3.1 Lorenz model

a)

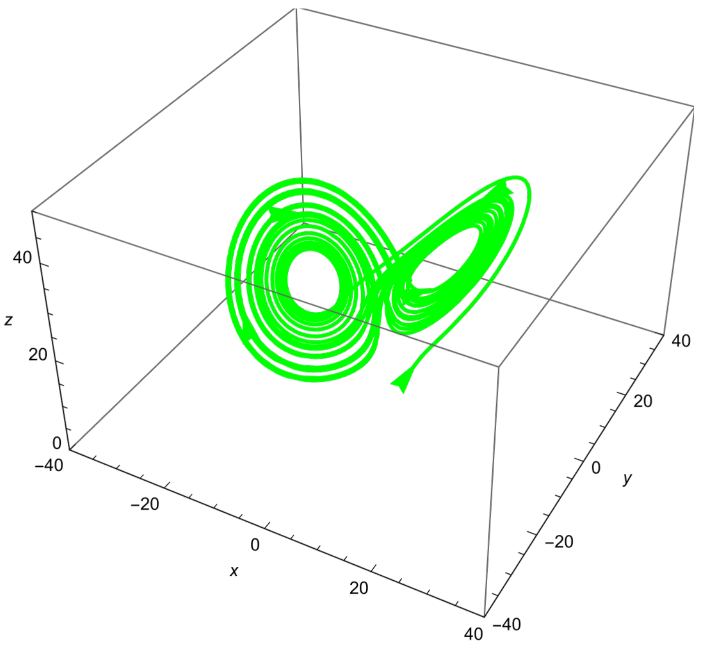
```
x =.; y =.; z =.; S =.; stabMat =.; f =.; g =.; h =.; sigma =.; r =.; b =.;
Remove[flow]

f = sigma * (y - x);
g = r * x - y - x * z;
h = x * y - b * z;
sigma = 10;
b = 8 / 3;
r = 28;
flow = {f, g, h};

S = NSolve[flow == 0, {x, y, z}];
stabMat = D[flow, {{x, y, z}}] /. %[[2]];
Eigenvalues[stabMat]

lim = 20;
StreamPlot3D[flow, {x, -lim, lim}, {y, -lim, lim}, {z, -lim, lim}];
```

```
Out[=]= {-13.8546 + 0. i, 0.0939556 + 10.1945 i, 0.0939556 - 10.1945 i}
```

b)

In[ ]:=
```
x =.;
y =.;
z =.;
S =.;
stabMat =.;
T =.;
t =.;
Tmax =.;
p0 =.;
sol =.;
t =.; T =.;
system = {x'[t] == sigma * (y[t] - x[t]),
   y'[t] == r * x[t] - y[t] - x[t] * z[t], z'[t] == x[t] * y[t] - b * z[t]};
sol[T_] := NDSolve[
   Join[system, Thread[{x[0], y[0], z[0]} == 5]], {x[t], y[t], z[t]}, {t, 0, T}];
Tmax = 20;
p0 = Show[ParametricPlot3D[
    Evaluate[{x[t], y[t], z[t]} /. sol[Tmax]], {t, 0, Tmax}, PlotStyle → Green,
    PlotRange → {{-40, 40}, {-40, 40}, {-0, 50}}, AxesLabel → {x, y, z}]];
(p0 // Normal) /.
 Line[x_] :→ {Arrowheads[{0.05, 0.05, 0.05, 0.05, 0.05}], Arrow[x], PlotStyle → Black}
```

Out[ ]=



3.2

a)

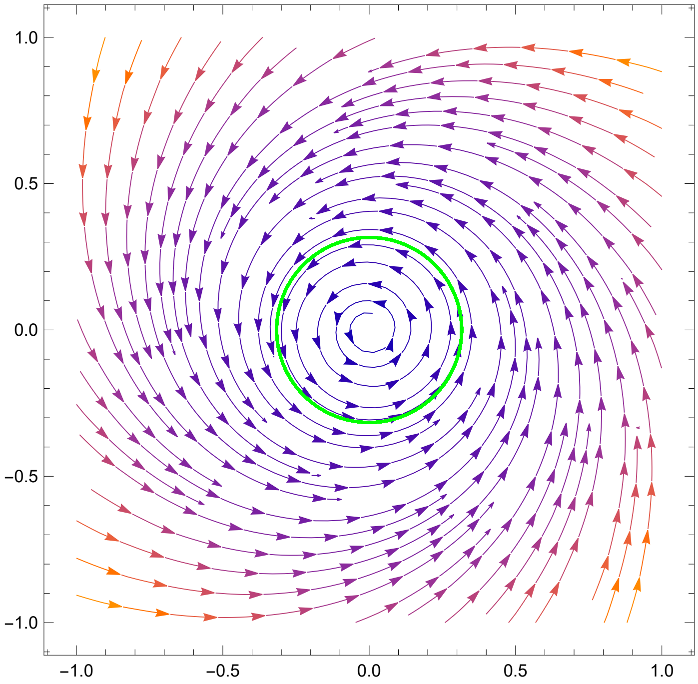
In[∘]:=
```
rdot =.; thetadot =.; μ =.;  ω =.; ν =.; r =.; S =.;

rdot = μ * r - r^3;
S = r /. NSolve[rdot == 0, {r}];
r0 = S[[3]];
r = r0;

thetadot = ω + ν * r^2;
Tperiod = 2 Pi / thetadot;

{r0, Tperiod}
```

Out[∘]= $\left\{\sqrt{\mu}, \frac{2\pi}{\mu\,\nu + \omega}\right\}$

b)

In[1]:=
```
X1 =.; X2 =.; DynSys =.; X1Dot =.; X2Dot =.; sol =.; flow =.;

X1Dot = .1 X1[t] - X2[t]^3 - X1[t] × X2[t]^2 - X1[t]^2 × X2[t] - X2[t] - X1[t]^3;
X2Dot = X1[t] + .1 X2[t] + X1[t] × X2[t]^2 + X1[t]^3 - X2[t]^3 - X1[t]^2 × X2[t] ;
DynSys = {X1'[t] == X1Dot, X2'[t] == X2Dot};
flow = {.1 X1 - X2^3 - X1 X2^2 - X1^2 X2 - X2 - X1^3,
   X1 + .1 X2 + X1 X2^2 + X1^3 - X2^3 - X1^2 X2};


sol[T_] := NDSolve[
   Join[DynSys, Thread[{X1[0], X2[0]} == {Sqrt[.1], 0}]], {X1[t], X2[t]}, {t, 0, T}];

lim = 1;
Tmax = 200;
p0 = Show[StreamPlot[flow, {X1, -lim, lim}, {X2, -lim, lim}],
  ParametricPlot[Evaluate[{X1[t], X2[t]} /. sol[Tmax]], {t, 0, Tmax},
   PlotStyle → Green, PlotRange → {{-lim, lim}, {-lim, lim}}]]
```

Out[9]=



c)

In[ ]:=
```
theta =.; r =.; X1 =.; X2 =.; flow =.;
X1 = r Cos[theta];
X2 = r Sin[theta];
flow = {.1 X1 - X2^3 - X1 X2^2 - X1^2 X2 - X2 - X1^3,
   X1 + .1 X2 + X1 X2^2 + X1^3 - X2^3 - X1^2 X2};
Simplify[flow] // MatrixForm
μ = .1;
ν = 1;
ω = 1;
```

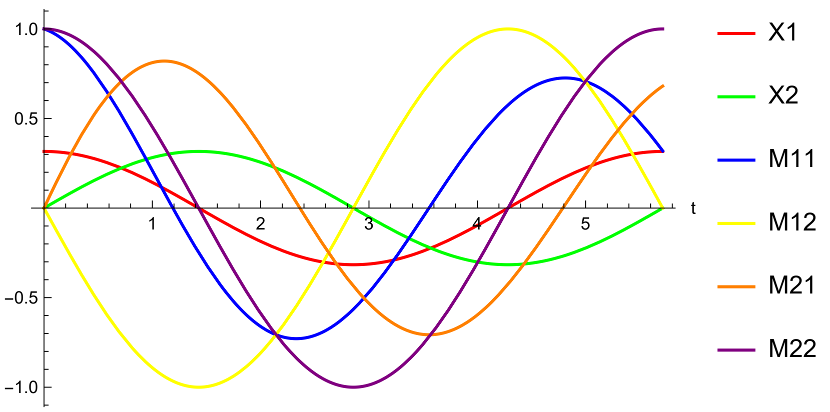
Out[ ]//MatrixForm=

$$\begin{pmatrix} (0.1\, r - 1.\, r^3)\, \mathrm{Cos}[theta] + r\, (-1. - 1.\, r^2)\, \mathrm{Sin}[theta] \\ (1.\, r + 1.\, r^3)\, \mathrm{Cos}[theta] + r\, (0.1 - 1.\, r^2)\, \mathrm{Sin}[theta] \end{pmatrix}$$

d)

In[ ]:=
```
X1 =.;
X2 =.;
X1Dot =.;
X2Dot =.;
DynSys =.;
ClearAll[X1];
ClearAll[X2];
ClearAll[M11];
ClearAll[M12];
ClearAll[M21];
ClearAll[M22]

X1Dot = .1 X1[t] - X2[t]^3 - X1[t] × X2[t]^2 - X1[t]^2 × X2[t] - X2[t] - X1[t]^3;
X2Dot = X1[t] + .1 X2[t] + X1[t] × X2[t]^2 + X1[t]^3 - X2[t]^3 - X1[t]^2 × X2[t] ;
stabMat = D[{X1Dot, X2Dot}, {{X1[t], X2[t]}}];

DynSys = {X1'[t] == X1Dot, X2'[t] == X2Dot};
sol =
  NDSolve[Join[{DynSys, M11'[t] == stabMat[[1]][[1]] * M11[t] + stabMat[[1]][[2]] * M21[t],
     M12'[t] == stabMat[[1]][[1]] * M12[t] + stabMat[[1]][[2]] * M22[t],
     M21'[t] == stabMat[[2]][[1]] * M11[t] + stabMat[[2]][[2]] * M21[t],
     M22'[t] == stabMat[[2]][[1]] * M12[t] + stabMat[[2]][[2]] * M22[t], X1[0] == 1 / Sqrt[10],
     X2[0] == 0, M11[0] == 1, M12[0] == 0, M21[0] == 0, M22[0] == 1}],
   {X1[t], X2[t], M11[t], M12[t], M21[t], M22[t]}, {t, 0, Tmax}];


p1 = Plot[{X1[t]} /. sol, {t, 0, Tperiod},
   PlotStyle → Red, PlotLegends → {"X1"}, AxesLabel → {"t", ""}];
p2 = Plot[X2[t] /. sol, {t, 0, Tperiod}, PlotStyle → Green, PlotLegends → {"X2"}];
p3 = Plot[M11[t] /. sol, {t, 0, Tperiod}, PlotStyle → Blue, PlotLegends → {"M11"}];
p4 = Plot[M12[t] /. sol, {t, 0, Tperiod}, PlotStyle → Yellow, PlotLegends → {"M12"}];
p5 = Plot[M21[t] /. sol, {t, 0, Tperiod}, PlotStyle → Orange, PlotLegends → {"M21"}];
p6 = Plot[M22[t] /. sol, {t, 0, Tperiod}, PlotStyle → Purple, PlotLegends → {"M22"}];
plot = Show[p1, p2, p3, p4, p5, p6, PlotRange → Automatic]
```

Out[ ]=



e)

In[ ]:=
```
t = Tperiod;
M11 = M11[t] /. sol
M12 = M12[t] /. sol
M21 = M21[t] /. sol
M22 = M22[t] /. sol
```

Out[ ]= {0.319053}

Out[ ]= $\{1.659 \times 10^{-8}\}$

Out[ ]= {0.680946}

Out[ ]= {0.999999}

f)

In[ ]:=
```
M = {{M11, M12}, {M21, M22}} ;
M // MatrixForm;
M = {{0.3190529478502174`, 1.6589956387813576`*^-8}, {0.6809461545562886`, 1}} ;
eig = Eigenvalues[M];
1 / Tperiod * Log[eig]
```

Out[ ]= $\{2.90441 \times 10^{-9}, -0.2\}$

g)

In[ ]:=
```
μ = 1 / 10;
r = Sqrt[μ];
stabMat = {{μ - 3 * r^2, 0}, {2 r, 0}};
Mexp = MatrixExp[Tperiod * stabMat];

G = {{Cos[θ], -r Sin[θ]}, {Sin[θ], r Cos[θ]}};
G.Mexp.Inverse[G] // MatrixForm;

a = G.Mexp.Inverse[G] /. {θ → 0, r → Sqrt[μ]}
eig = Eigenvalues[a];
Sort[1 / Tperiod * Log[eig]]
```

Out[ ]= $\left\{\left\{e^{-4\pi/11}, 0\right\}, \left\{e^{-4\pi/11}\left(-1 + e^{4\pi/11}\right), 1\right\}\right\}$

Out[ ]= $\left\{-\frac{1}{5}, 0\right\}$