Generalized Morse Theory for tubular neighborhoods in Euclidian spaces

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2023

Abstract

This paper tackles the problem of establishing new conditions over $X \subset \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$ to monitor the changes in topology of the sublevel sets $c \mapsto X \cap f^{-1}(-\infty, c]$. The main contribution of this work consists in using non-smooth analysis to prove Morse Theory theorems for compact subsets of $\mathbb{R}^d$ assuming weak regularity conditions over X and its complement set $\neg X$ while requiring f to be smooth. The class of compact subsets of $\mathbb{R}^d$ verifying those conditions is certainly vast, especially among tubular neighborhoods of compact sets. It contains all but a finite number of offsets of subanalytic sets in $\mathbb{R}^d$ or small offsets of sets with "positive μ-reach", a regularity notion of geometric inference.

1 Introduction

In his celebrated book [1], Milnor shows that taking a C^2 manifold X and $f : X \to \mathbb{R}$ smooth sufficiently generic called a *Morse Function*, the changes in topology of the closed sublevel sets $X_c = f^{-1}(-\infty, c]$ are completely known. It is shown that topological changes only happen around a handful of values c_i called *critical values*. These are determined by the values f takes at the *critical points* x_i, i.e $c_i = f(x_i)$. Critical points are exactly the points such that the differential of f vanishes, i.e $T_{x_i} f = 0$. Around a critical point x with $c = f(x)$, the topology of the sublevels sets $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by gluing a simple object around x when ε is small enough. To be Morse, a function is required to have a non-degenerate Hessian at every critical point. This second-order approximation ensures that the critical points are isolated. For the sake of conciseness, it is also often required that to each critical value c_i corresponds a unique critical point x_i, but most results stay true in the cases where a level set $f^{-1}(c)$ contains several critical points.Morse functions on X verify the two fundamental results of Morse Theory:

- In between critical values, the homotopy types of the sublevel sets X_c are the same, what is classically called the *Isotopy Lemma.*
- Around a critical value c, the homotopy type of $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by "gluing" a λ_i cell around each critical point $x_i \in f^{-1}(c)$, when ε is small enough. This is the *Handle Attachment Lemma.*

Those two results directly yield that X has the homotopy type of a CW-complex. Now we naturally wonder whether having a Morse Function on a manifold often happens, and the answer is yes. For any manifold X embedded in a Euclidian space $\mathbb{R}^d$, the function $d_{\{x\}|X}$ is Morse for $\mathcal{H}^d$-almost all x, and the height functions $x \mapsto \langle x, v\rangle$ restricted to X are Morse for $\mathcal{H}^{d-1}$-almost all $v \in \mathbb{S}^{d-1}$ [1]. Moreover, "almost all" functions are Morse in the Whitney topology on X, as the set they form is dense and open. A recent work from Monod, Song, Kim [2] showed that for a generic surface $S \subset \mathbb{R}^3$, the distance to S function d_S verifiy the two Morse Theory lemmas. When f is smooth, Fu [3] narrowed the assumptions to any set X with $C^{1,1}$ boundary, i.e and more generally to sets with positive reach. Our regularity assumptions are that $\mathrm{reach}_\mu(X) > 0$

for some $\mu \in (0, 1]$, its complement set $^\neg X = \overline{\mathbb{R}^d \setminus X}$ has positive reach and full-dimensionality condition on its tangent cones (notions described in Section 2.1). The notion of reach$_\mu$ has been mainly used in geometric inference to extend results to non-smooth and non-convex objects. In particular this class encompasses almost any tubular neighborhood of subanalytic sets.

Here is the major result of this paper formulated informally.

Theorem 1.1: Informal Generalized Morse Theory

Let $X \subset \mathbb{R}^d$ be such that $\mathrm{reach}_\mu(X) > 0$ *and* $\mathrm{reach}(^\neg X) > 0$ *for a certain* $\mu \in (0, 1]$*. Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a smooth function such that* $f_{|X}$ *admits only non degenerate critical points.*

Then for every regular $c \in \mathbb{R}$ *of* $f_{|X}$*,* $X_c = X \cap f^{-1}(-\infty, c]$ *has the homotopy type of a CW-complex with extra cells added independently in the sense of persistence theory at critical values, whose dimension depend explicitly on the curvatures of* X*.*

Outline

In Section 2 we define the objects used throughout this article. We show intermediate results required in Section 3 to prove the results of the paper.

- In Section 2.1 we define and illustrate the basic tools of our study. This includes the reach$_\mu$ of a compact subset of $\mathbb{R}^d$, basic vocabulary in persistence theory, normal and tangent cones of an object with positive reach, and currents.
- In Section 2.2 we define the Normal Cycle of sets with positive reach, or their complement set. We describe how they contain pointwise curvatures.
- Section 2.3 quickly summarizes the paper from Fu [3] which is the basis for our article. We recall definitions and notations of critical points and Hessian for a restricted function $f_{|X}$.
- Section 2.4 focuses on properties of locally Lipschitz functions for non-smooth analysis. We build a retraction between sublevel sets of such functions assuming their Clarke gradient stays away from zero.
- In Section 2.5 we establish a fundamental link between the Normal Bundle of a set X and the Clarke gradient of its distance function d_X. This crucial step allows us to use results from non-smooth analysis on assumptions about critical points of $f_{|X}$.

Section 3 articulates the previous results to establish the main theorem.

- In Section 3.1 we describe the regularity conditions we impose on X to prove Morse Theory results.
- In Section 3.2 we describe how to build a function $f_{r,c}$ such that $(X^{-r}, f_{r,c})$ are smooth surrogates for (X, f) such that the smooth sublevel set X_c^{-r} and the original X_c have the same homotopy type when c is a regular value. To that end we consider some locally Lipschitz functions and check that the results from Section 2.4 can be applied to them. The retractions obtained are used to build a homotopy equivalence.
- In Section 3.3 we show that in between critical values, the topology of sublevel sets does not evolve. We also apply Section 2.4 using computations from the previous section.
- Section 3.4 describes the topological changes happening around a critical value as long as it has only one corresponding critical point which is non-degenerate. We adapt the proof from Fu [3] to our setting.

- Section 3.5 describes topological changes around a critical value admitting several critical points that are all non-degenerate.

Finally, the small **??** describes the previous result in light of persistence theory. We count the number of topological events happening in the filtration $c \mapsto X_c$ and show that the topological events have to be independent in the sense of persistence theory.

2 Definitions and useful lemmas

2.1 Preliminaries

- Throughout this paper, the *complement set* of a closed $X \subset \mathbb{R}^d$ will denote $\neg X = \overline{\mathbb{R}^d \setminus X}$ the closure of the classical complement set, abusing notations for the sake of conciseness.

- For any set $A \subset \mathbb{R}^d$, $d_A : x \mapsto \inf\{||x - a||, a \in A\}$ is the *distance to* A function. It is 1-lip and thus differentiable almost everywhere. For any positive r and $X \subset \mathbb{R}^d$, define its r-tubular neighborhood
$$X^r = \{x \in \mathbb{R}^d \mid d_X(x) \leq r\}$$
It can also be extended for negative values. If $r > 0$,
$$X^{-r} = \{x \in \mathbb{R}^d \mid d_{\neg X}(x) \geq r\}$$

- A *Cone* A in $\mathbb{R}^d$ is a set stable under multiplication by a positive number, i.e for all $\lambda > 0$, $\lambda A \subset A$.

  Given any $B \subset \mathbb{R}^d$, denote $\operatorname{Cone} B$ the smallest cone containing B, defined as the image of $[0, \infty[\times B$ by the map $(\lambda, x) \mapsto \lambda x$.

  In the same vein, denote $\operatorname{Conv} B$ the *convex hull* of B the smallest convex set containing B, consisting in all convex combinations of elements of B.

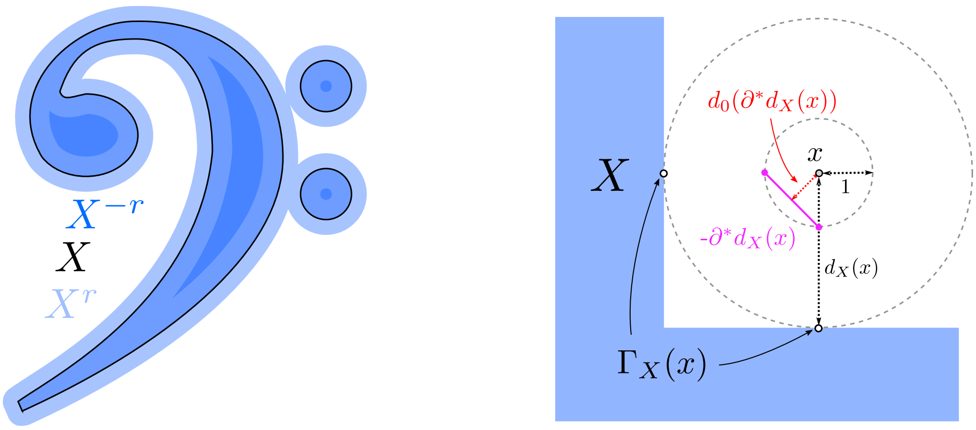
  Finally, a *Convex cone* is a subset of $\mathbb{R}^d$ which is both a cone and convex.

- Given a subset X of $\mathbb{R}^d$ define its *distance to 0* as
$$d_0(X) = \inf\{||x|| \mid x \in X\}$$
It measures how far X is from intersecting $\{0\}$. It is zero when $0 \in \overline{X}$.

- Given a function $f : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz, define $\partial^* f(x)$ its *Clarke Gradient at* x as the convex hull of limits $\lim_{h \to 0} \nabla f(x + h)$ - see section 2.4 for more details. In particular, if x lies outside of X, $-\partial^* d_X(x)$ is the convex hull of the directions to the points $z \in X$ realizing the distance to X, i.e such that $d_X(x) = ||x - z||$. Such z form the set $\Gamma_X(x)$ of *closest points* to x in X. Elements of $\Gamma_X(x)$ will often be denoted by the letter ξ. In particular, we denote $\xi_X(x)$ *the* closest point to x in X when $\Gamma_X(x)$ is a singleton.

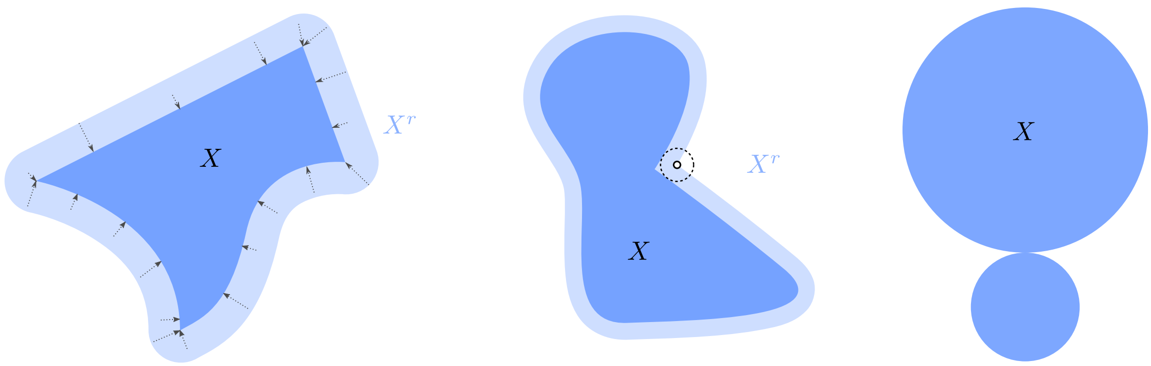
- Given μ in $(0, 1]$, define the μ*-reach* of a subset X of $\mathbb{R}^d$:
$$\operatorname{reach}_\mu(X) = \sup\left(\left\{s \in \mathbb{R} |\, d_X(x) \leq s \implies d_0(\partial^* d_X(x)) \geq \mu\right\}\right) \tag{2.1}$$
Having $\operatorname{reach}_\mu(X) > 0$ means that in a certain neighborhood of X, the angles between two closest point in X cannot be too flat. The lower the μ, the flatter allowed. Note that this definition coincides with the classical one found is geometric inference: $d_0(\partial^* d_X(x))$ is exactly the norm of $\nabla d_X(x)$ as defined by Lieutier in [4].

  Throughout this article, when no value of μ has been fixed, for any closed $X \subset \mathbb{R}^d$, *having a positive* μ*-reach* means that there is a certain $\mu \in)0, 1]$ with $\operatorname{reach}_\mu(X) > 0$. This class of sets is certainly broad, intuitively containing stratified sets whose corners are not infinitely pointy. An easy corollary from Fu, Lemma 1.6 [5] is that any subanalytic set $X \subset \mathbb{R}^d$, the set of value $r > 0$ such that X^r has not a positive μ reach is finite.



A bass clef X inflated (X^r) and eroded (X^{-r}) Clarke gradient of d_X outside of X

FIGURE 1: Offsets and Clarke gradient of d_X.



$\text{reach}(X) > r > 0$ $\text{reach}(X) = 0$ $\text{reach}_\mu > 0$ for some $1 > \mu > 0$. $\text{reach}_\mu(X) = 0$ for all $\mu \in (0, 1]$.

FIGURE 2: Sets with particular reach_μ.

- The *reach* of a subset of $\mathbb{R}^d$ is a quantity developed by Federer in [6] coinciding with reach_1. It is the largest number t such that $d_X(x) < t$ implies that x has a unique closest point in X. When X has a positive reach_μ the complementary set of small offsets of X have positive reach:

Theorem 2.1: Reach of complement of offsets (Chazal et al. [7], 4.1)

Let X be compact subset of $\mathbb{R}^d$, $\mu \in (0, 1]$ and $0 < r < \text{reach}_\mu(X)$.

Then $\text{reach}(^\neg(X^r)) \geq \mu r$*.*

- The *Tangent Cone* of X at x, $\text{Tan}(X, x)$ is defined as the cone generated by the limits $\lim_{n\to\infty} \frac{x_n - x}{||x_n - x||}$, where the sequence $(x_n)_{n\in\mathbb{N}}$ belongs in X and tends to x. We say that u is *represented* by the sequence $(x_n)_{n\in\mathbb{N}}$.

  When $X \subset \mathbb{R}^d$ has positive reach, Federer remarks that $\text{Tan}(X, x)$ is a convex cone and gives another characterization for the tangent cone for any $x \in X$,

$$\text{Tan}(X, x) = \left\{ u \in \mathbb{R}^d,\ \lim_{t\to 0^+} \frac{d_X(x + tu)}{t} = 0 \right\}$$

- When X has positive reach its *Normal Cone* $\text{Nor}(X, x)$ at x is defined as the set of directions dual to the tangent cone at x:

$$\mathrm{nor}(X,x) = \mathrm{Tan}(X,x)^{o} \quad \mathrm{Nor}(X,x) = \mathrm{nor}(X,x) \cap \mathbb{S}^{d-1}$$

Remark that $\mathrm{Nor}(X,x)$ is not a cone. We choose to keep this name in consistency with the literature in Geometric Measure Theory, especially Federer [8]. It is related to ξ_X *the projection to the closest point in* X by the following characterisation, for any $0 < t < \mathrm{reach}(X)$:

$$\mathrm{Nor}(X,x) = \mathrm{nor}(X,x) \cap \mathbb{S}^{d-1} = \left\{u \in \mathbb{S}^{d-1} \mid \xi_X(x+tu) = x\right\}$$

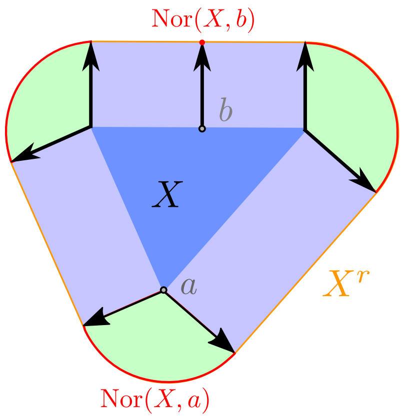


FIGURE 3: Normal cones when $\mathrm{reach}(X) > 0$

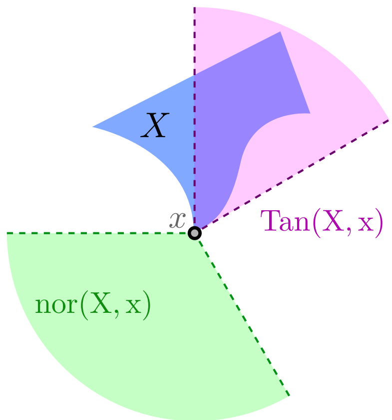


FIGURE 4: Tangent cone of X at x and its dual

- If X has positive reach, we say that X is *full dimensional* when every $\mathrm{Tan}(X,x)$ is full dimensional for every $x \in \partial X$, which is characterized by the following condition on the normal cones:
$$(x,n) \in \mathrm{Nor}(X) \implies (x,-n) \notin \mathrm{Nor}(X)$$

2.2 Normal bundle of sets with positive reach

The *Normal Bundle* $\mathrm{Nor}(X)$ of a closed set X with positive reach in $\mathbb{R}^d$ is defined by

$$\mathrm{Nor}(X) = \bigcup_{x \in \partial X} \{x\} \times \mathrm{Nor}(X,x)$$

For any $0 < r < \mathrm{reach}(X)$, ∂X^r is a $(d-1)$ smooth submanifold of $\mathbb{R}^d$ by the implicit function theorem. Now the application $\mathrm{Nor}(X) \to \partial X^r$, $(x,\nu) \to x + \nu r$ is bi-Lipschitz, giving $\mathrm{Nor}(X)$ a structure of $(d-1)$ *Lipschitz submanifold* of $\mathbb{R}^d \times \mathbb{S}^{d-1}$. As such, it admits tangent spaces $\mathrm{Tan}(\mathrm{Nor}(X),(x,n))$ of dimension $(d-1)$ $\mathcal{H}^{d-1}$-almost everywhere. When $\neg X$ has positive reach and is full dimensional, we define the normal cone of X at x as the opposite of the normal cone of its complement set. The normal bundle follows.

$$\mathrm{Nor}(X,x) = -\,\mathrm{Nor}(\neg X,x) \qquad \mathrm{Nor}(X) = \textstyle\bigcup_{x \in \partial X} \{x\} \times \mathrm{Nor}(X,x)$$

We say that X *admits a normal bundle* when either $\mathrm{reach}(X) > 0$ or $\mathrm{reach}(\neg X) > 0$ and $\neg X$ is full dimensional. In either case, remark that $\mathrm{Nor}(X)$ is a $(d-1)$ Lipschitz submanifold of $\mathbb{R}^d \times \mathbb{S}^{d-1}$. A pair $(x,n) \in \mathrm{Nor}(X)$ is called *regular* when $\mathrm{Tan}(\mathrm{Nor}(X),(x,n))$ is a $(d-1)$ dimensional vector space. Pairs inside $\mathrm{Nor}(X)$ are regular $\mathcal{H}^{d-1}$ almost everywhere.

The construction of $\mathrm{Nor}(X)$ stems from the more general concept of *Normal Cycle* N_X of a set X. While we do not need to write our hypothesis using this more involved language, in our case $\mathrm{Nor}(X)$ is the support of a $(d-1)$ Legendrian cycle N_X over $\mathbb{R}^d \times \mathbb{S}^{d-1}$, whose tangent

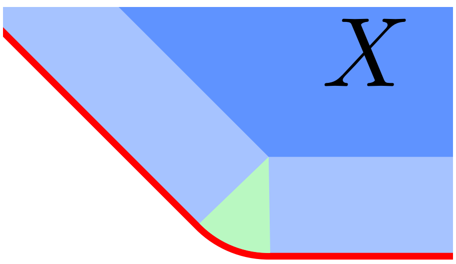


FIGURE 5: Normal Bundle of X with reach$(X) > 0$

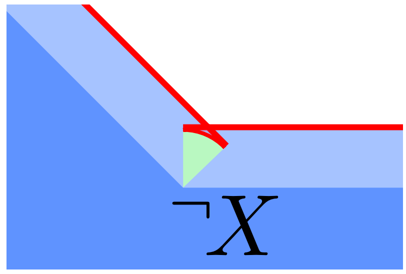


FIGURE 6: Normal Bundle of $\neg X$ with reach$(X) > 0$

spaces' structure is already known.

Proposition 2.2: Tangent spaces of Normal Bundles (Rataj & Zähle, 2019 [9])

Let X be a compact set admitting a normal bundle $\mathrm{Nor}(X)$*.*
Then for any regular pair $(x, n) \in \mathrm{Nor}(X)$, there exist

- *A family $\kappa_1, \dots, \kappa_{d-1}$ in $\mathbb{R} \cup \{\infty\}$ called the* principal curvatures at (x, n)
- *A family $b_1, \dots, b_{d-1} \in \mathbb{R}^d$ orthogonal to n called the* principal directions *at (x, n).*

such that the family $\left(\frac{1}{\sqrt{1+\kappa_i}^2} b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}} b_i\right)_{1 \le i \le d-1}$ *form an orthonormal basis of* $\mathrm{Tan}(\mathrm{Nor}(X), (x, n))$.

- *Principal curvatures are unique up to permutations*
- *Principal directions b_i associated to κ_i are unique up to the determination of an orthonormal basis of* $\ker(u, v \mapsto u - \kappa_i v)$ *if $\kappa_i < \infty$, or* $\ker(u, v \mapsto v)$ *if $\kappa_i = \infty$.*

This notion of curvatures coincide with the one found in differential geometry as eigenvalues of the second fundamental form. Indeed, assume that $X \subset \mathbb{R}^d$ is bounded by a $C^{1,1}$-hypersurface, i.e the boundary of X is an hypersurface such that the Gauss map $x \in \partial X \mapsto n(x) \in \mathbb{S}^{d-1}$ is Lipschitz. The pair $(x, n(x)) \in \mathrm{Nor}(X)$ is regular if and only if $n(x)$ is differentiable at x [3]. In that case, its differential is symmetric and its eigenvalues counted with multiplicity (resp. orthonormal basis of eigenvectors) are principal curvatures (resp. principal directions) at $(x, n(x))$.

2.3 Critical points and Hessians for $f_{|X}$

We recall the notations of Fu [3]. We give a brief summary of his work of Morse Theory on sets with positive reach, which form a basis of our future reasoning in Section 3. The projection $\mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ onto the first factor is denoted π_0.

Definition 2.3: Critical points and Hessian

Let $f : \mathbb{R}^d \to \mathbb{R}$ be smooth and X be a set of $\mathbb{R}^d$ admitting a normal bundle.

- *Let $(x, n) \in \partial X \times \mathbb{S}^{d-1}$ be regular as in Prop 2.2. Take (b_i) an orthonormal basis of $\pi_0(\mathrm{Tan}(N_X, (x, n)))$ consisting of all principal directions with finite associated curvatures. The second fundamental form $\mathrm{I\!I}_{x,n}$ is defined as the bilinear form on $\pi_0(\mathrm{Tan}(N_X, (x, n)))$ such that:*

$$\mathrm{I\!I}_{x,n}(b_i, b_j) = \kappa_i \delta_{i,j} \tag{2.2}$$

  which generalizes the classical fundamental form obtained when X has a smooth boundary.

- *$x \in X$ is a* critical point *of $f_{|X}$ when $\nabla f(x) \in \mathrm{nor}(X, x)$*

- *$c \in \mathbb{R}$ is a* critical value *of $f_{|X}$ when $f^{-1}(c)$ contains at least a critical point of $f_{|X}$. Otherwise, c is a* regular value *of $f_{|X}$.*

- *If x is a critical point of $f_{|X}$ with $\nabla f(x) \neq 0$, put $n = \frac{-\nabla f(x)}{||\nabla f(x)||}$.*
  If (x, n) is regular, the Hessian of $f_{|X}$ at x *is defined as a bilinear form over $\pi_0(\mathrm{Tan}(N_X, (x, n)))$:*

$$Hf_{|X}(x)(u, v) = Hf(x)(u, v) + ||\nabla f(x)||\, \mathrm{I\!I}_{x,n}(u, v)$$

- *The* index *of this Hessian is the dimension of the largest subspace on which $Hf_{|X}$ is negative definite.*

- *We say that a critical point x of $f_{|X}$ is* non-degenerate *when $\nabla f(x) \neq 0$ and its Hessian $Hf_{|X}(x)$ is not degenerate.*

Using these definitions, Fu showed that Morse Theory applies for sets with positive reach.

Theorem 2.4: ***Generalized Morse Theory for sets with positive reach*** **(Fu, 1989 [3])**

Let X be a compact subset of $\mathbb{R}^d$ with positive reach and let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $f_{|X}$ is Morse, i.e it has only non-degenerate critical points, and there is at most one critical value per level set.

Then for any regular value $c \in \mathbb{R}$, X_c has the homotopy type of a CW-Complex with one λ_p cell for each critical point p such that $f(p) < c$, where

$$\lambda_p = \textit{Index of } Hf_{|X} \textit{ at } p$$

2.4 Clarke gradients and approximate flows

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. It is differentiable almost everywhere thanks to Rademacher's Theorem. Consider $\partial^* \phi(x)$ its *Clarke gradient* at x. It is a subset of $\mathbb{R}^d$ generalizing the gradient of ϕ defined as the convex hull of limits of $\nabla\phi(x + h), h \to 0$. A key property of Clarke Gradients is its upper semicontinuity, leading to the following proposition.

Proposition 2.5: Semicontinuity Clarke Gradients

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function.
If a sequence $(x_i)_{i\in\mathbb{N}}$ converges to x, we have

$$\liminf_{i\to\infty} d_0\left(\partial^*\phi(x_i)\right) \geq d_0\left(\partial^*\phi(x)\right)$$

Assuming $\partial^*\phi(x)$ stays uniformly away from 0, we are able to build deformation retractions between the sublevel sets of ϕ using approximations of what would be the flow of $-\phi$ had it been smooth.

Proposition 2.6: Approximate flow of a Lipschitz function

Let $a < b \in \mathbb{R}$. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally lip function. Assume that

$$\inf\{d_0(\partial^*\phi(x)), x \in \phi^{-1}(a, b]\} = \mu > 0$$

. Then for every $\varepsilon > 0$, there exists a continuous flow

$$C_\phi : \begin{cases} [0,1] \times \phi^{-1}(]\infty, b]) & \to & \phi^{-1}(]-\infty, b]) \\ (t, x) & \mapsto & C_\phi(t, x) \end{cases}$$

- *For any $s > t$ and x such that $C(s, x) \in \phi^{-1}(a, b]$, we have*

$$\phi(C_\phi(s, x)) - \phi(C_\phi(t, x)) \leq (s - t)(b - a)$$

- *For any $t \in [0, 1]$, $x \in \phi^{-1}(\infty, a]$, $C_\phi(t, x) = x$*
- *For any $x \in \phi^{-1}(-\infty, b]$, the map $s \mapsto C_\phi(s, x)$ is $\frac{b-a}{\mu-\varepsilon}$-lipschitz.*

In particular, $C_\phi(1, \cdot)$ is a deformation retraction between $\phi^{-1}(-\infty, a]$ and $\phi^{-1}(-\infty, b]$.

Proof. This is a lemma from [10] except the we have optimized the constants. For the sake of completeness, we write the full proof. Let $\varepsilon > 0$. Let $x \in \phi^{-1}(a, b]$ and take by semicontinuity of the Clarke gradients B_x an open ball centered in x such that $\partial^*\phi(y) \subset \partial^*\phi(x)^\varepsilon$ for any $y \in B_x$. Since $\partial^*\phi(x)$ is a closed convex set, put $W(x)$ the vector in $\partial^*\phi(x)$ realising $||W(x)|| = d_0(\partial^*\phi(x))$ - that is, the closest point to 0 in $\partial^*\phi(x)$. By convexity,

$$\forall u \in \partial^*\phi(x), \langle u, W(x)\rangle \geq ||W(x)||^2 . \tag{2.3}$$

The family $\{B_x\}_{x\in\phi^{-1}(a,b]}$ is an open covering of $\phi^{-1}(a, b]$. Thanks to paracompactness, there exists a locally finite partition of unity $(\rho_i)_{i\in I}$ subordinate to this family. The support of each ρ_i has to be included in a certain $B(x_i)$, where $x_i \in \phi^{-1}(a, b]$.

Define the vector field V as a smooth interpolation of normalized $-W(x_i)$:

$$V(y) = -\sum_{i\in I} \rho_i(y) \frac{W(x_i)}{||W(x_i)||} \tag{2.4}$$

Obviously $||V(x)|| \leq 1$. Now by classical results write C_ϕ the flow of V defined on a maximal domain $\mathbb{D}$ in $\phi^{-1}(a, b] \times \mathbb{R}^+$. Define $\mathbb{D}_x$ via $\{x\} \times \mathbb{R}^+ \cap \mathbb{D} = \{x\} \times \mathbb{D}_x$. The set $\mathbb{D}_x$ is connected in $\mathbb{R}^+$ and we can thus put $s_x = \sup \mathbb{D}_x$. In particular remark that given for any $x \in \phi^{-1}(a, b]$ and any $\zeta \in \partial^*\phi(x)$,

$$\langle \dot{C}_x(0), \zeta\rangle \leq -\sum_i \rho_i(||W(x_i)|| - \varepsilon) \leq -\mu + \varepsilon \tag{2.5}$$

and there exists a $c_x > 0$ such that C_x is well defined on $[-c_x, c_x]$.

Function $\phi \circ C_x : [-c_x, c_x] \to \mathbb{R}$ is Lipschitz and thus differentiable almost everywhere on this interval. Without loss of generality we can assume that it is differentiable at 0. Since C_x has non-vanishing gradient $V(x)$ at 0, ϕ admits a directional derivative $\phi'(x, V(x))$ in direction $V(x)$. Now the work of Clarke [11] states that when the directional derivative exists, the Clarke gradients acts like a maxing support set, that is:

$$\phi'(x, V(x)) \leq \max \left\{ \langle \zeta, V(x) \rangle \mid \zeta \in \partial^* \phi(x) \right\} \leq -\mu + \varepsilon \tag{2.6}$$

Thus, as long as the trajectory does not leave $\phi^{-1}(a, b]$, absolute continuity implies

$$\phi(C(t, x)) - \phi(C(0, x)) \leq -(\mu - \varepsilon)t \tag{2.7}$$

Now put

$$s_x = \sup \{t \in \mathbb{R}, \phi(C(t, x)) > a\}$$

Remark that $x \mapsto s_x$ is continuous over $\phi^{-1}(a, b]$ since the flow is continuous and makes ϕ stricly decrease. The trajectory can be extended by continuity with $C(s_x, x) \in \phi^{-1}(\{a\})$. For any point $x \in \phi^{-1}(-\infty, a]$, put $s_x = 0$ so that the flow C is extended in all $\phi^{-1}(-\infty, a]$ by being constant. Finally, consider the following reparametrization C_ϕ of C:

$$C_\phi(t, x) = C\left(\min\left(\frac{b - a}{\mu - \varepsilon} t, s_x\right), x\right)$$

Which shows that $\phi^{-1}(-\infty, a]$ is a strong deformation retraction of $\phi^{-1}(-\infty, b]$. □

2.5 Relating the Normal Cones to Clarke Gradients of distance functions

The normal bundle of X is related to d_X in the following fashion.

Theorem 2.7: Normal cycle and the Clarke gradient of the distance function

Let $X \subset \mathbb{R}^d$ be such that $\mathrm{reach}(^\neg X) > 0$ *and full dimensional. Let $x \in \partial X$. Then the normal cone of X at x is determined by the Clarke gradient of d_X at x:*

$$\mathrm{Cone}\,\mathrm{Nor}(X, x) = \mathrm{Cone}\,\partial^* d_X(x)$$

Proof. Let $\mathrm{reach}(^\neg X) > r > 0$. First remark that

$$\begin{aligned} \partial^* d_{X^{-r}}(x) &= -\,\mathrm{Conv}\left\{\frac{x - z}{||x - z||}, z \in X^{-r} \text{ with } d_X^{-r}(x) = ||z - x||\right\} \\ &= -\,\mathrm{Conv}\{u \in \mathbb{S}^{d-1}, d_{\neg X}(x + ru) = r\}) \\ &= -\,\mathrm{Conv}\,\mathrm{Nor}(^\neg X, x) \end{aligned}$$

On the other hand by definition, the Clarke gradient of $d_{X^{-r}}$ at x is determined locally by the gradients around x in every direction:

$$\partial^* d_{X^{-r}}(x) = \mathrm{Conv}\left\{\lim \nabla d_{X^{-r}}(x_i) \mid (x_i) \in (\mathbb{R}^d)^{\mathbb{N}} \text{ converging to } x\right\}$$

Now compare to the Clarke Gradient of d_X for which the gradient contributing only come from directions outside of X (cf. [11], 2.5):

$$\partial^* d_X(x) = \mathrm{Conv}\{0, \lim \nabla d_X(x_i) \mid (x_i) \in (\mathbb{R}^d)^{\mathbb{N}} \text{ converging to } x \text{ such that for all } i, d_X(x_i) > 0\,\}$$

Note that in both definition we implictly require x_i to be points where d_X is differentiable. On those points the gradients of d_X and $d_{X^{-r}}$ coincide, yielding

$$\operatorname{Cone}\partial^* d_X(x) \subset -\operatorname{Cone}\operatorname{Nor}(\neg X, x). \tag{2.8}$$

The other inclusion $-\operatorname{nor}(\neg X, x) \subset \operatorname{Cone}\partial^* d_X(x)$ is Lemma 2.11 whose proof will be the remainder of this subsection. We will prove that

$$\partial^* d_X(x)^{\mathrm{o}} \subset -\operatorname{nor}(\neg X, x)^{\mathrm{o}} = -\operatorname{Tan}(\neg X, x) \tag{2.9}$$

□

Lemma 2.8: Tangent cone stability under addition with $\partial^* d_X(x)$

Let $X \subset \mathbb{R}^d$ and $x \in \partial X$. Let $u \in \partial^* d_X(x)^o$, Then for all $h \in \operatorname{Tan}(X, x)$, $u+h \in \operatorname{Tan}(X, x)$.

Proof. We use Clarke's [11] characterization of the dual cone to the Clarke gradient:

$$\partial^* d_X(x)^{\mathrm{o}} = \left\{ u, \lim_{\substack{x_h \to x \\ x_h \in X}} \lim_{\delta \to 0^+} \frac{1}{\delta} d_X(x_h + \delta u) = 0 \right\} \tag{2.10}$$

Consider the continuity module

$$\omega_u(\varepsilon, \lambda) = \sup_{\substack{x_h \in X \\ ||x - x_h|| \le \varepsilon}} \sup_{0 < \delta \le \lambda} \frac{d_X(x_h + \delta u)}{\delta}$$

By 2.10, u being in $\partial^* d_X(x)^{\mathrm{o}}$ implies $\omega_u(\varepsilon, \delta) \to 0$ as $\varepsilon, \delta \to 0$.

Now take $x_i \to x$ representing any $h \in \operatorname{Tan}(\partial X, x)$. Put $\varepsilon_i = ||x - x_i||$ and consider the sequence $x_i + \varepsilon_i u$. Take ξ_i in $\Gamma_X(x_i + \varepsilon_i u)$:

$$||\xi_i - x_i - \varepsilon_i u|| = d_X(x_i + \varepsilon_i u) \le \varepsilon_i \omega(\varepsilon_i, \varepsilon_i)$$

Thus we can write

$$\xi_i - x = \varepsilon_i(h + o(1) + u + O(\omega(\varepsilon_i, \varepsilon_i))) = \varepsilon_i(u + h + o(1))$$

showing that ξ_i is a sequence in X representing $u + h$. □

Lemma 2.9: Intersection of complement tangent spaces

Let $X \subset \mathbb{R}^d$. Then

$$\operatorname{Tan}(\partial X, x) = \operatorname{Tan}(X, x) \cap \operatorname{Tan}(\neg X, x)$$

Proof. We have to prove that $\operatorname{Tan}(X, x) \cap \operatorname{Tan}(\neg X, x)$ is included in $\operatorname{Tan}(\partial X, x)$.

Let $u \in \operatorname{Tan}(X, x) \cap \operatorname{Tan}(\neg X, x)$ be of norm 1. Take a sequence x_n (resp. $\neg x_n$) in X (resp. $\neg X$) representing u, i.e

$$\begin{aligned} x_n &= x + ||x_n - x||\,(u + o(1)\,) \\ \neg x_n &= x + ||\neg x_n - x||\,(u + o(1)\,) \end{aligned}$$

The segment $[x_n, \neg x_n]$ has to intersect ∂X. Thus there exists a $\lambda_n \in [0, 1]$ such that $\partial x_n = \lambda_n x_n + (1 - \lambda_n)\neg x_n$ belongs in ∂X. This yields

$$\begin{aligned} \partial x_n - x &= (\lambda_n\,||x_n - x|| + (1 - \lambda_n)\,||\neg x_n - x||\,)\,(u + o(1)\,) \\ &= ||\partial x_n - x||\,(u + o(1)\,) \end{aligned}$$

meaning that u is represented by a sequence in ∂X. □

Lemma 2.10: Complement tangent cone are tangent cone of complement

Let $X \subset \mathbb{R}^d$ be a closed set such that either X or $\neg X$ has positive reach and let $x \in \partial X$. We have

$$\neg\operatorname{Tan}(\neg X, x) = \operatorname{Tan}(X, x)$$

Proof. Without loss of generality, assume $\operatorname{reach}(\neg X) > 0$. Since $\operatorname{Tan}(X, x) \cup \operatorname{Tan}(\neg X, x) = \mathbb{R}^d$, we know that $\neg\operatorname{Tan}(\neg X, x) \subset \operatorname{Tan}(X, x)$. We will show the opposite inclusion by proving that $\operatorname{Tan}(X, x) \cap \operatorname{int}(\operatorname{Tan}(\neg X, x)) = \emptyset$.

Let $u \in \operatorname{Tan}(X, x) \cap \operatorname{int}(\operatorname{Tan}(\neg X, x))$. Then it belongs in $\operatorname{Tan}(\partial X, x)$ by Lemma 2.9. Take a sequence $x_n \in \partial X$ such that $\frac{x_n - x}{||x_n - x||} \to u$. Take a sequence $v_n \in \operatorname{Nor}(\neg X, x_n)$. Fix a $\lambda \in (0, \operatorname{reach}(\neg X))$. We have

$$\operatorname{int}(B(x_n + \lambda v_n, \lambda)) \cap \neg X = \emptyset \tag{2.11}$$

Since $u \in \operatorname{int}(\operatorname{Tan}(\neg X, x))$, there exists a $\lambda' \in (0, \lambda)$ such that for any n large enough

$$\frac{x_n - x}{||x_n - x||} + \lambda' v_n \in \operatorname{Tan}(\neg X, x)$$

Consider for any such n a sequence $(y_{m,n})_{m \in \mathbb{N}} \in \neg X$ representing the previous vector. We will now prove that $y_{m,n}$ cannot be in $\neg X$ for large m, n as it represents a infinitesimal version of vector of ∂X shifted in a direction normal to X. We can write

$$y_{m,n} = x + ||y_{m,n} - x|| \left(\frac{x_n - x}{||x_n - x||} + \lambda' v_n + \omega_{m,n} \right)$$

with $\omega_{m,n} \to_{m \to \infty} 0$ for every n.

$$\begin{aligned} ||y_{m,n} - x_n - \lambda' v_n|| =& \left|\left| \left(\lambda - \lambda')v_n + (x_n - x)(1 - \frac{||y_{m,n} - x||}{||x_n - x||} \right) + ||y_{m,n} - x||\, \omega_{m,n} \right|\right| \\ \leq& (\lambda - \lambda') + ||x_n - x|| + ||y_{m,n} - x||\, (\omega_{m,n} - 1) \end{aligned}$$

The last quantity is strictly smaller than λ for m, n large enough, contradicting 2.11. $\square$

Lemma 2.11: Relationship between normal cones and Clarke Gradients

Let $X \subset \mathbb{R}^d$ such that $\operatorname{reach}(\neg X) > 0$. Then if $\operatorname{Tan}(\neg X, x)$ has full dimension, we have:

$$\partial^* d_X(x)^o \subset -\operatorname{Tan}(\neg X, x)$$

In particular, this full-dimensional condition is verified for all $x \in \partial X$ when X is a Lipschitz submanifold.

Proof. Let $u \in \partial^* d_X(x)^{\mathrm{o}}$. By Lemma 2.8 we know that

$$u + \operatorname{Tan}(X, x) \subset \operatorname{Tan}(X, x)$$

which amounts to

$$\mathbb{R}^d \setminus (u + \operatorname{Tan}(X, x)) \supset \mathbb{R}^d \setminus \operatorname{Tan}(X, x)$$

Now by Lemma 2.10, this means

$$u + \operatorname{int}(\operatorname{Tan}(\neg X, x)) \supset \operatorname{int}(\operatorname{Tan}(\neg X, x))$$

now taking the closure of both sides, along with the full-dimensionality condition, ensures

$$u + \operatorname{Tan}(\neg X, x) \supset \operatorname{Tan}(\neg X, x)$$

which implies that u belongs in $-\operatorname{Tan}(\neg X)$. $\square$

3 Morse Theory for completentary regular sets

In this section, we use the previous tools and propositions to infer the two Morse Theory theorems when X is *complementary regular* (cf. section 3.1).

In this setting, the eroded sets X^{-r} are smooth surrogates for X in the sense that they converge in the Hausdorff sense: $\lim_{r\to 0} d_H(X^{-r}, X) = 0$ and that they are $C^{1,1}$ by the implicit function theorem when $r < \mathrm{reach}(^\neg X)$.

Let c be a regular value of $f_{|X}$. Our approach is as follows. Take any family of functions $f_{r,c}$ converging to f in a way we will later precise as r tends to 0. When $r = 0$, our notations are consistent with $f_{0,c} = f$. Consider the following sublevel sets:

$$X_c = X \cap f^{-1}(-\infty, c] \quad \text{and} \quad X_c^{-r} = X^{-r} \cap f_{r,c}^{-1}(-\infty, c]$$

Remark that they are the zero sublevel sets of the following functions:

$$\phi_r = d_{X^{-r}} + \max(f_{r,c} - c, 0)$$

- In Section 3.1, we define the regularity condition required on sets $X \subset \mathbb{R}^d$ to prove the Morse Theorems. Such sets are called *complementary regular*. **??** describes how most offsets are part of this class.
- In Section 3.2, we prove that there exists a $K > 0$ such that there exists a retraction of any tubular neighborhoods $(X_c^{-r})^K$ onto X when $r > 0$ is small enough. We prove a technical lemma to ensure that we can build an approximate inverse flow of $\phi_{r,c}$ using Prop 2.6.
- In Section 3.3 we fix $r = 0$ and prove that for $\varepsilon > 0$ small enough, the sets X_{c+a} can be retracted onto $X_{c-\varepsilon}$ also using Prop 2.6.
- In Section 3.4 we let c be a critical value and assume there is only one corresponding critical point which is be non-degenerate. We show that for any $\varepsilon > 0$ the change in homology between $X_{c+\varepsilon}$ and $X_{c-\varepsilon}$ is determined by the curvature of X at the pair $\left(x, \frac{\nabla f(x)}{||\nabla f(x)||}\right)$. We prove this by considering $f_{r,c}$ to be f translated with magnitude r in the direction $-\nabla f(x)$.
- In Section 3.5 we let c be a critical value and assume that the critical points in $f^{-1}(c)$ are non-degenerate although there might be several of them. We determine the topology changes between $X_{c-\varepsilon}$ and $X_{c+\varepsilon}$ through the curvature of X by considering a more involved $f_{r,c}$. A slightly altered version of this reasoning applies to the hypothesis made by Fu [3], thereby showing (unsurprisingly) that the results stand with several non-degenerate critical points sharing the same critical value.

3.1 Complementary regular sets

The regularity condition we impose on closed sets of $\mathbb{R}^d$ in order to prove the Morse Theorems is as follows.

Definition 3.1: Regularity condition on X

We say that $X \subset \mathbb{R}^d$ is a complementary regular set when it verifies:

- $\mathrm{reach}(^\neg X) > 0$
- $\exists\, \mu \in (0, 1]$ *such that* $\mathrm{reach}_\mu(X) > 0$
- X *is compact*
- X *has full dimension.*

The following proposition gives mild sufficient conditions for offsets to be complementary regular.

Proposition 3.2: Offsets among complementary regular sets

1. *Let $\mu \in (0,1], Y \subset \mathbb{R}^d$ and ε be such that* $\mathrm{reach}_\mu(Y) > \varepsilon \geq 0$. *Then $X = Y^\varepsilon$ is a complementary regular set.*

2. *Let Z be a compact subanalytic subset of $\mathbb{R}^d$. For all $\varepsilon > 0$ but a finite number, Z^ε is a complementary regular set.*

Proof. 1. Let Y be any set checking those conditions. Theorem 4.1 from Chazal et al. [7] states everything except the full dimensionality which is a consequence of Clarke's Inversion Theorem for Lipschitz functions.

2. From Fu [5], we know that the set of $\mathrm{Crit}(Z) = \{\varepsilon > 0 \mid \forall \mu \in (0,1], \mathrm{reach}_\mu(Z^\varepsilon) = 0\}$ is locally finite. If this set were to be unbounded, the diameter of Z would be infinite by the characterization of $\partial^* d_X$ in term of closest points. By the previous result, for any $\varepsilon \in \mathrm{int}(\mathbb{R} \setminus \mathrm{Crit}(Z)) = \mathbb{R}^+ \setminus \mathrm{Crit}(Z)$, the offset Z^ε is a complementary regular set.

□

3.2 Building a deformation retraction between X_c and its smooth surrogate

Let $c \in \mathbb{R}$ be a regular value of $f_{|X}$. We build a smoothed out version of X_c which we denote X_c^{-r}. We define it as the intersection between X^{-r} and the sublevel set of a modified f denoted $f_{r,c}$:

$$X_c^{-r} = X^{-r} \cap f_{r,c}^{-1}(-\infty, c]$$

Remark that X_c (resp. X_c^{-r}) can be written as the zero sublevel set of $\phi^c = d_X + \max(f - c, 0)$ (resp. $\phi_r^c = d_{X^{-r}} + \max(f_{r,c} - c, 0)$).

Typically think of $f_{r,c}$ as f translated in the direction of the gradient of f around a critical point with magnitude r. It will be denoted f_r to ease notation as c will be fixed. The direction of translation only matters around critical values as shows the following lemma.

Lemma 3.3: Deformation retractions around X_c, X_c^{-r}

Let X be a complementary regular set. Let c be a regular value of $f_{|X}$ and $f_r = f \circ \gamma_r$ be f composed with a smooth function γ_r such that $\gamma_r(x) = x + r\eta(x)$ where $\eta, \nabla\eta$ are bounded on $\mathbb{R}^d$. Put $\phi = d_X + \max(f - c, 0)$ and $\phi_r = d_{X^{-r}} + \max(f_r - c, 0)$.

Then there exists $K > 0, M \geq 1, L \geq 1$ and piecewise-smooth flows

$$C : [0,1] \times \phi^{-1}(] - \infty, K]) \to \phi^{-1}(] - \infty, K])$$

$$C^r : [0,1] \times \phi_r^{-1}(] - \infty, K]) \to \phi_r^{-1}(] - \infty, K])$$

such that:

- *For all $r > 0$ small enough, $(X_c)^{\frac{K}{M}} \subset \phi_r^{-1}(] - \infty, K])$ and $(X_c^{-r})^{\frac{K}{M}} \subset \phi^{-1}(] - \infty, K])$*
- *$C(0, \cdot), C^r(0, \cdot)$ are identity over their respective spaces of definition;*
- *$C(1, (X_c)^{\frac{K}{M}}) = X_c$ and $C^r(1, (X_c^{-r})^{\frac{K}{M}}) = X_c^{-r}$*
- *For any $t \in [0,1]$, $C(t, \cdot)_{|X_c}$, $C^r(t, \cdot)_{|X_c^{-r}}$ are the identity over X_c and X_c^{-r}.*
- *$C(\cdot, x)$ and $C^r(\cdot, x)$ are $2KL$-Lipschitz in the first parameter when $r > 0$ is small enough, with $L = \sup\{d_0(\partial^* \phi(y))^{-1} \mid y \in \phi^{-1}(0, K]\}$*

Proof. Remark that $X_c = \phi_c^{-1}(\{0\})$ (resp. X_c^{-r} with ϕ_c^r. We want to apply Prop 2.6.

Define

$$\omega(s,K) = \inf_{\substack{r\in[0,s]\\ x\in\phi_r^{-1}(0,K]}} d_0(\partial^*\phi_r(x))$$

The technical Lemma 3.5 describes the essential result. Admitting it for now, it states that

$$\liminf_{\substack{s\to 0^+\\ K\to 0^+}} \omega(s,K) > 0 \tag{3.1}$$

Take $K, s > 0$ small enough that for all $r \in [0,s]$, $\partial^*\phi_r$ does not vanish on $\phi_r^{-1}(0,K]$, allowing the offsets to be retracted by Prop 2.6. The first derivatives of the flow are bounded by $l_{r,K} = \sup\{d_0^{-1}(\partial^*\phi_s(y)), s\in[0,r], y\in\phi_r^{-1}(0,K]\}$ which is finite when r, K are taken small enough and tend to L when r, K go to zero. Reparametrizing the flow as in the proof of Prop 2.6, we can choose C, C^r to be $\frac{1+\varepsilon}{KL}$ Lipschitz for any $\varepsilon > 0$.

The functions $(\phi_r)_{r\in[0,s]}$ are uniformly Lipschitz. Consider $M = 1+\sup\{\mathrm{Lip}(\phi_r)_{r\in[0,s]}\}$. Since the sets X_c^{-t} converge to X_c in the Hausdorff sense when t goes to 0, and since $||\phi - \phi_r|| = O(r)$, we have

$$(X_c^{-t})^{\frac{K}{M}} \subset \phi_r^{-1}(0,K]$$

for any t, r small enough. □

Corollary 3.4: Homotopy Equivalence

Let c be a regular value of $f_{|X}$ such that $f_r = f\circ\gamma_r$ with $\gamma_r(x) = x + r\eta(x)$ with $\eta, \nabla\eta$ smooth and bounded on $\mathbb{R}^d$.
Then for all $r > 0$ small enough, X_c^{-r} and X_c have the same homotopy type.

Proof. Since $\lim_{r\to 0} d_H(X_c^{-r}, X_c) = 0$, the flows C, C_r are respectively well defined on X_c^{-r}, X_c for r small enough thanks to the previous lemma.

Then $C(1,\cdot)\circ C^r(1,\cdot)$ (resp. $C^r(1,\cdot)\circ C(1,\cdot)$ is homotopic to Id_{X_c} (resp. $\mathrm{Id}_{X_c^{-r}}$) via the homotopy $(t,x)\mapsto C(1, C(t, C_r(t,x)))$ □

Lemma 3.5: Non vanishing $\partial^*\phi_r$ around a critical value

Let $r_i, K_i \to 0^+$, $x_i \in \phi_{r_i}^{-1}(0,K_i]$ and c be a regular value of $f_{[X}$. Then,

$$\liminf_{i\to\infty} d_0(\partial^*\phi_{r_i}(x_i)) > 0$$

Proof. We distinguish 7 cases to compute $\partial^*\phi_{r_i}(x_i)$. By extracting subsequences we can assume that (x_i) lies in one of this case. They are depicted in Figure 7.

In fact, we will show that for any such sequence, we have

$$\liminf_{i\to\infty} d_0(\partial^*\phi_{r_i}(x_i)) \geq \min(\mu,\sigma,\kappa) > 0 \tag{3.2}$$

where

- $\kappa = \inf_{f^{-1}(c)\cap X} ||\nabla f||$. It is a positive quantity because c is a regular value of $f_{|X}$.

- $\sigma = \inf_{x \in \partial X \cap f^{-1}(c)} d_0(A_x)$ where $x \mapsto A_x$ is the upper semi-continuous set-valued application defined by:

$$\begin{aligned} A_x &= \left\{ \lambda u + \nabla f(x) \mid \lambda \in [0,1], u \in \partial^* d_X(x) \right\} \cup \left\{ u + \lambda f(x) \mid \lambda \in [0,1], u \in \partial^* d_X(x) \right\} \\ &= \left\{ [0,1] \cdot \partial^* d_X(x) + \{\nabla f(x)\} \right\} \cup \left\{ \partial^* d_X(x) + [0,1] \cdot \{\nabla f(x)\} \right\} \end{aligned}$$

  It is positive because c is a regular value of $f_{|X}$, $\partial X \cap f^{-1}(c)$ is a compact set and the map $x \mapsto d_0(A_x)$ is lower semicontinuous. If it were to be zero, there would be a point $x \in \partial X \cap f^{-1}(c)$ with $d_0(A_x) = 0$. Since $\operatorname{Cone} \partial^* d_X(x) = \operatorname{Cone} \operatorname{Nor}(X, x)$, this means that the direction of $\nabla f(x)$ meets $\operatorname{Nor}(X, x)$, which is a contradiction.

- $\mu \leq \inf_{t \to 0} \{ d_0(\partial^* d_X(x)) \mid 0 < d_X(x) < t \}$ is positive by hypothesis.

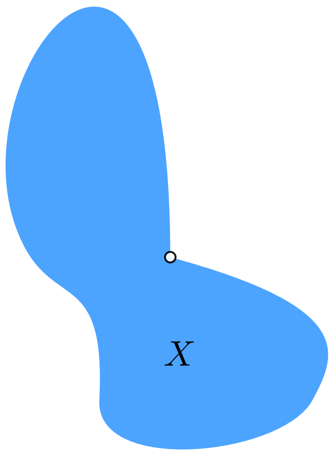
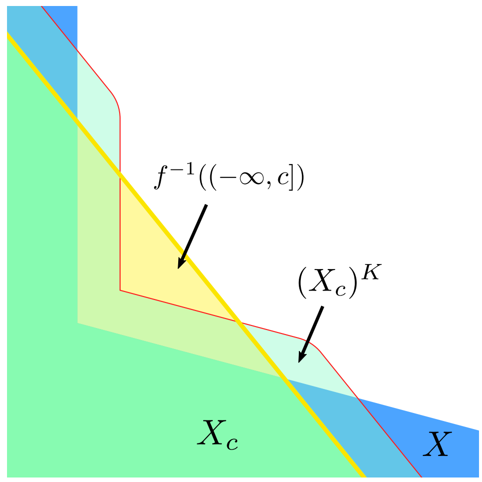


Illustration. Here X is a compact of $\mathbb{R}^2$ with $\operatorname{reach}_\mu(X) > 0$ for some $\mu > 0$.



Zoomed-in depiction of $X_c = X \cap f^{-1}(-\infty, c]$ and a tubular neighborhood $(X_c)^K$, $K > 0$ where f is a linear form.

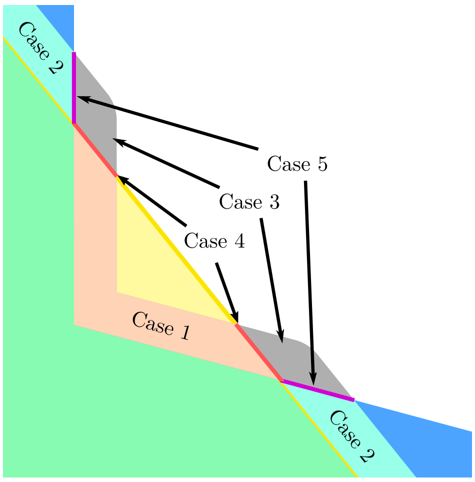
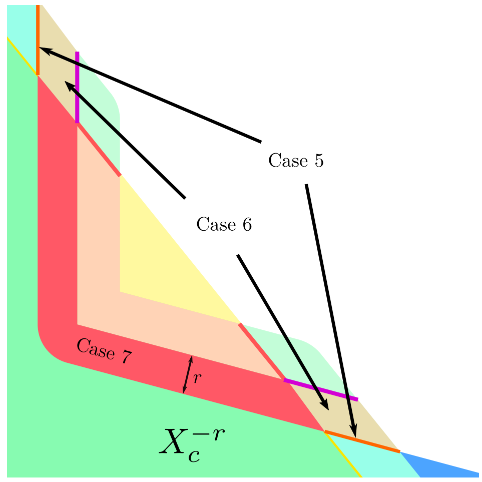


Illustration of cases 1 to 5 when $r = 0$. Cases 1 to 4 are defined independently of r.



Cases 5, 6 and 7 when $r > 0$.

FIGURE 7: Illustration of the 7 cases.

Idea. For each of the following cases, $\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i))$ is greater than one of the constants. The computation of $\partial^* \phi_{r_i}(x_i)$ shows that it either lies close to $\nabla f(x_i)$ or $\partial^* d_X(x_i)$ or close to be inside A_{x_i}. To ease some notations we write $\nu(x) = \frac{x}{||x||}$.

Case 1. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) < c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i)$ with $0 < d_X(x_i) < K_i$. By the μ-reach hypothesis, we have

$$\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \mu > 0 \tag{3.3}$$

Case 2. $x_i \in \operatorname{int}(X^{-r_i})$.

Then $\partial^* \phi_{r_i}(x_i) = \nabla f_{r_i}(x_i)$ and $0 < f_{r_i}(x_i) - c \leq K_i$. Since $||\nabla f_{r_i}(x_i) - \nabla f(x_i)|| = O(r_i)$, we have

$$\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \kappa > 0 \tag{3.4}$$

Case 3. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) > c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i) + \nabla f(x_i) \subset A_{x_i}$

$$\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0 \tag{3.5}$$

Case 4. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) = c$.

Remark that the Clarke gradient can be computed in a set of density 1 at x_i. Since $\nabla f_{r_i}(x_i)$ is non zero, $\{y, f_{r_i}(y) \neq c\}$ has density 1 at x. Now assume $x_i \to x \in \partial X \cap f^{-1}(c)$.

$$\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq d_0(A_x) \geq \sigma > 0 \tag{3.6}$$

Case 5. $x_i \in \partial X^{-r_i}$ and $f_{r_i}(x_i) > c$.

If $r_i > 0$, then $\partial^* d_{X^{-r_i}}(x_i) = [0,1] \cdot \nu(\xi_{\neg X}(x_i) - x_i)$. Adding the contribution of f_{r_i} we obtain

$$\partial^* \phi_{r_i}(x_i) \subset A_{\xi_{\neg X}(x_i)}$$

If $r_i = 0$, then $\partial^* \phi_{r_i}(x_i) = [0,1] \cdot \operatorname{Conv} \operatorname{Nor}(X, x_i) + \nabla f_{r_i}(x_i)$ and we obtain

$$\partial^* \phi_{r_i}(x_i) \subset A_{x_i}$$

Either way, $\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0$.

Now the remaining cases fit inside $0 < d_X^{-r}(x) \leq r$. Remark that we have

$$\partial^* d_{X^{-r}}(x) = \{\nu(x - \xi_X(x))\} \subset \operatorname{Conv} \operatorname{Nor}(X, \xi_{\neg X}(x))$$

Case 6. $0 < d_X^{-r_i}(x_i) \leq r_i$ and $f_{r_i}(x_i) \geq c$

$\partial^* \phi_{r_i}(x_i) \subset \operatorname{Conv} \operatorname{Nor}(X, \xi_{\neg X}(x)) + [0,1] \cdot \nabla f_{r_i}(x_i)$. Now by compactness assume that $x_i \to x$. Then $x \in \partial X \cap f^{-1}(c)$ and thus

$$\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0 \tag{3.7}$$

Case 7. $0 < d_X^{-r}(x_i) \leq r_i$ and $f_{r_i}(x_i) < c$

Then $\partial^* \phi_{r_i}(x_i) \subset \operatorname{Conv} \operatorname{Nor}(X, \xi_{\neg X}(x))$ and thus

$$\liminf_{i \to \infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \mu > 0 \tag{3.8}$$

$\square$

3.3 Isotopy Lemma

In this subsection we prove that under our assumptions the topology of the sublevel sets does not evolve in between critical values.

Theorem 3.6: Constant homotopy type in between critical values

Let $X \subset \mathbb{R}^d$ be a complementary regular set. Let $f : \mathbb{R}^d \to \mathbb{R}$ and $a < b \in \mathbb{R}$ be such that $[a, b]$ contains only regular values of $f_{|X}$.

Then X_a is a deformation retraction of X_b.

This theorem is a direct consequence of Lemma 3.8. We prove a technical lemma first.

Lemma 3.7: Locally non-vanishing Clarke gradients

Let c be a regular value of $f_{|X}$.
Then

$$\lim_{\substack{\varepsilon\to 0^+\\ K\to 0^+}} \inf \left\{ d_0(\partial^* \phi_{c+a}(x)), x \in \phi_{c+a}^{-1}(0, K] \mid a \in [-\varepsilon, \varepsilon] \right\} > 0$$

Proof. We proceed by contradiction. Assuming the inequality is false, there exist $a_i \to 0, K_i \to 0^+$ and $(x_i)_{i\in\mathbb{N}}$ a sequence in $\mathbb{R}^d$ such that

$$\lim_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) = 0$$

We keep the partition of $\phi_{c+a}^{-1}(0, K)$ as in the proof of Lemma 3.5. With $r = 0$, we obtain 5 cases to compute $\partial^* \phi_{c+a_i}$.

Case 1. $f(x_i) < c + a_i$. Then $\partial^* \phi_{c+a_i}(x_i) = \partial^* d_X(x_i)$ and thus

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) \geq \mu > 0$$

Case 2. $x_i \in \mathrm{int}(X)$. Then $\partial^* \phi_{c+a_i}(x_i) = \{\nabla f(x_i)\}$ and thus

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) \geq \sigma > 0$$

Cases 3, 4, 5.

$$\begin{cases} f(x_i) > c + a_i & \text{and} & d_X(x_i) > 0 \\ f(x_i) > c + a_i & \text{and} & x_i \in \partial X \\ f(x_i) = c + a_i & \text{and} & d_X(x_i) > 0 \end{cases}$$

In these 3 cases we have the inclusion $\partial^* \phi_{c+a_i}(x_i) \subset A_{x_i}$. As in the proof of Lemma 3.5, the map $y \mapsto A_y$ is lower semicontinuous. Now if (x_i) converges to x then it belongs to $\partial X \cap f^{-1}(c)$ and c being a regular value yields

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) \geq \kappa > 0$$

□

Lemma 3.8: Local deformation retractions

Let X be complementary regular, $f : \mathbb{R}^d \to \mathbb{R}$ smooth and let c be a regular value of $f_{|X}$. Then $X_{c-\varepsilon}$ is a deformation retraction of X_{c+a} for all $a \in [-\varepsilon, \varepsilon]$ for any $\varepsilon > 0$ small enough.

Proof. Put σ the positive constant obtained in Lemma 3.7. Thus for every $a \in [-\varepsilon, \varepsilon]$ there exists a $\frac{2K}{\sigma}$-Lipschitz approximate flow of ϕ_{c+a} on $\phi_{c+a}^{-1}(0, K]$ which we will denote $C_{c+a}(\cdot, \cdot)$. We also fix $M > 0$ such that the ϕ_{c+a} are all M Lipschitz over the sets we are considering.

Thus $C_{c-\varepsilon}$ is well-defined at any time in $X_{c-\varepsilon}^{\frac{K}{M}} \subset \phi_{c-\varepsilon}[0, K]$. Now since the $(\phi_{c+a})_{a\in[-\varepsilon,\varepsilon]}$ are Lipschitz functions whose constants are uniformly bounded, there is a constant $Q > 0$ such that $X_{c+a} \subset X_{c-\varepsilon}^{\varepsilon Q}$ for all $a \in [-\varepsilon, \varepsilon]$. For ε small enough we also have $X_{c+a} \subset \phi_{c-\varepsilon}^{-1}[0, K]$. Thus the approximate flow $C_{c-\varepsilon}(\cdot, \cdot)$ restricted to $[0, 1] \times X_{c+a}$ is well-defined for any $a \in [-\varepsilon, \varepsilon]$ when $\varepsilon > 0$ is small enough.

Now we show that for $\varepsilon > 0$ small enough, for any $a \in [-\varepsilon, \varepsilon]$ the end-flow $C_{c-\varepsilon}(1, \cdot)_{|X_{c+a}}$ is homotopic to $\mathrm{Id}_{X_{c+a}}$ via the homotopy

$$(t, x) \mapsto C_{c+a}(1, C_{c-\varepsilon}(t, x))$$

The homotopy is easy to check as long as it is well-defined.

$C_{c-\varepsilon}$ is $\frac{2K}{\sigma}$-Lipschitz in time parameter, yielding $C_{c-\varepsilon}(1, X_{c+a}) \subset X_{c+a}^{\frac{2K\varepsilon}{\sigma}}$. This last set is a subset of $\phi_{c+a}^{-1}[0, K]$ when $\varepsilon > 0$ is small enough. ☐

3.4 Handle attachment around critical values

First we describe how a cell is added around a unique critical point.

Proposition 3.9: Around unique critical values

Let X be complementary regular and $f : \mathbb{R}^d \to \mathbb{R}$. Assume $f_{|X}$ has only one critical point p in $f^{-1}(c)$ which is non degenerate.
Then for any $\varepsilon > 0$ small enough $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ attached with a λ-cell, where

$$\lambda_p = \textit{ indice of the Hessian at } p + \textit{number of infinite curvatures at } p$$

Proof. Let x_c be the sole critical point with value c and $n_c = \frac{\nabla f(x_c)}{||\nabla f(x_c)||} \in \mathrm{Nor}(\neg X, x_c)$ the normalized gradient of f at this point. Put $f_r(x) = f(x - rn_c)$ to be f translated in the direction n_c with magnitude r.

The pair $(x_c, n_c) \in \mathrm{Nor}(\neg X)$ is regular by non-degeneracy of f at x. Denote $(\kappa_i')_{1\le i\le d-1}$ the principal curvatures (cf. Prop 2.2) of $\neg X$ at (x_c, n_c) sorted in ascending order and put $m = \max\{i, \kappa_i' < \infty\}$. The regularity of (x_c, n_c) for X guarantees that the Gauss map $x \in \partial\neg X^{-r} \mapsto n(x) \in \mathbb{S}^{d-1}$ is differentiable at $x_c + rn_c$. The principal curvatures of $\neg X^{-r}$ at $(x_c + rn_c, n_c)$ can be obtained from the κ_i' via $\kappa_{i,r}' = \frac{\kappa_i'}{1+r\kappa_i'}$. Note that when a principal curvature κ_i' is infinite, the previous equality is valid with $\kappa_{i,r}' = \frac{1}{r}$.

The Gauss map of X^{-r} is the opposite of the previous one and thus also differentiable at $x_c + rn_c$. The principal curvatures $(\kappa_{i,r})_{1\le i\le d-1}$ of X^{-r} at $(x_c + rn_c, -n_c)$ are the opposite of that of $\neg X^{-r}$ $\kappa_{i,r} = -\kappa_{i,r}'$.

Let $a, b \in \mathrm{Tan}(X^{-r}, x_c + rn_c)$. The Hessian $H_r f_r$ of $(f_r)_{|X^{-r}}$ at $x_c + rn_c$ (cf. Def 2.3) is exactly

$$H_r f_r(a, b) = H f_r(a, b) + ||\nabla f_r(x_c + rn_c)||\, \mathrm{I\!I}_r(x_c + rn_c)(a, b)$$

where $\mathrm{I\!I}_r(x_c + rn_c)$ is the second fundamental form of X^{-r} at $x_c + rn_c$. Proceeding exactly in the same fashion as the proof of 4.6 in [3] we obtain that there exist matrices A_1, A_2, A_3, C, B such that in a good basis the Hessian $H_r f_r$ has the form

$$\begin{pmatrix} A_1 + rA_2 + r^2 A_3 & rC \\ rC^t & -r\,||\nabla f(p)||\, I_d + r^2 B \end{pmatrix}$$

where A_1 is the diagonal matrix of dimension m with diagonal $(-\kappa_i')_{1\leq i\leq m}$. It is the same computation as [3] except that we end up with a minus sign in front of the identity in the lower right corner. When $r > 0$ is small enough, the index of this matrix is that of A_1 plus the dimension of the identity matrix in the lower right corner. Then, we apply classical Morse Theory on sets bounded by a $C^{1,1}$ hypersurface to get the change in topology between $X^{-r}_{c-\varepsilon}$ and $X^{-r}_{c+\varepsilon}$. This is summarized in the following diagram.

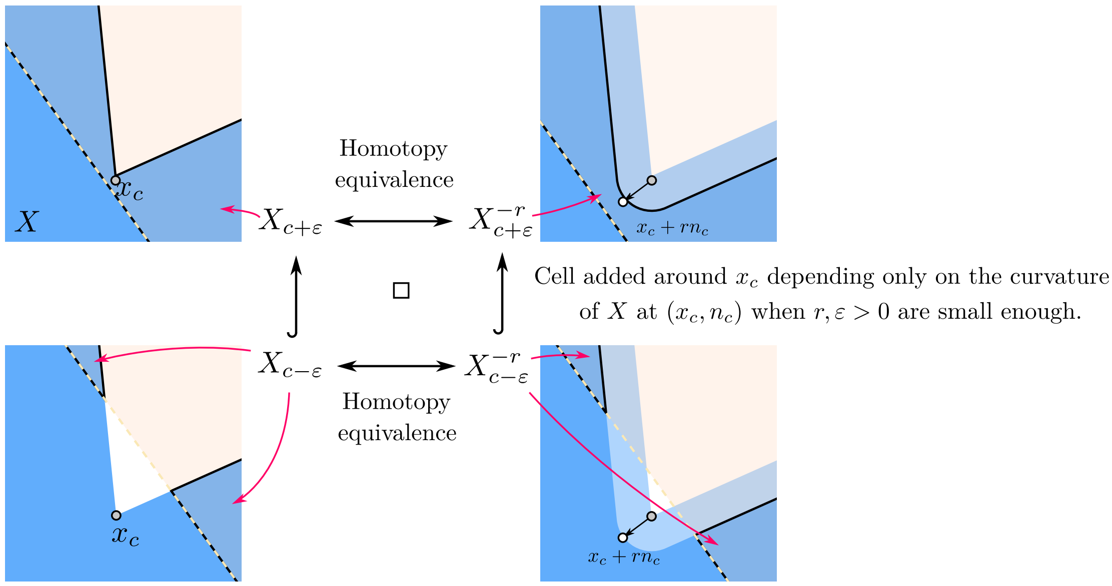


FIGURE 8: Commutative diagram in the proof of Prop 3.9

∎

3.5 Multi-handle attachement

Now we want to understand the change in topology when a critical value might have several corresponding critical points. We begin by showing that non-degenerate critical points of $f_{|X}$ have to be isolated.

Lemma 3.10: Correspondance between critical points of $f_{|X}$ and $f^r_{|X^{-r}}$

Let X be a subset of $\mathbb{R}^d$ and r such that $\mathrm{reach}(^\neg X) > r > 0$. *Assume x is a non-degenerate critical point of $f_{|X}$.*
Then $x^r = x + r\frac{\nabla f(x)}{||\nabla f(x)||}$ is a critical point of $f^r_{|X^{-r}}$ of the same value.
As a consequence, any non-degenerate critical point of $f_{|X}$ is isolated.

Proof. x^r being a critical point of $f^r_{X^{-r}}$ comes from a straightforward computation: we have $f^r(x^r) = f(x)$ and $\nabla f(x) = \nabla f^r(x^r)$. Moreover, we already know that $\mathrm{nor}(X^{-r}, x^r) = \mathrm{Cone}(\nabla f(x))$.

The last part follows from the isolatedness of critical points in X^{-r}.By the proof of Prop 3.9, x^r has to be a non-degenerate critical point for $f^r_{|X^{-r}}$ when $r > 0$ is small enough. Any non-degenerate critical point of a $C^{1,1}$ hypersurface has to be isolated. This forces x to be an isolated critical point by continuity of $y \mapsto y + rn_c$. ∎

Theorem 3.11: Morse Theory for sets whose complement set has positive reach

Let $X \subset \mathbb{R}^d$ *and* $\mu \in]0,1]$ *such that* $\mathrm{reach}_\mu(X) > 0$ *and* $\mathrm{reach}(^\neg X) > 0$.
Suppose $f_{|X}$ *has only non-degenerate critical points. Each critical level set* $X \cap f^{-1}(\{c\})$ *has a finite number* p_c *of critical points, whose indices (coming from Prop 3.9) we denote* $\lambda_1^c, \dots \lambda_{p_c}^c$.
Then

- *If* $[a,b]$ *does not contain any critical value,* X_a *is a deformation retract of* X_b.
- *If* c *is a critical value,* $X_{c+\varepsilon}$ *has the homotopy type of* $X_{c-\varepsilon}$ *with exactly* p_c *cells attached around the critical points in* $f^{-1}(c) \cap X$*, of respective dimension* $\lambda_{p_1}^c, \dots, \lambda_{p_c}^c$ *for all* $\varepsilon > 0$ *small enough.*

Proof. By Lemma 3.10 we know that the critical points in $f_{|X}$ have to be isolated. Put $x_1, \dots, x_p$ the critical points of $f_{|X}$ inside $f^{-1}(c)$. Put $n_i = \frac{\nabla f(x_i)}{||f(x_i)||}$ and $x_i^r = x_i + rn_i$. Let $n(x)$ be the normal n_i associated to the closest x_i of x. We will show that $\{x_1^r, \dots x_p^r\}$ is exactly the set of critical point of a certain $f^r_{|X^{-r}}$ with f^r a new function built in the following paragraphs.

Let $U_i \subset V_i$ be respectively closed and open balls containing x_i such that $\overline{V_i} \cap \overline{V_j} = \emptyset$ when $j \neq i$.

Let η_c be smooth function on $\mathbb{R}^d$ with values in $[0,1]$ such that η_c is constant of value 1 inside each U_i and 0 outside of $\bigcup V_i$. The map $\gamma_c : y \mapsto \eta_c(x)n(x)$ is well-defined and continuous when the U_i are small enough. When r is small enough, it is a diffeomorphism.

Finally, we keep the definition $X_c^{-r} = X^{-r} \cap f_r^{-1}(-\infty, c]$ but define a new f_r, which is f locally translated around the critical points:

$$f_r : x \mapsto f(x + r\gamma_c(x))$$

From Lemma 3.10 we know that the $(x_i^r)_{1 \le i \le p}$ are non-degenerate critical point of X^{-r} for $f_{r|X^{-r}}$ with corresponding index $(\lambda_i^c)_{1 \le i \le p}$. Moreover as in [3], Lemma 4.8, x_i^r is the only critical point inside $f_r(U_i)$ for $f_{r|X^{-r}}$ when r is small enough.

Now we prove that there are no critical points outside of $\cup_i f_r(U_i)$ when r is small enough. Remark that by classical theorems X^{-r} has a $C^{1,1}$ boundary. Since ∇f does not vanish in a neighborhood of $f^{-1}(c) \cap X$, we know that $x \in X^{-r}$ is a critical point of $f_{r|X^{-r}}$ if and only if $x \in \partial X^{-r}$, $\{\nu\} = \mathrm{Nor}(X^{-r}, x)$ (i.e ν is *the* normal at x) and $\left|\left|\frac{\nabla f_r(x)}{||\nabla f_r(x)||} - \nu\right|\right| = 0$.

Remark that we have both

- $\mathrm{Nor}(X^{-r}) = \{(x + r\nu, \nu), (x, \nu) \in \mathrm{Nor}(^\neg X)\}$
- $\sup_{(x,\nu)\in \mathrm{Nor}(X)} ||\nabla f(x) - \nabla f_r(x + r\nu)|| = O(r)$

leading to

$$\liminf_{r \to 0} \inf_{\substack{(x,\nu)\in \mathrm{Nor}(X^{-r}) \\ x \notin \cup_i f_r(U_i) \\ f_r(x)=c}} \left|\left|\frac{\nabla f_r(x)}{||\nabla f_r(x)||} - \nu\right|\right| \ge \inf_{\substack{(x,\nu)\in \mathrm{Nor}(^\neg X) \\ x \notin \cup_i U_i \\ f(x)=c}} \left|\left|\frac{\nabla f(x)}{||\nabla f(x)||} - \nu\right|\right| > 0 \tag{3.9}$$

Thereby showing that $\{x_1^r, \dots, x_p^r\}$ is exactly the set of critical points of $f_{r|X^{-r}}$ with value c. We obtain $X_{c+\varepsilon}^{-r}$ from $X_{c-\varepsilon}^{-r}$ by gluing cells locally around each critical point as in classical Morse Theory. □

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