13 Order to chaos in dissipative systems

This and the next lecture are mainly based on the books *From simple models to complex systems* by M. Cencini et al. and *Chaos In Dynamical Systems* by E. Ott, and on Chapter 10 in Stogatz.

Chaos, as discussed in previous lectures, denotes aperiodic deterministic motion that is sensitive to the initial conditions. In fully chaotic systems, trajectories are either attracted to strange attractors (dissipative systems) or form chaotic limit sets (volume conserving systems, i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{f}=0$ everywhere). Many systems show a mixture of regular and chaotic dynamics, some examples being:

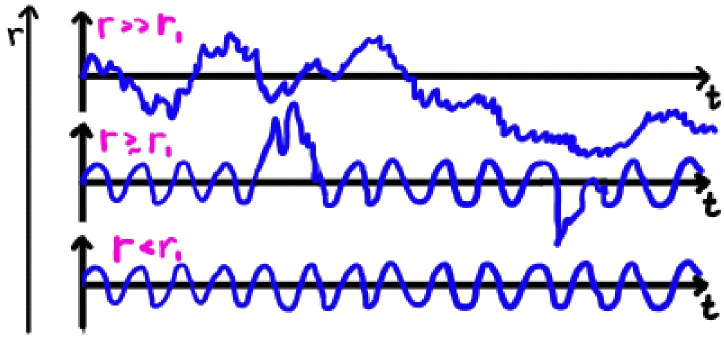
- System with a mixture of strange attractors and other types of dynamics, the dynamics chosen depends on the initial condition.
- A system where the maximal Lyapunov exponent is positive at finite times, but negative in the long run, leading to <u>transient chaos</u>. For example, a double pendulum shows chaotic dynamics if the energy is high, but the motion becomes regular as the energy of oscillations decrease due to friction. Another example is casting of dice.
- <u>Intermittent chaotic dynamics</u> often appears after a catastrophe, see Section 13.1 below.

There are several ways in which regular dynamics (stable fixed points or periodic/quasiperiodic motion) transforms into chaos as some system parameter r changes. The transition is very different in dissipative systems (this lecture) and in Hamiltonian systems (next lecture).

In numerical simulations and in experiments it is observed that the transition from regular motion to chaos in dissipative systems always pass through a strange attractor of low fractal dimension, before potential attractors of higher fractal dimension are reached. Below a few mechanisms for the transition from regular dynamics to strange attractors of low dimension are summarized.

13.1 Intermittency transition

Fixed point or periodic orbit becomes unstable at a single bifurcation r_1, giving chaos characterized by intermittency (Pomeau-Manneville).



Intermittency Nearly regular motion interrupted by occasional short irregular outbursts at irregular time intervals. As control parameter is increased, the bursts become more and more frequent until system becomes fully chaotic.

Explanation Just after the bifurcation ($r > r_1$ but $r \approx r_1$) the system has bottlenecks where the dynamics is regular (for example due to ghost of fixed point, or due to weakly unstable fixed point|periodic orbit). After leaving a bottleneck the dynamics becomes irregular (intermittent outburst) until a new bottleneck is reached.

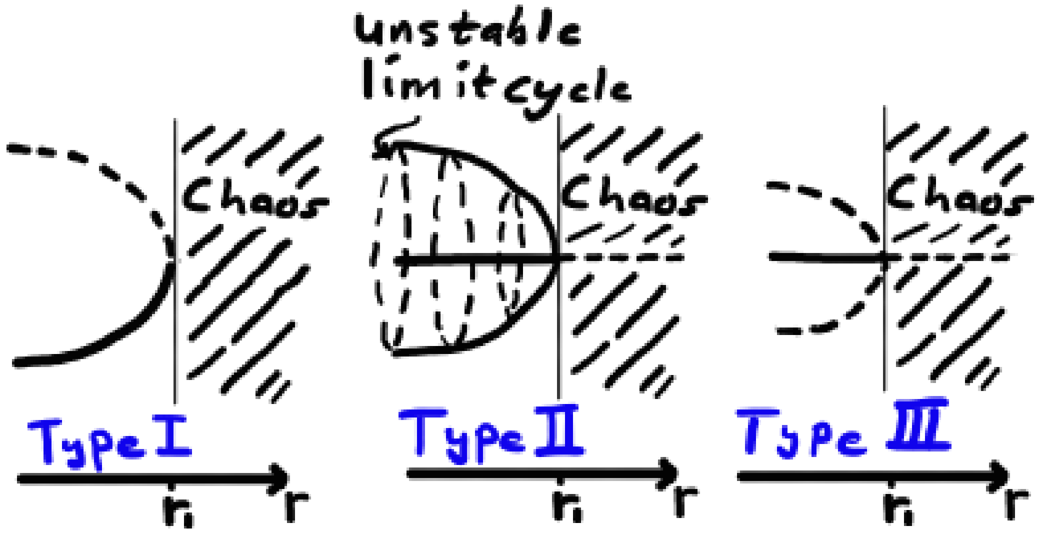
This often occurs in systems with saddle-node bifurcations between cycles: as the stable periodic orbit disappears in a saddle-node bifurcation a bottleneck is formed due to the slow dynamics close to the former limit cycle. In these systems the average time of regular motion is of order $1/\sqrt{r}$ (the time to pass the ghost of a saddle-node, see Lecture 2).

The intermittency bifurcation is classified in three types depending on how the stable attractor becomes unstable:

Type I Saddle-node bifurcation

Type II Subcritical Hopf bifurcation (stable fixed point and unstable periodic orbit merge into an unstable spiral)

Type III Subcritical pitchfork bifurcation

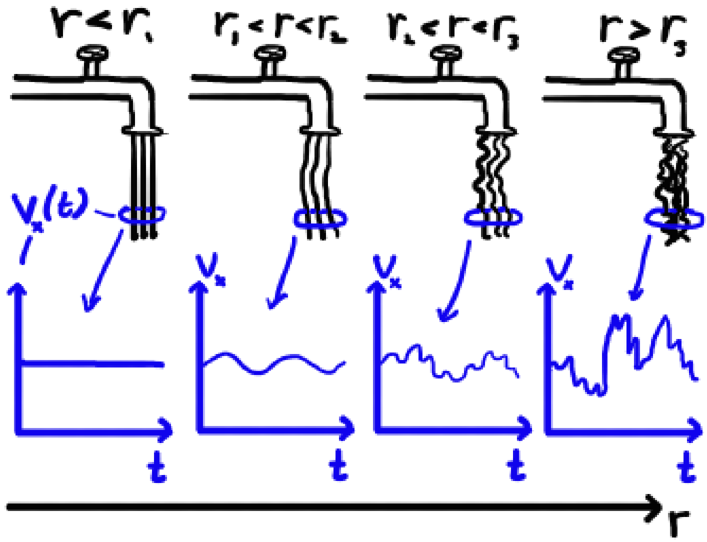


The intermittency transition can for example be found in Lorentz equations and some examples where it is found experimentally are in fluid flows, BZ-reaction, and driven non-linear semiconductors.

13.2 Ruelle-Takens

Typical example Transition from laminar to turbulent flow as the Reynolds number r (dimensionless flow speed) is increased, for example the water stream from a faucet flows regularly at small flow speeds and irregular (turbulent) for larger speeds.

Transition to chaos by a sequence of three bifurcations r_1, r_2, r_3:



r_1: Fixed point (constant flow velocity) to limit cycle with single frequency (supercritical Hopf bifurcation)
r_2: The limit cycle obtains two frequencies (periodic|quasiperiodic)

r_3: Chaos with strange attractor (Ruelle-Takens showed that periodic orbits with three frequencies are structurally unstable, i.e. extremely unlikely to be observed in real-world systems)

The Ruelle-Takens transition are experimentally found in various fluid systems.

13.3 Period-doubling bifurcation (Feigenbaum)

13.3.1 Example: Periodically driven pendulum

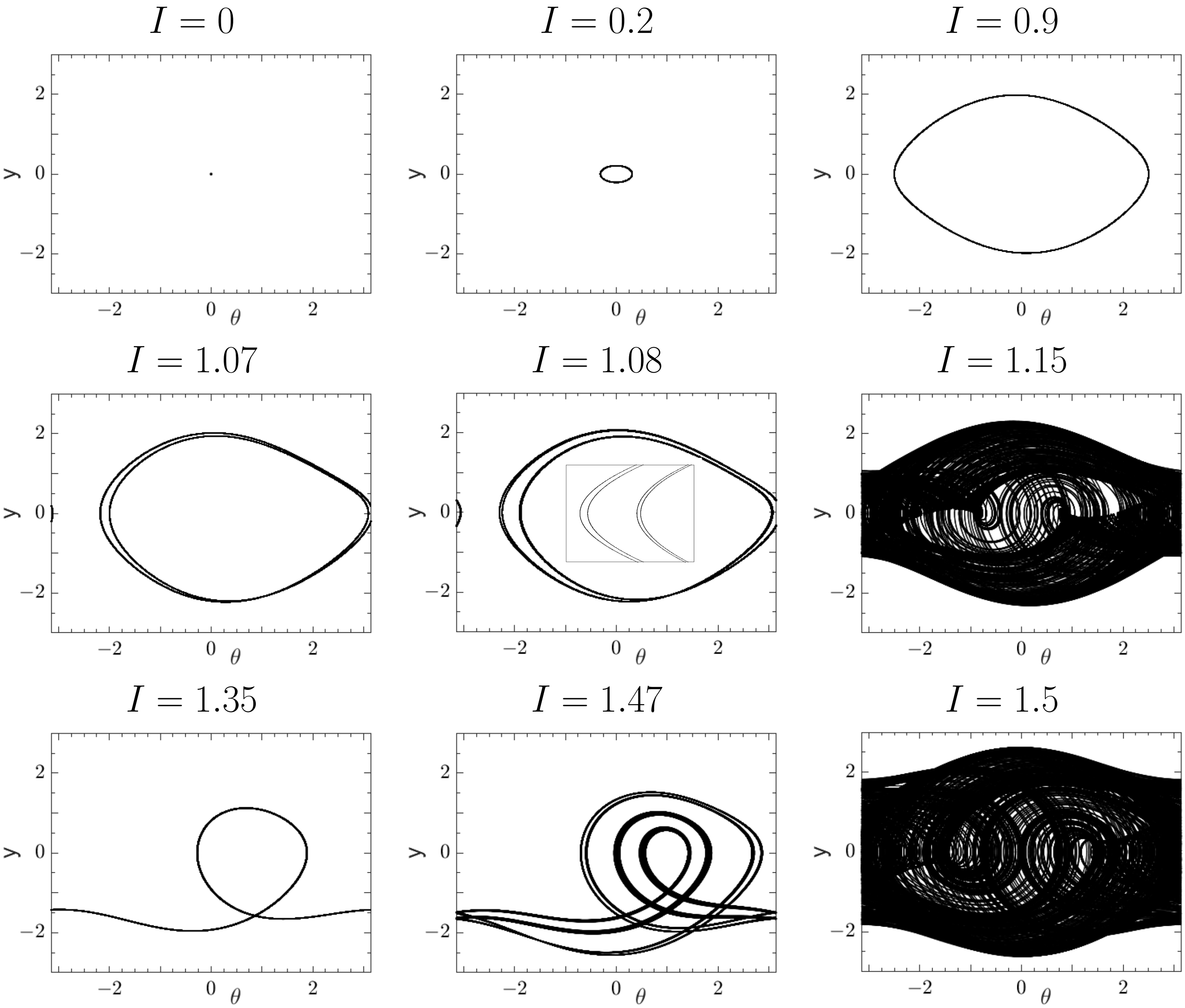
Consider a pendulum with a periodic torque in dimensionless units (c.f. Lecture 9)

$$\ddot{\theta} = -\underbrace{\alpha\dot{\theta}}_{\text{damping}} - \underbrace{\sin\theta}_{\text{gravity}} + \underbrace{I\cos(\omega_{\mathrm{F}}t)}_{\text{periodic forcing, angular frequency } \omega_{\mathrm{F}}}$$

Write as an autonomous dynamical system with $y = \dot{\theta}$ and $\tau = t$

$$\begin{aligned}
\dot{\theta} &= y \\
\dot{y} &= -\alpha y - \sin\theta + I\cos(\omega_{\mathrm{F}}\tau) \\
\dot{\tau} &= 1
\end{aligned}$$

Trajectories with $\alpha = 1/2$, $\omega_{\mathrm{F}} = 2/3$ (period time $T_{\mathrm{F}} = 2\pi/\omega_{\mathrm{F}}$) for some values of I projected on the θ-y plane for large times (neglecting an initial transient):



Note that the dynamics is three-dimensional:

- Trajectories are allowed to cross in the projection.
- Stable, periodic trajectories in the projection are not limit cycles in the three-dimensional dynamics (the time coordinate is not periodic).

Upon changing I, keeping α and ω_{F} fixed, we observe the following:

- When $I = 0$, trajectories are attracted to the stable fixed point at $\theta = y = 0$.
- When $I = 0.2$, linearisation theory applies and the system can be solved analytically: For large times the solution takes the

form $\theta(t) = A(\omega_{\rm F})\cos(\omega_{\rm F}t+\phi(\omega_{\rm F}))$, i.e. the pendulum oscillates with the frequency of the applied forcing. The amplitude A and phase shift ϕ depend on the applied frequency $\omega_{\rm F}$ and the eigenfrequency ω_0 (imaginary part of the eigenvalue at $\theta = y = 0$ when $I = 0$) of the pendulum. Period time $T_{\rm F}$.

- When $I = 0.9$ oscillations are larger and the linearized theory no longer applies. Period time $T_{\rm F}$.
- When $I = 1.07$ a period-doubling bifurcation has occurred and the pendulum oscillates with double period time $2T_{\rm F}$.
- When $I = 1.08$ another period-doubling bifurcation has occurred (see inset for zoom-in), pendulum oscillates with quadruple period time $4T_{\rm F}$.
- As I increases to around $I = 1.15$ period-doubling bifurcations happens at closer and closer values of the bifurcation parameter I. Eventually, the period becomes infinitely long at a finite bifurcation value (close to $I = 1.15$). The projected trajectory has an infinite period, i.e. it is aperiodic and the motion is chaotic. If we let $t \to \infty$ the projection plot would become uniformly black within the reachable part of phase space.
- When $I = 1.35$ a stable window appears. Period time $T_{\rm F}$.
- When $I = 1.45$ period-doubling bifurcation in the stable window. Period time $2T_{\rm F}$.
- When $I = 1.47$ period-doubling bifurcation in the stable window. Period time $4T_{\rm F}$.
- When $I = 1.5$ close to chaotic again

The behavior observed for the driven pendulum outlines a general period-doubling transition to chaos of periodic orbits (to have a period-doubling requires $d > 2$, otherwise trajectories cross).
The period-doubling cascade consists of an infinite sequence of bifurcations $r_1, r_2, \ldots, r_\infty$ where period time doubles. Assume the system

has a periodic orbit in the form of a loop for $r < r_1$.
At $r = r_1$ this bifurcates to a double loop (periodic orbit of period 2).
At $r = r_2$ the periodic orbit bifurcates to a periodic orbit of period 4.
At subsequent, increasingly denser values of r_i, the periodic orbit bifurcates into periodic orbits of growing period 2^i.
Finally, as $r \to r_\infty$ (r_∞ can be finite because bifurcation values r_i becomes denser with increasing i) the dynamics becomes aperiodic (chaotic).

For values of r larger than r_∞ the system typically exhibit chaos with 'windows' of periodic motion (c.f. the behaviour of the driven pendulum with $I > 1.15$ and $I \approx 1.35$).

The period-doubling bifurcation has been observed experimentally in for example lasers, plasmas, BZ reaction, and in fluid dynamics. It is also common in discrete dynamical systems.

13.3.2 Universality

It is possible to show that for certain subclasses of systems (systems that can be projected on a unimodal one-dimensional map) the period-doubling is universal. For example, for r_i with large i the difference between bifurcations shrinks with a constant factor (Feigenbaum constant)

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots$$

This is a constant of nature, it is independent of the form of the system.

This universality is easier to discuss in one-dimensional maps, and is discussed further in Computational Biology 1.