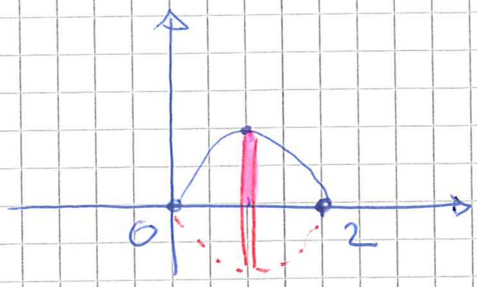
Kapitel tolv

12.1

a) $y = 2x - x^2$



Nollställen $y = x(2-x)$

$x = 0 \quad y = 0$

$x = 2 \quad y = 0$

$x = 1 \quad y = 2 - 1 = 1$

Rotation kring x-axeln ger en av figuren ovan som ser ut som en "american fotboll".

ΔV_{skiva} (cylinder) $= \pi r^2 \cdot$ tjockleken

$r = y$

Δx = tjockleken

$= \pi y^2 \Delta x$

$= \pi (2x - x^2)^2 \cdot \Delta x$

då $V = \int_0^2 \pi (2x - x^2) \, dx$

$V = \pi \int_0^2 (4x^2 + x^4 - 4x^3) \, dx$

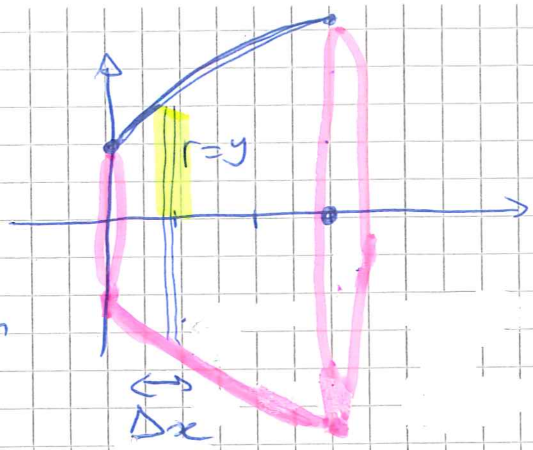
$= \pi \left[4\frac{x^3}{3} + \frac{x^5}{5} - \cancel{4}\frac{x^4}{\cancel{4}} \right]_0^2$

$= \pi \left[\frac{4}{3} \cdot 8 + \frac{1}{5} 32 - 16 - 0 - 0 + 0 \right]$

$= \pi \left[\frac{4 \cdot 8 \cdot 5}{3 \cdot 5} + \frac{32 \cdot 3}{5 \cdot 3} - \frac{16 \cdot 15}{15} \right]$

$= \frac{\pi}{15} \left[16 \right] = \frac{16\pi}{15}$ ve

b) $y = e^{x}$



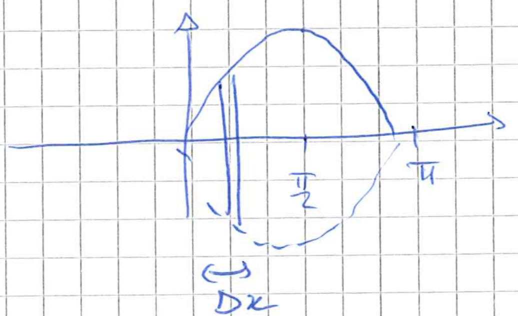
$$\Delta V_{skiva} = \pi r^2 \text{ tjockleken}$$
$$= \pi y^2 \cdot \Delta x$$
$$= \pi e^{2x} \cdot \Delta x$$

$x=0 \quad y=1$
$x=1 \quad y=e$

$$V = \pi \int_0^1 e^{2x} \, dx$$
$$= \pi \left[\frac{e^{2x}}{2} \right]_0^1$$
$$= \frac{\pi}{2} \left[e^{2x} \right]_0^1 = \frac{\pi}{2} \left[e^2 - e^0 \right]$$
$$= \frac{\pi}{2} \left[e^2 - 1 \right] \quad \text{ve} \quad \blacksquare$$

(Volym enheter)

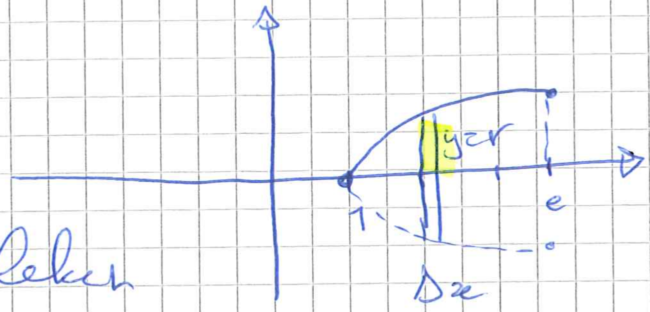
c) $y = \sin x$



$$\Delta V = \pi r^2 \cdot \text{ tjockleken}$$
$$= \pi y^2 \; \Delta x$$
$$= \pi \sin^2 x \; \Delta x$$

$$V = \pi \int_0^{\pi} \sin^2 x \, dx = \pi \int_0^{\pi} \frac{1 + \cos 2x}{2} \, dx$$
$$= \frac{\pi}{2} \int_0^{\pi} 1 + \cos 2x \, dx = \frac{\pi}{2} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi}$$
$$= \frac{\pi}{2} \left[\pi - 0 - 0 + 0 \right] = \frac{\pi^2}{2} \quad \text{Ve} \quad \blacksquare$$

d) $y = \ln x$



$\Delta V_{skiva} = \pi r^2$ tjockleken

$= \pi y^2 \, \Delta x$

$= \pi \ln^2 x \, \Delta x$

$$V = \pi \int_1^e \ln^2 x \, dx$$

partiell integration

$f: \ln^2 x \qquad g: 1$

$f': 2 \cdot \ln x \cdot \frac{1}{x} \qquad G: x$

$$V = \pi \left[x \ln^2 x - 2 \int \ln x \, dx \right]$$

Partiell integration <u>igen</u>

$f: \ln x \qquad g: 1$

$f': \frac{1}{x} \qquad G: x$

$$V = \pi \left[x \ln^2 x - 2 \left[x \ln x - \int 1 \, dx \right] \right]_1^e$$

$$V = \pi \left[x \ln^2 x - 2x \ln x + 2x \right]_1^e$$

$$= \pi \left[e \ln^2 e - 2e \ln e + 2e - 0 + 0 - 2 \right]$$

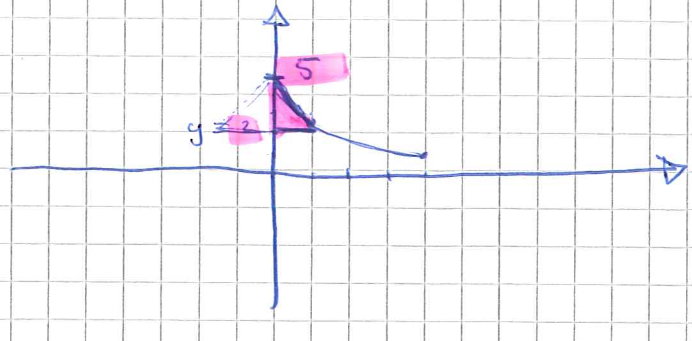
$$= \pi \left[e - 2e + 2e - 2 \right] = \pi \left[e - 2 \right]$$

V.e.

Här "står" skivan

∎

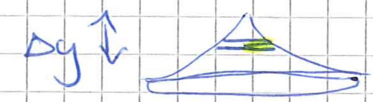
(12.2) $y = \frac{5}{1+x}$



$x = 0 \quad y = 5$

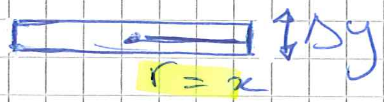
$x = 1 \quad y = 2.5$

$x = 4 \quad y = 1$



en skiva

$\Delta V_{skiva} = \pi r^2$ tjockleken



$\Delta V = \pi x^2 \Delta y$

Vad är x ?

$y + yx = 5$

$yx = 5 - y$

$x = \frac{5-y}{y} = \frac{5}{y} - 1$

$x^2 = \frac{25}{y^2} + 1 - \frac{10}{y}$

$$V = \pi \int_2^5 \frac{25}{y^2} + 1 - \frac{10}{y} \, dy$$

$$V = \pi \left[-\frac{25}{y} + y - 10 \ln y \right]_2^5$$

$$V = \pi \left[-\frac{25}{5} + 5 - 10 \ln 5 + \frac{25}{2} - 2 + 10 \ln 2 \right]$$

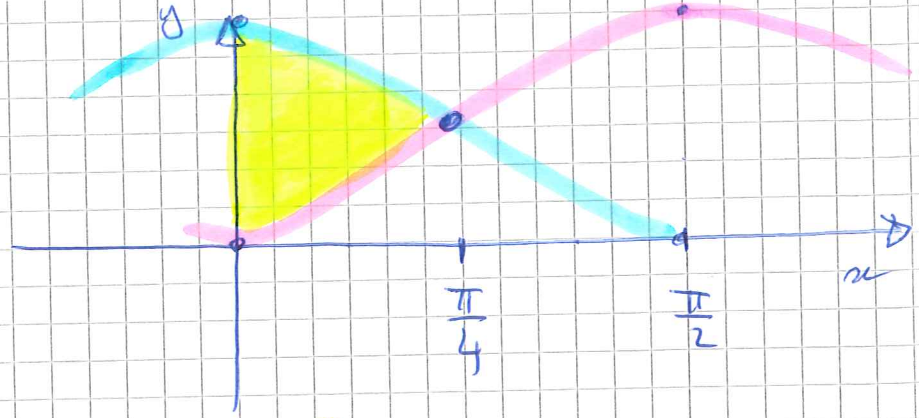
$$V = \pi \left[10 \ln \frac{2}{5} + \frac{21}{2} \right] \text{ ve}$$

Här "ligger" skivan

12.3

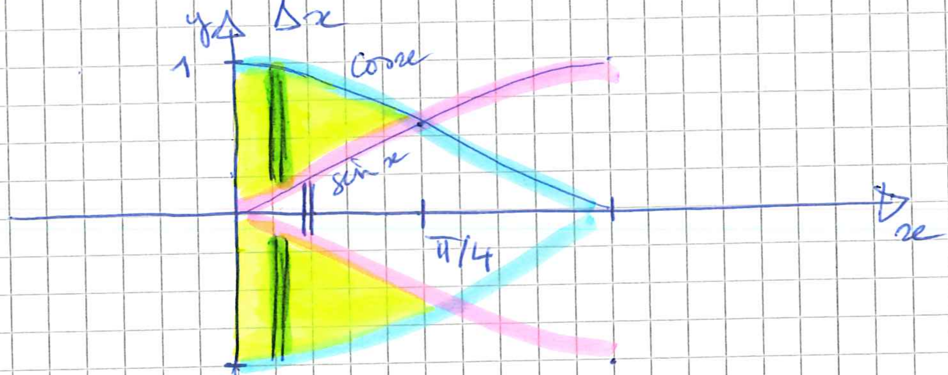
$y = \sin x$

$y = \cos x$



den gula kroppen går från
$y = \cos x$ till $y = \sin x$
och $x = 0$ och $x = \pi/4$

Rotationen kring x-axeln kommer att generera en "stående" skiva som ser ut:



och skivan ser ut:
hela skivan − det som är ihåligt

$\pi r_1^2 \Delta x - \pi r_2^2 \Delta x =$

$\pi \cos^2 x \, \Delta x - \pi \sin^2 x \, \Delta x$

$$V = \pi \int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx$$

$$V = \pi \int_0^{\pi/4} \cos 2x \, dx = \frac{\pi}{2} \left[\sin 2x \right]_0^{\pi/4}$$

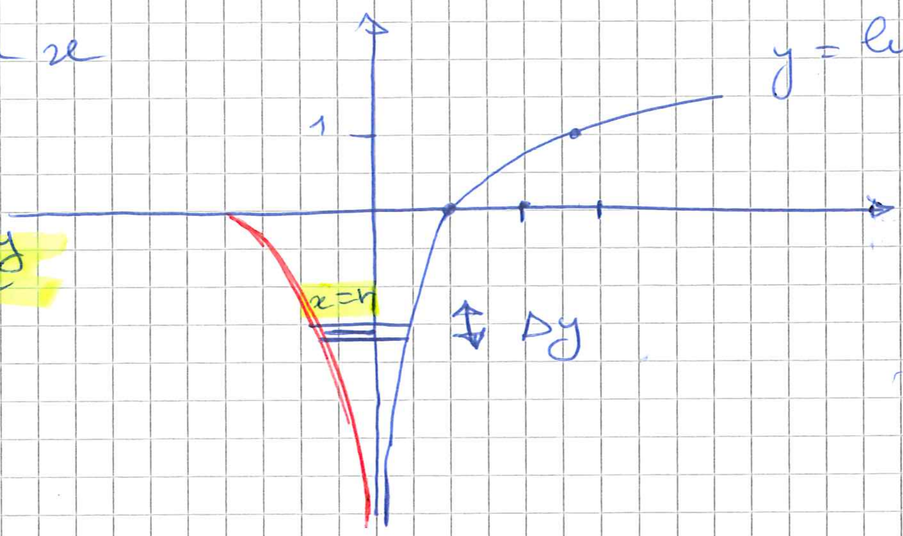
$$V = \frac{\pi}{2} \left[\sin \frac{\pi}{2} - \sin 0 \right] = \frac{\pi}{2} [1 - 0]$$

$$V = \pi/2 \text{ ve}$$

12.4

$y = \ln x$

$x = e^y$



rotation kring y-axeln genererar horizontella "liggande" skivor

$\Delta y \updownarrow$ $r = x$

$\Delta V_{\text{skiva}} = \pi r^2 \cdot \text{tjockleken}$

$= \pi x^2 \cdot \Delta y$

$= \pi e^{2y} \Delta y$

$$V = \pi \int_{-\infty}^{0} e^{2y}\, dy$$

eftersom alla $y \leq 0$

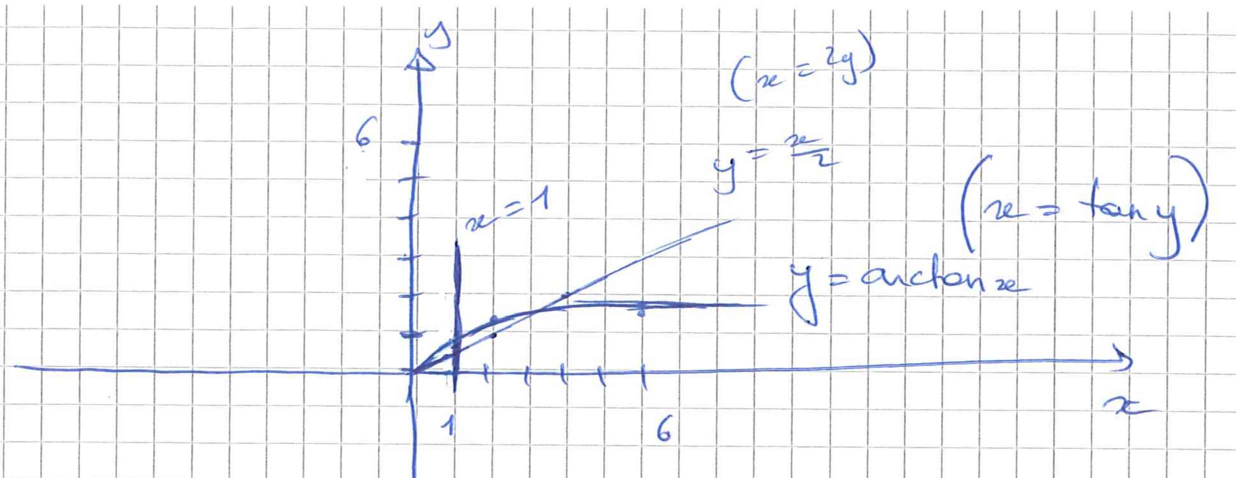
$$V = \frac{\pi}{2}\left[e^{2y}\right]_{-\infty}^{0} = \frac{\pi}{2}\left[e^{0} - \cancel{e^{-\infty}}\right]$$

$$V = \frac{\pi}{2}(1 - 0)$$

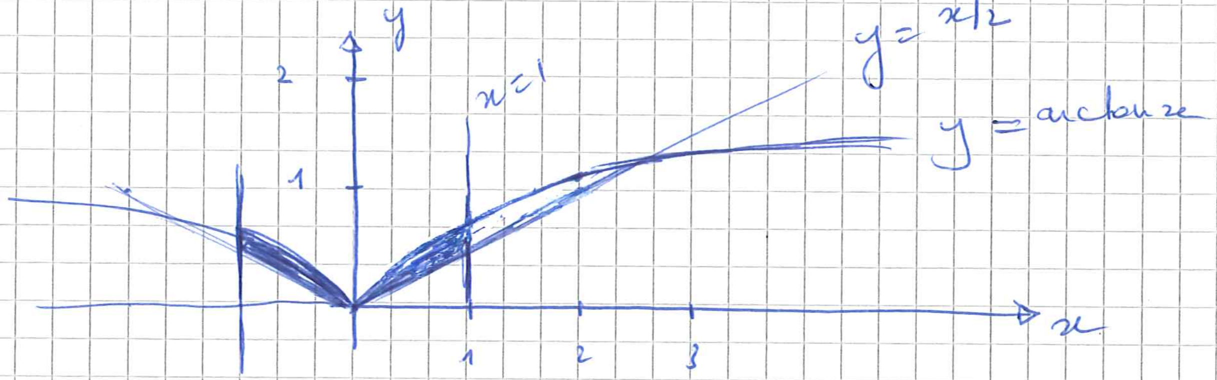
$$V = \frac{\pi}{2} \text{ ve}$$

$\blacksquare$

12.5

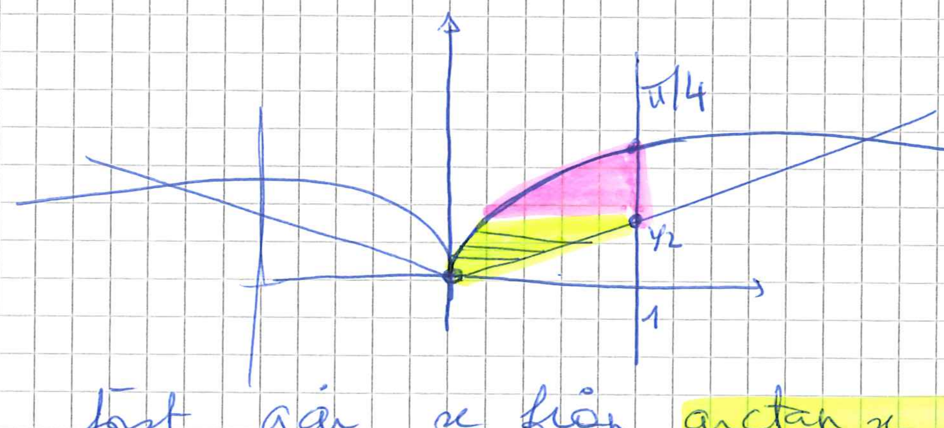


Vi ritar större



Det är en "liggande" skiva

hela skivan − det som är ihåligt.



först går x från arctan x till $\frac{x}{2}$

sedan från arctan x till $x = 1$

i slutändan:

$$\Delta V = \underbrace{\pi x^2 \Delta y}_{\text{hela}} - \underbrace{\pi x^2 \Delta y}_{\text{ihåligt}}$$

$$\Delta V_{skiva} = \pi (2y)^2 \Delta y - \pi (\tan y^2) \Delta y$$

från $y=0$ till $y = 1/2$

och

$$\Delta V_{skiva} = \pi (1)^2 \Delta y - \pi (\tan y)^2 \Delta y$$

från $y = 1/2$ till $y = \pi/4$

dvs

$$\Delta V = \pi \int_0^{1/2} 4y^2 - \tan^2 y \quad dy$$

$$+ \pi \int_{1/2}^{\pi/4} 1 - \tan^2 y \quad dy$$

$$\int (1+\tan^2 x)\, dx = \tan x$$

$$\Delta V = \pi \int_0^{1/2} 4y^2 - 1 - \tan^2 y + 1 \quad dy$$

$$+ \pi \int_{1/2}^{\pi/4} -1 - \tan^2 y + 1 + 1$$

$$\Delta V = \pi \left[\frac{4}{3} y^3 - \tan y + y \right]_0^{1/2}$$

$$+ \pi \left[- \tan y + 2y \right]_{1/2}^{\pi/4}$$

$$= \pi \left[\frac{4}{3} \frac{1}{8} - \cancel{\tan \frac{1}{2}} + \frac{1}{2} - 0 + 0 - 0 \right]$$

$$+ \pi \left[- \tan \frac{\pi}{4} + \frac{\pi}{2} + \cancel{\tan \frac{1}{2}} - 2 \cdot \frac{1}{2} \right]$$

$$= \pi \left[\frac{1}{6} + \frac{1}{2} - 1 + \frac{\pi}{2} - 1 \right] = \frac{\pi}{6} \left[-8 + 3\pi \right] \text{ ve}$$

12.6 Enligt formeln s. 319

$$S(a,b) = \int_a^b \sqrt{1+(f'(x))^2}\, dx$$

$\circledast$ $4x - 3y - 1 = 0 \quad x \in [-1, 5]$

$$3y = 4x - 1$$

$$y = \frac{4}{3}x - \frac{1}{3}$$

$$y' = \frac{4}{3} \qquad (y')^2 = \frac{16}{9}$$

$\underline{\underline{\text{dvs}}}$ $$S(-1,5) = \int_{-1}^{5} \sqrt{1 + \frac{16}{9}}\, dx$$

$$= \int_{-1}^{5} \sqrt{\frac{25}{9}}\, dx$$

$$= \frac{5}{3}\left[x\right]_{-1}^{5} = \frac{5}{3}\left[5+1\right] = \frac{5 \cdot 6}{3} = 10 \text{ le}$$

$\blacksquare$

12.8

a) $\bar{f} = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$

$y = \cos x$

$x \in [-\pi/2, \frac{\pi}{2}]$

dvs $\bar{f} = \dfrac{1}{\frac{\pi}{2} + \frac{\pi}{2}} \displaystyle\int_{-\pi/2}^{\pi/2} \cos x \; dx$

$\bar{f} = \dfrac{1}{\pi} \quad \sin x \Big]_{-\pi/2}^{\pi/2} = \dfrac{1}{\pi} \left[\sin \frac{\pi}{2} - \sin(-\pi/2) \right]$

$= \dfrac{1}{\pi} \left[1 + 1 \right] = 2/\pi$ ▨

b) $y = t^3 - 1 \qquad t \in [0, 3]$

$\bar{f} = \dfrac{1}{3-0} \displaystyle\int_0^3 t^3 - 1 \quad dt$

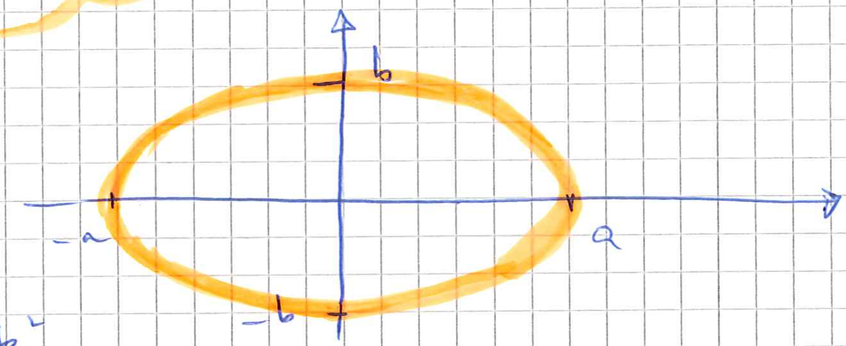
$= \dfrac{1}{3} \left[\dfrac{t^4}{4} - t \right]_0^3$

$= \dfrac{1}{3} \left[\dfrac{81}{4} - 3 - 0 + 0 \right]$

$= \dfrac{1}{3} \left[\dfrac{81 - 12}{4} \right] = \dfrac{69}{12} = \dfrac{23}{4}$ ▨

12.11

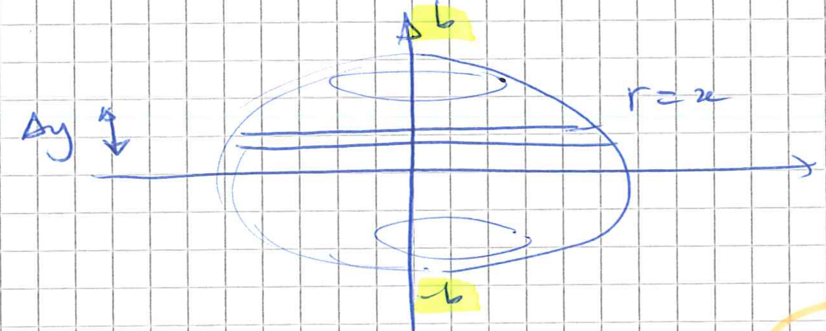
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad x > 0$$



om $x \Rightarrow y^2 = b^2$
$y = \pm b$

om $y \Rightarrow x^2 = a^2$
$x = \pm a$

Vi väljer den halva ellipsen $x > 0$
och låter den rotera kring y-axeln



$$\Delta V_{skiva} = \pi x^2 \Delta y \qquad \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$= \pi a^2 \left(1 - \frac{y^2}{b^2}\right) \Delta y$$

$$V = \pi a^2 \int_{-b}^{b} \left(1 - \frac{y^2}{b^2}\right) dy = \pi a^2 \left[y - \frac{y^3}{3b^2} \right]_{-b}^{b}$$

$$= \pi a^2 \left[b - \frac{b^3}{3b^2} + b - \frac{b^3}{3b^2} \right] = \pi a^2 \left(2b - \frac{2}{3} b\right)$$

$$= \pi a^2 b \left(2 - \frac{2}{3}\right) = \pi a^2 b \left(\frac{6-2}{3}\right) = \frac{4}{3} \pi a^2 b$$

om $a = 2{,}5$ och $b = 1{,}5$

$$V = \frac{4}{3} \pi (2{,}5)^2 \cdot 1{,}5 = 12{,}5\,\pi \text{ cm}^3$$

12.14

$$\underbrace{\int_{X_s}^{1} \frac{1}{x}\,dx}\ = \underbrace{\int_{T}^{T_{0,1}} \frac{\overline{\Delta H}_{fus,1}}{RT^2}\,dT}$$

$$\int_{X_s}^{1} \frac{1}{x}\,dx = \ln|x|\Big]_{X_s}^{1} = \ln 1 - \ln X_s$$
$$= \boxed{-\ln X_s}$$

saut

$$\int_{T}^{T_{0,1}} \frac{\overline{\Delta H}_{fus,1}}{RT^2}\,dT = \frac{\overline{\Delta H}_{fus,1}}{R}\int_{T}^{T_{0,1}} \frac{1}{T^2}\,dT$$

Konstanten

$$= \frac{\overline{\Delta H}_{fus,1}}{R}\left(-\frac{1}{T}\right)_{T}^{T_{0,1}}$$

$$= \boxed{\frac{\overline{\Delta H}_{fus,1}}{R}\left[-\frac{1}{T_{0,1}} + \frac{1}{T}\right]}$$

de ān likā rā ε

$$-\ln X_s = \frac{\overline{\Delta H}_{fus,1}}{R}\left[-\frac{1}{T_{0,1}} + \frac{1}{T}\right]$$

$$\frac{-R\cdot\ln X_s}{\overline{\Delta H}_{fus,1}} \quad \frac{1}{T_{0,1}} = \frac{1}{T}$$

$$\frac{-R\cdot\ln X_s\cdot T_{0,1} + \overline{\Delta H}_{fus,1}}{\overline{\Delta H}_{fus,1}\cdot T_{0,1}} = \frac{1}{T}$$

dus

$$T = \frac{T_{0,1}\cdot\overline{\Delta H}_{fus,1}}{\overline{\Delta H}_{fus,1} - R\cdot T_{0,1}\cdot\ln X_s}$$

11.8 $\int_2^3 \frac{x^4-4}{x^3-1}\,dx$ (fel uppgift är kopierad)

täljarens grad $\geq$ nämnaren grad

$\Longrightarrow$ polynomdivision

$$\begin{array}{l|l} \multicolumn{2}{l}{\quad x} \\ \hline x^4-4 & x^3-1 \\ x^4-x & \\ \hline x-4 & \end{array}$$

$$\int_2^3 \frac{x^4-4}{x^3-1}\,dx = \int_2^3 x\,dx + \int_2^3 \frac{x-4}{x^3-1}\,dx$$

jag undersöker/räknar integralen $\int \frac{x-4}{x^3-1}\,dx$

$$\frac{x-4}{x^3-1} = \frac{x-4}{(x-1)(x^2+x+1)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

dvs $A(x^2+x+1) + (Bx+C)(x-1) = x-4$

$Ax^2 + Ax + A + Bx^2 - Bx + Cx - C = x - 4$

$$\begin{cases} A+B=0 \\ A-B+C=1 \\ A-C=-4 \end{cases}$$

$A = -B$

$A + A + C = 1$

$$\begin{cases} 2A+C=1 \\ A-C=-4 \end{cases}$$

$3A = -3$ $\quad A = -1$

$A = -1 \qquad B = 1 \quad$ och $\quad C = A + 4$
$= -1 + 4 = 3$

$A = 1$

$B = 1$

$C = 3$

dvs $\displaystyle\int \frac{x-4}{x^3-1}\,dx = \int \frac{-1}{x-1} + \frac{(x+3)^2}{(x^2+x+1)^2}\,dx$

$\displaystyle = \int \frac{-1}{x-1}\,dx + \frac{1}{2}\int \frac{2x+6}{x^2+x+1}\,dx$

Varför?
för att
$(x^2+x+1)' = 2x+1$

$\displaystyle = \int \frac{-1}{x-1}\,dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1} + \frac{5}{x^2+x+1}\,dx$

$\displaystyle = \int \frac{-1}{x-1}\,dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1} + \frac{5}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$

$x^2 + x + \frac{1}{4} + \frac{3}{4}$

$\displaystyle = \int \frac{-1}{x-1}\,dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx$

$\displaystyle + \frac{5}{2}\cdot\frac{4}{3}\int \frac{1}{\cancel{\frac{3}{4}}\left(1 + \frac{(x+\frac{1}{2})^2\cdot 4}{3}\right)}\,dx$

$$= \int \frac{-1}{x-1}\,dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx$$

$$+ \frac{10}{3}\int \frac{1}{1+\left(\left(x+\frac{1}{2}\right)\frac{2}{\sqrt{3}}\right)^2}\,dx$$

$$= \int \frac{-1}{x-1}\,dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx$$

$$+ \frac{10}{3}\int \frac{1}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}\,dx$$

$$u = \frac{2x+1}{\sqrt{3}}$$

$$du = \frac{2}{\sqrt{3}}\,dx$$

$$dx = \frac{\sqrt{3}}{2}\,du$$

$$= \int \frac{-1}{x-1}\,dx + \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx$$

$$+ \frac{10}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{du}{1+u^2}\,du$$

$$= -\ln|x-1| + \frac{1}{2}\ln\left|x^2+x+1\right|$$

$$+ \frac{5}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

Nu återkommer vi till uppgiften med den bestämda integralen $\Longrightarrow$

$$\int_2^3 \frac{x^4-4}{x^3-1}\,dx = \int_2^3 x\,dx + \int_2^3 \frac{x-4}{x^3-1}\,dx$$

$$= \left[\frac{x^2}{2}\right]_2^3 + \left[\ln \frac{\sqrt{|x^2+x+1|}}{|x-1|} + \frac{5}{\sqrt{3}} \arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right]_2^3$$

$$= \left[\frac{9-4}{2}\right] +$$

$$\left[\ln \frac{\sqrt{9+3+1}}{3-1} + \frac{5}{\sqrt{3}} \arctan \frac{2.3+1}{\sqrt{3}} - \ln \frac{\sqrt{4+2+1}}{2-1} - \frac{5}{\sqrt{3}} \arctan \frac{2.2+1}{\sqrt{3}}\right]$$

$$= \frac{5}{2} + \ln \frac{\sqrt{13}}{2} + \frac{5}{\sqrt{3}} \arctan \frac{7}{\sqrt{3}} - \ln \frac{\sqrt{7}}{1} - \frac{5}{\sqrt{3}} \arctan \frac{5}{\sqrt{3}}$$