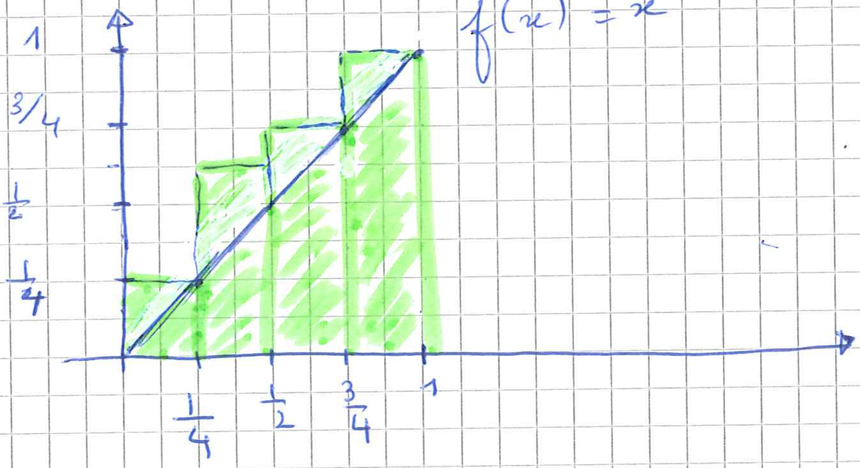
Kapitel tio:

(10.1) a) $f(x) = x$

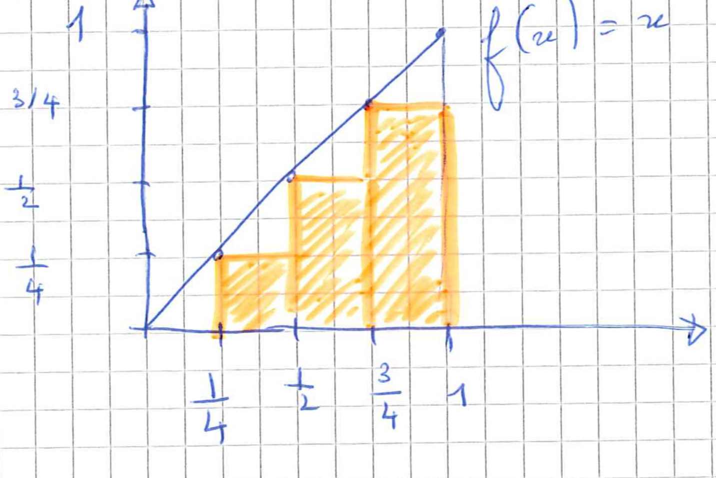


$I(\Psi_4)$ delar intervallet $[0,1]$ i 4 delar som ligger ovanpå kurvan $f(x) = x$.

$$I(\Psi_4) = \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{4}\cdot 1$$

$$= \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1\right)$$

$$= \frac{1}{4}\left(\frac{10}{4}\right) = \frac{10}{16} = \frac{5}{8}$$



$I(\Phi_4)$ delar intervallet $[0,1]$ i 4 delar som ligger under $f(x) = x$

$$I(\phi_4) = 0 + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{3}{4}$$

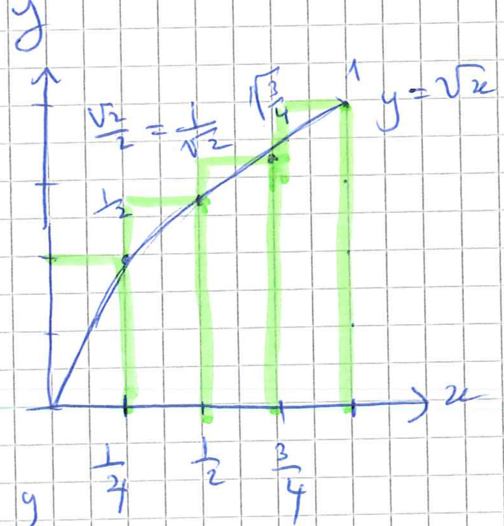
$$= \frac{1}{4}\left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4}\right)$$

$$= \frac{1}{4}\cdot\frac{6}{4} = \frac{6}{16} = \frac{3}{8}$$

$$\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big]_0^1 = \left[\frac{1}{2} - \frac{0}{2}\right] \quad ■$$

$$= 1/2$$

b)



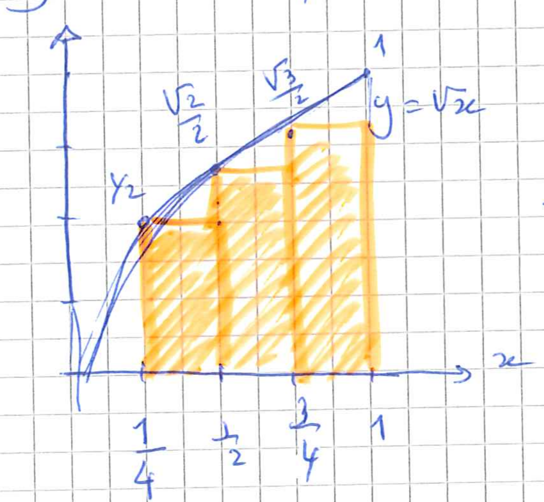
$$I(\psi_4) = \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{\sqrt{2}}{2} + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{4}\cdot 1$$

$$= \frac{1}{4}\left(\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + 1\right)$$

$$= \frac{1}{4}\left(\frac{1 + \sqrt{2} + \sqrt{3} + 2}{2}\right)$$

$$= \frac{1}{8}\left(1 + \sqrt{2} + \sqrt{3} + 2\right)$$

$$= 0{,}7682 \approx 0{,}77$$



$$I(\phi_4) = \frac{1}{4}\left(\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{8}\left(1 + \sqrt{2} + \sqrt{3}\right)$$

$$= 0{,}5182 \approx 0{,}52$$

$$\int_0^1 f(x)\,dx = \int_0^1 \sqrt{x}\,dx = \int_0^1 x^{1/2}\,dx$$

$$= \frac{x^{3/2}}{3/2}\Big]_0^1 = \frac{2}{3}\left[x^{3/2}\right]_0^1 = \frac{2}{3}\left[1 - 0\right] \quad ■$$

$$= 2/3 \approx 0{,}67$$

10.2 $$I(\phi_n) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\left(\frac{k}{n}\right)^2+1\right)$$

$$= \frac{1}{n}\left[0+1 + \frac{1^2}{n^2}+1 + \frac{2^2}{n^2}+1 + \frac{3^2}{n^2}+1 + \cdots + \frac{(n-1)^2}{n^2}+1\right]$$

(*) (*) $$= \frac{1}{n}\left[n + \frac{1}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\right]$$

$$= \left[1 + \frac{1}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}\right]$$

$$= 1 + \frac{1}{n^3}\left[1 + 2^2 + 3^2 + \cdots + (n-1)^2\right]$$

$$= 1 + \frac{1}{n^3}\sum_{k=1}^{n-1} k^2 \qquad \blacksquare$$

Jag tänkte att (*) (*) kan också skrivas som:

$$\frac{1}{n}\left[\underbrace{1+1+1+\cdots+1}_{\text{från } 0 \text{ till } n-1} + \frac{1}{n^2} + \frac{2^2}{n^2} + \cdots\right]$$

dvs. n gånger

därför ersätts alla "1" med "n"

$\blacksquare$

10.3

a) $\displaystyle\int_0^{\pi} \sin\frac{x}{2}\,dx = \frac{-\cos\frac{x}{2}}{\frac{1}{2}}\Big]_0^{\pi}$

$= -2\cos\frac{x}{2}\Big]_0^{\pi} = -2\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= 2\cdot[1-0] = 2$ ■

b) $\displaystyle\int_0^1 \frac{2}{1+x^2}\,dx = 2\int_0^1 \frac{1}{1+x^2}\,dx$

$= 2\arctan x\Big]_0^1 = 2\left[\arctan 1 - \arctan 0\right]$

$= 2\left[\frac{\pi}{4} - 0\right] = \pi/2$ ■

c) $\displaystyle\int_{-1}^{2} (x - 2x^3)\,dx = \frac{x^2}{2} - \frac{2}{4}x^4\Big]_{-1}^{2}$

$= \left(\frac{4}{2} - \frac{1}{2}\cdot 16\right) - \left(\frac{1}{2} - \frac{1}{2}(1)\right)$

$= \left(\frac{4}{2} - \frac{16}{2}\right) - \left(\cancel{\frac{1}{2} - \frac{1}{2}}\right)$

$= 2 - 8 = -6$ ■

d) $\displaystyle\int_{-1}^{1/2} \sqrt{2-4x}\,dx$

$u = 2-4x$

$du = -4\,dx$

$dx = -\frac{1}{4}\,du$

em $x = -1$

$u = 2+4 = 6$

em $x = 1/2$

$u = 2 - 4\cdot\frac{1}{2}$

$u = 0$

$= \displaystyle\int_6^0 \sqrt{u}\left(-\frac{1}{4}\,du\right)$

$= -\frac{1}{4}\displaystyle\int_6^0 u^{1/2}\,du$

$= -\frac{1}{4}\,\frac{2}{3}\,u^{\frac{3}{2}}\Big]_6^0 = -\frac{1}{6}\left[u\sqrt{u}\right]_6^0$

$= -\frac{1}{6}\left[0 - 6\sqrt{6}\right] = -\frac{1}{\cancel{6}}\left(-\cancel{6}\sqrt{6}\right) = \sqrt{6}$ ■

e) $\int_3^5 \frac{7}{x-1}\,dx = 7\int_3^5 \frac{1}{x-1}\,dx$

$= 7\left[\ln|x-1|\right]_3^5 = 7\left[\ln 4 - \ln 2\right]$

$= 7\ln\frac{4}{2} = 7\ln 2$ ■

f) $\int_0^{-2} \frac{1}{(1-x)^2}\,dx =$

$U = 1-x$

$dU = -dx$

$dx = -dU$

si $x = 0$
$U = 1$

si $x = -2$
$U = 1-(-2)$
$U = 3$

$\int_1^3 \frac{1}{U^2}(-dU) =$

$\int_1^3 -U^{-2}\,dU =$

$\left[\cancel{-}\frac{U^{-1}}{\cancel{-}1}\right]_1^3 = \left[\frac{1}{U}\right]_1^3$

$= \frac{1}{3} - \frac{1}{1} = \frac{1}{3} - \frac{3}{3} = -2/3$ ■

10.4

a) $\int_{\pi}^{0} \tan \frac{x}{4} \, dx$

$u = \frac{x}{4}$

om $x = 0$ $\quad u = 0$

om $x = \pi$ $\quad u = \frac{\pi}{4}$

$du = \frac{1}{4} dx$

$4\,du = dx$

$= \int_{\frac{\pi}{4}}^{0} \tan u \cdot 4\, du$

$= 4 \int_{\frac{\pi}{4}}^{0} \frac{\sin u}{\cos u} \, du$

$\int \frac{-\sin u}{\cos u} \, du$

$= \ln |\cos u|$

eftersom täljaren är derivatan av nämnaren

$= -4 \ln |\cos u| \Big]_{\frac{\pi}{4}}^{0}$

$= -4 \left[\ln \cos 0 - \ln \cos \frac{\pi}{4} \right]$

$= -4 \left[\cancel{\ln 1} - \ln \frac{\sqrt{2}}{2} \right]$

$= -4 \left[0 - \ln \frac{\sqrt{2}}{2} \right]$

$= + 4 \ln \frac{\sqrt{2}}{2}$

$= + \ln \left(\frac{\sqrt{2}}{2} \right)^4$

$= + \ln \frac{\sqrt{2}^4}{2^4}$

$= + \ln \frac{4}{16}$

$= + \ln \frac{1}{4} = + \left[\cancel{\ln 1} - \ln 4 \right]$

$= -\ln 4 = -\ln 2^2 = -2 \ln 2$ ▨

Kom ihåg:

$\cos 2x = 2\cos^2 - 1$

$\cos 2x + 1 = 2\cos^2 x$

$\frac{\cos 2x + 1}{2} = \cos^2 x$

b) $\int_0^{\pi/6} \cos^2 x \, dx =$

$\int_0^{\pi/6} \frac{\cos 2x + 1}{2} \, dx =$

$\frac{1}{2} \int_0^{\pi/6} \cos 2x + 1 \, dx =$

$\frac{1}{2} \left[\frac{\sin 2x}{2} + x \right]_0^{\pi/6} =$

$\frac{1}{2} \left[\frac{\sin 2\frac{\pi}{6}}{2} + \frac{\pi}{6} + \cancel{\frac{\sin 2 \cdot 0}{2}} \cancel{- 0} \right] =$

$\frac{1}{2} \left[\frac{\sin \frac{\pi}{3}}{2} + \frac{\pi}{6} \right] =$

$\frac{1}{2} \left[\frac{\sqrt{3}/2}{2} + \frac{\pi}{6} \right] =$

$\frac{1}{2} \left[\frac{\sqrt{3}}{4} + \frac{\pi}{6} \right] =$

$\frac{\sqrt{3}}{8} + \frac{\pi}{12}$ $\blacksquare$

c) $\int_0^1 t^3(1-t^4)^2 \, dt$

Variabel substitution

$u = 1 - t^4$

$du = -4t^3 \, dt$

om $t=0$ $u=1$

om $t=1$ $u=0$

$= \int -\frac{1}{4}(-4t^3)(1-t^4)^2 \, dt$

$= \int_1^0 -\frac{1}{4} u^2 \, du$

$= -\frac{1}{4} \int_1^0 u^2 \, du$

$= -\frac{1}{4} \frac{1}{3} [u^3]_1^0 = -\frac{1}{12} [u^3]_1^0$

$= -\frac{1}{12} [0-1] = \frac{1}{12}$ ∎

d) $\int_0^1 x \cdot \ln(x+1) \, dx$

partiell Integration

$f: \ln(x+1)$ $g: x$

$f': \frac{1}{x+1}$ $G: \frac{x^2}{2}$

gränserna använder jag bara i slutet

$= \frac{x^2}{2} \cdot \ln(x+1) - \frac{1}{2} \int \frac{x^2}{x+1} \, dx$

$= \frac{x^2}{2} \cdot \ln(x+1) - \frac{1}{2} \int \frac{x^2+2x+1-2x-2+1}{x+1} \, dx$

$= \frac{x^2}{2} \ln(x+1) - \frac{1}{2} \int \frac{(x+1)^2}{x+1} - \frac{2(x+1)}{x+1} + \frac{1}{x+1} \, dx$

$x+1 \quad -2 \quad + 1/(x+1)$

$= \left[\frac{x^2}{2} \ln(x+1) - \frac{1}{2} \left(\frac{x^2}{2} + x - 2x + \ln|x+1| \right) \right]_0^1$

$= \left[\cancel{\frac{1}{2} \ln 2} - \frac{1}{2} \left(\frac{1}{2} + 1 - 2 \cancel{+ \ln 2} \right) - 0 \right]$

$= -\frac{1}{2} \left(\frac{1}{2} - 1 \right) = 1/4$ ∎

e) $\int_1^e \frac{\ln x}{x^3}\, dx$

Partiell Integration

$f: \ln x \qquad g: x^{-3}$

$f': \frac{1}{x} \qquad G: \frac{1}{-2x^2}$

$= \frac{-\ln x}{2x^2} + \frac{1}{2}\int \frac{1}{x^3}\, dx$

$= -\frac{\ln x}{2x^2} + \frac{1}{2}\left(-\frac{1}{2x^2}\right)\Big]_1^e$

$= -\frac{\ln e}{2e^2} - \frac{1}{4e^2} + \cancel{\frac{\ln 1}{2 \cdot 1^2}} + \frac{1}{4 \cdot 1^2}$

$= \frac{-1^{\cdot 2}}{2 \cdot e^2_{\cdot 2}} - \frac{1}{4e^2} + \frac{1}{4} = \frac{-2}{4e^2} - \frac{1}{4e^2} + \frac{1}{4}$

$= -\frac{3}{4e^2} + \frac{1}{4} = \frac{1}{4}\left(1 - 3/e^2\right)$ ▨

f) $\int_1^{14} (2x-1)^{1/3}\, dx =$

Variabel substitution

$u = 2x - 1$

$x = 1 \quad u = 1$

$x = 14 \quad u = 28 - 1 = 27$

$du = 2\, dx$

$dx = \frac{du}{2}$

$\int_1 (u)^{1/3} \frac{du}{2} =$

$\frac{1}{2}\int_1^{27} u^{1/3}\, du =$

$\frac{1}{2}\, \frac{3}{4} u^{4/3}\Big]_1^{27} = \frac{3}{8}\left[u^{4/3}\right]_1^{27} =$

$\frac{3}{8}\left[27^{4/3} - 1^{4/3}\right] = \frac{3}{8}\left[3^4 - 1\right] =$

$\frac{3}{8}\left[81 - 1\right] = \frac{3 \cdot 80}{8} = 3 \cdot 10 = 30$ ▨

10.5

a) $$\int_0^{+\infty} e^{-2x}\,dx = \lim_{t\to\infty}\int_0^t e^{-2x}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{e^{-2x}}{-2}\right]_0^t = \lim_{t\to\infty} -\frac{1}{2}\left.\frac{1}{e^{2x}}\right]_0^t$$

$$= \lim_{t\to\infty}\left[\cancel{-\frac{1}{2e^{2t}}} + \frac{1}{2e^{2\cdot 0}}\right]$$

$$= \frac{1}{2\cdot e^0} = \frac{1}{2}$$ ■

b) $$\int_{-\infty}^{\infty} t^3 \cdot e^{-t^2}\,dt$$

jag börjar med integralen utan gränser dvs $\int t^3 e^{-t^2}\,dt = I$

$f: t^2$ $\qquad g: t e^{-t^2}$

$f': 2t$ $\qquad G: \dfrac{e^{-t^2}}{-2}$

$$I = \frac{-t^2}{2} e^{-t^2} \cancel{-} \int \frac{\cancel{2}t\, e^{-t^2}}{\cancel{-2}}\,dt$$

$$I = \frac{-t^2 e^{-t^2}}{2} + \int t e^{-t^2}\,dt$$

$$= \frac{-t^2 e^{-t^2}}{2} + \frac{e^{-t^2}}{-2}$$

$$= \frac{e^{-t^2}}{-2}\left[1 - t^2\right]$$

$$= \frac{+1}{2e^{t^2}}\left(t^2-1\right)$$

$$\int_{-\infty}^{\infty} t^3 e^{-t^2}\,dt = \lim_{a\to\infty}\int_{-a}^{a} t^3 e^{-t^2}\,dt$$

$$= \lim_{a\to\infty} \frac{1}{2e^{t^2}}\left(t^2-1\right)\Big]_{-a}^{a}$$

$$= \lim_{a\to\infty} \frac{1}{2e^{a^2}}\left(a^2-1\right) - \frac{1}{2e^{a^2}}\left(a^2-1\right)$$

$$= \lim_{a\to\infty} \frac{1}{2e^{a^2}}\left(\cancel{a^2}\cancel{-1}-\cancel{a^2}\cancel{+1}\right)$$

$$= \frac{1}{2e^{\infty}}(0) = 0$$

$$= 0.0 = 0 \quad \blacksquare$$

c) $\int_{-\infty}^{\pi} e^{x} \sin 2x \, dx$

$I = \int e^{x} \sin 2x \, dx$

$f: \sin 2x \qquad g: e^{x}$

$f': 2\cos 2x \qquad G: e^{x}$

$I = e^{x} \sin 2x - 2 \int e^{x} \cos 2x \, dx$

$f: \cos 2x \qquad g: e^{x}$

$f': -2 \sin 2x \qquad G: e^{x}$

$I = e^{x} \sin 2x - 2 \left[e^{x} \cos 2x + 2 \underbrace{\int e^{x} \sin 2x \, dx}_{I} \right]$

$I = e^{x} \sin 2x - 2 e^{x} \cos 2x - 4 I$

$5 I = e^{x} \sin 2x - 2 e^{x} \cos 2x$

$I = \dfrac{e^{x} (\sin 2x - 2 \cos 2x)}{5}$

Med gränser:

$-1 \leq \cos v \leq 1$
$-1 \leq \sin v \leq 1$
Alltid

$\int_{-\infty}^{\pi} e^{x} \sin 2x \, dx =$

$\lim_{t \to -\infty} \frac{e^{x}}{5} \left[\sin 2x - 2 \cos 2x \right]_{t}^{\pi} =$

$\frac{1}{5} e^{\pi} \left[\sin 2\pi - 2 \cos 2\pi \right] = -\frac{2}{5} e^{\pi}$

d) $\int_0^1 x \cdot \ln x \, dx$

$f: \ln x \qquad g: x$

$f': \frac{1}{x} \qquad G: \frac{x^2}{2}$

$$\int x \ln x \, dx =$$

$$\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx =$$

$$\frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx =$$

$$\frac{x^2}{2} \ln x - \frac{x^2}{4} = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right)$$

$$\int_0^1 x \cdot \ln x \, dx = \lim_{t \to 0} \left[\frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) \right]_t^1$$

$$= \lim_{t \to 0} \frac{1}{2} \left(0 - \frac{1}{2} \right) - \frac{t^2}{2} \left(\ln t - \frac{1}{2} \right)$$

$$= \lim_{t \to 0} -\frac{1}{4} \boxed{-\frac{t^2 \cdot \ln t}{2}} + \cancel{\frac{t^2}{4}}$$

?

undersök $\boxed{\lim_{t \to 0} t^2 \cdot \ln t}$ genom Hopitalsregel

$$\lim_{t \to 0} \boxed{\frac{\ln t}{t^{-2}}} \overset{l'H}{=} \lim_{t \to 0} \frac{\frac{1}{t}}{-2t^{-3}} = \lim_{t \to 0} \frac{1}{-2t^{-4}}$$

$$= \lim_{t \to 0} -\frac{t^4}{2} \longrightarrow 0$$

dvs $\lim_{t \to 0} -\frac{1}{4} \cancel{-\frac{t^2 \cdot \ln t}{2}} = -\frac{1}{4}$

$\blacksquare$

e) $\int_1^3 \frac{1}{\sqrt{x-1}}\,dx$

$$\int \frac{1}{\sqrt{x-1}}\,dx = \int (x-1)^{-1/2}\,dx$$

$$= \frac{(x-1)^{1/2}}{\frac{1}{2}} = 2(x-1)^{1/2} = 2\sqrt{x-1}$$

så egentligen behöver vi inte jobba med gränser och $\lim_{a\to 1}$

$$\int_1^3 \frac{1}{\sqrt{x-1}}\,dx = 2\sqrt{x-1}\Big]_1^3$$

$$= \cancel{-2\sqrt{1-1}} + 2\sqrt{3-1}$$

$$= 2\sqrt{2} \qquad \blacksquare$$

f) $\int_1^\infty \frac{1}{(x+1)^3}\,dx$

$$\int \frac{1}{(x+1)^3}\,dx = \int (x+1)^{-3}\,dx = \frac{(x+1)^{-2}}{-2}$$

$$= -\frac{1}{2}\,\frac{1}{(x+1)^2}$$

och $\int_1^\infty \frac{1}{(x+1)^3}\,dx = \lim_{t\to\infty} -\frac{1}{2}\left[\frac{1}{(x+1)^2}\right]_1^t$

$$= \lim_{t\to\infty} -\frac{1}{2}\left(\cancel{\frac{1}{(t+1)^2}} - \frac{1}{(1+1)^2}\right)$$

$$= -\frac{1}{2}\left(0 - \frac{1}{4}\right) = 1/8 \qquad \blacksquare$$

10.6

a) $\int_0^1 (x^2-2)^2\,dx$

$= \int_0^1 x^4 + 4 - 4x^2\,dx$

$= \left.\frac{x^5}{5} + 4x - 4\frac{x^3}{3}\right]_0^1$

$= \frac{1}{5} + 4 - \frac{4}{3} - 0 - 0 + 0 \Big] = \frac{3+60-20}{15}$

$= 43/15$ ■

b) $\int_1^1 \frac{dx}{(x^4+1)(x^5+1)} = 0$

utan att tänka :-) ■

f) $\int_0^{\frac{\pi}{4}} \tan x\,dx = \int_0^{\pi/4} \frac{\sin x}{\cos x}\,dx$

$= -\ln|\cos x|\Big]_0^{\pi/4} = -\ln\left|\frac{\sqrt{2}}{2}\right| + \ln|1|$

$= -\ln\frac{1}{\sqrt{2}} = \cancel{-\ln 1} + \ln\sqrt{2}$

$= \ln 2^{1/2} = \frac{1}{2}\ln 2$ ■

g) $\int_0^{\pi/6} \frac{1}{\cos^2 t}\,dt = \tan t\Big]_0^{\pi/6}$

$= \tan\frac{\pi}{6} - \tan 0$

$= \frac{\sqrt{3}}{3} - 0 = \sqrt{3}/3$ ■

eller $1/\sqrt{3}$

10.7

Partiell Integration

a) $\int_{-3}^{-2} t^2 \sqrt{t+3}\, dt$

$f: t^2 \qquad g: (t+3)^{1/2}$

$f': 2t \qquad G: \frac{2}{3}(t+3)^{3/2}$

$$I = \int t^2 \sqrt{t+3}\, dt = \frac{2t^2}{3}(t+3)^{3/2} - \int \frac{4t}{3}(t+3)^{3/2} dt$$

$f: \frac{4}{3}t \qquad g: (t+3)^{3/2}$

$f': \frac{4}{3} \qquad G: \frac{2}{5}(t+3)^{5/2}$

$$I = \frac{2t^2}{3}(t+3)^{3/2} - \left[\frac{8t}{15}(t+3)^{5/2} - \int \frac{8}{15}(t+3)^{5/2} dt\right]$$

$$= \frac{2t^2}{3}(t+3)^{3/2} - \frac{8t}{15}(t+3)^{5/2} + \frac{8}{15}\,\frac{2}{7}(t+3)^{7/2}$$

$$= \frac{2t^2}{3}(t+3)\sqrt{t+3} - \frac{8t}{15}(t+3)^2\sqrt{t+3}$$
$$+ \frac{16}{15 \cdot 7}(t+3)^3\sqrt{t+3}$$

och $\int_{-3}^{-2} t^2\sqrt{t+3}\, dt =$

$$\frac{2 \cdot 4}{3}(1)\sqrt{1} - \frac{8}{15}(-2)(1)^2\sqrt{1} + \frac{16}{15 \cdot 7}(1)^3\sqrt{1}$$

[resten är noll pgs $(-3+3)=0$]

$$\int_{-3}^{-2} t^2\sqrt{t+3} = \frac{5 \cdot 7 \cdot 8}{5 \cdot 7 \cdot 3} + \frac{16 \cdot 7}{15 \cdot 7} + \frac{16 \cdot 1}{15 \cdot 7}$$

$$= \frac{1}{5 \cdot 3 \cdot 7}(5 \cdot 7 \cdot 8 + 8 \cdot 16)$$

$$= \frac{8 \cdot (35+16)}{5 \cdot 3 \cdot 7} = \frac{8 \cdot 51}{3 \cdot 5 \cdot 7} = \frac{8 \cdot \cancel{3} \cdot 17}{\cancel{3} \cdot 5 \cdot 7}$$

$$= \frac{136}{35} = 3\ 31/35$$

b) $\int_0^1 \frac{\ln(3x+1)}{3x+1}\,dx$

Partielle Integration

$I = \int \frac{\ln(3x+1)}{3x+1}$

$f : \ln(3x+1)$ $\qquad$ $g : \frac{1}{3x+1}$

$f' : \frac{3}{3x+1}$ $\qquad$ $G : \frac{1}{3}\ln(3x+1)$

$I = \frac{1}{3}\ln^2(3x+1) - \int \frac{\cancel{3}}{\cancel{3}}\frac{\ln(3x+1)}{3x+1}$

$I = \frac{1}{3}\ln^2(3x+1) - I$

$2I = \frac{1}{3}\ln^2(3x+1)$

$I = \frac{1}{6}\ln^2(3x+1)$

$\int_0^1 \frac{\ln(3x+1)}{3x+1}\,dx$

$= \frac{1}{6}\left[\ln^2(3x+1)\right]_0^1$

$= \frac{1}{6}\left[\ln^2(3+1) - \cancel{\ln^2(1)}\right]$

$= \frac{1}{6}\ln 4 \cdot \ln 4 = \frac{1}{6}\ln 2^2 \cdot \ln 2^2$

$= \frac{\cancel{2}\cdot 2 \cdot \ln 2 \cdot \ln 2}{\cancel{6}\,3} = \frac{2}{3}\ln^2 2$

$= \frac{2}{3}(\ln 2)^2$

c) $\displaystyle\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$

$= \displaystyle\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-(2x)^2}}$ $\qquad u = 2x$

$du = 2\,dx$

$x = 0 \qquad u = 0$

$x = \frac{1}{4} \qquad u = 2 \cdot \frac{1}{4} = \frac{1}{2}$

$= \displaystyle\int_0^{1/2} \frac{1}{2} \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{2}\left[\arcsin u\right]_0^{1/2}$

$= \frac{1}{2}\left[\arcsin \frac{1}{2} - \cancel{\arcsin 0}\right]$

$= \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$

f) $\displaystyle\int_0^1 \arctan x\,dx$ $\qquad f : \arctan x \qquad g : 1$

$f' : \frac{1}{1+x^2} \qquad G : x$

$= x \cdot \arctan x - \frac{2}{2}\displaystyle\int \frac{x}{1+x^2}\,dx$

$= \left[x \cdot \arctan x - \frac{1}{2}\ln(x^2+1)\right]_0^1$

$= \arctan 1 - \frac{1}{2}\ln 2 \cancel{- 0 + \frac{1}{2}\ln(1)}$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2 \qquad = \frac{\pi}{4} - \ln 2^{1/2}$

$= \frac{\pi}{4} - \ln\sqrt{2}$

$= \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$

10.9

$f(x) = \int_1^x (t^2-1)e^t \, dt$ $\quad f: t^2-1 \quad g: e^t$

$\quad f': 2t \quad G: e^t$

$= (t^2-1)e^t - 2\int t \cdot e^t$ $\quad f: t \quad g: e^t$

$\quad f': 1 \quad G: e^t$

$= (t^2-1)e^t - 2[te^t - \int e^t]$

$= (t^2-1)e^t - 2te^t + 2e^t$

$= (t^2-1-2t+2)e^t \Big]_1^x$

$= (x^2-2x+1)e^x - e^1(1-1-2+2)$

$f(x) = (x^2-2x+1)e^x$

$f'(x) = (2x-2)e^x + (x^2-2x+1)e^x$

$= e^x(x^2 \cancel{-2x} +1 \cancel{+2x} -2)$

$= e^x(x^2-1)$

$e^x = 0$ aldrig $\quad x^2-1=0 \quad x = \pm 1$

$f(x) = (x^2-2x+1)e^x$ har max/min i $x = \pm 1$

$f(1) = (1-2+1)e^1 = 0$ Minimum

$f(-1) = (1+2+1)e^{-1} = 4/e$ Maximum

Kunde vi veta detta med mindre räkning?

$f(x) = \int_1^x (t^2-1)e^t \, dt$

Vad innebär att integrera sedan derivera igen?

10.10

$$\int_0^a \frac{\arctan \frac{x}{a}}{a^2 + x^2}\, dx =$$

$$\int_0^a \frac{\arctan \frac{x}{a}}{a^2 \left(1 + \frac{x^2}{a^2}\right)}\, dx =$$

$$\int_0^a \frac{\arctan \frac{x}{a}}{a^2 \left(1 + \left(\frac{x}{a}\right)^2\right)}\, dx$$

Variabelsubstitution

$\frac{x}{a} = u$

$\frac{1}{a}\, dx = du$

$dx = a\, du$

on $x = a \quad u = 1$

$x = 0 \quad u = 0$

$$\int_0^1 \frac{\arctan u}{a^2 \left(1 + u^2\right)}\, a \cdot du =$$

$$\frac{1}{a} \int_0^1 \frac{\arctan u}{\left(1 + u^2\right)}$$

Partiell Integration

$f: \frac{1}{a} \arctan u$ $\qquad$ $g: \frac{1}{1+u^2}$

$f': \frac{1}{a} \frac{1}{1+u^2}$ $\qquad$ $G: \arctan u$

$$\frac{1}{a}\int_0^1 \frac{\arctan u}{1+u^2}\,du$$

$$=\left[\frac{1}{a}\arctan^2 u\right]_0^1 - \frac{1}{a}\int_0^1 \frac{\arctan u}{1+u^2}\,du$$

$$\frac{2}{a}\int_0^1 \frac{\arctan u}{1+u^2}\,du = \frac{1}{a}\left[\arctan^2 u\right]_0^1$$

$$\frac{1}{a}\int_0^1 \frac{\arctan u}{1+u^2}\,du = \frac{1}{2a}\left[\arctan^2 u\right]_0^1$$

och det är det här som ska vara likamed "1"

dvs $\frac{1}{2a}\left[\arctan^2 1 - \cancel{\arctan^2 0}\right] = 1$

$$\frac{1}{2a}\left(\frac{\pi}{4}\cdot\frac{\pi}{4}\right) = 1$$

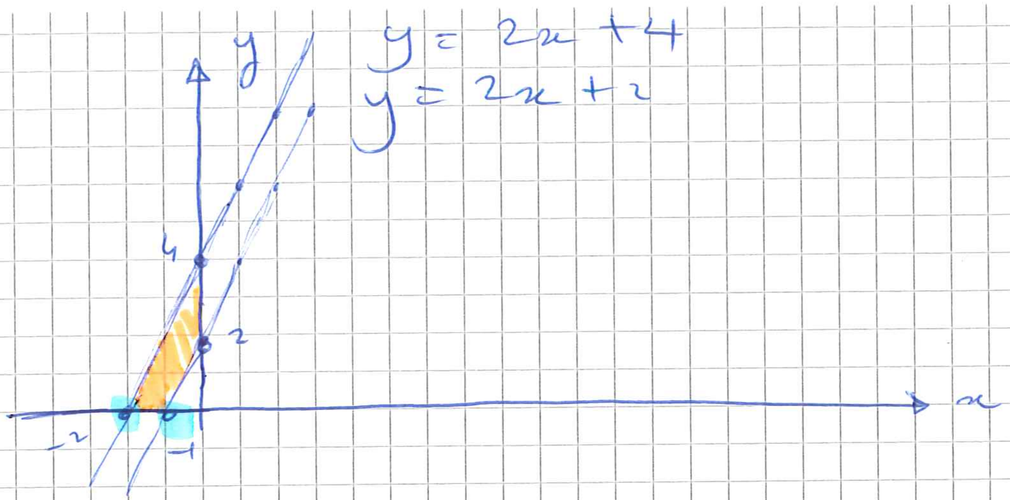
$$2a = \frac{\pi^2}{16}$$

$$a = \frac{\pi^2}{32}$$

■

10.11

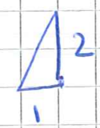
$y = 2x + 4$
$y = 2x + 2$



$x = -1$ och $x = -2$

hittar man genom att lösa
$y = 2x+4 = 0$
och $2x+2 = 0$

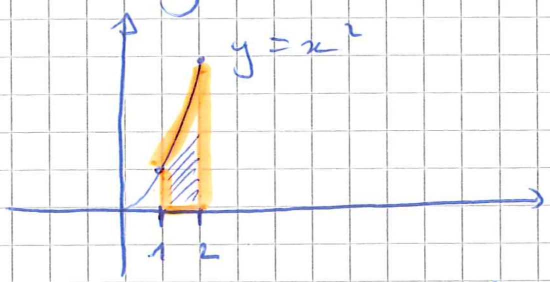
store triangeln
− lille triangeln



$\frac{4 \cdot \cancel{2}}{\cancel{2}} - \frac{\cancel{2} \cdot 1}{\cancel{2}} = 4 - 1$

$= 3$ ae

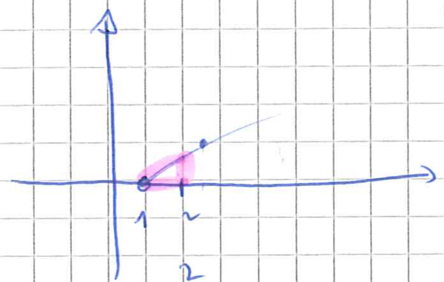
10.12 a) $y = x^2$ $\quad y = 0 \quad x = 1 \quad x = 2$



$$\text{Area} = \int_1^2 x^2 \, dx = \frac{x^3}{3}\Big]_1^2 = \frac{1}{3}\left[8 - 1\right] = 7/3$$

area enheter ■

b)



$$\text{Area} = \int_1^2 \ln x \, dx$$

$f: \ln x \qquad g: 1$

$f': \frac{1}{x} \qquad G: x$

$$= x \cdot \ln x - \int \frac{1}{x} \cdot x \, dx$$

$$= x \cdot \ln x - \int 1 \cdot dx$$

$$= x \cdot \ln x - x \Big]_1^2$$

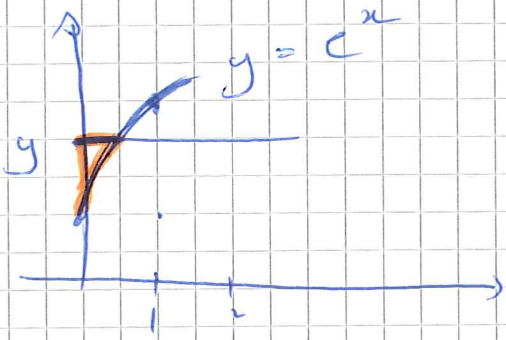
$$= 2 \ln 2 - 2 - 1 \cdot \ln 1 + 1$$

$$= 2 \ln 2 - 1$$

$$= \ln 2^2 - 1$$

$$= \ln 4 - 1$$

■

c) 

$y = 2$ $\qquad$ $y = e^x$

$e^x = 2$

$\underline{\underline{x = \ln 2}}$

x mellan $\ln 2$ och 0

$\int_0^{\ln 2}$ övre funktion − nedre funktion

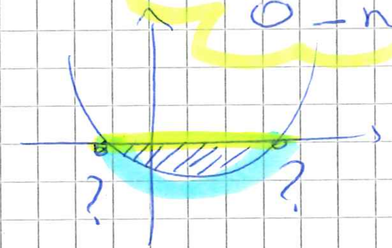
$$= \int_0^{\ln 2} 2 - e^x \; dx$$

$$= 2x - e^x \Big]_0^{\ln 2} = 2\cdot \ln 2 - e^{\ln 2} - 2\cdot 0 + e^0$$

$$= 2\ln 2 - 2 - 0 + 1$$

$$= 2\ln 2 - 1$$

övre − nedre
0 − nedre



d) $\int -x^2 + 2x + 3 \; dx$

$x^2 - 2x - 3 = 0$

p,q formel

$x = -1$

$x = 3$

$$= \int_{-1}^{3} -x^2 + 2x + 3 \; dx$$

$$= \left[\frac{-x^3}{3} + x^2 + 3x\right]_{-1}^{3}$$

$$= \left(\frac{-3\cdot 3\cdot \cancel{3}}{\cancel{3}} + 9 + 3\cdot 3\right) - \left(+\frac{1}{3} + 1 - 3\right)$$

$$= 9 - \frac{1}{3} + 2$$

$$= 32/3$$

10.14

$y_1 = x^2 - 4x + 3$ Nollstellen

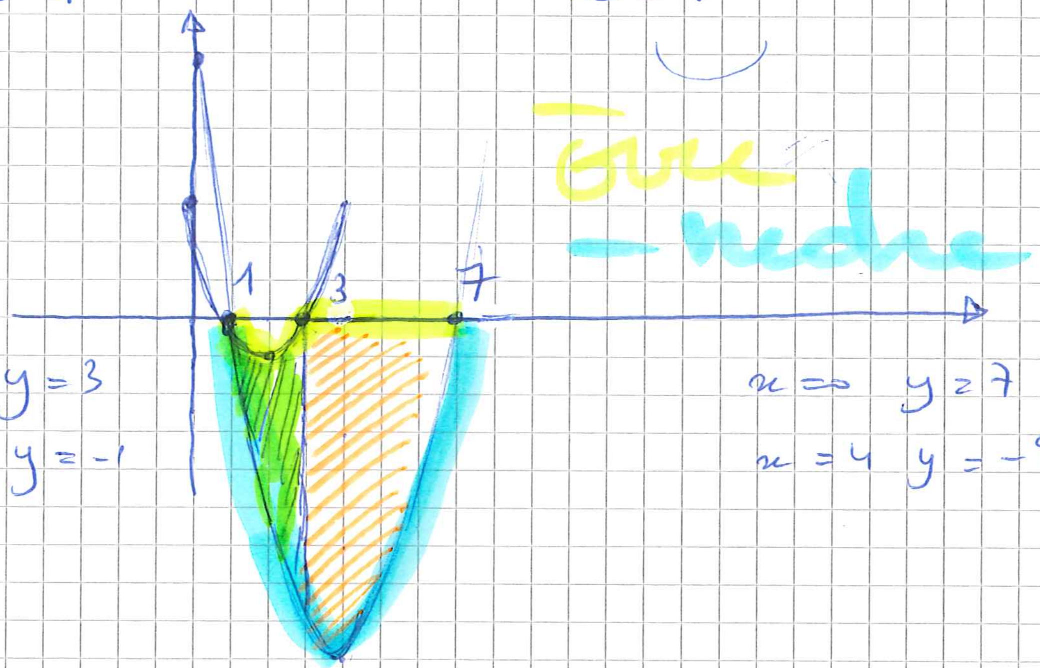
$x = +3$

$x = 1$

$y_2 = x^2 - 8x + 7$ Nollstellen

$x = 1$

$x = 7$



$x = 0 \quad y = 3$

$x = 2 \quad y = -1$

$x = 0 \quad y = 7$

$x = 4 \quad y = -9$

Dessuben $y_1 = y_2$

$\cancel{x^2} - 4x + 3 = \cancel{x^2} - 8x + 7$

$4x = 4 \qquad \underline{\underline{x = 1}}$

$$\text{Area} = \int_1^3 x^2 - 4x + 3 - x^2 + 8x - 7 \, dx$$

$$+ \int_3^7 0 - x^2 + 8x - 7 \, dx$$

$$= \int_1^3 4x - 4 \, dx + \int_3^7 -x^2 + 8x - 7 \, dx$$

$$= \left[2x^2 - 4x\right]_1^3 + \left[-\frac{x^3}{3} + 4x^2 - 7x\right]_3^7$$

$$= [12 - 12 - 2 + 4] + \left[-\frac{343}{3} + 4.49 - 49\right] - \left[-27/3 + 4.9 + 7.3\right]$$

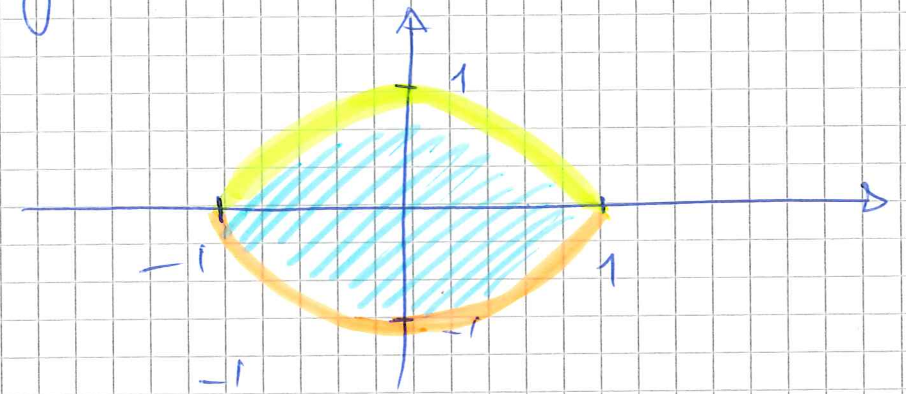
$$= 104/3$$

10.15

$$y^2 = (1-x^2)^2$$

$$y = \pm (1-x^2)$$

$$y_1 = 1-x^2 \qquad y_2 = x^2-1$$



$$\text{Area} = \int_{-1}^{1} \text{âre} - \text{neohe}$$

$$= \int_{-1}^{1} 1 - x^2 - x^2 + 1 \quad dx$$

$$= \int_{-1}^{1} -2x^2 + 2 \quad dx$$

$$= \left. -2\frac{x^3}{3} + 2x \right]_{-1}^{1} = \left(-\frac{2}{3} + 2\right) - \left(\frac{2}{3} - 2\right)$$

$$= -\frac{2}{3} + 2 - \frac{2}{3} + 2$$

$$= 4 - \frac{4}{3} = \frac{12-4}{3} = 8/3 \text{ ae}$$

10.17 $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin \pi \frac{k}{n}}{n}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \sin \pi \left(\frac{k}{n}\right)$$

$$= \int_0^1 \sin \pi x \, dx$$

$$= -\frac{\cos \pi x}{\pi} \Big]_0^1$$

$$= -\frac{1}{\pi} \left[\cos \pi - \cos 0\right]$$

$$= -\frac{1}{\pi} \left[-1 - 1\right]$$

$$= -\frac{1}{\pi} (-2) = 2/\pi$$

10.18 $\lim_{n\to\infty} \sum_{k=0}^{3n} \frac{k+1}{n^2} = \lim_{n\to\infty} \sum_{k=1}^{3n+1} \frac{k}{n^2}$

$= \lim_{n\to\infty} \sum_{k=1}^{3n+1} \frac{1}{n} \frac{k}{n}$

ledning i facit

$= \int_0^3 x \, dx$

$= \frac{x^2}{2} \Big]_0^3 = \frac{1}{2} [9 - 0] = 9/2$ ■

10.19 $\lim_{n\to\infty} \sum_{k=0}^{2n} \frac{n}{k^2 + 4n^2}$

$= \lim_{n\to\infty} \sum_{k=0}^{2n} \frac{\cancel{n^2} \left(\frac{1}{n}\right)}{\cancel{n^2} \left(\left(\frac{k}{n}\right)^2 + 4\right)}$

$= \lim_{n\to\infty} \sum_{k=0}^{2n} \frac{1}{n} \frac{1}{4 + \left(\frac{k}{n}\right)^2}$

$= \int_0^2 \frac{1}{4 + x^2} dx = \int_0^2 \frac{1}{4\left(1 + \left(\frac{x}{2}\right)^2\right)} dx$

$= \int \frac{2\, du}{4(1 + u^2)}$

$u = x/2$
$du = \frac{1}{2} dx$
$2du = dx$

$= \frac{1}{2} \arctan u \Big] = \frac{1}{2} \arctan\left(\frac{x}{2}\right) \Big]_0^2$

$= \frac{1}{2} [\arctan(1) - \cancel{\arctan(0)}] = \frac{\pi}{8}$ ■

10.20

$y = x^3 + 3x^2 - 4x$ Nollställen

$x(x^2+3x-4)=0$

$x=0 \quad x=1 \quad x=-4$

Tangentens ekvation i $x=-2$

$y = \cancel{-8} + 3\cdot 4 \cancel{-4(-2)} = 12$

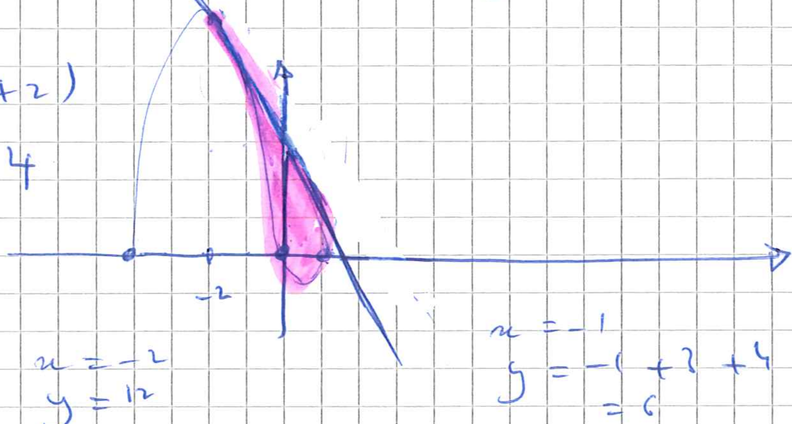
$y' = 3x^2 + 6x - 4$

$y'(-2) = 3\cdot 4 + 6(-2) - 4 = \cancel{12} \cancel{-12} - 4$

$y - 12 = -4(x+2)$

$y = -4x + 4$

tangenten



skärningspunkterna:

$x^3 + 3x^2 \cancel{-4x} = \cancel{-4x} + 4$

$x^3 + 3x^2 - 4 = 0$

$x = -2$ (det vet vi genom att de tangerar varandra i $x=-2$)

och $x = 1$ eftersom $1+3-4=0$

$$\text{Area} = \int_{-2}^{1} \text{övre} - \text{nedre} = \int_{-2}^{1} \cancel{-4x} + 4 - x^3 - 3x^2 \cancel{+4x}\, dx$$

$$= \int_{-2}^{1} -x^3 - 3x^2 + 4 \, dx = -\frac{x^4}{4} - x^3 + 4x \Big]_{-2}^{1}$$

$$= -\frac{1}{4} - 1 + 4 + \frac{16}{4} + \cancel{-8} \cancel{+8} =$$

$27/4$ ■

10.21

$$\int_2^\infty \frac{2}{1-x^2}\,dx = ?$$

$$\frac{2}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$\frac{2 \quad 0}{1-x^2} = \frac{A + Ax + B - Bx}{(1-x)(1+x)}$$

$A + B = 2$ $\qquad$ $A = 1$

$A - B = 0$ $\qquad$ $B = 1$

$$\int_2^\infty \frac{2}{1-x^2} = \int_2^\infty \frac{1}{1-x} + \frac{1}{1+x}$$

$$= -\ln|1-x| + \ln|1+x| \Big]_2^\infty$$

$$= \ln\left|\frac{1+x}{1-x}\right| \Big]_2^\infty$$

$$\lim_{x\to\infty} = \ln\left|\frac{\cancel{x}(1+\cancel{\frac{1}{x}})}{\cancel{x}(-1+\cancel{\frac{1}{x}})}\right| - \ln\left|\frac{1+2}{1-2}\right|$$

$$= \ln|-1| - \ln\left|\frac{3}{-1}\right|$$

$$= \ln|-1| - \ln|-3|$$

$$= 0 - \ln 3$$

$$= -\ln 3$$