Maclaurin utveckling.

Vi har haft olika funktioner.

- Polynom
- Sin, Cos, tan
- logaritm
- rationell funktion

Polynom funktion är lättaste på många sätt.

(1) $f(x) = 1 + x + x^2$

(2) $f(x) = \dfrac{x-1}{x^2+3}$

(3) $f(x) = \sin x$, $\cos x$

(4) $f(x) = \ln x$

(5) $f(x) = e^x$

(1) $f(1) = 1+1+1 = 3$

$f(x) = \sin x$

$f(1) = \sin 1$

Miniräknare ?

$f(x) = e^x$

$f(2) = e^2$?

$f(x) = \ln x$

$f(2) = \ln 2 = ?$

Maclaurin

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Säger att varje godt. funktion kan skrivas som ett polynom nära $x = 0$.

$\sin x$ = ett polynom

$\cos x$ = " annan poly.

e^x = — —

$\ln(1+x)$ = — —

Men hur?

Låt $f(x)$ vara en godt. funktion

ett Polynom ser ut som

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$* \quad f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Vad är koefficienter $a_0, a_1, a_2, a_3 \ldots$

Maclaurin svarar så här.

Sätt $x=0$ i $*$

$$\boxed{f(0) = a_0} \qquad \text{Svar } \boxed{a_0 = f(0)}$$

derivera $*$ överallt.

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$\boxed{\text{Sätt } x=0}$

$$\boxed{f'(0) = a_1} \qquad \boxed{a_1 = f'(0)}$$

För att veta vad a_2 är deriverar vi en gång till

$$f''(x) = 2a_2 + 2.3\,a_3 x + 3.4.a_4 x^2 + \dots$$

$\boxed{\text{Sätt } x = 0}$

$$f''(0) = 2a_2 \implies \boxed{a_2 = \frac{f''(0)}{2}}$$

$$f^{(3)}(x) = 2.3\,a_3 + 2.3.4\,a_4 x + \dots$$

$x = 0$

$$f^{(3)}(0) = 2.3.a_3 \implies a_3 = \frac{f^{(3)}(0)}{1.2.3}$$

$$\boxed{a_3 = \frac{f^{(3)}(0)}{3!}}$$

$$a_4 = \frac{f^{(4)}(0)}{4!}$$

$$a_{10} = \frac{f^{(10)}(0)}{10!}$$

$\left\{\begin{array}{l}\text{Maclaurin Polynom} \\ \text{Maclaurin utveckling} \\ \text{Maclaurin formel}\end{array}\right.$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Några ex.

Vad är Macl. formel för $f(x) = e^x$

$$f(x) = e^x \Longrightarrow f(0) = e^0 = 1$$

$$f'(x) = e^x \Longrightarrow f'(0) = e^0 = 1$$

$$f''(x) = e^x \Longrightarrow f''(0) = e^0 = 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\boxed{e = ?}$$

$$e^1 = 1 + 1 + \frac{1}{2} = 2.5$$

$$e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.7$$

Ex 2 Vad blir Mac. för

$$f(x) = \sin x$$

$$\begin{cases} f(x) = \sin x \longrightarrow f(0) = \sin 0 = 0 \\ f'(x) = \cos x \longrightarrow f'(0) = \cos 0 = 1 \\ f''(x) = -\sin x \longrightarrow f''(0) = -\sin 0 = 0 \\ f^{(3)}(x) = -\cos x \longrightarrow f^{(3)}(0) = -\cos 0 = -1 \end{cases}$$

$$f^{(4)}(0) = \sin x \longrightarrow f^{(4)}(0) = \sin 0 = 0$$

upprepas

$$f(x) = \cancel{f(0)} + \cancel{f'(0)}x + \frac{\cancel{f''(0)}}{2!}x^2 + \frac{\cancel{f^{(3)}(0)}}{3!}x^3 + \cdots$$

$$\quad 0 \qquad 1 \qquad 0 \qquad -1$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots}$$

Ex 3 $\quad f(x) = \cos x$

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots}$$

$$f(x) = \frac{1}{1+x}$$

$$f(x) = \frac{1}{1-x}$$

$$f(x) = \ln(1+x)$$

$$f(x) = \arctan x$$

$$f(x) = (1+x)^\alpha$$

Ex 4

$$f(x) = \frac{1}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

Vi kan visa denna utan att derivera

$$S = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$S = 1 + x\underbrace{(1 + x + x^2 + x^3 + \cdots)}_{S}$$

$$S = 1 + xS$$

$$S - xS = 1$$

$$S(1-x) = 1$$

$$\boxed{S = \frac{1}{1-x}}$$

$$f(x) = \frac{1}{1-x} \longrightarrow \boxed{f(0) = 1}$$

$$f(x) = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2} \longrightarrow \boxed{f'(0) = 1}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$\boxed{f''(0) = 2}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots}$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$$

$$\frac{1}{1+x} = ?$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

ersätt x med $-x$

(5) $$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \cdots}$$

Integrera

(6) $$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots}$$

$\ln 2 = ?$

$$\ln 2 = \underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}$$

$\arctan x = \ldots$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 \ldots$$

ersätt x med x^2

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 \ldots$$

Integrera

$$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots}$$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}$$

Anm $f(x) = \ln x$

Saknar Maclaurin utveckling

$f(0) = \ln 0$ ej definierad.

$f'(x) = \frac{1}{x}$ $f'(0) =$ finns inte.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$ 12

Några ex.

Beräkna Mac. Polynom av grad 2

till $f(x) = \dfrac{1}{1+4x}$

$$\boxed{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2}$$

$$f(x) = \frac{1}{1+4x} \quad (1+4x)^{-1} \longrightarrow \boxed{f(0) = 1}$$

$$f'(x) = -1(1+4x)^{-2}(4) \qquad \boxed{f'(0) = -4}$$

$$f''(x) = 2(1+4x)^{-3}4 \cdot 4 = 32(1+4x)^{-3}$$

$$\boxed{f''(0) = 32}$$

$$\boxed{\frac{1}{1+4x} = 1 - 4x + 16x^2}$$

$$f(x) = \frac{1}{1+4x}$$

Best Mac Polynom av grad 10. ?

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \cdots + x^{10} + \cdots$$

$\boxed{\text{ersätt } x \text{ med } 4x}$

$$\boxed{\frac{1}{1+4x} = 1 - 4x + 16x^2} - 64x^3 + \cdots$$

$$\boxed{\sin 2x = ?}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots$$

$$= 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 - \cdots$$

Viktigaste tillämpning av Maclaurin

är Beräkning av G.V.

Ex

$$\lim_{x \to 0} \frac{\sin x}{x}$$

$$\sin x = x \underbrace{- \frac{x^3}{6} + \frac{x^5}{5!} \cdots}$$

$$x - x^3 \underbrace{\left(\frac{1}{6} - \frac{x^2}{5!} \cdots \right)}_{B(x)}$$

$$\boxed{\sin x = x - x^3 B(x)}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - x^3 B(x)}{x} \quad \left(\frac{0}{0}\right)$$

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$$= \lim_{x \to 0} \frac{\cancel{x}(1 - x^2 B(x))}{\cancel{x}}$$

$$= \lim_{x \to 0} (1 - x^2 B(x)) = 1$$

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} \quad \left(\frac{e^0 - 1 - 0}{0} = \frac{0}{0} \right).$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$= \lim_{x \to 0} \frac{\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \cancel{-1} \cancel{-x}}{x^2}$$

$$= \lim_{x \to 0} \frac{\cancel{x^2}\left(\frac{1}{2} + x/6 + \cdots\right)}{\cancel{x^2}} = 1/2$$

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$$\lim_{x\to 0} \frac{x^3}{\sin 2x - 2x} \quad \left(\frac{0}{0}\right)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - -$$

$$\boxed{\sin 2x = 2x - \frac{8}{6}x^3 + \frac{32}{120}x^5 - - -}$$

$$\lim_{x\to 0} \frac{x^3}{\cancel{2x} - \frac{4}{3}x^3 + \frac{4}{15}x^5 - - - \cancel{-2x}}$$

$$= \lim_{x\to 0} \frac{\cancel{x^3}\ 1}{\cancel{x^3}\left[-\frac{4}{3} + \frac{4}{15}x^2 - - -\right]}$$

$$\frac{1}{-\frac{4}{3}} = -\frac{3}{4}$$

$$\lim_{x \to 0} \frac{x^3}{\sin 2x - 2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{3x^2}{2\cos 2x - 2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{6x}{-4\sin 2x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{6}{-8\cos 2x} = -6/8$$

$$= -3/4$$

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$$\lim_{x\to 0} \frac{\sin x - \arctan x}{x(\cos 2x - 1)} \quad \left(\frac{0}{0}\right)$$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$\frac{16x^4}{\cancel{24}}$

$2x^4/3$

$$\boxed{\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \cdots}$$

$\sin x = x - \frac{x^3}{3!} + x^5/5! - \cdots$

$\arctan x = x - x^3/3 + x^5/5 - \cdots$

$$= \lim_{x\to 0} \frac{\left(\cancel{x} - \frac{x^3}{3!} + \cdots\right) - \left(\cancel{x} - \frac{x^3}{3} + \cdots\right)}{x\left[\cancel{1} - 2x^2 + \cdots \cancel{-1}\right]}$$

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$$\lim_{x \to 0} \frac{-\frac{\cancel{x^3}}{6} + \frac{\cancel{x^3}}{3} \cdots}{-2\cancel{x^3} + \cdots}$$

$$= \lim_{x \to 0} \frac{-1/6 + 1/3 + \cancel{\cdots}}{-2 + \cancel{\cdots}}$$

$$= \frac{\frac{1}{3} - 1/6}{-2}$$