Kapitel tre Hsn

(1)
$$\begin{cases} \frac{dy}{dt} = -t^2 y & t \geq 0 \\ y(0) = e \end{cases}$$

Variabel separation

$$\frac{dy}{y} = -t^2\, dt$$

$$\ln|y| = -\frac{t^3}{3} + C$$

$$y = e^{-t^3/3 + C} = e^{-t^3/3} \cdot e^{C}$$

$$y = e^{C} e^{-t^3/3} \quad (*)(*)$$

$y(0) = e$ dvs i $(*)(*)$

$$e = e^{C} \cdot e^{-0} = e^{C} \cdot 1$$

$$\boxed{C = 1}$$

$$\boxed{y = e^{1 - t^3/3}}$$

■

(2) $y' + 2x e^{y} = 0 \qquad y(0) = 0$

$$\frac{dy}{dx} = -2x e^{y}$$

$$\frac{dy}{-e^{y}} = 2x \cdot dx$$

$$-e^{-y}\, dy = 2x\, dx$$

$$\int -e^{-y}\, dy = \int 2x\, dx$$

$$e^{-y} = x^{2} + C \qquad (*) \quad (*)$$

on $x = 0 \qquad y = 0 \qquad (y(0) = 0)$

$$e^{-0} = 0 + C \qquad C = 1$$

<u>dus</u> $(*)\ (*)$

$$e^{-y} = x^{2} + 1$$

$$-y = \ln(x^{2}+1)$$

$$y = -\ln(x^{2}+1)$$

$$\boxed{y = -\ln(x^{2}+1)}$$

③ $\frac{dv}{dt} = 1 - v^2$

(*) $\frac{dv}{(1-v)(1+v)} = dt$ Variabel-separation

$\frac{1}{(1-v)(1+v)} = \frac{A}{1-v} + \frac{B}{1+v}$ Partialbråk-uppdelning

$1 = A(1+v) + B(1-v) = A + Av + B - Bv$

$\begin{cases} 1 = A + B \\ 0 = A - B \end{cases}$ $A = \frac{1}{2}$ $B = \frac{1}{2}$

dvs (*) ger:

$\int \frac{1}{2} \frac{1}{1-v} + \frac{1}{2} \frac{1}{1+v} \, dv = dt$

$\frac{1}{2} \int \frac{1}{1-v} + \frac{1}{1+v} \, dv = dt$

$\int \frac{1}{1-v} + \frac{1}{1+v} \, dv = 2 \, dt$

$-\ln|1-v| + \ln|1+v| = 2t + C$

$\ln\left|\frac{1+v}{1-v}\right| = 2t + C$

och $v(0) = 0$ dvs:

$\ln\left|\frac{1+0}{1-0}\right| = 2 \cdot 0 + C$

$0 = 0 + C$ $C = 0$

i slutändan:

$$\ln \left| \frac{1+v}{1-v} \right| = 2t$$

$$\frac{1+v}{1-v} = e^{2t}$$

$$1 + v = e^{2t} - v \cdot e^{2t}$$

$$v(1 + e^{2t}) = e^{2t} - 1$$

och $v(t) = \dfrac{e^{2t} - 1}{1 + e^{2t}}$

$$v(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$$

$\blacksquare$

(4) $y' = x(1-y)$ $\quad y < 1$, $y(0) = 0$

Variabel separation

$$\frac{dy}{dx} = x(1-y)$$

$$\int \frac{dy}{1-y} = \int x \, dx$$

$$-\ln|1-y| = \frac{x^2}{2} + C$$

$y(0) = 0$ ger

$$-\ln|1-0| = 0 + C$$

$$C = 0$$

och $\quad -\ln|1-y| = \frac{x^2}{2}$

$$\ln|1-y| = \frac{-x^2}{2}$$

$$1-y = e^{-x^2/2}$$

$$y-1 = -e^{-x^2/2}$$

$$y = 1 - e^{-x^2/2}$$

$\blacksquare$

(5) $\begin{cases} yy' = e^{x+2y} \\ y(0) = 1 \end{cases}$

Variabel-separat. med $y(0)=1$

$$y\frac{dy}{dx} = e^{x} \cdot e^{2y}$$

$$\int e^{-2y} y \cdot dy = \int e^{x} dx$$

Partiell integrat. $f: y$ $g: e^{-2y}$

$f': 1$ $G: \frac{e^{-2y}}{-2}$

$$\frac{-y}{2} e^{-2y} + \frac{1}{2}\int e^{-2y} dy = e^{x} + C$$

$$\frac{-y}{2} e^{-2y} + \frac{1}{2}\frac{1}{-2} e^{-2y} = e^{x} + C$$

$$-e^{-2y}\left(\frac{y}{2} + \frac{1}{4}\right) = e^{x} + C \quad (*)$$

$y(0) = 1$ i $(*)$ ger:

$$-e^{-2}\left(\frac{1}{2} + \frac{1}{4}\right) = e^{0} + C$$

$$-1 - e^{-2}\left(\frac{3}{4}\right) = C \qquad C = -\left(1 + \frac{3}{4}e^{-2}\right)$$

dvs $(*)$ ger

$$-\frac{1}{4}e^{-2y}(2y+1) = e^{x} - \left(1 + \frac{3}{4}e^{-2}\right)$$

$$e^{-2y}(2y+1) = -4e^{x} + 4 + 3e^{-2}$$

(6) $y' \sin x = \cos x \,(1+y^2)$ $\quad y(\pi/2) = 1$

$\frac{dy}{dx} \sin x = \cos x \,(1+y^2)$ Variabel-separation

$$\int \frac{dy}{1+y^2} = \int \frac{\cos x}{\sin x}\, dx$$

$\arctan(y) = \ln|\sin x| + C$ (*)

villkoret $y(\pi/2) = 1$

$\arctan(1) = \ln|\sin \frac{\pi}{2}| + C$

$\frac{\pi}{4} = \cancel{\ln|1|} + C$

$C = \pi/4$

och (*) ger!

$$\arctan(y) = \ln|\sin x| + \frac{\pi}{4}$$

$$y = \tan\left(\ln|\sin x| + \frac{\pi}{4}\right)$$

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$$y' - \frac{1-y^2}{x-1} = 0 \qquad y(3) = 3$$
$$x > 2{,}5$$

$$\frac{dy}{dx} = \frac{1-y^2}{x-1}$$

$$\frac{dy}{1-y^2} = \frac{dx}{x-1} \qquad (*)$$

$$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y}$$ Partialbråk uppdelning

$$1 = A + Ay + B - By$$

$$\begin{cases} A + B = 1 \\ A - B = 0 \end{cases} \qquad A = 1/2 \quad B = 1/2$$

$(*)$ blir

$$\int \left(\frac{1}{2}\frac{1}{1-y} + \frac{1}{2}\frac{1}{1+y}\right) dy = \int \frac{1}{x-1}\, dx$$

$$\frac{1}{2}\left[-\ln|1-y| + \ln|1+y|\right] = \ln|x-1| + C$$

$$\ln\left|\frac{1+y}{1-y}\right| = 2\ln|x-1| + 2C$$

$$\ln\left|\frac{1+y}{1-y}\right| = \ln(x-1)^2 + 2C$$

$\ln\left|\frac{1+y}{1-y}\right|$ existerar om $\left|\frac{1+y}{1-y}\right| > 0$

y		-1		1	
$1+y$	$-$	0	$+$		$+$
$1-y$	$+$		$+$	0	$-$
$\frac{1+y}{1-y}$	$-$		$+$		$-$

om $-1 \leq y < 1$ ($y(3) = 3$?? fungerar det)

Vi fortsätter ändå ...

$$e^{\ln\left|\frac{1+y}{1-y}\right|} = e^{\ln (x-1)^2 + 2c}$$

$$\left|\frac{1+y}{1-y}\right| = (x-1)^2 \cdot e^{2c}$$

$$1 + y = (x-1)^2 e^{2c} - y(x-1)^2 e^{2c}$$

$$y\left[1 + (x-1)^2 e^{2c}\right] = (x-1)^2 e^{2c} - 1$$

$$y = \frac{(x-1)^2 e^{2c} - 1}{(x-1)^2 e^{2c} + 1}$$

förenkla med e^{2c}

$$y = \frac{(x-1)^2 - e^{-2c}}{(x-1)^2 + e^{-2c}}$$

Vi sätter $k = e^{-2c}$ $\quad k > 0$

$$y = \frac{(x-1)^2 - k}{(x-1)^2 + k}$$ med $y(3) = 3$

$$3 = \frac{(2)^2 - k}{(2)^2 + k}$$

$$12 + 3k = 4 - k$$

$$4k = -8$$

$$k = -2$$

(Motsägelse $k > 0$)

så $y = 3$ fungerar inte!

Men på facit har man fortsätt med $\underline{\underline{k = -2}}$

$$y = \frac{(x-1)^2 - (-2)}{(x-1)^2 + (-2)}$$

$$y = \frac{x^2 - 2x + 1 + 2}{1 + x^2 - 2x - 2}$$

$$y = \frac{x^2 - 2x + 3}{x^2 - 2x - 1}$$

Den uppgiften är <u>fel</u> formulerad från början!

(12) $xy' - 2y = x^2 + 1$ $\quad x > 0$

$y(1) = 0$

$y' - \frac{2}{x} y = x + \frac{1}{x}$ (*)

Integrerande faktor (I.F.)

$e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

I.F. är $\frac{1}{x^2}$

(*) multipliceras med I.F.

$\frac{1}{x^2} y' - \frac{2}{x^3} y = \frac{1}{x} + \frac{1}{x^3}$

$D\left[\frac{1}{x^2} y\right] = \frac{1}{x} + \frac{1}{x^3}$

$\frac{1}{x^2} y = \ln|x| - \frac{1}{2} x^{-2} + C$

$y = x^2 \ln|x| - \frac{1}{2} + Cx^2$

tillsammans med $y(1) = 0$

$0 = 1 \cdot \ln|1| - \frac{1}{2} + C \qquad C = 1/2$

$y = x^2 \ln|x| - \frac{1}{2} + \frac{x^2}{2}$ $\blacksquare$

(13) $2xy' - 4y = x^3$ $\quad x > 0$, $y(1) = 0$

$$y' - \frac{2}{x} y = \frac{1}{2} x^2 \quad (*)$$

$$\text{I.F}: \quad e^{\int -\frac{2}{x} dx} = e^{-2\ln|x|} = e^{\ln x^{-2}}$$

$$\text{I.f} = \frac{1}{x^2}$$

Multiplicera $(*)$ med I.F

$$\frac{1}{x^2} y' - \frac{2}{x^3} y = \frac{1}{2}$$

$$D\left[\frac{1}{x^2} y\right] = \frac{1}{2}$$

$$\frac{1}{x^2} y = \frac{1}{2} x + C$$

$$y = \frac{1}{2} x^3 + C x^2$$

tillsammans med $y(1) = 0$

$$0 = \frac{1}{2} + C \qquad C = -1/2$$

$$y = \frac{1}{2} x^3 - \frac{1}{2} x^2$$

$$y = \frac{1}{2} (x^3 - x^2)$$

$\blacksquare$

(14) $y' - 2y = x$

I.F: $e^{\int -2\,dx} = e^{-2x}$

$e^{-2x} y' - 2e^{-2x} y = e^{-2x} x$

$D[e^{-2x} \cdot y] = e^{-2x} \cdot x$

$e^{-2x} y = -\frac{x}{2} e^{-2x} + \frac{1}{2} \frac{e^{-2x}}{-2} + C$

$f = x \qquad g = e^{-2x}$

$f' = 1 \qquad G = \frac{e^{-2x}}{-2}$

$-\frac{x}{2} e^{-2x} + \frac{1}{2} \int e^{-2x}\,dx$

Partiell Integration

dvs

$y = -\frac{x}{2} - \frac{1}{4} + Ce^{2x}$

(15) $(1-x)y' - 2y = x^3 \qquad x > 1$

$$y' - \frac{2}{1-x}\,y = \frac{x^3}{1-x} \qquad (*)$$

IoF $e^{\int \frac{-2}{1-x}\,dx} = e^{2\ln|1-x|} = e^{\ln(1-x)^2} = (1-x)^2$

Multiplicera $(*)$ med IoF

$$(1-x)^2 y' - 2(1-x)y = x^3(1-x)$$

$$D[(1-x)^2 \cdot y] = x^3 - x^4$$

$$(1-x)^2 y = \frac{x^4}{4} - \frac{x^5}{5} + C$$

$$y = \frac{x^4}{(1-x)^2 \cdot 4} - \frac{x^5}{5(1-x)^2} + \frac{C}{(1-x)^2}$$

eller

$$y = \frac{\frac{x^4}{4} - \frac{x^5}{5} + C}{(1-x)^2}$$

$\blacksquare$

(1b) $y' + \frac{2}{\sin 2x} y = \frac{1}{\cos^2 x} \quad 0 < x < \frac{\pi}{2}$

I.F är $e^{\int \frac{2}{\sin 2x} dx}$

Vi undersöker $\int \frac{2}{\sin 2x} dx = I$

$I = \int \frac{2}{2 \sin x \cos x} dx = \int \frac{dx}{\sin x \cos x} \begin{matrix} /\sin^2 x \\ /\sin^2 x \end{matrix}$

$I = \int \frac{\frac{1}{\sin^2 x} dx}{\frac{\cos x}{\sin x}}$

Variabel substitution

$u = \frac{\cos x}{\sin x}$

$du = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} dx$

$du = -\frac{1}{\sin^2 x} dx$

$I = \int -\frac{du}{u}$

$I = -\ln |u|$ (behöver ingen konstant här!)

$I = -\ln \left| \frac{\cos x}{\sin x} \right| = \ln \left| \frac{\sin x}{\cos x} \right|$

och I.f är $e^{\int I} = e^{\ln \left| \frac{\sin x}{\cos x} \right|} = \frac{\sin x}{\cos x}$

ekvationen blir då :

$$\frac{\sin x}{\cos x} y' + \frac{\cancel{2}}{\cancel{2} \cancel{\sin x} \cos x} \frac{\cancel{\sin x}}{\cos x} y = \frac{\sin x}{\cos^3 x}$$

$$\frac{\sin x}{\cos x} y' + \frac{1}{\cos^2 x} y = \frac{\sin x}{\cos^3 x}$$

$$D\left[y \frac{\sin x}{\cos x} \right] = \frac{\sin x}{\cos^3 x}$$

$$y \frac{\sin x}{\cos x} = \int \frac{\sin x}{\cos^3 x} dx$$

$$t = \cos x$$

$$dt = -\sin x \, dx$$

$$= \int -\frac{dt}{t^3}$$

$$= \frac{1}{2t^2} + C = \frac{1}{2\cos^2 x} + C$$

och $y = \left(\frac{1}{2\cos^2 x} + C \right) \left(\frac{\cos x}{\sin x} \right)$

$$y = \frac{1}{2 \cos x \sin x} + C \frac{\cos x}{\sin x}$$

$$y = \frac{1}{\sin 2x} + C \frac{\cos x}{\sin x}$$

(17) $y' \sin x - 2y \cos x = \cos x$

$$y' - 2y \frac{\cos x}{\sin x} = \frac{\cos x}{\sin x} \quad (*)$$

I.F $e^{\int -2 \frac{\cos x}{\sin x} dx} = e^{-2 \ln|\sin x|}$

$$= e^{\ln|\sin x|^{-2}} = \frac{1}{(\sin x)^2}$$

(*) Multiplicera med I.F :

$$\Longrightarrow \frac{y'}{(\sin x)^2} - 2y \frac{\cos x}{(\sin x)^3} = \frac{\cos x}{(\sin x)^3}$$

$$D\left[y \cdot \frac{1}{(\sin x)^2} \right] = \frac{\cos x}{(\sin x)^3}$$

$$y \frac{1}{(\sin x)^2} = \int \frac{\cos x}{(\sin x)^3} dx \qquad \begin{aligned} t &= \sin x \\ dt &= \cos x \, dx \end{aligned}$$

$$= \int \frac{dt}{t^3} = \frac{t^{-2}}{-2} + C$$

$$= \frac{-1}{2 \sin^2 x} + C$$

$$y = -\frac{1}{2} + C (\sin x)^2$$

$\blacksquare$