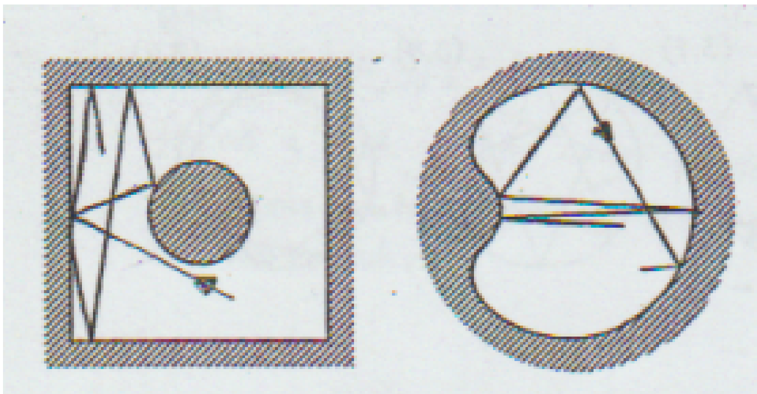
10 Chaos and Lyapunov exponents

10.1 Chaotic systems

Chaotic dynamics exhibit the following properties

- Trajectories have a finite probability to show aperiodic long-term behaviour. However, a subset of trajectories may still be asymptotically periodic or quasiperiodic in a chaotic system.
- System is deterministic, the irregular behavior is due to non-linearity of system and not due to stochastic forcing.
- Trajectories show sensitive dependence on initial condition (the 'butterfly effect'): Quantified by a positive Lyapunov exponent (this lecture).
- For continuous systems, must have dimensionality $n > 2$ (otherwise must reach fixed point or closed orbit according to Poincaré-Bendixon)

10.1.1 Illustrative example: Convex billiards



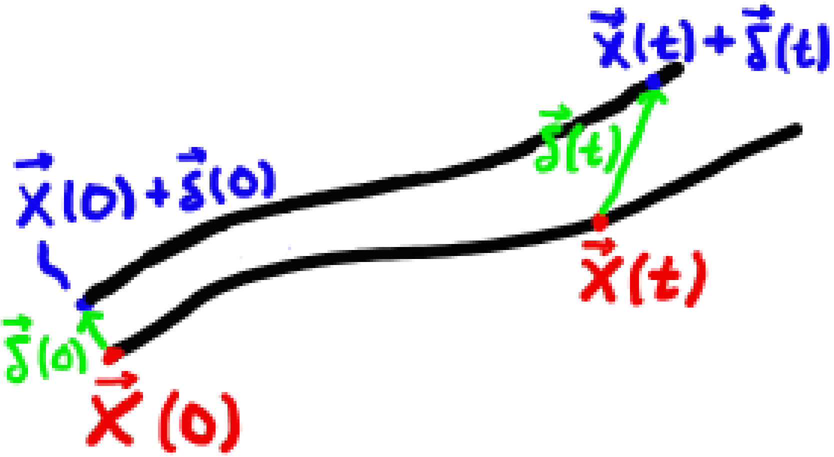
10.1.2 More examples of chaotic systems

It is more a rule than an exception that systems exhibit chaos (often in the form of a mixture between chaotic and regular motion). Examples:

- **Biology** Population dynamics, Arrythmia (hearth), Epilepsy (brain).
- **Physics** Double pendulum, helium atom, three-body gravitational problem, celestial mechanics, mixing of fluids, meteorological systems.
- **Computer science** Pseudo-random number generators, to send secret messages (Strogatz 9.6).

10.2 The maximal Lyaponov exponent

Let $\boldsymbol{\delta} \equiv \boldsymbol{x}' - \boldsymbol{x}$ be separation between two trajectories $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$:



Assume that small distance $\delta(t) \equiv |\boldsymbol{\delta}(t)|$ changes smoothly as $\delta \to 0$ ($\dot{\delta}$ approaches zero linearly as δ approaches zero) and neglect higher-order terms in $\delta(t)$ (assume that $\delta(0)$ is small enough so that $\delta(t)$ is small for all times of consideration):

$$\dot{\delta}(t) = h(t)\delta(t) \quad \Rightarrow \quad \delta(t) = \delta(0)\exp\left[\int_0^t \mathrm{d}t' h(t')\right] .$$

Define <u>maximal Lyapunov exponent</u> λ_1 as the long-time average of h:

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathrm{d}t' h(t')$$

and consider large t:

$$\delta(t) \sim e^{\lambda_1 t}\delta(0) \qquad \Rightarrow \qquad \lambda_1 \equiv \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}(0)|}\,.$$

λ_1 describes whether a system is sensitive to small deviations in initial conditions. Depending on the sign of λ_1, a small deviation between two trajectories either decreases ($\lambda_1 < 0$) or increases ($\lambda_1 > 0$) exponentially fast for large times.

10.2.1 Physical interpretation of λ_1

A positive λ_1 (and mixing) implies chaotic dynamics.
Magnitude of $1/|\lambda_1|$ is the Lyapunov time: when $\lambda_1 > 0$ it determines time horizon for which system is predictable. Examples:

- Motion of planets in our solar system is chaotic, but there is no problem in predicting planet motion on time scales of observation [Lyapunov time $\sim$ 50 million years for our solar system].
- Weather system: Lyapunov time (order of days) of same order as typical relevant time scale.
- Chaotic electric circuits (milliseconds)

Strogatz Example 9.3.1 An increase in the precision of initial condition $\delta(0) = \delta_0$ by factor $10^6 \Rightarrow$ system only predictable for 2.5 times longer (assuming a tolerance which is $10^4 \cdot \delta_0$) .

10.3 Deformation matrix

As before, consider a general flow $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ and a small separation $\boldsymbol{\delta} = \boldsymbol{x}' - \boldsymbol{x}$ with $|\boldsymbol{\delta}| \ll 1$ between two trajectories $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$. For the maximal Lyapunov exponent we only considered the distance $|\boldsymbol{\delta}|$, now we consider the full dynamics of $\boldsymbol{\delta}$. Linearized dynamics

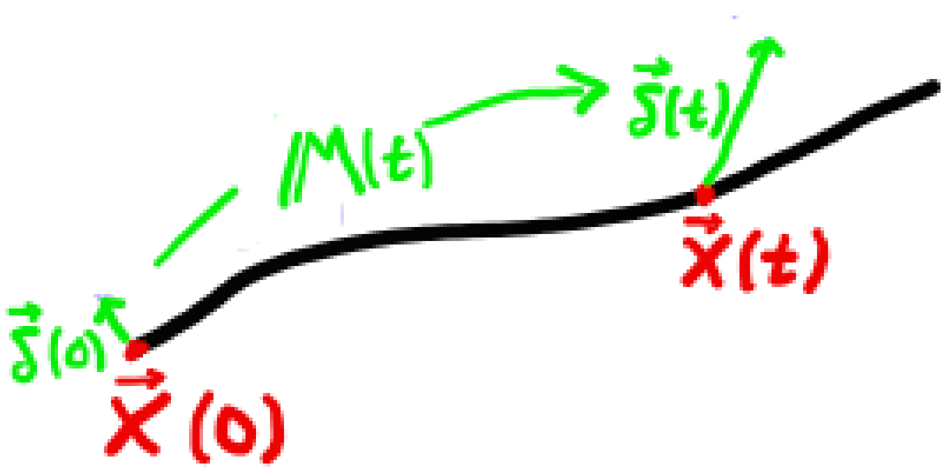
$$\begin{aligned}\dot{\boldsymbol{\delta}} &= \boldsymbol{f}(\boldsymbol{x}') - \boldsymbol{f}(\boldsymbol{x}) = [\boldsymbol{\delta} \text{ small} \Rightarrow \text{expand } \boldsymbol{f}(\boldsymbol{x}') \text{ around } \boldsymbol{x}] \\ &\approx [\boldsymbol{f}(\boldsymbol{x}) + \mathbb{J}(\boldsymbol{x})(\boldsymbol{x}' - \boldsymbol{x})] - \boldsymbol{f}(\boldsymbol{x}) = \mathbb{J}(\boldsymbol{x})\boldsymbol{\delta}\end{aligned}$$

with stability matrix $\mathbb{J}(\boldsymbol{x}) \equiv \partial \boldsymbol{f}/\partial \boldsymbol{x}$ evaluated along $\boldsymbol{x}(t)$.

The deformation matrix (deformation gradient tensor, Lyapunov matrix) $\mathbb{M}$ is defined such that

$$\boldsymbol{\delta}(t) = \mathbb{M}(t)\boldsymbol{\delta}(0)$$

with small initial separation $|\boldsymbol{\delta}(0)| \ll 1$. For a given trajectory $\boldsymbol{x}(t)$, $\mathbb{M}(t)$ transforms an initial separation $\boldsymbol{\delta}(0)$ to the separation $\boldsymbol{\delta}(t)$:



To derive an equation for the evolution of $\mathbb{M}$, differentiate $\boldsymbol{\delta}$ w.r.t. t

$$\dot{\boldsymbol{\delta}}(t) = \dot{\mathbb{M}}(t)\boldsymbol{\delta}(0)$$

But we also have from the linearisation

$$\dot{\boldsymbol{\delta}}(t) = \mathbb{J}(\boldsymbol{x})\boldsymbol{\delta}(t) = \mathbb{J}(\boldsymbol{x})\mathbb{M}(t)\boldsymbol{\delta}(0)$$

and consequently

$$\dot{\mathbb{M}}(t)\boldsymbol{\delta}(0) = \mathbb{J}(\boldsymbol{x})\mathbb{M}(t)\boldsymbol{\delta}(0)\,.$$

This equation is true for any initial separation $\boldsymbol{\delta}(0)$ $\Rightarrow$

$$\dot{\mathbb{M}}(t) = \mathbb{J}(\boldsymbol{x})\mathbb{M}(t)\,.$$

In summary, to find $\mathbb{M}(t)$ we need to integrate the joint equations

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x})\\ \dot{\mathbb{M}} &= \mathbb{J}(\boldsymbol{x})\mathbb{M}\,,\end{aligned} \tag{1}$$

with initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\mathbb{M}(0) = \mathbb{I}$ (identity matrix) for a time long enough that the initial conditions are 'forgotten'.

The eigenvalues of $\mathbb{M}$ define stability exponents of trajectory separations $\tilde{\sigma}_i \equiv \lim_{t\to\infty} t^{-1} \ln(\underset{i}{\mathrm{eig}}(\mathbb{M}))$.

Comparison to linearisation around fixed point The linearisation between closeby trajectories above, resembles the linearisation around a fixed point in Lecture 4:

Stability analysis of	fixed point	trajectory separation
Separation	$\boldsymbol{\delta} = \boldsymbol{x} - \boldsymbol{x}^*$	$\boldsymbol{\delta} = \boldsymbol{x}' - \boldsymbol{x}$
Dynamics $\dot{\boldsymbol{\delta}} = \mathbb{J}\boldsymbol{\delta}$	$\mathbb{J}(\boldsymbol{x}^*)$ const.	$\mathbb{J}(\boldsymbol{x}(t))$ along $\boldsymbol{x}(t)$
Solution $\boldsymbol{\delta}(t) = \mathbb{M}(t)\boldsymbol{\delta}(0)$	$\mathbb{M}(t) = \exp[\mathbb{J}(\boldsymbol{x}^*)t]$	$\mathbb{M}$ implicit from Eq. (1)
Stability exponents	$\sigma_i = \underset{i}{\mathrm{eig}}(\mathbb{J}(\boldsymbol{x}^*))$	$\tilde{\sigma}_i = \lim\limits_{t\to\infty} \frac{1}{t} \ln(\underset{i}{\mathrm{eig}}(\mathbb{M}))$

As a consequence, trajectories in the basin of attraction of a fixed-point attractor $\boldsymbol{x}^*$ have $\boldsymbol{x}(t) \to \boldsymbol{x}^*$ for large times and $\mathbb{M} \to \exp[\mathbb{J}(\boldsymbol{x}^*)t]$. Diagonalisation of $\mathbb{J}(\boldsymbol{x}^*) = \mathbb{P}\mathbb{D}\mathbb{P}^{-1}$ implies diagonalisation of $\mathbb{M}$: $\mathbb{M} = \mathbb{P}e^{\mathbb{D}t}\mathbb{P}^{-1} \Rightarrow$ the eigenvectors of $\mathbb{M}$ and $\mathbb{J}$ are the same. In this limit the stability exponents of separations are equal to the stability exponents σ_i of the fixed point (eigenvalues of $\mathbb{J}(\boldsymbol{x}^*)$), $\tilde{\sigma}_i = \sigma_i$. Note that we used λ_i for stability exponents in earlier lectures, but from now on we we use λ_i for Lyapunov exponents.

In general, the eigenvalues and eigenvectors of $\mathbb{M}$ are different from the eigenvalues and eigenvectors of $\mathbb{J}$ (eigensystem of $\mathbb{J}$ requires only local knowledge of system while eigensystem of $\mathbb{M}$ is influenced by all stability matrices along a trajectory via Eq. (1)). It is in general hard to solve the equations for $\mathbb{M}$ analytically, and one needs to use a numerical method (Section 10.5 below).

10.4 Lyapunov spectrum

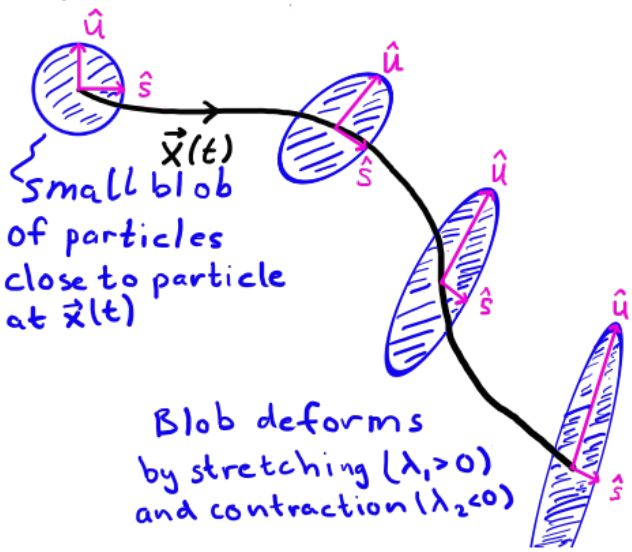
Using the deformation matrix $\mathbb{M}$ it is possible to generalize the maximal Lyapunov exponent in Section 10.2 that describes stretching rates of small separations to stretching rates of small K-dimensional subvolumes, $\mathcal{V}_K$, between groups of closeby trajectories for large t:

$$\mathcal{V}_K \sim e^{(\lambda_1+\lambda_2+\cdots+\lambda_K)t}\mathcal{V}_{K,0} \quad \Rightarrow \quad \lambda_1+\cdots+\lambda_K = \lim_{t\to\infty}\frac{1}{t}\ln\left(\frac{\mathcal{V}_K}{\mathcal{V}_{K,0}}\right) \tag{2}$$

Here $\mathcal{V}_{K,0}$ denotes a small initial volume and λ_i denotes ordered rates $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ defining a spectrum of Lyapunov exponents. Specifically, for $K = 1, 2, 3$:

- λ_1 determines exponential growth rate ($\lambda_1 > 0$) or contraction rate ($\lambda_1 < 0$) of small separations between two trajectories.

- $\lambda_1 + \lambda_2$ determines exponential growth rate ($\lambda_1 + \lambda_2 > 0$) or contraction rate ($\lambda_1 + \lambda_2 < 0$) of small areas between three trajectories.
  Example Consider the case $\lambda_1 > 0$, $\lambda_2 < 0$. A disk of trajectories around the trajectory $\boldsymbol{x}(t)$ grows exponentially fast in an unstable direction $\hat{\boldsymbol{u}}$ with rate λ_1 and shrinks exponentially fast along a stable direction $\hat{\boldsymbol{s}}$ with rate λ_2:



- $\lambda_1 + \lambda_2 + \lambda_3$ determines exponential growth/contraction rate of small volumes between four trajectories

and so on for sums over increasing number of Lyapunov exponents.

10.4.1 Analytical expression of Lyapunov exponents

To find an analytical expression for the Lyapunov exponents, first write $\mathbb{M} = \mathbb{M}\mathbb{I}$, with the unit matrix, $\mathbb{I} = [\hat{\boldsymbol{e}}^{(1)}\hat{\boldsymbol{e}}^{(2)}\cdots\hat{\boldsymbol{e}}^{(d)}]$, using a Cartesian coordinate system $\hat{\boldsymbol{e}}^{(i)}$ with elements $e_j^{(i)} = \delta_{ij}$, where $i, j = 1, \ldots, n$. The first K of these basis vectors span a K-dimensional hypercube with unit volume. Let $\boldsymbol{m}^{(i)}$ be column vectors in the deformation matrix, $\mathbb{M} = [\boldsymbol{m}^{(1)}\boldsymbol{m}^{(2)}\cdots\boldsymbol{m}^{(n)}]$, let $\mathbf{0}$ be the column vector of zeroes, and use $\mathbb{M}$ to deform a hypercube with small volume $\mathcal{V}_{K,0}$:

$$\mathbb{M}[\hat{\boldsymbol{e}}^{(1)}\hat{\boldsymbol{e}}^{(2)}\cdots\hat{\boldsymbol{e}}^{(K)}\mathbf{0}\cdots\mathbf{0}]\mathcal{V}_{K,0} = [\boldsymbol{m}^{(1)}\boldsymbol{m}^{(2)}\cdots\boldsymbol{m}^{(K)}\mathbf{0}\cdots\mathbf{0}]\mathcal{V}_{K,0}\,.$$

The column vectors of the resulting matrix, $\boldsymbol{m}^{(\alpha)}$ with $\alpha = 1, \ldots, K$, span a K-dimensional hyperparallelepiped (embedded in n dimensions). Its volume is $\mathcal{V}_K = \sqrt{|\det \mathbb{G}^{(K)}|}\mathcal{V}_{K,0}$, where $\mathbb{G}^{(K)}$ is the Gram matrix of the spanning vectors with elements $G_{\alpha\beta}^{(K)} = \boldsymbol{m}^{(\alpha)}\cdot\boldsymbol{m}^{(\beta)}$, with $\alpha, \beta = 1, \ldots, K$.

Now, factorize M using a QR-decomposition, $\mathbb{M} = \mathbb{Q}\mathbb{R}$, where $\mathbb{Q}\mathbb{Q}^{\mathrm{T}} = \mathbb{I}$ and $\mathbb{R}$ is right (upper) triangular, to obtain

$$G_{\alpha\beta}^{(K)} = [\boldsymbol{m}^{(\alpha)}]^{\mathrm{T}}\boldsymbol{m}^{(\beta)} = [\mathbb{M}^{\mathrm{T}}\mathbb{M}]_{\alpha\beta} = [\mathbb{R}^{\mathrm{T}}\mathbb{Q}^{\mathrm{T}}\mathbb{Q}\mathbb{R}]_{\alpha\beta} = [\mathbb{R}^{\mathrm{T}}\mathbb{R}]_{\alpha\beta}\,,$$

where $\alpha, \beta = 1, \ldots K$.

Using that the determinant of a right triangular matrix $\mathbb{R}$ is equal to the product of the diagonal elements gives:

$$\frac{\mathcal{V}_K}{\mathcal{V}_{K,0}} = \sqrt{|\det \mathbb{G}^{(K)}|} = \sqrt{|\det(\mathbb{R}_K^{\mathrm{T}}\mathbb{R}_K)|} = |\det \mathbb{R}_K| = \prod_{\alpha=1}^{K} |R_{\alpha\alpha}|\,,$$

where $\mathbb{R}_K$ is the first $K \times K$ matrix in $\mathbb{R}$:

$$\mathbb{R}_K \equiv \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1K} \\ 0 & R_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_{K-1,K} \\ 0 & \cdots & 0 & R_{KK} \end{pmatrix}.$$

Inserting into the definition of the Lyapunov spectrum (2) gives

$$\lambda_1 + \cdots + \lambda_K = \lim_{t\to\infty} \frac{1}{t} \ln \left(\prod_{\alpha=1}^{K} |R_{\alpha\alpha}| \right) = \sum_{\alpha=1}^{K} \lim_{t\to\infty} \frac{1}{t} \ln(|R_{\alpha\alpha}|)$$

By inserting $K = 1$, $K = 2$, $\ldots K = n$ in this relation, it follows that the ordered Lyapunov exponents are given by the diagonal elements of $\mathbb{R}$:

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln |R_{ii}| \,. \tag{3}$$

As is shown in the next section, this relation is useful for evaluating Lyapunov exponents numerically. The Lyapunov exponents are real numbers. Due to the singular nature of $\mathbb{M}$ at large t, we generically have $\lambda_i = \operatorname{Re} \tilde{\sigma}_i$, but counterexamples can be constructed (see for example Goldhirsch et al, Physica D **27** 311 (1987)).

The Lyapunov exponents (and stability exponents of separations) are useful to understand the behavior of regular and chaotic systems. As we will see in the next lectures, their signs quantifies the attractors of a system, and their relative magnitudes gives information on the degree of fractal clustering in chaotic systems.

10.5 Numerical evaluation of Lyapunov exponents

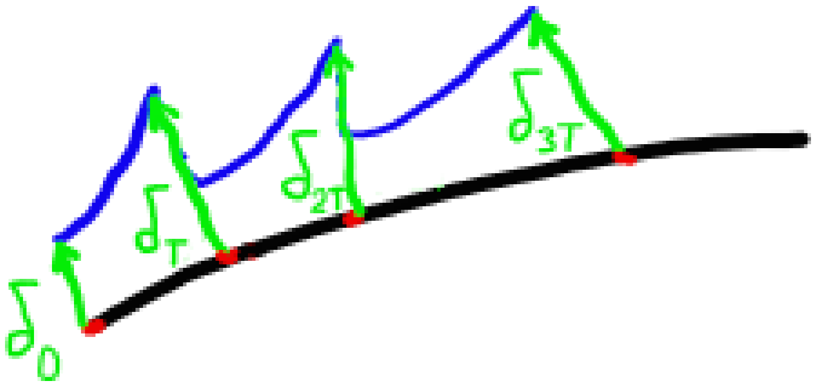
The Lyapunov exponents are in general hard to calculate analytically and one needs to rely on numerical methods.

10.5.1 Naive evaluation of λ_1 using separations

A naive approach is to solve the dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

numerically for two trajectories starting at $\boldsymbol{x}(0)$ and $\boldsymbol{x}(0) + \boldsymbol{\delta}(0)$.



Since the separation must remain small, rescale the separation vector to original length at regular time intervals T

$$\boldsymbol{\delta}(nT) \to \frac{1}{\alpha_n}\boldsymbol{\delta}(nT)\,, \qquad \alpha_n = \frac{|\boldsymbol{\delta}(nT)|}{|\boldsymbol{\delta}(0)|}$$

and use scaling factors α_n to evaluate

$$\lambda_1 = \frac{1}{t}\ln\frac{|\boldsymbol{\delta}(t)|}{|\boldsymbol{\delta}(0)|} = \frac{1}{NT}\sum_{n=1}^{N}\ln\alpha_n$$

with the total number of rescalings, N, large.

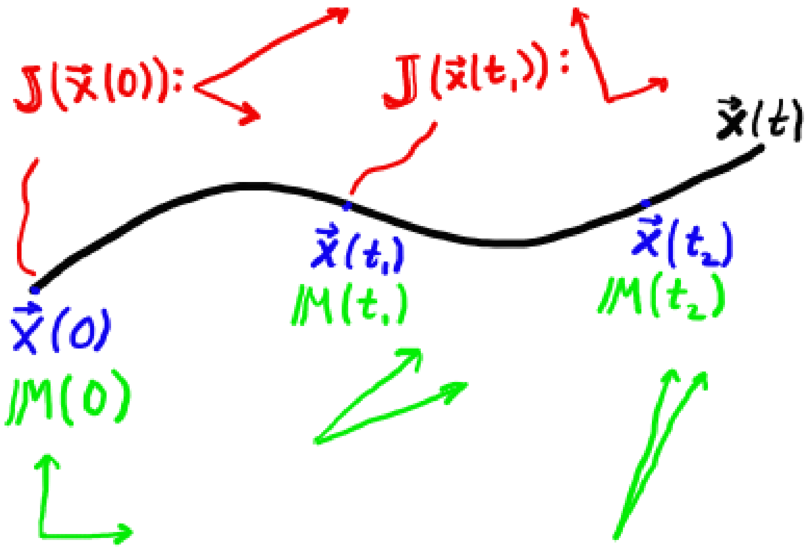
This often works! But it is unreliable: what are good values for $|\boldsymbol{\delta}(0)|$ and T? Another problem is that the method does not give the stretching rate in directions other than the maximal, needed to calculate $\lambda_2, \ldots, \lambda_n$. In order to calculate these, one would need to follow $n + 1$ trajectories and rescale and reorthonormalize the volume spanned between the trajectories. This is quite complicated.

10.5.2 Evaluation using the deformation matrix

In principle, the Lyapunov exponents can be obtained from Eq. (3) using QR-decomposition of $\mathbb{M}$ following Eq. (1). However, direct evaluation of Eq. (1) is in general numerically problematic (the elements in $\mathbb{M}$ blow up exponentially with increasing t). A workaround is obtained by first discretizing time $t \to t_n \equiv n\delta t$ (n integer and δt small time step):

$$
\begin{aligned}
&\frac{\mathbb{M}(t_n) - \mathbb{M}(t_{n-1})}{\delta t} = \mathbb{J}(\boldsymbol{x}(t_{n-1}))\mathbb{M}(t_{n-1}) \\
\Rightarrow \mathbb{M}(t_n) &= [\mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_{n-1}))\delta t]\mathbb{M}(t_{n-1}) \\
&= [\mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_{n-1}))\delta t][\mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_{n-2}))\delta t]\mathbb{M}(t_{n-2}) \\
&= [\mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_{n-1}))\delta t][\mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_{n-2}))\delta t]\dots[\mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_0))\delta t]\underbrace{\mathbb{M}(t_0)}_{\mathbb{I}}
\end{aligned}
$$

i.e. $\mathbb{M}(t_n)$ consists of product of n matrices $\mathbb{M}(t_n) = \mathbb{M}^{(n-1)}\mathbb{M}^{(n-2)}\cdots\mathbb{M}^{(0)}$ where $\mathbb{M}^{(i)} \equiv \mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_i))\delta t$.
The time evolution of the deformation matrix $\mathbb{M}$ driven by stability matrices $\mathbb{J}(\boldsymbol{x}(t_n))$ along a trajectory $\boldsymbol{x}(t)$:



Arrows show eigensystems of $\mathbb{M}$ (green) and $\mathbb{J}$ (red). At each time step, the eigendirections of $\mathbb{M}$ strives against the maximal direction of $\mathbb{J}$ and becomes longer if maximal eigenvalue of $\mathbb{J}$ is positive. $\Rightarrow$ eigenvectors of $\mathbb{M}$ tend to become very long and almost aligned. $\Rightarrow$ hard numerics

Numerical implementation We evaluate the QR-decomposition of $\mathbb{M}(t_n) = \mathbb{M}^{(n-1)}\cdots\mathbb{M}^{(0)}$ without numerical overflow as follows

- Before first time step, QR-decompose $\mathbb{M}^{(0)} = \mathbb{Q}^{(0)}\mathbb{R}^{(0)}$, i.e. $\mathbb{Q}^{(0)} = \mathbb{R}^{(0)} = \mathbb{I}$ because $\mathbb{M}^{(0)} = \mathbb{I}$.

- After first time step, rewrite $\mathbb{M}^{(1)}\mathbb{M}^{(0)} = \underbrace{\mathbb{M}^{(1)}\mathbb{Q}^{(0)}}_{\mathbb{Q}^{(1)}\mathbb{R}^{(1)}}\mathbb{R}^{(0)} = \mathbb{Q}^{(1)}\mathbb{R}^{(1)}\mathbb{R}^{(0)}$
- After second time step, rewrite $\mathbb{M}^{(2)}\mathbb{M}^{(1)}\mathbb{M}^{(0)} = \underbrace{\mathbb{M}^{(2)}\mathbb{Q}^{(1)}}_{\mathbb{Q}^{(2)}\mathbb{R}^{(2)}}\mathbb{R}^{(1)}\mathbb{R}^{(0)} = \mathbb{Q}^{(2)}\mathbb{R}^{(2)}\mathbb{R}^{(1)}\mathbb{R}^{(0)}$
- Repeat for each time step: $\mathbb{M}^{(n-1)}\cdots\mathbb{M}^{(0)} = \mathbb{Q}^{(n-1)}\mathbb{R}^{(n-1)}\cdots\mathbb{R}^{(1)}\mathbb{R}^{(0)}$

In conclusion, we have a QR-decomposition $\mathbb{M}(t_n) = \mathbb{Q}\mathbb{R}$, where $\mathbb{Q} = \mathbb{Q}^{(n-1)}$ and $\mathbb{R} = \mathbb{R}^{(n-1)}\cdots\mathbb{R}^{(1)}\mathbb{R}^{(0)}$. Eq. (3) gives the Lyapunov spectrum for large N, corresponding to a final time $t_N = N\delta t$:

$$\lambda_i = \lim_{N\to\infty}\frac{1}{N\delta t}\ln|R_{ii}| = \lim_{N\to\infty}\frac{1}{N\delta t}\sum_{n=0}^{N-1}\ln|R_{ii}^{(n)}|\,. \tag{4}$$

The $\ln|R_{ii}^{(n)}|$ can be added one at a time to avoid overflow.

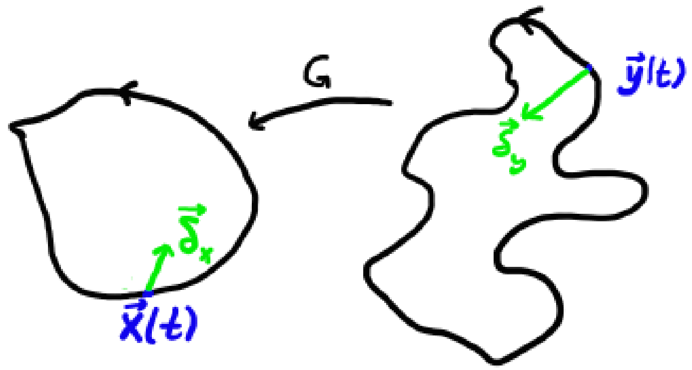
What you should implement:

1. Solve the equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ for some time to end up close to the fractal attractor (Lorenz system in Problem 3.2).
2. Start with matrix $\mathbb{Q} = \mathbb{I}$ and zero-valued variables λ_i for the sums in Eq. (4)
3. At each time step you get a new matrix $M^{(n)} = \mathbb{I} + \mathbb{J}(\boldsymbol{x}(t_n))\delta t$ where $\boldsymbol{x}(t_n)$ is taken from solution of the $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ equation.
4. At each time step QR-decompose $\mathbb{M}^{(n)}\mathbb{Q}_{\rm old} = \mathbb{Q}_{\rm new}\mathbb{R}_{\rm new}$
OBS: QRDecomposition[M] in Mathematica gives matrices named Q and R, but $\mathbb{M} = \mathbb{Q}^{\rm T}\mathbb{R}$, i.e. one must transpose $\mathbb{Q}$.
5. At each time step add the diagonal elements of $\mathbb{R}_{\rm new}$ to λ_i in Eq. (4)
6. Repeat from step 3 with $\mathbb{Q} = \mathbb{Q}_{\rm new}$ (total of N iterations)

10.6 Coordinate transform of $\mathbb{M}$ for closed orbits

At multiples of the period time of a closed orbit, the deformation matrix $\mathbb{M}$ is (similarity-) invariant under general coordinate transformations. This property can sometimes be useful for analytical calculations of eigenvalues of the deformation matrix (Problem set 3.1).

Start from the equation defining $\mathbb{M}$ (subscripts denote original coordinates $\boldsymbol{x}$), $\boldsymbol{\delta_x}(t) = \mathbb{M}_{\boldsymbol{x}}(t)\boldsymbol{\delta_x}(0)$. Change coordinates $\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{y})$



For small separations $\boldsymbol{\delta_x}$ and $\boldsymbol{\delta_y}$ we have

$$\begin{aligned}\boldsymbol{\delta_x}(t) &= \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}}\boldsymbol{\delta_y}(t) \equiv \mathbb{J}_G(\boldsymbol{y}(t))\boldsymbol{\delta_y}(t)\\ \boldsymbol{\delta_x}(0) &= \mathbb{J}_G(\boldsymbol{y}(0))\boldsymbol{\delta_y}(0)\end{aligned}$$

where $\mathbb{J}_G$ is the gradient matrix of the transformation $\boldsymbol{G}$. Consequently

$$\begin{aligned}\boldsymbol{\delta_y}(t) &= \mathbb{J}_G^{-1}(\boldsymbol{y}(t)) \quad \underbrace{\boldsymbol{\delta_x}(t)}_{\mathbb{M}_{\boldsymbol{x}}(t)\boldsymbol{\delta_x}(0)}\\ &= \mathbb{J}_G^{-1}(\boldsymbol{y}(t))\mathbb{M}_{\boldsymbol{x}}(t)\mathbb{J}_G(\boldsymbol{y}(0))\boldsymbol{\delta_y}(0)\,.\end{aligned}$$

But from the definition of the deformation matrix $\mathbb{M}_{\boldsymbol{y}}(t)$ in the $\boldsymbol{y}$-system we also have $\boldsymbol{\delta_y}(t) = \mathbb{M}_{\boldsymbol{y}}(t)\boldsymbol{\delta_y}(0)$.

$$\Rightarrow \mathbb{M}_{\boldsymbol{y}}(t) = \mathbb{J}_G^{-1}(\boldsymbol{y}(t))\mathbb{M}_{\boldsymbol{x}}(t)\mathbb{J}_G(\boldsymbol{y}(0))$$

For a closed orbit at multiples of the period time (so that $\boldsymbol{y}(t) = \boldsymbol{y}(0)$), eigenvalues of $\mathbb{M}_{\boldsymbol{y}}$ = eigenvalues of $\mathbb{M}_{\boldsymbol{x}}$. This can be seen by diagonalisation $\mathbb{M}_{\boldsymbol{x}}(t) = \mathbb{P}^{-1}\mathbb{D}\mathbb{P} \Rightarrow \mathbb{M}_{\boldsymbol{y}}(t) = [\mathbb{P}\mathbb{J}_G^{-1}(\boldsymbol{y}(0))]^{-1}\mathbb{D}[\mathbb{P}\mathbb{J}_G(\boldsymbol{y}(0))]$, i.e. also $\mathbb{M}_{\boldsymbol{y}}(t)$ is diagonalized with the same diagonal matrix $\mathbb{D}$.