CHALMERS, GÖTEBORGS UNIVERSITET

EXAM for
DYNAMICAL SYSTEMS

COURSE CODES: **TIF 155, FIM770GU, PhD**

Time:	August 16, 2017, at $08^{30} - 12^{30}$
Place:	Johanneberg
Teachers:	Kristian Gustafsson, 070-050 2211 (mobile), visits once at 09^{30}
Allowed material:	Mathematics Handbook for Science and Engineering
Not allowed:	any other written material, calculator

Maximum score on this exam: 12 points (need 5 points to pass).
Maximum score for homework problems: 24 points (need 10 points to pass).
CTH $\geq$20 passed; $\geq$27 grade 4; $\geq$32 grade 5,
GU $\geq$20 grade G; $\geq$ 29 grade VG.

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1. Short questions [2 points] For each of the following questions give a concise answer within a few lines per question.

a) Give a definition for what a dynamical system is.

b) A nonautonomous system can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t),$$

i.e. the flow $\boldsymbol{f}$ depends explicitly on time. Is a nonautonomous system a dynamical system? Explain your answer.

c) What does a transcritical bifurcation mean?

d) What are the stable manifolds of a fixed point?

e) Give an example of how the knowledge of stable manifolds of a fixed point could be used to understand the dynamics in a dynamical system.

f) What is a quasiperiodic flow? Give an example!

g) In the problem sets the Lyapunov exponents were evaluated using a QR-decomposition method. Why is this method preferred over direct numerical evaluation of the eigenvalues of $\mathbb{M}^{\mathrm{T}}\mathbb{M}$ where $\mathbb{M}$ is the deformation matrix, or over evaluation of the Lyapunov exponent using separations between a number of particles?

h) Sketch the typical shape of the generalized dimension spectrum D_q against q for a mono fractal and for a multi fractal.

2. Quadfurcation [2 points]

a) Give/construct an example of a one-dimensional dynamical system showing a pitchfork bifurcation as a parameter r passes 0.

b) Sketch the bifurcation diagram for your system in subtask a).

c) Pitchfork bifurcations are examples of 'trifurcations', meaning a division into three branches of fixed points as r passes 0. Construct an example of a 'quadfurcation', in which no fixed points exist for $r < 0$ and four fixed points exist for $r > 0$.

d) Sketch the bifurcation diagram for your system in subtask c).

3. Phase portrait [2 points] Consider the system

$$\begin{aligned} \dot{x} &= x(ax - y) \\ \dot{y} &= y(2x - y)\,. \end{aligned} \tag{1}$$

a) Find all fixed points of the system (1).

b) What does linear stability analysis predict about the fixed point(s)?

c) For $a = 2$, sketch the nullclines and the phase-plane dynamics (phase portrait) in the region $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

4. Trapping regions for the van der Pol oscillator [2 points] Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{2}$$

with μ a real parameter.

a) Give physical interpretations or explanations of the different terms in Eq. (2).

b) Consider the dynamics in the phase-plane (x, y) with $y = \dot{x}$. Knowing that this dynamical system shows an attractive limit cycle when $\mu > 0$, show that it has a repelling limit cycle when $\mu < 0$.

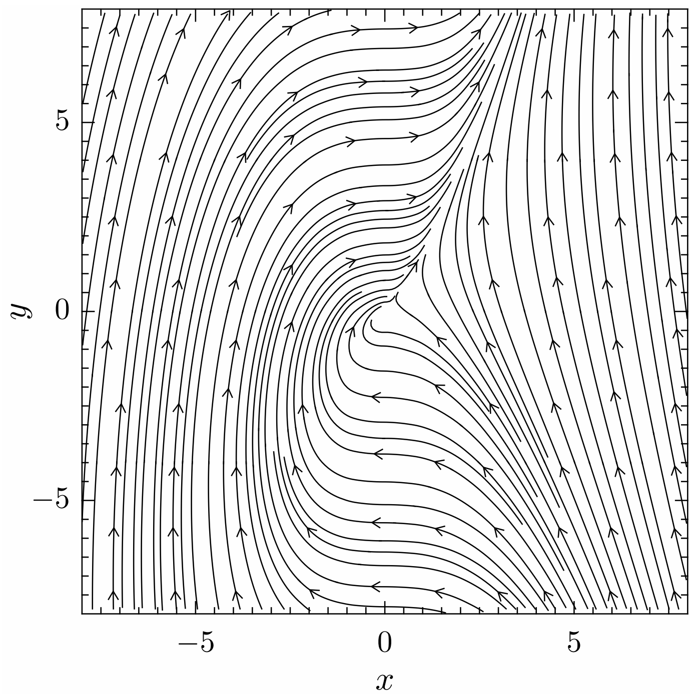
c) Let $r = \sqrt{x^2 + y^2}$ and derive an equation for $\dot{r}$ in terms of x and y.

d) When $\mu < 0$, show that there exist 'trapping regions' in the form of circles of radii $r < r_c$ such that all solutions starting from initial conditions inside these circles tend to the origin. Determine r_c.

5. Indices and bifurcations [2 points] The phase portraits of two dynamical systems are plotted in subtasks a) and b) below.

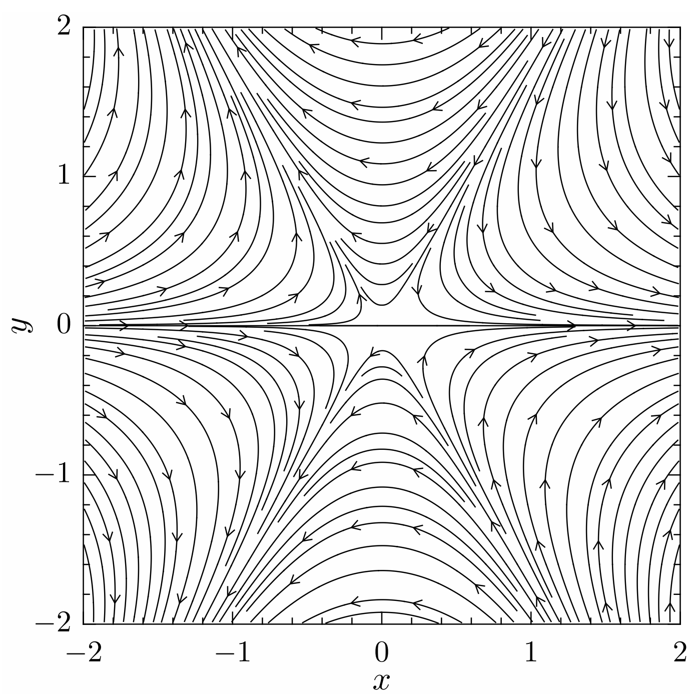
a) What is the index of the fixed point of the following dynamical system?

$$\dot{x} = y - x$$
$$\dot{y} = x^2$$



b) What is the index of the fixed point of the following dynamical system?

$$\dot{x} = x^2 - y^2$$
$$\dot{y} = -2xy$$



c) Add a perturbation term μ to the x-component of the flow in subtask b). Describe the bifurcation (if any) that occurs when μ passes through zero in the perturbed system:

$$\dot{x} = x^2 - y^2 + \mu$$
$$\dot{y} = -2xy\,.$$

d) Is the bifurcation in subtask c) consistent with the indices of involved fixed points and with the result you obtained in subtask b)?

6. Box-counting dimension [2 points] The two figures below show the first few generations in the construction of two fractals. The fractal set is obtained by iterating to generation S_n with $n \to \infty$.

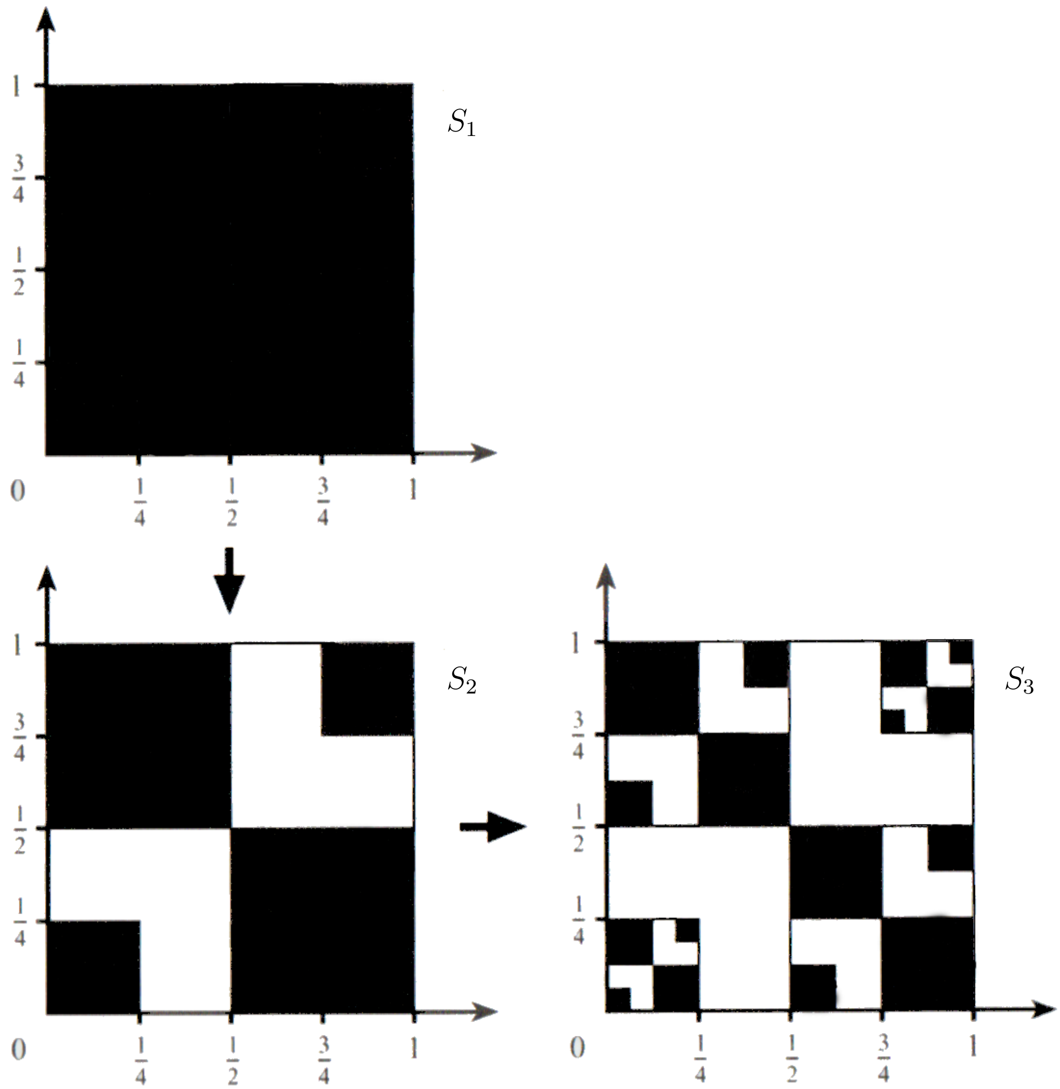
a) Analytically find the box-counting dimension D_0 (explicitly if possible, otherwise implicitly) of the Koch curve, obtained by at each generation replacing the middle third interval of all lines of length L with two new lines. The two replacing lines both have length $L/3$ and form a wedge:

S_1

S_2

S_3

b) Analytically find the box-counting dimension D_0 (explicitly if possible, otherwise implicitly) of the fractal constructed by infinite iteration of the sequence illustrated below:


S_1
S_2
S_3
0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1