11.1
.2
.3

Kapitel elva

11.1.a)

a) $$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

dvs $A(x+1) + Bx = 1$

om $x = 0$ $\quad A(0+1) + B.0 = 1$ (1)

om $x = -1$ $\quad A(0) + B(-1) = 1$ (2)

dvs (1) $A = 1$

(2) $-B = 1 \qquad B = -1$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

11.2a) Nu kan vi bestämma en primitiv

$$\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx$$

$$= \ln|x| - \ln|x+1| + C$$

$$= \ln\left|\frac{x}{x+1}\right| + C$$

11.3.a) Nu räknar vi $\int_2^\infty f(x)\,dx$

$$\int_2^\infty \frac{1}{x(x+1)}\,dx = \left[\ln\left|\frac{x}{x+1}\right|\right]_2^\infty$$

$$= \lim_{x_1 \to \infty} \ln\left|\frac{\cancel{x_1}(1)}{\cancel{x_1}(1+\cancel{\frac{1}{x_1}})}\right| - \ln\left|\frac{2}{2+1}\right|$$

$$= \cancel{\ln 1} - \ln 2/3 = \ln\frac{3}{2}$$

11.4 $\displaystyle\int \frac{dx}{a^2x^2+b^2}$ $a>0$ $b>0$

$$I = \int \frac{dx}{b^2\left(1+\frac{x^2\cdot a^2}{b^2}\right)} = \int \frac{dx}{b^2\left(1+\left(\frac{x\cdot a}{b}\right)^2\right)}$$

Variabelsubstitution

$$\frac{ax}{b} = u$$

$\frac{a}{b}\,dx = du$ och $dx = \frac{b}{a}\,du$

dvs $$I = \int \frac{\frac{b}{a}\,du}{b^2(1+u^2)}$$

$$= \frac{\frac{b}{a}}{b^2}\int \frac{du}{1+u^2}$$

$$= \frac{\cancel{b}}{a}\cdot\frac{1}{b^{\cancel{2}}}\cdot \arctan u + C$$

tillbaka till "x"

$$I = \frac{1}{ab}\arctan\left(\frac{ax}{b}\right) + C$$

■

11.5

a) $$\int_0^\infty \frac{dx}{2+3x^2} = \int_0^\infty \frac{dx}{2\left(1+\frac{3}{2}x^2\right)}$$

$$= \frac{1}{2}\int_0^\infty \frac{dx}{1+\left(\sqrt{\frac{3}{2}}\,x\right)^2}$$

$$u = \sqrt{\frac{3}{2}}\,x$$
$$du = \sqrt{\frac{3}{2}}\,dx$$
$$dx = \sqrt{\frac{2}{3}}\,du = \frac{\sqrt{2}}{\sqrt{3}}\,du$$

$$= \frac{1}{2}\int_0^\infty \frac{\frac{\sqrt{2}}{\sqrt{3}}\,du}{1+u^2}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{\sqrt{3}}\int_0^\infty \frac{du}{1+u^2}$$

$x=\infty \quad u=\infty$
$x=0 \quad u=0$
de ändras inte ! :-)

$$= \frac{1}{\sqrt{6}}\arctan u\Big]_0^\infty$$

$$= \frac{1}{\sqrt{6}}\left[\arctan\infty - \arctan 0\right]$$

$$= \frac{1}{\sqrt{6}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2\sqrt{6}}$$

$$= \frac{\pi\cdot\sqrt{6}}{2\sqrt{6}\cdot\sqrt{6}} = \frac{\pi\sqrt{6}}{12}$$

b) $$\int_0^{\sqrt{2}} \frac{x+1}{x^2+2}\,dx = \int_0^{\sqrt{2}} \frac{1}{2}\frac{2x}{x^2+2} + \int_0^{\sqrt{2}} \frac{1}{2\left(1+\left(\frac{x}{\sqrt{2}}\right)^2\right)}$$

$$= \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)\Big]_0^{\sqrt{2}}$$

$$= \frac{1}{2}\ln(2+2) + \frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}}{\sqrt{2}}\right) - \frac{1}{2}\ln 2 - \arctan 0$$

$$= \frac{1}{2}\ln 4 + \frac{\sqrt{2}}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\ln 2 - 0$$

$$= \frac{1}{2}\ln 2^2 + \frac{\sqrt{2}\pi}{8} - \frac{1}{2}\ln 2 = \ln 2 - \frac{1}{2}\ln 2 + \sqrt{2}\frac{\pi}{8}$$
$$= \frac{1}{2}\ln 2 + \sqrt{2}\pi/8$$

11.6

a) $\frac{3}{9-x^2} = \frac{3}{(3-x)(3+x)}$

$= \frac{A}{3-x} + \frac{B}{3+x}$

dvs $A(3+x) + B(3-x) = 3$

$3A + Ax + 3B - Bx = 3 + 0x$

$\begin{cases} 3A + 3B = 3 \\ A - B = 0 \end{cases} \Rightarrow \begin{cases} A + B = 1 \\ A - B = 0 \end{cases}$

$\Rightarrow A = \frac{1}{2} \quad B = 1/2$

od $\frac{3}{9-x^2} = \frac{1}{2}\frac{1}{3-x} + \frac{1}{2}\frac{1}{3+x}$ ■

b) $\frac{1}{6x-x^2-5} = \frac{-1}{x^2-6x+5}$

$1-6+5=0$
$x_1 = 1$
$x_2 = 5$

$= \frac{-1}{(x-1)(x-5)}$

$= \frac{A}{x-1} + \frac{B}{x-5}$

dvs $A(x-5) + B(x-1) = -1$

$Ax - 5A + Bx - B = -1 + 0x$

$\begin{cases} A + B = 0 \\ -5A - B = -1 \end{cases} \Rightarrow -4A = -1$

$A = +1/4$

od

$B = -1/4$

$\frac{1}{6x-x^2-5} = \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x-5}$

■

11.6

c) kom ihåg:

$$x^3 - 1 = x^3 - 1^3$$
$$= (x-1)(x^2 + x + 1)$$

och

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)}$$

$$= \frac{A}{x-1} + \frac{Bx + C}{x^2+x+1}$$

dvs $A(x^2 + x + 1) + (Bx + C)(x-1) = 1$

$$Ax^2 + Ax + A + Bx^2 - Bx + Cx - C = 1 + 0x^2 + 0x$$

$$\begin{cases} A + B = 0 & \Rightarrow A = -B \\ A - B + C = 0 & \Rightarrow A + A + C = 0 \\ A - C = 1 & \end{cases} \Longrightarrow \begin{cases} 2A + C = 0 \\ A - C = 1 \end{cases}$$

$$\Longrightarrow 3A = 1 \qquad A = 1/3, \quad B = -1/3$$

och $C = A - 1 = 1/3 - 3/3 = -2/3 = C$

dvs $\dfrac{1}{x^3-1} =$

$$\frac{1}{3}\frac{1}{x-1} + \frac{-1/3\,x - 2/3}{x^2+x+1} =$$

$$\frac{1}{3}\left(\frac{1}{x-1} + \frac{-x-2}{x^2+x+1}\right) =$$

$$\frac{1}{3}\left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right)$$

11.7

kolla upp 11.6

a) $\int \frac{3}{9-x^2}\,dx$

$= \int \frac{1}{2}\frac{1}{3-x} + \frac{1}{2}\frac{1}{3+x}\,dx$

$= \int -\frac{1}{2}\frac{1}{x-3} + \frac{1}{2}\frac{1}{3+x}\,dx$

$= -\frac{1}{2}\ln|x-3| + \frac{1}{2}\ln|x+3| + C$

$= \frac{1}{2}\left[\ln|x+3| - \ln|x-3|\right] + C$

$= \frac{1}{2}\ln\left|\frac{x+3}{x-3}\right| + C$

$= \ln\sqrt{\left|\frac{x+3}{x-3}\right|} + C$ ■

b) $\int \frac{1}{6x-x^2-5}\,dx = \int \frac{1}{4}\frac{1}{x-1} - \frac{1}{4}\frac{1}{x-5}\,dx$

$= \frac{1}{4}\ln|x-1| - \frac{1}{4}|x-5| + C$

$= \frac{1}{4}\left[\ln|x-1| - |x-5|\right] + C$

$= \frac{1}{4}\ln\left|\frac{x-1}{x-5}\right| + C$ ■

OBS: Om ert svar skiljer sig från facit kom ihåg att

$|a-b| = |b-a|$:-)

c) $\displaystyle\int \frac{1}{x^3-1}\,dx = \frac{1}{3}\int \frac{1}{x-1} - \frac{x+2}{x^2+x+1}\,dx$

jag börjar med $\displaystyle\int \frac{x+2}{x^2+x+1}\,dx =$

$\displaystyle\int \frac{x+2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx =$ $\quad x^2+x+\frac{1}{4}+\frac{3}{4} =$

$\displaystyle\frac{1}{2}\int \frac{2x+4}{x^2+x+1} = \frac{1}{2}\int \frac{2x+1+3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

Varför? för att $(x^2+x+1)' = 2x+1$

$\displaystyle = \frac{1}{2}\int \frac{2x+1}{x^2+x+1} + \frac{1}{2}\int \frac{3}{\frac{3}{4}\left(1+\frac{\left(x+\frac{1}{2}\right)^2}{\frac{3}{4}}\right)}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{4}{\cancel{3}}\cdot\frac{\cancel{3}}{2}\int \frac{1}{1+\left(\frac{\left(x+\frac{1}{2}\right)2}{\sqrt{3}}\right)^2}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + 2\int \frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}\,dx$

$\displaystyle = \frac{1}{2}\int \frac{2x+1}{x^2+x+1} + \cancel{2}\cdot\frac{\sqrt{3}}{\cancel{2}}\int \frac{du}{1+u^2}$ $\quad \frac{2x+1}{\sqrt{3}} = u$ $\quad \frac{2}{\sqrt{3}}\,dx = du$ $\quad dx = \frac{\sqrt{3}\,du}{2}$

$\displaystyle = \frac{1}{2}\ln|x^2+x+1| + \sqrt{3}\arctan u + C$

$\displaystyle = \ln\sqrt{|x^2+x+1|} + \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$

I slutet:

$$\int \frac{1}{x^3-1}\,dx = \frac{1}{3}\int \frac{1}{x-1} - \frac{x+2}{x^2+x+1}\,dx$$

$$= \frac{1}{3}\Big[\ln|x-1| - \ln\sqrt{|x^2+x+1|} - \sqrt{3}\,\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\Big] + C$$

(11.8) $\displaystyle\int_2^3 \frac{x^4-4}{x^3-x}\,dx$

grad täljaren $\geqslant$ grad nämnaren

$\Longrightarrow$ polynomdivision

$$\begin{array}{r|l} & x \\ \hline x^4-4 & x^3-x \\ x^4-x^2 & \\ \hline x^2-4 & \end{array}$$

<u>dvs</u> $\displaystyle\int_2^3 \frac{x^4-4}{x^3-x}\,dx = \int_2^3 x + \frac{x^2-4}{x^3-x}\,dx$

$$= \int_2^3 x\,dx + \underline{\int_2^3 \frac{x^2-4}{x^3-x}\,dx}$$

$$\int \frac{x^2-4}{x^3-x}\,dx = \frac{1}{3}\int \frac{3(x^2-4)}{x^3-x}\,dx$$

Varför? derivera x^3-x

$\Rightarrow (x^3-x)' = 3x^2-1$

vi måste "se" $3x^2-1$ i täljaren

$$= \frac{1}{3}\int \frac{3x^2-12}{x^3-x}\,dx$$

$$= \frac{1}{3} \int \frac{3x^2 - 1 \quad - 11}{x^3 - x} \, dx$$

$$= \frac{1}{3} \int \frac{3x^2 - 1}{x^3 - x} dx - \frac{11}{3} \int \frac{1}{x^3 - x} \, dx$$

och hela integralen $\int_2^3 \frac{x^4 - 4}{x^3 - x} \, dx$

skrivs nu som :

$$\int_2^3 x \, dx + \frac{1}{3} \int_2^3 \frac{3x^2 - 1}{x^3 - x} dx - \frac{11}{3} \underbrace{\int_2^3 \frac{1}{x^3 - x} \, dx}$$

Nu vill vi kolla/räkna $\int \frac{1}{x^3 - x} \, dx$ genom att partial bråkuppdela

$$\frac{1}{x^3 - x} = \frac{1}{x(x^2 - 1)} = \frac{1}{x(x-1)(x+1)}$$

$$= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

dvs $A(x-1)(x+1) + B(x+1)x + Cx(x-1) = 1$

om $x = 0$ $\quad A(-1)(1) + 0 + 0 = 1 \quad A = -1$

om $x = 1$ $\quad A(0) + B(2) \cdot 1 + C(0) = 1 \quad B = 1/2$

om $x = -1$ $\quad A(0) + B(0) + C(-1)(-2) = 1 \quad C = 1/2$

och $\int \frac{1}{x^3 - x} \, dx = \underbrace{\int \frac{-1}{x} + \frac{1}{2} \frac{1}{x-1} + \frac{1}{2} \frac{1}{x+1} \, dx}$

Hele integralen blir:

$$\int_2^3 x\,dx + \frac{1}{3}\int_2^3 \frac{3x^2-1}{x^3-x}\,dx$$

$$- \frac{11}{3}\int_2^3 \frac{-1}{x} + \frac{1}{2}\,\frac{1}{x-1} + \frac{1}{2}\,\frac{1}{x+1}\,dx$$

$$= \Big[\frac{x^2}{2} + \frac{1}{3}\ln|x^3-x| - \frac{11}{3}\Big(-\ln|x| + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1|\Big)\Big]_2^3$$

$$= \Big[\frac{x^2}{2} + \frac{1}{3}\ln|x^3-x| + \frac{11}{3}\ln|x| - \frac{11}{6}\ln|x-1| - \frac{11}{6}\ln|x+1|\Big]_2^3$$

$$= \frac{x^2}{2} + \frac{1}{3}\ln|x^3-x| + \frac{11}{3}\ln|x| - \frac{11}{6}\ln(x-1)(x+1)\Big]_2^3$$

$$= \Big[\frac{x^2}{2} + \frac{1}{3}\ln|x^3-x| + \frac{11}{3}\ln|x| - \frac{11}{6}\ln|x^2-1|\Big]_2^3$$

$$= \frac{9}{2} + \frac{1}{3}\ln|27-3| + \frac{11}{3}\ln|3| - \frac{11}{6}\ln|9-1|$$
$$- \frac{4}{2} - \frac{1}{3}\ln|8-2| - \frac{11}{3}\ln|2| + \frac{11}{6}\ln|4-1|$$

$$= \frac{5}{2} + \frac{1}{3}\ln\frac{24}{6} + \frac{11}{3}\ln\frac{3}{2} + \frac{11}{6}\ln\frac{3}{8}$$

$$= \frac{5}{2} + \frac{1}{3}\ln 2^2 + \frac{11}{3}\ln 3 + \frac{11}{6}\ln 3 - \frac{11}{6}\ln 2^3$$
$$- \frac{11}{3}\ln 2$$

$$= \frac{5}{2} + \frac{2}{3}\ln 2 - \frac{11}{3}\ln 2 - \frac{33}{6}\ln 2 + \frac{11}{3}\ln 3$$
$$+ \frac{11}{6}\ln 3$$

$$= \frac{5}{2} + \ln 2 \left(\frac{2}{3} - \frac{11}{3} - \frac{33}{6} \right) + \ln 3 \left(\frac{11}{3} + \frac{11}{6} \right)$$

$$= \frac{5}{2} + \ln 2 \left(\frac{4 - 22 - 33}{6} \right) + \ln 3 \left(\frac{22 + 11}{6} \right)$$

$$= 5/2 + \ln 2 \left(-\frac{51}{6} \right) + \frac{33}{6} \ln 3$$

$$= \frac{5}{2} - \frac{17}{2} \ln 2 + \frac{11}{2} \ln 3$$

$$= \frac{1}{2} \left(5 - 17 \ln 2 + 11 \ln 3 \right)$$

$\blacksquare$

11.9

a) $$\int_0^1 \frac{dx}{(4-x^2)^2}$$

$$\frac{1}{(4-x^2)^2} = \frac{1}{(2-x)^2\,(2+x)^2}$$

$$= \frac{A}{2-x} + \frac{B}{(2-x)^2} + \frac{C}{2+x} + \frac{D}{(2+x)^2}$$

dvs $A(2-x)(2+x)^2 + B(2+x)^2 + C(2-x)(2+x)^2$
$+ D(2-x)^2 = 1$

om $x = 2$ $\quad A\cdot 0 + B(4)^2 + 0 + 0 = 1 \quad B = 1/16$

om $x = -2$ $\quad A\cdot 0 + 0 + 0 + D(2+2)^2 = 1 \quad D = 1/16$

$A(2-x)(2+x)^2 + C(2+x)(2-x)^2 = 1 - \frac{1}{16}(2+x)^2$
$- \frac{1}{16}(2-x)^2$

$16A(2-x)(2+x)^2 + 16C(2+x)(2-x)^2$
$= 16 - 4 - x^2 - \cancel{4x} - 4 - x^2 + \cancel{4x}$

$16(2-x)(2+x)[2A + Ax + 2C - Cx] = 8 - 2x^2$
$= 2(4-x^2)$

$2A + 2C + Ax - Cx = \frac{2}{16} \qquad A - C = 0$
$\qquad A = C$

$2A + 2C = \frac{2}{16}$

$4A = \frac{2}{16} = \frac{1}{8} \qquad D = 1/16$

$A = \frac{1}{32} \qquad C = 1/32 \qquad$ och $\quad B = 1/16$

$$\int_0^1 \frac{dx}{(4-x^2)^2} = \int_0^1 \frac{1}{32}\frac{1}{2-x} + \frac{1}{16}\frac{1}{(2-x)^2}$$
$$+ \frac{1}{32}\frac{1}{2+x} + \frac{1}{16}\frac{1}{(2+x)^2}\,dx$$

slutligen räknar vi:

$$\int_0^1 -\frac{1}{32}\frac{1}{x-2}\,dx + \int_0^1 \frac{1}{16}\frac{1}{(x-2)^2}\,dx$$

$$+\int_0^1 \frac{1}{32}\frac{1}{x+2}\,dx + \int_0^1 \frac{1}{16}\frac{1}{(x+2)^2}\,dx$$

$$= -\frac{1}{32}\ln|x-2| - \frac{1}{16}\frac{1}{x-2}$$

$$\left. + \frac{1}{32}\ln|x+2| - \frac{1}{16}\frac{1}{x+2}\right]_0^1$$

$$= \left[\frac{1}{32}\ln\left|\frac{x+2}{x-2}\right| - \frac{1}{16}\left(\frac{1}{x-2}+\frac{1}{x+2}\right)\right]_0^1$$

$$= \frac{1}{32}\ln\left|\frac{3}{-1}\right| - \frac{1}{16}\left(\frac{1}{1-2}+\frac{1}{3}\right)$$

$$\cancel{-\frac{1}{32}\ln\left|\frac{2}{-2}\right|} + \frac{1}{16}\left(\frac{1}{0-2}+\frac{1}{0+2}\right)$$

$$= \frac{1}{32}\ln 3 - \frac{1}{16}\left(-\frac{3}{3}+\frac{1}{3}\right) + \cancel{\frac{1}{16}\left(-\frac{1}{2}+\frac{1}{2}\right)}$$

$$= \frac{1}{32}\ln 3 - \frac{1}{16}\left(-\frac{2}{3}\right)$$

$$= \frac{1}{32}\ln 3 + \frac{1}{24}$$

$$= \frac{1}{8}\left(\frac{\ln 3}{4} + \frac{1}{3}\right)$$

■

11.1c $t = \int_{N_0}^{N} \frac{K}{n(M-n)} \, dn$

$$\frac{k}{n(M-n)} = \frac{A}{n} + \frac{B}{M-n}$$

$K, M > 0$
$M > N$

$A(M-n) + Bn = K$

$AM - An + Bn = K$

$AM = K$

$A = \frac{K}{M}$

$B - A = 0$

$A = B = \frac{K}{M}$

$$t = \int_{N_0}^{N} \frac{K}{M} \frac{1}{n} + \frac{k}{M} \frac{1}{M-n} \, dn$$

$$= \frac{k}{M} \int_{N_0}^{N} \frac{1}{n} + \frac{1}{M-n} \, dn$$

$$= \frac{k}{M} \ln|n| - \ln|M-n| \Big]_{N_0}^{N}$$

$$= \frac{k}{M} \ln\left|\frac{n}{M-n}\right| \Big]_{N_0}^{N}$$

$$= \frac{k}{M} \left[\ln \frac{N}{M-N} - \ln \frac{N_0}{M-N_0} \right]$$

$$= \frac{k}{M} \left[\ln \frac{N}{M-N} \cdot \frac{M-N_0}{N_0} \right]$$

$$t = \frac{k}{M} \ln \frac{N(M-N_0)}{N_0(M-N)}$$

N som en funktion av t:

$$t = \frac{K}{M} \ln \frac{N(M - N_0)}{N_0(M - N)}$$

$$\frac{Mt}{K} = \ln \frac{N(M - N_0)}{N_0(M - N)}$$

$$e^{\frac{Mt}{K}} = \frac{N(M - N_0)}{N_0(M - N)}$$

$$N_0 e^{\frac{Mt}{K}} M - N_0 \cdot N e^{\frac{Mt}{K}} = MN - NN_0$$

$$N_0 e^{\frac{Mt}{K}} M = N\left(M - N_0 + N_0 e^{\frac{Mt}{K}}\right)$$

och

$$N = \frac{M N_0 e^{\frac{Mt}{K}}}{M - N_0 + N_0 e^{\frac{Mt}{K}}}$$

$$N = \frac{\cancel{e^{\frac{Mt}{K}}}\left(M N_0\right)}{\cancel{e^{\frac{Mt}{K}}}\left(N_0 - \cancel{\frac{N_0}{e^{\frac{Mt}{K}}}} + \cancel{\frac{M}{e^{\frac{Mt}{K}}}}\right)}$$

om $t \longrightarrow \infty$

$$N \longrightarrow \frac{M N_0}{N_0} = M$$

då $N \longrightarrow M$
när $t \longrightarrow \infty$

$\blacksquare$