Kapitel nio!

9.1

a) $\int (x^2 - x)\,dx = \frac{x^3}{3} - \frac{x^2}{2} + C$

b) $\int 3e^x\,dx = 3e^x + C$

c) $\int \left(\frac{1}{2} - \frac{x}{2}\right)dx = \frac{x}{2} - \frac{x^2}{4} + C$

d) $\int (\sin x + 2\cos x)\,dx$

$= -\cos x + 2\sin x + C$

e) $\int 6\sqrt{x}\,dx = \int 6 \cdot x^{+\frac{1}{2}}\,dx$

$= 6\frac{x^{3/2}}{\frac{3}{2}} + C = 6 \cdot \frac{2}{3} x^{3/2} + C$

$= 4x\sqrt{x} + C$

f) $\int \frac{3}{x^3}\,dx = \int 3x^{-3}\,dx$

$= 3\frac{x^{-2}}{-2} + C = -\frac{3}{2}\frac{1}{x^2} + C$

g) $\int (x-3)(2-x)\,dx = \int 2x - x^2 - 6 + 3x\,dx$

$= \int -x^2 + 5x - 6\,dx = -\frac{x^3}{3} + 5\frac{x^2}{2} - 6x + C$

h) $\int \frac{1}{(1+x)^2}\,dx = \int (1+x)^{-2}\,dx = \frac{(1+x)^{-1}}{-1} + C$

$= \frac{-1}{1+x} + C$

i) $\int 2e^{-x}\,dx = \frac{2e^{-x}}{-1} + C = -2e^{-x} + C$

9.2

a) $\int x^{3/5}\,dx = \dfrac{x^{3/5+1}}{3/5+1} + C$ $\qquad \dfrac{3}{5}+1 = \dfrac{8}{5}$

$= \dfrac{x^{8/5}}{8/5} + C = \dfrac{5}{8} x^{8/5} + C$

b) $\int \left(5 \sin 2x - x^{-3/2}\right) dx =$

$\dfrac{-5}{2} \cos 2x - \dfrac{x^{-1/2}}{-\frac{1}{2}} + C =$ $\qquad -\dfrac{3}{2}+1 = \dfrac{-3+2}{2}$

$\dfrac{-5}{2} \cos 2x + 2 \dfrac{1}{\sqrt{x}} + C$

c) $\int \arctan 2 \, dx = x \cdot \arctan 2 + C$

(arctan 2 är en konstant)

d) $\int \dfrac{x^5 - x^{-3}}{5x^3}\,dx = \int \dfrac{1}{5} x^2 - \dfrac{1}{5} x^{-6}\,dx$

$= \dfrac{1}{5}\dfrac{x^3}{3} - \dfrac{1}{5}\dfrac{x^{-5}}{-5} + C$

$= \dfrac{x^3}{15} + \dfrac{1}{25x^5} + C$

$\blacksquare$

9.3 a) $\int e^{2x}(2x-5)^2\,dx =$

$\int e^{2x}(4x^2+25-20x)\,dx$

partiell Integration

$f: 4x^2+25-20x$
$= (2x-5)^2$

$f': 8x-20$

$g: e^{2x}$

$G: \frac{e^{2x}}{2}$

$\int e^{2x}(4x^2+25-20x)\,dx =$

$(2x-5)^2\,\frac{e^{2x}}{2} - \int (8x-20)\,\frac{e^{2x}}{2}\,dx =$

$\frac{(2x-5)^2}{2}\cdot e^{2x} - \int e^{2x}(4x-10)\,dx$

Partiell Integrat-igen

$f: 4x-10$

$f': 4$

$g: e^{2x}$

$G: \frac{e^{2x}}{2}$

$\frac{(2x-5)^2}{2}\cdot e^{2x} - \left[\frac{(4x-10)}{2}e^{2x} - \int 4\,\frac{e^{2x}}{2}\,dx\right]$

$= \frac{(2x-5)^2 e^{2x}}{2} - \left[(2x-5)e^{2x} - \int 2e^{2x}\,dx\right]$

$= \frac{(2x-5)^2 e^{2x}}{2} - e^{2x}(2x-5) + e^{2x} + C$

$= e^{2x}\left[\frac{4x^2}{2} + \frac{25}{2} - \frac{20x}{2} - 2x + 5 + 1\right] + C$

$= e^{2x}\left[2x^2 - 12x + \frac{37}{2}\right] + C$

b) $\int (3x^3 - 5) \ln x \, dx$

partiell Integration

$f: \ln x \qquad g: 3x^3 - 5$

$f': \frac{1}{x} \qquad G: \frac{3x^4}{4} - 5x$

$\int (3x^3 - 5) \ln x \, dx =$

$\ln x \cdot \left(\frac{3}{4}x^4 - 5x\right) - \int \frac{1}{\cancel{x}}\left(\frac{3x^{\cancel{4}}}{4} - 5\cancel{x}\right) dx =$

$\ln x \left(\frac{3}{4}x^4 - 5x\right) - \int \left(\frac{3x^3}{4} - 5\right) dx =$

$x \cdot \ln x \left(\frac{3}{4}x^3 - 5\right) - \left(\frac{3x^4}{16} - 5x\right) + C =$

$x \ln x \left(\frac{3}{4}x^3 - 5\right) - \frac{3x^4}{16} + 5x + C$ ■

c) $\int \ln(x-1) \, dx$

$f: \ln(x-1) \qquad g: 1$

$f': \frac{1}{x-1} \qquad G: x$

$\int \ln(x-1) \, dx = x \cdot \ln(x-1) - \int \frac{x}{x-1} \, dx$

$= x \cdot \ln(x-1) - \int \frac{x-1}{x-1} + \frac{1}{x-1} \, dx$

$= x \cdot \ln(x-1) - \int 1 + \frac{1}{x-1} \, dx$

$= x \cdot \ln(x-1) - x - \ln(x-1) + C$

$= (x-1) \cdot \ln(x-1) - x + C$ ■

d) $\int e^{-x} \cos x \, dx$

$f: e^{-x}$ $\qquad g: \cos x$

$f': -e^{-x}$ $\qquad G: \sin x$

$$\int e^{-x} \cdot \cos x \, dx = e^{-x} \cdot \sin x + \int e^{-x} \cdot \sin x \, dx$$

$f: e^{-x}$ $\qquad g: \sin x$

$f': -e^{-x}$ $\qquad G: -\cos x$

$$= e^{-x} \sin x + \underbrace{e^{-x}(-\cos x) - \int e^{-x} \cdot \cos x \, dx}$$

dvs:

$$\int e^{-x} \cos x \, dx = e^{-x}(\sin x - \cos x) - \int e^{-x} \cos x \, dx$$

$+\int e^{-x}\cos x\,dx$ $\qquad +\int e^{-x}\cos x\,dx$

$$2\int e^{-x} \cos x \, dx = e^{-x}(\sin x - \cos x)$$

$$\int e^{-x} \cdot \cos x \, dx = \frac{e^{-x}(\sin x - \cos x)}{2} + C$$

$\blacksquare$

f) $\int \frac{\ln x}{x} \, dx$

$f: \ln x \qquad g: \frac{1}{x}$

$f': \frac{1}{x} \qquad G: \ln x$

$\int \frac{\ln x}{x} \, dx = (\ln x)^2 - \int \frac{\ln x}{x} \, dx$

$+\int \frac{\ln x}{x} \, dx \qquad +\int \frac{\ln x}{x} \, dx$

$2\int \frac{\ln x}{x} \, dx = (\ln x)^2$

$\int \frac{\ln x}{x} \, dx = \frac{(\ln x)^2}{2} + C$ ▨

g) $\int \ln(x^2+1) \, dx$

$f: \ln(x^2+1) \qquad g: 1$

$f': \frac{2x}{1+x^2} \qquad G: x$

$\int \ln(x^2+1) \, dx = x \cdot \ln(x^2+1) - \int \frac{2x^2}{1+x^2} \, dx$

$= x \cdot \ln(x^2+1) - \int \frac{2(x^2+1)}{1+x^2} - \frac{2}{1+x^2} \, dx$

$= x \cdot \ln(x^2+1) - \int 2 - \frac{2}{1+x^2} \, dx$

$= x \cdot \ln(x^2+1) - 2x + 2\arctan(x) + C$

▨

9.4

a) $\int (x^3 - 2x)\,dx = \frac{x^4}{4} - x^2 + C$ $\blacksquare$

b) $\int \left(\frac{1}{3} x^{-3} + \frac{4}{5} \frac{1}{x} + 1\right) dx$

$= \frac{1}{3} \frac{x^{-2}}{-2} + \frac{4}{5} \ln|x| + x + C$

$= -\frac{1}{6x^2} + \frac{4}{5} \ln|x| + x + C$ $\blacksquare$

c) $\int (2x-3)^4\,dx$

$u = 2x - 3$

$du = 2\,dx$

$= \int u^4 \frac{du}{2}$

$= \frac{u^5}{5} \frac{1}{2} = \frac{u^5}{10} + C = \frac{(2x-3)^5}{10} + C$

d) $\int \frac{1}{x^3} + \frac{5x}{x^3}\,dx = \int x^{-3} + 5x^{-2}\,dx$

$= \frac{x^{-2}}{-2} + 5\frac{x^{-1}}{-1} + C = -\frac{1}{2x^2} - \frac{5}{x} + C$ $\blacksquare$

9.5

a) $\int \sqrt{4-x^2}\, dx$

$f : (4-x^2)^{\frac{1}{2}}$ $\qquad g : 1$

$f' : -2x \cdot \frac{1}{2}(4-x^2)^{-\frac{1}{2}}$ $\qquad G : x$

$= -\frac{x}{\sqrt{4-x^2}}$

$$\int \sqrt{4-x^2}\, dx = x\sqrt{4-x^2} - \int \frac{-x^2}{\sqrt{4-x^2}}\, dx$$

$$= x \cdot \sqrt{4-x^2} - \int \frac{-x^2+4-4}{\sqrt{4-x^2}}\, dx$$

$$= x\sqrt{4-x^2} - \int \frac{4-x^2}{\sqrt{4-x^2}} - \frac{4}{\sqrt{4-x^2}}\, dx$$

$$= x\sqrt{4-x^2} - \int \sqrt{4-x^2}\, dx + 4\int \frac{1}{2\sqrt{1-(\frac{x}{2})^2}}\, dx$$

$$2\int \sqrt{4-x^2}\, dx = x\sqrt{4-x^2} + 2\int \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\, dx$$

$$\int \sqrt{4-x^2}\, dx = \frac{x}{2}\sqrt{4-x^2} + \int \frac{1}{\sqrt{1-t^2}}\, 2\,dt$$

$\frac{x}{2} = t$

$dx\, \frac{1}{2} = dt$

$dx = 2\,dt$

$$= x\sqrt{\frac{4}{4} - \frac{x^2}{4}} + 2 \arcsin t$$

$$= x\sqrt{1-\frac{x^2}{4}} + 2\arcsin\frac{x}{2} + C$$

$$\int \sqrt{4-x^2}\, dx = x\sqrt{1-\frac{x^2}{4}} + 2\arcsin\frac{x}{2} + C$$

b) $\int \sqrt{1-\cos x}\, dx$

$= \int \sqrt{\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)}}\, dx$ förläng!

$= \int \frac{\sqrt{1-\cos^2 x}}{\sqrt{1+\cos x}}\, dx$

$\sin^2 x + \cos^2 x = 1$

$= \int \frac{\sqrt{\sin^2 x}}{\sqrt{1+\cos x}}\, dx$

$= \int \frac{|\sin x|\, dx}{\sqrt{1+\cos x}}$

$\sin x \geq 0$ om $x \in [0;\pi]$

$\sin x \leq 0$ om $x \in [\pi; 2\pi]$

vi delar i 2 fall!!

$= \begin{cases} \int \frac{\sin x\, dx}{\sqrt{1+\cos x}}\, dx \text{ eller} \\ \int \frac{-\sin x\, dx}{\sqrt{1+\cos x}}\, dx \end{cases}$

då letar vi efter $\int \frac{\pm \sin x\, dx}{\sqrt{1+\cos x}}$

Variabelsubstitution

Låt $u = 1+\cos x$

$du = -\sin x\, dx$

och integralen blir $\int \frac{\pm\, du}{\sqrt{u}}$

$$\int \pm \frac{du}{\sqrt{u}} = \int \pm u^{-1/2}\, du$$

$$= \pm \frac{u^{1/2}}{1/2} + C$$

$$= \pm 2\sqrt{u} + C$$

$$= \pm 2\sqrt{1+\cos x} + C$$

(*)(*)

Vi kan sluta här eller fortsätta

Vi vet att $\cos 2x = \cos^2 x - \sin^2 x$

$$= 2\cos^2 x - 1$$

dvs $\cos 2x + 1 = 2\cos^2 x$

jämför $\cos x + 1 = 2\cos^2 \frac{x}{2}$ (halvera vinkeln)

och $\sqrt{1+\cos x} = \sqrt{2}\sqrt{\cos^2 \frac{x}{2}}$

$$\sqrt{1+\cos x} = \sqrt{2}\left|\cos \frac{x}{2}\right|$$

$$\sqrt{1+\cos x} = \pm\sqrt{2}\cos\left(\frac{x}{2}\right)$$

samma som innan

från (*)(*) skriver vi integralen som:

$$\pm 2\left(\pm\sqrt{2}\cos\frac{x}{2}\right) + C$$

$$= 2\sqrt{2}\cos\frac{x}{2} + C$$

c) $\int \frac{x(x+1)}{1+x^2}\,dx =$

$\int \frac{x^2+x+1-1}{1+x^2}\,dx =$

$\int \frac{x^2+1+x-1}{1+x^2}\,dx =$

$\int \frac{x^2+1}{x^2+1} + \frac{x}{1+x^2} - \frac{1}{1+x^2}\,dx =$

$\int 1 + \frac{2x}{2(1+x^2)} - \frac{1}{1+x^2}\,dx =$

$x + \frac{1}{2}\ln(1+x^2) -$

$\arctan x + C$ ∎

d) $\int \frac{\arctan x}{1+x^2}\,dx = \int \arctan x \cdot \frac{1}{1+x^2}\,dx$

$u = \arctan x$

$du = \frac{1}{1+x^2}\,dx$

$= \int u \cdot du = \frac{u^2}{2} + C$

$= \frac{(\arctan x)^2}{2} + C$ ∎

c) $\int \cos x \cdot \ln|\sin x| \, dx$

$u = \ln|\sin x|$

$du = \frac{\cos x}{\sin x} \, dx \qquad dx = \frac{\sin x}{\cos x} \, du$

$\int \cos x \cdot \ln|\sin x| \, dx =$

$\int \cancel{\cos x} \cdot u \cdot \frac{\sin x}{\cancel{\cos x}} \, du =$

och $\sin x = e^{u}$

$\int u \cdot e^{u} \, du$

$f: u \qquad g: e^{u}$

$f': 1 \qquad G: e^{u}$

$\int u \cdot e^{u} \, du = u \cdot e^{u} - \int e^{u} \, du$

$= u \cdot e^{u} - e^{u} + C$

$= \ln|\sin x| \cdot \sin x$

$- \sin x + C$

■

9.6

a) $$\int \frac{dx}{2x^2+4}$$

$$= \int \frac{dx}{\left(\frac{x^2}{2}+1\right)}$$

$$= \frac{1}{4} \int \frac{dx}{1+\left(\frac{x}{\sqrt{2}}\right)^2}$$

Variabelsubstitution:

$$u = \frac{x}{\sqrt{2}}$$

$$du = \frac{1}{\sqrt{2}}\, dx \qquad dx = \sqrt{2}\, du$$

och integralen blir

$$\frac{1}{4} \int \frac{\sqrt{2}\, du}{1+u^2}$$

$$= \frac{\sqrt{2}}{4} \int \frac{du}{1+u^2} = \frac{\sqrt{2}}{4} \arctan u + C$$

$$= \frac{\sqrt{2}}{4} \arctan \frac{x}{\sqrt{2}} + C \qquad \blacksquare$$

b) $\displaystyle\int \frac{dx}{\sqrt{3-4x^2}} = \int \frac{dx}{\sqrt{3\left(1-\frac{4}{3}x^2\right)}}$

$$= \int \frac{dx}{\sqrt{3}\,\sqrt{1-\frac{4}{3}x^2}} = \frac{1}{\sqrt{3}} \int \frac{dx}{\sqrt{1-\left(\frac{2}{\sqrt{3}}x\right)^2}}$$

løt $u = \frac{2}{\sqrt{3}}x$

$$du = \frac{2}{\sqrt{3}}\,dx$$

$$dx = \frac{\sqrt{3}}{2}\,du$$

Integralen blir :

$$\frac{1}{\cancel{\sqrt{3}}} \int \frac{\frac{\cancel{\sqrt{3}}}{2}\,du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{2} \arcsin u + C$$

$$= \frac{1}{2} \arcsin\left(\frac{2}{\sqrt{3}}x\right) + C \quad \blacksquare$$

c) $\displaystyle\int \frac{x^3}{\sqrt{1-x^4}}\,dx$

Låt $1-x^4 = u$

$-4x^3\,dx = du$

Integralen blir

$$-\frac{1}{4}\int \frac{-4x^3}{\sqrt{1-x^4}}\,dx =$$

$$-\frac{1}{4}\int \frac{du}{\sqrt{u}} = -\frac{1}{4}\int u^{-1/2}\,du$$

$$= -\frac{1}{4}\left[\frac{u^{1/2}}{\frac{1}{2}}\right] + C$$

$$= -\frac{1}{4}\cdot 2\sqrt{u} + C$$

$$= -\frac{1}{2}\sqrt{u} + C$$

$$= -\frac{1}{2}\sqrt{1-x^4} + C \quad \blacksquare$$

9.7 $h(t) = 3t^2 + 6t$ → t = dagar

$p(t) = \int h(t)\,dt$ → population

$= t^3 + \frac{6t^2}{2} + C$

$p(t) = t^3 + 3t^2 + C$

från början, när $t = 0$ då $p(t) = 100$

$\Rightarrow 100 = 0 + 3 \cdot 0 + C$

$C = 100$

och modellen blir:

$$\boxed{p(t) = t^3 + 3t^2 + 100}$$

Efter 5 dagar, $t = 5$

hitta $p(5)$:

$p(5) = 5^3 + 3 \cdot 5^2 + 100$

$= 125 + 3 \cdot 25 + 100$

$= 125 + 175$

$= 300$

Det finns 300 insekter efter 5 dagar.

■

9.8

$$\frac{dV}{dt} = \frac{1}{20} t^2 (5-t)$$

$$\frac{dV}{dt} = \frac{5t^2}{20} - \frac{t^3}{20}$$

$$\frac{dV}{dt} = \frac{t^2}{4} - \frac{t^3}{20}$$

V: Volym i liter
t: sekunder
$\frac{dV}{dt}$: hastighet

$$V(t) = \frac{t^3}{12} - \frac{t^4}{80} + C$$

från början: $t = 0$ $V(0) = 0,5$

dvs $0,5 = 0 - 0 + C$

$$C = 0,5$$

och $V(t) = \frac{t^3}{12} - \frac{t^4}{80} + 0,5$

Vad blir det när $t = 5$ sek ?
(Max volym förekommer när inandnings perioden är slut efter 5 sekunder)

$$V(5) = \frac{5^3}{12} - \frac{5^4}{80} + 0,5$$

$$V(5) = V_{max} = 3,104$$

Ca 3 liter luft är den maximala volymen