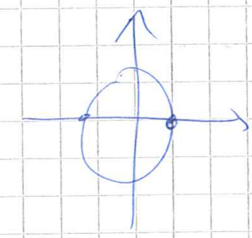
Kapitel fem:

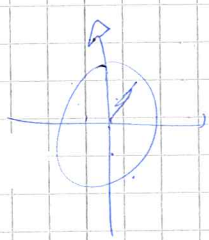
5.1

a) $\frac{2-2}{1} = 0$



b) $\frac{2\ln 2}{3\cos 2\pi} = \frac{2}{3}\ln 2$

c) $\frac{\sin \frac{3\pi}{9}}{3} = \frac{\sin \pi/3}{3} = \frac{\sqrt{3}}{6}$



d) $\frac{\arccos \frac{\sqrt{2}\cdot 2}{4}}{\arcsin \frac{-2}{4}} = \frac{\pi/4}{-\pi/6} = \frac{-4}{6}$

$= -2/3$ $\blacksquare$

5.2

a) $\frac{\sin(x-1)(x+1)}{x+1} = \frac{\sin(x^2-1)\ (x-1)}{(x^2-1)}$

$\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

$\Rightarrow \lim\limits_{x \to -1} \frac{\sin(x^2-1)}{(x^2-1)}\ (x-1) = 1.\ (-1\ \ -1)$

$= -2$ $\blacksquare$

b) $\frac{3\ln 2}{1^2} = 3\ln 2$ $\blacksquare$

c) $\frac{(x-3)(x+3)}{3-x} = -(x+3) \longrightarrow -6$

når $x \longrightarrow 3$ $\blacksquare$

d) $\frac{(x-1)\cancel{(x+2)}}{(x-2)\cancel{(x+2)}} \longrightarrow \frac{-2-1}{-2-2}$ når $x \longrightarrow -2$

$\longrightarrow \frac{-3}{-4} = 3/4$ $\blacksquare$

e) $\lim_{x \to 0} \frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2}$

$= \lim_{x \to 0} \frac{(1-x^2) - (1+x^2)}{x^2 \left(\sqrt{1-x^2} + \sqrt{1+x^2}\right)}$

$= \lim_{x \to 0} \frac{\cancel{1} - x^2 - \cancel{1} - x^2}{x^2 \left(\sqrt{1-x^2} + \sqrt{1+x^2}\right)}$

$= \lim_{x \to 0} \frac{-2 \cancel{x^2}}{\cancel{x^2}\left(\sqrt{1-x^2} + \sqrt{1+x^2}\right)} = \frac{-2}{\sqrt{1} + \sqrt{1}} = \frac{-2}{2}$

$= -1$ ■

f) $\lim_{x \to 1} \frac{\sqrt{x} - 1}{1 - x} = \lim_{x \to 1} \frac{\sqrt{x} - 1}{-(x-1)}$

$= \lim_{x \to 1} - \frac{\cancel{\sqrt{x} - 1}}{(\cancel{\sqrt{x}-1})(\sqrt{x} + 1)}$

$= \frac{-1}{\sqrt{1} + 1} = \frac{-1}{2}$

■

(53) a) $\lim_{x\to\infty} \frac{\sin x^2}{x^2} = \frac{A}{\infty} \to 0$

deoarece $-1 \le \sin x \le 1$ ▨

b) $\lim_{x\to 0} x \ln x^2 = 0$ (x mai slab decât ln x)

l'Hôpital $2x \ln x = \frac{2\ln x}{\frac{1}{x}} = \lim \frac{2\frac{1}{x}}{-\frac{1}{x^2}} = -2x \to 0$ ▨

c) $\lim_{x\to\infty} \frac{5x^3}{2e^{2x}} = 0$ (exp mai slab e)

ele $\frac{15x^2}{4e^{2x}} \to \frac{30x}{8e^{2x}} \to \frac{30}{16e^{2x}} \to 0$ ▨

d) $\lim_{x\to -\infty} x^3 e^x = 0$ e^{-x} mai slab e

$\frac{x^3}{e^{-x}} \to \frac{3x^2}{-e^{-x}} \to \frac{6x}{e^{-x}} \to \frac{6}{-e^{-x}} - 6e^x \to 0$ ▨

e) $\lim_{x\to\infty} \frac{\cancel{x^2}(1-\frac{3}{x})}{\cancel{x^2}(\frac{3}{x^2}-2)} = -1/2$ ▨

f) $\lim_{x\to\infty} \frac{x^3(1-2/x^3)}{x^3(\cancel{3}/x^3-2)} = -1/2$

(1) exp e^x

(2) putere x^p ▨

(3) log $\ln x$

5.4

a) $\lim_{x\to1} \frac{\sin(x-1)}{x(x-1)(x+1)} = \lim_{x\to1} \frac{\sin(x-1)}{x-1} \frac{1}{x(x+1)}$

$= 1 \cdot \frac{1}{1.2} = \frac{1}{2}$ $\blacksquare$

b) $\lim_{x\to0} \frac{\sin(x-\pi)}{\cos(x-\pi)} \frac{1}{x} =$

$\lim_{x\to0} \frac{-\sin(\pi-x)}{\cos(x-\pi)} \frac{1}{x} =$

$\lim_{x\to0} -\frac{\sin x}{\cos(x-\pi)} \cdot \frac{1}{x} =$

$\lim_{x\to0} -\frac{\sin x}{x} \cdot \frac{1}{\cos(x-\pi)} =$

$-1 \cdot \frac{1}{\cos(-\pi)} = -1 \cdot -1$

$= 1$ $\blacksquare$

c) $\lim_{x\to2} \frac{4-6+4}{4-4} \longrightarrow \infty$ $\blacksquare$

d) $\frac{-1+1}{1+1+1} \longrightarrow 0$ $\blacksquare$

e) Hôpital $1_{x\to0} \frac{\frac{1}{2\sqrt{x+1}} - \cancel{0}}{\frac{-1}{2\sqrt{1-x}} - \cancel{0}} = \frac{\frac{1}{1}}{\frac{-1}{1}} = -1$ $\blacksquare$

f) $\lim_{x\to\infty} \frac{\cancel{x^3}\left(\frac{\ln x}{x}\right)}{\cancel{x^3}(1-0)} = 0$ poleus moblear

Hôpital $\frac{\frac{1}{x}}{1} \xrightarrow[x\to\infty]{} 0$ $\blacksquare$

5.5

a) $$\lim_{x\to\infty} \frac{\arctan x - \arctan \frac{1}{x}}{\arctan(-2x)}$$

$$= \frac{\frac{\pi}{2} - 0}{-\frac{\pi}{2}} = \frac{\cancel{\frac{\pi}{2}}\,(1)}{\cancel{\frac{\pi}{2}}\,(-1)} = -1$$

∎

b) $$\lim_{x\to\infty} \frac{x \ln x}{2^x}$$ förläng med "x"!

$$= \lim_{x\to\infty} \frac{\ln x \cdot x^2}{2^x \cdot x}$$

$$= \lim_{x\to\infty} \frac{\ln x}{x} \cdot \lim_{x\to\infty} \frac{x^2}{2^x}$$

$$= 0 \cdot 0 = 0$$

∎

c) $$\lim_{x\to\infty} \frac{x^3 e^x + 1}{x^2 e^{2x} - 2x}$$

$$= \lim_{x\to\infty} \frac{x^{\cancel{3}} e^x \left(1 + \cancel{\frac{1}{x^3 e^x}}\right)}{\cancel{x^2} e^{2x} \left(1 - \cancel{\frac{2}{x e^{2x}}}\right)}$$

$$= \lim_{x\to\infty} \frac{x}{e^x}$$ exponential är starkare än potens

$$= 0$$

∎

d) $\lim_{x \to 0} \frac{\sin 4x}{\sin x} =$

$\lim_{x \to 0} \frac{2 \sin 2x \cos 2x}{\sin x} =$

$\lim_{x \to 0} \frac{4 \cancel{\sin x} \cos x \cos 2x}{\cancel{\sin x}} = 4$

e) $\lim_{x \to 0} \frac{1 - \cos x}{x^2} =$

$\lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2 (1 + \cos x)} =$

$\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x^2 (1 + \cos x)}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{x} \cdot \frac{1}{1 + \cos x}$

$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{1 + \cos x}$

$= 1 \cdot 1 \cdot \frac{1}{2} = \frac{1}{2}$

f) $\lim_{x \to 0} \frac{\cos^2 x - 1}{x^2} =$

$\lim_{x \to 0} - \frac{\sin^2 x}{x^2} =$

$- \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{\sin x}{x} =$

$- (1 \cdot 1) = -1$

5.6

(a) $\lim_{x \to 0} \frac{e^{2x}-1}{x} =$

$\lim_{x \to 0} \frac{e^{2x}-1}{\frac{1}{2} 2x} =$

$\frac{1}{\frac{1}{2}} \lim_{x \to 0} \frac{e^{2x}-1}{2x} =$

$2 \cdot 1 = 1$

utnyttja att $\lim_{x \to 0} \frac{e^{x}-1}{x} = 1$

(b) $\lim_{x \to 0} \frac{\ln(1-2x)}{x}$

Hopital: $\lim_{x \to 0} \frac{\frac{-2}{1-2x}}{1} = \frac{-2}{1} = -2$

eller

$\lim_{x \to 0} \frac{\ln(1-2x)}{\frac{-2x}{-2}} = -2 \cdot \lim_{x \to 0} \frac{\ln(1-2x)}{-2x}$

$= \lim_{t \to 0} (-2) \frac{\ln(1+t)}{t}$ $\quad (t = -2x$, $x \to 0$, $t \to 0)$

$= (-2) \cdot 1 = -2$

eller Maclaurin

$\ln(1-2x) \approx -2x$

$\frac{\ln(1-2x)}{x} \approx \frac{-2x}{x} = -2$

5.7 a) $\lim_{x \to 0} \ln \frac{\sin x}{x} = \ln 1$ (kaärd formel)
$= 0$

eller Hôpital $\frac{\cos x}{1} \xrightarrow[x \to 0]{} 1$ ▨

b) $\lim_{x \to 2} \ln\left(\frac{x^2 - 2x}{2x - 4}\right)$

$= \lim_{x \to 2} \ln \frac{x(x-2)}{2(x-2)}$

$= \lim_{x \to 2} \ln\left(\frac{x}{2}\right) = \ln 1 = 0$ ▨

c) $\lim_{x \to \infty} \ln\left(\frac{2x^2 - x}{x^2 - 1}\right)$

$= \lim_{x \to \infty} \ln\left(\frac{\cancel{x^2}(2 - \cancel{\frac{1}{x}})}{\cancel{x^2}(1 - \cancel{\frac{1}{x^2}})}\right)$

$= \ln 2$ ▨

d) $\lim_{x \to -\infty} \ln\left(\sqrt{x^4 - 1} \cdot \frac{1}{x^2}\right)$

$= \lim_{x \to -\infty} \ln\left(\frac{\cancel{|x^2|}\sqrt{1 - \frac{1}{x^4}}}{\cancel{x^2}}\right)$

$= \lim_{x \to -\infty} \ln 1 = 0$ ▨

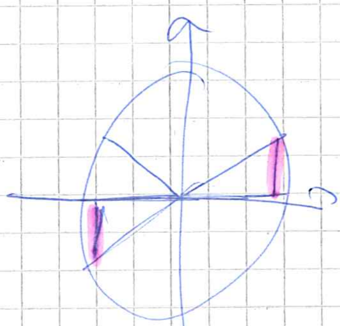
e) $\lim_{x \to \infty} \frac{\arctan x}{x} = \frac{\frac{\pi}{2}}{\infty} = 0$ ■

f) $\lim_{x \to \pi} \frac{(x-\pi)\sin x}{\tan^2 x}$

$= \lim_{x \to \pi} (x-\pi)\sin x \quad \frac{\cos^2 x}{\sin^2 x}$

$= \lim_{x \to \pi} \frac{(x-\pi)\cos^2 x}{\sin x}$

$= \lim_{x \to \pi} \frac{(x-\pi)\cos^2 x}{-\sin(x-\pi)}$



$= \lim_{t \to 0} \frac{t \cdot \cos^2(t+\pi)}{-\sin t}$ $\quad t = x - \pi$, $t \to 0$

$= \lim_{t \to 0} \frac{-t}{\sin t} \cdot \cos^2(t+\pi)$

$= -1 \cdot \cos^2(\pi)$

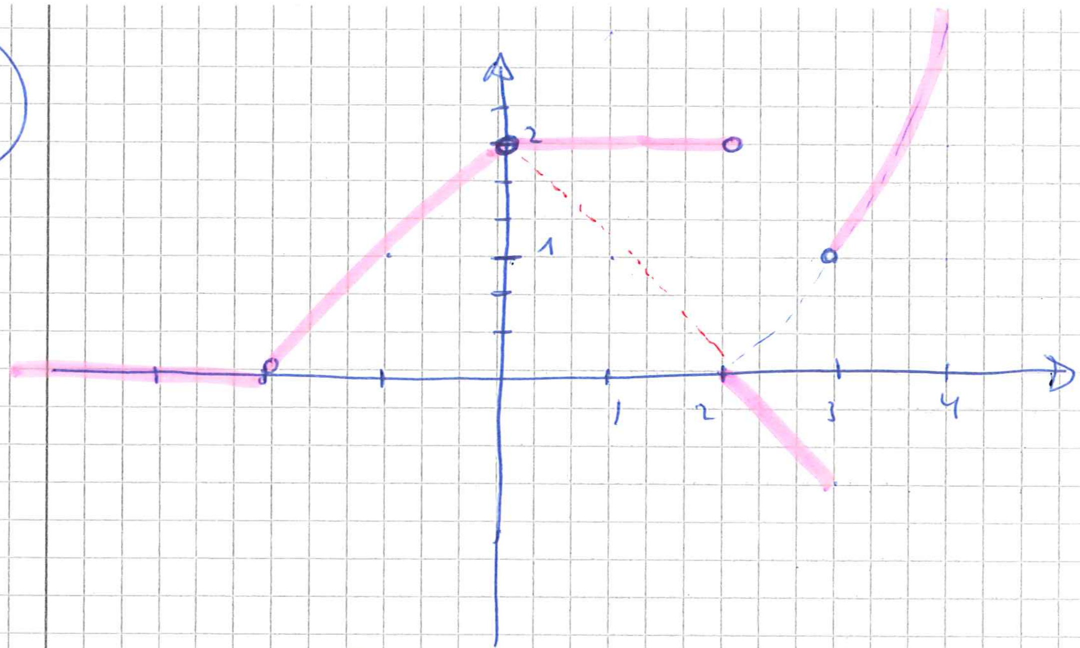
$= -1 \cdot -1 \cdot -1 = -1$

Kom ihåg $\sin(x-\pi) = -\sin x$

och om $t = x - \pi$

$x = t + \pi$ ■

5.8



$x = 0$ (tom cirkel för båda sidor)

$x = 2$

$x = 3$

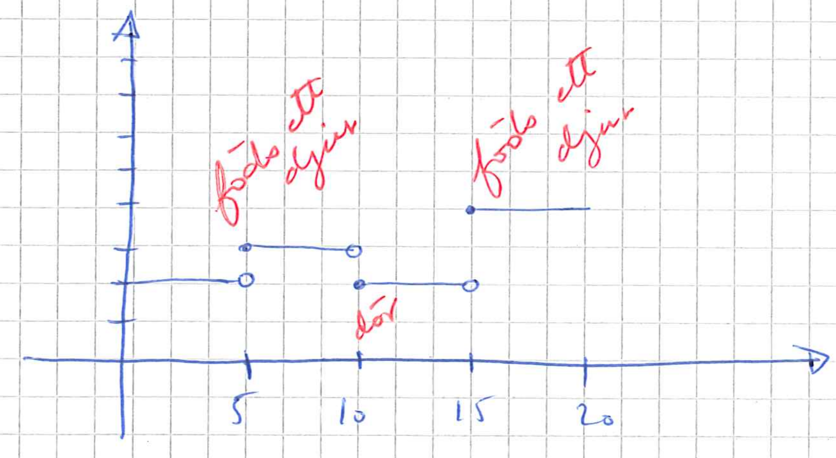
$x = 0$ är hävbara genom att framtvinga villkoret

$$f(x) = \begin{cases} x+2 & \text{om } -2 < x \le 0 \\ 2 & \text{om } 0 < x < 2 \end{cases}$$

eller

$$f(x) = \begin{cases} x+2 & \text{om } -2 < x < 0 \\ 2 & \text{om } 0 \le x < 2 \end{cases}$$

5.9



Diskontinuiteten beror på att djur föds eller dör, annars är funktionen konstant.

5.10 $f(x) = \begin{cases} ax & \text{om } x \leq 1 \\ x^2 - 1 & \text{om } x > 1 \end{cases}$

kontinuerlig om $ax = x^2 - 1 \quad (x = 1)$

$$a \cdot 1 = 1^2 - 1$$

$$a = 0 \quad \blacksquare$$

5.11 $f(x) = \begin{cases} ae^{x+1} & \text{om } x \leq -1 \\ 1 + ax & \text{om } x > -1 \end{cases}$

Kontinuerlig om $ae^{x+1} = 1 + ax \quad (x = -1)$

$$ae^{-1+1} = 1 + a(-1)$$

$$ae^0 = 1 - a$$

$$a + a = 1$$

$$a = 1/2 \quad \blacksquare$$

5.12 $y = \dfrac{|x^2 - 1|}{x+1}$

$y = \dfrac{|(x-1)(x+1)|}{(x+1)}$ $\quad x \neq -1$

Hade det varit $y = \dfrac{x^2 - 1}{x+1} = \dfrac{(x+1)(x-1)}{x+1}$

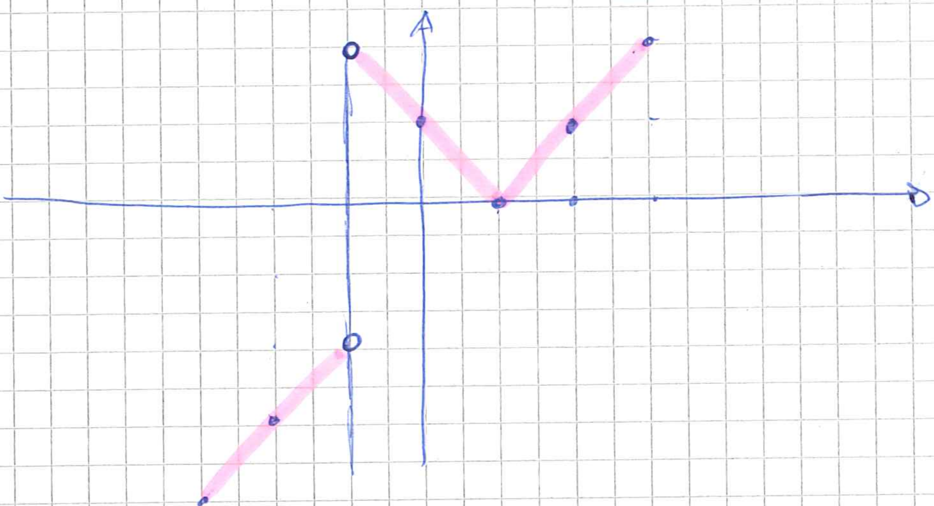
så skulle det vara hävbart.

I det här fallet är diskontinuiteten inte hävbar. Diskontinuiteten är vid $x = -1$.

$\Rightarrow$

Rita grafen : $y = \frac{|x^2-1|}{x+1}$

$x \neq -1$



$x = 1 \qquad y = 0$

$x = 2 \qquad y = \frac{|4-1|}{2+1} = \frac{3}{3} = 1$

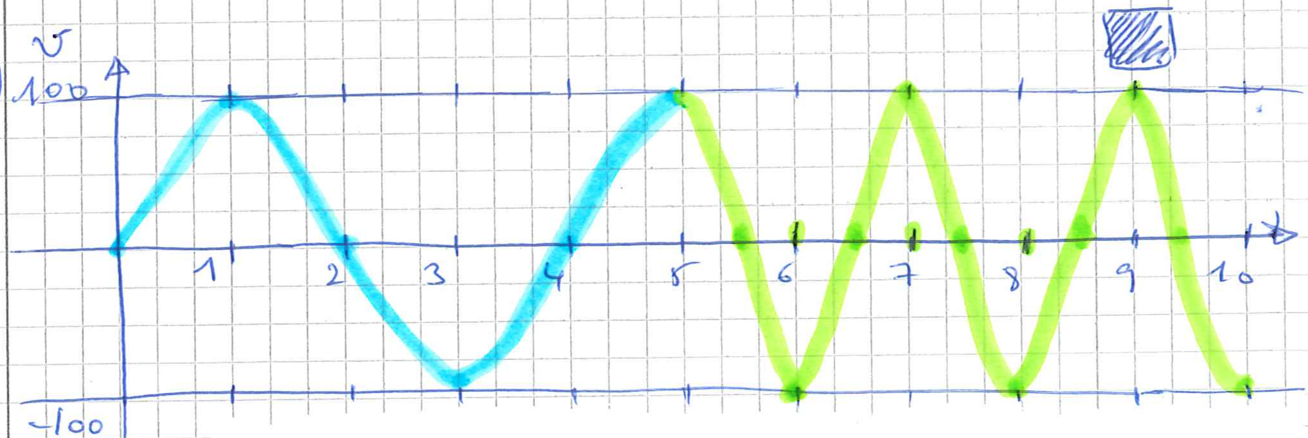
$x = 0 \qquad y = \frac{|-1|}{1} = 1$

$x = 3 \qquad y = \frac{|9-1|}{3+1} = \frac{8}{4} = 2$

$x = -2 \qquad y = \frac{|4-1|}{-2+1} = \frac{3}{-1} = -3$

$x = -3 \qquad y = \frac{|9-1|}{-3+1} = \frac{8}{-2} = -4$

5.13



$$v = f(t) = \begin{cases} 100 \sin \frac{\pi}{2} t & 0 \leq t < 5 \\ 100 \sin \pi t & 5 \leq t < 10 \end{cases}$$

funktionen är kontinuerlig

5.14

a) $y = \frac{1}{x-1}$

$x = 1$ är den lodräta asymptoten

när $x \to \pm\infty$ $y \to 0$

$y=0$ är den vågräta asymptoten

b) $y = \frac{x}{x^2-1}$

$x = \pm 1$ är de lodräta asymptoterna

när $x \to \pm\infty$ $\frac{\cancel{x}(1)}{x^2(1-\cancel{\frac{1}{x^2}})} \to \frac{1}{x} \to 0$

$y=0$ är den vågräta asymptoten

c) $y = \frac{x^2-1}{x^2+1}$

$x^2+1 = 0$ Aldrig
dvs det finns ingen
lodrät asymptot.

när $x \to \pm\infty$ $y = \frac{\cancel{x^2}(1-\cancel{\frac{1}{x^2}})}{\cancel{x^2}(1+\cancel{\frac{1}{x^2}})} \to 1$

$y=1$ är den vågräta asymptoten

d) $y = \dfrac{x^3+1}{x^2-1} = \dfrac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}(x-1)}$ Faktorisera

$x = 1$ är den vertikala asymptoten

e) $y = \dfrac{x^2+2}{x^2+x-6}$

faktorisera nämnaren $(x-2)(x+3)$

$\left.\begin{array}{l} x=2 \\ x=-3 \end{array}\right\}$ vertikala asymptoter

när $x \longrightarrow \pm\infty$ $y = \dfrac{\cancel{x^2}\left(1+\cancel{\frac{2}{x^2}}\right)}{\cancel{x^2}\left(1+\cancel{\frac{1}{x}}-\cancel{\frac{6}{x^2}}\right)}$

$y \longrightarrow 1$

$y = 1$ är den horizontella asymptoten

f) $y = \dfrac{x-2}{x^2-3x+2}$

Faktorisera

$x^2-3x+2 = (x-1)(x+2)$

$x = 1$ och $x = -2$ är de vertikala asymptoterna

när $x \longrightarrow \pm\infty$

$y = \dfrac{\cancel{x}\left(1-\cancel{\frac{2}{x}}\right)}{x^{\cancel{2}}\left(1-\cancel{\frac{3}{x}}+\cancel{\frac{2}{x^2}}\right)} \longrightarrow \dfrac{1}{x} \longrightarrow 0$

$y = 0$ är den horizontella asymptoten

5.15

$$\frac{t+1}{t} = \frac{4N}{3N+4000} \qquad t \geq 4$$

$$(3N+4000)(t+1) = 4Nt$$

$$3Nt + 3N + 4000t + 4000 = 4Nt$$

$$-3Nt \qquad -3N \qquad -3Nt \qquad -3N$$

$$4000t + 4000 = Nt - 3N$$

$$4000(t+1) = N(t-3)$$

$$N = \frac{4000(t+1)}{(t-3)} \qquad t \geq 4$$

$$N = \frac{\cancel{t}\left(1+\frac{1}{\cancel{t}}\right)\cdot 4000}{\cancel{t}\left(1-\frac{3}{\cancel{t}}\right)}$$

när $t \to \infty$

$$\frac{1}{t} \to 0$$

och $N \to \dfrac{4000 \cdot 1}{1}$ N går mot 4000.

5.17

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= f'(x) \quad \text{dvs} \quad = \cos x$$

5.16

a) $\lim\limits_{n \to \infty} \frac{2^n}{3^n}$

$= \lim\limits_{n \to \infty} \left(\frac{2}{3}\right)^n$ $\qquad \frac{2}{3} < 1$

$\left(\frac{2}{3}\right)^n \xrightarrow[n \to \infty]{} 0$

$= 0$

b) $\lim\limits_{n \to \infty} \frac{n}{n - \frac{1}{\sqrt{n}}}$

$= \lim\limits_{n \to \infty} \frac{\cancel{n}(1)}{\cancel{n}(1-0)} = 1$

c) $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} 2 \cdot \left(\frac{1}{3}\right)^k$

$= \frac{2}{3} + 2\frac{1}{3^2} + 2 \cdot \frac{1}{3^3} + \dots$

Summen: $\lim\limits_{n \to \infty} S = \frac{\frac{2}{3}\left(1 - \cancel{\left(\frac{1}{3}\right)^n}\right)}{1 - \frac{1}{3}}$

$a = 2/3$

$q = 1/3$

$= \frac{\frac{2}{3}(1-0)}{2/3} = 1$

d) $\lim\limits_{n \to \infty} \sum\limits_{k=0}^{n} 2\left(\frac{1}{n}\right)^k$

$a = 2$

$q = \frac{1}{n}$

$\lim\limits_{n \to \infty} S = \frac{2\left(1 - \cancel{\left(\frac{1}{n}\right)^{n+1}}\right)}{1 - \cancel{\frac{1}{n}}} = \frac{2(1-0)}{(1-0)} = 2$

e) $\lim\limits_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n} = \ ?$

kolla B8 $\lim\limits_{x \to \infty} \left(1 + \frac{b}{x}\right)^{x} = e^{b}$

jämför och $b = -1$

dvs $\lim\limits_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n} = e^{-1}$ ■

f) $\lim\limits_{n \to \infty} \left(1 + \frac{2}{n}\right)^{3n}$

$= \lim\limits_{n \to \infty} \left(1 + \frac{2 \cdot 3}{3n}\right)^{3n}$

$= \lim\limits_{3n \to \infty} \left(1 + \frac{6}{3n}\right)^{3n}$ $\left(\begin{array}{l} n \to \infty \\ 3n \to \infty \end{array}\right)$

$= e^{6}$ ■