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Géométrie persistante

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Persistent Geometry

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Lorsqu'on me demandait ce que j'espérais trouver en Extrême-Amont, cette question banale posée mille fois, je répondais maintenant : "J'espère trouver mon visage. Quelqu'un là-haut le sculpte à coup de salves dures. Chaque acte que je fais le modifie et l'affine. Mes fautes le balafrent."

Les sujets de thèses m'ont toujours amusé et attendri : c'est mignon, ces étudiants qui, pour imiter les grands, écrivent des sottises dont les titres sont hypersophistiqués et dont les contenus sont la banalité même, comme ces restaurants prétentieux qui affublent les œufs mayonnaise d'appellations grandioses.

Géométrie persistante

Résumé

Cette thèse est dédiée à l'inférence géométrique et plus particulièrement à l'estimation de courbures d'un objet dans un espace euclidien à partir d'une approximation proche suivant la distance de Hausdorff. Nous développons le concept de *géométrie persistante*, destiné à étendre les propriétés filtrantes de l'homologie persistante au domaine de la géométrie. La géométrie persistante consiste à utiliser les relations entre la topologie et les courbures d'un sous-ensemble de $\mathbb{R}^d$ fournies par la géométrie intégrale, comme la formule cinématique principale. À l'aide d'une construction appelée persistance image, nous mêlons ces relations à l'homologie persistante pour estimer les volumes intrinsèques d'un objet à partir d'une approximation quelconque, et ce à un taux linéaire suivant la distance de Hausdorff qui les sépare. Parmi les volumes intrinsèques, on trouve notamment la caractéristique d'Euler, la courbure moyenne et l'aire du bord de l'objet. Nous montrons que cette approximation est valide tant que l'objet approximé a des courbures totales bornées et un μ-reach positif pour un certain μ dans $]0, 1]$, le μ-reach étant une quantité généralisant le reach de Federer. Elle a été définie pour généraliser certains résultats d'inférence géométrique à des objets potentiellement ni lisses, ni convexes. Les objets compacts de $\mathbb{R}^d$ à μ-reach positif pour un certain $\mu \in]0, 1]$ et à courbures totales bornées forment une vaste classe, contenant notamment les sous-variétés C^1 compactes, les compacts convexes, les polyèdres, et même plus généralement la plupart des compact stratifiés. Nous utilisons et obtenons de nouveaux résultats dans différentes théories mathématiques comme l'analyse non-lisse, la théorie géométrique de la mesure et la théorie de Morse. En particulier, un résultat crucial à notre approche est le développement d'une notion de fonctions de Morse pour des voisinages tubulaires de parties de $\mathbb{R}^d$ à des valeurs régulières de leur fonction distance. Nous montrons que la topologie des sous-niveaux d'une fonction lisse restreinte à un tel objet, qui n'est pourtant pas une variété C^2, évolue généralement par l'attachement d'une cellule autour de chaque point critique, comme dans la théorie de Morse sur des variétés C^2.

Mots-clés : Inférence géométrique, Homologie persistante, Théorie géométrique de la mesure, Géométrie intégrale, Analyse topologique des données

Persistent Geometry

Abstract

This thesis is dedicated to geometric inference, and more specifically to the estimation of curvatures of objects in Euclidean space from an approximating set that is close in the Hausdorff distance. In order to extend the filtering property of persistent homology to the realm of geometry, we introduce the framework of *persistent geometry*. It consists in combining connections between the topology and the curvatures of a subset of $\mathbb{R}^d$ provided by integral geometry, such as the principal kinematic formula, with persistence theory thanks to the so-called image persistence modules. We develop a new method to estimate the intrinsic volumes of a set, which are global quantities built from the curvatures of the set; particular intrinsic volumes include boundary area, Euler characteristic, and mean curvature. Our method allows for the recovery of the intrinsic volumes of a set from any approximating set up to an error that is linear with respect to the Hausdorff distance between them. We show that this approximation is valid as long as the estimated set has bounded total curvature and a positive μ-reach for some $\mu \in (0, 1)$. The μ-reach is a relaxation of the reach of Federer defined to extend geometric inference results to possibly non-smooth, non-convex sets. The class of compact sets of $\mathbb{R}^d$ having bounded total curvatures and a positive μ-reach for an arbitrary μ in $(0, 1)$ is broad, containing compact C^1 submanifolds, compact convex sets, polyhedra, and more generally many compact stratified subsets of $\mathbb{R}^d$. To deal with these possibly singular sets, we use tools from different fields of mathematics, such as non-smooth analysis, geometric measure theory, and Morse theory. In particular, a crucial step in our reasoning consists in the development of Morse theory for offsets of a set at regular values of its distance function. We show that the topology of sublevel sets of smooth maps restricted to such objects — which are not C^2 manifolds — typically evolves by the gluing of cells around each critical point, just as in the classical Morse theory on C^2 manifolds.

Keywords: Geometric inference, Persistent homology, Geometric measure theory, Integral geometry, Topological data analysis

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Contents

Annexes

CHAPTER 1

Introduction

1.1 English Introduction

Geometric inference deals with the retrieval of information about a geometric object from a close, approximating set. This thesis aims at contributing to this field by introducing the concepts of *persistent geometry*, which we use to develop a new method for recovering the curvatures of subsets of Euclidean spaces verifying mild regularity conditions. We will prove that this method enjoys a linear accuracy with respect to the Hausdorff distance. We employ the numerous relations between the topology of a set and its geometry to extend the noise filtering properties of persistent homology to the realm of geometry. To that end, we use tools from various fields of mathematics such as non-smooth analysis, Morse theory, geometric measure theory and persistent homology.

In this introduction, we explain the ideas of persistent geometry and introduce the core principles of the above-mentioned fields, while focusing on their interactions with persistent geometry. We then present the outline of this thesis and highlight our contributions.

Basic concepts. When studying the topology of a subset of $\mathbb{R}^d$, we mostly consider the so-called *homology* of the set seen as a topological space, its topology being induced by the distance inherited from $\mathbb{R}^d$. To any topological space X and any field $\mathbb{K}$, singular homology associates vector spaces over $\mathbb{K}$ denoted by $(H_i(X, \mathbb{K}))_{i\in\mathbb{N}}$. These vector spaces are *homotopy invariant*, which intuitively means that they are constant under continuous deformations of X. We interpret their dimension to be the number of i-th dimensional *topological features* of X. More precisely, the dimension of $H_0(X, \mathbb{K})$ counts the number of connected components of X, the dimension of $H_1(X, \mathbb{K})$ the number of cycles, that is, the number of independent, circle-like holes; and generally the dimension of $H_i(X, \mathbb{K})$ corresponds to the number of independent i-th dimensional "voids". An example of set admitting a 2-dimensional void is the sphere of $\mathbb{R}^3$, which has also one connected component and no cycle. The dimension of $H_i(X, \mathbb{K})$ is called the i-th *Betti number* of X. Most of the time, we reason with a fixed field and omit it from the notation. The Euler characteristic $\chi(X)$ of X is defined the alternating sum of the Betti numbers, i.e., $\chi(X) = \sum_i(-1)^i \dim H_i(X, \mathbb{K})$ - when this expression makes sense. Remarkably, this quantity does not depend on the field $\mathbb{K}$.

Thanks to the homotopy invariance of homology, one can recover the Betti numbers of a shape in $\mathbb{R}^d$ by deforming it — when possible — into a union of glued, deformed balls of finite dimension, called a CW-Complex. For instance, we can deform a t-shirt into a patch of three circles, which has clearly three holes and one connected component.

To compare subsets of $\mathbb{R}^d$, we use the so-called *Hausdorff distance*. Let $d_A : x \mapsto \inf_{a\in A} ||x - a||$ be the distance function associated to any subset A of $\mathbb{R}^d$. The *offsets* of A

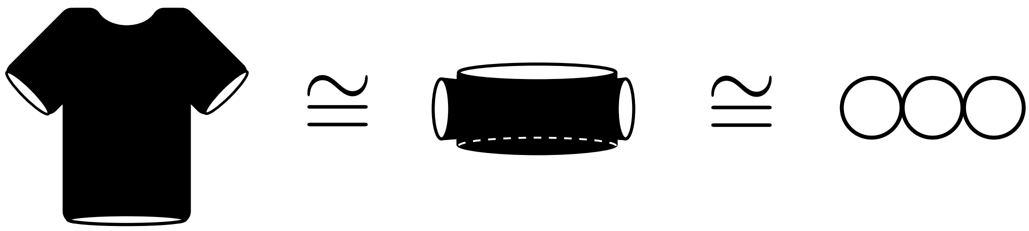
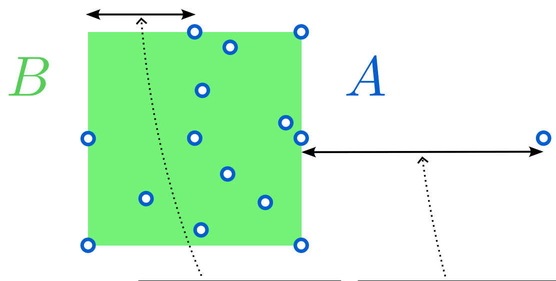


Figure 1.1 – Cell decomposition of a t-shirt



$$d_H(A,B) = \max\left(\overbrace{\inf\{t \mid B \subset A^t\}}, \overbrace{\inf\{t \mid A \subset B^t\}}\right)$$

Figure 1.2 – Hausdorff distance between a square B and a point cloud A with an outlier.

are the positive sublevel sets of d_A, which we denote by $A^t := d_A^{-1}(-\infty, t]$. The Hausdorff pseudo-distance between two subsets A and B of $\mathbb{R}^d$ is defined as

$$d_H(A,B) := \inf\{t \in \mathbb{R} \mid A \subset B^t, B \subset A^t\}.$$

This quantity can be infinite, for instance when A is bounded and B is not, or zero for two distinct sets, when A is open and B is the closure of A. Nevertheless, d_H endows the set of compact subsets of $\mathbb{R}^d$ with a complete metric.

Remark also that the definition of the Hausdorff distance does not involve any assumptions on the sets. This allows to compare subsets of $\mathbb{R}^d$ no matter their regularity, as opposed to other distance-like definitions such as the Fréchet distance or the volume of the symmetric difference of two subsets. A sequence of point clouds can converge in the Hausdorff sense to a smooth object in $\mathbb{R}^d$. It is, however, sensitive to outliers.

Persistent Geometry. The idea of persistent geometry is to associate geometrical concepts to a pair of nested sets $X \subset Z$ in a Euclidean space. The inclusion $\iota : X \hookrightarrow Z$ factors through any set Y in between X and Z, and this factorization can be further restricted to any other set A in the same Euclidean space. Now, continuous functions between topological spaces induce linear maps between the homology vector spaces commuting with the composition*, to which the previous factorization extends.

$$\begin{tikzcd} X \cap A \arrow[rr, hook, "\iota"] \arrow[dr, hook, "j"'] & & Z \cap A \\ & Y \cap A \arrow[ur, hook, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} H_\bullet(X \cap A) \arrow[rr, "\iota_*"] \arrow[dr, "j_*"'] & & H_\bullet(Z \cap A) \\ & H_\bullet(Y \cap A) \arrow[ur, "h_*"'] & \end{tikzcd}$$

$*$. This is the *functoriality* of singular homology.

The rank of $\iota_* : H_\bullet(X \cap A) \to H_\bullet(Z \cap A)$ is smaller than the dimension of $H_\bullet(Y \cap A)$. We think of this fact as the *topological filtering* property of the inclusion map $X \hookrightarrow Z$.

On the other hand, there exist so-called *integral geometry* formulas relating classical geometric quantities to topological quantities. Crofton's formula relates for instance the boundary area of X, which we will denote by $V_{d-1}(X)$, to the average number of crossings of X with lines in $\mathbb{R}^d$. Remark that this coincides with the Euler characteristic, as the sole topological features of a subset of a line are connected components. Equipping the space of lines of $\mathbb{R}^d$ with its so-called invariant measure, there is a constant c_d such that for every reasonable set X, we have:

$$V_{d-1}(X) = c_d \int \chi(X \cap L)\,\mathrm{d}L. \tag{1.1}$$

For any line L of $\mathbb{R}^d$, we let $\iota^L_{X,Z} : H_0(X \cap L) \to H_0(Z \cap L)$ be the inclusion induced map. The boundary area of the nested pair (X, Z), which we will denote by $V_{d-1}(X, Z)$, could be defined directly from Crofton's formula:

$$V_{d-1}(X, Z) \coloneqq c_d \int \mathrm{rank}(\iota^L_{X,Z})\,\mathrm{d}L. \tag{1.2}$$

The filtering property tells us that $V_{d-1}(X, Z) \leq V_{d-1}(Y)$ whenever $X \subset Y \subset Z$. From these considerations arise two natural questions.

— Does there exist an object Y^* containing X, included in Z, such that $V_{d-1}(X, Z) = V_{d-1}(Y^*)$?

— When are $V_{d-1}(Y)$ and $V_{d-1}(X, Z)$ close ?

The first one is an optimization problem: if such an object Y^* were to exist, then it would be the set sandwiched between X and Z minimizing its boundary area; and this property would be characterized by having for almost-all lines L the equality $\chi(Y \cap L) = \mathrm{rank}(\iota^L_{X,Z})$. It is not clear that such an object exists, but if it does, it is expected to enjoy some form of regularity as it satisfies an area-minimizing problem. This intuition gives an idea for the second problem : if Y is regular as well, then $V_{d-1}(Y)$ should not be exceeding $V_{d-1}(X, Z)$ by a too large margin, although there remains to see how this translates to a quantitative bound.

These ideas can be used for geometric inference with respect to the Hausdorff distance. Take for instance a point cloud X in a Euclidean space approximating a shape Y up to Hausdorff distance ε as in Figure 1.3. Both offsets X^ε, $X^{3\varepsilon}$ consist in a finite union of balls. Their boundaries have many corners, and their boundary areas generally overestimate that of Y as illustrated in two dimensions in Figure 1.4. However, following our intuition, the pair of nested sets $(X^\varepsilon, X^{3\varepsilon})$ could be used to approximate any regular object in-between them, which notably includes $Y^{2\varepsilon}$. To infer the boundary area of Y itself, we also need to control the difference in boundary areas between Y and $Y^{2\varepsilon}$.

From Crofton's formula, we obtained a candidate for the boundary area of the nested pair (X, Z) defined using the inclusions $X \cap L \to Z \cap L$ where L ranges among every line of $\mathbb{R}^d$. There exist other integral geometric formulas linking the topology of a set to its geometry, the most notable example being the *principal kinematic formula*. One of its special case allows us to use the inclusions $X \cap B \to Z \cap B$, where B runs among all bounded balls of $\mathbb{R}^d$, to determine a candidate for the area boundary of the pair (X, Z). The bounded balls of $\mathbb{R}^d$ can be parametrized as sublevel sets of maps $d_x : y \mapsto ||y - x||$, where x runs among $\mathbb{R}^d$; which is particularly suited to the use of persistent homology, as will be described in the next paragraph. It is not clear whether these area boundary candidates coincide or not, raising the question of the consistence of persistent

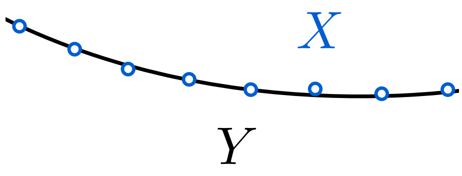


Figure 1.3 – Point cloud lying close to a curve.

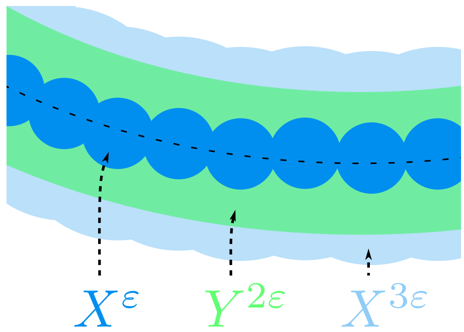


Figure 1.4 – $X^\varepsilon \subset Y^{2\varepsilon} \subset X^{3\varepsilon}$.

geometry. Can we give sense to the geometry of a nested pair (X, Z) using the aforementioned concepts ? Does it depend on the choice of a family of subsets of $\mathbb{R}^d$, such as the family of lines or balls as above, or the sublevel sets of a family of functions $\mathbb{R}^d \to \mathbb{R}$? In any case, our use of the framework of persistent geometry will settle for the family of bounded balls.

Persistent Homology. Persistent homology consists in the study of the evolution of the topological features of a filtration $(X_t)_{t\in\mathbb{R}}$, that is, a non-decreasing family of subsets of $\mathbb{R}^d$. It associates to any filtration the values of birth and death of features in a so-called *persistence diagram*. In particular, if $X_t = f^{-1}(-\infty, t]$ is the closed sublevel set filtration of a map $f : \mathbb{R}^d \to \mathbb{R}$, we speak of the persistence diagram of f. Diagrams can be represented as barcodes, as illustrated below for the height sublevel-set filtration of a t-shirt.

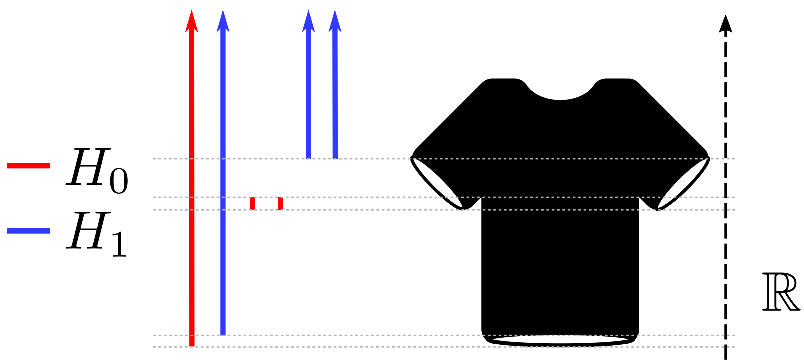


Figure 1.5 – Barcode associated to a height sublevel-set filtration of a t-shirt

Now recall the context of persistent geometry, with $X \subset Z \subset \mathbb{R}^d$. The existence of the inclusion map $X \subset Z$ allows to build from the persistent diagrams of $f_{|X}, f_{|Z}$ another object called the *image persistence diagram* of $f_{|(X,Z)}$. This approach systemizes the factorizing property mentioned in the ideas behind persistent geometry; indeed, we will see that the image persistence diagram of $f_{|(X,Z)}$ is in a sense simpler than the one of $f_{|Y}$ for any set Y in between X and Z. Furthermore, the space of persistence diagrams is equipped with the so-called *bottleneck distance*, which is built by comparing the birth and death of features between diagrams. Under some mild regularity conditions on Y, we will use this metric to compare the diagrams of $f_{|Y}$ and $f_{|(X,Z)}$, thereby justifying the intuition of persistent geometry.

Morse theory. Classically, *Morse theory* deals with the analysis of the topology of a manifold X through the study of the so-called *Morse functions*, which form the class of C^2 maps $X \to \mathbb{R}$ whose Hessian at critical points is non-degenerate. Such maps are not out of the ordinary as Morse functions form a dense, open subset of all C^2 maps on X equipped with a certain norm. Furthermore, if X is a submanifold of $\mathbb{R}^d$, it can be shown that almost all height functions restricted to X are Morse. When f is a proper Morse function, the two core results of Morse theory are the following:

(1) Let $a < b$. If for every $x \in f^{-1}[a, b]$, we have $\nabla f(x) \neq 0$, then the closed sublevel set $f^{-1}(-\infty, a]$ is a deformation retract of $f^{-1}(-\infty, b]$;

(2) If there exists a $\delta > 0$, and $x \in X$ a critical point of f with $f(x) = c$ such that x is the unique critical point of f within $f^{-1}[c - \delta, c + \delta]$, then for all $0 < \varepsilon \leq \delta$ the topology* of $f^{-1}(-\infty, c + \varepsilon]$ is that of $f^{-1}(-\infty, c - \varepsilon]$ with a λ-cell (that is, a set homeomorphic to a unit ball in R^λ) glued around x, where λ is the index of the Hessian of f at x.

The locality of the gluing ensures that assertion (2) (also called *handle-attachment lemma*) generalizes to any finite number of critical points sharing a critical value. In particular, the persistence diagram of any such Morse function has exactly as many birth/deaths in an interval I as there are critical points within $f^{-1}(I)$.

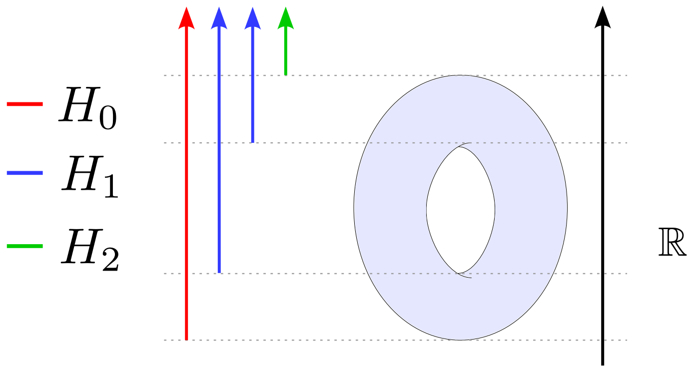


Figure 1.6 – Height function on a torus embedded in $\mathbb{R}^3$. The homotopy type of the sublevel sets does not evolve between critical values, and there is exactly one homological event in its associated barcode per critical point.

The classical proof of (1) (also known as the *constant homotopy lemma*) consists in following the flow of $-f$, which provides a homotopy whose trajectories make f decrease at a lower bounded rate by assumption. To see (2), observe that the graph of the second-order approximation of f around the critical point x is a multidimensional parabola going down in λ directions and going up in $d - \lambda$ directions. Note that this second order approximation is also related to the curvature of the manifold, hinting a relationship between curvatures and topological events that we will exploit in great details throughout the thesis.

Such proofs require X to be a manifold. As we work with sets satisfying only mild regularity conditions arises naturally the question of possible extensions of Morse theory to classes containing some singular sets, requiring tools from non-smooth analysis. The handle attachment lemma requires a precise understanding of the geometry of the sublevel sets around its critical points,

$*$. Morse precisely, the homotopy type.

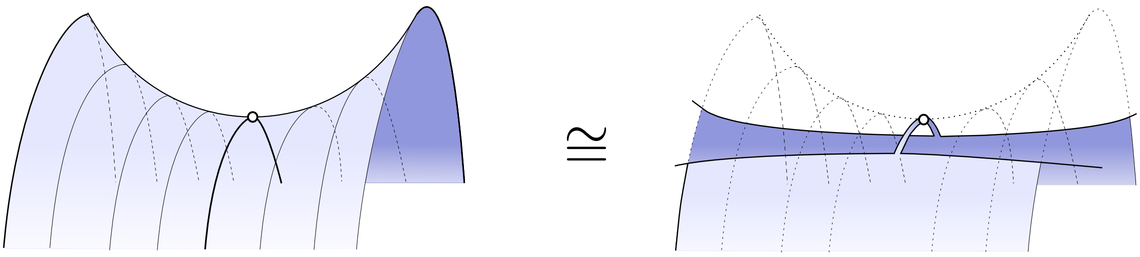


Figure 1.7 – Idea of the proof of assertion (2) studying the topological event around the second critical point of the height filtration of Figure 1.6. Here, a 1-dimensional cell is glued around the critical point.

whose behavior is vastly wilder when giving up the manifold assumption. For instance, a generic behavior of smooth functions restricted to polyhedra of $\mathbb{R}^d$ may consist in the gluing of several cells at once around a single critical point, as illustrated in Figure 1.8.

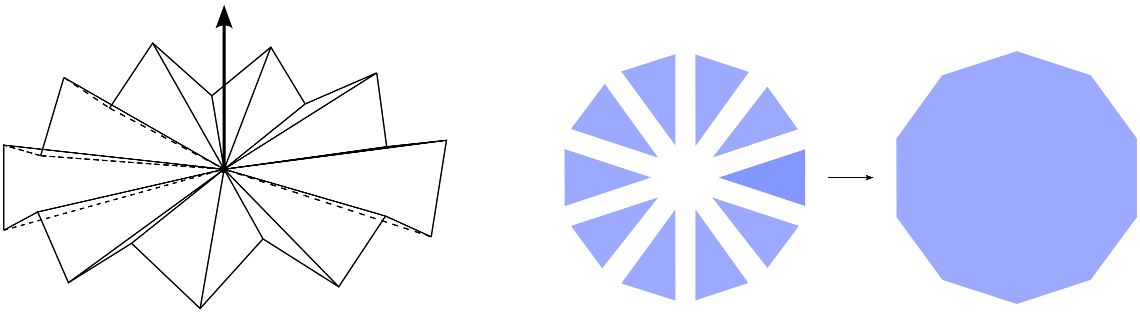


Figure 1.8 – A 20 triangles, symmetric ruff in $\mathbb{R}^3$. The sublevel set filtration of the height function indicated with an arrow goes from 10 connected component to 1 when encountering the center.

Stratified Morse theory was developed in the eighties to extend the notion of Morse functions to stratified subsets of Riemannian manifolds; yet in this context the number of topological events in the persistent homology diagrams of a Morse function can be arbitrarily larger than the number of critical points of the functions. Nevertheless, we will prove that the original handle attachment lemma typically holds for smooths functions restricted to offsets of sets, using a careful comparison with smooth sets.

Non-smooth analysis. To compare persistence diagrams, we will build continuous deformations between certain subsets of $\mathbb{R}^d$. Examples of such deformations are given by the flow of smooth functions, as mentioned in assertion (1) of the previous paragraph. However, we will need similar properties for maps that are merely Lipschitz, for which flows do not exist. For instance, we will work with maps built from elementary operations involving distance maps, which are generally not C^1 but always 1-Lipschitz. Thankfully, the class of Lipschitz maps has already been studied, with the notable concept of *Clarke gradient* $\partial^* f$ of a locally Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}$. It is a set-valued map defined at x as the convex hull of limits of gradients of f around x. It generalizes the classical gradient of differentiable functions, in the sense that when f is differentiable at x, we have $\partial^* f(x) = \{\nabla f(x)\}$.

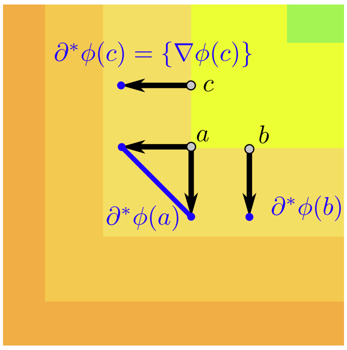


Figure 1.9 – Clarke gradient of a Lipschitz map $\mathbb{R}^2 \to \mathbb{R}$ whose increasing sublevel sets are represented from green to orange.

Several results on C^1 maps have been generalized to Lipschitz maps with the use of the Clarke gradient. For instance, when the Clarke gradient of f does not contain the point 0 on a level set $f^{-1}(c)$, then the level set is a *Lipschitz submanifold*, that is, a set which is locally the graph of a Lipschitz application from a Euclidean space. This is a Lipschitz extension to the classical local inversion theorem. Going further, we will see that other functional properties of a Lipschitz map translate into geometrical properties for its associated sublevel sets, with distance functions playing an important role.

In the smooth setting, the flow of $-f$ has trajectories making f decrease exactly at speed $||\nabla f||$. We will construct *approximate inverse flows* of a Lipschitz map f using its Clarke gradients and a smooth approximation of the map itself. Up to an arbitrarily small constant, there exist an approximate inverse flow with the same bound on the decreasing speed as in the smooth case, except that $||\nabla f||$ is replaced by the distance between 0 and $\partial^* f$, which we will denote by $\Delta(\partial^* f)$. Note that when f is differentiable, these two quantities coincide. This approximate flow result underlines the importance of $\Delta(\partial^* f)$ in studying sublevel sets of f. For instance, when $f = d_A$, we define the *μ-reach* of A to be the largest t such that $\Delta(\partial^* d_A) \geq \mu$ on $A^t \setminus A$; this quantity will prove to be central in our geometric inference result.

Curvatures and integral geometry. In this thesis, we will infer quantities relative to the *curvatures* of certain subsets of Euclidean spaces, which measure how unflat a set is. For instance, oriented surfaces of $\mathbb{R}^3$ are flat around a point x when the map $y \mapsto \nu(y)$ associating outward pointing normals to the surface, called the *Gauss map*, is constant in a neighborhood of x. The pointwise *principal curvatures* of a surface at x are given by the eigenvalues of the differential of the Gauss map at x.

With the same idea, it can be shown that the curvature of a smooth curve is determined by its first and second derivatives. However, when the curve is piecewise linear, it is obviously not flat despite being flat almost everywhere; meaning that the curvature is concentrated at the corner of the curves. The same conclusion holds for any polyhedron of $\mathbb{R}^3$ compared to oriented surfaces: its curvatures are located at corners and edges, that is, at non-smooth points of the set.

Federer gave a united presentation of the apparent distinction between smooth and piecewise affine objects. As long as there is a number $r > 0$ such that for any x in $\mathbb{R}^d$ with $d_X(x) < r$, there exists a unique closest point $\xi_X(x)$ to x in X, he showed that for any Borelian subset U of $\mathbb{R}^d$, the

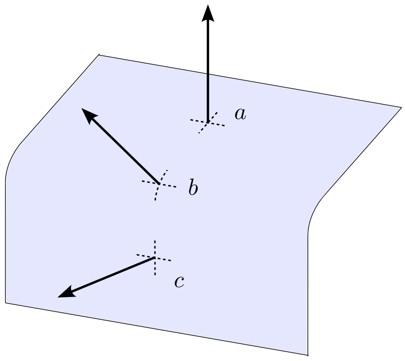
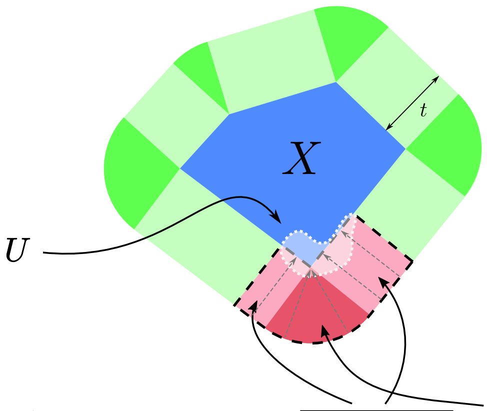


Figure 1.10 – Pointwise curvature of a surface. The Gauss map is constant around a and c, but not around b.

map

$$t \in [0, r] \mapsto \mathrm{Vol}(\xi_X^{-1}(U) \cap X^t) = \sum_{i=0}^{t} \omega_i t^i C_{d-i}(X, U)$$

was a polynomial of degree d, and that its $(d+1)$ coefficients, rescaled by the volume ω_i of a unit ball in $\mathbb{R}^i$, defined measures as a function of U which he called *curvature measures*. At a corner of a convex polyhedron, these measures coincide with multiples of the solid angles made by the cones of outward normals, as illustrated in Figure 1.11.



$$\mathrm{Vol}\left(\xi_X^{-1}(U) \cap (X^t \setminus X)\right) = \overbrace{2tC_1(X, U)} + \overbrace{\pi t^2 C_0(X, U)}$$

Figure 1.11 – Curvature measures of the convex hull of some points in $\mathbb{R}^2$.

The consistent definition of curvatures for the so-called *singular sets*, i.e., sets containing singular, non-convex points, is much more complex and has been the subject of numerous extensions over the last decades. Perhaps the most intuitive extension is the one involving generic, finite unions of sets for which there exists such an r *. Indeed, on the one hand generic intersections of sets with positive reach have positive reach; and on the other hand if $X, Y, X \cap Y$ admit curvatures measures, then so does $X \cup Y$, with $C_i(X \cup Y, \cdot) + C_i(X \cap Y, \cdot) = C_i(X, \cdot) + C_i(Y, \cdot)$ as illustrated in Figure 1.12.

$*$. Called sets of *positive reach*

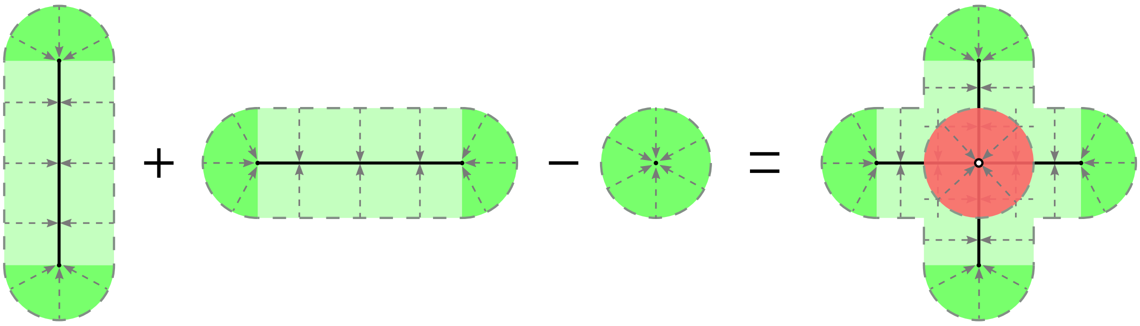


Figure 1.12 – Example of the additivity of curvature measures with a cross.

General outline and contributions

Chapter II. This chapter begins by fixing terminology before focusing on the study of Lipschitz maps in Euclidean spaces. We describe the major concepts of Lipschitz analysis that we use in the following chapters, such as the Clarke gradients, or the co-area formula between rectifiable sets, which generalizes the classical change of variables formula. The contribution of this chapter lies in the construction of *approximate inverse flows* between sublevel sets of Lipschitz maps, assuming that their Clarke gradients stay uniformly away from zero.

Chapter III. The third chapter is devoted to the study of the relationship between the geometry of a set and its associated distance function. We recall the definition of classical notions about the geometry of subsets of Euclidean spaces, such as the reach of a set, and a relaxation of the reach called the μ-reach. We also recall the definitions of tangent and normal cones. To prove that a set with a positive μ-reach for some $\mu > 0$ is not necessarily regular, we introduce a fractal-like, injective curve $K(\theta)$ in $\mathbb{R}^2$, whose construction depends on a sequence of parameters $(\theta_i)_{i\in\mathbb{N}}$. We prove that for every $0 < \mu < 1$, there exists a choice of parameters such that $K(\theta)$ has an infinite μ-reach. We show that further choices of θ can yield an unrectifiable curve, or a rectifiable curve such that the normal bundle of its offset $K(\theta)^\varepsilon$ has length diverging to ∞ as ε tends to 0.

We define the class of *complementary regular sets* as subsets of $\mathbb{R}^d$ having a positive μ-reach for an arbitrary $\mu \in (0, 1)$ verifying other regularity conditions on their complement set. We prove that complementary regular sets coincide with offsets of sets at regular value of their distance function. We show further that this condition is equivalent to being the sublevel set of a semi-concave function at one of its regular values. Another contribution is an identity linking the normal cones of a complementary regular set and the Clarke gradient of its associated distance function.

Chapter IV. In the fourth chapter, we extend the classical results of Morse theory to C^2, real-valued functions on Euclidean spaces restricted to complementary regular sets as defined in Chapter 3. This result obtained through a careful study of certain Lipschitz maps built by elementary operations involving distance functions, and is independent of persistent geometry. It will be used as a crucial lemma for the inference bounds of Chapter 7. Another exposition can be found in the article *Generalized Morse theory for tubular neighborhoods*.

Chapter V. The fifth chapter is devoted to persistence theory. It begins with an introductory section containing reminders about classical results of homology used in persistence theory, as well as an exposition of the classical framework of persistent homology. The second section serves as an introduction to the fundamental concepts of persistence theory to non-specialists. In the third section, we are interested in the concept of *image persistence*. Our contribution lies in the proof of a stability theorem on image persistence diagrams.

Chapter VI. The sixth chapter is dedicated to the curvatures of subsets of Euclidean spaces. It consists in an introduction to the concept of curvatures, including the definition of curvatures for the class of complementary regular sets. Using considerations from geometric measure theory, we prove that restricted to a complementary regular set, almost all distance-to-a-point functions and almost all height functions are Morse. We also compare the properties of curvature of complementary regular sets with the existing literature on curvatures using the language of the so-called normal cycles described in the annex.

Chapter VII. This chapter gathers the results and definitions of the previous chapters to finally follow the precepts of persistent geometry. We approximate the *intrinsic volumes* $V_0(X), \ldots, V_d(X)$ of a set $X \subset \mathbb{R}^d$, which are quantities related to its curvatures. More precisely, let X be a set of positive μ-reach and let Y be an approximating set. We build quantities $V_i^\varepsilon(Y)$ from Y^ε, $Y^{3\varepsilon}$ such that when $d_H(X, Y) \leq \varepsilon \leq \frac{1}{4}\,\mathrm{reach}_\mu(X)$, we have:

$$\left| V_i(X^{2\varepsilon}) - V_i^\varepsilon(Y) \right| = O(\varepsilon/\mu),$$

where the constant in the big O is a function of the curvatures of $X^{2\varepsilon}$. Using a similar method, we are able to show under additional regularity conditions on X that we also have:

$$\left| V_i(X^{2\varepsilon}) - V_i(X) \right| = O(\varepsilon/\mu),$$

thereby showing that we can effectively estimate the intrinsic volumes of X from the knowledge of an approximating set Y. These proofs are followed by a discussion on the minimal regularity conditions necessary to guarantee this rate of convergence following the same method. We also discuss the tractability of the construction. The same results are obtained in the paper *Persistent intrinsic volumes* co-written with David Cohen-Steiner.

Annex. Although not required to obtain the geometric inference result of Chapter 7, currents provide a concise framework to represent the curvatures of Euclidean subsets. The annex is dedicated to the description of basic concepts surrounding currents, which we use at the end of Chapter 6 and Chapter 7.

1.2 Introduction en français

L'inférence géométrique est l'étude d'objets de nature géométrique à partir d'approximations. Dans cette thèse, nous ambitionnons d'y contribuer en introduisant le concept de *géométrie persistante*. Nous utiliserons ces idées pour développer une nouvelle méthode d'inférence de courbures valable pour une classe d'objets vérifiant de faibles conditions de régularité. Cette approche permet d'approximer des quantités globales liées aux courbures, appelées *volumes intrinsèques*, et ce à un taux linéaire - qui se trouve être optimal - en la distance de Hausdorff entre les objets approximants et approximés. À cette fin, nous employons des outils de différents pans des mathématiques, comme l'analyse de fonctions non lisses, la théorie de Morse, la théorie géométrique de la mesure, la théorie de Morse, et, bien sûr, l'homologie persistante.

Cette introduction est consacrée aux principes généraux de la géométrie persistante ainsi qu'à l'exposition des notions centrales des théories mathématiques sus-citées. Nous terminerons par décrire la structure de la thèse en soulignant nos diverses contributions.

Concepts élémentaires. Pour étudier la topologie de parties de $\mathbb{R}^d$, nous nous intéresserons à leur *homologie*, en considérant sur ces parties la topologie induite par la distance euclidienne de l'espace ambiant. L'homologie singulière associe à toute paire espace topologique X/ corps $\mathbb{K}$ une famille d'espaces vectoriels $(H_i(X, \mathbb{K}))_{i\in\mathbb{N}}$ sur $\mathbb{K}$. En tant que fonction de X, cette famille est *invariante par homotopie*, ce qui signifie intuitivement que les déformations continues appliquées sur l'espace topologique ne modifient pas la famille d'espaces vectoriels. D'autre part, la dimension de $H_i(X, \mathbb{K})$ est interprétée comme le nombre de "trous" de dimension i que compte X : par exemple, pour $i = 0$ se voient dénombrées les composantes connexes ; pour $i = 1$ le nombre de boucles non-triviales indépendantes. Les entiers $(\dim H_i(X, \mathbb{K})_{i\in\mathbb{N}})$ sont appelés les *nombres de Betti* de X. Dans la suite de cette thèse, nous omettrons la dépendance des nombres de Betti au corps $\mathbb{K}$ - dûe à des phénomènes homologiques dits *de torsion* ; cela est d'autant plus justifié que l'homologie nous servira d'outil pour étudier la *caractéristique d'Euler* $\chi(X) = \sum_i \dim H_i(X, \mathbb{K})$ (quand cette somme a un sens) qui se trouve être une quantité indépendante du choix du corps $\mathbb{K}$.

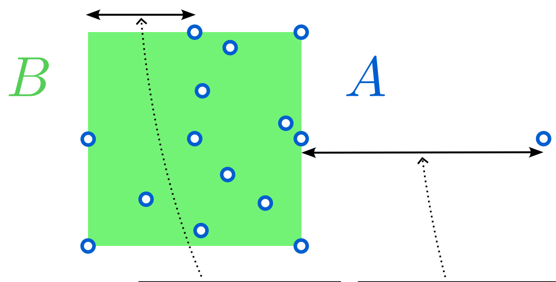
L'invariance par homotopie des groupes d'homologie nous permet de déterminer les nombres de Betti d'une forme de $\mathbb{R}^d$ en la déformant, lorsque c'est possible, en union de cellules (des boules déformées de dimension finie) collées entre elles. On peut par exemple compter le nombre de trous d'un t-shirt en le déformant continûment en une union de trois cercles collés entre eux.

Pour comparer des parties de $\mathbb{R}^d$, nous utilisons la *distance de Hausdorff*. Pour ce faire, on note $d_A : x \mapsto \inf_{a\in A} ||x - a||$ la fonction distance associée à une partie A de $\mathbb{R}^d$. On appelle *offsets* (ou *voisinages tubulaires*) de A les sous-niveaux fermés de A, qu'on note $A^t := d_A^{-1}(-\infty, t]$. On définit la pseudo-distance de Hausdorff entre deux parties A et B de $\mathbb{R}^d$ via la formule

$$d_H(A, B) := \inf\{t \in \mathbb{R} \mid A \subset B^t, B \subset A^t\}.$$

Cette quantité peut être infinie (par exemple, lorsque A est borné mais pas B), ou nulle pour deux ensembles pourtant différents (quand B est l'adhérence d'un ouvert A). Néanmoins, la pseudo-distance de Hausdorff est une métrique complète lorsqu'elle est restreinte aux compacts de $\mathbb{R}^d$.

Contrairement à d'autres distances entre ensembles, comme celle de Fréchet ou le volume de la différence symétrique, la définition de la distance de Hausdorff ne fait pas intervenir d'hypothèses



$$d_H(A,B) = \max\left(\overline{\inf\{t \mid B \subset A^t\}}, \overline{\inf\{t \mid A \subset B^t\}}\right)$$

Figure 1.13 – Distance de Hausdorff entre un carré B et un nuage de points A avec une observation aberrante.

(explicitement ou implicitement) pour comparer les objets considérés ; une suite de nuages de points peut converger au sens de Hausdorff vers un objet aux bords lisses. En revanche, cette comparaison est sensible aux "outliers" ("observations aberrantes" en français).

Géométrie persistante. L'idée de la géométrie persistante est d'associer des concepts géométriques à une paire d'objets imbriqués $X \subset Z$ contenus dans un espace euclidien. L'inclusion $\iota : X \hookrightarrow Y$ est factorisée par tout objet Y situé "entre" X et Y pour l'inclusion. Cette factorisation reste vraie en considérant tous ces objets intersectés avec un autre objet A de cet espace euclidien. En outre, les fonctions continues entre espaces topologiques induisent des applications linéaires qui commutent avec la composition [*]

$$\begin{tikzcd} X \cap A \arrow[rr, hook, "\iota"] \arrow[dr, hook, "j"'] & & Z \cap A \\ & Y \cap A \arrow[ur, hook, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} H_\bullet(X \cap A) \arrow[rr, "\iota_*"] \arrow[dr, "j_*"'] & & H_\bullet(Z \cap A) \\ & H_\bullet(Y \cap A) \arrow[ur, "h_*"'] & \end{tikzcd}$$

Le rang de $\iota_* : H_\bullet(X \cap A) \to H_\bullet(Z \cap A)$ est plus petit que la dimension de $H_\bullet(Y \cap A)$; nous interprétons cette propriété comme le *filtrage topologique* de l'inclusion $X \hookrightarrow Y$.

D'autre part, les formules dites de *géométrie intégrale* relient des quantités géométriques classiques à des quantités topologiques : la formule de Crofton relie par exemple l'aire du bord de X, qu'on note $V_{d-1}(X)$, au nombre moyen d'intersections de X avec les droites de $\mathbb{R}^d$, nombre qui coincide avec la caractéristique d'Euler de l'intersection avec la droite. Plus précisément, si l'on équipe l'espace des droites de $\mathbb{R}^d$ avec la mesure invariante sous l'action du groupe des isométries rigides, il existe une constante c_d telle que pour une large classe de parties de $\mathbb{R}^d$, on ait :

$$V_{d-1}(X) = c_d \int \chi(X \cap L)\, \mathrm{d}L. \tag{1.3}$$

Pour chaque droite L de $\mathbb{R}^d$, posons $\iota^L_{X,Z} : H_0(X \cap L) \to H_0(Z \cap L)$ l'application linéaire induite par l'inclusion. Par la formule de Crofton, on pourrait définir le volume du bord de la paire (X, Z) par la formule suivante :

$$V_{d-1}(X, Z) := c_d \int \mathrm{rang}(\iota^L_{X,Z})\, \mathrm{d}L. \tag{1.4}$$

$*$. On parle de la *fonctorialité* de l'homologie.

La propriété de filtrage implique que pour tout triplet $X \subset Y \subset Z$, on a $V_{d-1}(X, Z) \leq V_{d-1}(Y)$. Des considérations précédentes se posent deux questions :

— Existe-t-il un ensemble Y^* entre X et Z tel que $V_{d-1}(X, Z) = V_{d-1}(Y^*)$?

— Existe-t-il un moyen de borner la différence entre $V_{d-1}(Y)$ et $V_{d-1}(X, Z)$?

La première question est un problème d'optimisation. Si un tel Y^* existe, ce serait l'objet entre X et Z qui minimiserait l'aire de son bord ; et cette propriété serait caractérisée par l'égalité presque partout (pour la mesure invariante) $\chi(Y \cap L) = \mathrm{rank}(\iota^L_{X,Z})$. Il n'est pas clair qu'un tel objet existe, mais si c'est le cas on s'attend à ce que celui-ci soit régulier. Dans une veine similaire, il semble raisonnable que des hypothèses de régularité (à déterminer) sur Y impliquent que $V_{d-1}(Y)$ ne dépasse pas trop $V_{d-1}(X, Z)$ - reste à voir de quelle manière.

Ces idées peuvent être utilisées dans le cadre d'inférence géométrique par rapport à la distance de Hausdorff. Prenons par exemple un nuage de point X approximant une forme Y à une distance de Hausdorff ε comme dans la Figure 1.14. Les offsets X^ε, $X^{3\varepsilon}$ forment une union finie de boules. Leurs bords contiennent de nombreux coins, et l'aire de ses bords dépassent (généralement) celle de Y comme suggéré en dimension deux par la Figure 1.15. Suivant les concepts de géométrie persistante, la paire imbriquée $(X^\varepsilon, X^{3\varepsilon})$ pourrait être utilisée pour approximer n'importe quel objet lisse situé entre eux, comme $Y^{2\varepsilon}$.

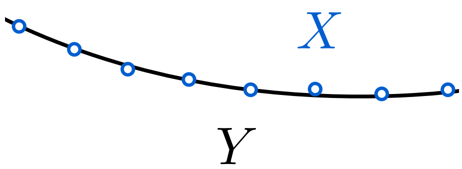


Figure 1.14 – Nuage de points proche d'une courbe.

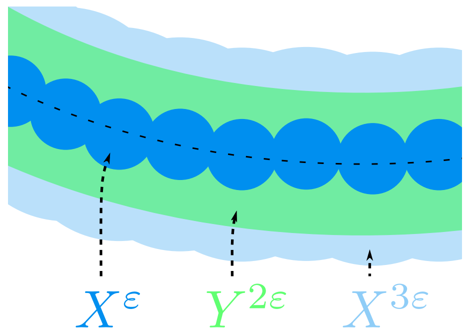


Figure 1.15 – $X^\varepsilon \subset Y^{2\varepsilon} \subset X^{3\varepsilon}$.

La formule de Crofton nous offre un candidat pour l'aire du bord d'une paire imbriquée (X, Z), grâce aux inclusions $X \cap L \hookrightarrow Z \cap L$, où L parcourt les droites de $\mathbb{R}^d$. D'autres formules de géométrie intégrale relient la topologie d'un ensemble à sa géométrie, comme la non moins célèbre *formule cinématique principale*. Un cas particulier de celle-ci nous permet d'utiliser les inclusions $X \cap B \hookrightarrow Z \cap B$, où B parcourt toutes les boules bornées de $\mathbb{R}^d$ afin de définir un nouveau candidat pour l'aide du bord de (X, Z). Les boules bornées de $\mathbb{R}^d$ correspondent aux différents sous-niveaux des applications $d_x : y \mapsto ||y - x||$, où x, le centre de la boule, parcourt $\mathbb{R}^d$. C'est ce qui rend l'homologie persistante (décrite un peu plus loin) pertinente dans ce contexte. Il n'est pas clair que ces deux candidats pour $V_{d-1}(X, Z)$ coincident, soulevant la question de la cohérence géométrie persistante à travers les différentes relations entre la topologie et la géométrie d'un objet. Est-ce que la géométrie persistante dépend du choix d'une famille de sous-ensemble de $\mathbb{R}^d$, comme parcourus par B et L plus haut, ou encore la famille des sous-niveaux de fonctions lisses $\mathbb{R}^d \to \mathbb{R}$? Dans cette thèse, nous étudierons la géométrie persistante avec le choix de familles de boules de $\mathbb{R}^d$.

Homologie persistante L'homologie persistante consiste en l'étude de l'évolution des caractéristiques topologiques d'une filtration $(X_t)_{t\in\mathbb{R}}$, c'est-à-dire une famille de parties de $\mathbb{R}^d$ indexée sur $\mathbb{R}$ et croissante pour l'inclusion. L'homologie persistante associe à chaque filtration les valeurs de naissance de mort des caractéristiques pour composer ce qu'on appelle un *diagramme de persistance*. En particulier, si $X_t = f^{-1}(-\infty, t]$ est la filtration des sous-niveaux de $f : \mathbb{R}^d \to \mathbb{R}$, on parle du diagramme de persistance de f. Une façon de représenter les diagrammes de persistance est d'utiliser des codes-barres (*barcodes*, en anglais), comme ci-dessous pour la filtration d'un t-shirt par une fonction hauteur.

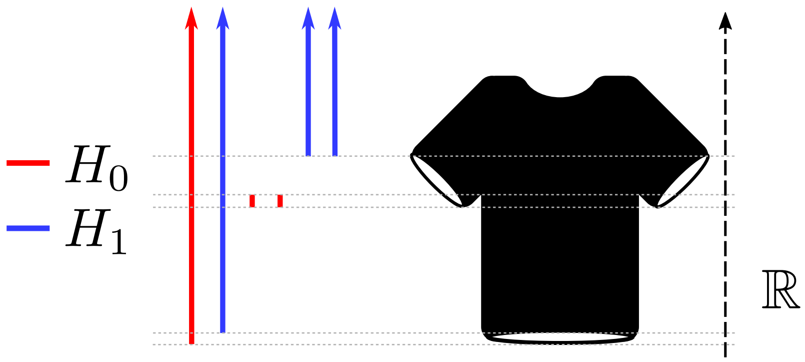


Figure 1.16 – Code-barres associé à une filtration hauteur sur un t-shirt.

Rappelons nous l'idée de géométrie persistante, avec $X \subset Z \subset \mathbb{R}^d$. L'existence de l'inclusion $X \subset Z$ permet de construire un *diagramme de persistance image* $f_{|(X,Z)}$ à partir de ceux de $f_{|X}$ et $f_{|Z}$. Cette approche systématise la propriété de filtrage mentionnée plus haut ; nous verrons en effet que le diagramme de persistance image est, en un certain sens, plus simple que celui de $f_{|Y}$ quel que soit l'ensemble Y entre X et Z. D'autre part, il existe sur les diagrammes de persistance une métrique appelée *distance bottleneck* construite en comparant les naissance et morts des caractéristiques topologiques entre diagrammes. Sous certaines hypothèses sur Y, nous utiliserons cette métrique pour comparer les diagrammes $f_{|Y}$ et $f_{|(X,Z)}$, justifiant ainsi l'intuition de la géométrie persistante.

Théorie de Morse. La *théorie de Morse* classique étudie la topologie d'une variété X via les propriétés de fonctions dites *de Morse*, consistant en l'ensemble des applications $C^2 : X \to \mathbb{R}$ dont la hessienne est non-dégénérée aux points critiques. Les fonctions de Morse sont légions, dans le sens où elles forment un ensemble ouvert et dense pour une certaine norme. De plus, si X est une sous-variété de $\mathbb{R}^d$, on peut montrer que presque toutes les formes linéaires restreintes à X sont Morse. Lorsque f est une fonction de Morse, les deux résultats centraux de la théorie de Morse sont les suivants :

(1) Soient $a < b$. Si $\nabla f(x) \neq 0$ pour tout $x \in f^{-1}[a, b]$, alors le plus petit sous-niveau $f^{-1}(-\infty, a]$ est un rétract par déformation du plus gros $f^{-1}(-\infty, b]$;

(2) S'il existe $\delta > 0$, x un point critique de f, en posant $c = f(x)$, de sorte que x est le seul point critique de f dans $f^{-1}[c - \delta, c + \delta]$, alors pour tout $0 < \varepsilon \leq \delta$ la topologie* de $f^{-1}(-\infty, c + \varepsilon]$ est celle $f^{-1}(-\infty, c - \varepsilon]$ à laquelle a été recollée une λ-cellule (c'est-à-

$*$. Plus précisément, le type d'homologie.

dire un ensemble homéomorphe à une boule unité de $\mathbb{R}^d\lambda$) autour de x, avec λ l'indice de la hessienne de f en x.

Grâce au caractère local du recollement de cellule, l'assertion (2) se généralise à un nombre fini de points critiques partageant une même valeur critique. En particulier, le diagramme de persistance d'une fonction de Morse a autant de naissance/mort dans un intervalle I qu'il y a de points critiques de f dans $f^{-1}(I)$.

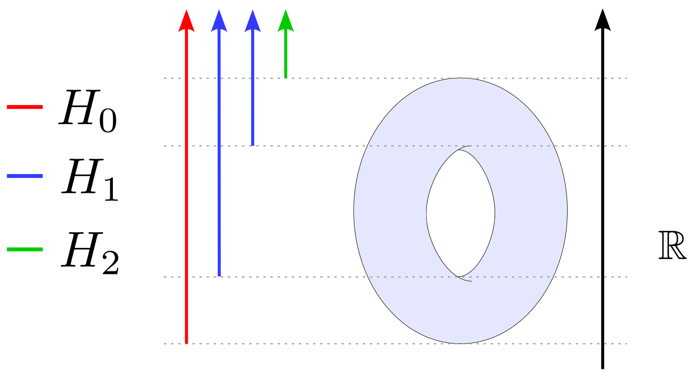


Figure 1.17 – Fonction hauteur restreinte à un tore dans $\mathbb{R}^3$. Le type d'homotopie des sous-niveaux ne change pas entre les points critiques, et il y a exactement un évènement homologique dans son code-barres par point critique.

La preuve classique de (1) consiste à suivre le flot de la fonction $-f$, ce qui fournit une homotopie dont les trajectoires font décroître f à un taux minoré par une borne strictement positive. Pour comprendre (2), on peut observer que le graphe d'une approximation d'ordre deux de f autour d'un point critique est une parabole multidimensionnelle dont le nombre de directions descendantes (resp. ascendantes) est λ (resp. $d - \lambda$). Remarquons que l'approximation d'ordre deux est liée à la courbure de la variété, intuitant une relation entre les courbures de l'objet et sa topologie que nous utiliserons à notre avantage.

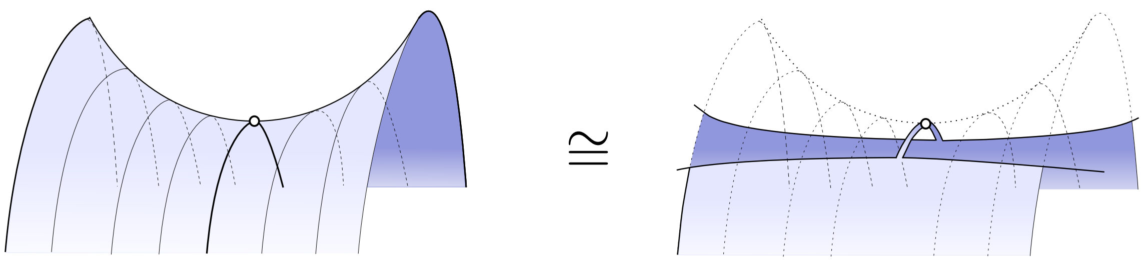


Figure 1.18 – Idée de la preuve de l'assertion (2) étudiant l'évènement topologique autour du second point critique de la Figure 1.17. Une cellule de dimension 1 est collée autour du point critique.

De telles preuves nécessitent que X soit une variété. Comme nous travaillons avec des ensembles vérifiant de plus faibles conditions de régularité se pose la question d'étendre la théorie de Morse à des classes d'objets contenant des objets singuliers, requérant possiblement d'utiliser de l'analyse non-lisse. Le lemme (2) requiert une compréhension précise de la géométrie des sous-niveaux de f autour de ses points critiques, mais celle-ci est peut-être particulièrement sauvage.

Par exemple, le comportement de fonctions lisses restreintes à des polyèdres de $\mathbb{R}^d$ peut consister en le recollement de plusieurs cellules autour d'un seul point, comme illustré en Figure 1.19.

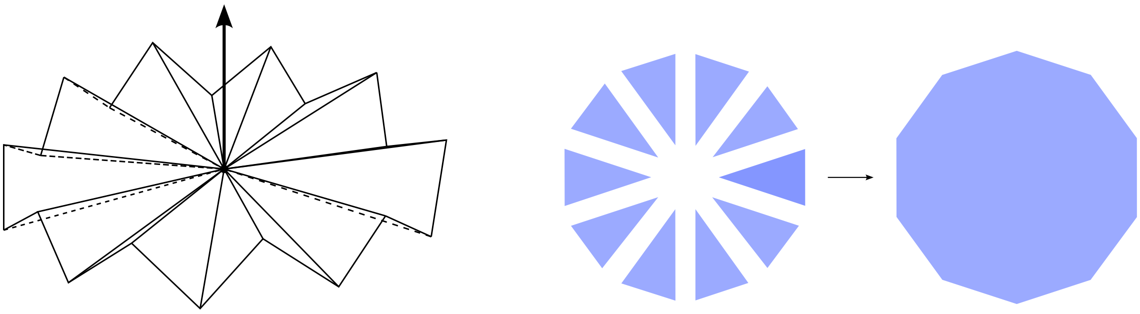


Figure 1.19 – Une collerettes de 20 triangles dans $\mathbb{R}^3$. Les sous-niveaux de la fonction hauteur passent de 10 composantes connexes à 1 lorsqu'ils croisent le point central.

La théorie de Morse stratifiée, développée à partir des années 80, a permis d'étendre la notion de fonction de Morse aux parties straifiées de variétés riemanniennes. Malheureusement, dans ce contexte il peut y avoir arbitrairement plus d'évènements topologiques dans le diagrammes de persistance d'une fonction de Morse qu'elle n'admet de points critiques. Nous prouverons que la version originale du lemme (2) est vraie pour des fonctions lisses restreintes à des offsets d'autres ensembles, en comparant ses offsets à des objets lisses bien choisis.

Analyse non-lisse. Pour comparer des diagrammes de persistance, nous construirons des déformations continuent entre certaines parties de $\mathbb{R}^d$. On peut obtenir des déformations continues en suivant le flot de fonctions lisses comme dans la preuve de l'assertion (1) du paragraphe précédent. Néanmoins, nous aurons besoin de déformations avec des propriétés similaires pour des fonctions simplement lipschitz, pour lesquelles il n'existe pas de flot. Nous considérerons à de nombreuses reprises des fonctions construites à l'aide de fonctions distance, qui ne sont pas lisses mais s'avèrent 1-lipschitz. Fort heureusement pour nous, l'analyse de fonctions lipschitz est un sujet qui a déjà été creusé. Nous utiliserons le concept de *gradient de Clarke* $\partial^* f$ d'une fonction locallement lipschitz $f : \mathbb{R}^d \to \mathbb{R}$. C'est une application qui prend pour valeurs des ensembles, et qui est définie au point x comme l'enveloppe convexe des limites de gradients de f autour de x. Cela généralise la notion classique de gradient de fonctions différentiables, dans le sens où on a pour une telle fonction $\partial^* f(x) = \{\nabla f(x)\}$.

Différent résultats portant sur des fonctions C^1 ont été généralisés à des applications lipschitz via le gradient de Clarke. Par exemple, lorsque le gradient de Clarke de f ne contient pas le point 0 sur le niveau $f^{-1}(x)$, alors celui-ci sera une sous-variété lipschitz de $\mathbb{R}^d$, c'est-à-dire un ensemble qui est localement le graphe d'une application lipschitz depuis un espace euclidien. C'est une extension Lipschitz au théorème classique d'inversion locale. Nous verrons que d'autres propriétés des fonctions lipschitz sont liées aux propriétés géométriques de leurs sous-niveaux ou niveaux, et que les fonctions distance joueront un rôle important dans cette analyse.

Dans le cadre lisse, le flot de $-f$ a des trajectoires qui font décroître f exactement à vitesse $||\nabla f||$. Nous contruirons des *flots inverses approximés*[*] d'applications lipschitz à l'aide de son gradient de Clarke et d'approximations lisses de la fonction. À une constante arbitrairement petite

$*$. *Approximate inverse flows* dans la thèse.

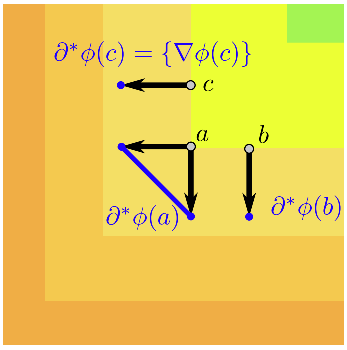


Figure 1.20 – Gradient de Clarke d'une application lipschitz $\mathbb{R}^2 \to \mathbb{R}$ dont les sous-niveaux croissants sont représentés du vert à l'orange.

près, il existe des flots inverses approximés faisant décroître la fonction f à la vitesse du cas lisse, en remplaçant $||\nabla f||$ par la distance de 0 à $\partial^* f$, quantité qu'on notera $\Delta(\partial^* f)$. Ces deux quantités coincident lorsque f est différentiable. Ces concepts appliqués à une fonction distance $f = d_A$ permettent de définir le μ-*reach* de A comme étant le plus grand t tel que $\Delta(\partial^* d_A) \geq \mu$ sur $A^t \setminus A$. Cette quantité sera centrale dans notre résultat d'inférence géométrique.

Courbure et géométrie intégrale. Dans cette thèse, nous inférerons des quantités liées aux *courbures* de certaines parties d'un espace euclidien. Grossièrement, les courbures d'un objet mesurent à quel point l'objet n'est *pas plat*. Les surfaces orientées de $\mathbb{R}^3$ sont plates autour d'un point x lorsque l'application $y \mapsto \nu(y)$ (dite *de Gauss*) qui a chaque point associe la normale, est constante dans un voisinage de x. Ce qu'on appelle les *courbures principales* de la surface au point x sont définies - lorsqu'elles existent - comme les valeurs propres de la différentielle de l'application de Gauss en x.

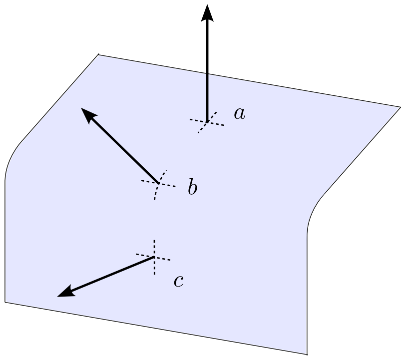


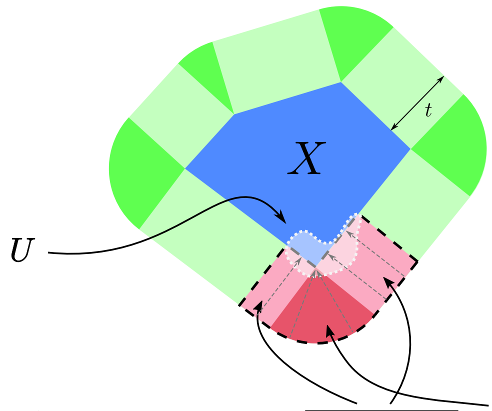
Figure 1.21 – Courbure ponctuelle sur une surface. L'application de Gauss est constante autour de a c, mais pas de b.

Suivant cette définition, on peut montrer que la courbure d'une courbe lisse est déterminée par ses dérivées première et seconde. Lorsque la courbe est linéaire par morceau elle n'est généralement pas plate alors qu'elle est plate *presque partout* au sens de la longueur de la courbure, montrant que la courbure doit être concentrée en ses coins. Le même raisonnement s'applique aux polyèdres de $\mathbb{R}^d$ comparés à des surfaces orientées : la courbure est concentrée aux arêtes ou aux sommets, c'est-à-dire à aux points non-lisses.

Dans son article fondateur *Curvature Measures*, Federer expose une théorie unifiant les objets lisses et affines par morceaux. Tant qu'il existe un $r > 0$ tel que pour tout x de $\mathbb{R}^d$ avec $d_X(x) < r$, il y a un unique plus proche voisin $\xi_X(x)$ de x dans X, il montra que pour tout Borélien U l'application

$$t \in [0, r] \mapsto \mathrm{Vol}(\xi_X^{-1}(U) \cap X^t) = \sum_{i=0}^{t} \omega_i t^i C_{d-i}(X, U)$$

est un polynôme de degré d, et que ses coefficients définissent des mesures signées en fonction de U qu'il appelle, après mise à l'échelle, *mesures de courbures*. La mesure $C_0(X, s)$ d'un sommet s d'un polyèdre convexe coincide avec la mesure de l'angle solide des normales pointant vers l'extérieur de ce sommet, comme illustré dans la Figure 1.22.



$$\mathrm{Vol}\left(\xi_X^{-1}(U) \cap (X^t \setminus X)\right) = 2tC_1(X, U) + \pi t^2 C_0(X, U)$$

Figure 1.22 – Mesures de courbures de l'enveloppe convexe de quelques points de $\mathbb{R}^2$.

La définition des courbures pour des *singuliers*, terme qu'on utilisera maintenant pour qualifier des objets contenant au moins un point non-lisse, autour duquel l'objet n'est pas localement convexe, est bien plus complexe et a fait l'objet de différentes extensions dans les dernières décennies. Une extension intuitive est celle concernant les unions finies, génériques d'ensembles pour lesquels il existe un tel r [*]. En effet, d'une part une intersection générique d'objets à reach positif est aussi à reach positif, et d'autre part si X, Y et $X \cap Y$ admettent des mesures de courbures, alors $X \cup Y$ en admet aussi, avec $C_i(X \cup Y, \cdot) + C_i(X \cap Y, \cdot) = C_i(X, \cdot) + C_i(Y, \cdot)$, comme illustré dans la Figure 1.12.

Squelette et contributions

Chapitre II. On commence par fixer la terminologie générale suivie dans cette thèse avant d'exposer des concepts classiques des fonctions lipschitz, comme le gradient de Clarke et la formule de la co-aire de Federer entre ensembles rectifiables, qui généralise la formule de changement de variable. La contribution de ce chapitre réside dans la construction de *flots inverses approximés*

$*$. On parle d'ensemble à *reach positif*.

entre les sous-niveaux de fonctions lipschitz, sous l'hypothèse que leur gradient de Clarke reste uniformément éloigné de zéro.

Chapitre III. Le chapitre trois est dédié aux relations entre la géométrie d'un objet et les propriétés de la fonction distance qui lui est associée. Nous rappelons des définitions classiques à propos de la géométries de parties d'un espace euclidian, comme le *reach* ainsi que le μ-reach qui en est une relaxation. Nous rappelons les définitions de cônes tangents et normaux. Pour montrer qu'un ensemble avec un μ-reach strictement positif pour un certain $\mu > 0$ n'est pas nécessairement lisse, nous construisons une courbe fractale $K(\theta)$ dans $\mathbb{R}^2$ libre d'intersections, dépendant d'une suite décroissante de paramètres positifs $(\theta_i)_{i\in\mathbb{N}}$. Nous démontrons que pour tout μ dans $]0,1[$, il existe des choix de paramètres tel que $K(\theta)$ a un μ-reach infini, et qu'on peut affiner pour que $K(\theta)$ soit une courbe non-rectifiable, ou alors une courbe rectifiable dont la longueur du fibré normal des offets $K(\theta)^\varepsilon$ tend vers l'infini lorsque ε tend vers 0.

Nous définissons la classe des *complementary regular sets* comme étant constitué des parties de $\mathbb{R}^d$ ayant un μ-reach positif pour un certain μ dans $]0,1[$ vérifiant certains conditions de régularité sur leur complémentaire. Nous montrons que cette classe coincide avec celle des ensembles qui sont offsets d'un autre ensemble, à un offset qui est valeur régulière de leur fonction distance. Nous montrons que cette condition équivaut à être le sous-niveau d'une fonction semi-concave à une valeur régulière. Une autre contribution est l'obtention d'une identité reliant les cônes normaux d'un ensemble complementary regular au gradient de Clarke de sa fonction distance.

Chapitre IV. Dans le quatrième chapitre, nous étendons les résultats classiques de la théorie de Morse à des fonctions C^2 à valeurs réelles sur un espace euclidien, qu'on a restreintes à un ensemble complementary regular du chapitre III. Ce résultat est obtenu après une étude minutieuse de certaines applications lipschitz construites à partir d'opérations élémentaires impliquant des fonctions distance. Il sera utilisé comme lemme important pour les bornes d'inférence du chapitre VII. Ce travail est indépendant du concept de géométrie persistante est disponible dans l'article *Generalized Morse theory for tubular neighborhoods*.

Chapitre V. Le cinquième chapitre s'intéresse à la théorie de la persistante. Il commence par une section introductrice contenant des rappels de résultats classiques en homologie utilisés en persistance, ainsi qu'une introduction - se voulant intuitive - aux concepts généraux d'homologie persistante. La deuxième section est une exposition mathématique des définitions et résultats de base de la persistance pour des non-spécialistes. Enfin, la troisième section s'intéresse à *l'image persistance*. Notre contribution réside dans l'obtention d'un théorème de stabilité sur les diagrammes de persistance image.

Chapitre VI. Le sixième chapitre est dédié aux courbures de parties d'un ensemble euclidien. Cela consiste en une introduction au concept général de courbure, incluant une définition pour la classe des ensemble complementary regular. Avec des considérations de théorie géométrique de la mesure, nous montrerons que, restreintes à un objet complementary regular, presque toutes les fonctions distance à un point, et presque toutes les fonctions hauteurs, sont de Morse. Nous comparons aussi les propriétés des courbures d'ensembles complementary regular à la littérature existante sur les courbures d'objets singuliers grâce au vocabulaire des cycles normaux, décrit en annexe.

Chapitre VII. Le chapitre final rassemble les résultats des chapitres précédents pour enfin suivre les préceptes de géométrie persistante. Nous approximons les *volumes intrinsèques* $V_0(X), \ldots, V_d(X)$ d'un ensemble $X \subset \mathbb{R}^d$, qui sont des quantités liées à ses courbures. Plus précisément, prenons X un ensemble de μ-reach strictement positif et Y un ensemble approximant X. Nous construisons des quantités $V_i^\varepsilon(Y)$ à partir des offsets Y^ε, $Y^{3\varepsilon}$, telles que lorsque $d_H(X,Y) \leq \varepsilon \leq \frac{1}{4}\operatorname{reach}_\mu(X)$, on a

$$\left|V_i(X^{2\varepsilon}) - V_i^\varepsilon(Y)\right| = O(\varepsilon/\mu),$$

où la constante dans le grand O est une fonction des courbures de $X^{2\varepsilon}$. En utilisant une méthode similaire, nous montrons qu'avec une condition de régularité supplémentaire sur X, on a aussi

$$\left|V_i(X^{2\varepsilon}) - V_i(X)\right| = O(\varepsilon/\mu),$$

ce qui montre que nous pouvons effectivement estimer les volumes intrinsèques de X à partir d'une approximation Y. Toutes ces considérations peuvent aussi être lues dans l'article *Persistent intrinsic volumes* co-écrit avec David Cohen-Steiner. À la suite des preuves, nous discutons des hypothèses de régularité minimales pour garantir ce taux de convergence. Nous discutons aussi de la computabilité de notre estimateur.

Annexe. Bien qu'ils ne soient pas nécessaires pour obtenir les résultats d'inférence géométrique du chapitre VII, les courants fournissent un cadre concis pour décrire les courbures de parties d'un espace euclidien. L'annexe contient une introduction suffisante pour comprendre les notions de courants apparaissant à la fin des chapitres VI et VII.

CHAPTER 2

General terminology and tools from Lipschitz analysis

We fix various notations and define some terminology of several mathematical fields used throughout this thesis. We focus in particular on Lipschitz applications, listing a handful of lemmas used in further chapters. Using the concept of Clarke gradients, we build so-called approximate inverse flows *of a Lipschitz map. Parametrized to the arc-length, their trajectories provide continuous deformations between sublevel sets of the Lipschitz map, at a speed controlled by the distance to zero of the Clarke gradient.*

2.1 General notations and terminology

Notations for Euclidean spaces. Let $d \geq 1$. In this thesis, we equip $\mathbb{R}^d$ with its canonical scalar product $\langle \cdot, \cdot \rangle$. The Euclidean norm it induces will be denoted by $||\cdot||$, while the absolute value over $\mathbb{R}$ will be denoted by $|\cdot|$. The Euclidean metric provides a topology to $\mathbb{R}^d$ and any of its subsets. For any set X included in $\mathbb{R}^d$, we denote by $\mathrm{int}(X)$ the largest open subset of X and call it the *interior* of X. In the same vein, $\overline{X}$ is the *closure* of X, which is the smallest closed subset of $\mathbb{R}^d$ containing X. By a slight abuse of notation, we call *complement set* of X the closure of the classical complement set of X, and we denote it by $\neg X := \overline{\mathbb{R}^d \setminus X} = \mathbb{R}^d \setminus \mathrm{int}(X)$.

Distance functions and offsets. For any subset X of $\mathbb{R}^d$, its *distance function* d_X is $d_X : x \mapsto \inf\{||x - a|| \mid a \in A\}$. For any positive real r, we define the r and $-r$ *tubular neighborhoods* (also respectively called *offsets* and *counter offsets*) of X as follows:

$$\begin{aligned} X^r &:= \left\{ x \in \mathbb{R}^d \mid d_X(x) \leq r \right\} \\ X^{-r} &:= \left\{ x \in \mathbb{R}^d \mid d_{\neg X}(x) \geq r \right\}. \end{aligned}$$

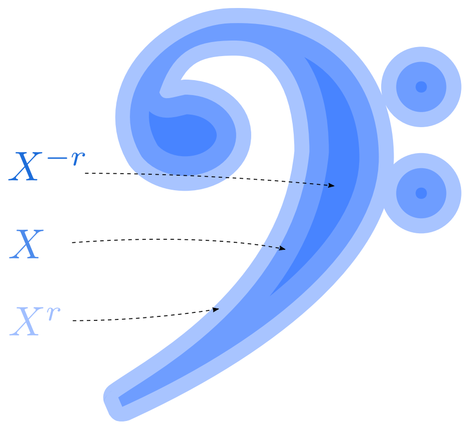


Figure 2.1 – A bass clef inflated (X^r) and eroded (X^{-r})

Convex sets and functions A set X in $\mathbb{R}^d$ is said to be *convex* when, for every pair x, y in X, the segment $[x, y] := \{\lambda x + (1 - \lambda)y \mid \lambda \in [0, 1]\}$ is included in X. When $d = 1$, this notation coincides with the classical interval notation. For any interval I of $\mathbb{R}$, a map $f : I \to \mathbb{R}$ is said to be convex if, for every x, yin I and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{2.1}$$

When f is differentiable, convexity is characterized by its derivative f' being non-decreasing. A map $f : \mathbb{R}^d \to \mathbb{R}$ is said to be convex on a subset U of $\mathbb{R}^d$ when the map $f_{|I}$ is convex for every segment I included in U. A map is *concave* when its opposite is convex.

Cone, dual cones and convex hulls. A *cone* A in $\mathbb{R}^d$ is a set stable under multiplication by a positive number, i.e., such that for all $\lambda > 0$, we have $\lambda A \subset A$. Given any subset B of $\mathbb{R}^d$, we denote by $\operatorname{Cone} B$ the smallest (for the inclusion) cone containing B, defined as the image of $[0, \infty) \times B$ by the scalar multiplication map $(\lambda, x) \mapsto \lambda x$. We denote by $\operatorname{Conv} B$ the *convex hull* of B, consisting in the smallest convex set of $\mathbb{R}^d$ containing B. The *dimension* of a cone or a convex set is the dimension of the affine vector space it spans. The *polar cone* or *dual cone* of a set $B \subset \mathbb{R}^d$, denoted by B^{o}, is the convex cone defined by:

$$B^{\mathrm{o}} := \{u \in \mathbb{R}^d \mid \langle u, b \rangle \leq 0 \quad \forall\, b \in B\}.$$

The polar cone operation is idempotent on convex cones, as it notably verifies the identity $(B^{\mathrm{o}})^{\mathrm{o}} = \operatorname{Conv}(\operatorname{Cone} B)$ for any subset B of $\mathbb{R}^d$.

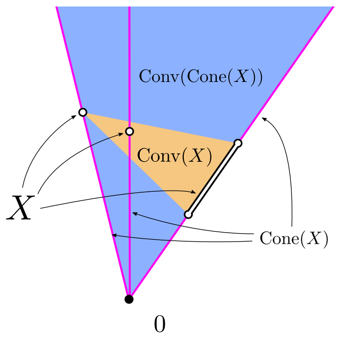


Figure 2.2 – Cone, convex hull and convex cone of a set $X \subset \mathbb{R}^2$.

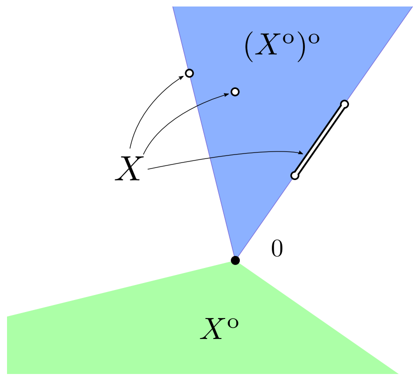


Figure 2.3 – Dual cone and illustration of the identity $\operatorname{Conv}(\operatorname{Cone}(X)) = (X^{\mathrm{o}})^{\mathrm{o}}$.

Distance to zero. Given a subset X of $\mathbb{R}^d$, its *distance to zero* measures how far it is from intersecting $\{0\}$. It is defined by $\Delta(X) := \inf\{\, ||x|| \mid x \in X\}$. In particular, when X is closed and convex, the infimum $\Delta(X)$ is attained by a unique point in X which is the closest point to 0 in X.

Homotopy equivalences. Given two topological spaces X, Y, we say that two functions $f, g : X \to Y$ are *homotopic* when there exists a continuous map $H : [0,1] \times X \to Y$ such that for all $x \in X$, $H(0,x) = f(x)$ and $H(1,x) = g(x)$. X and Y are said to be *homotopy equivalent* or to *share the same homotopy type* when there exist two continuous maps $\psi : X \to Y$, $\phi : Y \to X$ such that $\phi \circ \psi$ and $\psi \circ \phi$ are respectively homotopic to the identity maps of X and Y. Furthermore, when $X \subset Y$ and ψ is the inclusion map $X \hookrightarrow Y$, we say that Y *deformation retracts* onto X. When a topological space is homotopy equivalent to a point, we say that it is *contractible*. Every convex set in a normed vector space is contractible.

Singular and stratified sets. This document assumes basic knowledge of differential geometry of subsets or $\mathbb{R}^d$, i.e., the study of submanifolds of Euclidean spaces. We say that a subset of $\mathbb{R}^d$ is *singular* when it is not a C^1 submanifold of $\mathbb{R}^d$ nor a convex set.

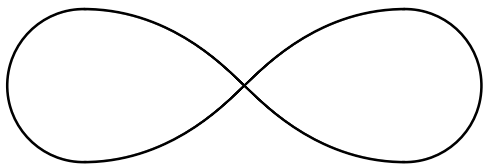


Figure 2.4 – A singular stratified set.

A classical example is the class of *stratified* sets. Although those are not studied *per se* in this thesis, their properties will be compared to those of sets we study.

In $\mathbb{R}^d$, a set is said to be stratified when it can be written as a disjoint, locally finite union of C^1 submanifolds of $\mathbb{R}^d$ of possibly varying dimensions satisfying Whitney's condition A. More precisely, for any sequence (x_i) in of these submanifolds converging to a point y in another submanifold, with tangent spaces Tx_i converging to a subspace T of $\mathbb{R}^d$, the space T is included in the tangent space at y.

Spatial and normal coordinates. Throughout this thesis, we will encounter several subsets of $\mathbb{R}^d \times \mathbb{S}^{d-1}$. We denote the respective projection maps onto the first and second coordinates by π_0 and π_1. We also call them *spatial* and *normal* coordinates.

$$\begin{tikzcd} & \mathbb{R}^d \times \mathbb{S}^{d-1} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ \mathbb{R}^d & & \mathbb{S}^{d-1} \end{tikzcd}$$

Figure 2.5 – Projections onto spatial and normal coordinates

Properties satisfied almost-everywhere. We assume basic knowledge of measure theory. Recall that a property is said to be true *almost everywhere* in a set X for a measure ν when there exists a ν-measurable set $E \subset X$ with $\nu(E) = 0$ containing the points of X which do not satisfy the property. In this thesis, ν will either be the Lebesgue measure on $\mathbb{R}^d$, in which case its mention is implied, or the m-dimensional Hausdorff measure $\mathcal{H}^m$ restricted to a subset of Euclidean space.

2.2 Analysis of Lipschitz functions

2.2.1 Lipschitz maps

When E, F are two normed vector spaces, a map $f : U \subset E \to F$ is said to be C-*Lipschitz* when we have for every $a, b \in U$:

$$||f(a) - f(b)||_F \leq C\,||a - b||_E \tag{2.2}$$

The map f is said to be *Lipschitz* when there exists a constant $C \geq 0$ such that f is C-Lipschitz. The best Lipschitz constant possible for f is:

$$\sup_{\substack{a,b \in U \\ a \neq b}} \frac{||f(a) - f(b)||_F}{||a - b||_E}.$$

The map f is said to be *locally Lipschitz* when, for every compact set $K \subset A$, the restriction $f_{|K}$ is Lipschitz.

The case where $E = \mathbb{R}^d$, U is an open set of E and $F = \mathbb{R}^m$ is of particular interest. Such a map f is said to be *differentiable* at $x \in U$ when there is a linear map $D_x f$ realizing the first order expansion

$$f(x+h) = f(x) + D_x f(h) + o(h),$$

where $o(h)$ is a function of h such that $\lim_{h \to 0} ||o(h)|| \,/\, ||h|| = 0$. When $F = \mathbb{R}$ and f is differentiable at x, by Riesz' representation theorem, there exists a vector $\nabla f(x)$, called the *gradient* of f at x, such that the first-order expansion can be written in the following way

$$f(x+h) = f(x) + \langle \nabla f(x), h \rangle + o(h).$$

When the map $x \mapsto D_x f$ is continuous, we say that f *is* C^1 *over* U. When this map is Lipschitz, we say that f is $C^{1,1}$. The concept of differentiable maps is related to that of Lipschitz maps, as a C^1 function is Lipschitz if and only if its gradient ∇f is bounded. The following theorem states that the converse is true in a measure theoretical sense.

Theorem 2.1 (Rademacher's Theorem [Fed69, Theorem 3.1.6])**.** *Let U be an open subset of $\mathbb{R}^d$ and let $f : U \to \mathbb{R}^m$ be a Lipschitz map. Then f is differentiable almost everywhere in U.*

When $d = m = 1$, any Lipschitz map is *absolutely continuous*, i.e., the map $f' := \nabla f$ is Lebesgue integrable and we have $f(b) - f(a) = \int_a^b f'(t)\, \mathrm{d}t$.

Definition 2.2 (Semi-convexity and semi-convexity)**.** Let U be an open subset of $\mathbb{R}^d$ and let $f : U \to \mathbb{R}$. We say that f is K*-semi-convex* when for every convex subset (or equivalently, on any segment) V of U, the map

$$x \mapsto f(x) + K\, ||x||^2$$

is convex. It is semi-convex when there exists a $K > 0$ such that it f is K-semi-convex. A map f is said to be K-semi-concave (resp. semi-concave) when $-f$ is K-semi-convex (resp. semi-convex).

Semi-concave and semi-convex maps are locally Lipschitz, since concave/convex maps have this property. Remark that a C^2 map f is K semi-convex (resp. K semi-concave) if and only if the matrix $H_x f - K\,\mathrm{Id}$ (resp. $K\,\mathrm{Id} - H_x f$) is positive semi-definite for every point x in the domain of f.

2.2.2 Clarke gradient

Thanks to Rademacher's theorem, we are in position to define the *Clarke gradient* of a locally Lipschitz, real-valued function on an open set of $\mathbb{R}^d$. The Clarke gradient is a set-valued map first defined by Clarke in [Cla75] to generalize classical properties of differentiable functions to Lipschitz ones.

Definition 2.3 (Clarke gradients of locally Lipschitz functions)**.** Let U be an open set of $\mathbb{R}^d$ and let $\phi : U \to \mathbb{R}$ be a locally Lipschitz function. Its *Clarke gradient* at x is the subset of $\mathbb{R}^d$ defined as the convex hull of limits of the form $\nabla\phi(x+h), h \to 0$:

$$\partial^* \phi(x) := \mathrm{Conv}\left(\lim_{i \to \infty} \nabla \phi(x_i) \mid x_i \in \mathbb{R}^d \to x, \phi \text{ differentiable at } x_i \text{ for all } i \right).$$

Every time we refer to the explicit definition of the Clarke gradient, the fact that ϕ needs to be differentiable at any x_i will be implied.

Proposition 2.4 (Basic properties of the Clarke Gradient)**.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function.*

- *By Rademacher's theorem, $\partial^*\phi(x)$ is non-empty for all x;*
- *If ϕ is R-Lipschitz around x, $\partial^*\phi(x) \subset B(0, R)$;*
- *The Clarke gradient of ϕ is a singleton at a point x if and only if $\nabla\phi$ is C^1 at x, and in this case we have*
$$\partial^*\phi(x) = \{\nabla\phi(x)\};$$
- *The map $x \mapsto \partial^*\phi(x)$ is upper semi-continuous, i.e., for each $x \in U$, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $||x - y|| \leq \delta$ implies $\partial^*\phi(y) \subset \left(\partial^*\phi(x)\right)^{\varepsilon}$.*

Definition 2.3 is illustrated in Figure 2.6 below. Here the map $\phi : \mathbb{R}^2 \to \mathbb{R}$ is represented in a portion of $\mathbb{R}^2$ by 5 encapsulating sublevel sets, growing towards the orange. Each boundary between shades represents a level set. At any point where the sublevel set is smooth (such as b and c) the gradient of f has direction perpendicular to the level set; at point a, the Clarke gradient is the convex hull of the neighboring gradient.

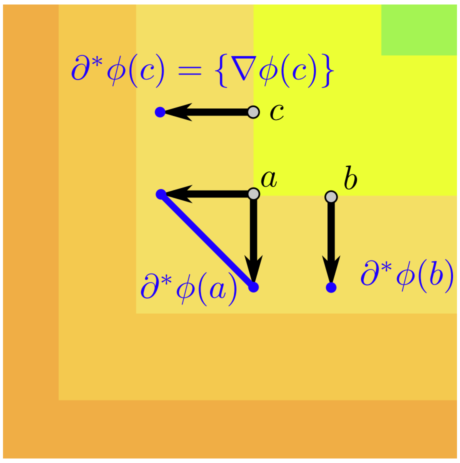


Figure 2.6 – Clarke gradient from sublevel sets. The map ϕ grows towards the more orange regions.

Critical points of differentiable functions are usually defined as points where the gradient vanishes. This notion can be consistently extended to Lipschitz maps as having a Clarke gradient containing 0.

Definition 2.5 (Critical points of Lipschitz functions)**.** Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. We say that $x \in \mathbb{R}^d$ is a *critical point* of ϕ when $0 \in \partial^*\phi(x)$. A real c is a *critical value* of ϕ when $\phi^{-1}(c)$ contains a critical point of ϕ. Otherwise, we say that c is a *regular value* of ϕ.

This terminology will be justified by the approximate inverse flows of Proposition 2.9. The condition $0 \in \partial^*\phi(x)$ is equivalent to $\Delta(\partial^*\phi(x)) = 0$. In practice, we prefer working with the second characterization, as the map $x \mapsto \Delta(\partial^*\phi(x))$ has practical properties. Indeed, the upper semi-continuity of the Clarke gradient leads to the following proposition.

Proposition 2.6 (Semi-continuity of Clarke gradients)**.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. If a sequence $(x_i)_{i\in\mathbb{N}}$ converges to x, we have*

$$\liminf_{i\to\infty} \Delta\left(\partial^* \phi(x_i)\right) \geq \Delta\left(\partial^* \phi(x)\right).$$

Mimicking the definition of C^k-submanifold of $\mathbb{R}^d$ as local graphs of C^k functions for Lipschitz and $C^{1,1}$ maps, we obtain *Lipschitz submanifolds* and *Lipschitz domains*.

Definition 2.7 (Lipschitz and $C^{1,1}$ Domains)**.** We say that $X \subset \mathbb{R}^d$ is a *m-Lipschitz submanifold* (resp. a *m-$C^{1,1}$ submanifold*) of $\mathbb{R}^d$ when around any point x of X, there exists an open subset U of $\mathbb{R}^d$ containing x such that $U \cap X$ is the graph of a Lipschitz (resp. $C^{1,1}$) function $\mathbb{R}^m \to \mathbb{R}^{d-m}$.

We say that $X \subset \mathbb{R}^d$ is a *Lipschitz domain* (resp. *$C^{1,1}$ domain)* when $\overline{\mathrm{int}(X)} = X$ and ∂X is $(d-1)$-Lipschitz (resp. $C^{1,1}$) submanifold of $\mathbb{R}^d$.

The classical implicit function theorem states that when the map $\phi : \mathbb{R}^d \to \mathbb{R}$ is C^1, the condition that at a point x we have $\nabla\phi(x) \neq 0$ is sufficient for the level set of ϕ around x to be the graph of a C^1 function. The same results hold when ϕ is Lipschitz, replacing the condition with $\Delta(\partial^*\phi(x)) > 0$ and with the level set locally being the graph of a Lipschitz map. A global version of this fact is the following.

Theorem 2.8 (Clarke's Lipschitz implicit function theorem)**.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a proper, locally Lipschitz function. Let $\xi \in \mathbb{R}$ be such that*

$$\inf_{x\in\phi^{-1}(\xi)} \Delta(\partial^*\phi(x)) > 0. \tag{2.3}$$

Then $\phi^{-1}(-\infty, \xi]$ is a Lipschitz domain.

2.2.3 Approximate inverse flow of Lipschitz maps

Another property of smooth functions that we want to generalize to Lipschitz maps is the existence of maps akin to flows. Indeed, assume that ϕ is smooth and has a non-vanishing gradient. When parametrized to the arc-length, the flow of $-\phi$ makes ϕ decrease at a rate $||\nabla\phi||$ as a function of the time parameter. This fact, which easily extends to Riemannian geometry, is crucial in Morse theory as it shows that $\phi^{-1}(-\infty, a]$ is a deformation retract of $\phi^{-1}(-\infty, b]$ when $\nabla\phi$ does not vanish in $\phi^{-1}[a, b]$, with $a < b$. The following proposition, which is one of the contribution of this thesis, describes an extension of this result to the case where ϕ is Lipschitz.

Proposition 2.9 (Approximate inverse flow of a Lipschitz function)**.** *Let $a < b \in \mathbb{R}$. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function on $\overline{\phi^{-1}(a, b]}$. Assume that*

$$\inf\{\Delta(\partial^*\phi(x)) \mid x \in \phi^{-1}(a, b]\} = \mu > 0.$$

Then for every $\varepsilon > 0$, there exists a continuous function

$$C_\phi : \left\{ \begin{array}{ccc} [0,1] \times \phi^{-1}(\infty, b] & \to & \phi^{-1}(-\infty, b] \\ (t, x) & \mapsto & C_\phi(t, x) \end{array} \right.$$

such that

— *For any $s > t$ and x such that $C(s,x) \in \phi^{-1}(a,b]$, we have*

$$\phi(C_\phi(s,x)) - \phi(C_\phi(t,x)) \leq -(s-t)(b-a)$$

— *For any $t \in [0,1]$ and $x \in \phi^{-1}(\infty, a]$, we have $C_\phi(t,x) = x$*
— *For any $x \in \phi^{-1}(-\infty, b]$, the map $s \mapsto C_\phi(s,x)$ is $\frac{b-a}{\mu-\varepsilon}$-Lipschitz.*
In particular, $C_\phi(1,\cdot)$ is a deformation retraction between $\phi^{-1}(-\infty,a]$ and $\phi^{-1}(-\infty,b]$.

Proof.

Approximate inverse flows of Lipschitz maps have been used implicitly in several works (see [Fu94]) without the need for explicit constants. A weaker form of our claim can be found in section D of [KSC+20] and the method can be traced back to [Gro93]. Here the constants have been optimized, and the proposition generalized from distance functions to Lipschitz functions. For the sake of completeness, we display a full proof. Note that a result of the same type can be found in [PRZ19, Lemma 3.1].

Let $\varepsilon > 0$. By semi-continuity of the Clarke gradient for any $x \in \phi^{-1}(a,b]$ we can consider B_x an open ball centered in x such that $\partial^*\phi(y) \subset \partial^*\phi(x)^\varepsilon$ for any $y \in B_x$. Since $\partial^*\phi(x)$ is a closed convex set, there is a unique point $W(x)$ in $\partial^*\phi(x)$ realising the distance to 0 i.e., $||W(x)|| = \Delta(\partial^*\phi(x))$. This is the closest point to 0 in $\partial^*\phi(x)$. From the convexity of $\partial^*\phi(x)$, we have:

$$\forall\, u \in \partial^*\phi(x), \langle u, W(x)\rangle \geq ||W(x)||^2 \,. \tag{2.4}$$

The family $\{B_x\}_{x\in\phi^{-1}(a,b]}$ is an open covering of $\phi^{-1}(a,b]$. By paracompactness, there exists a locally finite partition of unity $(\rho_i)_{i\in I}$ subordinate to this family, i.e., such that the support of each ρ_i is included in one of the balls $B(x_i)$ with $x_i \in \phi^{-1}(a,b]$. Use them to define the vector field V as a smooth interpolation of normalized $-W$:

$$V(y) := -\sum_{i\in I} \rho_i(y) \frac{W(x_i)}{||W(x_i)||}. \tag{2.5}$$

Obviously $||V(x)|| \leq 1$ and V is locally Lipschitz. Now by classical results there is a flow C of V defined on a maximal open domain $\mathbb{D}$ in $\mathbb{R}^+ \times \phi^{-1}(a,b]$. For any $x \in \phi^{-1}(a,b]$ and any $\zeta \in \partial^*\phi(x)$, we have:

$$\left\langle \frac{\partial}{\partial t} C(0,x), \zeta \right\rangle = \langle V(x), \zeta\rangle \leq -\sum_{i\in I} \rho_i(x)\,(\,||W(x_i)|| - \varepsilon\,) \leq -\mu + \varepsilon. \tag{2.6}$$

Define $\mathbb{D}_x$ via $(\mathbb{R}^+ \times \{x\}) \cap \mathbb{D} =: \mathbb{D}_x \times \{x\}$ the maximal subset of $\mathbb{R}^+$ for which the flow starting at x is defined. The set $\mathbb{D}_x$ is connected in $\mathbb{R}^+$ and we put $s_x = \sup \mathbb{D}_x$, assuming this is finite. Now the trajectory $C(\cdot, x)$ is 1-Lipschitz, meaning that the curve $s \mapsto C(s,x)$ is rectifiable. We can thus define $C(s_x, x)$ as the endpoint of this curve, that is, $C(s_x,x) = \lim_{s\to s_x} C(s,x)$.

The function $\phi(C(\cdot,x)) : \overline{\mathbb{D}_x} \to [a,b]$ is Lipschitz and thus differentiable almost everywhere. Let (s,x) be in $\mathbb{D}$ with $\phi(C(\cdot,x))$ differentiable at s. Since we have $C(s+h,x) = C(s, C(h,x))$, we can assume $s = 0$ without loss of generality. Since $C(\cdot,x)$ has non-vanishing gradient $V(x)$ at 0, ϕ has a directional derivative $\phi'(x, V(x))$ in direction $V(x)$. From the work of Clarke (Proposition 1.4, [Cla75]) we know that when the directional derivative exists, the Clarke gradients acts like a maxing support set, that is:

$$\phi'(x, V(x)) \leq \max\left\{\langle \zeta, V(x)\rangle \mid \zeta \in \partial^* \phi(x)\right\} \leq -\mu + \varepsilon. \tag{2.7}$$

Any Lipschitz function is absolutely continuous, thus when $s \leq t \in \mathbb{D}_x$ we can integrate the previous inequality to obtain:

$$\phi(C(s,x)) - \phi(C(t,x)) \leq -(\mu - \varepsilon)(s - t) \tag{2.8}$$

This yields $\phi(C(s_x, x)) = a$ and $s_x \leq \frac{b-a}{\mu-\varepsilon}$ for all $x \in \phi^{-1}(a, b]$.
We extend the flow to $\mathbb{R}^+ \times \phi^{-1}(-\infty, b]$ by putting

$$C(t,x) := \begin{cases} C(\min(t, s_x), x) & \text{when } a < \phi(x) \leq b, \\ x & \text{else.} \end{cases}$$

We will now show that C is continuous at every point $(s,x) \in \mathbb{R}^+ \times \phi^{-1}(-\infty, b]$. C is obviously continuous inside its original domain $\mathbb{D}$. C is continuous inside the open set $\mathbb{R}^+ \times \phi^{-1}(-\infty, a)$ since in this set $C(t,x) = x$. We now turn our attention to the other points. Let k be a Lipschitz constant for ϕ over $\overline{\phi^{-1}(a,b]}$.
Let $x \in \phi^{-1}(a, b]$ and let $s \geq s_x$. Let $0 < c < s_x$. For every $\delta > 0$, there exists a radius $\rho_x(\delta) > 0$ such that for all $y \in B(x, \rho_x(\delta))$ and for any $t \in [0, s_x - c]$,

$$\begin{cases} s_y > s_x - c \\ |\phi(C(t,y)) - \phi(C(t,x))| \leq \delta. \end{cases}$$

This implies $\phi(C(s_x - c, y)) \leq a + \delta + kc$, in turn yielding $s_y \leq s_x - c + \frac{kc+\delta}{\mu-\varepsilon}$. Now for any (y,t) such that $|s - t| \leq c$ and $||y - x|| \leq \rho_x(\delta)$, we have:

$$\begin{aligned} ||C(t,y) - C(s,y)|| \leq\ & ||C(\min(t, s_y), y) - C(s_x - c, y)|| \\ & + ||C(s_x - c, y) - C(s_x - c, x)|| \\ & + ||C(s_x - c, x) - C(s_x, x)|| \\ \leq\ & \frac{\delta + kc}{\mu - \varepsilon} + \delta + c. \end{aligned}$$

The only case left is when $\phi(x) = a$. Then $C(s,x) = x$ for all $s \in \mathbb{R}^+$. Since $u \mapsto \max(a, \phi(u))$ is k-lipschitz, we have $s_y \leq \frac{k||x-y||}{\mu-\varepsilon}$. We can write:

$$||C(s,y) - C(s,x)|| \leq ||C(s,y) - y|| + ||y - x|| \leq \left(\frac{k}{\mu - \varepsilon} + 1\right) ||x - y||\,.$$

and thus C is continuous at (s,x). Finally, let $C_\phi(t,x) = C\left(\frac{(b-a)t}{\mu-\varepsilon}, x\right)$ to obtain an homotopy such that $\phi^{-1}(-\infty, a]$ is a strong deformation retraction of $\phi^{-1}(-\infty, b]$.

□

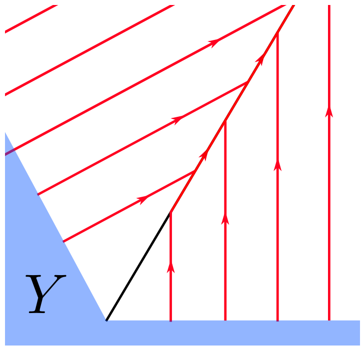


Figure 2.7 – Exact flow of the locally semi-concave map d_Y.

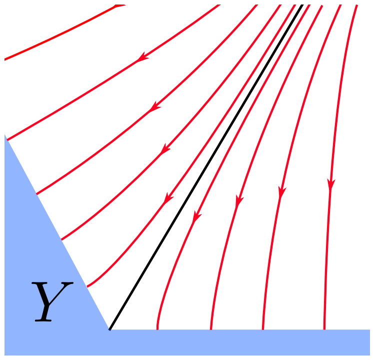


Figure 2.8 – Approximate inverse flow of d_Y outside of Y.

Remark 2.10 – It is a well-known fact in Alexandrov's geometry [Pet07] and in the optimal control community [CS04] that any semiconcave map $f : \operatorname{dom} f \subset \mathbb{R}^d \to \mathbb{R}$ admits a continuous flow $\operatorname{dom} f \times \mathbb{R}^+ \to \operatorname{dom} f$ whose trajectories make f increase at a rate $\Delta(\partial^* f)$ when parametrized by the arc-length. Semiconcave maps notably include functions of the form $(d_K)^2 + \phi$ where K is a compact subset of $\mathbb{R}^d$ and ϕ any C^2 map, or $(d_K)_{|\mathbb{R}^d \setminus V}$, where V is an open neighborhood of K. However, when $-f$ is not semiconcave this exact flow is not bijective, and it cannot be reversed to obtain a continuous flow making f decrease at rate $\Delta(\partial^* f)$, as in Figure 2.7. Proposition 2.9 shows that under some hypothesis on the Clarke gradient, such "ideal inverse flows" can be approximated as illustrated in Figure 2.8.

We will see that any compact sublevel set of a Lipschitz function at a regular value are always Lipschitz domains. The concept of *weak regular value* allows sublevel sets with empty interior.

Definition 2.11 (Weak regular values of a Lipschitz function)**.** Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. We say that α is a *weak regular value* of f when there exist $\mu > 0, \delta > 0$ such that $\alpha < f(x) \leq \alpha + \delta \implies \Delta(\partial^* f(x)) \geq \mu$.

2.2.4 Co-area formula for rectifiable sets

The last property of smooth functions that we want to generalize to Lipschitz maps is that their integrals are subject to change of variable formulas. The formula we use applies to any Lipschitz maps, non-necessarily injective, between sets that are called *rectifiable*.

Definition 2.12 (Rectifiability)**.** A set A included in $\mathbb{R}^d$ is said to be m-rectifiable when it can be written $A = \bigcup_{i=0}^{\infty} A_i$ with $\mathcal{H}^m(A_0) = 0$ and where each A_i, $i \geq 1$ is the image of a bounded subset of $\mathbb{R}^m$ by a Lipschitz map.

Any m-Lipschitz submanifold of $\mathbb{R}^d$ is m-rectifiable as it is the union of a countable number of patches of graphs of bounded subsets of $\mathbb{R}^m$. The converse is not true, since m-rectifiable sets contain m-submanifolds with self-intersections, or any set with $\mathcal{H}^m$-measure zero.

A k-rectifiable set admits $\mathcal{H}^k$-almost everywhere so-called approximate tangent spaces of dimension k [Fed69, 3.2.16] which coincide almost everywhere with the tangent cones defined later in Section 3.3. Now let f be a Lipschitz map between two open subsets of Euclidean spaces containing respectively V, W rectifiable subsets of respective dimension d and m, such that $f(V) \subset W$. Further assume that f is differentiable at x and that V admits an approximate tangent space $\mathrm{Tan}(V,x)$ of dimension d. We let $J_i f_{|V}(x)$ be the supremum of the volume of the convex hull of $D_x f(e_1), \dots, D_x f(e_i)$ when $e_1, \dots, e_i$ run among the families of i orthonormal vectors of $\mathrm{Tan}(V,x)$. In particular, when $i = d$, $J_d f_{|V}(x)$ is equal to the absolute value of the determinant of $D_x f$ restricted to $\mathrm{Tan}(V,x)$.

Then, the Co-area formula from Federer [Fed69, Theorem 3.2.22] states the following generalization of the classical change of variable formula.

Theorem 2.13 (Co-area formula)**.** *Let V, W be respectively d and m-rectifiable subsets in Euclidean spaces with $d \geq m$, let f be a Lipschitz function defined on an neighborhood of V such that $f(V) \subset W$, and let g be a $\mathcal{H}^d$-measurable function on V. Then, $\mathcal{H}^m$-almost everywhere on W, the set $f^{-1}(z)$ is $\mathcal{H}^{d-m}$-measurable, and we have*

$$\int_V g(x)\, J_m f_{|V}(x)\, \mathrm{d}\mathcal{H}^d(x) = \int_{z \in W} \left(\int_{u \in f^{-1}(z)} g(u)\, \mathrm{d}\mathcal{H}^{d-m}(u) \right) \mathrm{d}\mathcal{H}^m(z). \tag{2.9}$$

When the context is clear, me might write $J_i f$ instead of $J_i f_{|V}$. A consequence of Co-area formula is the following weak version of Sard's theorem on Lipschitz maps:

Theorem 2.14 (Weak Sard's theorem)**.** *Let V, W be d-rectifiable subsets in Euclidean spaces, and let f be a Lipschitz function between the previous Euclidean spaces such that $f(V) = W$. Let $A := \{x \in V \mid J_d f_{|V}(x) = 0\}$. Then*

$$\mathcal{H}^d(f(A)) = 0. \tag{2.10}$$

Proof.

By the previous theorem, with $g = 1$, for any $\mathcal{H}^d$-measurable set U we have:

$$\begin{aligned} \int_{V \cap U} J_d f(x)\, \mathrm{d}\mathcal{H}^d(x) &= \int_W \left(\sum_{z \in f^{-1}(y) \cap U} 1 \right) \mathrm{d}\mathcal{H}^d(y) \\ &\geq \int_{f(U)} 1\, \mathrm{d}\mathcal{H}^d(y) = \mathcal{H}^d(f(U)). \end{aligned}$$

With $U = A$, the left-hand side is zero.

□

Chapter 3

Geometric tools, μ-reach and complementary regular sets

La géométrie est l'art de raisonner juste sur des figures fausses.

The geometry of subsets of Euclidean spaces has been studied for a long time, and the range of concepts at our disposal is broad. We describe several tools, such asdistance functions associated to closed sets, which allow the use of Lipschitz analysis concepts in the study of their geometry and lead to the definitions of the reach and the μ-reach of a set. We show that having a positive μ-reach does not necessarily entail regularity properties, as we give explicit constructions of pathological, fractal-like compact sets satisfying this assumption. Other tools include tangent and normal cones, which are generalizations of tangent spaces and their associated orthogonal complements, to sets which are not necessarily differential manifolds. We notably prove an identity between the normal cones of a set with positive μ-reach and the Clarke gradient of its distance function. Using the previous tools, we define the class of complementary regular sets*, which are the sets for which we will extend Morse theory in Chapter 4. We characterize them as being the offsets of compact sets at a regular value of their distance functions.*

3.1 Distance functions and their Clarke gradients

3.1.1 Distance functions and sets with positive reach

For any subset A of $\mathbb{R}^d$, we define the *distance to* A function $d_A : \mathbb{R}^d \to \mathbb{R}^+$ as:

$$d_A : x \mapsto \inf_{a \in A} ||x - a|| \,. \tag{3.1}$$

We saw earlier that closed positive sublevel sets of d_A were called *offsets* of A, denoted by $A^r := d_A^{-1}(-\infty, r]$ for any positive r. The class of compact subsets of $\mathbb{R}^d$ is endowed with a metric d_H * called the *Hausdorff distance*, defined as follows:

$$d_H(A, B) := \inf\{t \in \mathbb{R} \mid B \subset A^t \text{ and } A \subset B^t\}. \tag{3.2}$$

This quantity coincides with $||d_A - d_B||_\infty = \sup_{x \in \mathbb{R}^d} |d_A(x) - d_B(x)|$.

We denote by $\Gamma_A(x) := \{a \in A \mid d_A(x) = ||x - a||\}$ the set of closest points to x in A. This set is non-empty when A is closed. When $\Gamma_A(x)$ contains a unique point, we denote this point by $\xi_A(x)$. The *medial axis* $\mathcal{M}_A$ of a closed set A consists in the points of $\mathbb{R}^d$ with strictly more than one point in $\Gamma_A(x)$, i.e., such that

$$\mathcal{M}_A := \left\{x \in \mathbb{R}^d \mid \operatorname{Card}(\Gamma_A(x)) > 1\right\}.$$

As a consequence of the triangular inequality, distance maps are 1-Lipschitz, and thus differentiable almost everywhere in $\mathbb{R}^d$ by Rademacher's theorem. The differentiability of d_A is related to projection onto A in the following sense:

Proposition 3.1 (Differentiability of distance maps [Fed59, Theorem 4.8])**.** *Let A be a closed subset of $\mathbb{R}^d$. Then d_A is differentiable at a point $x \notin A$ if and only if x has a unique closest point $\xi_A(x)$ in A, and we have:*

$$\nabla d_A(x) = \frac{\xi_A(x) - x}{||\xi_A(x) - x||}. \tag{3.3}$$

The map d_A is differentiable at $x \in A$ if and only if $x \in \operatorname{int}(A)$, in which case $\nabla d_A(x) = 0$.

In particular, this shows that the Lebesgue measure of the medial axis of any subset of $\mathbb{R}^d$ is zero. Note that there exists compact subsets of $\mathbb{R}^d$ whose medial axis is dense in their complement set (see [Mér09, Lemma I.2]).

The differentiability of d_A inside an offset A^t (but outside A) has strong consequences on the regularity of A, notably endowing it with well-defined curvatures, as we will see in Chapter 6. This assumption is equivalent to the existence of a map $\xi_A : A^t \to A$ associating each point to its closest point.

Definition 3.2 (Reach of a set)**.** The reach of a subset A of $\mathbb{R}^d$ is defined as the supremum of the size of offsets for which the projection onto the closest point is well-defined.

$$\operatorname{reach}(A) := \sup\{t \in \mathbb{R} \mid \text{For all } x \in A^t \setminus A, \operatorname{Card}(\Gamma_A(x)) = 1\}. \tag{3.4}$$

$*$. The notation d_H will always refer to the Hausdorff distance, and never to the distance to a set called H.

The 1959 article *Curvature Measures* of Federer [Fed59] was the first to underline the importance of the class of sets with positive reach. This class contains compact submanifolds of $\mathbb{R}^d$ and $C^{1,1}$-domains with compact boundaries. Intuitively, any compact stratified subset of $\mathbb{R}^d$ whose corners are convex has positive reach. Compact convex sets of $\mathbb{R}^d$ are characterized by having reach $+\infty$.

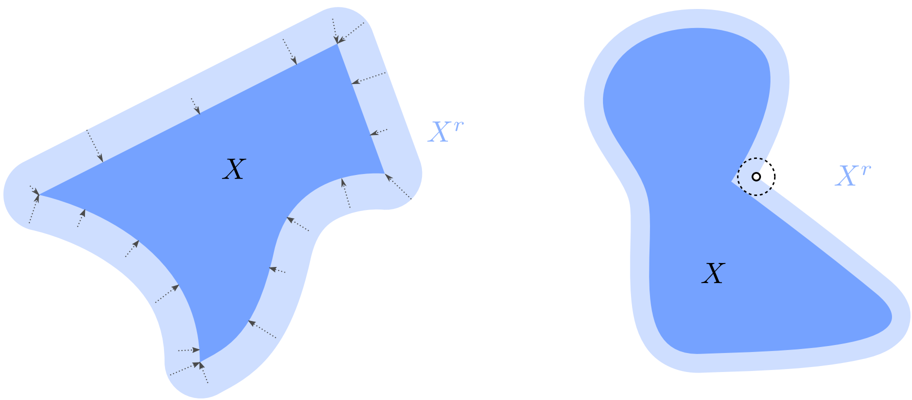


Figure 3.1 – The set on the left has reach greater than r. The one on the right has reach 0 because of the concave corner.

Non-empty intersections of balls of a bounded size with a set of positive reach are necessarily contractible.

Proposition 3.3 (Non-empty intersection of small balls with sets with positive reach are contractible [RZ19, Lemma 5.5])**.** *Let* $A \subset \mathbb{R}^d$ *and* $r \in \mathbb{R}$ *be such that* $0 < r < \mathrm{reach}(A)$*. Then* $A \cap B(x, r)$ *is contractible when* $x \in A^r$ *and empty otherwise.*

Another consequence of a set A having positive reach is that the associated distance map d_A is semi-convex on small neighborhoods of A [Kle81, Satz 2.8]. We precise this fact by giving explicit bounds on the semi-convexity constant and the size of neighborhood, which, to the best of our knowledge, is an elementary new result.

Proposition 3.4 (Reach and semi-convexity of the distance function)**.** *Let* $A \subset \mathbb{R}^d$ *and let* $0 \leq q < r \leq \mathrm{reach}(A)$*. Then on* $\overline{A^q \setminus A}$*, the function* $x \mapsto d_A(x) + \frac{1}{r-q} \left|\left|x\right|\right|^2$ *is convex.*

Proof.

Let x, y be in $A^q \setminus A$ and let a, b be their respective projection on A, with c, c' their distance to A. Since $\mathrm{reach}(A^c) \geq r - c$ and the same with c', we have $\langle \nabla d_A(x), x - y\rangle \geq -\frac{||y-x||^2}{2(r-c)}$ and $\langle \nabla d_A(y), y - x\rangle \geq -\frac{||y-x||^2}{2(r-c')}$. These two inequalities, along with $c, c' \leq q$, yield

$$\langle \nabla d_A(x) - \nabla d_A(y), x - y\rangle \geq -\frac{1}{r-q} \left|\left|x - y\right|\right|^2 . \tag{3.5}$$

Now we want to show that for any segment $[x, y]$ lying in $A^q \setminus A$, the map

$$\phi_A^{r-q} : t \mapsto d_A(ty + (1-t)x) + \frac{||ty + (1-t)x||^2}{2(r-q)} \tag{3.6}$$

is convex. Assuming that the segment $[x, y]$ lies inside $A^q \setminus A$, this map is C^1 and Equation (3.5) shows that its derivative is non-decreasing.

□

Federer proved that below the reach, the gradient of d_A was Lipschitz.

Proposition 3.5 (Small offsets of a set with positive reach are $C^{1,1}$ domains [Fed59, Theorem 4.8, (8)])**.** *Let A be a set with $\mathrm{reach}(A) = r > 0$. Then for any $0 < q < r$, the restriction of ξ_A to A^q is $\frac{r}{r-q}$-Lipschitz. As a consequence, A^q is a $C^{1,1}$-domain.*

Remark 3.6 – Sets of positive reach have been the subjects of many studies, and have been proven to enjoy many geometric properties generalizing that of manifolds and convex sets, such as their curvatures (see Chapter 6) or their geodesics [BLW19]. Similar properties are satisfied by stratified sets (defined Section 2.1). However, these classes are distinct and there is no inclusion of one into the other. Think for example of a piecewise, half-spiral, with ever-increasing precision around its center. This objet is convex, but its stratification cannot be locally finite. On the other hand, any stratified with a non-convex corner has reach zero.

3.1.2 Clarke gradients of distance functions

Sets with reach zero are plentiful, as any set with a non-convex corner has reach zero. At any point x outside A, it is well-known (see [Fu85, Cla75]) that one can describe the Clarke gradient $\partial^* d_A(x)$ as a function of the closest points $\Gamma_A(x)$, as illustrated in Figure 3.2.

Proposition 3.7 (Clarke gradient of d_A outside A)**.** *Let A be a closed subset of $\mathbb{R}^d$. Then for any $x \notin A$, we have*

$$\partial^* d_A(x) = \mathrm{Conv}\left\{ \frac{z - x}{||z - x||} \;\middle|\; z \in \Gamma_A(x) \right\}. \tag{3.7}$$

Proof.

Let x_n be a sequence of points where d_A is differentiable converging to x such that $\nabla d_A(x_n)$ has a limit v. For n big enough, $x_n \notin A$ and we have $\nabla d_A(x_n) = \frac{x_n - \xi_A(x_n)}{d_A(x_n)}$. Since $d_A(x_n)$ converges to $d_A(x)$ and A is closed, any accumulation point z of the sequence $\xi_A(x_n)$ converges belongs to A. Furthermore, $v = \frac{x-z}{d_A(x)}$ is a unit vector, yielding $z \in \Gamma_A(x)$.

□

When $\Delta(\partial^* d_A(x))$ is bounded away from 0 in a neighborhood of a set X, the approximate inverse flows of Proposition 2.9 show that there are continuous trajectories in $A^t \backslash A$ whose endpoints are in A, as illustrated in Figure 3.3.

The Clarke gradient of d_A is related to the generalized gradient ∇_A of Lieutier [Lie04], which is defined as follows.

Definition 3.8 (Generalized gradient of the distance function)**.** Let A be a closed subset of $\mathbb{R}^d$. The generalized gradient of d_A, which we denote by ∇_A, is defined by:

$$\nabla_A(x) := \frac{x - c_A(x)}{d_A(x)}, \tag{3.8}$$

where $c_A(x)$ is the center of the smallest ball containing $\Gamma_A(x)$.

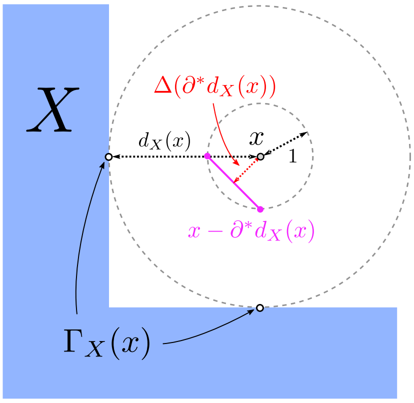


Figure 3.2 – Configuration of $\partial^* d_X(x)$.

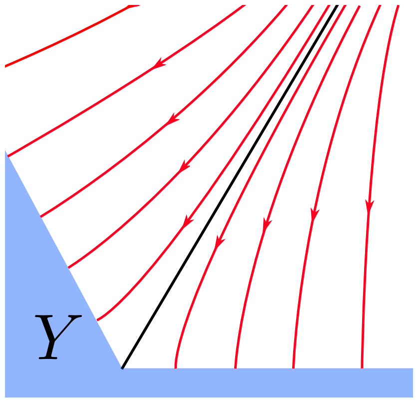


Figure 3.3 – Approximate flow of $-d_Y$.

The norm of $\nabla_A(x)$ is the cosine of the angle between $c_A(x), x$ and any point at distance $d_A(x)$ of x lying in the affine hyperplane orthogonal to $x - c_A(x)$ containing $c_A(x)$, as illustrated in Figure 3.4. Moreover, the generalized gradient $\nabla_A(x)$ coincides with the closest point to 0 in $\partial^* d_A(x)$.

The main point of [Lie04] was that the Euler scheme of the map $\nabla_A : \mathbb{R}^d \setminus A \to \mathbb{R}^d$ converges to a continuous flow. Parametrized to the arc length, this flow has 1-Lipschitz trajectories that increase d_A at the rate $||\nabla_A||$. However, this flow cannot be reversed as it is not injective, explaining the use of approximate inverse flows of Proposition 2.9 to build distance-decreasing Lipschitz trajectories.

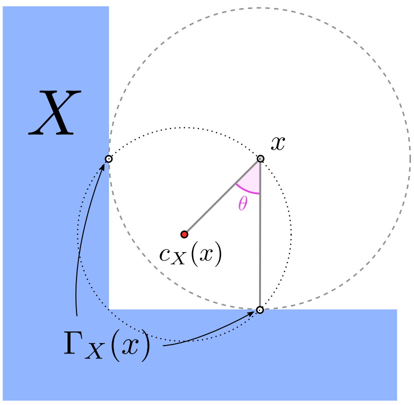


Figure 3.4 – Configuration of $\nabla_X(x)$.

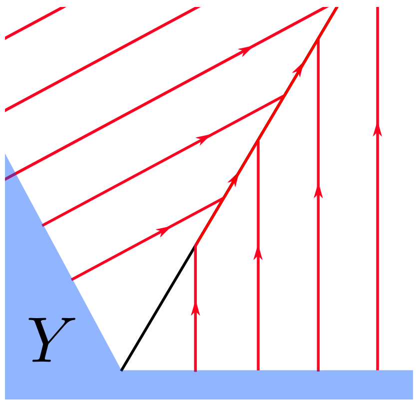


Figure 3.5 – Flow of ∇_Y.

3.2 Sets with positive μ-reach

3.2.1 Definition and history

The reach of a subset A of $\mathbb{R}^d$ can equivalently be defined as the maximum of the sizes of offset for which the Clarke gradient of d_A stands at distance 1 to zero. Replacing 1 by an arbitrary threshold $\mu \in (0, 1]$ in the previous definition gives the μ-reach of A.

Definition 3.9 (μ-reach of a set)**.** Let μ in $(0, 1]$ and $A \subset \mathbb{R}^d$. The μ-reach of A is:

$$\mathrm{reach}_\mu(A) := \sup\left\{t \in \mathbb{R} \mid \forall x \in A^t \setminus A,\, \Delta(\partial^* d_A(x)) \geq \mu\right\}. \tag{3.9}$$

For any A, the map $\mu \mapsto \mathrm{reach}_\mu(A)$ is non-increasing, and this quantity coincides with the previously defined reach when $\mu = 1$.

Graphs of Lipschitz functions all have positive μ-reach for some $\mu > 0$ depending on their Lipschitz constant.

Proposition 3.10 (μ-reach of the graph of Lipschitz function)**.** *Let $f : \mathbb{R}^{d-1} \to \mathbb{R}$ be a k-Lipschitz function and let $l(k) = (1+k^2)^{-1}$. Then $\mathrm{reach}_{l(k)}(\mathrm{graph} f) = +\infty$.*

As a consequence, compact Lipschitz submanifolds and Lipschitz domains with compact boundaries of $\mathbb{R}^d$ all have a positive μ-reach for some $\mu > 0$.

Proof.

Let $x \in \mathbb{R}^d \setminus \mathrm{graph} f$ and let a, b be two closest points to x in the graph. Let θ be the half-angle between a, x and b. The graph of f lies outside the ball centered in x with radius $d_{\mathrm{graph} f}(x)$, but contains a and b. Around each point $(z, f(z))$, the graph of f is contained in the cone $\{(z_1, z_2) \mid ||z_2 - f(z)|| \leq k\, ||z_1 - z||\}$, as illustrated in green around the point b of Figure 3.6.

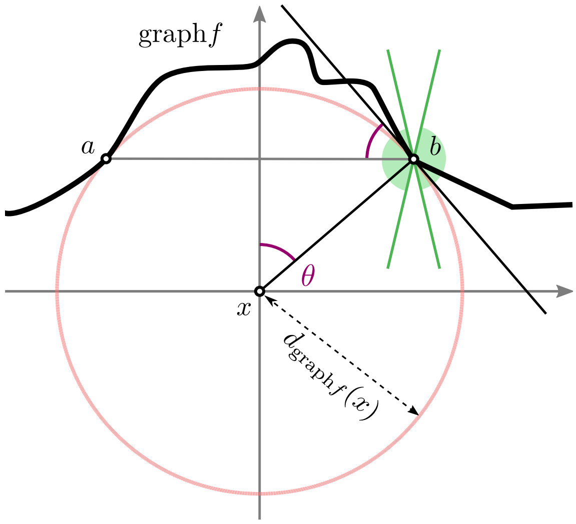


Figure 3.6 – Bound on the angle between two closest point of the graph of a Lipschitz map.

The intersection of the cone with the complement set of the ball of radius $d_{\mathrm{graph} f}(x)$ is non-empty, as it contains the graph of the function. This yields $\tan(\theta) \leq k$ which is equivalent to $\frac{1}{\sqrt{1+k^2}} \leq \cos(\theta)$.

□

Although easily defined, the μ-reach has only been recently studied in detail with the emergence of geometric inference [CCLT07, ABL23, Lie04, KSC+20]. It is closely related to the β-reach of [Cot24]. The problem of estimating the μ-reach of a set from a sampling was studied in [ABL23].

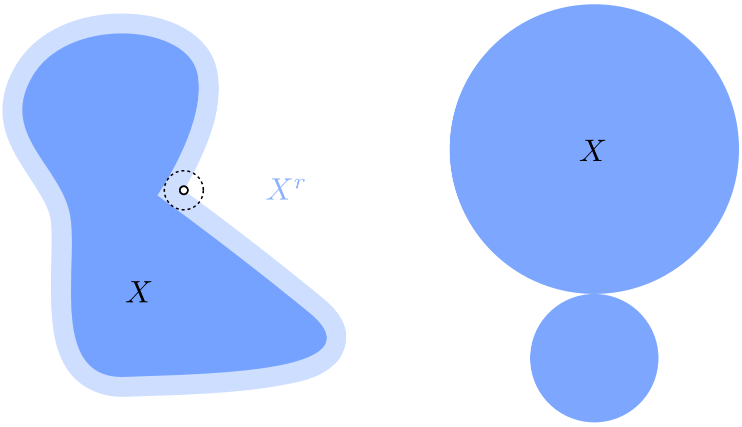


Figure 3.7 – The set on the left has positive μ reach for μ small enough. On the right, the set has μ-reach 0 for all μ, but a positive weak feature size.

The main property of sets with a positive μ-reach that we will use is the existence of approximate inverse flows of it distance function Proposition 2.9 linking A to its tubular neighborhoods A^t when $0 < t < \mathrm{reach}_\mu(A)$. In the geometric inference literature, one may also encounter the so-called the *weak feature size*:

Definition 3.11 (Weak feature size of a set)**.** Let A be a closed subset of $\mathbb{R}^d$. Its *weak feature size* [CL05] is:

$$wfs(A) := \sup\left\{t \in \mathbb{R}^+ \mid 0 < d_A(x) \le t \implies \Delta(\partial^* d_A(x)) > 0\right\}. \tag{3.10}$$

This quantity has been used to infer topological information on small offsets of A [CL05, CSEH05]. Sets with positive weak feature size might have μ-reach zero for every $\mu > 0$, such as the union of two balls intersecting in one point. Any subanalytic set of $\mathbb{R}^d$ has a positive weak feature size [Fu94]. The difference between having a positive μ-reach for an arbitrary μ in $(0, 1)$ and having a positive weak feature size lies close to A: for any $0 < t \le s < wfs(A)$, by the semi-continuity of the Clarke gradient $\Delta \circ \partial^* d_A$ is uniformly bounded from below on $A^s \setminus A^t$, implying that A^s deformation retracts onto A^t. However, A^s might not deformation retracts onto A (see *Warsaw's circle*, [Spa66, Example 2.4.8], which does not have the homotopy type of a circle contrary to its small offsets).

Any set with a positive μ-reach must have a "well-behaved" topology, in the following sense.

Proposition 3.12 (Homotopy type of sets with positive μ-reach)**.** *Let $\mu > 0$ and $A \subset \mathbb{R}^d$ be such that $\mathrm{reach}_\mu(A) > 0$. Then A has the homotopy type a CW-Complex. In particular, its homology groups are finitely-generated.*

Proof.

The tubular neighborhood A^t deformation retracts into A for t small enough, and every Euclidean neighborhood retract has the homotopy type of a CW-Complex [Hat02].

□

The complement of small offsets of sets with positive μ-reach or weak-feature size have positive reach.

Proposition 3.13 (Complement of offsets have positive reach)**.** *Let $A \subset \mathbb{R}^d$ and let $0 < t \leq wfs(A)$. Then $\neg(A^t)$ has a positive reach. Moreover, if $t \leq \mathrm{reach}_\mu(A)$ for some $\mu \in (0,1]$, then we have*

$$\mathrm{reach}(\neg A^t) \geq \mu t. \tag{3.11}$$

Proof.

For the first part, remark that d_A^2 is semi-concave Proposition 3.40. The result follows from the classical fact that sublevel sets of semi-convex functions at weak regular values have positive reach, which we prove later at Corollary 3.42. The second assertion, more involved, can be obtained using the flow of Lieutier as [CCLT07, Theorem 4.1] or using a characterization of the reach as the supremum of certain functionals as in [RWZ23, Lemma 6.3].

□

However, A having a positive μ-reach for some positive μ is not a strong regularity assumption, in the sense that for any $\mu \in (0,1)$ there exist fractal-like objects with positive μ-reach. The next section is dedicated to an explicit construction of such objects.

3.2.2 Construction of a non-rectifiable curve with positive μ-reach.

In this section, we prove that for any $\mu \in (0,1)$, there exists a compact, unrectifiable continuous curve K in $\mathbb{R}^2$ such that $\mathrm{reach}_\mu(K) = \infty$. This set is obtained as the limit of a sequence of compact piecewise linear curves $(K_n)_{n\in\mathbb{N}}$ built by induction. The induction steps depend on a sequence of angle parameters $(\theta_i)_{i\in\mathbb{N}}$. We also show that for another choice of parameters, the limit curve is rectifiable, but its small offsets $K(\theta)^\varepsilon$ have total curvature going to infinity as $\varepsilon \to 0$.

Definition 3.14 (Inductive definition of K)**.** The sequence of compact subsets K_n of $\mathbb{R}^2$ is obtained by induction as follows.

- The set K_0 consists in a segment of length 1, oriented by the choice of the starting point $(0,0)$ and the ending point $(1,0)$.
- Each set K_{n+1} is obtained from K_n by either folding each segment of K_n in half upwards when n is even or downwards when it is not, each with a folding angle of θ_n. This is illustrated in Figure 3.8.

We name some points, segments and triangles further pertaining to our analysis.

Definition 3.15 (Further definitions)**.** Let n be a natural number.

- We define $(L_{i,n})_{1\leq i\leq 2^n}$ to be the set of all segments among the piecewise linear curve K_n, with order induced by its starting point $(0,0)$ and its ending point $(1,0)$.
- We define $\{p_{i,n} \mid 0 \leq i \leq 2^n\}$ to be the ordered set of extremities of the segments, with $p_{i-1,n}$ (resp. $p_{i,n}$) being the starting (resp. ending) point of $L_{i,n}$.
- When $n \geq 1$, we define a collection T_n of 2^{n-1} triangles $T_{i,n}$ of $\mathbb{R}^2$ defined by

$$T_{i,n} := \mathrm{Conv}(p_{2i,n}, p_{2i+1,n}, p_{2(i+1),n}).$$

  It is bounded by the three segments $L_{i+1,n-1}, L_{2i+1,n}, L_{2i+2,n}$ and any segment $[p_{k,n}, p_{k+1,n}]$ is included in $T_{\lfloor k/2\rfloor,n}$.

The length of K_n can be iteratively computed.

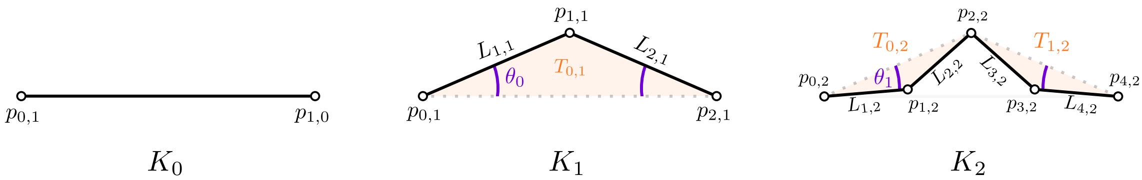


Figure 3.8 – Folding process to obtain K_2 from K_0.

Proposition 3.16 (Length of K_n). *The length of the piecewise linear curve K_n is:*

$$\mathcal{H}^1(K_n) = \prod_{i=0}^{n-1} \frac{1}{\cos(\theta_i)}.$$

This quantity is non-decreasing as a function of n. It is bounded as n goes to infinity if and only if the sum $\sum \theta_i^2$ diverges. Each segment of K_n has the same length $l_n := \mathcal{H}^1(L_{i,n}) = \frac{1}{2^n}\mathcal{H}^1(K_n)$.

Proof.

By elementary trigonometry, the sum of the length of the two segments obtained by folding a segment from K_n is $\cos(\theta_n)^{-1}$ times the length of the previous segment.

□

When the angles bounded above by $\pi/4$ and non-increasing, triangles in T_{n+1} are subtriangles of the ones in T_n, as precised in the following proposition.

Proposition 3.17 (Partial order on triangles). *If $\theta_{n+1} < \theta_n \leq \pi/4$, then the intersection $T_{2i,n+1} \cap T_{2i+1,n+1}$ consists in one point, and both $T_{2i,n+1}, T_{2i+1,n+1}$ are included in $T_{i,n}$ for any $0 \leq i \leq 2^n - 1$.*

Moreover, if θ is non-increasing and strictly bounded from above by $\pi/4$, two triangles in T_n have non-empty intersection if and only if they have consecutive indices, and the partial order induced by inclusions $T_{2i,n+1}, T_{2i+1,n+1} \subset T_{i,n}$ for $n \in \mathbb{N}, 0 \leq i \leq 2^n - 1$ endows $T := \cup_{n\in\mathbb{N}} T_n$ with a binary tree structure.

In particular, if $n \leq m$ the curve K_m is included into the unions of triangles at step n, i.e., $K_m \subset \bigcup_{i=0}^{2^n-1} T_{i,n}$.

Proof.

The upper angle of a triangle $T_{i,n}$ at a step n is $\pi - 2\theta_n$. If $\theta_{n+1} \leq \theta_n \leq \pi/4$, then θ_{n+1} is smaller than the half-angle $\frac{\pi-2\theta_n}{2}$, yielding that $T_{2i,n+1}, T_{2i+1,n+1}$ are respectively included into the left and right half of $T_{i,n}$ with respect to the bisector of the upper vertex. This is illustrated in Figure 3.9.

If θ is non-increasing and strictly bounded from above by $\pi/4$, this reasoning can be iterated. Further subtriangles which are not consecutive do not keep their intersection point.

□

Under these assumptions, the sequence of curves K_n converges to a compact continuous curve K.

Proposition 3.18 (Convergence properties). *If θ is non-increasing and bounded from above by $\pi/4$, the sequence K_i converges to a compact continuous curve K of $\mathbb{R}^2$ which does not self-intersect.*

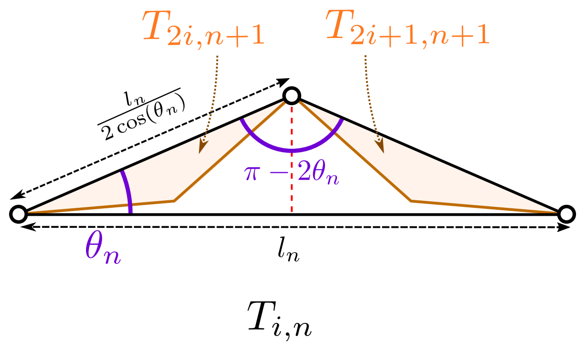


Figure 3.9 – Inclusions $T_{2i,n+1}, T_{2i+1,n+1} \subset T_{i,n}$.

Proof.

By assumption, $c := \limsup_{i\to\infty}(2\cos(\theta_i))^{-1} < 1/$ For every $n \in \mathbb{N}$, the set K_n is a continuous curve which can be parametrized as a piecewise linear function $f_n : [0,1] \to \mathbb{R}^2$ interpolating linearly between $f_n\left(\frac{k}{2^n}\right) = p_{k,n}$ for every $0 \leq k \leq 2^n$. By the iterated inclusions of the previous lemma, if $m > n$, the set $f_m\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)$ is included in the triangle $T_{k,n}$, while $f_n\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right)$ is exactly the segment $\mathrm{Conv}(p_{k,n}, p_{k+1,n})$. All in all, for any $t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$, we have:

$$||f_n(t) - f_m(t)|| \leq \sup\{||y - z|| \mid y \in [p_{k,n}, p_{k+1,n}], z \in T_{\lfloor k/2\rfloor,n}\} \leq \mathrm{diam}(T_{\lfloor k/2\rfloor,n}).$$

By the assumption we have $c < 1$, which implies that the largest side of a triangle at step n is $l_n = O(c^n)$ and so is its diameter.
The sequence of continuous functions f_n is thus Cauchy with respect to the infinity norm over maps $[0,1] \to \mathbb{R}^2$. Letting f be its limit, the set $K := f([0,1])$ is compact. Moreover, for any $s < t$ in $[0,1]$, points $f(s)$ and $f(t)$ belong to non-intersecting triangles at any step n such that $t - s > 2^{-n}$, which implies that f is injective.

□

When the sequence $\mathcal{H}^1(K_n)$ diverges to infinity, the limit curve K is not rectifiable.

Proposition 3.19 (Unrectifiablity of K)**.** *If θ is non-increasing, bounded from above by $\pi/4$ and the sum $\sum \theta_i^2$ diverges, then the limit continuous curve K is non-rectifiable, i.e., $\mathcal{H}^1(K) = \infty$.*

Proof.

Since $K = f([0,1])$, its 1-Hausdorff measure is equal to the total variation of f. Recalling that $f(k/2^n) = p_{k,n}$ for $0 \leq k \leq 2^n$, we have:

$$\begin{aligned}\mathcal{H}^1(K) &= \sup_{\substack{0\leq a_0<\cdots<a_n\leq 1\\ n\in\mathbb{N}}} \sum_{i=0}^{n-1} ||f(a_{i+1}) - f(a_i)|| \\ &\geq \sup_{n\in\mathbb{N}} \left(\sum_{k=0}^{2^n-1} ||p_{k+1,n} - p_{k,n}||\right) \\ &\geq \sup_{n\in\mathbb{N}} \mathcal{H}^1(K_n) = \infty.\end{aligned}$$

□

Nevertheless, when the sequence of angles tends to 0, the limit curve has necessarily Hausdorff dimension 1.

Proposition 3.20 (Hausdorff dimension 1)**.** *Assume that the sequence (θ_i) tends to 0. Then for every $s > 1$,*

$$\mathcal{H}^s(K) = 0. \tag{3.12}$$

Proof.

Let $s \in \mathbb{R}$. The limit set K is included in $\cup_{0 \leq i \leq 2^n - 1} T_{i,n}$ for every $n \in \mathbb{N}$, and as such each point of K belongs to infinitely many triangles of the form $T_{i,n}$. Per classical result on the upper-bounding of Hausdorff dimensions, we know that:

$$\sum_{n \geq 1, 0 \leq i \leq 2^n - 1} \operatorname{diam}(T_{i,n})^s < \infty \text{ implies } \mathcal{H}^s(K) = 0.$$

Thanks to previous computations, the left-hand side can be expressed as

$$\sum_{n \in \mathbb{N}} 2^n l_n^s = \sum_{n \in \mathbb{N}} 2^{n(1-s)} \prod_{i=1}^{n} \cos(\theta_i)^{-1}.$$

Terms in the right-hand side sum are equal to $\exp\left(\ln(2) n(1-s) - s \sum_{i=1}^n \ln(\cos(\theta_i)\right)$. Assuming $\theta_i \to 0$ we have as functions of n

$$\sum_{i=1}^{n} \ln(\cos(\theta_i)) \sim \frac{1}{2} \sum_{i=1}^{n} \theta_i^2 = o(n)$$

which yields the desired result.

□

Using the folding process, we can bound the angles of two closest points a, b in K to any point $x \notin K$.

Proposition 3.21 (Choice of θ to obtain a non-rectifiable curve with positive μ-reach)**.** *Let $\mu \in (0, 1)$. If the sequence θ is non-increasing, is bounded from above by $\left(\frac{1-\mu}{8}\right)^{1/2}$ and if the sum $\sum_{i \in \mathbb{N}} \theta_i^2$ diverges, then the compact set K obtained by the folding process is a non-rectifiable continuous curve such that* $\operatorname{reach}_\mu(K) = \infty$.

Proof.

By Proposition 3.18, the limit curve K is properly defined as θ is bounded from above by $\left(\frac{1-\mu}{8}\right)^{1/2} < \pi/4$. Let $x \in \mathbb{R}^2$ lie in the medial axis of K, and let a, b be two distinct elements of the set of closest point to x in K. Let

$$n := \sup\left\{m \mid \exists\, j \text{ such that } a, b \in T_{j,m}\right\}$$

be the largest step m for which a and b belongs to a same triangle in T_m. This number is well-defined since a, b belong to T_0 and is finite as the diameter of each $T_{i,m}$ goes to 0 when $m \to \infty$. Let i be the index of this triangle in T_n. Without loss of generality, we can assume that $a \in T_{2i,n+1}$ and $b \in T_{2i+1,n+1}$.

Now the curve K inside $T_{2i,n+1}, T_{2i+1,n+1}$ is symmetric around the bisector of $T_{i,n}$ at the point of intersection $T_{2i,n+1} \cap T_{2i+1,n+1}$, meaning that x lies in this bisector. We want to bound the cosine of the half-angle between the three points a, x and b. Without loss of generality, after translating and rotating, we obtain the configuration of Figure 3.10.

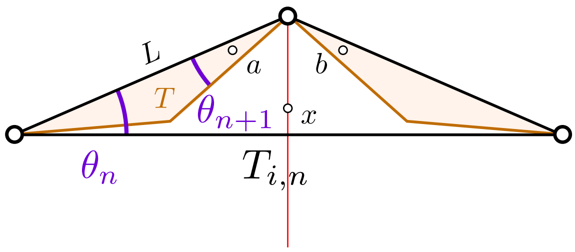


Figure 3.10 – Schematic configuration of a, b and x.

Let $B_{\max}$ be the largest disk centered in x which does not go above the segment L. By elementary trigonometry, $B_{\max}$ has radius $\cos(\theta)\,||x||$. Since K is a continuous curve included in the triangle T which connects the extremities of the segment L,K crosses $B_{\max}$ and $B_{\max} \cap T$ contains a.
Since the sequence θ is non-increasing, the set $B_{\max} \cap T$ is included in $B_{\max} \cap C$ where C is the cone of points lying in between the lines Δ_1, Δ_2 with respective angles θ and 2θ to the horizontal line. Without loss of generality, we can take the top of the triangle to be the origin as in Figure 3.11. For any $u \in B_{\max} \cap C$, the cosine of the half angle between u, x and the symmetric of u by the bisector is exactly

$$\frac{\pi_2(u-x)}{||u-x||} \tag{3.13}$$

where $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the second coordinate.
If C intersects $B_{\max}$ only in its upper left quarter, the minimum of this quantity is attained at the second intersection of $B_{\max}$ with the line Δ_2, as this point has the largest distance to x and the difference between its second coordinate and that of x is the smallest among $B_{\max} \cap C$. We denote this point v. Its coordinates are obtained solving for the largest solution t^+ of the equation $||tw - x||^2 = \cos(\theta)^2\,||x||^2$ where w is the unit vector with the same direction as Δ_2 going downward. Since $\langle w, x\rangle = \sin(2\theta)\,||x||$, this amounts to solving

$$t^2 - 2t\sin(2\theta)\,||x|| + \sin^2(\theta)\,||x||^2 = 0. \tag{3.14}$$

From this we obtain

$$\frac{\pi_2(v)}{||x||} = -\frac{\sin(2\theta)t^+}{||x||} = -\sin(2\theta)^2 - \sin(2\theta)\sqrt{\sin(2\theta)^2 - \sin(\theta)^2}$$

and the previous condition is valid if and only if this last quantity is larger than -1.
Since $||v - x|| = \cos(\theta)\,||x||$ as v belongs to $\partial B_{\max}$, we are in position to compute and bound the half angle between v, x and the symmetric of v through the bisector.

$$\frac{\pi_2(v-x)}{||v-x||} = \frac{1 - \sin(2\theta)^2 - \sin(2\theta)\sqrt{\sin(2\theta)^2 - \sin(\theta)^2}}{\cos(\theta)} \geq 1 - 8\theta^2. \tag{3.15}$$

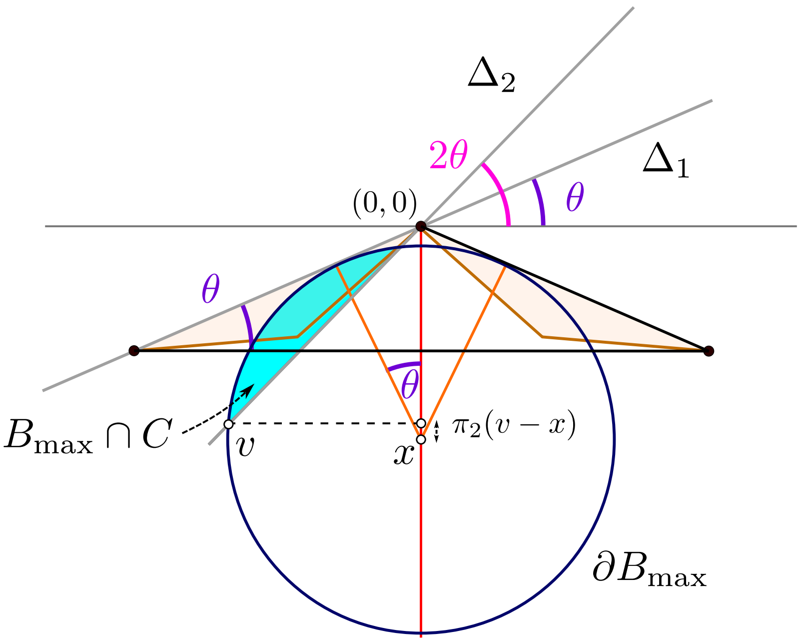


Figure 3.11 – Configuration of v and $B_{\max} \cap C$.

This quantity is larger than μ when $\theta \leq \left(\frac{1-\mu}{8}\right)^{1/2}$. In this case, the cone C intersects the disk $B_{\max}$ only in its upper left quarter which means, as argued before, that the cosine of the half angle between any point of $C \cap B_{\max}$ and x and its symmetric point through the bisector is bounded below by the one of v, which is bounded by μ.

□

In particular, for any $\mu \in (0,1)$, letting $\theta_n := \left(\frac{1-\mu}{8(n+1)}\right)^{1/2}$ for every natural number n we obtain the desired result on K:

Theorem 3.22 (Unrectifiable compact set with positive reach.). *Let* $\mu \in (0,1)$. *Then there exists a compact unrectifiable curve* K *in* $\mathbb{R}^d$ *such that* $\text{reach}_\mu(K) = \infty$.

Finally, we show that the length of the graph of $\partial^* d_X$ on ∂K^δ tends to ∞ when δ goes to 0. First, we prove the following lemma.

Lemma 3.23 (Corners of small offsets of K). *Let θ be a non-increasing sequence of positive reals bounded by $\pi/4$. Then for all $n \in \mathbb{N}$, there exists a $\delta_n > 0$ such that for every $0 < \delta \leq \delta_n$, for every $0 \leq i \leq n$ there are sets $\mathcal{M}_i^\delta$ containing 2^i points in ∂K^δ such that, with $\nu : x \mapsto \frac{x}{||x||}$, we have*

- $\mathcal{H}^1\left(\nu(\partial^* d_K(x))\right) \geq 2(2-\sqrt{3})\theta_i$ *for each* $i \in \mathcal{M}_i^\delta$;
- $\mathcal{M}_i^\delta \cap \mathcal{M}_j^\delta = \emptyset$ *if* $i \neq j$.

Proof.

Let $n \in \mathbb{N}$ and $0 \leq i \leq n$. Any point on a bisector of a triangle at step i has closest points in K symmetric to the bisector, meaning that $\nu(\partial^* d_K(x))$ is a subset of $\mathbb{S}^1$ with length at least $m(\theta)$

given by the first intersection of the line Δ_2 with $\partial B_{\max}$, see Figure 3.11.

$$\begin{aligned} m(\theta) &= 2\cos(\theta)^{-1}\left(\sin(2\theta) - \sqrt{\sin^2(2\theta) - \sin^2(\theta)}\right) \\ &\geq 2(2-\sqrt{3})\theta. \end{aligned}$$

Now for any δ small enough compared to the height of a triangle at step n, these bisectors intersect with K^δ in every triangle built before step n.

□

Proposition 3.24 (Normal bundle of K^δ). *Let θ be a non-increasing sequence of positive reals bounded by $\pi/4$ such that $\sum_{i=0}^{n} 2^i\theta_i$ diverges. Then $\mathcal{H}^1(\mathrm{Nor}(K^\delta))$ diverges to ∞ when δ tends to 0.*

Proof.

When $\delta \leq \delta_n$, by the previous lemma we have

$$\begin{aligned} \mathcal{H}^1(K^\delta) &\geq \sum_{x\in\bigcup_{i\leq n}\mathcal{M}_i^\delta} \mathcal{H}^1(\mathrm{Nor}(X,x)\cap\mathbb{S}^1) \\ &\geq \sum_{i=0}^{n}(2-\sqrt{3})2^{i+1}\theta_i. \end{aligned}$$

□

Remark 3.25 – The fact that for each $\mu \in (0,1)$ there exists a non-rectifiable curve with positive μ-reach was obtained in [PRZ19, Example 3.5] using self-similar fractals. This more direct construction is equivalent to ours with the choice of a constant sequence of parameters. Our construction was obtained independently, and we believe that we can adapt it to build an unrectifiable curve of $\mathbb{R}^2$ with positive μ reach for any $\mu \in (0,1)$.

3.3 Tangent cones, normal cones and normal bundles.

This section is devoted to three classical notions in geometric measure theory, namely tangent cones, normal cones and normal bundles. Normal cones and tangent cones were first introduced by Federer in [Fed59] in an effort to generalize the concept of tangent spaces and normal bundles of submanifolds of Euclidean spaces to any set of $\mathbb{R}^d$.

3.3.1 Definitions

The *tangent cone* of $X \subset \mathbb{R}^d$ at x, denoted by $\mathrm{Tan}(X,x)$, is defined by Federer as the cone of $\mathbb{R}^d$ generated by 0 and limits of the form $\lim_{n\to\infty}\frac{x_n-x}{||x_n-x||}$ where $(x_n)_{n\in\mathbb{N}}$ runs among sequences belonging in X, converging to x and never taking the value x.

$$\mathrm{Tan}(X,x) := \mathrm{Cone}\left\{\lim_{n\to\infty}\frac{x_n-x}{||x_n-x||}\right\}. \tag{3.16}$$

For any such sequence, we say that the vector $u = \lim_{n\to\infty} \frac{x_n - x}{||x_n - x||}$ is *represented* by the sequence $(x_n)_{n\in\mathbb{N}}$. This is equivalent to the following first order expansion of x_n:

$$x_n = x + ||x_n - x||\,(u + o(1)).$$

When $X \subset \mathbb{R}^d$ has positive reach, the *normal cone* of X at x, denoted by $\mathrm{Nor}(X, x)$, is the cone dual to the tangent cone:

$$\mathrm{Nor}(X, x) := \mathrm{Tan}(X, x)^{\mathrm{o}}.$$

In this case, $\mathrm{Tan}(X, x)$ is convex and we have $\mathrm{Tan}(X, x) = \mathrm{Nor}(X, x)^{\mathrm{o}}$. The normal cone is related to the projection onto the closest point ξ_X by the following characterization, for any $0 < t < \mathrm{reach}(X)$:

$$\mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1} = \left\{ u \in \mathbb{S}^{d-1} \,\middle|\, \xi_X(x + tu) = x \right\}.$$

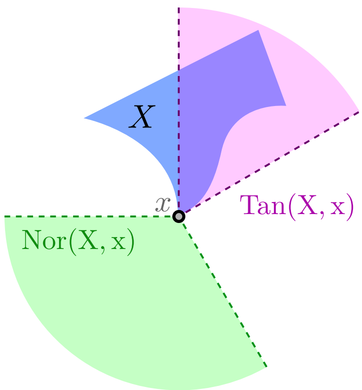


Figure 3.12 – Tangent and normal cones of X at x when $\mathrm{reach}(X) > 0$.

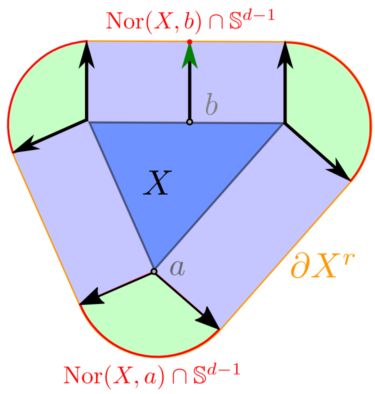


Figure 3.13 – Some unit normal cones (in red) when $0 < r < \mathrm{reach}(X)$.

When $^\neg X$ has positive reach, we let $\mathrm{Nor}(X, x) := -\mathrm{Nor}(^\neg X, x)$ be the normal cone of X at x. When both X and $^\neg X$ have positive reach, these definitions coincide.

Proposition 3.26 (Normal cones when X, $^\neg X$ both have positive reach)**.** *Let $X \subset \mathbb{R}^d$ be such that both $\mathrm{reach}(X)$ and $\mathrm{reach}(^\neg X)$ are positive. Then $\mathrm{Tan}(X, x)$ and $\mathrm{Tan}(^\neg X, x)$ are respectively the lower and upper half-spaces associated to the same non-zero linear form. In particular, their associated normal cones have dimension 1 and are opposite.*

Proof.

Corollary 3.31, which is proved independently, shows that $\mathrm{Tan}(X, x)$ and $\mathrm{Tan}(^\neg X, x)$ are complementary convex cones. Since none of them is the whole space, the complement set of a convex cone is itself a tangent cone if and only if both are opposite half-spaces.

□

Definition 3.27 (Normal bundle of a subset of $\mathbb{R}^d$)**.** Let $X \subset \mathbb{R}^d$. When $\mathrm{reach}(^\neg X) > 0$ and $\partial X = \partial(^\neg X)$, or $\mathrm{reach}(X) > 0$, we define $\mathrm{Nor}(X)$ as follows:

$$\mathrm{Nor}(X) := \left\{ (x, n) \mid x \in \partial X, n \in \mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1} \right\}. \tag{3.17}$$

We say that any set X is *fully dimensional* when $\mathrm{Tan}(X, x)$ has dimension d for every $x \in \partial X$. This is equivalent to having the set equality $\overline{\mathrm{int}(\mathrm{Tan}(X, x))} = \mathrm{Tan}(X, x)$ for all $x \in \partial X$. In particular, a Lipschitz domain is fully dimensional.

Remark 3.28 – Clarke [Cla75, Chapter 2] gave the name "tangent cones" to another closely related construction, which is necessarily convex and included into the tangent cone from Federer — which Clarke calls *contingent cone*. When X has positive reach, these two notions coincide. We prefer working with the definition of Federer as it behaves well on complement sets, as will be seen in Lemma 3.29 and Corollary 3.31. Nevertheless, we will see in Theorem 3.32 that Clarke's definition of the normal cone as the cone generated by the Clarke gradient of the distance function coincides with ours among the class of complementary regular sets.

3.3.2 Technical lemmas involving tangent cones

We prove several results on the tangent cones on compact sets of $\mathbb{R}^d$ satisfying weak regularity assumptions, leading to Theorem 3.32 which relates normal cones to the Clarke gradient of the distance function. These assumptions are verified by all *complementary regular sets* as defined in Section 3.4, which is the class for which we will prove the Morse theorems in the next chapter.

Lemma 3.29 (Tangent cone of the boundary)**.** *Let $X \subset \mathbb{R}^d$. Then for every $x \in \partial X$,*

$$\mathrm{Tan}(\partial X, x) = \mathrm{Tan}(X, x) \cap \mathrm{Tan}(^\neg X, x).$$

Proof.

The cone $\mathrm{Tan}(\partial X, x)$ being included in both $\mathrm{Tan}(X, x)$ and $\mathrm{Tan}(^\neg X, x)$, we have to prove that $\mathrm{Tan}(X, x) \cap \mathrm{Tan}(^\neg X, x)$ is included in $\mathrm{Tan}(\partial X, x)$.

Let $u \in \mathrm{Tan}(X, x) \cap \mathrm{Tan}(^\neg X, x)$ be of norm 1. Take a sequence x_n (resp. a sequence $^\neg x_n$) in X (resp. $^\neg X$) representing u, i.e., such that

$$\begin{aligned} x_n &= x + ||x_n - x||\,(u + o(1)\,), \\ ^\neg x_n &= x + ||^\neg x_n - x||\,(u + o(1)\,). \end{aligned}$$

The segment $[x_n, ^\neg x_n]$ has to intersect ∂X, which means that there exists a $\lambda_n \in [0, 1]$ such that $\partial x_n = \lambda_n x_n + (1 - \lambda_n)^\neg x_n$ belongs in ∂X. This yields

$$\partial x_n - x = (\lambda_n\,||x_n - x|| + (1 - \lambda_n)\,||^\neg x_n - x||\,)\,(u + o(1)\,)$$

Taking the norm of this equality yields

$$||\partial x_n - x|| = (\lambda_n\,||x_n - x|| + (1 - \lambda_n)\,||^\neg x_n - x||\,) + o(||\partial x_n - x||).$$

This quantity is strictly positive when n is large enough, and we have

$$\partial x_n - x = ||\partial x_n - x||\,(u + o(1)\,)$$

meaning that u is represented by the sequence ∂x_n, which lies in ∂X.

□

Lemma 3.30. *Let $A \subset \mathbb{R}^d$ be a closed set with positive reach. Then for any x in A and any non-zero $u \in \operatorname{int}(\operatorname{Tan}(A,x))$, we have*

$$\liminf_{t \to 0^+} \frac{1}{t} d_{\neg A}(x+tu) > 0. \tag{3.18}$$

As a consequence, for any $A \subset \mathbb{R}^d$ with positive reach, we have:

$$\operatorname{int}(\operatorname{Tan}(A,x)) \cap \operatorname{Tan}(\neg A, x) \cap \mathbb{S}^{d-1} = \emptyset. \tag{3.19}$$

Proof.

Let $\lambda > 0$ be such that $\operatorname{Tan}(A,x)$ contains the closed ball of radius λ centered at u. By Federer's [**?**] characterization of tangent cones for sets with positive reach, using Landau's notation we have for any $0 \leq \lambda' \leq \lambda$:

$$d_A(x + t(u + \lambda' v)) = o(t). \tag{3.20}$$

For any $t \geq 0$, let $x(t)$ be a point in $\neg A$ realizing $d_{\neg A}(x+tu) = ||x + tu - x(t)|| =: d(t)$. For $t > 0$ small enough, the point $x + tu$ lies well within the reach of A, so that we can take $\delta > 0$ small enough and a unit $v(t) \in \operatorname{Nor}(A, x(t))$ (with the convention that $\operatorname{Nor}(A,x) = \mathbb{R}^d$ if $x \notin A$) such that $d_A(x(t) + \delta' v(t)) \geq \delta'$ for every $0 \leq \delta' \leq \delta$. Distance functions being 1-Lipschitz, we have obtained

$$d_A(x + tu + t\delta v(t)) \geq t\delta - d(t). \tag{3.21}$$

Now consider a positive sequence t_n converging to 0. Extracting a subsequence we can assume that $v(t_n)$ converges to a unit vector v, leading to

$$d_A(x + t_n(u + \delta v)) \geq t_n \delta - d(t_n) - o(t_n). \tag{3.22}$$

By Equation (3.20), the right hand-side must be $o(t_n)$ as n tends to $+\infty$, which shows that $\liminf_{n\to\infty} \frac{d(t_n)}{t_n} \geq \delta$. Since this applies to any sequence t_n going to 0, we have obtained the desired result Equation (3.18).

Now to obtain Equation (3.19) let $y_n = x + a_n(u + o(1))$ be a sequence representing $u \in \operatorname{int}(\operatorname{Tan}(A,x))$. By Equation (3.18) there is a $\delta > 0$ such that for n large enough, $d_{\neg A}(x + a_n u) \geq \delta a_n$, implying that $d_{\neg A}(y_n) \geq a_n(\delta + o(1))$. The right hand-side quantity is strictly positive when n is large enough. As such, u cannot be represented by a sequence in $\neg A$.

□

Taking $A = \neg X$ in the previous result yields the following.

Corollary 3.31 (Complement of tangent cones are tangent cones of complements)**.** *Let $X \subset \mathbb{R}^d$ be a compact set such that $\neg X$ has positive reach. Then for any $x \in \partial X$,*

$$\neg\operatorname{Tan}(\neg X, x) = \operatorname{Tan}(\neg(\neg X), x). \tag{3.23}$$

In particular, if $X = \overline{\operatorname{int}(X)}$, we have

$$\neg\operatorname{Tan}(\neg X, x) = \operatorname{Tan}(X, x). \tag{3.24}$$

Proof.

We write $Y = \neg(\neg X)$ to ease notations. Since $\neg X \cup Y = \mathbb{R}^d$, we have $\operatorname{Tan}(\neg X, x) \cup \operatorname{Tan}(Y,x) = \mathbb{R}^d$ and thus $\neg\operatorname{Tan}(\neg X, x) \subset \operatorname{Tan}(Y,x)$. The reverse inclusion is equivalent to $\operatorname{Tan}(Y,x) \cap \operatorname{int}(\operatorname{Tan}(\neg X, x))$ containing no non-unit vector, something we obtained in the previous lemma.

□

3.3.3 Relation between Clarke gradients of distance functions and normal cones

We are now in position to link the normal cones of a set X whose complement $^\neg X$ has positive reach and the Clarke gradient of its distance function d_X under some technical assumptions.

Theorem 3.32 (Normal cones and the Clarke gradient of the distance function)**.** *Let $X \subset \mathbb{R}^d$ be such that* $\operatorname{reach}(^\neg X) > 0$ *and such that $^\neg X$ is fully dimensional. Then the normal cone of X at any $x \in \partial X$ is determined by the Clarke gradient of d_X at x:*

$$\operatorname{Nor}(X, x) = \operatorname{Cone}\left(\partial^* d_X(x)\right).$$

Proof.

Let $\operatorname{reach}(^\neg X) > r > 0$. First remark that we have

$$\begin{aligned}
\partial^* d_{X^{-r}}(x) &= -\operatorname{Conv}\left\{ \frac{x-z}{||x-z||} \;\middle|\; z \in X^{-r} \text{ with } d_X^{-r}(x) = ||z-x|| \right\} \\
&= -\operatorname{Conv}\left\{ u \in \mathbb{S}^{d-1} \mid d_{^\neg X}(x + ru) = r \right\} \\
&= -\operatorname{Conv}\left(\operatorname{Nor}(^\neg X, x) \cap \mathbb{S}^{d-1}\right).
\end{aligned}$$

On the other hand, by definition the Clarke gradient of $d_{X^{-r}}$ at x is determined locally by the gradients around x in every direction:

$$\partial^* d_{X^{-r}}(x) = \operatorname{Conv}\left\{ \lim_{i\to\infty} \nabla d_{X^{-r}}(x_i) \;\middle|\; (x_i) \in (\mathbb{R}^d)^{\mathbb{N}} \text{ converging to } x \right\}.$$

Now compare to the Clarke gradient of d_X at x, for which the gradient contributing only come from directions outside X (cf. [Cla75, Corollary 2.5]):

$$\partial^* d_X(x) = \operatorname{Conv}\left(\{0\} \cup \left\{ \lim_{i\to\infty} \nabla d_X(x_i) \;\middle|\; (x_i) \underset{i\to\infty}{\rightarrow} x \text{ with } d_X(x_i) > 0 \right\} \right).$$

Note that in both definition we implicitly require x_i to be points where d_X is differentiable. On those points the gradients of d_X and $d_{X^{-r}}$ coincide, yielding

$$\operatorname{Cone}\left(\partial^* d_X(x)\right) \subset \operatorname{Cone}\left(\partial^* d_{X^{-r}}(x)\right) = -\operatorname{Nor}(^\neg X, x). \tag{3.25}$$

The other inclusion $-\operatorname{Nor}(^\neg X, x) \subset \operatorname{Cone} \partial^* d_X(x)$ is Lemma 3.34 whose proof will be the remainder of this section. We will prove the opposite inclusion on their polar cones, that is:

$$\partial^* d_X(x)^{o} \subset -\operatorname{Nor}(^\neg X, x)^{o} = -\operatorname{Tan}(^\neg X, x). \tag{3.26}$$

□

Lemma 3.33 (Tangent cone stability under addition with $\partial^* d_X(x)$)**.** *Let $X \subset \mathbb{R}^d, x \in \partial X$ and $u \in \partial^* d_X(x)^o$. Then for all $h \in \operatorname{Tan}(X, x)$, $u + h \in \operatorname{Tan}(X, x)$.*

Proof.

We use Clarke's [Cla75, Proposition 3.7] characterization of the dual cone to the Clarke gradient:

$$\partial^* d_X(x)^{\mathrm{o}} = \left\{ u \;\middle|\; \lim_{\substack{x_h \to x \\ x_h \in X}} \lim_{\delta \to 0^+} \frac{1}{\delta} d_X(x_h + \delta u) = 0 \right\}. \tag{3.27}$$

Consider the following modulus of continuity:

$$\omega_u(\varepsilon, \lambda) := \sup_{\substack{x_h \in X \\ ||x - x_h|| \le \varepsilon}} \sup_{0 < \delta \le \lambda} \frac{d_X(x_h + \delta u)}{\delta}.$$

When u belongs $\partial^* d_X(x)^{\mathrm{o}}$, by Clarke's characterization 3.27 we have

$$\lim_{\substack{\varepsilon \to 0^+ \\ \lambda \to 0^+}} \omega_u(\varepsilon, \lambda) = 0.$$

Now take a sequence $x_i \to x$ representing any $h \in \operatorname{Tan}(\partial X, x)$. Put $\varepsilon_i = ||x - x_i||$ and consider the sequence $x_i + \varepsilon_i u$. Take ξ_i in $\Gamma_X(x_i + \varepsilon_i u)$, i.e a point in X realizing the distance of $x_i + \varepsilon_i u$ to X. By the definition of ω_u, we have:

$$||\xi_i - x_i - \varepsilon_i u|| = d_X(x_i + \varepsilon_i u) \le \varepsilon_i \omega(\varepsilon_i, \varepsilon_i).$$

Thus we can write

$$\xi_i - x = \varepsilon_i(h + o(1) + u + O(\omega(\varepsilon_i, \varepsilon_i))) = \varepsilon_i(u + h + o(1))$$

which shows that ξ_i is a sequence in X representing $u + h$.

□

Lemma 3.34 (Relationship between normal cones and Clarke gradients)**.** *Let $X \subset \mathbb{R}^d$ such that $\operatorname{reach}(\neg X) > 0$. Then, if $\operatorname{Tan}(\neg X, x)$ has full dimension, we have:*

$$\partial^* d_X(x)^o \subset -\operatorname{Tan}(\neg X, x).$$

Proof.

Let $u \in \partial^* d_X(x)^{\mathrm{o}}$. By Lemma 3.33 we know that

$$u + \operatorname{Tan}(X, x) \subset \operatorname{Tan}(X, x),$$

which is equivalent to

$$u + \mathbb{R}^d \setminus \operatorname{Tan}(X, x) \supset \mathbb{R}^d \setminus \operatorname{Tan}(X, x).$$

By Corollary 3.31 we have $\neg \operatorname{Tan}(X, x) = \operatorname{Tan}(\neg X, x)$. Thanks to the full dimensionality condition, taking the closure of the previous inclusion yields:

$$u + \operatorname{Tan}(\neg X, x) \supset \operatorname{Tan}(\neg X, x),$$

which implies that u belongs in $-\operatorname{Tan}(\neg X)$.

□

3.4 Complementary regular sets

In this section, we define *complementary regular sets* as a class of sets satisfying for every point on their boundary the assumptions of every lemma of the previous Section 3.3 on geometric grounds.

3.4.1 Definition

Definition 3.35 (Complementary regular sets)**.** We say that a compact subset X of $\mathbb{R}^d$ is a *complementary regular set* when it verifies the following three conditions:
(A_1) $\overline{\text{int}(X)} = X$;
(A_2) $\exists\, \mu \in (0, 1]$ such that $\text{reach}_\mu(X) > 0$;
(A_3) $\text{reach}(^\neg X) > 0$.

By (A_1) the topological boundaries ∂X and $\partial(^\neg X)$ coincide, and by (A_3) the set X has a normal bundle. From the assumption (A_2) on the Clarke gradient of d_X on a neighborhood of X, we have that $^\neg X$ is fully dimensional in a quantitative way.

Lemma 3.36 (Tangent cones of complementary regular sets contain a ball)**.** *Let $\mu \in (0, 1]$ and let X be complementary regular with* $\text{reach}_\mu(X) > 0$ *Let $x \in \partial X$. Then* $\text{Tan}(^\neg X, x)$ *contains a ball of radius μ centered around a unit vector.*

Proof.

By [CCLT07, Section 3], we know that for each $0 < r < \text{reach}_\mu(X)$ there exists a point x_r such that $d_X(x_r) = r$ and $||x_r - x|| \leq \frac{r}{\mu}$. Let r_n be any sequence converging to 0, and consider a sequence x_n such that $||x_n - x|| \leq \frac{r_n}{\mu}$ and $d_X(x_n) = r_n$. Extracting a subsequence, we can assume that $\frac{x_n - x}{||x_n - x||}$ converges to a unit vector $u \in \text{Tan}(^\neg X, x)$, i.e., that we have

$$x_n = x + \varepsilon_n(u + o(1)) \tag{3.28}$$

where $\varepsilon_n = ||x_n - x|| \to 0^+$. For any unit vector v the sequence $x_n + \mu\varepsilon_n v$ lies in $^\neg X$, and moreover we have

$$x_n + \varepsilon_n \mu v = x + \varepsilon_n(u + \mu v + o(1)) \tag{3.29}$$

which implies that $u + \mu v$ belongs in $\text{Tan}(^\neg X, x)$.

□

A consequence of the previous lemma is that convex combination of unit vectors in normal cones are bounded away from zero, a fact that will be useful in characterizing complementary regular sets.

Corollary 3.37 (Normal cones of $^\neg X$ are thin)**.** *Let $\mu \in (0, 1]$ and let X be complementary regular with* $\text{reach}_\mu(X) > 0$ *Then for any $x \in \partial X$, we have* $\Delta(\text{Conv}(\text{Nor}(^\neg X, x) \cap \mathbb{S}^{d-1})) \geq \mu$.

Proof.

By the previous lemma, take a unit vector u such that $B(u, \mu) \subset \text{Tan}(^\neg X, x)$. This yields the opposite inclusion on their dual cones $\text{Nor}(X, x) \subset B(u, \mu)^{\text{o}}$. Take any unit vector $w \in B(u, \mu)^{\text{o}}$. For any $v \in \mathbb{S}^{d-1}$, we have

$$0 \geq \langle w, u + \mu v \rangle = \langle u, w \rangle + \mu \langle w, v \rangle .$$

Letting $v = w$, we see that any such w lies in the half space $H_u^{-\mu} = \{u' \in \mathbb{R}^d \mid \langle u, u' \rangle \leq -\mu\}$ which is a convex set such that $\Delta(H_u^{-\mu}) \geq \mu$.

□

3.4.2 Characterizations of complementary regular sets

Now, we prove that a subset X of $\mathbb{R}^d$ is complementary regular if and only if it is the offset of some $Y \in \mathbb{R}^d$ at a regular value of d_Y. First, we characterize the assumptions of Definition 3.35 with a fixed $\mu > 0$

Lemma 3.38 (Characterization of complementary regular sets)**.** *Let X be a compact subset of $\mathbb{R}^d$ and let $\mu \in (0, 1]$. Then the three conditions*

(A_1) $\overline{\mathrm{int}(X)} = X$*;*
(A_2') $\mathrm{reach}_\mu(X) > 0$*;*
(A_3) $\mathrm{reach}(^\neg X) > 0$*;*

are equivalent to the existence of $\varepsilon, \delta > 0$ and of a compact subset Y of $\mathbb{R}^d$ such that $X = Y^\varepsilon$ with $\inf\{\Delta(\partial^* d_Y(x)) \mid d_Y(x) \in [\varepsilon, \varepsilon + \delta]\} \geq \mu$*. The quantity* $\mathrm{reach}_\mu(X)$ *is the supremum of δ such that the previous inequality holds.*

Proof.

On the one hand, assume that the three conditions (A_i) are satisfied by X. Then for any $0 < r < \mathrm{reach}(^\neg X)$ we have $(X^{-r})^r = X$ thanks to (A_1). Further assuming that $r < \mathrm{reach}_\mu(X)$, any such X^{-r} will provide a suitable Y with $\varepsilon = r$. Now let $\delta \in (0, \mathrm{reach}_\mu(X))$. For any $x \in \mathbb{R}^d$ such that $d_X(x) > 0$ we have $d_{X^{-r}} = d_X + r$ on a neighborhood of x. Thus, we have:

$$\mu \leq \inf\{\Delta(\partial^* d_{X^{-r}}(x)) \mid d_{X^{-r}}(x) \in (r, r + \delta]\}.$$

We now bound $\Delta(\partial^* d_{X^{-r}}(x))$ from below for points x such that $d_{X^{-r}}(x) = r$. Those points are exactly the set $\partial {}^\neg X$ when $r < \mathrm{reach}(^\neg X)$. For such x, we have $\partial^* d_{X^{-r}}(x) = -\mathrm{Conv}(\mathrm{Nor}(^\neg X, x) \cap \mathbb{S}^{d-1})$, and Corollary 3.37 yields the desired bound $\Delta(\partial^* d_{X^{-r}}(x)) \geq \mu$. On the other hand, assume that there exist $\varepsilon > 0$ and $Y \subset \mathbb{R}^d$ such that $X = Y^\varepsilon$ with

$$\inf\{\Delta(\partial^* d_Y(x)) \mid d_Y(x) \in [\varepsilon, \varepsilon + \delta] \geq \mu.$$

By Clarke's Lipschitz local inversion theorem, the set X is a Lipschitz domain of $\mathbb{R}^d$, which implies that $\overline{\mathrm{int}(X)} = X$ (condition (A_1)). Since $d_X = d_Y - \varepsilon$ around any point at distance to Y strictly greater than ε, the Clarke gradients of Y and X coincide at any such point. We thus have $\mathrm{reach}_\mu(X) \geq \delta$, implying condition (A_2'), and moreover $\mathrm{reach}_\mu(X)$ is equal to the supremum of such δ. Finally, by lower semi-continuity of the Clarke gradient and compactness of Y, there exists a $\sigma > 0$ such that

$$\inf\{\Delta(\partial^* d_Y(x)) \mid d_Y(x) \in [\varepsilon - \sigma, \varepsilon + \delta]\} \geq \frac{\mu}{2} \tag{3.30}$$

which yields $\mathrm{reach}(^\neg X) \geq \sigma \frac{\mu}{2} > 0$ by Proposition 3.13 combined with the equality $(Y^{\varepsilon - \sigma})^\sigma = X$, and condition (A_3) is verified.

□

We are now in position to obtain the following characterization of complementary regular sets.

Theorem 3.39 (Complementary regular sets are offsets of sets with regular value)**.** *A set is complementary regular if and only if it is the offset of a compact set at a regular value of its distance function.*

Proof.

This is a consequence of the previous lemma along with the semi-continuity of the Clarke gradient, since if $\text{reach}_\mu(X) > 0$ and $X = Y^\varepsilon$, there is a $\sigma > 0$ such that on $d_Y^{-1}[\varepsilon - \sigma, \varepsilon + \sigma]$, $\Delta(\partial^* d_Y)$ is greater than $\frac{\mu}{2}$ and thus positive. From the set equality $d_Y^{-1}(\varepsilon, \varepsilon + \sigma] = d_X^{-1}(0, \sigma]$ and the fact that in this set $\partial^* d_Y$ and $\partial^* d_X$ coincide, we have the desired result.

□

Before extending this characterization to regular sublevel sets of semi-concave functions, we need the following elementary result on the semi-concavity of squared distance functions.

Proposition 3.40 (Semi-concavity of squared distance functions)**.** *For any $X \subset \mathbb{R}^d$, the map d_X^2 is 1-semi-concave on $\mathbb{R}^d$.*

Proof.

The map $d_X^2 - ||\cdot||^2 : z \mapsto \inf_{x \in X} ||x - z||^2 - ||z||^2 = \inf_{x \in X} -2\langle z, x\rangle + ||x||^2$ is the infimum of affine functions, and thus concave.

□

Moreover, regular sublevel sets of semi-convex functions have positive reach. This is a well-known fact that generalizes to the Riemaniann setting (see [Ban82]), for which we give an elementary proof in the Euclidean setting.

Lemma 3.41 (Lower bound on the reach of sublevel sets)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz map. Let $\alpha \in \mathbb{R}$ be such that $X = f^{-1}(-\infty, \alpha]$ is non-empty. For any $\rho, K > 0$, let*

$$t(\rho, K) := \sup\Big\{t \in \mathbb{R}^+ \textit{ such that } (\Delta \circ \partial^* f)_{|X^t \setminus X} \geq \rho, \quad \textit{and } f + K\,||\cdot||^2 \textit{ is convex for any segment of length } t \textit{ in } (\partial X)^t\Big\}.$$

Then for any $\rho, K > 0$,

$$\text{reach}_\mu(X) \geq \min\left(\frac{\rho}{K(1+\mu)}, t(\rho, K)\right). \tag{3.31}$$

In particular,

$$\text{reach}(X) \geq \min\left(\frac{\rho}{2K}, t(\rho, K)\right). \tag{3.32}$$

Proof.

Let ρ, K be such that $t := t(\rho, K)$ is positive. Let x be a point on the medial axis of X within X^t and let $a, b \in \Gamma_X(x)$ be two distinct closest points of x in X. Remark that $(a + b)/2$ also belongs to X^t. Since $a \neq b$, $(a + b)/2$ is strictly closer to x than both a and b, which implies that $f\left(\frac{a+b}{2}\right) > 0$. Yet $f(a) = f(b) = 0$, which by the K-semi-convexity assumption inside X^t yields $f\left(\frac{a+b}{2}\right) \leq K\left|\left|\frac{a-b}{2}\right|\right|^2$. Now by the inverse flow theorem Proposition 2.9, for any

$\varepsilon > 0$ small enough there is a 1-Lipschitz trajectory starting from x making f decrease at a speed greater than $\rho - \varepsilon$, implying that $(a+b)/2$ lies at a distance less than $\frac{1}{\rho-\varepsilon} f\left(\frac{a+b}{2}\right)$ from X. Letting ε tend to 0, we obtain:

$$d_X\left(\frac{a+b}{2}\right) \leq \frac{1}{\rho} f\left(\frac{a+b}{2}\right) \leq \frac{K}{\rho}\left|\left|\frac{a-b}{2}\right|\right|^2. \tag{3.33}$$

Denoting by $\theta \in (0, \pi/2]$ the half-angle between a, x and b, we have $\left|\left|\frac{a-b}{2}\right|\right| = \sin(\theta) d_X(x)$ and $\left|\left|x - \frac{a+b}{2}\right|\right| = \cos(\theta) d_X(x)$. Since $d_X(x) \leq \left|\left|x - \frac{a+b}{2}\right|\right| + d_X(\frac{a+b}{2})$ we finally obtain

$$\begin{array}{rrcl} & d_X(x) & \leq & \cos(\theta) d_X(x) + \frac{K}{\rho} \sin^2(\theta) d_X(x)^2 \\ \Longrightarrow & \rho & \leq & K(1+\cos(\theta)) d_X(x). \end{array}$$

□

Corollary 3.42 (Weakly regular sublevel sets of semi-convex maps have positive reach)**.** *Let $\alpha \in \mathbb{R}$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz map such that α is a weakly regular value of f, with $f^{-1}(-\infty, \alpha]$ compact, and such that f is semi-convex in a neighborhood of $f^{-1}(\alpha)$. Then the sublevel set $f^{-1}(-\infty, \alpha]$ has positive reach.*

Proof.

By the semi-convex hypothesis, there is a $K > 0$ such that $f + K\,||\cdot||^2$ is convex on a neighborhood of $f^{-1}(\alpha)$. Since α is a weak regular value, there is a $\rho > 0$ such that the quantity $t(\rho, K)$ of Lemma 3.41 is positive.

□

Theorem 3.43 (Complementary regular sets are sublevel sets of semi-concave functions at regular value)**.** *A compact subset of $\mathbb{R}^d$ is complementary regular if and only if is the sublevel set $f^{-1}(-\infty, \alpha]$ of a semi-concave map $f : \mathbb{R}^d \to \mathbb{R}$ at a regular value α of f.*

Proof.

On the one hand, a complementary regular set is the offset of a set Y at a regular value ε of its distance function. Recall that from Proposition 3.40 that d_Y^2 is semi-concave, showing that $Y^\varepsilon = (d_Y^2)^{-1}(-\infty, \varepsilon^2]$ is a sublevel set of a semi-concave function. The value ε^2 is regular for d_Y^2 as we have for any $x \in \partial Y^\varepsilon$:

$$\begin{array}{rrcl} & \partial^* d_Y^2(x) & = & \left\{2(x-z) \,\middle|\, z \in \Gamma_Y(x)\right\} \\ \Longrightarrow & \Delta(\partial^* d_Y^2(x)) & = & 2\varepsilon \Delta(\partial^* d_Y(x)). \end{array}$$

Now assume that X is the sublevel set of a semi-concave function f at regular value α. Since α is a regular value of f, the sublevel set $f^{-1}(-\infty, \alpha]$ is a Lipschitz domain and we have thus $\overline{\mathrm{int}(X)} = X$. The complement set $^\neg X$ is the compact sublevel set of the semi-convex function $-f$ at a regular value, and thus has positive reach. Finally, it is a Lipschitz domain with compact boundary and as such, has a positive μ-reach for some $\mu > 0$.

□

CHAPTER 4

Generalized Morse theory for complementary regular sets

In this chapter, we study the sublevel sets of smooth functions restricted to complementary regular sets as defined in Chapter 3. We show that the topology of the sublevel sets of some of those functions, which we call Morse functions, evolves by the gluing of cells around the critical points of the restricted function. The dimension of these cells is determined by the curvature of the sets and the Hessian of the ambient function at the critical point, exactly as in the classical Morse theory over C^2 manifolds. Using non-smooth analysis, we adapt the main ideas of the article Curvature measures and generalized Morse theory *[Fu89a] to complementary regular sets.*

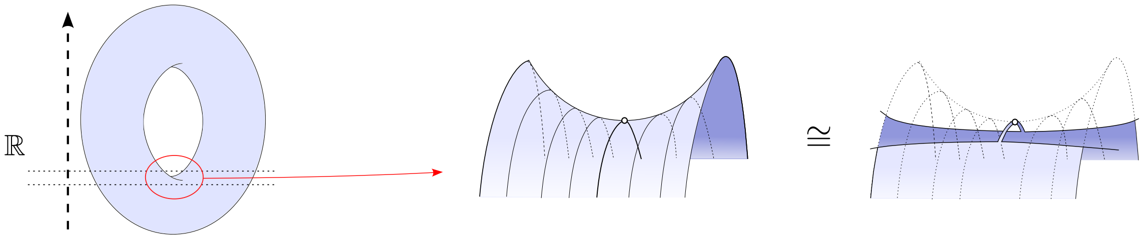


Figure 4.1 – Handle attachment lemma for a height filtration around a saddle point of a surface in $\mathbb{R}^3$.

4.1 Introduction

Morse theory was born in the first half of the twentieth century to analyze the topology of a manifold by studying differentiable, real-valued functions on that manifold. The intuition is that for a generic function f, the homotopy type of the sublevel sets $X_t := f^{-1}(-\infty, t]$ should change only around a finite number of filtration value t, with the topology changes following the combinatorial rules of CW-Complexes. This is explicited by the two core results of Morse theory:

Theorem 4.1 (Constant homotopy type lemma)**.** *Let X be a smooth manifold and let f be a C^1 function on X. Then if the segment $[a, b]$ contains no critical value of f, X_a is a deformation retract of X_b.*

Theorem 4.2 (Handle-attachment lemma)**.** *Let X be a smooth manifold and let f be a C^2 function on X. Suppose that there is a neighborhood V of $c \in \mathbb{R}$ such that c is the only critical value of f, and that the set $\mathrm{crit}_c(f)$ of critical points of f within $f^{-1}(c)$ is finite. Assume that at each point of $\mathrm{crit}_c(f)$, the Hessian of f is non-degenerate. Then for any $\varepsilon > 0$ small enough, $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ with a cell glued around each point of $\mathrm{crit}_c(f)$, and the dimension of each cell is the index of the Hessian f at this critical point.*

In this setting, we say that a C^2 function is Morse when its Hessian is non-degenerate at its critical points; in this case, its critical points are isolated within X. This condition is generic, as the set of Morse functions is open and dense among C^2 maps of X with the corresponding C^2-Fréchet metric. Notably, when X is equipped with a Riemannian metric, the subset of points $x \in X$ such that d_x^2 is Morse is open and dense, and when X is a submanifold of a Euclidean space, the linear forms $h_\nu : u \mapsto \langle u, \nu \rangle$ restricted to X are Morse for $\mathcal{H}^{d-1}$-almost every ν in $\mathbb{S}^{d-1}$.

Several works aimed at adapting the constant homotopy and handle-attachment lemmas to other classes of sets or functions. During the eighties, consequent works extended Morse theory to C^2 functions restricted to a stratified subset of a Riemannian manifold, culminating in the monograph of Goresky and MacPherson [GM88]. In the context of stratified Morse theory, the change of topology around a non-degenerate critical point is obtained by the gluing of the so-called "local Morse data", which might not be a cell. With a similar weak version of the handle-attachment lemma, Morse theory was extended to broader classes of functions such as so-called "min-type functions" on a manifold [GR97], which are functions which can be locally written as the minimum of a finite number of C^2 functions, and for distance functions to a set in a o-minimal structure [GLRS24]. An adaption of the Morse lemmas to the combinatorial settings of functions on CW-Complexes can be found in the discrete Morse theory of Forman [For95].

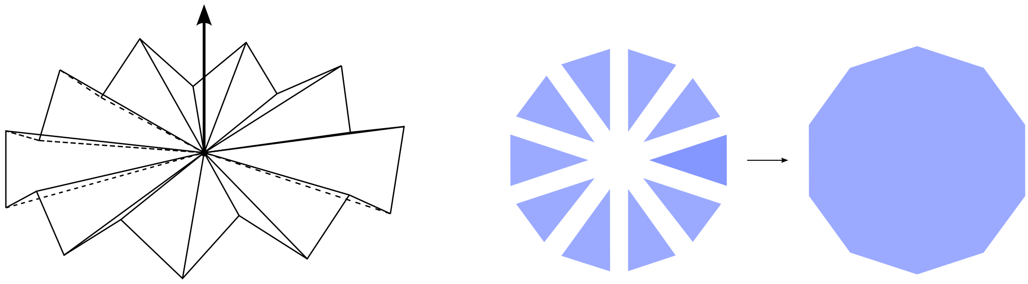


Figure 4.2 – Consider the height filtration, whose direction is indicated by the arrow, on a ruff as in the left figure. The topological event around the central point is not equivalent to the gluing of a cell as the number of connected components changes by more than one.

In 1989, Joseph Fu [Fu89a] extended Morse theory to smooth maps restricted to sets with positive reach. He proved that the topological changes happening around a critical point are as in the original handle-attachment lemma. Given $X \subset \mathbb{R}^d$ of positive reach and a generic smooth function $f : \mathbb{R}^d \to \mathbb{R}$, he considered some functions f_r, which consists in f precomposed by a pertaining translation, such that $f_{r_{|X^r}}$ is Morse when $r \to 0$ is small enough. He showed that the topological events of their associated sublevel set filtration $(X^r \cap f_r^{-1}(-\infty, t])$ converged to those of $(X_t)_{t\in\mathbb{R}}$ and used this limit to define the critical points of $f_{|X}$, their Hessians and thereby Morse functions on X. Finally, he obtained Morse theorems for compact sets with positive reach.

Theorem 4.3 (*Generalized Morse theory for sets with positive reach*)**.** *Let X be a compact subset of $\mathbb{R}^d$ with positive reach and let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $f_{|X}$ is Morse with at most one critical point per level set.*

Then for any regular value $c \in \mathbb{R}$, X_c has the homotopy type of a CW-complex with one λ_p cell for each critical point p such that $f(p) < c$, where

$$\lambda_p = \textit{Index of } Hf_{|X} \textit{ at } p.$$

Remark 4.4 – Following Fu's definitions - which will soon follow - when X is domain bounded by a $C^{1,1}$-hypersurface, the maps $f_{|X}$ and $(-f)_{|\neg X}$ share the same critical points. One is Morse if and only if the other is, and the change of topology around a critical point x in the sublevel set filtration in one determines the change of topology around x in the other, with

$$\mathrm{dimcell}(f_{|X}, x) = d - 1 - \mathrm{dimcell}((-f)_{|\neg X}, x), \tag{4.1}$$

where $\mathrm{dimcell}(f_{|X}, x)$ (resp. $\mathrm{dimcell}((-f)_{|\neg X}, x)$) denotes the dimension of the cell glued around x in the filtration $(f^{-1}(-\infty, t] \cap X)_{t\in\mathbb{R}}$ (resp. $(f^{-1}[-t, \infty) \cap (\neg X))_{t\in\mathbb{R}}$). This equality follows from the fact that around x, the second order approximation of f around x in ∂X goes down in $\mathrm{dimcell}(f_{|X}, x)$ directions (whence the gluing of a cell of dimension $\mathrm{dimcell}(f_{|X}, x)$ around x in the classical handle-attachment lemma) and up in $d - 1 - \mathrm{dimcell}(f_{|X}, x)$ directions. It is the opposite for the second-order approximation of $(-f)_{|\neg X}$, yielding Equation (4.1).

However, for general $X \subset \mathbb{R}^d$ the change of topology of the sublevel set filtration of $f_{|X}$ can have a Morse behavior - that is, a change of the form of the gluing of a cell around x - while not that of $(-f)_{|\neg X}$. Such a fact is illustrated in Figure 4.3 below with a stratified set $X \subset \mathbb{R}^2$ such that $\overline{\mathrm{int}(X)} = X$.

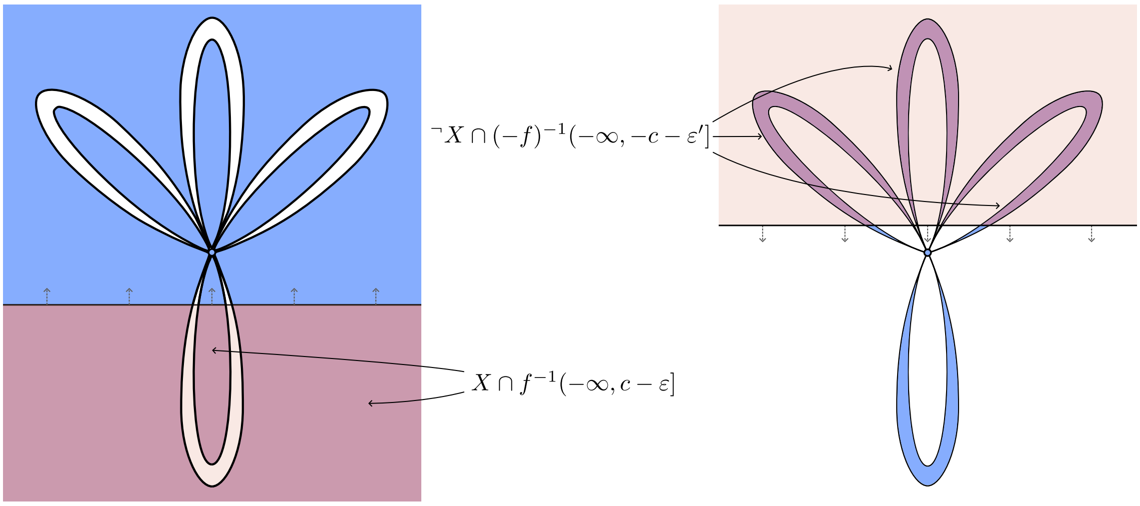


Figure 4.3 – $\neg X$ is a bouquet of cuspy handles around a central point x and X is its complement set. Below (resp. above) the horizontal line of the left (resp. right) figure is an half-space of the form $f^{-1}(-\infty, c-\varepsilon]$ (resp. $(-f)^{-1}(-\infty, -c-\varepsilon']$) where f is a vertical height function and $c = f(x)$.

Indeed, when encountering the central point, the sublevel set filtration of $f_{|X}$ evolves by gluing one cell around the central point, going from 2 connected components (in purple in the left-handside of Figure 4.3) to 1. In that of $(-f)_{|\neg X}$ there are 4 changes in homology: going from 3 connected components (in purple in the right-handside of Figure 4.3) to 1 connected component and 3 cycles. Note that the number of gluing of cells to describe the changes of homotopy type of the sublevel sets filtration of $(-f)_{|\neg X}$ around x can be arbitrarily high while maintaining a Morse behavior for $f_{|X}$ around x, as one can consider another set X with more cuspy handles above the central point's height.

4.2 Morse vocabulary and outline of the chapter

The method of Fu uses the existence in a neighborhood of X of a continuous map ξ_X projecting onto its closest points, which does not exist for complementary regular sets. To circumvent this problem, we use the tools of Lipschitz analysis developed in Chapter 2 and Chapter 3. We recall below the definitions of critical points of smooth function restricted to a set with positive reach, as well as Hessians and non-degenerate critical points of restricted functions found in [Fu89a]. We will use these definitions as they naturally extend to any complementary regular set via their normal bundle.

Definition 4.5 (Critical points and Hessian)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth and X be a set of positive reach or a complementary regular set of $\mathbb{R}^d$.

— Let $(x, n) \in \mathrm{Nor}(X)$ be a regular pair . The second fundamental form $\mathrm{II}_{x,n}$ of X at (x, n) is defined as the bilinear form on $\pi_0(\mathrm{Tan}(\mathrm{Nor}(X), (x, n)))$ such that for every pair

$(u,v),(u',v')$ in $\mathrm{Tan}(\mathrm{Nor}(X),(x,n))$,

$$\mathrm{II}_{x,n}(u,u') := \langle u, v' \rangle \,. \tag{4.2}$$

Taking (b_i) an orthonormal basis of $\pi_0(\mathrm{Tan}(\mathrm{Nor}(X),(x,n)))$ consisting of all principal directions with finite associated principal curvatures, this definition is equivalent to:

$$\mathrm{II}_{x,n}(b_i,b_j) := \kappa_i \delta_{i,j} \tag{4.3}$$

and generalizes the classical fundamental form obtained when X has a smooth boundary;

— We say that $x \in X$ is a *critical point* of $f_{|X}$ when $\nabla f(x) \in -\mathrm{Nor}(X,x)$;

— We say that $c \in \mathbb{R}$ is a *critical value* of $f_{|X}$ when $f^{-1}(c)$ contains at least a critical point of $f_{|X}$. Otherwise, c is a *regular value* of $f_{|X}$;

— If x is a critical point of $f_{|X}$ with $\nabla f(x) \neq 0$, let $n := \frac{-\nabla f(x)}{||\nabla f(x)||}$. When (x,n) is a regular pair, *the Hessian of f restricted to X at x* denoted by $H_x f_{|X}$ is defined over $\pi_0(\mathrm{Tan}(N_X,(x,n)))$ by:

$$H_x f_{|X}(u,u') := H_x f(u,u') + ||\nabla f(x)||\, \mathrm{II}_{x,n}(u,u');$$

— The *index* of this Hessian is the dimension of the largest subspace on which $Hf_{|X}$ is negative definite;

— We say that a critical point x of $f_{|X}$ is *non-degenerate* when $\nabla f(x) \neq 0$, (x,n) is a regular pair of $\mathrm{Nor}(X)$, and the Hessian $H_x f_{|X}$ is not degenerate;

— $f_{|X}$ is said to be *Morse* when its critical points are non-degenerate.

Adapting the techniques of [Fu89a] to understand whole topological events of the sublevel filtrations of a Morse function can be summarized as follows. Let $f : \mathbb{R}^d \to \mathbb{R}$ be such that $f_{|X}$ is Morse, and a, b be two critical values of $f_{|X}$ such that (a,b) contains only regular values. When $\varepsilon > 0$ is small enough, we will obtain the following diagram:

$$\begin{tikzcd}
X_{b+\varepsilon} \arrow[r, shift left, "(1)"] & X^{-r}_{b+\varepsilon} \arrow[l, shift left] \\
X_{b-\varepsilon} \arrow[u] \arrow[r, shift right, "(1)"'] & X^{-r}_{b-\varepsilon} \arrow[u, "(3)"] \arrow[l, shift right] \\
X_{a+\varepsilon} \arrow[u, "(2)"]
\end{tikzcd}$$

— In Section 4.3, we prove that the maps (1) are homotopy equivalences when $r > 0$ is small enough;

— Using computations from the previous section, we prove in Section 4.4 that $X_{b-\varepsilon}$ deformation retracts onto $X_{a+\varepsilon}$, thereby proving that arrow (2) is an homotopy equivalence;

— In Section 4.5 we work on arrow (3). We prove that for $r, \varepsilon > 0$ small enough, the topology of $X^{-r}_{b+\varepsilon}$ is obtained from that of $X^{-r}_{b-\varepsilon}$ by gluing cells whose dimension is the index of Hessian of f at critical points with value b, around which they are glued. We also prove that non-degenerate critical points are isolated.

4.3 Smooth surrogates for X_c

For the remainder of this chapter, we let $X \subset \mathbb{R}^d$ be a complementary regular set and $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function. Following Fu's technique, we want to apply classical Morse theory to the eroded sets $X^{-r} = \{x \in \mathbb{R}^d \mid d_{\neg X}(x) \geq r\}$, in a way that the topological events of the filtration $(X_t^{-r})_{t\in\mathbb{R}}$ converge when $r \to 0$ to that of $(X_t)_{t\in\mathbb{R}}$. To that end, we will consider the restriction to X^{-r} of the sublevel sets filtration of f slightly translated by a smooth function $\eta : \mathbb{R}^d \to \mathbb{R}$ with $||\eta||_\infty \leq 1$.

Definition 4.6 (Smooth surrogates for sublevel sets of X)**.** Let c be a regular value of $f_{|X}$ and let f_r be f precomposed by the translation by $r\eta$:

$$f_r : x \mapsto f(x + r\eta(x)).$$

We define the smooth surrogates for X_c set as:

$$X_c^{-r} := X^{-r} \cap f_r^{-1}(-\infty, c]$$

and non-negative, locally Lipschitz functions

$$\phi^c := d_X + \max(f - c, 0) \qquad \phi_r^c := d_{X^{-r}} + \max(f_r - c, 0).$$

verifying $X_c = (\phi^c)^{-1}(0)$ and $X_c^{-r} = (\phi_r^c)^{-1}(0)$. When the value of c is clear from the context, we write ϕ_r instead of ϕ_r^c to ease notations.

By definition, when $r > 0$ is small enough, the set X^{-r} is a $C^{1,1}$ domain. We will show in this section that X_c^{-r} and X_c have the same homotopy type when $r > 0$ is small enough and c is a regular value, explaining the name *smooth surrogate* for X_c^{-r}. This reasoning begins by showing that the following convergence of sublevel sets holds.

Lemma 4.7 (Hausdorff convergence of sublevel sets)**.** *Let X be a complementary regular set, let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth and let c be a regular value of $f_{|X}$. Then in the Hausdorff topology we have:*

$$\lim_{r\to 0} X_c^{-r} = X_c. \tag{4.4}$$

Proof.

Since $||\eta|| \leq 1$, we have $X_c^{-r} \subset (X_c)^r$ for any $r > 0$. Assume that there is no Hausdorff convergence. Then there is a point $x \in X$ and a real $t > 0$ such that $B(x,t)$ and X_c^{-r} have empty intersection for any $r > 0$ small enough. Let u be in $\mathrm{Tan}(X, x)$ and consider a sequence $x_n \in \mathrm{int}(X)$ (possible since $\overline{\mathrm{int}\, X} = X$ representing u, i.e., such that $x_n = x + \varepsilon_n(u + o(1))$ with the sequence ε_n in $\mathbb{R}^+ \setminus 0$ converging to 0. For n big enough, x_n lies in $B(x,t)$ and x_n belongs to X^{-r} for any $0 < r < d_{\neg X}(x_n)$. However, by assumption it does not belong in X_c^{-r} and we have

$$f_r(x_n) > c.$$

Letting r go to zero yield $c \leq f(x_n)$. Since $f(x) \leq c$, the first order expansion of f at x yields $\langle \nabla f(x), u\rangle \geq 0$. This holds for any u in $\mathrm{Tan}(X, x)$, which amounts to the following inclusion in a half-space:

$$\mathrm{Tan}(X, x) \subset -\nabla f(x)^{\mathrm{o}}. \tag{4.5}$$

Now $\mathrm{Tan}(X, x)$ is the complement set of the convex cone $\mathrm{Tan}(^\neg X, x)$. The previous inclusion thus yields $\mathrm{Tan}(^\neg X, x) = \nabla f(x)^{\mathrm{o}}$, which is equivalent to the equality $\mathrm{Cone}(\nabla f(x)) = \mathrm{Nor}(^\neg X, x)$, contradicting the fact that c is a regular value of f.

□

When c is a regular value, the following lemma gives a uniform lower bound on $\Delta \circ \partial^* \phi_r$ over neighborhoods of X_c^{-r} of fixed size when r tends to 0.

Lemma 4.8 (Non-vanishing $\partial^* \phi_r$ around a regular value)**.** *Let c be a regular value of $f_{|X}$. Then there exists a positive constant α such that for any sequences of positive reals $(r_i), (K_i)$ such that $r_i, K_i \to 0^+$, and any sequence (x_i) of points within $\phi_{r_i}^{-1}(0, K_i]$ for all $i \in \mathbb{N}$, we have:*

$$\liminf_{i \to \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \alpha.$$

Proof.

The map $\phi_{r_i} = d_{X^{-r_i}} + \max(0, f_{r_i} - x)$ is the sum of a Lipschitz function and the positive part of a smooth function. We distinguish seven cases to compute the Clarke gradient $\partial^* \phi_{r_i}(x_i)$, each with different contributions from $d_{X^{-r_i}}$ and $\max(0, f_{r_i} - c)$. By extracting subsequences, we can assume that the sequence (x_i) lies in one of these cases. They are depicted in Figure 4.4. In fact, we will show that for any such sequence, we have:

$$\liminf_{i \to \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \min(\mu, \sigma, \kappa) > 0 \tag{4.6}$$

where

— $\kappa := \inf_{f^{-1}(c) \cap X} ||\nabla f||$ is a positive quantity because c is a regular value of $f_{|X}$.

— $\mu \leq \inf_{t \to 0} \{\Delta(\partial^* d_X(x)) \mid 0 < d_X(x) < t\}$ is positive by hypothesis.

— $\sigma := \inf_{x \in \partial X \cap f^{-1}(c)} \Delta(A_x)$ where $x \mapsto A_x$ is the upper semi-continuous set-valued map defined by:

$$A_x := \Big([0,1] \cdot \partial^* d_X(x) + \{\nabla f(x)\}\Big) \cup \Big(\partial^* d_X(x) + [0,1] \cdot \{\nabla f(x)\}\Big).$$

For any point $x \in \partial X$, keep in mind that from Theorem 3.32 we have the identity

$$\operatorname{Cone} \partial^* d_X(x) = \operatorname{Nor}(X, x),$$

which means that any direction in $\partial^* d_X(x)$ is a direction in $\operatorname{Nor}(X, x)$. The constant σ is positive when c is a regular value of $f_{|X}$. The set $\partial X \cap f^{-1}(c)$ is compact, and the map $x \mapsto \Delta(A_x)$ is lower semi-continuous. Assume that σ is zero. Then there is a point $x \in \partial X \cap f^{-1}(c)$ with $\Delta(A_x) = 0$. This means that the direction of $\nabla f(x)$ meets $\operatorname{Nor}(X, x)$, and thus c is a critical value of $f_{|X}$.

Idea behind the proof. For each of the seven cases, we will show that $\liminf_{i \to \infty} \Delta(\partial^* \phi_{r_i}(x_i))$ is greater than one among σ, κ, μ, depending on the contributions of $d_{X^{-r_i}}$ and f_{r_i}. Computations will show that $\partial^* \phi_{r_i}(x_i)$ either lies close to $\nabla f(x_i)$, $\partial^* d_X(x_i)$ or close to be inside A_{x_i}, each being bounded away from zero respectively by the non-vanishing of κ, μ and σ.

To ease some notations, we write $\nu(x) := \frac{x}{||x||}$ and $||\nabla f_{r_i} - \nabla f||_{\infty, X^1} =: \varepsilon_i$ the infinity norm of $\nabla f_{r_i} - \nabla f$ over the 1-offset of X * Remark that by elementary computations we have $\varepsilon_i = O(r_i)$.

∗. We could have taken the infinity norm over any bounded neighborhood of X without altering the line of reasoning.

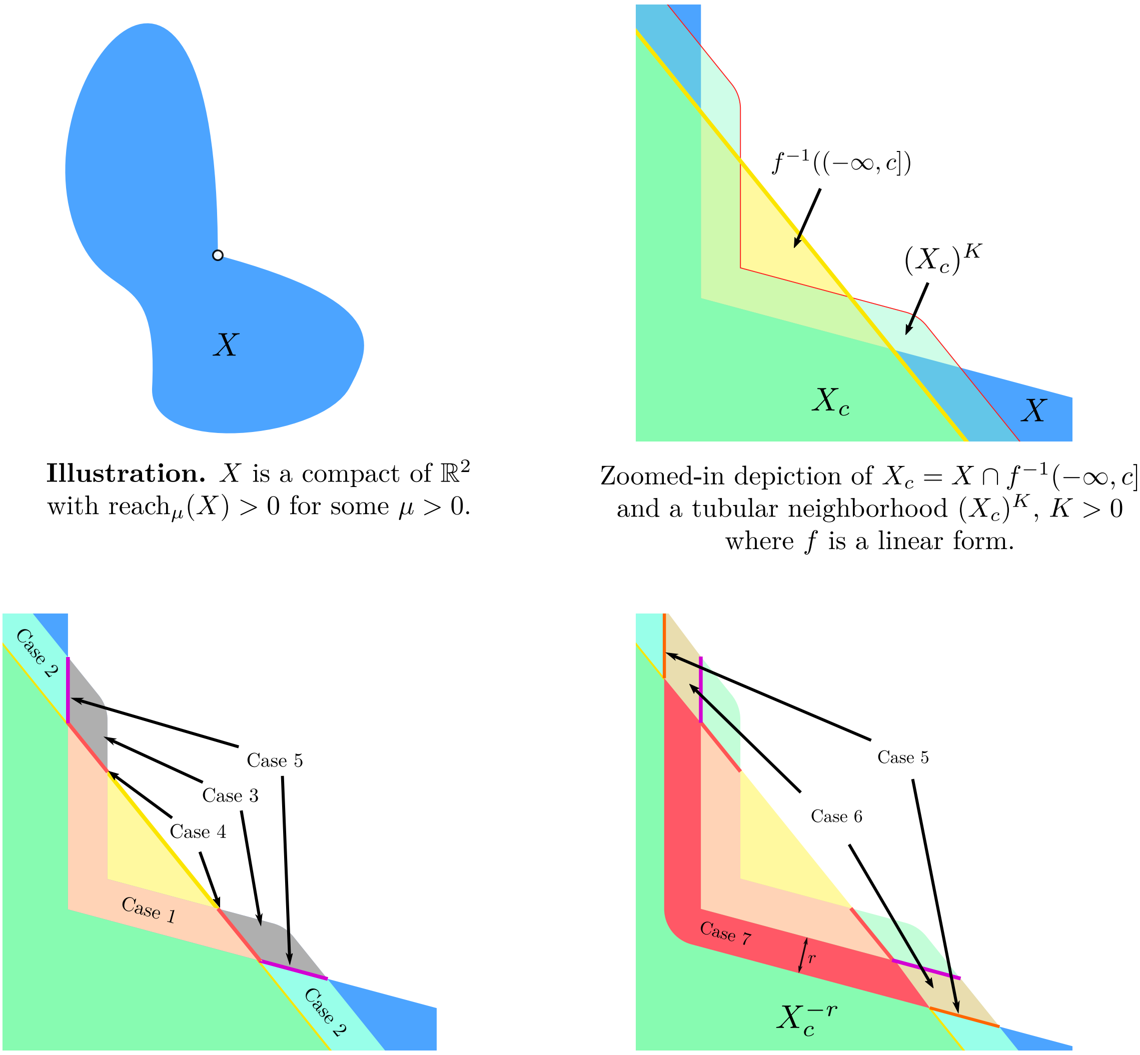


Illustration. X is a compact of $\mathbb{R}^2$ with $\text{reach}_\mu(X) > 0$ for some $\mu > 0$.

Zoomed-in depiction of $X_c = X \cap f^{-1}(-\infty, c]$ and a tubular neighborhood $(X_c)^K$, $K > 0$ where f is a linear form.

Illustration of cases 1 to 5 when $r = 0$. Cases 1 to 4 are defined independently of r.

Cases 5, 6 and 7 when $r > 0$.

Figure 4.4 – Illustration of the 7 cases of Lemma 4.8.

Case 1. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) < c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i)$ with $0 < d_X(x_i) < K_i + d_H(X^{-r_i}, X)$ which tends to 0 as $i \to \infty$. By the μ-reach hypothesis, we have

$$\liminf_{i \to \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \mu > 0. \tag{4.7}$$

Case 2. $x_i \in \text{int}(X^{-r_i})$.

Then $\partial^* \phi_{r_i}(x_i) = \{\nabla f_{r_i}(x_i)\}$ and $0 < f_{r_i}(x_i) - c \leq K_i$. As such, we have the inclusion $\partial^* \phi_{r_i}(x_i) \subset \{\nabla f(x_i)\}^{\varepsilon_i}$ and we obtain

$$\liminf_{i \to \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \kappa > 0. \tag{4.8}$$

Case 3. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) > c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i) + \nabla f_{r_i}(x_i) \subset (A_{x_i})^{\varepsilon_i}$, which yields

$$\liminf_{i\to\infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0. \tag{4.9}$$

Case 4. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) = c$.

First remark that since $d_{X^{-r_i}}(x_i) > r_i$ we have $\partial^* d_{X^{-r_i}}(x_i) = \partial^* d_X(x_i)$, $d_X(x_i) \to 0$ since $\lim_{r\to 0} X^{-r} = X$, and $d_X(x_i) > 0$. Now without loss of generality by extracting we can assume x_i converges to a point x in $\partial X \cap f^{-1}(c)$.
Now $\nabla f_{r_i}(x_i)$ has to be non-zero for i big enough as $\varepsilon_i = O(r_i)$ and

$$\liminf_{i\to\infty} ||\nabla f(x_i)|| \geq \inf_{x\in X\cap f^{-1}(c)} ||\nabla f(x)|| = \kappa$$

which yields that the set $\{y \mid f_{r_i}(y) \neq c\}$ has density 1 at x_i by the local inverse function theorem. As the Clarke gradient can be computed in a set of density 1 at x_i (see [Cla75]), we have for any x_i where $\nabla f_{r_i}(x_i) \neq 0$:

$$\partial^* \phi_{r_i}(x_i) = \mathrm{Conv}\left\{ \lim_{n\to\infty} \nabla \phi_{r_i}(z_n) \mid z_n \to x_i, f_{r_i}(z_n) \neq c \right\}.$$

We can decompose this set as

$$\partial^* \phi_{r_i}(x_i) = \mathrm{Conv}(A_+ \cup A_-)$$

where

$$\begin{aligned} A_+ &:= \left\{ \lim_{n\to\infty} \nabla \phi_{r_i}(z_n) \,\middle|\, z_n \to x_i, f_{r_i}(z_n) > c \right\} \\ A_- &:= \left\{ \lim_{n\to\infty} \nabla \phi_{r_i}(z_n) \,\middle|\, z_n \to x_i, f_{r_i}(z_n) < c \right\}. \end{aligned}$$

Now only $d_{X^{-r_i}}$ contributes to the gradients of A_- whereas f_{r_i} also contributes in A_+. Thus any point in $\mathrm{Conv}(A_+ \cup A_-)$ can be written as $u + \lambda \nabla f_{r_i}(x)$ where $u \in \partial^* d_{X^{-r_i}}(x_i) = \partial^* d_X(x_i)$ and $\lambda \in [0,1]$. This finally yields:

$$\liminf_{i\to\infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \Delta(A_x) \geq \sigma > 0. \tag{4.10}$$

Case 5. $x_i \in \partial X^{-r_i}$ and $f_{r_i}(x_i) > c$.

If $r_i > 0$, then $\partial^* d_{X^{-r_i}}(x_i)$ is the convex set generated by 0 and the direction normal to X^{-r_i} at x_i, that is $[0,1] \cdot \nu(\xi_{\neg X}(x_i) - x_i)$. Note that this direction belongs in the normal cone $\mathrm{Nor}(X, \xi_{\neg X}(x_i))$ as illustrated in Figure 4.5. Adding the contribution of f_{r_i} we obtain

$$\partial^* \phi_{r_i}(x_i) \subset (A_{\xi_{\neg X}(x_i)})^{\varepsilon_i}.$$

If $r_i = 0$, then $\partial^* \phi_{r_i}(x_i) = [0,1] \cdot \partial^* d_X(x_i) + \nabla f_{r_i}(x_i)$ and we obtain

$$\partial^* \phi_{r_i}(x_i) \subset (A_{x_i})^{\varepsilon_i}.$$

Either way,

$$\liminf_{i\to\infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \Delta(A_x) \geq \sigma > 0. \tag{4.11}$$

Now the remaining cases fit inside sequences of points (x, r) such that $0 < d_{X^{-r}}(x) \leq r$. Remark that $\mathrm{reach}(X^{-r}) \geq r$. If $d_{X^{-r}}(x) < r$ we know that x has only one closest point $\xi_{X^{-r}}(x)$ in X, which yields $\partial^* d_{X^{-r}}(x) = \{\nu(x - \xi_X(x))\}$. If $d_{X^{-r}}(x) = r$, x belongs to ∂X and the Clarke gradient $\partial^* d_{X^{-r}}(x)$ is $\mathrm{Conv}(\mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1})$ which is $\mathrm{Conv}(\mathrm{Cone}\, \partial^* d_X(x) \cap \mathbb{S}^{d-1})$ by Theorem 3.32. These considerations are illustrated in Figure 4.5 with $0 < d_{X^{-r}}(x_1) < r$ and $d_{X^{-r}}(x_2) = r$. In any case, this leads to

$$\partial^* d_{X^{-r}}(x) \subset \mathrm{Conv}(\partial^* d_X(\xi_{\neg X}(x)) \cup \mathbb{S}^{d-1}) \tag{4.12}$$

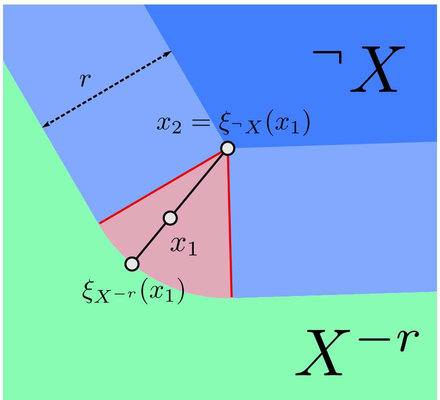


Figure 4.5 – Visualisation of the inclusion $\partial^* d_{X^{-r}}(x) \subset \partial^* d_X(\xi_{\neg X}(x))$ for two points x_1 and x_2, with $0 < r < \mathrm{reach}(\neg X, x)$. The translated unit cone $x_2 + \mathrm{Nor}(\neg X, x_2) \cap B(x_2, r)$ is depicted in red.

Case 6. $0 < d_X^{-r_i}(x_i) \leq r_i$ and $f_{r_i}(x_i) \geq c$

$\partial^* \phi_{r_i}(x_i) \subset \mathrm{Conv}\left(\mathrm{Nor}(X, \xi_{\neg X}(x)) \cap \mathbb{S}^{d-1}\right) + [0,1] \cdot \nabla f_{r_i}(x_i)$. Now by compactness assume that $x_i \to x$. Then $x \in \partial X \cap f^{-1}(c)$ and thus

$$\liminf_{i \to \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \Delta(A_x) \geq \sigma > 0. \tag{4.13}$$

Case 7. $0 < d_X^{-r}(x_i) \leq r_i$ and $f_{r_i}(x_i) < c$

Then $\partial^* \phi_{r_i}(x_i) \subset \mathrm{Conv}\left(\partial^* d_X(\xi_{\neg X}(x_i)) \cap \mathbb{S}^{d-1}\right)$ which yields

$$\liminf_{i \to \infty} \Delta(\partial^* \phi_{r_i}(x_i)) \geq \mu > 0. \tag{4.14}$$

□

We are now able to build homotopies in neighborhoods of fixed size of both X_c and X_c^{-r} when r is small enough.

Lemma 4.9 (Deformation retractions around X_c and X_c^{-r})**.** *Let c be a regular value of $f_{|X}$. Using the notations of Definition 4.6, there exists $K > 0, M \geq 1$ as well as continuous, piecewise-smooth flows*

$$C : [0,1] \times \phi^{-1}(-\infty, K] \to \phi^{-1}(-\infty, K]$$

$$C^r : [0,1] \times \phi_r^{-1}(-\infty, K] \to \phi_r^{-1}(-\infty, K]$$

such that:

— $L := \sup\{\Delta(\partial^* \phi(y))^{-1} \mid y \in \phi^{-1}(0, K]\}$ *is finite;*
— *For all* $r > 0$ *small enough,* $(X_c)^{\frac{K}{M}} \subset \phi_r^{-1}(-\infty, K]$ *and* $(X_c^{-r})^{\frac{K}{M}} \subset \phi^{-1}(-\infty, K]$*;*
— $C(0, \cdot), C^r(0, \cdot)$ *are the identity over their respective spaces of definition;*
— $C(1, \phi^{-1}(-\infty, K]) = X_c$ *and* $C^r(1, \phi_r^{-1}(-\infty, K]) = X_c^{-r}$*;*
— *For any* $t \in [0,1]$*,* $C(t, \cdot)_{|X_c}$*,* $C^r(t, \cdot)_{|X_c^{-r}}$ *are the identity over* X_c *and* X_c^{-r}*;*
— $C(\cdot, \cdot)$ *and* $C^r(\cdot, \cdot)$ *are* $2KL$*-Lipschitz in the first variable when* $r > 0$ *is small enough.*

Proof.

Remark that $X_c = \phi^{-1}(0)$ and $X_c^{-r} = (\phi_r)^{-1}(0)$. We want to bound $\Delta \circ \partial^* \phi_r$ and $\Delta \circ \partial^* \phi)$ from below to apply Proposition 2.9.

Let

$$\omega(s, K) := \inf_{\substack{r \in [0,s] \\ x \in \phi_r^{-1}(0,K]}} \Delta(\partial^* \phi_r(x)).$$

Lemma 4.8 states that

$$\liminf_{\substack{s \to 0^+ \\ K \to 0^+}} \omega(s, K) > 0. \tag{4.15}$$

When $K, s > 0$ are small enough, for all $r \in [0, s]$, $\Delta\, \partial^* \phi_r$ is uniformly bounded below by a positive number in $\phi_r^{-1}(0, K]$, allowing the offsets to be retracted by the approximate inverse flows C, C^r of respectively ϕ and ϕ_r by Proposition 2.9. For any positive ε, the flows can be chosen so that the gradients of the flows in the time parameter are bounded by $(1 + \frac{1}{2}\varepsilon) l_{r,K} = (1 + \frac{\varepsilon}{2}) K \sup\{\Delta(\partial^* \phi_s(y))^{-1} \mid s \in [0, r], y \in \phi_r^{-1}(0, K]\}$ which is finite when r, K are taken small enough, and the supremum tends to a positive number L when r, K go to zero. When these numbers are small enough, the whole quantity is bounded by $(1 + \varepsilon) KL$.

Since the functions $(\phi_r)_{r \in [0,s]}$ are uniformly Lipschitz, we let $M := 1 + \sup\{\mathrm{Lip}(\phi_r)_{r \in [0,s]}\}$. As the sets X_c^{-t} converge to X_c when t goes to 0 by Lemma 4.7, and $||\phi - \phi_r||_\infty = O(r)$, we have

$$(X_c^{-t})^{\frac{K}{M}} \subset \phi_r^{-1}(0, K]$$

for any t, r small enough.

□

Corollary 4.10 (Homotopy Equivalence)**.** *Let* $X \subset \mathbb{R}^d$ *be a complementary regular set and* $f : \mathbb{R}^d \to \mathbb{R}$ *be a smooth function. Let* c *be a regular value of* $f_{|X}$*, let* η *be a smooth function with* $||\eta||_\infty \leq 1$ *and let* $f_r : x \mapsto f(x + r\eta(x))$*. Then for all* $r > 0$ *small enough,* $X_c^{-r} = X^{-r} \cap f_r^{-1}(-\infty, c]$ *and* X_c *have the same homotopy type.*

Proof.

Since $\lim_{r \to 0} d_H(X_c^{-r}, X_c) = 0 < K/M$, the flows C, C^r are respectively well-defined on X_c^{-r}, X_c for r small enough thanks to Lemma 4.9. Letting $\psi := C(1, \cdot)_{|X_c^{-r}} : X_c^{-r} \to X_c$ and $\psi^r := C^r(1, \cdot)_{|X_c} : X_c \to X_c^{-r}$, their composition $\psi \circ \psi^r$ is homotopic to Id_{X_c} via the map

$$\begin{cases} X_c \times [0,1] & \to & X_c \\ (x, t) & \mapsto & C(1, C(t, C^r(t, x))). \end{cases}$$

In the same fashion, $\psi^r \circ \psi$ is homotopic to $\mathrm{Id}_{X_c^{-r}}$ via $(t,x) \mapsto C^r(1, C^r(t, C(t,x)))$.
□

4.4 Constant homotopy type lemma

In this section, we prove that the topology of the sublevel sets of a smooth map restricted to a complementary regular set does not evolve between critical values.

Theorem 4.11 (Constant homotopy type between critical values)**.** *Let $X \subset \mathbb{R}^d$ be a complementary regular set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth map and $a < b \in \mathbb{R}$ be such that $[a,b]$ contains only regular values of $f_{|X}$. Then X_a is a deformation retract of X_b.*

This theorem is a direct consequence of the compactness of $[a,b]$ and Lemma 4.13, which we will prove using the following technical lemma.

Lemma 4.12 (Regular values of the family $(\phi^c)_{c\in\mathbb{R}}$ are open.)**.** *Let c be a regular value of $f_{|X}$ and let $\phi^s := d_X + \max(f - s, 0)$ for any $s \in \mathbb{R}$. Then we have:*

$$\lim_{\substack{\varepsilon\to 0^+\\ K\to 0^+}} \inf \left\{ \Delta(\partial^* \phi^{c+a}(x)) \mid x \in (\phi^{c+a})^{-1}(0,K], a \in [-\varepsilon, \varepsilon] \right\} > 0.$$

Proof.

We proceed by contradiction. Assuming the inequality is false, there exist two real sequences $a_i \to 0, K_i \to 0^+$, and $(x_i)_{i\in\mathbb{N}}$ a sequence in $\mathbb{R}^d$ such that:

$$\forall i \in \mathbb{N},\ 0 < \phi^{c+a_i}(x_i) \le K_i \quad \text{and} \quad \lim_{i\to\infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) = 0.$$

We use the same distinction of sequences of $\phi^{-1}_{c+a_i}(0, K_i]$ into cases as in the proof of Lemma 4.8. Since $r = 0$, we distinguish 5 cases to compute $\partial^* \phi^{c+a_i}$.

Case 1. $f(x_i) < c + a_i$ and $d_X(x_i) > 0$.

Then $\partial^* \phi^{c+a_i}(x_i) = \partial^* d_X(x_i)$ and since $d_X(x_i) \le K_i \to 0$, we have:

$$\liminf_{i\to\infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) \ge \mu > 0.$$

Case 2. $x_i \in \mathrm{int}(X)$ and $f(x_i) > c + a_i$.

Then $\partial^* \phi^{c+a_i}(x_i) = \{\nabla f(x_i)\}$ and thus

$$\liminf_{i\to\infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) \ge \sigma > 0.$$

Cases 3, 4, 5.

$$\begin{cases} f(x_i) > c + a_i & \text{and} \quad d_X(x_i) > 0 \\ f(x_i) > c + a_i & \text{and} \quad x_i \in \partial X \\ f(x_i) = c + a_i & \text{and} \quad d_X(x_i) > 0. \end{cases}$$

In these 3 cases we have the inclusion $\partial^* \phi^{c+a_i}(x_i) \subset A_{x_i}$. As in the proof of Lemma 4.8, the map $y \mapsto A_y$ is semi-continuous. Now if (x_i) converges to a point x then this point belongs to $\partial X \cap f^{-1}(c)$. Since c is a regular value, we have:

$$\liminf_{i\to\infty} \Delta(\partial^* \phi^{c+a_i}(x_i)) \ge \kappa > 0.$$

□

Lemma 4.13 (Local deformation retractions)**.** *Let X be complementary regular, $f : \mathbb{R}^d \to \mathbb{R}$ smooth and let c be a regular value of $f_{|X}$. Then for all $\varepsilon > 0$ small enough and any $-\varepsilon \leq a \leq b \leq \varepsilon$, X_{c+a} is a deformation retract of X_{c+b}.*

Proof.

By Lemma 4.12 there exist $\sigma, \varepsilon, K > 0$ such that for every $a \in [-\varepsilon, \varepsilon]$ we have

$$\Delta(\partial^* \phi^{c+a}(x)) \geq \sigma \text{ for all } x \text{ in } (\phi^{c+a})^{-1}(0, K]. \tag{4.16}$$

Thus by Proposition 2.9 for every $\alpha \in [-\varepsilon, \varepsilon]$ there exists a continuous $\frac{2K}{\sigma}$-Lipschitz approximate flow of $\phi^{c+\alpha}$ on $(\phi^{c+\alpha})^{-1}(0, K]$ which we will denote $C_{c+\alpha}(\cdot, \cdot)$. By elementary computations one has for every $a < b \in [-\varepsilon, \varepsilon]$:

$$\phi^{c+a}(X_{c+b}) \subset [0, b-a] \subset [0, 2\varepsilon] \tag{4.17}$$

meaning that $X_{c+b} \subset (\phi^{c+a})^{-1}(0, K]$ when $\varepsilon > 0$ is small enough. The flow C_{c+a} makes ϕ^{c+a} decrease, leading to the following inclusions for $\varepsilon > 0$ small enough and any $t \in [0, 1]$:

$$C_{c+a}(t, X_{c+b}) \subset (\phi^{c+a})^{-1}[0, 2\varepsilon] \subset (\phi^{c+b})^{-1}[0, K]. \tag{4.18}$$

Consequently, the composition $C_{c+b}(s, C_{c+a}(t, x))$ is well-defined for any $t, s \in [0, 1]$ and $x \in X_{c+b}$ and is continuous in every of these variables. Now letting i be the inclusion $X_{c+a} \to X_{c+b}$ and $\psi := C_{c+a}(1, \cdot) : X_{c+b} \to X_{c+a}$, one clearly has $\psi \circ i = \mathrm{Id}_{X_{c+a}}$. The map $i \circ \psi$ is homotopic to $\mathrm{Id}_{X_{c+b}}$ via the homotopy

$$\left\{ \begin{array}{ccc} X_{c+b} \times [0, 1] & \to & X_{c+b} \\ (x, t) & \mapsto & C_{c+b}(1, C_{c+a}(t, x)). \end{array} \right. \tag{4.19}$$

□

4.5 Handle attachment around critical values

We now want to study the evolution of the topology of the sublevel set filtration of a Morse function and prove the handle attachment lemma. We begin by showing that non-degenerate critical points are isolated.

Proposition 4.14 (Critical points of a Morse function are isolated)**.** *Let $X \subset \mathbb{R}^d$ be a set with positive reach or a complementary regular set and let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function. Then in the set of critical points of $f_{|X}$, the non-degenerate critical points are isolated.*

Proof.

Let x be a non-degenerate critical point and assume that there is a sequence x_i in ∂X of critical points of $f_{|X}$ all distinct from x and converging to x. This means that for every $i \in \mathbb{N}$, the unit vector $n_i := -\frac{\nabla f(x_i)}{||\nabla f(x_i)||}$ lies in $\mathrm{Nor}(X, x_i)$. The sequence (x_i, n_i) lies in $\mathrm{Nor}(X)$ and converges to (x, n) where $n := -\frac{\nabla f(x)}{||\nabla f(x)||}$. Extracting a subsequence, we can assume that $\frac{(x_i - x, n_i - n)}{||(x_i - x, n_i - n)||}$

converges to $(u, v) \in \mathrm{Tan}(\mathrm{Nor}(X), (x, n))$. Since x is non-degenerate, $\mathrm{Tan}(\mathrm{Nor}(X), (x, n)$ is a vector space and both u and $-u$ belong in $\pi_0(\mathrm{Tan}(\mathrm{Nor}(X), (x, n)) \subset \mathrm{Tan}(X, x)$, yielding $\langle u, n \rangle = 0$. Moreover, the second fundamental form of X at (x, n) in the direction u is given by:

$$\mathrm{I\!I}_{x,n}(u, u) = \langle u, v \rangle. \tag{4.20}$$

Since $\nu := -\frac{\nabla f}{||\nabla f||}$ is smooth around x, we have $||n_i - n|| = ||\nu(x_i) - \nu(x)|| = O(||x_i - x||)$. This entails $||(x_i - x, n_i - n)|| = O(||x_i - x||)$ which ensures that the spacial component u of the limit is non-zero. We can further assume that $\frac{x_i - x}{||x_i - x||}$ converges to $\frac{u}{||u||}$. The first order expansion of $n_i = \nu(x_i)$ gives

$$n_i - n = ||x_i - x||\, D_x\nu\left(\frac{u}{||u||}\right) + o(||x_i - x||). \tag{4.21}$$

If $D_x\nu(u) = 0$, we have $||n_i - n|| = o(||x_i - x||)$ meaning that $v = 0 = D_x\nu(u)$ and $\mathrm{I\!I}_x(u, u) = 0$. Otherwise, $||n_i - n|| \sim C\, ||x_i - x||$ for some $C > 0$. By elementary computations this also yields $D_x\nu(u) = v$ and we thus have in any case

$$D_x\nu(u) = v. \tag{4.22}$$

Now we can write the first order expansion of $\nabla f(x_i) + ||\nabla f(x_i)||\, n_i$:

$$\begin{aligned} 0 =& \nabla f(x_i) + ||\nabla f(x_i)||\, n_i \\ =& ||x_i - x||\, (H_x f(u) + ||\nabla f(x)||\, D_x\nu(u) - n\, \langle n, H_x f(u) \rangle) + o(||x_i - x||). \end{aligned}$$

Taking the scalar product of this vector with u yields:

$$H_x f_{|X}(u, u) = H_x f(u, u) + ||\nabla f(x)||\, \langle u, v \rangle = 0. \tag{4.23}$$

This contradicts the non-degeneracy of $H_x f_{|X}$ in the direction u which belongs to $\pi_0(\mathrm{Tan}(\mathrm{Nor}(X)), (x, n)) \setminus \{0\}$.

□

When c is critical value of $f_{|X}$ with only one corresponding critical point x, we choose the map $\eta : \mathbb{R}^d \to \mathbb{R}$ as follows.

Definition 4.15 (Choice of surrogates when there is at most one critical point per sublevel set)**.** Let $X \subset \mathbb{R}^d$ be a complementary regular set and let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth. If $c \in \mathbb{R}$ is such that $f^{-1}(c)$ contains only one critical point x of $f_{|X}$ which is non-degenerate, we put for any $r > 0$:

$$\gamma_r^c := y \mapsto y - r\frac{\nabla f(x)}{||\nabla f(x)||} \qquad\qquad f_{r,c} := f \circ \gamma_r^c$$

When the value c is clear from the context, we write γ_r and f_r instead to ease notations.

The following two lemmas focus on the properties of the critical points of $f_{r_{|X^{-r}}}$.

Lemma 4.16 (Local correspondence between critical points of $f_{|X}$ and $f_{r|X^{-r}}$)**.** *Let X be a complementary regular subset of $\mathbb{R}^d$. Assume x is a non-degenerate critical point of $f_{|X}$ and let ind_x be the index of the Hessian of $f_{|X}$ at x. Then $x^r = \gamma_r(x)$ is a critical point of $f_{r|X^{-r}}$ such that*

$f_r(x^r) = f(x)$ *for all* $0 < r < \operatorname{reach}(\neg X)$. *When* r *is small enough,* x^r *is a non-degenerate critical point of* $f_{r|X^{-r}}$*, whose Hessian at point* x_r *has index*

$$\operatorname{ind}_x^r := \operatorname{ind}_x + \textit{number of infinite curvatures at } \left(x, \frac{\nabla f(x)}{||\nabla f(x)||}\right).$$

Proof.

Let $n = \frac{\nabla f(x)}{||\nabla f(x)||} \in \operatorname{Nor}(\neg X, x)$ the normalized gradient of f at this point. Keep in mind that $f_r : x \mapsto f(x - rn)$ is f translated in the direction n with magnitude r.

The pair $(x, n) \in \operatorname{Nor}(\neg X)$ is regular by non-degeneracy of f at x. Denote by $(\kappa_i')_{1\leq i\leq d-1}$ the principal curvatures (defined in Definition 6.4) of $\neg X$ at (x, n) sorted in ascending order and put $m := \max\{i \,|\, \kappa_i' < \infty\}$. From there we follow the reasoning of Fu [Fu89a]. When $0 < r < \operatorname{reach}(\neg X)$, X^{-r} is as $C^{1,1}$-domain and the regularity of the pair (x, n) in X guarantees that the Gauss map $x \in \partial\neg X^{-r} \mapsto n(x) \in \mathbb{S}^{d-1}$ is differentiable at $x + rn$. We have the following linear correspondence between tangent spaces:

$$\operatorname{Tan}(\operatorname{Nor}(\neg(X^{-r})), (x + rn, n)) = \{(\tau + r\sigma, \sigma) \mid (\tau, \sigma) \in \operatorname{Tan}(\operatorname{Nor}(\neg X), (x, n))\}.$$

Since $\operatorname{Nor}(X^{-r}) = \{(z, -n) \,|\, (z, n) \in \operatorname{Nor}(\neg(X^{-r})\}$ we have:

$$\pi_0(\operatorname{Tan}(\operatorname{Nor}(X^{-r}), (x + rn, n))) = \{\tau - r\sigma \mid (\tau, \sigma) \in \operatorname{Tan}(\operatorname{Nor}(X), (x, n))\}.$$

This vector space is identifiable with the classical tangent space of differential geometry since and thus has dimension d. Proceeding exactly in the same fashion as the proof of [Fu89a, 4.6], we can write, for any $\tau - r\sigma, \tau' - r\sigma'$ in $\pi_0(\operatorname{Tan}(\operatorname{Nor}(X^{-r}), (x + rn, n)))$:

$$\begin{aligned}
&H_{x+rn} f_{r|X^{-r}}(\tau - r\sigma, \tau' - r\sigma') \\
&\qquad = H_{x+rn} f_r(\tau - r\sigma, \tau' - r\sigma') + ||\nabla f_r(x^r)||\, \mathrm{II}_{x+rn}(\tau - r\sigma, \tau - r\sigma') \\
&\qquad = H_x f(\tau - r\sigma, \tau' - r\sigma') + ||\nabla f(x)||\, \langle \tau - r\sigma, \sigma' \rangle\,.
\end{aligned}$$

We can decompose $\pi_0(\operatorname{Tan}(\operatorname{Nor}(X^{-r}), (x + rn, n)))$ as the direct sum of $F := \{\sigma \,|\, (0, \sigma) \in \operatorname{Tan}(\operatorname{Nor}(X), (x, n))\}$ and a supplementary subspace E. E has dimension m and F dimension $d - m$. By the structure theorem of tangent spaces, E and F are orthogonal. From the previous computation, identifying coefficients in front of the r-monomials, there are square matrices A_1, A_2, A_3 of size m, a square matrix B of size $d - m$ and a rectangular matrix C such that the Hessian $H_{x+rn} f_{r|X^{-r}}$ has the form

$$\begin{pmatrix} A_1 + rA_2 + r^2 A_3 & rC \\ rC^t & -r\,||\nabla f(p)||\, Id + r^2 B \end{pmatrix}$$

where A_1 is similar to the matrix of $H_x f_{|X}$. It is the same computation as [Fu89a] except that we end up with a minus sign in front of the identity in the lower right corner. When $r > 0$ is small enough, this matrix is non-degenerate and its index is that of A_1 plus the dimension of the identity matrix in the lower right corner.

□

Lemma 4.17 (Critical points of $f_{r_{|X^{-r}}}$ when r is small enough)**.** *Let $X \subset \mathbb{R}^d$ be a complementary regular set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $f_{|X}$ is Morse. Assume x is the only critical point in $X \cap f^{-1}(c)$. Then for $\varepsilon, r > 0$ small enough, $x^r = x + r\frac{\nabla f(x)}{||\nabla f(x)||}$ is the only critical point of $f_{r_{|X^{-r}}}$ inside $f^{-1}_{r|X^{-r}}(c-\varepsilon, c+\varepsilon)$, and $f_r(x^r) = c$.*

Proof.

First remark that $f_r(x^r) = f(x)$ and $\nabla f_r(x^r) = \nabla f(x)$. For $r > 0$ small enough, the normal consists in a sole line, in the same fashion as x_1 in Figure 4.5. More precisely, we have $\mathrm{Nor}(X^{-r}, x^r) = -\,\mathrm{Cone}(\nabla f(x))$ and x^r is a critical point of $f_{r|X^{-r}}$. Assuming the claim of Lemma 4.17 is false, there are sequences $\varepsilon_i, r_i > 0$ converging to 0, and y_i a sequence in ∂X such that:

— $d_{\neg X}(y_i) = r_i$

— $c - \varepsilon_i \leq f_{r_i}(y_i) \leq c + \varepsilon_i$

— $y_i \neq x^{r_i}$

— $n_i := -\frac{\nabla f_{r_i}(y_i)}{||\nabla f_{r_i}(y_i)||} \in \mathrm{Nor}(X, y_i)$.

By semi-continuity of the normal cones as functions of ∂X, which is a consequence of the identity $\mathrm{Nor}(X,x) = \mathrm{Cone}(\partial^* d_X(x))$, any accumulation point $\bar{x}$ of the sequence $(y_i)_{i\in\mathbb{N}}$ is a critical point of $f_{|X}$ with $f(\bar{x}) = c$, thus showing that y_i converges to x. Now put $x_i := \xi_{\neg X}(y_i)$. If we assume that $x_i = x$ for all i, then $y_i = x + r_i n_i$. Since $y_i \neq x^{r_i}$, n_i and n are not equal, and we can also assume that $\frac{n_i - n}{||n_i - n||}$ converges to some unit vector $v' \in \mathbb{R}^d$ by extracting a subsequence. Then we would have

$$\begin{aligned} n_i - n &= -\frac{\nabla f(x + r_i(n_i - n))}{||\nabla f(x + r_i(n_i - n))||} + \frac{\nabla f(x)}{||\nabla f(x)||} \\ &= -\,r_i\,||n_i - n||\,(D_x\nu)(v) + o(r_i\,||n_i - n||) \\ &= o(||n_i - n||) \end{aligned}$$

which is absurd. We can thus assume without loss of generality that x_i is different from x for all $i \in \mathbb{N}$. Reasoning exactly as in the proof of Proposition 4.14, we can extract a subsequence such that the sequence $\frac{(x_i - x, n_i - n)}{||(x_i - x, n_i - n)||}$ converges to $(u, v) \in \mathrm{Tan}(\mathrm{Nor}(X), (x, n))$. The same computations yield that the restricted Hessian $H_x f_{|X} = H_x f + ||\nabla f(x)||\,\mathrm{I\!I}_{x,n}$ is degenerate in the direction $u \in \pi_0(\mathrm{Tan}(\mathrm{Nor}(X)), (x, n)) \setminus \{0\}$.

□

With these two results, the homotopy equivalence with the surrogates sublevel sets X_t^{-r} proved in Section 4.3 allows to study the evolution of the topology of X_t around the critical value c.

Theorem 4.18 (Handle attachment around unique critical values)**.** *Let X be complementary regular and $f : \mathbb{R}^d \to \mathbb{R}$. Assume $f_{|X}$ has only one critical point x in $f^{-1}(c)$ which is non-degenerate. Then for any $\varepsilon > 0$ small enough, $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ with a λ_x-cell attached, where*

$$\begin{aligned} \lambda_x := \ & \textit{index of the Hessian of } f_{|X} \textit{ at } x \\ & + \textit{number of infinite curvatures at } \left(x, \frac{\nabla f(x)}{||\nabla f(x)||}\right). \end{aligned}$$

Proof.

By Lemma 4.17, when $\varepsilon, r > 0$ are small enough, there is only one critical point x_r in $f^{-1}_{r|X^{-r}}((c-\varepsilon, c+\varepsilon))$. By $C^{1,1}$ Morse theory, $X^{-r}_{c+\varepsilon}$ has the homotopy type of $X^{-r}_{c-\varepsilon}$ with a cell added around x^r. The dimension of the cell is λ_x for all $r > 0$ small enough by Lemma 4.16. Now by Corollary 4.10, when $r > 0$ is small enough, $X^{-r}_{c+\varepsilon}$ and $X_{c+\varepsilon}$ are homotopy equivalent, and so are $X^{-r}_{c-\varepsilon}$ and $X_{c-\varepsilon}$. This is summarized by Figure 4.6.

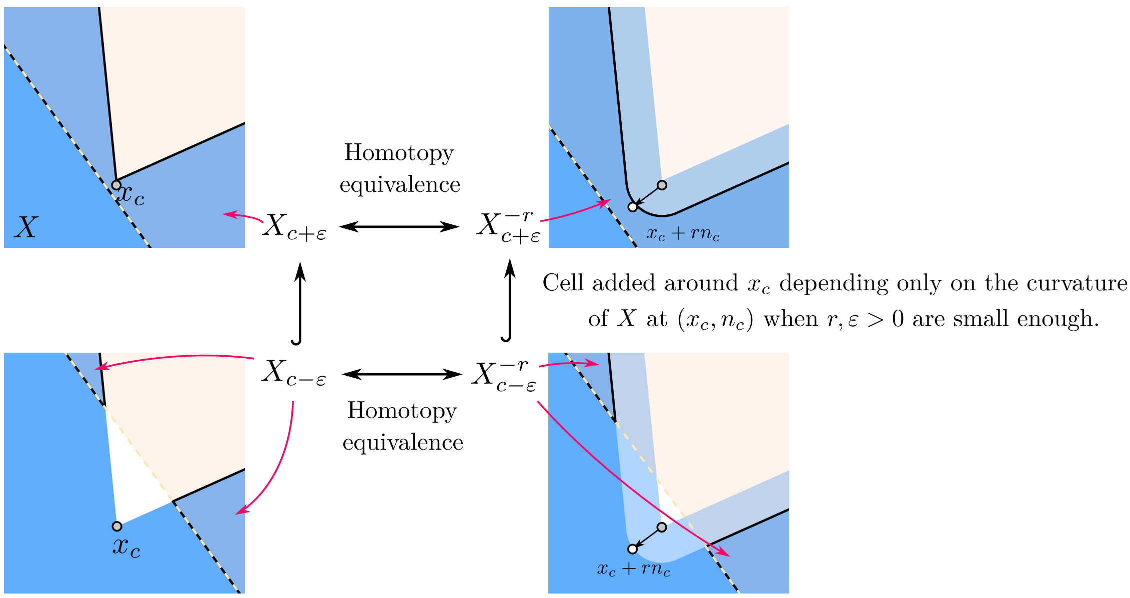


Figure 4.6 – Commutative diagram in the proof of Theorem 4.18.

□

Finally, we prove that the previous result holds when there might be several critical points of $f_{|X}$ sharing the same critical value.

Theorem 4.19 (Morse Theory for complementary regular sets). *Let $X \subset \mathbb{R}^d$ be a complementary regular set. Suppose $f_{|X}$ has a finite number of critical points, which are all non-degenerate. Each critical level set $X \cap f^{-1}(\{c\})$ has a finite number p_c of critical points, whose indices (defined in Theorem 4.18) we denote by $\lambda^c_1, \ldots \lambda^c_{p_c}$. Then:*

- *If $[a, b]$ does not contain any critical value, X_a is a deformation retract of X_b.*
- *If c is a critical value, $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ with exactly p_c cells attached around the critical points in $f^{-1}(c) \cap X$, of respective dimension $\lambda^c_{p_1}, \ldots, \lambda^c_{p_c}$ for all $\varepsilon > 0$ small enough.*

Proof.

The first point is Theorem 4.11. We turn our attention to the second point, which is a generalization of Theorem 4.18 to the case where several critical points of $f_{|X}$ have the same value. Let c be a critical value of $f_{|X}$. Put $x_1, \ldots, x_p$ the critical points of $f_{|X}$ inside $f^{-1}(c)$. Put $n_i := -\frac{\nabla f(x_i)}{||f(x_i)||}$ and $x^r_i = x_i - rn_i$. Let $n(x)$ be the function mapping x to the n_i associated to the closest critical point x_i of x. This map is piecewise constant and defined almost everywhere. Let $U_i \subset V_i$ be respectively closed and open balls containing x_i such that $\overline{V_i} \cap \overline{V_j} = \emptyset$ when

$j \neq i$. Let η_c be a smooth function on $\mathbb{R}^d$ with values in $[0, 1]$ such that η_c is constant of value 1 inside each U_i and 0 outside $\bigcup V_i$. The map $n_c : y \mapsto \eta_c(y)n(y)$ is well-defined and smooth when the U_i are small enough. When r is small enough, the map $\gamma_r : y \mapsto y + rn_c(y)$ is a diffeomorphism. Now define f_r to be f locally translated around the critical points:

$$f_r = f \circ \gamma_r : y \mapsto f(y + rn_c(y)).$$

From Lemma 4.16 we know that the $(x_i^r)_{1\leq i\leq p}$ are non-degenerate critical points of X^{-r} for $f_{r|X^{-r}}$ with corresponding index $(\lambda_i^c)_{1\leq i\leq p}$. From Lemma 4.17, we know that x_i^r is the only critical point of $f_{r|X^{-r}}$ inside $\gamma_r(U_i)$ when r is small enough.
Now we prove that there are no critical points outside $\bigcup_i \gamma_r(U_i)$ when r is small enough. On the one hand, outside this set, the sets $\mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1}$ and $\frac{\nabla f(x)}{||\nabla f(x)||}$ have a fixed distance separating them. On the other hand, when r goes to 0, the sets $\mathrm{Nor}(X^{-r}, x) \cap \mathbb{S}^{d-1}$ (resp. $\left\{\left(x, \frac{\nabla f_r(x)}{||\nabla f_r(x)||}\right)\right\}$) converge uniformly in x (as will soon be precised) in the Hausdorff distance to $\mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1}$ (resp. $\frac{\nabla f(x)}{||\nabla f(x)||}$) meaning by semi-continuity that for r small enough, the two still cannot intersect.
More quantitatively, by the inverse function theorem X^{-r} has a $C^{1,1}$ boundary. Since ∇f does not vanish in a neighborhood of $f^{-1}(c) \cap X$, we know that $x \in X^{-r}$ is a critical point of $f_{r|X^{-r}}$ if and only if $x \in \partial X^{-r}$, $\{\nu\} = \mathrm{Nor}(X^{-r}, x) \cap \mathbb{S}^{d-1}$ (i.e. ν is *the* normal at x) and $\left|\left|\frac{\nabla f_r(x)}{||\nabla f_r(x)||} - \nu\right|\right| = 0$.
Remark that we have both

$$\mathrm{Nor}(X^{-r}) = \{(x + r\nu, -\nu) \mid (x, \nu) \in \mathrm{Nor}(^\neg X)\}$$

and

$$\sup_{(x,\nu)\in\mathrm{Nor}(X)} ||\nabla f(x) - \nabla f_r(x + r\nu)|| = O(r)$$

leading to

$$\liminf_{r\to 0} \inf_{\substack{(x,\nu)\in\mathrm{Nor}(X^{-r})\\ x\notin\cup_i\gamma_r(U_i)\\ f_r(x)=c}} \left|\left|\frac{\nabla f_r(x)}{||\nabla f_r(x)||} - \nu\right|\right| \geq \inf_{\substack{(x,\nu)\in\mathrm{Nor}(^\neg X)\\ x\notin\cup_i U_i\\ f(x)=c}} \left|\left|\frac{\nabla f(x)}{||\nabla f(x)||} - \nu\right|\right| > 0. \tag{4.24}$$

This shows that $\{x_1^r, \ldots, x_p^r\}$ is exactly the set of critical points of $f_{r|X^{-r}}$ with value c. We obtain $X_{c+\varepsilon}^{-r}$ from $X_{c-\varepsilon}^{-r}$ by gluing cells locally around each critical point as in classical Morse theory.

□

Remark. The results proved in this section also hold when $X \subset \mathbb{R}^d$ has positive reach, essentially because there is a correspondence between the critical points of $f_{|X}$ and those of $(-f)_{|^\neg X}$ since $\mathrm{Nor}(^\neg X, x) = -\mathrm{Nor}(X, x)$. The proofs have to be adapted by taking $\gamma_r^c : y \mapsto y + r\frac{\nabla f(x)}{||\nabla f(x)||}$ around any critical points.

Chapter 5

Image persistence

Some crucial tools of this thesis come from the field of persistence. In this chapter, we expose the classical motivations and definitions of persistence theory, before introducing the notion of image persistence. Using results from Chapter 2, we prove an image stability theorem in the following sense. If $f : \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz map and $A \subset X \subset B \subset \mathbb{R}^d$ are sets such that A, B both lie at Hausdorff distance less than ε to X, we can build a persistence diagram from $f_{|A}, f_{|B}$ at distance bounded by a multiple of ε to the persistence diagram of $f_{|X}$, provided that X is a sublevel set of a Lipschitz map satisfying certain conditions. This adds to the traditional stability theorem which bounds the distance between the diagrams of $f_{|X}$ and $g_{|X}$ by $||f - g||$. Furthermore, we prove an inequality on the average Euler characteristics of close persistent homology diagrams satisfying some injection properties, which are verified by image persistence diagrams. These two results will be essential in obtaining the geometric inference results of Chapter 7.

Persistence theory is the mathematical field devoted to the study of functors from the poset category $\mathbb{R}$ to the vector spaces over some field, which are called *persistence modules*. Appearing in the nineties (see [Per18] for an historical summary) as a tool in computational geometry destined to quantitatively compare the topology of data sets, persistence theory went on to become its own field of research, leading to the proofs of core results such as the structure theorem [ZC05], and stability theorem [CSEH05]. The third pillar of persistence is the computability of the persistent homology of Vietoris-Rips or Alpha filtrations of datasets in high-dimensions, which put the field at the center of topological machine learning - also called *topological data analysis*. It has found a considerable number of applications in various fields such as biology [RCK+17, RB19], study of time series [PdM15, BCMR24], medicine [ACC+20, LSB+19], graph theory [AAF19]. It has also proven to be a useful tool in symplectic geometry [PRSZ20] and statistical geometry [NSW08], making it an asset in both applied and pure mathematics.

This chapter is divided in three sections. In the first, we broadly describe what we mean by homology,, and we explain the original motivation of persistent homology. The second section is an exposition of the classical notions of persistence theory, namely the decomposition of tame modules in intervals, and the stability theorem. No proof will be given, and we refer the hungry-for-evidence reader to the book [CdSGO16]. The third section will build on these ideas to construct *image persistence* using definitions appearing first in [CSEHM09]. Thanks to Chapter 3 and results from [BL15], we prove results of which we will make good in in chapter Chapter 7 : the image stability theorem for Lipschitz functions (Theorem 5.12) and the χ-averaging lemma (Lemma 5.16).

5.1 Context and motivation for persistence homology

5.1.1 A few reminders about homology

Let $\mathbb{K}$ be a field. By *homology* of a subset X of $\mathbb{R}^d$ over $\mathbb{K}$, we mean the singular homology of X equipped with the topology induced by the metric of $\mathbb{R}^d$. It associates to X a $\mathbb{K}$-vector space $H_i(X)$ for every $i \in \mathbb{N}$, with $H_i(X) = 0$ when $i > d$. We omit the dependence on $\mathbb{K}$ as it will not matter in our study. As the definition of singular homology is out of the scope of the present document, we refer the curious reader to introductory texts such as [Hat02]. This reference also contains the definitions of CW-Complexes and cellular homology, which gives another way to define the homology of CW-complexes.

Proposition 5.1 (Homotopy invariance)**.** *Let X, Y be two subsets of Euclidean spaces sharing the same homotopy type. Then they have isomorphic homology groups in every dimension. In particular, when X has the homotopy type of a CW-complex, the homology of X coincides with the cellular homology of said CW-complex.*

Recall that singular homology is functorial.

Proposition 5.2 (Functoriality of the singular homology)**.** *For any continuous map h between two subsets of Euclidean spaces X and Y, for every $i \in \mathbb{N}$, there exists an associated linear map $h_* : H_i(X) \to H_i(Y)$. Moreover, if A, B, C are three subsets of Euclidean spaces, with continuous maps $f : A \to B$ and $g : B \to C$, we have:*

$$(g \circ f)_* = g_* \circ f_*.$$

In particular, if $(X_t)_{t\in\mathbb{R}}$ is a family of subsets of $\mathbb{R}^d$ such that $s \leq t$ implies $X_s \subset X_t$ - also called a filtration of $\mathbb{R}^d$ *- then we have linear maps $\phi_s^t : H_i(X_s) \to H_i(X_t)$ such that $\phi_t^r \circ \phi_s^t = \phi_s^r$ for every triplet $s \leq t \leq r$.*

Singular homology gives a formal and topologically consistent definition of what we intuitively perceive as the holes of a set; with the number of i-dimensional holes of X being equal to $\dim H_i(X)$. Because of the so-called torsion phenomenon in homology, this number might depend on the choice of the field $\mathbb{K}$. However, the *Euler characteristic* of X, defined as $\chi(X) = \sum_{i=0}^{\infty}(-1)^i \dim(H_i(X))$ when the sum is well-defined, is independent of the field $\mathbb{K}$.

Proposition 5.3 (Additivity of Euler-Characteristic and inclusion-exclusion principle)**.** *Let X, Y be two compact subsets of $\mathbb{R}^d$ such that the homology of $X, Y, X \cap Y$ and $X \cup Y$ all have finite dimensions. Then we have*

$$\chi(X \cup Y) + \chi(X \cap Y) = \chi(X) + \chi(Y). \tag{5.1}$$

We also speak of the inclusion-exclusion principle *when additivity is used iteratively. Letting $X_1, \dots X_n$ be compact subsets of $\mathbb{R}^d$, we have*

$$\chi(X_1 \cup \dots \cup X_n) = \sum_{\substack{I \subset [\![1,n]\!] \\ I \neq \emptyset}} (-1)^{\mathrm{Card}(I)-1} \chi(\cap_{i\in I} X_i). \tag{5.2}$$

The following proposition shows that the number of features of reasonable shapes is finite and that the Euler characteristic is well-defined. Recall that a *Euclidean neighborhood retract* is a subset X of some $\mathbb{R}^d$, such that there is a neighborhood $V \subset \mathbb{R}^d$ of X such that V deformation retracts to X

Proposition 5.4 (Euclidean neighborhood retracts have finitely generated homology)**.** *Let $X \subset \mathbb{R}^d$ be a Euclidean neighborhood retract. Then it has the homotopy type of a CW-Complex of dimension less than d. In particular, we have $\dim H_i(X) < \infty$ for all $0 \leq i \leq d$ and $H_i(X) = 0$ else.*

5.1.2 Motivation

Before delving into algebraic considerations, we want to motivate and give intuition to the concept of *barcodes*. Say we have at our hands data that is of topological nature, e.g., a collection X of points lying on a circle with reasonable noise, as in Figure 5.1.

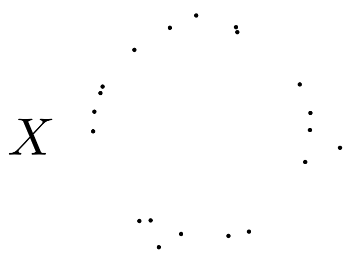


Figure 5.1 – Point cloud lying close to a circle

While a point cloud does not have an interesting topology, one can guess the topology of the (supposedly) underlying set by considering the union of balls of a certain radius t around those points, which forms the offset X^t. Admittedly, the topology obtained is indeed that of a circle for a good choice of a radius (X^{t_3} in Figure 5.2); but should t be too low, the topology of X^t consist in too many connected components (X^{t_1}) or some small cycles (X^{t_2}); and should it be too large, X^t would be homeomorphic to a ball (X^{t_4}). The number of cycles of X^t as a function of t can be large at first, with the many small circles, then stays at one until X^t has a contractile homotopy type.

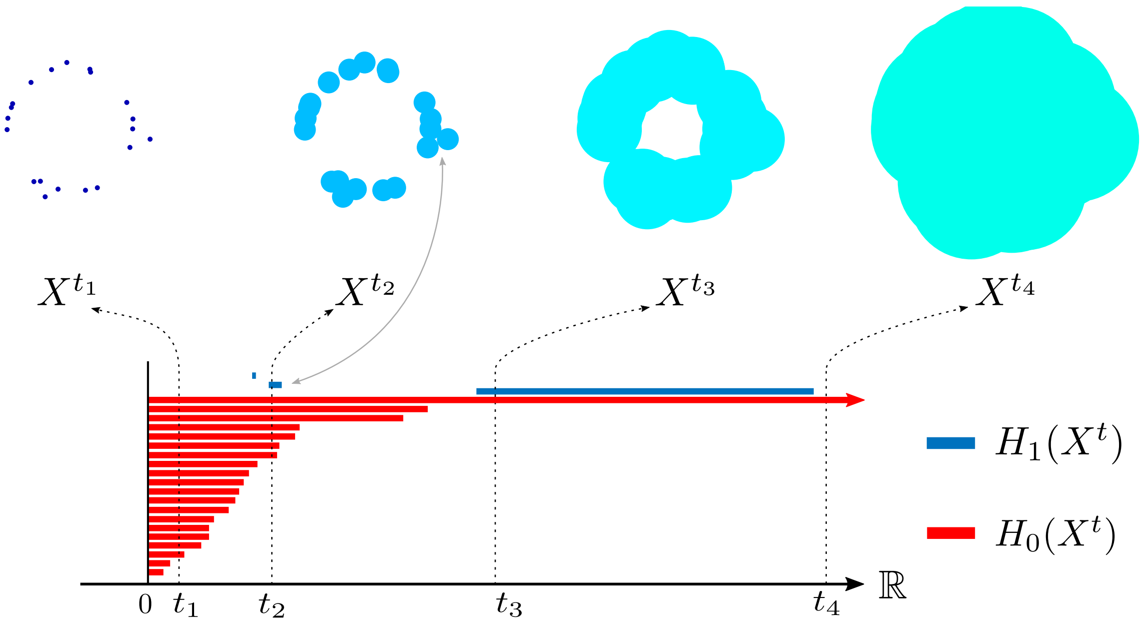


Figure 5.2 – Evolution of the homology of X^t when t varies.

The idea of persistent homology is to keep track of the values at which features are *born*, i.e., at which radius the feature appears, and values at which they *die*. Since offsets $X^r = d_X(-\infty, r]$ are sublevel sets of the distance function, the previous approach can first be generalized by studying the evolution of the topology of the closed sublevel sets filtration $r \mapsto f^{-1}(-\infty, r]$ for some map $f : \mathbb{R}^d \to \mathbb{R}$. When f is a Morse function, we already know that the topology of the filtration changes only at critical values, and we know that its homotopy type evolves by gluing a cell of suitable dimension around critical points of said value. *Persistent homology* consists in the study of maps of the form $t \mapsto H_i(f^{-1}(-\infty, t])$. Compared to Morse theory, the interests are twofold:

— The theory should apply to sublevel sets of functions with regularity less than C^2, such as distance functions to a set X;

— The birth and the death of a feature are critical values should be linked.

We will see in the following section how to construct a rigorous theory on algebraic ground from these principles.

5.2 Basics in persistence theory

5.2.1 Persistence modules

Definition 5.5 (Persistence modules)**.** Let $\mathbb{K}$[*] be a field. A persistence module M is a functor $\mathbb{R} \to \mathrm{Vect}_{\mathbb{K}}$, that is a collection of vector spaces $(M_t)_{t\in\mathbb{R}}$ and maps $\phi_s^t : M_s \to M_t$ for any $s \leq t \in \mathbb{R}$, such that $\phi_t^u \circ \phi_s^t = \phi_s^u$ for any ordered triple $s \leq t \leq u$ in $\mathbb{R}$.

From the functoriality of singular homology (Proposition 5.2), we can associate to any real-valued map f the persistent homology modules $(H_i(f^{-1}(-\infty, t]))_{t\in\mathbb{R}}$ for any $i \in \mathbb{N}$. Given any interval I in $\mathbb{R}$, another simple example is obtained by letting $\mathbb{1}_I$ be the persistence module defined by

$$(\mathbb{1}_I)_t := \begin{cases} \mathbb{K} & \text{when } t \in I \\ 0 & \text{otherwise} \end{cases}$$

and such that linear maps between $(\mathbb{1}_I)_s \to (\mathbb{1}_I)_t$ are the identity $\mathbb{K} \to \mathbb{K}$ when $s < t \in I$. We will see that direct sums of interval modules play an important role in persistence theory. Before that, we define a pseudo-distance over the space of persistence modules.

Definition 5.6 (Morphisms, δ-interleavings and interleaving distance)**.** Let M, N be two persistence modules and let $\delta \geq 0$.

- M^δ denotes the persistence module $(M_{t+\delta})_{t\in\mathbb{R}}$, i.e, the module M shifted by δ.
- A morphism j between two persistence modules N and M is a natural transformation between functors, that is, a collection of linear maps $(j_t)_{t\in\mathbb{R}} : N_t \to M_t$ such that the left diagram in Figure 5.3 commutes.
- The modules N and M are said to be δ-*interleaved* when there exist two morphisms u, v respectively from M to N^δ and from N to M^δ such that the right diagram in Figure 5.3 commutes.
- The *interleaving distance* between M and N is defined as:

$$d_I(M, N) := \inf\{\delta \in \mathbb{R}^+ \mid M \text{ and } N \text{ are } \delta\text{-interleaved}\}.$$

$$\begin{tikzcd} M_s \arrow[r] & M_t \\ N_s \arrow[u, "j_s"] \arrow[r] & N_t \arrow[u, "j_t"'] \end{tikzcd} \qquad \begin{tikzcd} M_s \arrow[r] \arrow[dr, "u_s" near start] & M_{s+\delta} \arrow[r] \arrow[dr, "u_{s+\delta}" near start] & M_{s+2\delta} \\ N_s \arrow[r] \arrow[ur, "v_s"' near start] & N_{s+\delta} \arrow[r] \arrow[ur, "v_{s+\delta}"' near start] & N_{s+2\delta} \end{tikzcd}$$

Figure 5.3 – Commutative diagrams in the definition of morphisms (left) and δ-interleaving (right).

The interleaving distance is in fact only a pseudo distance: for any compact interval I of $\mathbb{R}$, is it easy to see that $d_I(\mathbb{1}_I, \mathbb{1}_{\mathrm{int}(I)})) = 0$ in spite of these two modules being non-isomorphic. The following structure theorem shows that interval persistence modules are the building blocks of persistence theory with respect to this pseudo-metric.

∗. Again, the choice of the field $\mathbb{K}$ has no importance.

Theorem 5.7 (Interval decomposition theorem [CdSGO16]). *Let M be a persistence module such that for all $s < t$ in $\mathbb{R}$, the rank of the linear map $M_s \to M_t$ is finite. Then there exists a multiset $\mathcal{I}$ of closed intervals of $\mathbb{R}$ with positive length such that:*

$$d_I(M, \bigoplus_{I \in \mathcal{I}} \mathbb{1}_I) = 0.$$

Moreover, for any $\sigma > 0$ and compact segment K of $\mathbb{R}$, the numbers of intervals of length greater than σ intersecting K is finite. When(M_t) is the persistent homology module of a bounded function such that $\dim M_t$ is uniformly bounded, the set $\mathcal{I}$ is finite. The multiset $\mathcal{I}$ is unique up to permutations, and its elements are the intervals decomposing M.

5.2.2 Persistence diagrams

The *persistence diagram* $\mathrm{dgm}(M)$ associated to the persistence module M is the multiset of $\mathbb{R}^2$ whose coordinates are the bounds the intervals decomposing M:

$$\mathrm{dgm}(M) := \bigsqcup_{I \in \mathcal{I}} \{(\inf I, \sup I)\}.$$

More generally, we call persistence diagram any locally finite multiset of $\{(t, s) \in \mathbb{R}^d \mid t < s\}$. We choose to visually represent persistence diagrams as *barcodes*, that is, as a pile of intervals, as in Figure 5.2. To compare them, we use partial bijections between barcodes, also called *matchings*.

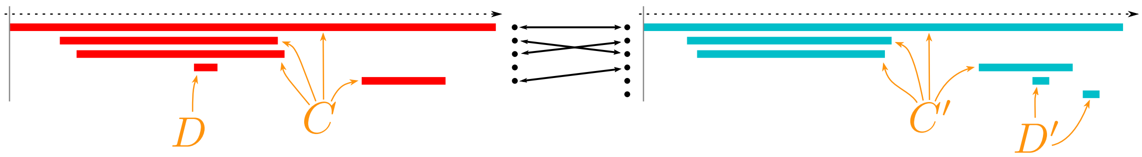


Figure 5.4 – A matching between two diagrams.

The notion of δ-interleaving between persistence modules transfers to their associated diagrams via the notion of δ-matching.

Definition 5.8 (δ-matchings and bottleneck distance). A δ-matching between two persistence diagrams D, D' is a bijective map $\gamma : C \to C'$, between subsets of D, D' such that for any $c \in C$, $||\gamma(c) - c||_\infty \leq \varepsilon$, and such that for any $(a, b) \in (D \setminus C) \cup (D' \setminus C')$, $|a - b| \leq 2\varepsilon$.

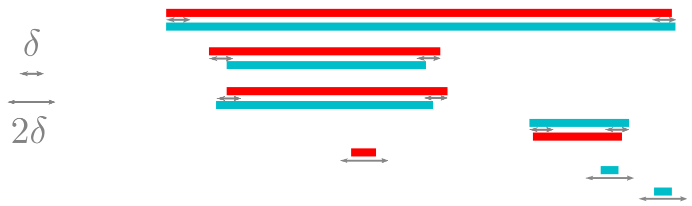


Figure 5.5 – The previously depicted matching is a δ-matching.

The *bottleneck distance* between two diagrams D, D' is defined as:

$$d_B(D, D') := \inf\{\delta \mid \text{There exists a } \delta\text{-matching between } D \text{ and } D'\}.$$

Theorem 5.9 (Isometry theorem). *For any pair of persistence modules M, N, we have:*

$$d_I(M, N) = d_B(\mathrm{dgm}(M), \mathrm{dgm}(N)).$$

5.2.3 Comparison with Morse theory

We now precise how Morse theory and persistent homology are linked.

Let $X \subset \mathbb{R}^d$ and $f : X \to \mathbb{R}$ be a bounded, locally Lipschitz map. As long as the filtration $X_t = X \cap f^{-1}(-\infty, t]$ has a finitely generated homology for all but a finite number of filtration values, the family $(H_i(X_t))_{t\in\mathbb{R}}$ forms a persistence module decomposable into intervals for every $i \in \{0, \dots, d\}$. Assume further that there are a finite number of critical points $\{x_1, \dots, x_p\} = \operatorname{crit}(f_{|X})$ of $f_{|X}$ with associated values $c_i = f(x_i)$, such that for $\varepsilon > 0$ small enough, in a neighborhood of x_i, the topology of $X_{c_i+\varepsilon}$ is obtained by gluing a cell of dimension λ_i around x_i in $X_{c_i-\varepsilon}$, as in Morse theory. As we have seen in the previous Chapter 4, this ensures that X_t has the homotopy type of a CW-complex for every regular value t of f. From cellular homology, we are able to show that among all dimensions of persistent homology diagrams, there is one topological event per critical point. This is illustrated by Figure 5.6.

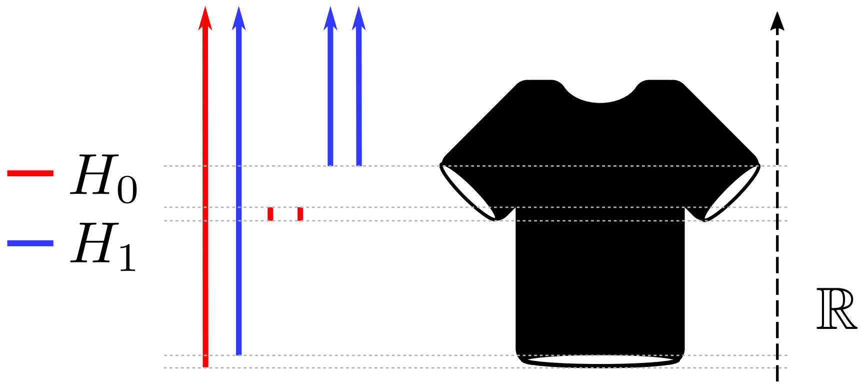


Figure 5.6 – Morse height filtration on a t-shirt. Critical points can have the same height, but they each contribute to independent topological events.

Proposition 5.10 (Morse theory and persistence diagrams)**.** *In the previous setting, the number of topological events at filtration value t is exactly* $\operatorname{Card}(\operatorname{crit}(f_{|X} \cap f^{-1}(t))$ *the number of critical points having value t, and each such critical point x_i is associated to either the birth of a feature of dimension λ_i or the death of a feature of dimension $\lambda_i - 1$.*

Proof.

Let $y_1, \dots, y_k$ be the critical points of $f_{|X}$ with value t, and λ_i the dimension of the cell glued around y_i in the filtration. For every $\varepsilon > 0$ small enough, there is a CW-complex A^- with the same homotopy type as $X_{t-\varepsilon}$, such that $X_{t+\varepsilon}$ has the homotopy type of the CW-complex A^+ obtained from A^- by the gluing of cells of dimension λ_i in A^-. Recall that cellular homology and singular homology coincide. The cellular homology of A^- is obtained by the cellular complex consisting of free Abelian groups $C_i(A^\pm)$ generated by the cells and a boundary operator $\partial_i^\pm : C_i(A^\pm) \to C_{i-1}(A^\pm)$, and $H_i(A^\pm) = \ker(\partial_i^\pm)/\operatorname{im}(\partial_{i+1}^\pm)$. Decompose the gluing process by the following filtration of CW-complexes:

$$A^- = A_1 \subset A_2 \subset \cdots \subset A_k = A^+.$$

Here A_{j+1} is obtained from A_j by gluing one cell of dimension λ_j. Either the dimension of the image of the boundary operator $\operatorname{im} \partial_{\lambda_j}$ grows by one, in which case $\dim H_{\lambda_j - 1}(A_j) =$

$\dim H_{\lambda_j-1}(A_{j-1}) - 1$, which marks the *death* of a feature; or it does not, and $\dim H_{\lambda_j}(A_j) = \dim H_{\lambda_j}(A_{j-1}) + 1$, which marks a *birth*. In both cases, these are the only changes among the homology groups, and there are at most k changes in homology between A^+ and A^-. Since ∂_i^+ has image in $C_{i-1}(A^-)$ by assumption, any non-trivial homology class of dimension λ appearing between A^j and A^{j-1} cannot be killed by the emergence of a feature at a later stage of the filtration, and thus there are exactly k changes in homology classes between A^- and A^+.

□

5.3 Image Persistence

5.3.1 Definitions

Definition 5.11 (Image Persistence)**.** Let M, N be two persistence modules and $f : M \to N$ be a morphism between them. The *image persistence module* $\operatorname{im} f$ *is the persistence module with vector space* $(\operatorname{im} f)_t = f(M_t)$ at value $t \in \mathbb{R}$ and whose connecting maps $(\operatorname{im} f)_a \to (\operatorname{im} f)_b$ are the restrictions of $N_a \to N_b$ to $(\operatorname{im} f)_a$ for every pair $a \leq b \in \mathbb{R}$.

We will only deal with image persistence modules arising in the following situation. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ and $A \subset B \subset \mathbb{R}^d$. For every $a \in \mathbb{R}$, there is an inclusion

$$A \cap \phi^{-1}(-\infty, a] = A_a \xrightarrow{\iota^a} B_a = B \cap \phi^{-1}(-\infty, a],$$

which yields for every dimension $0 \leq j \leq d$ a morphism of persistence modules $\iota_j^\bullet$:

$$\iota_j^\bullet : H_j(A_\bullet) \to H_j(B_\bullet).$$

We write $\operatorname{dgm}(\phi, A, B) = \bigsqcup_{j=0}^d \operatorname{dgm}(\operatorname{im}\, \iota_j^\bullet)$ for the persistence diagram obtained by taking the direct sum of the homology induced modules in every dimension. When the decomposition into interval is finite, its *Euler characteristic* $\chi(\operatorname{dgm}(\phi, A, B)(r))$ is the alternating sum of the ranks of $\iota_j^\bullet$ at filtration value r.

5.3.2 Image stability theorem

We are now in position to prove a stability theorem for image persistence modules associated with sublevel set filtrations of locally Lipschitz functions.

Theorem 5.12 (Image persistence stability theorem)**.** *Let $h, \tilde{h} : \mathbb{R}^d \to \mathbb{R}$ be two real-valued function such that $\|h - \tilde{h}\|_\infty \leq \varepsilon$ and $f : \mathbb{R}^d \to \mathbb{R}$ be a κ-Lipschitz function. Denote $\tilde{X}^a = \tilde{h}^{-1}(-\infty, a]$ and $X^a = h^{-1}(-\infty, a]$. Suppose that there exists $\mu > 0$ such that on $X^{2\varepsilon} \setminus X^{-2\varepsilon}$, h is locally Lipschitz and $\Delta(\partial^* h(x)) \geq \mu$. Then we have:*

$$d_B(\operatorname{dgm}(f, \tilde{X}^{-\varepsilon}, \tilde{X}^{\varepsilon}), \operatorname{dgm}(f_{|X})) \leq \frac{2\kappa\varepsilon}{\mu}. \tag{5.3}$$

Proof.

This is an extension of the stability theorem for noisy domains of [CSEHM09] from distance functions to Lipschitz functions. We adapt this proof to our setting using approximate inverse flows of Lipschitz functions obtained in Proposition 2.9. For any $\sigma > 0$, take $C_\sigma(\cdot,\cdot)$ a continuous deformation retraction between $X^{2\varepsilon}$ and X given by Proposition 2.9. Let $x \in X^{2\varepsilon}$. Each trajectory $C_\sigma(\cdot, x)$ is $\frac{2\varepsilon}{\mu-\sigma}$-Lipschitz and needs at most time 1 to send x to X. For every $a \in \mathbb{R}$, $C_\sigma(1,\cdot) : X_a^{2\varepsilon} \to X_{a+c}$ is a continuous map with $c := \frac{2\kappa\varepsilon}{\mu-\sigma}$. With the same reasoning, we obtain a continuous map $X_a \to X_{a+c}^{-2\varepsilon}$. The homotopies induced by the flows yield the following commutative diagram, where the vertical and horizontal maps are induced by inclusions:

$$\begin{tikzcd}
{} \arrow[r, dashed] & H_*(X_a^{2\varepsilon}) \arrow[r] \arrow[rdd, red] & H_*(X_{a+c}^{2\varepsilon}) \arrow[r] \arrow[rdd] & H_*(X_{a+2c}^{2\varepsilon}) \arrow[r, dashed] & {} \\
{} \arrow[r, dashed] & H_*(\tilde{X}_a^{\varepsilon}) \arrow[r] \arrow[u, red] & H_*(\tilde{X}_{a+c}^{\varepsilon}) \arrow[r] \arrow[u] & H_*(\tilde{X}_{a+2c}^{\varepsilon}) \arrow[r, dashed] \arrow[u] & {} \\
{} \arrow[r, dashed] & H_*(X_a) \arrow[r] \arrow[u] \arrow[rdd] & H_*(X_{a+c}) \arrow[r] \arrow[u] \arrow[rdd, blue] & H_*(X_{a+2c}) \arrow[r, dashed] \arrow[u] & {} \\
{} \arrow[r, dashed] & H_*(\tilde{X}_a^{-\varepsilon}) \arrow[r] \arrow[u] & H_*(\tilde{X}_{a+c}^{-\varepsilon}) \arrow[r] \arrow[u] & H_*(\tilde{X}_{a+2c}^{-\varepsilon}) \arrow[r, dashed] \arrow[u] & {} \\
{} \arrow[r, dashed] & H_*(X_a^{-2\varepsilon}) \arrow[r] \arrow[u] & H_*(X_{a+c}^{-2\varepsilon}) \arrow[r] \arrow[u] & H_*(X_{a+2c}^{-2\varepsilon}) \arrow[r, dashed] \arrow[u, blue] & {}
\end{tikzcd}$$

As in [CSEHM09], the two colored arrow paths provide interleavings showing that, for every $\sigma \in (0,\mu)$:

$$d_B(\mathrm{dgm}(f, \tilde{X}^{-\varepsilon}, \tilde{X}^{\varepsilon}), \mathrm{dgm}(f_{|X})) \leq \frac{2\kappa\varepsilon}{\mu-\sigma}.$$

□

Now let X, Y be two compact sets of $\mathbb{R}^d$ and $f = d_x : z \mapsto ||z - x||$ be the distance function to any point x. Applying the previous theorem with $h = d_X + 2\varepsilon$, $\tilde{h} = d_Y + 2\varepsilon$ yields the following statement:

Corollary 5.13 (Image stability theorem for compact sets)**.** *Let $\mu \in (0,1]$, $\varepsilon > 0$ and X, Y be two compact subsets of $\mathbb{R}^d$ such that $d_H(X,Y) \leq \varepsilon \leq \frac{1}{4}\mathrm{reach}_\mu(X)$. Then for any $x \in \mathbb{R}^d$,*

$$d_B(\mathrm{dgm}(d_x, Y^\varepsilon, Y^{3\varepsilon}), \mathrm{dgm}(d_{x_{|X^{2\varepsilon}}})) \leq \frac{2\varepsilon}{\mu}. \tag{5.4}$$

5.3.3 Injecting property and χ-averaging lemma

Now we use results from Bauer & Lesnick [BL15] to show that a persistence module sandwiched between two persistence modules cannot be smaller in a certain sense than the image persistence module. We formalize that by saying that a persistence diagram D' *injects into* another persistence diagram D when there is an injective map $\phi : D' \to D$ such that $\phi((a',b')) = (a,b)$ with $a \leq a'$ and $b' \leq b$ for all (a',b') in D'.

Theorem 5.14 (Interleaving of bars in image persistence)**.** *Let A, B, C be persistent modules that are decomposable into intervals. Let furthermore be φ, ψ morphisms of persistent modules and write $j = \psi \circ \varphi$:*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[rr, bend left, "j"] & B \arrow[r, "\psi"] & C \end{tikzcd}$$

Then $\mathrm{dgm}(\mathrm{im}\, j)$ injects into $\mathrm{dgm}(B)$. In particular, if B has a finite decomposition in intervals, then so does $\mathrm{im}\, j$.

Proof.

The morphisms of persistence modules $im(\varphi) \to B$, $im(\varphi) \to im(j)$ are respectively monomorphism and an epimorphism of persistence modules. By Lemma 4.2 in [BL15], we know that there exist injections of barcodes $\mathrm{dgm}(\mathrm{im}\ \varphi) \hookrightarrow \mathrm{dgm}(B)$ and $\mathrm{dgm}(\mathrm{im}\ j) \hookrightarrow \mathrm{dgm}(\mathrm{im}\ \varphi)$ respectively extending the intervals to the left and to the right.

□

Remark 5.15 – For a persistence diagram D and for two real numbers $a < b$, define $N_a^b(D)$ to be the total number of bars of D intersecting with $[a, b]$. The theorem above implies that, with the same notations, $N_a^b(\mathrm{dgm}(\mathrm{im}\, j)) \leq N_a^b(\mathrm{dgm}(B))$. This is a generalization to persistence modules of the fact that the rank of a linear map cannot exceed the dimension of a vector space it factors through.

The next lemma bounds the average difference of the Euler characteristics of close persistent diagrams.

Lemma 5.16 (χ-averaging lemma)**.** *Let D, D' be two homology persistent diagrams with $d_B(D, D') \leq \varepsilon$. Then for any $a < b \in \mathbb{R}$ we have:*

$$\int_a^b |\chi(D(t)) - \chi(D'(t))|\, \mathrm{d}t \leq 2\varepsilon(N_a^b(D) + N_a^b(D')). \tag{5.5}$$

If D' injects into D, we have:

$$\int_a^b |\chi(D(t)) - \chi(D'(t))|\, \mathrm{d}t \leq 2\varepsilon N_a^b(D). \tag{5.6}$$

Proof.

A look at an ε-matching between barcodes, such as illustrated in Figure 5.5, might give a good intuition for these inequalities. The first inequality is a slight extension of an argument obtained in [CSE07]. Let $\mathcal{I}_i$ (resp. $\mathcal{I}'_i$) be the set of intervals of the decomposition of D (resp. D') in dimension i. We have:

$$\chi(D(t)) = \sum_{i=0}^{d} (-1)^i \sum_{I_i \in \mathcal{I}_i} \mathbb{1}_{I_i}(t).$$

Let γ be an ε-matching between D and D'. Define C_i and C'_i to be the respective largest subsets of $\mathcal{I}_i$ and $\mathcal{I}'_i$ matched bijectively by γ. We have:

$$\int_a^b |\chi(D(t)) - \chi(D'(t))|\, \mathrm{d}t \leq \sum_{i=0}^{d} \Big(\sum_{I_i \in C_i} \int_a^b \left| \mathbb{1}_{I_i} - \mathbb{1}_{\gamma(I_i)} \right| \mathrm{d}t \\ + \sum_{I_i \in \mathcal{I}_i \setminus C_i} \int_a^b \mathbb{1}_{I_i}(t)\, \mathrm{d}t + \sum_{I'_i \in \mathcal{I}'_i \setminus C'_i} \int_a^b \mathbb{1}_{I'_i}(t)\, \mathrm{d}t \Big).$$

Since γ is an ε-matching, each of these integrals is bounded by 2ε. Now for the first term in the right-hand side, the support of the map $\left|\mathbb{1}_{I_i} - \mathbb{1}_{\gamma(I_i)}\right|$ is included in $I_i \cup \gamma(I_i)$, meaning that its integral over $[a, b]$ vanishes if none of these intervals intersect with (a, b). As for the remaining terms, only intervals intersecting with $[a, b]$ contribute to the sum. Overall, there is at most one non-vanishing contribution per interval in $D \cup D'$ whose intersection with (a, b) is non-empty, and no contribution otherwise. The total number of non-vanishing integrals is bounded by $N_a^b(D) + N_a^b(D')$.

In case D' injects into D, there is a ε-matching such that $C'_i = \mathcal{I}'_i$ and such that $\gamma(I_i) \subset I_i$ for every interval I_i of D, meaning there is at most one non-vanishing contribution per interval of D whose intersection with (a, b) is non-empty, leading to the desired bound $2\varepsilon N_a^b(D)$.

□

CHAPTER 6

Curvatures of subsets of Euclidean spaces

In this chapter, we define the curvatures of various classes of subsets of $\mathbb{R}^d$ pertaining to our study, notably the curvatures of complementary regular sets. This culminates with the principal kinematic formula*, from which one can recover the curvatures of a set from the Euler characteristic of intersections with balls of fixed radius. This forms the explicit bridge between topology and geometry on which this thesis rests. Aiming at being as self-contained as possible, our presentation includes the classical definition of curvatures of sets with positive reach, and some of their properties. Moreover, we highlight various results related to curvatures, such as their connections with Morse theory. We compare the properties of the curvatures of complementary regular sets to the ones of other classical classes of subsets of $\mathbb{R}^d$, using the terminology of the so-called* normal cycles.

6.1 Introduction

The object of this chapter is to define the geometric quantities studied in this thesis, namely the *curvatures* of certain subsets of Euclidean spaces. The idea of measuring how far a set is from being flat is fundamental across various fields of geometry. In Riemaniann geometry, this is measured by the curvature tensor, whose study provides great hindsight on the geometry of the manifold and has been the subject of numerous studies for decades. Taking inspiration from characterizations using geodesic triangles on manifolds, the notion of curvatures can be extended in a weaker sense to any geodesic metric space. Research around this concept has proven to be rich and complex, with numerous applications e.g., in geometric group theory [Gro87, Ago13] or geometric topology [Per02].

In the context of this thesis, we are interested in measuring the curvatures of subsets of Euclidean spaces. While geodesics lying in the interior of such sets are obviously flat, the geometry of their boundaries can be curved. In 1929, Weyl [Wey39] already remarked that the curvatures of a submanifold with Riemannian metric inherited from $\mathbb{R}^d$ could be retrieved through the properties of close tubular neighborhoods. Interestingly enough, Steiner showed as early as 1842 [Ste82] that the same fact holds for convex polyhedra. Unifying these two points of view in a coherent theory was first done by Federer [Fed59] through the study of sets of positive reach. Since then, several works aimed at extending this theory to broader classes of sets while keeping its structural properties. These extensions were obtained using various arguments, such as the additiveness of curvature measures [Sch88, Zä87], properties of the sublevel sets of certain classes of functions [Fu94, PR13], Morse theory [BK00] and o-minimal structures [Ber07]. We add that the recent monograph of Rataj & Zähle [RZ19] provides a self-contained presentation describing the curvatures of set among several of the aforementioned classes.

6.2 Curvatures of sets admitting a normal bundle

6.2.1 Curvatures of $C^{1,1}$ domains

Let $X \subset \mathbb{R}^d$ be a $C^{1,1}$ domain. Recall that on ∂X there exists a Lipschitz map $n : \partial X \mapsto \mathbb{S}^{d-1}$ of outward pointing normals called the *Gauss map*. This map is differentiable $\mathcal{H}^{d-1}$-almost everywhere on ∂X and, when it exists, its differential $D_y n$ at $y \in \partial X$ is a symmetric linear map called the *Weingarten map*.

Definition 6.1 (Principal curvatures and directions for $C^{1,1}$ domains)**.** Let $X \subset \mathbb{R}^d$ be a $C^{1,1}$ domain. Vectors $(b_i)_{1 \leq i \leq d-1}$ forming orthonormal basis of eigenvectors of $D_y n$ are called *principal directions*, and their associated eigenvalues $(k_i)_{1 \leq i \leq d-1}$ are called *principal curvatures*. Principal directions are orthogonal to $n(y)$.

Proposition 6.2 (Bound on the principal curvatures)**.** *Let X be a $C^{1,1}$ domain. Then any principal curvature $k(y)$ of its differential at a point y where n is differentiable is bounded from below:*

$$k(y) \geq -\frac{1}{\mathrm{reach}(X)} . \tag{6.1}$$

Proof.

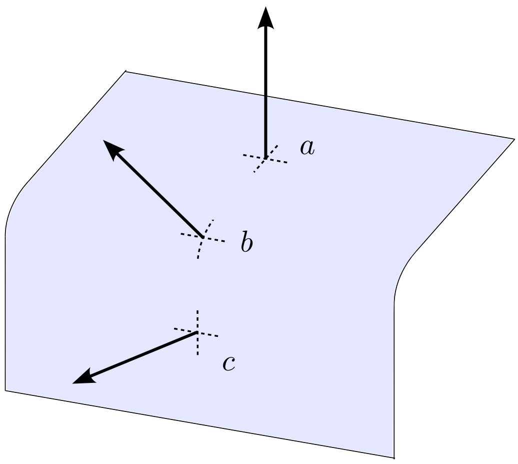


Figure 6.1 – Gauss map on the boundary of a $C^{1,1}$-domain. Dotted lines around points a, b, c represent principal directions.

Let $x, y \in X$. For any $0 < r < \text{reach}(X)$, we have $||y - x - rn(x)||^2 \geq r^2$ and $||x - y - rn(y)||^2 \geq r^2$. Expanding and combining these inequalities and then letting $r \to \text{reach}(X)$, we obtain :

$$\langle n(y) - n(x), x - y\rangle \leq \frac{||x-y||^2}{\text{reach}(X)}. \tag{6.2}$$

from which the desired inequality follows.

□

From these definitions, we want to extend the definition of principal curvatures and directions to sets which have positive reach or which are complementary regular. Before doing so, we remark that the two concepts are mutually exclusive among sets which are not $C^{1,1}$ domains.

Proposition 6.3 (Sets which are of positive reach and complementary regular are $C^{1,1}$ domains.)**.** *Let X be a compact subset of $\mathbb{R}^d$ which is both complementary regular and of positive reach. Then X is a $C^{1,1}$-domain.*

Proof.

Recall from the characterization of complementary regular set that for every r in $(0, \text{reach}(^\neg X))$, we have $X = (X^{-r})^r$. We will prove that $\text{reach}(X^{-r}) = r + \text{reach}(X)$, which implies that $X = d_{X^{-r}}(-\infty, r]$ with r a regular value of $d_{X^{-r}}$, which is $C^{1,1}$.

Even though the map $d_{X^{-r}}$ might not be differentiable, recall from Definition 3.8 there is a continuous flow on $\mathbb{R}^d \setminus X^{-r}$ starting from any point outside X^{-r} defined by Lieutier in [Lie04] whose arc-length trajectories make $d_{X^{-r}}$ increase at speed $||\nabla_{X^{-r}}||$, where $\nabla_{X^{-r}}$ is the generalised gradient of $d_{X^{-r}}$. Outside X, it coincides with ∇_X. Now if a point in ∂X had distinct closest points in X^{-r}, then so would have any point in its trajectory by [Lie04, Lemma 4.17]. Since X is complementary regular, r is regular value of d_X^{-r} and the flow of $d_{X^{-r}}$ is strictly increasing around ∂X, and thus leaves X; however, inside $X^{\text{reach}(X)}$, the generalized gradient $\nabla_{X^{-r}}$ coincide with ∇_X and has thus norm one. This contradicts the fact that the trajectory keeps having distinct closest points in X^{-r}.

□

6.2.2 Curvatures of sets with positive reach

Let X be a subset of $\mathbb{R}^d$ with positive reach. Contrary to $C^{1,1}$ domains, ∂X is not necessarily a hypersurface and can even have zero $(d-1)$-dimensional Hausdorff measure, with the example of submanifolds of $\mathbb{R}^d$ with codimension greater than 2. To extend the definitions of principal curvatures and directions consistently with the ones of $C^{1,1}$ domains, we use the normal bundle of X. Recall from Definition 3.27 that $\mathrm{Nor}(X) = \left\{(x,n) \,|\, x \in \partial X, n \in \mathrm{Nor}(X,x) \cap \mathbb{S}^{d-1}\right\}$ is a $(d-1)$-Lipschitz submanifold of $\mathbb{R}^d \times \mathbb{S}^{d-1}$ for any X. This entails that $\mathcal{H}^{d-1}$-almost everywhere in $\mathrm{Nor}(X)$, pairs (x,n) are regular, i.e., their tangent cone $\mathrm{Tan}(\mathrm{Nor}(X),(x,n))$ is a $(d-1)$-dimensional space.

Definition 6.4 (Principal curvatures and principal directions)**.** Let $X \subset \mathbb{R}^d$ be a set of positive reach and (x,n) be a regular pair of $\mathrm{Nor}(X)$. A family $b_1,\dots,b_{d-1}$ of unit vectors of $\mathbb{R}^d$ orthogonal to n is said to be *principal directions* of X at (x,n) when there exists a family $k_1,\dots,k_{d-1}$ in $\mathbb{R}\cup\{\infty\}$ such that

$$\left(\frac{1}{\sqrt{1+k_i^2}}b_i, \frac{k_i}{\sqrt{1+k_i^2}}b_i\right)_{1\leq i\leq d-1}$$

is an orthonormal basis of $\mathrm{Tan}(\mathrm{Nor}(X),(x,n))$, with the convention that $\frac{1}{\sqrt{1+\infty^2}} = 0$ and $\frac{\infty}{\sqrt{1+\infty^2}} = 1$.

Definition 6.5 (Sets admitting a normal bundle)**.** The previous definitions of principal directions and principal curvatures can be straightforwardly extended to complementary regular sets. As these two classes share similar properties, we say that any set which is either of positive reach or complementary regular *admits a normal bundle*.

Now we want to see how having a positive reach ensures the existence of principal directions and principal curvatures. The offset X^r is a $C^{1,1}$ domain, and thus admits a Gauss map $n_r : \partial X^r \to \mathbb{S}^{d-1}$ coinciding with ∇d_X, which is differentiable $\mathcal{H}^{d-1}$-almost everywhere in ∂X^r. By the closed formula for d_X, the projection map $\xi_X : X^r \setminus X \to X$ coincides with $Id - \frac{1}{2}\nabla(d_X^2)(x)$. The map

$$\Xi_X : \left\{\begin{array}{ccc} X^r \setminus X & \to & \mathrm{Nor}(X) \\ x & \mapsto & (\xi_X(x), \nabla d_X(x)) \end{array}\right.$$

has the same set of differentiable points as ∇d_X. Furthermore, its differential is a symmetric isomorphism.

Proposition 6.6 (Tangent spaces of normal bundles of sets with positive reach [Fu89a])**.** *Let X be a subset of $\mathbb{R}^d$ and r be a real such that $0 < r < \mathrm{reach}(X)$. Let $y \in \partial X^r$. Then $\mathrm{Tan}(\mathrm{Nor}(X),\Xi_X(y))$ is a $(d-1)$-vector space if and only if ∇d_X is differentiable at y, and*

$$(D_y \Xi_X)_{|\,\mathrm{Tan}(\partial X^r, y)} = (Id - rD_y\nabla d_X, D_y \nabla d_X)$$

is a symmetric isomorphism between $\mathrm{Tan}(\partial X^r, y)$ and $\mathrm{Tan}(\mathrm{Nor}(X),\Xi_X(y))$.

From this proposition, one can obtain explicit pointwise principal curvatures and principal directions of a set with positive reach.

Proposition 6.7 (Principal curvatures and principal directions of a set with positive reach.)**.** *Let X be a set of positive reach with $0 < r < \operatorname{reach}(X)$ and assume that ∇d_X is differentiable at the point $y \in \partial X^r$. Let $(b_i^r)_{1\leq i\leq d-1}$ principal directions of X^r at y and $(k_i^r)_{1\leq i\leq d-1}$ be their associated principal curvatures at point y. Then $-\frac{1}{\operatorname{reach}(X)-r} \leq k_i^r \leq \frac{1}{r}$ for any $1 \leq i \leq d-1$.*

Moreover, let $b_i = b_i^r$ and

$$k_i := \begin{cases} \frac{k_i^r}{1-rk_i^r} & \text{if } k_i^r < \frac{1}{r} \\ \infty & \text{if } k_i^r = \frac{1}{r}. \end{cases}$$

Then $(b_1, \dots b_{d-1})$ are principal curvatures of X at (x, n), and their associated principal curvatures $(k_1, \dots, k_{d-1})$ belong to $[-\frac{1}{\operatorname{reach}(X)}, \infty]$.

Proof.

Thanks to Proposition 6.6, the only claim left to prove is $-\frac{1}{\operatorname{reach}(X)-r} \leq k_i^r \leq \frac{1}{r}$ from which the bounds on k_i follow. The left-hand side inequality comes directly from Proposition 6.2 and $\operatorname{reach}(X^r) \geq \operatorname{reach}(X) - r$. As for the right-hand side, remark that $\operatorname{reach}(\neg X^r) \geq r$ and that $\operatorname{Nor}(\neg X^r) = \rho(\operatorname{Nor}(X^r))$ with $\rho : (x, n) \mapsto (x, -n)$, as will be exposed in more details in Section 6.2.3. Notably, this implies that $-k_i^r$ is a principal curvature of $\neg X^r$ at the pair $(x, x + rn)$, which is regular by Proposition 6.6. Applying Proposition 6.2 once again yields $-k_i^r \geq -\frac{1}{r}$.

□

Proposition 6.8 (The map $t \mapsto \operatorname{Vol}(\xi_X^{-1}(U) \cap X^t)$ is a polynomial for $t \in [0, \operatorname{reach}(X)]$)**.** *Let $X \subset \mathbb{R}^d$ be a set with positive reach. Let U be an open subset of $\mathbb{R}^d$. The map $t \mapsto \operatorname{Vol}(\xi_X^{-1}(U) \cap X^t)$ is a polynomial for $t \in [0, \operatorname{reach}(X)]$.*

Proof.

Let $\operatorname{Nor}(X, U) := \{(x, n) \,|\, x \in U, (x, n) \in \operatorname{Nor}(X)\}$. This set is $\mathcal{H}^{d-1}$-measurable. Since the map

$$\phi : \begin{cases} \operatorname{Nor}(X, U) \times [0, r] & \to & \xi_X^{-1}(U) \cap (X^r \setminus X) \\ (x, n, t) & \mapsto & x + tn \end{cases}$$

is bilipschitz, by the change of variable formula, we have:

$$\begin{aligned} \int_{\xi_X^{-1}(U)\cap(X^r\setminus X)} 1 \,\mathrm{d}\mathcal{H}^{d-1} &= \int_{\operatorname{Nor}(X,U)\times[0,r]} J_d\phi(x, n, t) \,\mathrm{d}\mathcal{H}^{d-1}(x, n) \,\mathrm{d}t \\ &= \int_{\operatorname{Nor}(X,U)\times[0,r]} \prod_{i=1}^{d-1} \frac{1 + tk_i}{\sqrt{1 + k_i^2}} \,\mathrm{d}\mathcal{H}^{d-1}(y) \,\mathrm{d}t. \end{aligned}$$

which is indeed a polynomial in t. Adding the volume of $(\xi^{-1}(U) \cap X) = U \cap X$ to the previous quantity yields the desired result.

□

Definition 6.9 (Curvature measures and intrinsic volumes of a set with positive reach)**.** As a function of U, the coefficients of the polynomial of Proposition 6.8 define signed measures which we call *curvature measures* $C_k(X, \cdot)$ of X:

$$\mathrm{Vol}(\xi_X^{-1}(U) \cap X^t) =: \sum_{i=0}^{d} \omega_i t^i C_{d-i}(X, U). \tag{6.3}$$

The *intrinsic volumes* $(V_i(X))_{0 \le i \le d}$ of a set with positive reach are defined as the full measure of the corresponding curvature measures, i.e., $V_i(X) = C_{d-i}(X, X)$. Equivalently, they can be defined as the coefficients of the volume of t-offsets with $t \in [0, \mathrm{reach}(X)]$.

$$\mathrm{Vol}(X^t) =: \sum_{i=0}^{d} \omega_i t^i V_{d-i}(X). \tag{6.4}$$

Equation (6.4) is called *Steiner's formula*.

From the proof of Proposition 6.8, one infers the following explicit representations of curvature measures.

Proposition 6.10 (Explicit representation of the curvature measures of sets with positive reach)**.** *Let $X \subset \mathbb{R}^d$ be a set with positive reach. For $0 \le i \le d-1$, the i-th curvature measure is a function of the principal curvatures of X, via the formula*

$$C_i(X, U) = \frac{1}{(d-i)\omega_{d-i}} \int_{\mathrm{Nor}(X,U)} \prod_{i=1}^{d-1} \frac{1}{(1+k_i^2)^{1/2}} \Sigma_{d-i+1}\,(k_1\,, \ldots,\, k_{d-1})\, \mathrm{d}\mathcal{H}^{d-1}(x, n), \tag{6.5}$$

where

$$\Sigma_j : (u_1, \cdots, u_{d-1}) \mapsto \sum_{\substack{I \subset \{1,\ldots,d-1\} \\ \mathrm{Card}(I)=j}} \prod_{i \in I} u_i$$

is the canonical j-homogeneous symmetric polynomial in $(d-1)$ variables.

When $i = d$, we have

$$C_d(X, U) = \mathrm{Vol}(X \cap U). \tag{6.6}$$

Moreover, if X is a C^1 domain, the curvature measures can be represented as integrals on ∂X:

$$C_i(X, U) = \frac{1}{(d-i)\omega_{d-i}} \int_{\partial X \cap U} \Sigma_{d-i}\,(k_1, \ldots k_{d-1})\, \mathrm{d}\Omega \tag{6.7}$$

where Ω is the volume form on ∂X compatible with the orientation of the domain.

Proof.

The first two equations are direct computations from Proposition 6.8. When X is a $C^{1,1}$ domain, Equation (6.7) is a consequence of Equation (6.5) along with the bijective change of variable induced by the projection onto the spatial coordinate $\mathrm{Nor}(X) \to X, (x, n) \mapsto x$.

□

6.2.3 Curvatures of complementary regular sets

Let $X \subset \mathbb{R}^d$ be a complementary regular set. By definition, its complement set $\neg X = \overline{\mathbb{R}^d \setminus X}$ has positive reach, and they share the same boundaries, i.e., $\partial X = \partial \neg X$. Recall that the normal bundle of X is a $(d-1)$-Lipschitz submanifold of $\mathbb{R}^d \times \mathbb{S}^{d-1}$ defined by:

$$\mathrm{Nor}(X) = \rho(\mathrm{Nor}(\neg X)) = \bigcup_{x \in \partial X} \{x\} \times (\mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1}), \tag{6.8}$$

where $\rho : (x, n) \mapsto (x, -n)$ is the antipodal map of $\mathbb{R}^d \times \mathbb{S}^{d-1}$. This entails a correspondence between principal directions and curvatures of X and $\neg X$, which flips the sign of the principal curvatures.

Proposition 6.11 (Tangent spaces of complementary regular sets)**.** *Let X be a complementary regular set. Then (x, n) is a regular pair of* $\mathrm{Nor}(X)$ *if and only if $(x, -n)$ is a regular pair of* $\mathrm{Nor}(\neg X)$. *Moreover, the map $(u, v) \to (u, -v)$ is an isomorphism between* $\mathrm{Tan}(\mathrm{Nor}(X), (x, n))$ *and* $\mathrm{Tan}(\mathrm{Nor}(\neg X), (x, -n))$.

In particular, a family $\left(\frac{1}{\sqrt{1+k_i^2}} b_i, \frac{k_i}{\sqrt{1+k_i^2}} b_i\right)_{1 \le i \le d-1}$ *is a basis of orthogonal unit vectors of* $\mathrm{Tan}(\mathrm{Nor}(X), (x, n))$ *if and only if* $\left(\frac{1}{\sqrt{1+k_i^2}} b_i, \frac{-k_i}{\sqrt{1+k_i^2}} b_i\right)_{1 \le i \le d-1}$ *is an orthogonal basis of unit vectors in* $\mathrm{Tan}(\mathrm{Nor}(\neg X), (x, -n))$.

Remark 6.12 – Even though infinite curvature are invariant under the map $(u, v) \mapsto (u, -v)$ we chose the convention that infinite curvatures of complementary regular sets are of negative sign. Since sets which are complementary regular and of positive reach are $C^{1,1}$ domain, they have finite principal curvatures, which ensures that this convention is consistent.

Definition 6.13 (Curvature measures of complementary regular sets)**.** Let $X \subset \mathbb{R}^d$ be a complementary regular set. The curvature measures of X are defined from $\neg X$ for a Borelian set U via :

$$\left\{ \begin{array}{lcll} C_i(X, U) & := & (-1)^{d-i-1} C_i(\neg X, U) & \text{when } 0 \le i \le d-1 \\ C_i(X, U) & := & \mathrm{Vol}(X \cap U) & \text{when } i = d. \end{array} \right. \tag{6.9}$$

For $0 \le i \le d-1$, this is equivalent to the explicit representation

$$C_i(X, U) = ((d-i)\omega_{d-i})^{-1} \int_{\mathrm{Nor}(X,U)} \prod_{i=1}^{d-1} \left(1 + k_i^2\right)^{-1} \Sigma_{d-i+1} \left(k_1, \ldots k_{d-1}\right) \mathrm{d}\mathcal{H}^{d-1}(x, n) \tag{6.10}$$

where $k_i = k_i(x, n)$ are principal curvatures of $\mathrm{Nor}(X)$ at (x, n).

6.2.4 R-mass of a set

We are now able to define the R-curvature mass of a set admitting a normal bundle. This quantity will appear in the explicit convergence bounds of our main results in Chapter 7.

Definition 6.14 (R-curvature mass of a compact set admitting a normal bundle)**.** Let $X \subset \mathbb{R}^d$ be a compact subset of $\mathbb{R}^d$ admitting a normal bundle and let $R > 0$. The R-curvature mass of X is defined by:

$$M_R(X) := \int_0^R \int_{\mathrm{Nor}(X)} \prod_{i=1}^{d-1} \frac{|1 + t\kappa_i(x, n)|}{\sqrt{1 + \kappa_i(x, n)^2}} \, \mathrm{d}\mathcal{H}^{d-1}(x, n) \, \mathrm{d}t.$$

The R-mass of a set can be bounded by a multiple of $\mathcal{H}^{d-1}(\mathrm{Nor}(X))$.

Proposition 6.15 (Bounds on $M_R(X)$)**.** *Let X admit a normal bundle.*
If $R \le \mathrm{reach}(X)$, we have:

$$M_R(X) = \sum_{i=1}^{d} R^i \omega_i V_{d-i}(X). \tag{6.11}$$

For any $R > 0$, we have:

$$M_R(X) \le C_R(d)\mathcal{H}^{d-1}(\mathrm{Nor}(X)), \tag{6.12}$$

where

$$C_R(d) := \int_0^R (1+t^2)^{\frac{d-1}{2}}\,\mathrm{d}t.$$

Proof.

The inequality of Equation (6.11) comes from the fact that for any $0 \le t < \mathrm{reach}(X)$, $1+t\kappa_i \ge 0$ and thus the coefficient in front of t^{i-1} integrates over $\mathrm{Nor}(X)$ to $iV_{d-i}(X)\omega_i$ for any $1 \le i \le d$. The inequality of Equation (6.12) comes from the elementary fact that $\frac{|1+tx|}{\sqrt{1+x^2}} \le \sqrt{1+t^2}$ for any $t, x \in \mathbb{R}$.

□

6.2.5 Properties of curvature measures

Proposition 6.16 (Some properties of the curvature measures)**.** *Let X be a compact set of $\mathbb{R}^d$ admitting a normal bundle.*

(1) For any $\lambda > 0$, $C_i(\lambda X, \lambda U) = \lambda^i C_i(X, U)$;
(2) $C_i(X, U) = C_i(g(X), g(U))$ for any $0 \le i \le d$ and any isometry $g : \mathbb{R}^d \to \mathbb{R}^d$;
(3) When X is a hypersurface, $V_{d-1}(X) = \mathcal{H}^{d-1}(\partial X)$. When X is a domain, $V_{d-1}(X) = \frac{1}{2}\mathcal{H}^{d-1}(\partial X)$.
(4) Let $\iota : \mathbb{R}^d \to \mathbb{R}^m$ be an isometry. Then the intrinsic volumes of $\iota(X)$ are the same as those of X.

Proof.

Items (1) and (2) are immediate consequences of the change of variable formula, respectively with λ-dilatation and the isometry g. Item (3) comes from the explicit representation of the curvature measures, as $V_{d-1}(X) = \frac{1}{\omega_1}\mathcal{H}^{d-1}(\partial X)$ when X is $C^{1,1}$. This extends to domains with positive reach or complementary regular set by the convergence of Steiner's formula applied to offset/counter offsets when $t \to 0$. The other assertion of (3) is a consequence of the soon-to-be introduced additivity of the curvature measures, seeing a hypersurface as the intersection of two domains. Finally, item (4) is the result of a more technical computation. By isometry invariance, we can take $\iota : \mathbb{R}^d \to \mathbb{R}^m$ to be the identity on the first d coordinates and zero on the others. From the co-area formula induced by the map $\mathbb{R}^d \times \mathbb{R}^{m-d} \to \mathbb{R}^{m-d}$, $(x, y) \mapsto y$, we have

$$\begin{aligned}
\mathrm{Vol}(\iota(X)^t) &= \int_{||y|| \le t} \mathrm{Vol}\left(X^{\sqrt{t^2-||y||^2}}\right) \mathrm{d}\mathcal{H}^{m-d}(y) \\
&= \int_{||y|| \le t} \sum_{i=0}^{d} V_{d-i}(X)\omega_i t^i \left(1 - \frac{||y||^2}{t^2}\right)^{i/2} \mathrm{d}\mathcal{H}^{m-d}(y).
\end{aligned}$$

Now consider the bijective change of variables $B(0,t) \to [0,1] \times \mathbb{S}^{m-d-1}, y \mapsto \left(\frac{||y||}{t}, \frac{y}{||y||}\right)$. Its inverse is the map $(r,u) \mapsto tru$, which has Jacobian $t^{m-d}r^{m-d-1}$. The Co-area formula (Theorem 2.13) thus yields:

$$\begin{aligned}\mathrm{Vol}(\iota(X)^t) &= \sum_{i=0}^{d} V_{d-i}(X)(m-d)\omega_{m-d}\omega_i t^{i+m-d} \int_0^1 (1-r^2)^{i/2} r^{m-d-1}\,\mathrm{d}r \\ &= \sum_{i=0}^{d} V_{d-i}(X)(m-d)\omega_{m-d}\omega_i t^{i+m-d} B\left(\frac{i+1}{2}, \frac{m-d}{2}\right)\end{aligned}$$

where B is the Beta function. Linking the closed form $\omega_i = \frac{\pi^{i/2}}{\Gamma(i/2+1)}$ with the expression of B as a function of Euler's Gamma function ($B(x,y)\Gamma(x+y) = \Gamma(x)\Gamma(y)$) yields the following.

$$\mathrm{Vol}(\iota(X)^t) = \sum_{i=0}^{d} t^{m-d+i} V_{d-i}(X)\omega_{m-d+i}. \tag{6.13}$$

□

Among the class of sets with positive reach, the curvature measures enjoy a combinatorial property called *additivity*.

Proposition 6.17 (Additivity of the curvature measures of sets with positive reach [Fed59, 5.16])**.** *Let A, B be two compact subsets of $\mathbb{R}^d$ with positive reach such that $A \cup B$ also has positive reach. Then so does $A \cap B$, and we have for every $0 \leq i \leq d$:*

$$C_i(A,\cdot) + C_i(B,\cdot) = C_i(A \cup B,\cdot) + C_i(A \cap B,\cdot).$$

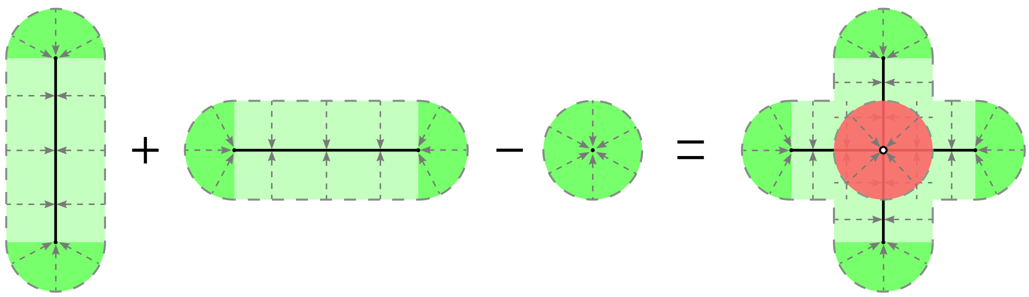


Figure 6.2 – Inclusion-exclusion principle on a cross. Remark that the central point has negative curvatures.

This property is crucial in extending the definition of curvature measures to union of sets with positive reach, as one expects the inclusion-exclusion principle to hold. In fact, along isometry invariance and Hausdorff continuity over the class $\mathcal{K}$ of compact convex sets of $\mathbb{R}^d$, this property characterizes the intrinsic volumes.

Theorem 6.18 (Hadwiger's Theorem)**.** *Let $f : \mathcal{K} \to \mathbb{R}$ be such that:*

— *f is isometry-invariant, that is, for every isometry g of $\mathbb{R}^d$, $f \circ g = f$;*

— *f is additive, i.e., if A, B and $A \cup B$ are in $\mathcal{K}$, then*

$$f(A) + f(B) = f(A \cup B) + f(A \cap B); \tag{6.14}$$

— *f is continuous for the Hausdorff topology;*
Then f is a linear combination of the intrinsic volume functionals on $\mathcal{K}$.

As a consequence, one can construct *integral geometric formulas* which are homogeneous, real valued functionals of compact convex subsets of $\mathbb{R}^d$ satisfying the assumptions of Hadwiger's theorem. These are obtained as integral over invariant measures of well-chosen sets, as defined below.

Definition 6.19 (Invariant measure over isometries and affine subspaces)**.** Let $\overline{SO(d)}$ be the group of affine isometries of $\mathbb{R}^d$. It is isomorphic to $SO(d) \times \mathbb{R}^d$ via $\alpha : (u, v) \mapsto \tau_v \circ u$, where $\tau_v : x \mapsto x + v$ is the translation by v. There is a unique measure μ on $SO(d)$, called the Haar measure, such that $\mu(u(U)) = \mu(U)$ for any Borelian of $SO(d)$, and $\mu(SO(d)) = 1$. We equip $\overline{SO(d)}$ with the image measure of $\mu \otimes \mathcal{H}^d$ by α. This measure is $\overline{SO(d)}$ invariant.

In the same vein, let $A_{d,j}$ be the sets of affine j-dimensional subspaces of $\mathbb{R}^d$. Such a subspace a unique representation as $V + z$ where V is j-dimensional subspace of $\mathbb{R}^d$ and z in $V^\perp$. Letting ν_j^d be the sole $O(d)$-invariant measure on the set $G(d, j)$ of j-dimensional subspaces of $\mathbb{R}^d$ with measure 1. We equip $A_{d,j}$ with the image of the product measure $\nu_j^d \otimes \mathcal{H}^{d-j}$.

The two main kinds of integral geometric formulas are given by the following two propositions. By invariance of the respective measure of integration, it is straightforward to see that these formulas satisfy the assumptions of Hadwiger's theorem as functions of A and B, except the continuity in the Hausdorff distance. This continuity is a consequence of Lebesgue's integration theorem and measure geometric considerations, which can be found in [RZ19, Chapter 6]. The constants of the functionals can be explicitly computed by considering A, B to be two balls of varying radii. Their exact values are $c_{d,r,s} = \gamma(r)\gamma(s)/\gamma(r + s - d)\gamma(d)$, where $\gamma : z \mapsto \Gamma((z + 1)/2)$ is obtained from Euler's Gamma function.

Proposition 6.20 (Principal Kinematic Formula for convex sets)**.** *Let A, B be compact convex subsets of $\mathbb{R}^d$. Then there are constants $c_{d,r,s} \in \mathbb{R}$ such that:*

$$\int_{g \in \overline{SO(d)}} V_k(A \cap gB)\,\mathrm{d}g = \sum_{\substack{r+s=d+k \\ 0 \le r,s \le d}} c_{d,r,s} V_r(A) V_s(B). \tag{6.15}$$

Proposition 6.21 (Crofton's formula)**.** *Let A be a compact convex subset of $\mathbb{R}^d$. Then we have*

$$\int_{A_{d,j}} V_k(A \cap E)\,\mathrm{d}E = c_{d,d+k-j,j} V_{d+k-j}(X). \tag{6.16}$$

As V_0 is the Euler-characteristic by the Gauss-Bonnet theorem, these two formulas relate topological quantities to geometrical ones. Further taking B to be a ball of radius t in the principal kinematic formula, we recover Steiner's formula thanks to the local contractability of any set with positive reach Proposition 3.3. We will see in Chapter 7 that this very case can be combined with the persistent homology framework.

6.3 Connections to Morse theory

In this section, we present how the class of sets admitting a normal bundle can be linked to previous concepts found in the literature of curvatures of subsets of Euclidean spaces. Let $\phi_{X,r}$ be the following map:

$$\phi_{X,r} := \begin{cases} \mathrm{Nor}(X) \times (0,r) & \to & \mathbb{R}^d \\ (z,n,t) & \mapsto & z+nt \end{cases}$$

We write ϕ_X for $\phi_{X,\infty}$.

6.3.1 Almost all linear forms and distance to a point functions are Morse

Definition 6.22 (Linear forms and distance-to-a-point functions on $\mathbb{R}^d$)**.** For any $\nu \in \mathbb{S}^{d-1}$, let $h_\nu : x \mapsto \langle x, \nu \rangle$ be the linear form associated to ν. We denote by $H_{\nu,t}$ the half-space $h_\nu^{-1}(-\infty, t] = \{x \mid \langle x, \nu \rangle \leq t\}$.

For any $x \in \mathbb{R}^d$, we let $d_x : y \mapsto ||x-y||$ be the distance to x. Its sublevel sets are the balls centered around x. The family $(d_x)_{x \in \mathbb{R}^d}$ forms the class of *distance to a points* functions.

It is sometimes more convenient to work with d_x^2 as it is more regular. Indeed, the map d_x^2 is C^∞ for all $x \in \mathbb{R}$, with gradient $\nabla d_x^2(y) = 2(y-x)$ and Hessian $H_y d_x^2 = 2\,\mathrm{Id}$.

Theorem 6.23 (Almost all linear forms are Morse)**.** *Let X be a set admitting a normal bundle. Then for $\mathcal{H}^{d-1}$-almost all ν in $\mathbb{S}^{d-1}$, the map $(h_\nu)_{|X}$ is Morse. Moreover, if y is a critical point of the Morse function $(h_\nu)_{|X}$, then the dimension of the cell added around y is the number of negative principal curvatures of the pair $(y, -\nu)$ in* $\mathrm{Nor}(X)$.

Proof.

A point y in X is critical for $(h_\nu)_{|X}$ if and only if $-\nu \in \mathrm{Nor}(X, y)$. Since almost all pairs of $\mathrm{Nor}(X)$ are regular, the set π_{reg} consisting of the ν in $\mathbb{S}^{d-1}$ such that the fiber $\pi_1^{-1}(\nu)$ with $\pi_1 : \mathrm{Nor}(X) \to \mathbb{S}^{d-1}$ contains only regular pairs has full $\mathcal{H}^{d-1}$-measure. Now let $y \in \pi_{ref}$ be a critical point of $(h_\nu)_{|X}$. The Hessian of $(h_\nu)_{|X}$ at y is exactly the second fundamental form at $(y, -\nu)$, whose index is the number of finite negative curvatures. We obtain the desired result on the dimension of the cell glued around y by adding the eventual number of negative infinite curvatures in case X is complementary regular.

Now there remains to prove that such a second fundamental form is non-degenerate for almost all ν in π_{reg}. Remark that $J_{d-1}\pi_{1|\,\mathrm{Nor}(X)}(y, -\nu) = \prod_{i=1}^{d-1} \frac{|k_i|}{\sqrt{1+k_i^2}}$ is zero if and only if the second fundamental form is degenerate. Let Degen_X be the set of regular pairs $(y, -\nu)$ in $\mathrm{Nor}(X)$ such that the Hessian $H_y(h_{-\nu})_{|X}$ is degenerate. By the weak version of Sard's theorem (Theorem 2.14), $\mathcal{H}^{d-1}(\pi_1(\mathrm{Degen}_X)) = 0$. The set of ν such that $(h_{-\nu})_{|X}$ is Morse is exactly $\pi_{reg} \setminus \pi_1(\mathrm{Degen}_X)$, which has full $\mathcal{H}^{d-1}$-measure.

□

We now turn our attention to distance to a point functions. The definition of Morse functions in Chapter 4 is not adapted to distance to a point function. Indeed, remark that $d_x^2 : y \mapsto ||x-y||^2$ has vanishing gradient at $y = x$ although the point x plays the role of a critical point of $(d_x^2)_{|X}$ when X is a complementary regular set containing x. To deal with this class of maps, we adapt (for this class only!) the definition of a Morse function.

Definition 6.24 (Morse distance to a point function)**.** Let X admit a normal bundle. We say that d_x^2 is a Morse function when either:

— $x \notin X$ and $(d_x^2)_{|X}$ is Morse in the sense of Chapter 4;
— $x \in X$ and every critical point of $(d_x^2)_{|X}$ different from x is non-degenerate.

With this definition, distance to a point functions are indeed almost-all Morse.

Theorem 6.25 (Distance to a point functions & Morse theory)**.** *Let X be a set admitting a normal bundle. Then $(d_x^2)_{|X}$ is a Morse function for $\mathcal{H}^d$-almost all x in $\mathbb{R}^d$.*

When $(d_x^2)_{|X}$ is a Morse function, the dimension of the cell added around a critical point y in the filtration $(X \cap B(x,t))_{t\in\mathbb{R}}$ is the number of principal curvatures of X at $(y, \frac{x-y}{||x-y||})$ lower than $-\frac{1}{||x-y||}$. Furthermore, the number of homological events of the filtration $(X \cap B(x,t))_{t\in\mathbb{R}}$ with critical value strictly less than $r \in \mathbb{R}^+$ is $\mathbb{1}_X(x) + \operatorname{Card}(\phi_{X,r}^{-1}(x))$.

Proof.

A point $y \neq x$ is critical for $(d_x^2)_{|X}$ when $n := \frac{x-y}{||x-y||}$ belongs to $\operatorname{Nor}(X,y)$, i.e., $x = y + nt$ for some (y,n) in $\operatorname{Nor}(X)$ and $t = ||x-y||$. The map $(d_x^2)_{|X}$ is Morse if and only if $\phi_{X,\infty}^{-1}$ contains only regular pairs (y,n) in the first two coordinates, with $H_y(d_x^2)_{|X}$ non-degenerate. Let Unreg_X be the set of non-regular pairs of $\operatorname{Nor}(X)$ and let x belong to $\mathbb{R}^d \setminus \phi_X(\operatorname{Unreg}_X \times \mathbb{R}^+)$. For any (y,n,t) in ϕ_X^{-1}, let $(b_i)_{1\le i\le d-1}$ (resp. $(k_i)_{1\le i\le d-1}$) be principal directions (resp. associated principal curvatures) at (y,n), such that they have associated finite curvatures if and only if $1 \le i \le p$. The restricted Hessian on $\pi_0(\operatorname{Tan}(\operatorname{Nor}(X),(y,n))) = \operatorname{Vect}((b_i)_{1\le i\le p})$ writes

$$
\begin{aligned}
H_y(d_x^2)_{|X}(b_i,b_j) &= H_y(d_x^2)(b_i,b_j) + t\,\mathrm{I\!I}_{y,n}(b_i,b_j)\\
&= \begin{cases} 2(1+tk_i) & \text{if } i = j;\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

□

When it is non-degenerate, the index of this bilinear form is $\operatorname{Card}\left\{1 \le i \le p \mid k_i \le -\frac{1}{t}\right\}$ and the dimension of the cell glued around y is that plus the eventual negative infinite curvatures.

Now the previous Hessian is non-degenerate if and only if the product

$$
\prod_{1\le i\le d-1} \frac{|1+tk_i|}{\sqrt{1+k_i^2}} = J_d\phi_X(y,n,t)
$$

vanishes. Letting Degen_X be the set consisting of triplets (y,n,t) where (y,n) is a regular pair and t is such that the previous product is zero, by the weak Sard's theorem (Theorem 2.14) we have $\mathcal{H}^d(\phi_X(\operatorname{Degen}_X)) = 0$. In the end, the set

$$
\mathbb{R}^d \setminus \Big(X \cup \phi_X\left(\operatorname{Unreg}_X \times \mathbb{R}^+\right) \cup \phi_X\left(\operatorname{Degen}_X\right)\Big)
$$

has full Lebesgue measure in $\mathbb{R}^d \setminus X$ and contains only points x such points $(d_x^2)_{|X}$ is Morse.

Since any set admitting a normal bundle is locally contractible, when $x \in X$, the sets $X \cap B(x,t)$ have the homotopy type of a point when t is small enough or zero, and there is only one topological change of the filtration $(X \cap B(x,t))_{t\in\mathbb{R}}$ happening at $t = 0$. The remaining changes happen as per usual Morse theory as long as x does not belong to $\phi_X\left(\operatorname{Unreg}_X \times \mathbb{R}^+ \cup \operatorname{Degen}_X\right)$ which has $\mathcal{H}^d$-measure zero.

6.3.2 Curvatures and Morse theory

Let X be a set admitting a normal bundle. We saw in Chapter 4 that the normal bundle was closely related to the topology of the sublevel sets of Morse functions on X. In particular, recall that the maps $h_\nu : x \mapsto \langle \cdot, \nu \rangle$ restricted to X are Morse for $\mathcal{H}^{d-1}$-almost every ν in $\mathbb{S}^{d-1}$, and that $y \in X$ is critical for h_ν when $(y, -\nu) \in \mathrm{Nor}(X)$. Analogously, the squared distance to a point maps $d_x^2 = ||x - \cdot||^2$ restricted to X are Morse for $\mathcal{H}^d$-almost every x in $\mathbb{R}^d$. A point $y \in X$ distinct from x is critical for d_X^2 when $(y, \frac{x-y}{||x-y||}) \in \mathrm{Nor}(X)$. Inspired by the ideas of *Integral geometry for tame sets* [BK00], we define the index functions of linear forms and distance to a points functions as follows.

Definition 6.26 (Index of particular Morse functions)**.** Let $X \subset \mathbb{R}^d$ be a set admitting a normal bundle, $\nu \in \mathbb{S}^{d-1}$ and $x \in \mathbb{R}^d$. Assume that h_ν restricted to X is a Morse function and let y be a critical point of h_ν for a certain $\nu \in \mathbb{S}^{d-1}$. The *index of h_ν at a point y* is:

$$\alpha(y,\nu) := \begin{cases} (-1)^{\lambda(y,h_\nu)} & \text{if } y \text{ is a critical point of } h_\nu, \\ 0 & \text{else.} \end{cases} \tag{6.17}$$

where $\lambda(y, \nu)$ is the dimension of the cell added around y in the sublevel set filtration $(X \cap H_{\nu,t})_{t\in\mathbb{R}}$.

Similarly, assume that d_x^2 restricted to X is a Morse function. The index of d_x^2 at a point y is:

$$g(y,x) := \begin{cases} 1 & \text{if } y = x; \\ (-1)^{\lambda(y,d_x^2)} & \text{if } y \neq x \text{ is a critical point of } d_x^2; \\ 0 & \text{else.} \end{cases} \tag{6.18}$$

where $\lambda(y, d_x^2)$ is the dimension of the cell added around y in the sublevel set filtration $(X \cap B(X,t))_{t\in\mathbb{R}}$.

The indices of these two families of functions are pertinent for our study. We begin by showing that these families are Morse almost everywhere. Using the co-area formula, one infers the following.

Proposition 6.27 (Curvature measure C_0 from indices of linear forms)**.** *Let X be a set admitting a normal bundle. Then*

$$C_0(X,U) = \frac{1}{d\omega_d} \int_{\mathbb{S}^{d-1}} \left(\sum_{y\in U} \alpha(y,\nu) \right) \mathrm{d}\mathcal{H}^{d-1}(\nu). \tag{6.19}$$

Proof of Proposition 6.27.

Let $\pi_1 : \mathrm{Nor}(X) \to \mathbb{S}^{d-1}, (x,n) \mapsto n$ be the projection onto the normal coordinate. When (y,ν) is regular, one infers from the structure of tangent spaces that

$$J_{d-1}\pi_1(y,\nu) = \left| \prod_{i=1}^{d-1} \frac{k_i}{\sqrt{1+k_i^2}} \right|.$$

For almost every (y,ν) in $\mathrm{Nor}(X)$, the number of negative principal curvatures coincides with the index $\alpha(y,\nu)$. Using the Co-area formula on the explicit representation of C_0, one has

$$\begin{aligned} d\omega_d C_0(X,U) &= \int_{\mathrm{Nor}(X,U)} \prod_{i=1}^{d-1} \frac{k_i}{\sqrt{1+k_i^2}} \,\mathrm{d}\mathcal{H}^{d-1}(x,n) \\ &= \int_{\mathrm{Nor}(X,U)} \alpha(y,n) J_{d-1}\pi_1(y,n) \,\mathrm{d}\mathcal{H}^{d-1}(x,n) \\ &= \int_{\mathbb{S}^{d-1}} \left(\sum_{\pi_2^{-1}(n)\cap U} \alpha(x,n) \right) \mathrm{d}\mathcal{H}^{d-1}(n). \end{aligned}$$

□

In particular, this yields the famous Gauss-Bonnet theorem.

Corollary 6.28 (Gauss-Bonnet theorem). *Let X be a complementary regular set. Then $V_0(X) = \chi(X)$.*

Proof.

Let $U = X$ in the previous formula Proposition 6.27. As X is compact, if h_ν is Morse the sum $\sum_{x\in\pi_2^{-1}(n)} \alpha(x,n)$ is finite and equals the Euler characteristic of X. As this is the case for $\mathcal{H}^{d-1}$-almost all ν in $\mathbb{S}^{d-1}$, we have:

$$V_0(X) = \frac{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})}{d\omega_d}\chi(X) = \chi(X).$$

□

Using similar methods, we show that the indices $g(x,y)$ determine the curvature measure of X.

Theorem 6.29 (Curvatures measure and index function g). *Let $X \subset \mathbb{R}^d$ be a set admitting a normal bundle and U be a Borelian subset of $\mathbb{R}^d$. For any $0 \le r < \infty$, let*

$$\Lambda_U(r) := \int_{x\in\mathbb{R}^d} \Big(\sum_{\substack{y\in U \\ ||x-y||\le r}} g(y,x) \Big) \,\mathrm{d}x. \tag{6.20}$$

Then Λ_U is the Steiner polynomial localized at U, i.e.,

$$\Lambda_U(r) = \sum_{i=0}^{d} \omega_i r^i C_{d-i}(X,U). \tag{6.21}$$

Proof.

Let $\phi : \mathrm{Nor}(X) \times [0,r], (z,n,t) \mapsto z + nt$ and $\psi(z,n,t) = (-1)^{m(z,n,t)}$, where at a regular pair (z,n), $m(z,n,t)$ is the number of principal curvatures at (z,n) smaller than $\frac{-1}{t}$, so that $\psi(z,n,t)$ is the sign of the product of $(1+tk_i)$. At a regular pair (z,n), for any $t \in (0,r)$, the Jacobian of ϕ is

$$J_d\phi(z,n,t) = \prod_{i=1}^{d-1} \frac{|1+tk_i|}{\sqrt{1+k_i^2}}. \tag{6.22}$$

On the one hand, remark that $\psi(z,n,t)J_d\phi(z,n,t) = \prod_{i=1}^{d-1} \frac{1+tk_i}{\sqrt{1+k_i^2}}$. Thus, we end up with the same polynomial as in the proof of the explicit representation of the curvature measures, without the constant coefficient:

$$\int_{\mathrm{Nor}(X,U)\times(0,r)} \psi(z,n,t)J_d\phi(z,n,t)\,\mathrm{d}\mathcal{H}^{d-1}(z,n)\,\mathrm{d}t = \sum_{i=1}^{d} \omega_i r^i C_{d-i}(X,U) \tag{6.23}$$

On the other hand, letting $\phi_{X,U,r}$ be the restriction of $\phi_{X,r}$ to $\mathrm{Nor}(X,U)\times(0,r)$, the co-area formula yields

$$\int_{\mathrm{Nor}(X,U)\times[0,r]} \psi J_d\phi_X\,\mathrm{d}\mathcal{H}^d = \int_{\mathbb{R}^d}\left(\sum_{v\in\phi_U^{-1}(x)} \psi(v)\right)\mathrm{d}x. \tag{6.24}$$

When $(d_x^2)_{|X}$ is Morse, the inside sum ranges over (z,n,t) in $\mathrm{Nor}(X,U)\times(0,r)$ such that $x = z+nt$, i.e., over critical points y at distance to x in $[0,r]$ with normal $n = \frac{x-y}{||x-y||}$. As a consequence, $\mathcal{H}^d$-almost everywhere in x one has:

$$\sum_{v\in\phi_{X,r}^{-1}(x)} \psi(v) = \sum_{\substack{y\in U\setminus\{x\}\\ ||x-y||\le r}} g(y,x).$$

The desired equality is obtained by allowing $y = x$ in the inside sum, which adds $\mathrm{Vol}(X,U) = C_d(X,U)$ to the quantity of Equation (6.23).

□

Remark that for a fixed $r>0$, when d_x^2 restricted to X is Morse and r is not a critical value of d_x^2 (which is the case $\mathcal{H}^d$-almost everywhere in x), the Euler characteristic of $X\cap B(x,r)$ is the sum of the indices of d_x^2 of critical points within $B(x,r)$. Along Theorem 6.29, this yields the following particular case of the kinematic formula.

Corollary 6.30 (Particular case of the Principal Kinematic Formula)**.** *Let X be a set admitting a normal bundle and let $0<r<\infty$. Then we have*

$$\int_{x\in\mathbb{R}^d} \chi(X\cap B(x,r))\,\mathrm{d}x = \sum_{i=0}^{d-1}\omega_i r^i V_{d-i}(X). \tag{6.25}$$

6.4 The theory of normal cycles

In this section, we compare the properties of the curvatures of complementary regular sets to the literature on curvatures of subsets of Euclidean spaces. The concept of *normal cycle* has been key in the study of curvatures of subsets or $\mathbb{R}^d$ since its first definition in [Win82, Zä86]. It consists in a $(d-1)$-current N_X with support in $X\times\mathbb{S}^{d-1}$ associated to each "reasonably geometric" subset X of $\mathbb{R}^d$. As this section is not necessary to obtain the final inference result on the intrinsic volumes, we do not include a full introduction to the concept of currents, which are dual to smooth differential forms just as distributions are duals to smooth, real-valued maps. We refer the reader to [Fed69, Chapter 4] or [RZ19, 1.3] for an introduction to the concept of currents. Nevertheless, basic currents terminology can be found in Appendix A.

Normal cycles are currents containing all the previously defined informations about the curvatures of a set. We will see for instance that every curvature measure can be retrieved as the normal cycle against particular differential forms. Using the terminology of currents allows to represent in concise and general ways some results about the curvatures.

6.4.1 Axiomatic definition of the normal cycle of a compact subset of $\mathbb{R}^d$

Definition 6.31 (Legendrian currents)**.** A current T on $\mathbb{R}^d \times \mathbb{R}^d$ is said to be a Legendrian cycle when:

— $\partial T = 0$ (T is a cycle);

— $T \llcorner \alpha = 0$ where α is the canonical contact form with $\langle (u,v), \alpha(x,n) \rangle = \langle v, n \rangle$;

— T is a locally integral current, i.e., there exists W_T is a $(d-1)$ rectifiable subset of $\mathbb{R}^d \times \mathbb{S}^{d-1}$ a $\mathcal{H}^{d-1}$-integrable integer valued function i_T on W_T and a $\mathcal{H}^{d-1}$-integrable field of simple unit covectors a_T on W_T such that for any differential form ϕ with compact support, we have:

$$T(\psi) = \int_{W_T} i_T(x,n) \, \langle a_T(x,n), \psi(x,n) \rangle \, \mathrm{d}\mathcal{H}^{d-1}(x,n). \tag{6.26}$$

From the two first conditions, we have that $T \llcorner \omega = 0$, where $\omega = \mathrm{d}\alpha$ is the canonical symplectic form on $\mathbb{R}^d \times \mathbb{R}^d$. In fact, $T \llcorner \omega$ being zero summarizes the structure of principal directions and curvatures as explicited by the following theorem about the covector field a_T [RZ19, Theorem 9.2].

Proposition 6.32 (Structure of tangent spaces of Legendrian currents)**.** *If T is a Legendrian current, then for $\mathcal{H}^{d-1}$-almost all (x,n) in W_T, the tangent cones* $\mathrm{Tan}(W_T,(x,n))$ *are $(d-1)$-vector spaces represented by a_T. Furthermore, there exist unit vectors $(b_i)_{1 \le i \le d-1}$ in $\mathbb{R}^d$ and $(k_i)_{1 \le i \le d-1}$ in $\mathbb{R} \cup \{\infty\}$ such that the following vectors*

$$a_i(x,n) = \left(\frac{1}{\sqrt{1+k_i^2}} b_i, \frac{k_i}{\sqrt{1+k_i^2}} b_i \right)$$

form an orthonormal basis of $\mathrm{Tan}(W_T,(x,n))$.

Fu [Fu89b] proved a unicity theorem for normal cycles. Indeed, he showed that establishing the validity of the Gauss-Bonnet theorem for almost-all intersection with half-spaces as a principle was sufficient in characterizing the curvatures of a compact subset of $\mathbb{R}^d$.

Definition 6.33 (Normal cycles)**.** A compact set X in $\mathbb{R}^d$ is said to *admit a normal cycle* when there exists a $(d-1)$-Legendrian cycle N_X on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ such that:

— The support W_X of N_X is compact;

— For $\mathcal{H}^d$ almost all (ν, t) in $\mathbb{S}^{d-1} \times \mathbb{R}$,

$$\sum_{x, \langle x, \nu \rangle \le t} (-1)^{\lambda(x,-\nu)} i_X(x,-\nu) = \chi(X \cap H_{\nu,t}). \tag{6.27}$$

where $\lambda(x,-\nu)$ is the number of negative principal curvatures at $(x,-\nu)$

From this definition and the additivity of the Euler characteristic, we obtain the inclusion-exclusion principle over normal cycles:

Proposition 6.34 (Additivity of the normal cycle)**.** *Let X, Y be two compact subsets of $\mathbb{R}^d$. If three among $X, Y, X \cup Y, X \cap Y$ admit a normal cycle, then so does the fourth, and we have*

$$N_X + N_Y = N_{X\cup Y} + N_{X\cap Y}. \tag{6.28}$$

Remark 6.35 – The left-hand side quantity of Equation (6.27) can be expressed by classical operations on currents plus the so-called *slicing* operateor. Indeed, almost everywhere, we have:

$$\sum_{x,\langle x,\nu\rangle\leq t}(-1)^{\lambda(x,-\nu)}i_X(x,-\nu) = (\Lambda_\# N_X, \pi_0, -\nu)(\mathbb{1}_{(-\infty,t]})$$

with $\Lambda : (x,\nu) \mapsto (\nu, \langle x,\nu\rangle)$.

The representation as a sum has a topological interpretation : if the map h_ν is Morse, then the equality $\sum_{x,\langle x,\nu\rangle\leq t}(-1)^{\lambda(x,-\nu)}i_X(x,-\nu) = \chi(X\cap H_{\nu,t})$ states that the Euler characteristics of the sublevel sets $h_\nu(-\infty,t]$ are the sum of indices of the critical points of h_ν. The condition of eq. (6.27) can be seen as measure theoretical generalization of this fact without the need for the maps h_ν to be Morse for almost every $\nu \in \mathbb{S}^{d-1}$.

Definition 6.36 (Curvature measures & normal cycles)**.** Let ϕ_k be the differential forms on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ defined by

$$\langle a_1\wedge\cdots\wedge a_{d-1}, \phi_k(x,n)\rangle \coloneqq \frac{1}{(d-k)\omega_{d-k}} \sum_{\substack{j_1+\cdots+j_{d-1}=d-k-1\\ j_i\in\{0,1\}}} \det(\pi_{j_1}(a_1),\ldots,\pi_{j_{d-1}}(a_{d-1}),n). \tag{6.29}$$

The curvature measures $(C_k(X,\cdot))_{0\leq i\leq d-1}$ of a compact set $X \subset \mathbb{R}^d$ admitting a normal cycle are defined by:

$$C_k(X,U) \coloneqq N_X(\phi_k \llcorner \mathbb{1}_{U\times\mathbb{S}^{d-1}}). \tag{6.30}$$

Remark 6.37 – As opposed to most other works in the literature, which constructed the curvatures measures or normal cycles of ever broader classes of sets either explicitly or as the consistent limits of the curvatures of more regular objects, the axiomatic definition constrains the classes of sets which can admit a normal cycle. For instance, it is hopeless to build the normal cycle of compact, fractal-like objects. Indeed, the Euler characteristic of such a set intersected with half-planes may not be well-defined, as its Betti numbers might be infinite.

Moreover, the axiomatic definition does not cover the normal cycle of non-compact sets, although we will see that their definition for e.g. the class of sets with positive reach is exactly the same as in the compact case. This is due to the fact that the Gauss-Bonnet theorem fails for non-compact set. For instance, the 0-intrinsic volume of the complement set of a ball in $\mathbb{R}^2$ - whose reach is positive - is -1, whereas its Euler characteristic is 0.

6.4.2 Normal cycles of sets admitting a normal bundle

In this subsection, we give an explicit construction of the normal cycle of sets admitting a normal cycle.

Definition 6.38 (Normal cycles of sets admitting a normal bundle)**.** Let X be a set admitting a normal bundle. For any regular pair (x,n) in $\mathrm{Nor}(X)$, let b_i (resp. k_i) be principal directions (resp.

associated principal curvatures) of X at (x, n) such that the family $b_1, \dots, b_{d-1}, n$ is positively oriented. When X is a complementary regular set, we let infinite curvatures to be $-\infty$. We let a_X be the $\mathcal{H}^{d-1}$-measurable vector field of $(d-1)$-covectors on $\mathrm{Nor}(X)$ defined by:

$$a_X^-(x, n) := \bigwedge_{i=1}^{d-1} \left(\frac{1}{\sqrt{1+k_i^2}} b_i, \frac{k_i}{\sqrt{1+k_i^2}} b_i \right). \tag{6.31}$$

The *normal cycle* N_X of a complementary regular set X is the $(d-1)$-dimensional current on $\mathbb{R}^d \times \mathbb{R}^d$ with support in $\mathbb{R}^d \times \mathbb{S}^{d-1}$, defined by

$$N_X(\psi) := \int_{\mathrm{Nor}(X)} \left\langle a_X^-(x, n), \psi(x, n) \right\rangle \mathrm{d}\mathcal{H}^{d-1}(x, n). \tag{6.32}$$

Remark 6.39 – In Proposition 6.32 principal curvatures and directions were defined from the normal cycle N_X, whereas Definition 6.38 did the opposite. In particular, when X is a complementary regular set, the definition of Proposition 6.32 only allows for positive infinite curvatures. With this convention, the previous factors of a_X^- with infinite curvature become $(0, b_i)$ instead of $(0, -b_i)$, yielding a new vector field $a_X = i_X a_X^-$, where i_X is 1 if the number of infinite curvatures at (x, n) is even and -1 else. Thus the definition of N_X is consistent, with

$$N_X(\psi) = \int_{\mathrm{Nor}(X)} i_X(x, n) \left\langle a_X(x, n), \psi(x, n) \right\rangle \mathrm{d}\mathcal{H}^{d-1}(x, n). \tag{6.33}$$

When X is complementary regular, its complement set $\neg X$ is not compact. Yet, its boundary is compact, and we can define $N_{\neg X}$ by integration on its boundary just as in Definition 6.38 thanks to the structure of $\mathrm{Nor}(\neg X)$. The fact that $\mathrm{Nor}(X) = \rho(\mathrm{Nor}(\neg X))$ translates into the following relation between normal cycles.

Proposition 6.40 (Normal cycle of complementary regular sets)**.** *Let* $\rho : (x, n) \mapsto (x, -n)$ *and let* $X \subset \mathbb{R}^d$ *be a complementary regular set. Then we have*

$$N_{\neg X} = -\rho_\sharp N_X. \tag{6.34}$$

Remark that the equalities $C_k(X, \cdot) = (-1)^{d-k-1} C_k(\neg X, \cdot)$ for any $0 \le k \le d-1$ can be seen as consequences of the fact that $\rho^* \phi_k(x, n) = (-1)^{d-k} \phi_k(x, -n)$.

Proposition 6.41 (N_X is the normal cycle of X in the sense of Fu)**.** *Let* X *be a compact subset of* $\mathbb{R}^d$ *admitting a normal bundle. Then the current* N_X *defined in Definition 6.38 is the normal cycle of* X *in the sense of Definition 6.33.*

Proof.

If X is a compact $C^{1,1}$-domain, N_X represent the integration over a manifold without boundary, which implies that $\partial N_X = 0$. The correspondence between $\mathrm{Nor}(X)$ and $\mathrm{Nor}(X^{\varepsilon r})$ where $\varepsilon = 1$ (resp. -1) if X is of positive reach (resp. is complementary regular) yields, when $r < \mathrm{reach}(X)$ (resp. $< \mathrm{reach}\,\neg X$):

$$N_X = (f^{\varepsilon r})_\#(N_{X^{\varepsilon r}})$$

with $f^t = (z, n) \mapsto (z + tn, n)$. The right-hand side is the normal cycle of a $C^{1,1}$ domain.

Recall that α is the 1-form defined by $\alpha(x,n)(u,v) = \langle n, v\rangle$. Let (x,n) be a regular pair of $\mathrm{Nor}(X)$. Since $a_X(x,n) = \bigwedge_{i=1}^{d-1} a_i(x,n)$ with the two components of $a_i(x,n) \in \mathbb{R}^d \times \mathbb{R}^d$ orthogonal to n, we have

$$\begin{aligned}\langle a_X(x,n), \alpha(x,n) \wedge \psi(x,n)\rangle &= \sum_{i=1}^{d-1} (-1)^i \alpha(a_i(x,n)) \left\langle a_X^i(x,n), \psi(x,n)\right\rangle \\ &= 0.\end{aligned}$$

for some $(d-2)$-covectors $a_X^i(x,n)$. Integrating over $\mathrm{Nor}(X)$ yields $N_X(\alpha\wedge\psi) = 0$.
Now there remains to prove that $\sum_{x,\langle x,\nu\rangle\le t}(-1)^{\lambda(x,-\nu)} i_X(x,-\nu) = \chi(X\cap H_{\nu,t})$ for $\mathcal{H}^d$-almost everywhere in $\mathbb{S}^{d-1}\times\mathbb{R}$. In our case, $i_X(x,n) = 1$ in $\mathrm{Nor}(X)$. By Theorem 6.23, almost everywhere in $\mathbb{S}^{d-1}$ the linear form h_ν is Morse, and for any regular value t pf h_ν, $\chi(X\cap H_{\nu,t})$ is the sum of the index $(-1)^{\lambda(x,-\nu)}$ of the critical points x with $\langle x,\nu\rangle \le t$. Since there are finitely many critical points, this equality is true $\mathcal{H}^d$-almost everywhere in ν, t.

□

6.4.3 Comparison with other classes of sets admitting a normal cycle

In the geometric measure theory framework, normal cycles were first defined by Wintgen in [Win82] for submanifolds of $\mathbb{R}^d$ and by Zähle in [Zä86] for sets with positive reach. The extension to complementary regular sets presented in this chapter is a simple consequence of the fact that almost all linear forms are Morse for the generalized Morse theory presented in Chapter 4 and the fact that $\mathrm{Nor}(X) = \rho(\mathrm{Nor}(^\neg X))$ by putting $N_X = -\rho_\# N_{^\neg X}$. We compare this construction to some other constructions of normal cycles or curvature measures found in the litterature.

Normal cycles of stratified sets In [BK00], the authors defined the curvature measures of any stratified set as the coefficients of the polynomial Λ_U, just as in Theorem 6.29, using stratified Morse theory. Although complementary regular sets are not necessarily stratified - and reciprocally - we adapted the ideas of [BK00] to our context thanks to the Morse theory for complementary regular sets developed in Chapter 4. The class of complementary regular sets is morse simple, as the index of Morse functions at a critical points are either 1 or -1. In particular, when not vanishing, the index function i_X of the normal cycle of a complementary regular set always takes value 1 or -1.

Lipschitz domains In [RZ03], Rataj and Zähle obtained that all elements of the class of Lipschitz domains with locally bounded inner curvatures (LBIC), which contains complementary regular sets, admitted a normal cycle. They did so defining N_X as the limit in the flat norm of the normal cycle of the eroded sets $X^{-\varepsilon}$, and showed that it was the Euler characteristic condition of Definition 6.33 was verified by N_X by local retraction method. In [RZ05], this result was extended to Lipschitz submanifold of lower dimensions, replacing the normal bundle $\mathrm{Nor}(X)$ with $\mathcal{N}_X := \bigcup_{x\in\partial X}\{x\}\times\left(\partial^* d_X(x)\cap\mathbb{S}^{d-1}\right)$.

Locally WDC-sets Another class of sets for which the existence of a normal cycle is guaranteed is the class of sets which are locally sublevel sets of maps which can be written as the difference of two convex functions at weak regular value. Such sets are called *WDC sets*. The class of sets admitting a normal bundle is contained in the class of WDC sets.

Proposition 6.42 (Sets admitting a normal bundle are WDC)**.** *Every set in* $\mathbb{R}^d$ *admitting a normal bundle is WDC.*

Proof.

If $\mathrm{reach}(X) > 0$, then d_X is semi-convex on a neighborhood of X by Proposition 3.4 and 0 is a weak regular value of d_X. If X is complementary regular set, then X is sublevel set of a semi-concave function at a regular value by Section 3.4.2. Moreover, both semi-concave and semi-convex maps are obviously dc by definition.

□

In particular, this implies that the principal kinematic formula does apply to the class of sets admitting a normal bundle:

Theorem 6.43 (Principal Kinematic Formula for sets admitting a normal bundle)**.** *Let* $A, B \subset \mathbb{R}^d$ *be sets admitting a normal bundle, and* U, V *be two Borelians of* $\mathbb{R}^d$*. Then for every* $0 \leq i \leq d-1$ *we have:*

$$\int_{g \in \overline{SO(d)}} C_i(A \cap gB, U \cap gV)\,\mathrm{d}g = \sum_{r+s=i+d} c_{d,r,s} C_r(A,U) C_s(B,V). \tag{6.35}$$

Chapter 7

Persistent geometry

We develop a new method to estimate the area, and more generally the intrinsic volumes, of a compact subset X of $\mathbb{R}^d$ from a set Y that is close in the Hausdorff distance. This estimator enjoys a linear rate of convergence as a function of the Hausdorff distance under mild regularity conditions on X. Our approach combines tools from both geometric measure theory and persistent homology, extending the noise filtering properties of persistent homology from the realm of topology to geometry. Along the way, we obtain a stability result for intrinsic volumes.

7.1 Introduction

7.1.1 Previous work

Previous results focused mostly on the estimation of the area $V_{d-1}(X)$. One approach is to estimate the area by the area of a piecewise linear reconstruction of the data. For example, the tangent complex triangulation [BG14] guarantees that this estimator converges to $V_{d-1}(X)$ at a linear rate in the Hausdorff distance $d_H(X,Y)$ when X is a smooth submanifold of $\mathbb{R}^d$ and Y a noise-free sample. Using Crofton's formula, [ACF22] builds an estimator for the surface area $\mathcal{H}^{d-1}(\partial X)$ from a point cloud sample Y and obtains a square-root rate of convergence $O(d_H(X,Y)^{1/2})$ in the general case. Other works have focused on the retrieval of *curvatures measures*, which are local, more informative versions of the intrinsic volumes. A convergence rate of $d_H(X,Y)^{1/2}$ for the curvature measures was obtained in [CCSM10] under the condition $d_H(X,Y) \leq \mathrm{reach}(X)$. Chazal et al. [CCSLT09] obtain the convergence of the curvature measures of Y^r to that of X^r at a square root rate, for any fixed $r > 0$.

Linear convergence rates were obtained for the estimation of the first intrinsic volume using persistent homology of height functions and Crofton's formula. Authors in [CSE07] showed a linear rate of convergence of $V_1(Y)$ to $V_1(X)$ with respect to the Fréchet distance between X and Y when they are both compact surfaces of $\mathbb{R}^3$ or both curves in $\mathbb{R}^d$ assuming their total absolute curvature is bounded. Building on these ideas, Edelsbrunner et al. [EP16] obtained more recently a linear rate of convergence for the first intrinsic volume of voxelizations of smooth sets, in addition to showing the convergence of all intrinsic volumes of spheres voxelized with ever-increasing precision.

Another line of research has focused on non-deterministic geometric inference from uniform samples of convex sets. Notably, the authors of [BHH08, Rei04] worked with convex sets with a C^k boundary, where $k \geq 2$. The expected intrinsic volumes of the convex hull of the sample were shown to converge each to the ideal intrinsic volume at a rate of $C_X n^{-2/(d+1)}$ where n is the number of sample points and C_X a constant depending on X. It was also proven that this result does not hold with a mere C^1 boundary condition, suggesting that finding an estimator that is robust to the lack of regularity is difficult.

7.1.2 Contributions

We define quantities $V_i^\varepsilon(Y)$ depending on a parameter ε, that approach $V_i(X)$ at a linear rate in $d_H(X,Y)$ assuming only mild regularity conditions on X. This rate is easily seen to be optimal. To the best of our knowledge, these are the first estimators that come with theoretical guarantees beyond sets with positive reach. Even for the basic problem of estimating the boundary area of three-dimensional object with reach zero, we are not aware of any other provably correct method.

Theorem (Main Results)**.** *Let X, Y be two compact sets of $\mathbb{R}^d$ and let $\mu \in (0,1], \varepsilon > 0$ be such that $d_H(X,Y) \leq \varepsilon \leq \frac{1}{4}\mathrm{reach}_\mu(X)$. Then we have:*

$$\left| V_i^\varepsilon(Y) - V_i(X^{2\varepsilon}) \right| \leq C_d K(X^{2\varepsilon}) \frac{\varepsilon}{\mu}, \tag{7.1}$$

where C_d is a constant depending on d, $K(X^{2\varepsilon}) := \mathcal{H}^d(X^{2\varepsilon}) + \mathcal{H}^{d-1}(\mathrm{Nor}(X^{2\varepsilon}))$ and $\mathrm{Nor}(X^{2\varepsilon})$ is the unit normal bundle of $X^{2\varepsilon}$.

Further assuming that $\operatorname{reach}(X) > 0$*, we prove a linear rate of convergence for the intrinsic volumes of offsets:*

$$\left|V_i(X) - V_i(X^{2\varepsilon})\right| \leq C_d \left(K(X) + K(X^{2\varepsilon})\right) \frac{\varepsilon}{\mu}. \quad (7.2)$$

It is worth noting that the second claim holds even when $\operatorname{reach}(X)$ is arbitrarily smaller than ε, a case for which, to the best of our knowledge, no quantitative convergence result between the curvatures of $X^{2\varepsilon}$ and X was known. We also conjecture that this claim holds when X is subanalytic. Taken together, the two claims above provide a way to estimate the intrinsic volumes of an unknown shape X with positive reach from a Hausdorff approximation Y. From this point of view, the condition that $\operatorname{reach}(X)$ is positive is not restrictive since sets with positive reach form a dense family of compact subsets for the Hausdorff distance.

A byproduct of our methods is an answer to the second open question asked by Milnor in [Mil94]: in which sense do X and Y have to be close to guarantee that their intrinsic volumes are close? It turns out that the existence of a C^0-controlled homotopy equivalence (see [Cha83] for a related notion) is sufficient, assuming a bound on the volume of the unit normal bundle of both sets. More precisely, say that X and Y are (ε, δ)-*homotopy equivalent* if there exist two continuous maps $f : X \to Y$, $g : Y \to X$ such that $||f - \mathrm{Id}||_\infty \leq \varepsilon$, $||g - \mathrm{Id}||_\infty \leq \varepsilon$ and such that there exist homotopies H_1 (resp. H_2) between $f \circ g$ and Id_Y (resp. $g \circ f$ and Id_X) satisfying $||H_1(t, \cdot) - \mathrm{Id}_Y||_\infty \leq 2\delta$ (resp. $||H_2(t, \cdot) - \mathrm{Id}_X||_\infty \leq 2\delta$) for all $t \in [0, 1]$.

Theorem 7.1 (Intrinsic volumes & (ε, δ)-homotopy equivalence)**.** *Let X and Y be two compact subsets of $\mathbb{R}^d$ with positive reach. If X and Y are (ε, δ)-homotopy equivalent for ε and δ positive, we have:*

$$|V_i(X) - V_i(Y)| \leq C_d \max(\varepsilon, \delta) \left(K(X) + K(Y)\right). \quad (7.3)$$

For our inference problem, a naive approach would be to estimate the intrinsic volumes of X by the intrinsic volumes of small offsets of Y. However, this leads to a trade-off between the bias induced by too large offset parameters and the spurious geometric details that come with small offset parameters. Optimizing this trade-off yields a sublinear rate of convergence. We use the noise-filtering properties of persistent homology to improve over this sublinear behavior by studying the inclusion $Y^\varepsilon \subset Y^{3\varepsilon}$ instead, similar to the usual method for estimating Betti numbers [CL05, CSEH05]. Our approach uses the principal kinematic formula from integral geometry to express the intrinsic volumes as integrals of certain Euler characteristics. We then define our persistent intrinsic volumes by replacing these Euler characteristics with persistent Euler characteristics associated with the pair $Y^\varepsilon \subset Y^{3\varepsilon}$. A stability theorem for image persistence then allows us to prove a linear rate of convergence for our estimators.

7.2 Persistent Intrinsic Volumes

7.2.1 Definition

In this section, we define the persistent intrinsic volumes $V_i^{\varepsilon,R}(Y)$, where ε, R are positive numbers, and state our main results. Recall from Section 3.1.1 that $d_x : z \mapsto ||z - x||$ is the distance to a point $x \in \mathbb{R}^d$, and from Definition 5.11 that for any pair of nested subsets $A \subset B$ of $\mathbb{R}^d$, $\operatorname{dgm}(d_x, A, B)$ is the image persistence diagram of the map d_x induced by the inclusion $A \hookrightarrow B$.

Definition 7.2 (Persistent Intrinsic Volumes)**.** Let $Y \subset \mathbb{R}^d$ be closed, $x \in \mathbb{R}^d$ and $\varepsilon \geq 0$. We let $D_Y^{\varepsilon,x} := \mathrm{dgm}(d_x, Y^\varepsilon, Y^{3\varepsilon})$. When $\varepsilon = 0$, we write $D_Y^x := \mathrm{dgm}(d_{x|Y})$.

— The *persistent Steiner function* Q_Y^ε is defined by:

$$Q_Y^\varepsilon(r) := \int_{x\in\mathbb{R}^d} \chi(D_Y^{\varepsilon,x}(r))\mathrm{d}x.$$

— Given $R > 0$, the persistent Steiner polynomial is the orthogonal projection of Q_Y^ε restricted to $[0, R]$ on the space of polynomials of degree at most d for the scalar product of $L^2([0, R])$.

— Writing the persistent Steiner polynomial as $\sum_{i=0}^d \omega_i V_{d-i}^{\varepsilon,R}(Y)r^i$, the rescaled coefficients $(V_i^{\varepsilon,R}(Y))_{0\leq i\leq d}$ are the *persistent intrinsic volumes* of Y.

Remark 7.3 – When Y is compact and ε is positive, one can always find a finite simplicial complex C such that $Y^\varepsilon \subset C \subset Y^{3\varepsilon}$. By Theorem 5.14, the diagrams $D_Y^{\varepsilon,x}$ inject into D_C^x for every x, implying that their size is bounded by the number of simplices of C. In particular, the persistent Steiner function is locally bounded, and persistent intrinsic volumes are well-defined without any regularity condition on Y.

7.2.2 Bounds on Steiner polynomial

The following is an immediate consequence of Corollary 5.13 and Theorem 5.14.

Proposition 7.4 (Diagram approximation)**.** *Let X, Y be compact subsets of $\mathbb{R}^d$ and $\varepsilon, \mu > 0$ be such that $d_H(X, Y) \leq \varepsilon \leq \frac{1}{4}\,\mathrm{reach}_\mu(X)$. For all $x \in \mathbb{R}^d$, we have:*

— $d_B(D_Y^{\varepsilon,x}, D_{X^{2\varepsilon}}^x) \leq \frac{2\varepsilon}{\mu}$;

— *$D_Y^{\varepsilon,x}$ injects into $D_{X^{2\varepsilon}}^x$.*

We use the previous lemmas to establish a linear rate of convergence of Q_Y^ε to $Q_{X^{2\varepsilon}}$ over $[0, R]$ for any positive real R.

Theorem 7.5 (Estimating the Steiner polynomial in the L^1 norm)**.** *Let $X, Y \subset \mathbb{R}^d$ be compact sets and $\varepsilon, \mu > 0$ be such that $d_H(X, Y) \leq \varepsilon \leq \frac{1}{4}\,\mathrm{reach}_\mu(X)$. Then we have:*

$$||Q_Y^\varepsilon - Q_{X^{2\varepsilon}}||_{L^1([0,R])} \leq \frac{4\varepsilon}{\mu}\int_{\mathbb{R}^d} N_0^R(D_{X^{2\varepsilon}}^x)\mathrm{d}x. \tag{7.4}$$

Proof.

The result is an immediate consequence of Proposition 7.4 along with the second case of the χ-averaging lemma Lemma 5.16.

□

The next lemma relates the integral term in the bound of Theorem 7.5 to the geometry of $X^{2\varepsilon}$.

Lemma 7.6 (Number of critical points of d_x)**.** *Let $X \subset \mathbb{R}^d$ be a compact set and $\varepsilon, \mu > 0$ be such that $0 < 2\varepsilon < \mathrm{reach}_\mu(X)$. For any $R > 0$, define $\phi_{X^{2\varepsilon},R} : \mathrm{Nor}(X^{2\varepsilon}) \times [0, R] \to \mathbb{R}^d, (y, n, t) \mapsto y + tn$.*

Then $\mathcal{H}^d$-almost everywhere in x,

— $N_0^R(D_{X^{2\varepsilon}}^x) \leq \mathbb{1}_{X^{2\varepsilon}}(x) + \mathrm{card}\ \phi_{X^{2\varepsilon},R}^{-1}(x)$.

— card $\phi^{-1}_{X^{2\varepsilon},R}(x)$ *is finite;*
Moreover, we have:

$$\int_{\mathbb{R}^d} N_0^R(D^x_{X^{2\varepsilon}})\,\mathrm{d}x \le \mathrm{Vol}(X^{2\varepsilon}) + M_R(X^{2\varepsilon}).$$

Proof.

The map $\phi_{X^{2\varepsilon},R}$ is Lipschitz between two rectifiable sets of dimension d, $\mathbb{R}^d$ and $\mathrm{Nor}(X^{2\varepsilon}) \times [0,R]$. By the Co-area formula (Theorem 2.13), we have:

$$\int_{\mathbb{R}^d} \mathrm{card}\ \phi^{-1}_{X^{2\varepsilon},R}(x)\,\mathrm{d}x = \int_{\mathrm{Nor}(X^{2\varepsilon})\times[0,R]} J_d\phi_{X^{2\varepsilon},R}(y,n,t)\,\mathrm{d}\mathcal{H}^d(y,n,t). \tag{7.5}$$

and the right-hand side quantity is $M_R(X^{2\varepsilon})$. This yields the integral inequality from the first point. Since $\phi_{X^{2\varepsilon},R}$ is Lipschitz, its Jacobian is bounded on the compact set $\mathrm{Nor}(X^{2\varepsilon}) \times [0,R]$, the right-hand side of Equation (7.5) is finite, and thus the $\mathrm{Card}\,\phi^{-1}_{X^{2\varepsilon},R}(x)$ is finite almost everywhere.

Recall that by Theorem 3.39, $X^{2\varepsilon}$ is a complementary regular set. By Theorem 6.25, $\mathcal{H}^d$-almost everywhere in x, the map $(d_x^2)_{|X^{2\varepsilon}}$ is Morse in the generalized sense. Furthermore, its set of critical points within the ball $B(x,R)$ of radius R centered in x is exactly $\phi^{-1}_{X^{2\varepsilon},R}$. Adding the eventual case where $x \in X^{2\varepsilon}$, this shows that the number of topological events in the filtration $(X^{2\varepsilon} \cap B(x,t))_{t\in\mathbb{R}}$ with filtration value less or equal than R is $\mathbb{1}_{X^{2\varepsilon}}(x) + \mathrm{card}\ \phi^{-1}_{X^{2\varepsilon},R}(x)$. In case $(d_x^2)_{|X^{2\varepsilon}}$ is Morse, there is exactly one homological event per critical point (cf. Chapter 4), and the number of bars $N_0^R(D^x_{X^{2\varepsilon}})$ is bounded by $\mathbb{1}_{X^{2\varepsilon}}(x) + \mathrm{card}\ \phi^{-1}_{X^{2\varepsilon},R}(x)$.

□

7.2.3 Bounds on intrinsic volumes

We are now in position to prove our main theorems.

Theorem 7.7 (Linear convergence of the persistent intrinsic volumes)**.** *Let X, Y be compact subsets of $\mathbb{R}^d$ and let $\varepsilon, \mu > 0$ be such that $d_H(X,Y) \le \varepsilon \le \frac{1}{4}\,\mathrm{reach}_\mu(X)$. There exist constants $P(i,d)$ such that, for any $R > 0$, we have:*

$$\left|V_i^{\varepsilon,R}(Y) - V_i(X^{2\varepsilon})\right| \le \frac{\varepsilon P(i,d)}{\mu R^{i+1}}\left(\mathrm{Vol}(X^{2\varepsilon}) + M_R(X^{2\varepsilon})\right). \tag{7.6}$$

Proof.

Applying Theorem 7.5 and Lemma 7.6 we obtain for any positive R:

$$||Q_Y^\varepsilon - Q_{X^{2\varepsilon}}||_{L^1([0,R])} \le \frac{4\varepsilon}{\mu}(\mathrm{Vol}(X^{2\varepsilon}) + M_R(X^{2\varepsilon})).$$

Let $i \le j$ and let $(L_n)_{n\in\mathbb{N}}$ be the Legendre polynomials on $[0,1]$. We obtain after renormalization and reparametrization an orthonormal basis of the polynomials of degree at most d on $[0,R]$ formed by the following polynomials:

$$P_j^R(X) := \sqrt{\frac{2j+1}{R}}\,L_j\left(\frac{X}{R}\right).$$

The fact that $||L_i||_{\infty,[0,1]} \leq 1$ yields $\left|\left|P_j^R\right|\right|_{\infty,[0,R]} \leq \sqrt{\frac{2j+1}{R}}$ for any positive R. Denoting by $c_i(P_j^R)$ the i-th coefficient of P_j^R, we have for any $i \leq j$:

$$\left|c_i(P_j^R)\right| = \frac{1}{R^{i+\frac{1}{2}}}\sqrt{2j+1}\binom{j}{i}\binom{i+j}{i}$$

and $c_i(P_j^R) = 0$ otherwise. Now decomposing Q_Y^ε in the basis P_j^R and taking the coefficient of X^i yields

$$V_i^{\varepsilon,R}(Y) = \frac{1}{\omega_{d-i}}\int_0^R \int_{\mathbb{R}^d} \chi(D_Y^{\varepsilon,x}(r)) \sum_{j=i}^{d} P_j^R(r) c_i(P_j^R)\,\mathrm{d}x\,\mathrm{d}r.$$

Since we have

$$\left|\left|P_j^R c_i(P_j^R)\right|\right|_{\infty,[0,R]} \leq \frac{(2j+1)}{R^{i+1}}\binom{j}{i}\binom{i+j}{i},$$

we see that:

$$\left|V_i^{\varepsilon,R}(Y) - V_i(X^{2\varepsilon})\right| \leq \frac{1}{R^{i+1}}\,||Q_Y^\varepsilon - Q_{X^{2\varepsilon}}||_{L^1([0,R])}\,\frac{1}{\omega_{d-i}}\sum_{j=i}^{d}(2j+1)\binom{j}{i}\binom{i+j}{i}.$$

We now obtain the desired inequality by applying Lemma 7.6 and Theorem 7.5 and putting

$$P(i,d) := \frac{4}{\omega_{d-i}}\sum_{j=i}^{d}(2j+1)\binom{j}{i}\binom{i+j}{i}.$$

□

Remark 7.8 – Our approach consists in bounding the L^1 norm of $Q_Y^\varepsilon - Q_{X^{2\varepsilon}}$ (Theorem 7.5 and Lemma 7.6):

$$||Q_Y^\varepsilon - Q_{X^{2\varepsilon}}||_{L^1([0,R])} \leq \frac{4\varepsilon}{\mu}\left(\mathrm{Vol}(X^{2\varepsilon}) + M_R(X^{2\varepsilon})\right)$$

and retrieving quantities that are close to the rescaled coefficients $(V_i(X^{2\varepsilon}))_{0\leq i\leq d}$ of the polynomial $Q_{X^{2\varepsilon}}$ from the persistent Steiner function Q_Y^ε, which is in $L^2([0,R])$ but is not a priori a polynomial. We chose in Definition 7.2 the rescaled coefficients $V_i^{\varepsilon,R}(Y)$ of its orthogonal projection on the space $\mathbb{R}_d[X]$ of polynomials of degree at most d. From this definition and from the explicit formulas for Legendre polynomials (see the proof of Theorem 7.7) we obtain:

$$\left|V_i^{\varepsilon,R}(Y) - V_i(X^{2\varepsilon})\right| \leq \frac{P(i,d)}{4R^{i+1}}\,||Q_Y^\varepsilon - Q_{X^{2\varepsilon}}||_{L^1([0,R])}\,. \tag{7.7}$$

Taking the i-th coefficients of orthogonal projections is one way to build linear forms $\phi_{i,d}^R : L^2([0,R]) \to \mathbb{R}$ such that, restricted to the space $\mathbb{R}_d[X]$ of polynomials with degree at most d, $\phi_{i,d}^R$ is the map $\sum_{0\leq j\leq d} a_j X^j \mapsto a_i$. In fine, we defined $V_i^{\varepsilon,R}(Y)$ as $\phi_{i,d}^R(Q_Y^\varepsilon)/\omega_{d-i}$. By Equation (7.7) the linear forms $\phi_{i,d}^R$ are $\omega_{d-i}P(i,d)/4R^{i+1}$-Lipschitz for the L^1 norm over $[0,R]$. The bound we infer on $V_i^{\varepsilon,R}(Y) - V_i(X^{2\varepsilon})$ from $||Q_Y^\varepsilon - Q_{X^{2\varepsilon}}||_{L^1([0,R])}$ comes from a bound on the Lipschitz constant of $\phi_{i,d}^R$ obtained using Legendre polynomials. Yet, by the Hahn-Banach extension theorem, the best Lipschitz constant possible for a linear form $\phi^R_{i,d}$ whose restriction to $\mathbb{R}_d[X]$ is the i-th coefficient map is exactly the Lipschitz constant of $(\phi^R_{i,d})_{|\mathbb{R}_d[X]}$ which we denote by $l^R_{i,d}$. When $i = d \geq 0$ by classical work in optimization (e.g., [Ric64, 4.9, p. 117]), we have:

$$l^R_{d,d} = \frac{4^d}{R^{d+1}} \leq \frac{(2d+1)}{R^{d+1}}\binom{2d}{d}$$

where the right-hand side is the bound on the Lipschitz constant we explicitly obtained by the Legendre polynomials method. This shows that there exist alternate ways of defining the persistent intrinsic volumes from the persistent Steiner function that lead to a strictly better bound than the one in Theorem 7.7.

Theorem 7.9 (Convergence of the intrinsic volumes of an offset)**.** *Let* $X \subset \mathbb{R}^d$ *and* $\mu, \varepsilon > 0$ *be such that* $\varepsilon < \frac{1}{2}\operatorname{reach}_\mu(X)$. *If* $\operatorname{reach}(X) > 0$, *we have:*

$$\left|V_i(X) - V_i(X^{2\varepsilon})\right| \leq \frac{\varepsilon P(i,d)}{\mu R^{i+1}}\left(\operatorname{Vol}(X) + \operatorname{Vol}(X^{2\varepsilon}) + M_R(X) + M_R(X^{2\varepsilon})\right).$$

Proof.

Let $x \in \mathbb{R}^d$ and $\delta > 0$. For any sufficiently small $\sigma > 0$, set $c = \frac{2\varepsilon-\delta}{\mu-\sigma}$. By the μ-reach hypothesis, there exists a continuous flow between $X^{2\varepsilon}$ and X^δ which is c-Lipschitz in the time parameter thanks to **??**. This yields the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, dashed] & H_*(X^{2\varepsilon}_a) \arrow[r] \arrow[dr] & H_*(X^{2\varepsilon}_{a+c}) \arrow[r] \arrow[dr] & H_*(X^{2\varepsilon}_{a+2c}) \arrow[r, dashed] & \cdots \\
\cdots \arrow[r, dashed] & H_*(X^{\delta}_a) \arrow[r] \arrow[u] & H_*(X^{\delta}_{a+c}) \arrow[r] \arrow[u] & H_*(X^{\delta}_{a+c}) \arrow[r, dashed] \arrow[u] & \cdots
\end{tikzcd}$$

This gives a $\frac{2\varepsilon-\delta}{\mu-\sigma}$ interleaving between the two persistence modules, which implies $d_B(D^x_{X^{2\varepsilon}}, D^x_{X^\delta}) \leq \frac{2\varepsilon}{\mu}$ by letting σ go to zero. Using the same reasoning as in the proof of Theorem 7.7, except that we have to use the first inequality of Lemma 5.16 since a priori there is no injection between the two diagrams D^x_X and $D^x_{X^{2\varepsilon}}$, we get for any positive R:

$$\left|V_i(X^{2\varepsilon}) - V_i(X^\delta)\right| \leq \frac{\varepsilon P(i,d)}{\mu R^{i+1}} \int_{\mathbb{R}^d} (N^R_0(D^x_{X^{2\varepsilon}}) + N^R_0(D^x_{X^\delta}))\,\mathrm{d}x. \tag{7.8}$$

By Lemma 7.6:

$$\left|V_i(X^{2\varepsilon}) - V_i(X^\delta)\right| \leq \frac{\varepsilon P(i,d)}{\mu R^{i+1}}\left(\operatorname{Vol}(X^{2\varepsilon}) + M_R(X^{2\varepsilon}) + \operatorname{Vol}(X^\delta) + M_R(X^\delta)\right). \tag{7.9}$$

We already know that $\operatorname{Vol}(X^\delta)$ converges to $\operatorname{Vol}(X)$. Let us prove the first statement and assume that $\operatorname{reach}(X) > 0$. By the tube formula, the intrinsic volumes of X^δ are the (scaled) coefficients of the Steiner polynomial of X translated by δ, and thus $V_i(X^\delta)$ converges to $V_i(X)$ when δ goes to zero. We are left to prove that $\lim_{\delta\to 0} M_R(X^\delta) = M_R(X)$ to obtain the desired inequality. If $0 \leq \delta < \operatorname{reach}(X)$, writing $h_\delta : (x,n) \mapsto (x+\delta n, n)$, we have:

$$\operatorname{Nor}(X^\delta) = \bigcup_{(x,n)\in\operatorname{Nor}(X)} \{x+\delta n\} \times \{n\} = h_\delta(\operatorname{Nor}(X)).$$

Denoting by $\kappa_i(z,n)$ the principal curvatures of X at a regular pair (z,n) of X and $\kappa_{\delta,i}(z',n')$ the principal curvatures of X^δ at a regular pair (z',n') of $\mathrm{Nor}(X^\delta)$, we have:

$$\kappa_{\delta,i}(z+\delta n, n) = f_\delta(\kappa_i(z,n)),$$

where $f_\delta(s) = \frac{s}{1+\delta s}$. Since $\kappa_i(x,n) \geq -\frac{1}{\mathrm{reach}(X)}$ the quantities $f_\delta(\kappa_i(x,n))$ are well-defined for any $\delta < \mathrm{reach}(X)$. The change of variable formula yields for any positive t:

$$\begin{aligned} M_R(A^\delta) &= \int_{\mathrm{Nor}(A^\delta)} \prod_{i=1}^{d-1} \frac{|1+t\kappa_{\delta,i}|}{\sqrt{1+\kappa_{i,\delta}^2}} \,\mathrm{d}\mathcal{H}^{d-1}(x,n) \\ &= \int_{\mathrm{Nor}(A)} J_{d-1}(h_\delta) \prod_{i=1}^{d-1} \frac{|1+tf_\delta(\kappa_i)|}{\sqrt{1+f_\delta(\kappa_i)^2}} \,\mathrm{d}\mathcal{H}^{d-1}(x,n). \end{aligned}$$

Since $h_\delta - \mathrm{Id}$ is δ-Lipschitz, we have

$$(1-\delta)^{d-1} \leq J_{d-1}(h_\delta(x,n)) \leq (1+\delta)^{d-1}.$$

Hence, as δ tends to 0, $J_{d-1}h_\delta$ tends to 1 and $f_\delta(\kappa)$ tend to κ for all $\kappa \in \mathbb{R}$. Lebesgue dominated convergence theorem gives:

$$\begin{aligned} \lim_{\delta\to 0} M_R(X^\delta) &= \int_0^R \int_{\mathrm{Nor}(A)} \prod_{i=1}^{d-1} \frac{|1+t\kappa_i|}{\sqrt{1+\kappa_i^2}} \,\mathrm{d}\mathcal{H}^{d-1}(x,n)\,\mathrm{d}t \\ &= M_R(X). \end{aligned}$$

□

Combining Theorem 7.9 and Theorem 7.7 gives us a way to estimate $V_i(X)$ from Y:

Corollary 7.10 (Linear rate of approximation for the intrinsic volumes)**.** *Let* $X, Y \subset \mathbb{R}^d$ *and* $\mu, \varepsilon > 0$ *be such that* $d_H(X,Y) \leq \varepsilon < \frac{1}{4}\,\mathrm{reach}_\mu(X)$*. If* $\mathrm{reach}(X) > 0$*, we have:*

$$\left|V_i(X) - V_i^{\varepsilon,R}(Y)\right| \leq \frac{\varepsilon P(i,d)}{\mu R^{i+1}} \left(\mathrm{Vol}(X) + 2\,\mathrm{Vol}(X^{2\varepsilon}) + M_R(X) + 2M_R(X^{2\varepsilon})\right). \tag{7.10}$$

We prove Theorem 7.1 using similar methods as before, except that the interleavings between persistence modules stem from the existence of a (ε,δ)-homotopy equivalence.
Proof of Theorem 7.1.

Let x be any point of $\mathbb{R}^d$ and let $M_a := M \cap B(x,a)$ for any subset M of $\mathbb{R}^d$ and $a \in \mathbb{R}$. The map $H_1 : [0,1] \times Y \to Y$ is a homotopy between $f \circ g$ and Id_Y, and by assumption its restriction to Y_a is a continuous map $H_1^a : [0,1] \times Y_a \to Y_{a+2\delta}$. This yields a homotopy between $f \circ g : Y_a \to Y_{a+2\delta}$ and the inclusion $Y_a \hookrightarrow Y_{a+2\delta}$. Letting δ' be a positive real such that $2\varepsilon + 2\delta' \geq 2\delta$, we obtain this commutative diagram of continuous maps:

$$\begin{tikzcd} & X_{a+\varepsilon} \arrow[r, hook] & X_{a+\varepsilon+\delta'} \arrow[dr, "g"] \arrow[drr, "\psi"] & & \\ Y_a \arrow[ur, "f"] \arrow[urr, "\phi"'] & & & Y_{a+2\varepsilon+\delta'} \arrow[r, hook] & Y_{a+2\varepsilon+2\delta'} \end{tikzcd}$$

Since the same holds symmetrically for H_2^a, we can apply the homology functor to obtain the following commutative diagram thanks to the homotopy between $\psi \circ \phi$ (resp. $\phi \circ \psi$) and the inclusion $Y_a \hookrightarrow Y_{a+2\varepsilon+2\delta'}$ (resp. $X_a \hookrightarrow X_{a+2\varepsilon+2\delta'}$):

$$\begin{tikzcd}
{} \arrow[r, dashed] & H_*(X_a) \arrow[r] \arrow[dr] & H_*\left(X_{a+\varepsilon+\delta'}\right) \arrow[r] \arrow[dr] & H_*\left(X_{a+2\varepsilon+2\delta'}\right) \arrow[r, dashed] & {} \\
{} \arrow[r, dashed] & H_*(Y_a) \arrow[r] \arrow[ur] & H_*\left(Y_{a+\varepsilon+\delta'}\right) \arrow[r] \arrow[ur] & H_*\left(Y_{a+2\varepsilon+2\delta'}\right) \arrow[r, dashed] & {}
\end{tikzcd}$$

Optimizing on δ', this yields $d_B(D_X^x, D_Y^x) \leq \max(\varepsilon, \delta)$ for any $x \in \mathbb{R}^d$. Now following the same line of reasoning as the proof of Theorem 7.9, first bounding the L^1 norm of $Q_X - Q_Y$, and then retrieving its coefficients, we obtain the bound:

$$|V_i(X) - V_i(Y)| \leq \frac{\max(\varepsilon, \delta) P(i, d)}{R^{i+1}} \left(\mathrm{Vol}(X) + M_R(X) + \mathrm{Vol}(Y) + M_R(Y)\right).$$

□

Remark 7.11 – The bounds obtained in Theorem 7.7, Theorem 7.9, Corollary 7.10 and Theorem 7.1 depend on the parameter R and explode when $R \to 0$ or ∞. While this is not critical for the asymptotic rate, it could be interesting to find ways of guessing good values of R a priori. The main results stated in the introduction are obtained taking $R = 1$ and using Proposition 6.15.

7.2.4 Optimality of the linear rate

When the class of approximated sets contain non-smooth sets, the rate of convergence of an estimator for intrinsic volumes cannot be better than linear. To see it, let Y be a point cloud on a 2D grid, on a square, with neighbours at distance ε of one another. It is straightforward that Y lies at Hausdorff distance to the square X less than $\text{‰}(eps)$, but also at distance $O(\varepsilon)$ to the square with a truncated triangle of length ε (X' on Figure 7.1). However, the codimension one intrinsic volume of X and X' are different by a quantity proportional to ε, showing the desired result. This reasoning easily extends to any dimension.

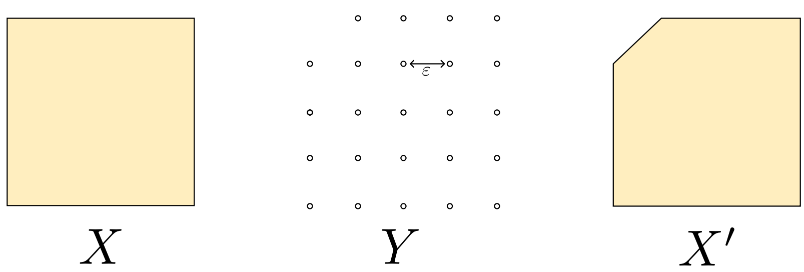


Figure 7.1 – The rate of convergence with respect to the Hausdorff distance is optimal cannot be better than linear.

7.2.5 Computing persistent intrinsic volumes

We conclude this paper by discussing how one may compute our estimators in practice, assuming Y is given as a finite set of points. From the proof of Theorem 7.7 the $V_i^{\varepsilon,R}(Y)$ are given by:

$$V_i^{\varepsilon,R}(Y) = \int_0^R \int_{\mathbb{R}^d} \chi(D_Y^{\varepsilon,x}(r)) S_{i,d}(r)\,\mathrm{d}x\,\mathrm{d}r,$$

where $S_{i,d}$ is a polynomial.

Computing this integral exactly would involve computing a d-dimensional family of persistence diagrams, which is computationally daunting. We may however easily approximate $V_i^{\varepsilon,R}(Y)$ with arbitrary accuracy using a Monte-Carlo method by sampling points x uniformly in the R-offset of Y and computing the image persistence diagrams of their distance function d_x using e.g., [BS23]. The number of trials required to reach accuracy δ is of the order of δ^{-2} times the variance of the random variable $V = \int_0^R \chi(D_Y^{\varepsilon,x}(r)) S_{i,d}(r)\,\mathrm{d}r$.

The variance of V is bounded by $\sup |V|^2$ and we have:

$$\sup |V| \leq \sup_{x\in\mathbb{R}^d} N_0^R(D_Y^{\varepsilon,x}) \sup_{r\in[0,R]} R S_{i,d}(r)$$

that is, a polynomial in R times the maximum size of the image persistence diagram of d_x. Now by Proposition 7.4, the sizes of these image persistence diagrams are at most the sizes of the persistence diagrams of $d_{x_{|X^{2\varepsilon}}}$.

One situation where we can uniformly bound the size of these diagrams is when X is a semi-algebraic set, since by Thom-Milnor [Mil64] the number of critical points of $d_{x_{|X^{2\varepsilon}}}$ is bounded by a function of the degrees of the polynomial equalities and inequalities defining X. Another bound is provided by Remark 7.3. This bound is pessimistic in general as the smallest simplicial complex sandwiched between Y^ε and $Y^{3\varepsilon}$ may have numerous simplices, yet one expects the number of critical points of $d_{x_{|C}}$ to be much smaller on average. We leave more in-depth investigations of the computational aspects of our method for future work.

7.3 Validity of the approximation results among broader classes of sets

Interestingly enough, along the way to Theorem 7.9, we obtained bounds on $||Q_{A^{2\varepsilon}} - Q_{A^\delta}||_{1,R}$ when $\max(\delta/2, d_H(A,B)) \leq \varepsilon \leq \mathrm{reach}_\mu(A)/4$. Our method is applicable to some cases where $A^{2\varepsilon}$ is outside the reach of A^δ , that is, in a case where the normals of $A^{2\varepsilon}$ are not related in a straightforward manner to those of A^δ. This illustrates the robustness of persistent geometry compared to more classical methods of estimating the normal cycle by comparing normals, such as [LRT22], which require the approximating set to be within the reach of the approximated set.

However, to ensure that the last bound $||Q_{A^{2\varepsilon}} - Q_{A^\delta}||_{1,R}$ actually transfers to a bound involving the Steiner polynomial of A when $\delta \to 0$, we assumed that A had positive reach. As we have seen in Chapter 6, there are many kinds of subsets admitting curvatures which have zero reach, with examples as simple as non-convex polyhedra. The aim of this section is to discuss how our quantitative results can still hold in various contexts. First remark that the R-mass of a current can be extended to sets admitting a normal cycle.

7.3.1 R-mass of a set admitting a normal cycle

Definition 7.12 (R-Mass of a set)**.** Let $R > 0$. The R-mass is extended to any set $X \subset \mathbb{R}^d$ admitting a normal cycle N_X by putting

$$M_R(X) \coloneqq \int_0^R M(N_X \llcorner P_\phi(t))\,\mathrm{d}t \tag{7.11}$$

where $P_\phi : t \mapsto \sum_{i=0}^{d-1} t^i (d-i)\omega_{d-i}\phi_{d-i} \in \bigwedge_{d-1}(\mathbb{R}^d \times \mathbb{S}^{d-1})$ is a $(d-1)$ differential form, polynomial in t involving the Lipschitz-Killing forms, with action on $(d-1)$-covectors

$$\langle a_1 \wedge \cdots \wedge a_{d-1}, P_\phi(t)(x,n)\rangle \coloneqq \det(\pi_0(a_1) + t\pi_1(a_1), \ldots, \pi_0(a_{d-1}) + t\pi_1(a_{d-1}), n).$$

Thanks to the factorized expression, we can see that $||P_\phi(t)||_\infty \leq (1+t^2)^{(d-1)/2}$, thereby showing that the bound of Proposition 6.15 holds in the context of normal cycles.

Proposition 7.13 (Bounds on R-mass)**.** *Let X be a set admitting a normal cycle. Then for every $R > 0$, we have*

$$M_R(X) \leq \left(\int_0^R (1+t^2)^{(d-1)/2}\,\mathrm{d}t\right) M(N_X). \tag{7.12}$$

Remark 7.14 – A part of the proof of Theorem 7.9 was to show that when X has positive reach, $M_R(X^\delta)$ was converging to $M_R(X)$ when δ tends to 0^+ for any positive R. Using the language of currents, the proof becomes straightforward. Let $0 < \delta < \mathrm{reach}(X)$. Then $N_{X^\delta} = (h_\delta)_\#(N_X)$. Since h_δ is $(1+\delta)$-Lipschitz, for any $0 \leq t \leq R$, we have $M(N_{X^\delta} \llcorner P_\phi(t)) \leq (1 + O(\delta))M(N_X \llcorner P_\phi(t))$. We obtain the same inequality with X, X^δ switched considering that $N_X = (h_{-\delta})_\#(N_{X^\delta})$.

7.3.2 Generalization of the convergence results

To generalize Theorem 7.9 to sets with $\mathrm{reach}\ 0$ with our method, we have to assume that X admits a normal cycle and that it has a positive μ-reach. We need the Steiner polynomial of small offsets Q_{X^δ} to converge to Q_X when δ tends to 0. This is equivalent to the convergence of the intrinsic volumes of X^δ to those of X. However, the existence of a link between the normal cycle of X^δ, which is defined as the integration over the submanifold $\mathrm{Nor}(X^\delta)$, and the normal cycle of X defined axiomatically is not clear. If we assume that N_{X^δ} converges to N_X in the flat norm, the weak convergence implies

$$V_i(X) = N_X(\phi_i) = \lim_{\delta\to 0^+} N_{X^\delta}(\phi_i) = \lim_{\delta\to 0^+} V_i(X^\delta). \tag{7.13}$$

The bounds we obtained on $||Q_{A^{2\varepsilon}} - Q_{A^\delta}||_{1,R}$ involves $\mathrm{Vol}(X^\delta)$, which converges to $\mathrm{Vol}(X)$ when δ tends to 0, and $M_R(X^\delta)$. When δ tends to 0, this bound becomes $\liminf_{\delta\to 0} M_R(X^\delta)$. One can wonder if this quantity is finite, and if so, if it can be replaced by one only involving the set X itself.

Without further informations, Theorem 7.9 generalizes as follows:

Theorem 7.15 (Convergence of the intrinsic volumes of an offset)**.** *Let $X \subset \mathbb{R}^d$ and $\mu, \varepsilon > 0$ be such that $\varepsilon < \frac{1}{2}\operatorname{reach}_\mu(X)$. Assume the normal cycles X^δ converge to those of X as δ tends to zero. Then we have*

$$\left|V_i(X) - V_i(X^{2\varepsilon})\right| \leq \frac{\varepsilon P(i,d)}{\mu R^{i+1}} \left(\operatorname{Vol}(X) + \operatorname{Vol}(X^{2\varepsilon}) + \liminf_{\delta \to 0} M_R(X^\delta) + M_R(X^{2\varepsilon})\right).$$

Now we discuss when this convergence is true. We begin by a counter-example to illustrate how N_{X^δ} and $M_R(X^\delta)$ might diverge.

7.3.3 Counter-example to the convergence of intrinsic volumes

The assumption that X has a positive μ-reach is not enough to guarantee that X admits a normal cycle, as we have seen in Section 3.2.2 that such a set might have infinite area boundary.

Let us now consider a sequence of positive, non-increasing angles $(\theta_i)_{i\in\mathbb{N}}$ bounded by $\pi/4$, such that the sum $\sum_i \theta_i^2$ converges but not $\sum_i 2^i\theta_i$. The limit set $X \subset \mathbb{R}^2$ has positive μ-reach for some $\mu > 0$, a finite length $\mathcal{H}^1(X)$ and is a non-intersecting curve (thus $\chi(X) = 1$). If the normal cycle of X were to exist, these last two quantities could be considered respectively $V_1(X)$ and $V_0(X)$ by the classical interpretation of the intrinsic volumes.

Small offsets X^δ are complementary regular sets, and admit as such normal cycles N_{X^δ}. They can be obtained as the functionals integrating on the Lipschitz submanifold $\operatorname{Nor}(X^\delta)$. By Theorem 3.32, this coincides with the graph of the normalized Clarke gradient of d_X, on the δ-level set:

$$\operatorname{graph}_\delta(d_X) := \bigcup_{d_X(x)=\delta} \{x\} \times \nu(\partial^* d_X(x))$$

where $\nu : \mathbb{R}^d \setminus \{0\} \to \mathbb{S}^{d-1}$ is the normalizing map. By the assumption on $\sum_i 2^i\theta_i$, the normal cycles N_{X^δ} do not converge to any current in the flat sense, as $|N_{X^\delta}(\phi_0)|$ diverges to ∞ when $\delta \to 0$ per Proposition 3.24. When $d = 2$, the differential form $P_\phi(t)$ can be written $a\phi_0 + tb\phi_1$ for some positive constants a, b, which implies that $M_R(X^\delta)$ diverges to ∞ as δ tends to 0.

7.3.4 Distance auras

The fact that N_{X^δ} is the integration over the oriented Lipschitz submanifold $\operatorname{graph}_\delta(d_X)$ was formalized and generalized by Fu [Fu94, Definition 1.1] using the following terminology of *auras*.

Definition 7.16 (Monge-Ampère maps and auras)**.** A locally Lipschitz, proper map $f : \mathbb{R}^d \to \mathbb{R}$ is said to be *Monge-Ampère* when there exists an integral Legendrian cycle T representing the integration over the graph of $\partial^* f$, i.e., for any smooth function $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

$$T(\phi \cdot \pi_0^*(\mathrm{d}x)) = \int_{\mathbb{R}^d} \phi(x, \nabla f(x))\,\mathrm{d}x \tag{7.14}$$

where $\pi_0^*(\mathrm{d}x)$ is pullback of the canonical volume form of $\mathbb{R}^d$ by π_0. When it exists, the current T is uniquely defined. Any $C^{1,1}$, semi-concave or semi-convex map is Monge-Ampère [Fu89b, 2.4].

An *aura* for X is a proper, non-negative Monge-Ampère map such that $f^{-1}(0) = X$. It is said to be *non-degenerate* when there exists a $\delta > 0$ such that on $f^{-1}(0, \delta]$, we have $\Delta(\partial^* f(x)) \geq \delta$.

Every set admitting a non-degenerate aura admits a normal cycle defined as the limit (in the flat norm) of the current integrating over the graph of $\nu(\partial^* f)$ over the boundary of the oriented Lipschitz submanifold $f^1(-\infty, r]$, thereby showing that this limit does not depend on the choice of the non-degenerate aura f [Fu94, Section 3]. The Monge-Ampère condition ensures that these currents are all Legendrian cycles, and that there is indeed convergence. The previous counter-example of sets with positive μ-reach with no normal cycle shows that d_X might be non-degenerate and not Monge-Ampère.

Yet by conception, d_X is a good candidate to be an aura for X as it is proper, non-negative and satisfy $d_X^{-1}(0) = X$. The non-degeneracy condition for d_X is equivalent to having $\mathrm{reach}_\mu(X) > 0$ for some $\mu > 0$. We can define "distance auras" so that Theorem 7.15 apply.

Definition 7.17 (Distance auras)**.** We say that a compact X of $\mathbb{R}^d$ admits a *distance aura* when X admits a normal cycle N_X and when there exists a $\mu \in (0, 1]$ such that
— $\mathrm{reach}_\mu(X) > 0$;
— The currents N_{X^δ} converge to N_X in the flat norm as δ tends to 0.
— $\liminf_{\delta \to 0} M(N_{X^\delta}) < \infty$.
When the condition $\mathrm{reach}_\mu(X) > 0$ is replaced by $wfs(X) > 0$, we say that X admit a *weak distance aura*.

We have already seen that compact sets with positive reach have a distance aura, with the added property that $M_R(X) = \lim_{\delta \to 0^+} M_R(X^\delta)$. Subanalytic sets with positive μ-reach for some $\mu > 0$ form another class of sets admitting a distance aura.

Proposition 7.18 (Subanalytic sets admit a weak distance aura)**.** *Every compact subanalytic set has a weak distance aura.*

Proof.

This is a rephrasing of Fu's main result in [Fu94] : subanalytic sets have necessarily a positive weak feature size as the set of critical values of d_X is locally finite; and N_{X^δ} converges to N_X as d_X is a subanalytic aura for X. By subanalycity of the map $\delta \to M_R(X^\delta)$, the liminf is necessarily finite.

□

It is not known whether other sets which have been proven to admit normal cycles and have a positive μ-reach admit a distance aura. For example, a WDC-set X can locally be written as a sublevel set of a DC function at a weak regular value. Determining whether this translates to d_X is not clear, as Fu, Rataj and Pokorny pointed out in [FPR17, Section 5.1]. Pokorny and Zajicek established that it was true for $d = 2$ in [PZ21], while the problem remains open when $d \geq 3$.

CHAPTER 8

Conclusion and future prospects

Donnez-moi de la terre... Donnez-moi de la terre à contrer !

Conclusion. In this thesis, we have used notions from different areas of mathematics to obtain a result in geometric inference concerning the so-called intrinsic volumes. Following the principles of what we call persistent geometry, we have exploited the relationships between the topology of an object and its geometry, obtained thanks to integral geometric formuals (Chapter 6). Thanks to a quantitative result connecting sublevel sets of Lipschitz functions (Chapter 2), to the links between an object satisfying weak regularity conditions and its distance function (Chapter 3), as well as the use of image persistence modules (Chapter 5), these relationships enabled us to transfer the filtering properties of persistence homology to the geometric inference framework (Chapter 7). This required a bound on the number of critical points of distance to a point functions in $\mathbb{R}^d$, which we obtained by constructing a Morse theory adapted to tubular neighborhoods (Chapter 4).

Future prospects. However, we only estimated the intrinsic volumes of an object, which are global quantities obtained from the curvatures, and not the curvatures themselves. Indeed, we were estimating the total measure of the curvature measures, but not where the mass of these measures lies. We conjecture that by exploiting more finely the properties of the persistence diagrams of the distance-to-a-point functions, we can localize the curvature measures, leading to their retrieval up to an error that is proportional to the square root of the Hausdorff distance in the Wasserstein distance. This would corroborate the results of [CCM10], while generalizing them to the case where $d_H(X, Y)$ exceeds the reach of the approximated object. The challenge resides in circumventing the lack of well-defined projection map onto the closest point above the reach, which we believe can be done with the help of image persistence. Estimating the normal cycle of the approximated object is more complex than estimating curvature measures. Following the previous conjecture, it seems reasonable to think that the normal cycle can be estimated at a rate of $O(\sqrt{d_H(X, Y)})$ for the flat norm.

On another note, our works focused on the inference of intrinsic volumes but did not explore the full framework of persistent geometry. Even restricted to the sole use of the principal kinematic formula as a link between topology and geometry, our method yields different estimators depending on a parameter $R > 0$. It would be interesting to focus on the concepts of persistent geometry per se to see to what extent we can make sense of the geometry of a nested pair, of which we have only explored one facet.

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Appendix

Appendix A

Basic terminology surrounding currents

We fix the basic notations for the space of covectors, differential forms and give general vocabulary for currents. While the theory of currents is not necessary to understand the main result in inference geometry in Chapter 7, its terminology is particularly adapted to the study of curvatures as seen in the end of Chapter 6.

A.1 Covectors and differential forms

We assume that basic knowledge about covectors and differential forms is already known to the reader; if not, we refer to either [Fed69, Chapter 1] or [RZ19, Section 1.2]. We fix $d \in \mathbb{N}$ for the remainder of the annex.

Definition A.1 (Covectors and m-alternating forms)**.** We denote by $\bigwedge_m(\mathbb{R}^d)$ the space of m-covectors of $\mathbb{R}^d$, and $\bigwedge^m(\mathbb{R}^d)$ the space of m-linear, alternating forms on $\mathbb{R}^d$.

Recall that $\bigwedge_1(\mathbb{R}^d) = \mathbb{R}^d$ and that there exists a wedge operator $\wedge : \bigwedge_i(\mathbb{R}^d) \times \bigwedge_j(\mathbb{R}^d) \to \bigwedge_{i+j}(\mathbb{R}^d)$ for every $i, j \in \mathbb{N}$. For every basis $e_1, \dots, e_d$ of $\mathbb{R}^d$ and $I = \{i_1, \dots i_r\} \subset \{1, \dots d\}$, we let $e_I \coloneqq e_{i_1} \wedge \dots \wedge e_{i_r} \in \bigwedge^r(\mathbb{R}^d)$. Equivalently, we also write $e_I = \bigwedge_{i \in I} e_i$. It can be seen that the m-covectors e_J, where J ranges among all subsets of $\{1, \dots, d\}$ of cardinal m, form a basis of $\bigwedge_m(\mathbb{R}^d)$. When $e_1, \dots, e_d$ is the canonical basis of $\mathbb{R}^d$, we equip $\bigwedge_m(\mathbb{R}^d)$ with the scalar product corresponding to the decomposition into the basis of m-covectors of the form e_J.

A great interest of covectors lies in the geometrical meaning they carry. For instance, the norm of $e_1 \wedge \dots \wedge e_m$ is the $\mathcal{H}^m$-measure of the convex hull of $e_1, \dots, e_m$, which is also equal to the absolute value of the determinant of the vectors $e_1, \dots, e_m$ in any orthonormal basis of the vector space they generate. We can also think m-covectors as a way to equip subspaces of $\mathbb{R}^d$ of dimension m with a structure of vector space. Indeed, we may think of $e_1 \wedge \dots \wedge e_m$ as the vector space generated by the vectors $e_1, \dots, e_m$ of $\mathbb{R}^d$ - keeping in mind that this correspondence is bijective up to a linear factor. Note, however, that not all covectors can be written that way.

Definition A.2 (Simple covectors)**.** An element of $\bigwedge_m(\mathbb{R}^d)$ is said to be *simple* when it can be written as the wedge product of m vectors of $\mathbb{R}^d$.

The space of alternating m-linear maps is dual to m-covectors of $\mathbb{R}^d$. Following the convention of geometric measure theory, for every pair (v, u) of $\bigwedge^m(\mathbb{R}^d) \times \bigwedge_m(\mathbb{R}^d)$, we write $\langle u, v \rangle$ for the evaluation $v(u)$. The graded vector spaces of alternating multilinear forms is also endowed with a wedge operator $\wedge$.

Definition A.3 (Differential forms)**.** For every open set U of $\mathbb{R}^d$, we denote by $\mathcal{D}^k(U)$ the space of smooth k-differential forms with compact support in U.

Let $f : U \to V$ be a C^1 map and $\phi \in \mathcal{D}^k(V)$. The pullback of $\phi^* f$ is defined as

$$(\phi^* f)_x : e_1 \wedge \cdots \wedge e_k \mapsto \phi_{f(x)} \left(D_x f(e_1) \wedge \cdots \wedge D_x f(e_k) \right) \tag{A.1}$$

A.2 Currents

Although properly defined in functional analysis as linear forms over smooth, compactly supported differential forms exactly as distributions are defined as continuous linear forms on the space of compactly smooth functions, *currents* can be used to encode great a many geometric concepts [Fed69]. We omit their precise definition and simply denote by $\mathcal{D}_k(U)$ the space of k-currents over an open subset U or $\mathbb{R}^d$. However, we will work with currents *representable by integration* and their derivates, which are easier to represent.

Definition A.4 (Current representable by integration)**.** T is said to be a *k-current representable by integration* on $U \subset \mathbb{R}^d$ when there exists a k-rectifiable subset of $\mathbb{R}^d$ W_T, called the *support* of T - uniquely defined up to set of $\mathcal{H}^k$-measure zero - and $\mathcal{H}^k$-integrable field of covectors a_T, such that for any $\psi \in \mathcal{D}^k(U)$,

$$T(\psi) := \int_{W_T} \langle a_T(x), \psi(x) \rangle \, \mathrm{d}\mathcal{H}^k(x). \tag{A.2}$$

Thanks to this expression, $T(\psi)$ can be defined for any measurable, bounded vector field of k-alternating linear forms ψ. Moreover, when $||a_T(x)||$ is an integer $\mathcal{H}^k$-almost everywhere in W_T, we say that T is an *integral* k-current.

Definition A.5 (Restriction of a current)**.** Let T be a k-current on $U \subset \mathbb{R}^d$ and let ω be in $\mathcal{D}^m(U)$ with $m \leq k$. Then the restriction $T \llcorner \omega \in \mathcal{D}_{k-m}(U)$ is defined by

$$(T \llcorner \omega)(\psi) := T(\omega \wedge \psi). \tag{A.3}$$

In particular, for any Borelian subset A of $\mathbb{R}^d$, we let $T \llcorner \mathbb{1}_A$ be the k-current T restricted to A, which is defined as:

$$(T \llcorner \mathbb{1}_A)(\psi) := \int_{W_T} \mathbb{1}_A(x) \, \langle a_T(x), \psi(x) \rangle \, \mathrm{d}\mathcal{H}^k(x). \tag{A.4}$$

Definition A.6 (Pushforward of a current)**.** Let $f : U \to V$ be a C^1 map. The pushforward of T by f is:

$$f_\# T : \phi \mapsto T(\phi^* f). \tag{A.5}$$

When $f : U \to V$ is merely Lipschitz and $\phi \in \mathcal{D}^k(V)$, the pullback of ϕ by f exists almost everywhere thanks to Rademacher's theorem. When T is representable by integration, the pushforward can be further defined by:

$$f_\# T : \phi \mapsto \int_{W_T} \langle a_T(x), (\phi^* f)(x) \rangle \, \mathrm{d}\mathcal{H}^k(x). \tag{A.6}$$

Definition A.7 (Boundary of a current)**.** Using the classical exterior derivative on the space of differential forms we define the *boundary* ∂T of a k-current T, which is a $(k-1)$-current, as follows.

$$\partial T(\psi) := T(\mathrm{d}\psi). \tag{A.7}$$

In particular, we say that T is a *cycle* when $\partial T = 0$.

There are two main metrics on the space of currents.

Definition A.8 (Mass of a current)**.** The *mass* of a current T is

$$M(T) := \sup\{\, T(\psi) \mid ||\psi||_\infty \leq 1\}. \tag{A.8}$$

When T is representable by integration as in Definition A.4, the mass of T has an explicit integral representation:

$$M(T) = \int_{W_T} ||a_T(x)|| \, \mathrm{d}\mathcal{H}^d(x). \tag{A.9}$$

Example A.9 – Let X be a compact, oriented, m-dimensional C^1-submanifold (possibly with boundary) of $\mathbb{R}^d$. Consider the field a_X of unit simple covectors induced by its tangent spaces with a consistent orientation. Let $[[X]]$ be the current integrating on X, i.e.:

$$[[X]](\psi) := \int_X \langle a_X(x), \psi(x) \rangle \, \mathrm{d}\mathcal{H}^m(x). \tag{A.10}$$

The infamous Stokes theorem states that $\partial[[X]] = [[\partial X]]$, and we have furthermore $M([[X]]) = \mathcal{H}^m(X)$. In particular, the current integrating over a submanifold without boundary is a cycle.

The topology induced by the mass metric is restrictive, as even infinitesimally translated currents $\tau(t)_\#(T)$ fail to converge to T in the general setting when $\tau(t) : x \mapsto x + t\nu$ is the translation in some direction $\nu \in \mathbb{S}^{d-1}$ of magnitude t.

Definition A.10 (Flat norm of a current)**.** The *flat norm* of a current T with compact support is:

$$F(T) := \sup\{\, T(\psi) \mid ||\psi||_\infty, ||\mathrm{d}\psi||_\infty \leq 1\}. \tag{A.11}$$

This coincides with the following dual definition:

$$F(T) = \inf\{\, M(S) + M(Q) \mid T = Q + \partial S\}, \tag{A.12}$$

where Q, S range among compactly supported currents of suitable dimensions.

The topology induced by this metric is related to the weak convergence of currents.

Proposition A.11 (Flat norm metrizes weak convergence)**.** *Let U be a bounded open subset of $\mathbb{R}^d$. Then a sequence $(T_n)_{n\in\mathbb{N}}$ of $\mathcal{D}^k(U)$ with uniformly bounded mass converges to T for the flat norm, (i.e. $F(T_n - T) \to 0$) if and only if for every k-differential form ψ of $\mathbb{R}^d$, we have*

$$\lim_{n\to\infty} T_n(\psi) = T(\psi). \tag{A.13}$$

From these considerations, we can bound the mass of a flat limit of currents.

Proposition A.12 (Mass of flat limits)**.** *Let U be a bounded subset of $\mathbb{R}^d$ and let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}^k(U)$ with bounded mass converging to T with respect to the flat norm. Then $M(T) \leq \liminf_{n\to\infty} M(T_n)$.*

Géométrie persistante

Antoine COMMARET

Abstract

This thesis is dedicated to geometric inference, and more specifically to the estimation of curvatures of objects in Euclidean space from an approximating set that is close in the Hausdorff distance. In order to extend the filtering property of persistent homology to the realm of geometry, we introduce the framework of *persistent geometry*. It consists in combining connections between the topology and the curvatures of a subset of $\mathbb{R}^d$ provided by integral geometry, such as the principal kinematic formula, with persistence theory thanks to the so-called image persistence modules. We develop a new method to estimate the intrinsic volumes of a set, which are global quantities built from the curvatures of the set; particular intrinsic volumes include boundary area, Euler characteristic, and mean curvature. Our method allows for the recovery of the intrinsic volumes of a set from any approximating set up to an error that is linear with respect to the Hausdorff distance between them. We show that this approximation is valid as long as the estimated set has bounded total curvature and a positive μ-reach for some $\mu \in (0, 1)$. The μ-reach is a relaxation of the reach of Federer defined to extend geometric inference results to possibly non-smooth, non-convex sets. The class of compact sets of $\mathbb{R}^d$ having bounded total curvatures and a positive μ-reach for an arbitrary μ in $(0, 1)$ is broad, containing compact C^1 submanifolds, compact convex sets, polyhedra, and more generally many compact stratified subsets of $\mathbb{R}^d$. To deal with these possibly singular sets, we use tools from different fields of mathematics, such as non-smooth analysis, geometric measure theory, and Morse theory. In particular, a crucial step in our reasoning consists in the development of Morse theory for offsets of a set at regular values of its distance function. We show that the topology of sublevel sets of smooth maps restricted to such objects — which are not C^2 manifolds — typically evolves by the gluing of cells around each critical point, just as in the classical Morse theory on C^2 manifolds.

Keywords: Geometric inference, Persistent homology, Geometric measure theory, Integral geometry, Topological data analysis