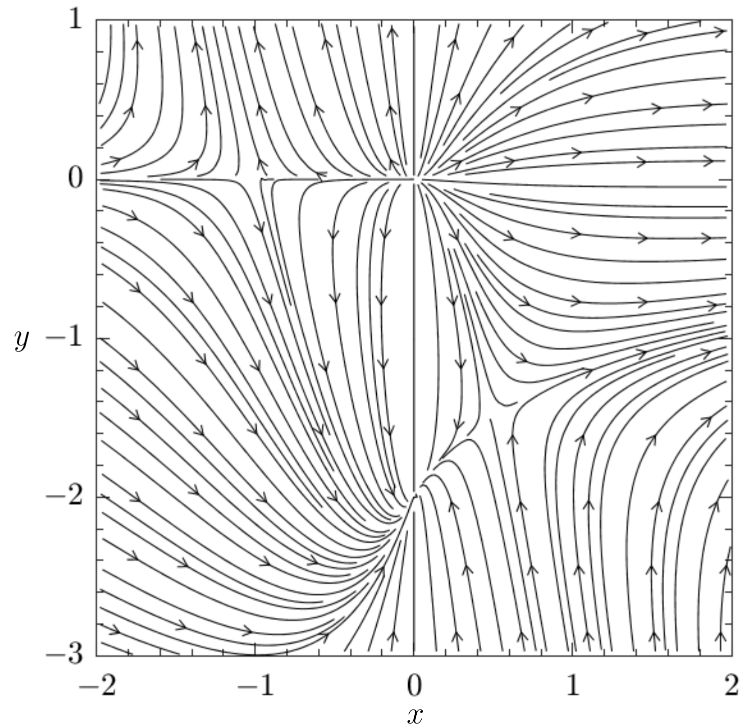
With a small amount of damping, the centers become stable spirals. The unstable manifolds of the saddle point become attracted to the spirals (blue curves in figure below). The stable manifolds of the saddle corresponds to trajectories coming from larger initial oscillations, being damped to just reach the saddle point as $t \to \infty$ (green curves below). The stable manifolds of the spirals consist of their basin of attraction, indicated by magenta color for the spiral at $(x, y) = (1, 0)$ and white for the spiral at $(x, y) = (-1, 0)$. The unstable manifolds of the spirals simply consist of the fixed points themselves.



6. Index theory [2 points] Consider the system

$$\begin{aligned}\dot{x} &= x(1 + x + y)\\ \dot{y} &= y(2 - x + y)\end{aligned} \tag{5}$$

with the following phase portrait



a) Find all fixed points of the system (5) and determine their indices. Use the phase portrait above if you do not want to calculate the indices using analytical calculations.

Solution
The system has fixed points at $(x_1^*, y_1^*) = (0,0)$ $[I_1 = +1]$, $(x_2^*, y_2^*) = (0,-2)$ $[I_2 = +1]$, $(x_3^*, y_3^*) = (-1,0)$ $[I_3 = -1]$, $(x_4^*, y_4^*) = (1,-3)/2$ $[I_4 = -1]$. The indices in brackets were obtained by traversing a small curve around the fixed points.

b) What is the index of a curve encircling the entire phase portrait above?

Solution
The index of the curve is the sum of the indices of the fixed points, i.e. $I = 0$. This can be confirmed by traversing the boundary of the phase portrait above.

c) Use index theory and the structure of the flow in Eq. (5) to show that the system cannot have closed orbits.

Solution
Any closed orbits must encircle fixed points whose indices sum up to $+1$. The only possibility of such closed orbit would be to encircle at least one of the fixed points (x_1^*, y_1^*) or (x_2^*, y_2^*), but these lie along the $x = 0$ axis which is an invariant set: $\dot{x} = 0$ along $x = 0$ implies that trajectories can only move vertically at $x = 0$ which rules out that they can be encircled by a closed orbit.

d) In the problem sets you were asked to calculate the index of the fixed point of the system

$$\dot{x} = (x^2 + y^2)^{|n|/2} \cos[n \arctan(y/x)]$$
$$\dot{y} = (x^2 + y^2)^{|n|/2} \sin[n \arctan(y/x)]$$

where n is a non-zero integer. Explain why the index takes the value it does.

Solution

The only fixed point of this system is the origin (assuming $n \neq 0$). Using polar coordinates, $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$, the flow is simply $r^{|n|}(\cos(n\theta), \sin(n\theta))$. If we traverse a curve one lap counter-clockwise in Cartesian space, θ moves from 0 to 2π. At the same time, the flow rotates from 0 to $2\pi n$, i.e. the flow vector makes n revolutions. Thus, the index is n.