Kapitel åtta:

8.1

$f(x) = \ln(1+x)$ $\qquad f(0) = 0$

$f'(x) = \dfrac{1}{1+x}$ $\qquad f'(0) = 1$

$f''(x) = \dfrac{-1}{(1+x)^2} = -(1+x)^{-2}$ $\qquad f''(0) = -1$

$f'''(x) = \dfrac{2}{(1+x)^3} = 2(1+x)^{-3}$ $\qquad f'''(0) = 2$

$f^{(4)}(x) = \dfrac{-6}{(1+x)^4} = -6(1+x)^{-4}$ $\qquad f^{(4)}(0) = -6$

$f^{(5)}(x) = \dfrac{24}{(1+x)^5}$ $\qquad f^{(5)}(0) = 24$

osv ----

Enligt definitionen av MacLaurin serien:

$$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)\cdot x^2}{2!} + \frac{f'''(0)\, x^3}{3!} + \frac{f^{(4)}(0)\, x^4}{4!} + \frac{f^{(5)}(0)\, x^5}{5!} + \cdots +$$

dvs

$$\ln(1+x) = 0 + 1.x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3 + \frac{(-6)\, x^4}{4!} + \frac{24}{5!}x^5 + \cdots + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{2!\cdot x^3}{3\cdot 2!} - \frac{3!\, x^4}{4\cdot 3!} + \frac{4!\, x^5}{5\cdot 4!}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots +$$

8.2

Två metoder: Metod I

$f(x) = e^{2x}$ $\qquad f(0) = 1$

$f'(x) = 2e^{2x}$ $\qquad f'(0) = 2$

$f''(x) = 4e^{2x}$ $\qquad f''(0) = 4 = 2^2$

$f'''(x) = 8e^{2x}$ $\qquad f'''(x) = 8 = 2^3$

$f^{(4)}(x) = 16e^{2x}$ $\qquad f^{(4)}(x) = 16 = 2^4$ osv...

Definitionen ger:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \dots +$$

$$e^{2x} = 1 + 2x + \frac{2^2x^2}{2} + \frac{2^3 \cdot x^3}{3!} + \frac{2^4 \cdot x^4}{4!} + \dots +$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{8x^3}{6} + \frac{\cancel{16}^{4}x^4}{\cancel{4}\cdot 3 \cdot 2 \cdot 1}$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \dots$$

Metod II utnyttja den kända formeln

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!} + \dots$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \dots$$

8.3

$f(x) = (1+x)^5 \qquad f(0) = 1$

$f'(x) = 5(1+x)^4 \qquad f'(0) = 5$

$f''(x) = 20(1+x)^3 \qquad f''(0) = 20$

$f'''(x) = 60(1+x)^2 \qquad f'''(0) = 60$

$f^{(4)}(x) = 120(1+x) \qquad f^{(4)}(0) = 120$

$f^{(5)}(x) = 120 \qquad f^{(5)}(0) = 120$

$f^{(6)}(x) = 0 \quad \text{----} \qquad f^{(6)}(0) = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$+ \frac{f^{(4)}(0)x^4}{4!} + \frac{f^{(5)}(0)x^5}{5!} + 0 + 0 + \cdots$$

$$(1+x)^5 = 1 + 5 \cdot x + \frac{20x^2}{2} + \frac{60x^3}{6}$$

$$+ \frac{120x^4}{24} + \frac{120x^5}{120} + 0 + \cdots$$

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3$$

$$+ 5x^4 + x^5$$

resten är noll

$\blacksquare$

8.4 Taylor serier lyder: (Maclaurin fast i $x=a$ inte $x=0$)

a) $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$

när $a = \frac{\pi}{4}$

$f(x) = \sin x$ $\qquad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$ $\qquad f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{2} = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $\qquad f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{-\sqrt{2}}{2}$

$f'''(x) = -\cos x$ $\qquad f'''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = \frac{-\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$ $\qquad f^{(4)}\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

osv ...

$n = 4 \Longrightarrow$ fjärde grad med $a = \frac{\pi}{4}$

$$\sin x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right) + \frac{-\frac{\sqrt{2}}{2}}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{-\frac{\sqrt{2}}{2}}{3!}\left(x-\frac{\pi}{4}\right)^3 + \frac{\frac{\sqrt{2}}{2}}{4!}\left(x-\frac{\pi}{4}\right)^4$$

$$\sin x = \frac{\sqrt{2}}{2}\left(1 + \left(x-\frac{\pi}{4}\right) - \frac{\left(x-\frac{\pi}{4}\right)^2}{2} - \frac{\left(x-\frac{\pi}{4}\right)^3}{6} + \frac{1}{24}\left(x-\frac{\pi}{4}\right)^4\right)$$

b) $f(x) = \sqrt{1+x} = (x+1)^{\frac{1}{2}}$ $\qquad f(1) = \sqrt{2}$

$f'(x) = \frac{1}{2} \frac{1}{\sqrt{x+1}} = \frac{1}{2}(x+1)^{-\frac{1}{2}}$ $\qquad f'(1) = \frac{1}{2\sqrt{2}}$

$f''(x) = -\frac{1}{4}(x+1)^{-3/2}$ $\qquad f''(1) = \frac{-1}{8\sqrt{2}}$

$= \frac{-1}{4(x+1)\sqrt{x+1}}$

$f'''(x) = \frac{3}{8}(x+1)^{-5/2}$ $\qquad f'''(1) = \frac{3}{8 \cdot 2^2 \sqrt{2}}$

$= \frac{3}{8(x+1)^2\sqrt{x+1}}$ $\qquad = \frac{3}{32\sqrt{2}}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$+ \frac{f'''(a)}{3!}(x-a)^3$$

$n = 3$ och $a = 1 \Longrightarrow$

$$\sqrt{1+x} = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1) + \frac{-1}{8\sqrt{2}} \frac{(x-1)^2}{2}$$

$$+ \frac{\cancel{3}}{32\sqrt{2}} \frac{(x-1)^3}{\cancel{6}\ 2}$$

$$\sqrt{1+x} = \sqrt{2} + \frac{1}{2\sqrt{2}}(x-1)$$

$$- \frac{1}{16\sqrt{2}}(x-1)^2$$

$$+ \frac{(x-1)^3}{64\sqrt{2}}$$

c) $f(x) = \tan 2x$ $\qquad f(-\frac{\pi}{2}) = 0$

$f'(x) = (1 + \tan^2 2x) \cdot 2 \qquad f'(-\frac{\pi}{2}) = 2$

$f''(x) = 2 \cdot \left[2 \tan 2x \cdot (2 + 2\tan^2 2x) \right]$

$f''(-\frac{\pi}{2}) = 0$

$n = 2$ och $a = -\pi/2$

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

$\tan 2x = 0 + 2(x + \frac{\pi}{2}) + \frac{0}{2}(x-a)^2$

$\tan 2x = 2(x + \frac{\pi}{2})$

$\tan 2x = \overline{2x + \pi}$ $\blacksquare$

8.5

a) $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$

$\sin 0{,}2 = 0{,}2 - \frac{0{,}2^3}{6} + \frac{0{,}2^5}{120}$

$= 0{,}198669$

$\approx 0{,}199$

Alla formler finns på s. 221

b) $\cos x = 1 - \frac{x^3}{2} + \frac{x^4}{24}$

$\cos 0{,}1 = 1 - \frac{0{,}1^3}{2} + \frac{0{,}1^4}{24}$

$= 0{,}99950$

$\approx 0{,}1$

c) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

$\arctan 0{,}1 = 0{,}1 - \frac{0{,}1^3}{3} + \frac{0{,}1^5}{5} - \frac{0{,}1^7}{7}$

$= 0{,}099668$

$\approx 0{,}1$

d) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$

$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!}$

$= 2{,}718$

8.6

$y = \cos x$ och

$y = 1 - \frac{x^2}{2}$ och $y' = -\frac{2x}{2} = -x$

$x^2 = 2 \quad x = \pm\sqrt{2}$

$y = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

$y' = -x + \frac{4x^3}{24} = 0$

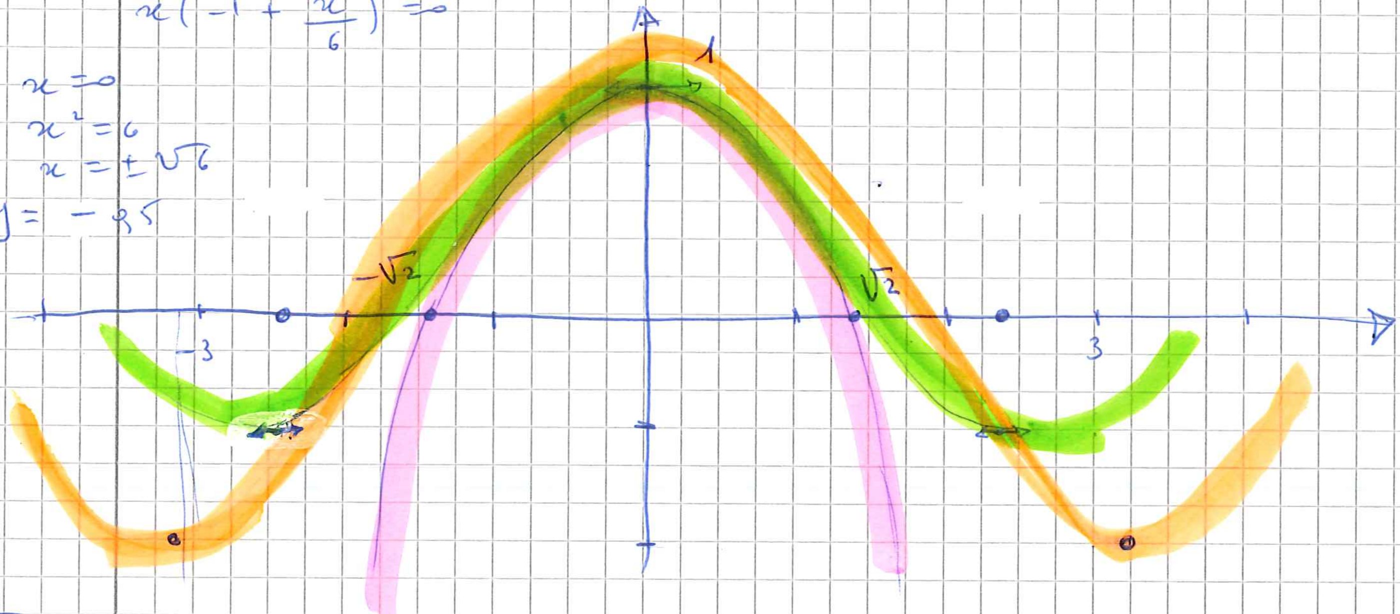
$x\left(-1 + \frac{x^2}{6}\right) = 0$

$x = 0$

$x^2 = 6$

$x = \pm\sqrt{6}$

$y = -0{,}5$



8.7

$e^{0,5} = 1{,}6487212\,71$

$e^{0,5} = 1 + 0{,}5 + \frac{0{,}5^2}{2} + \frac{0{,}5^3}{6} + \frac{0{,}5^4}{24}$

$= 1{,}6484375$

$R_4 = 2{,}8377 \cdot 10^{-4}$ (skillnaden)

$e^{0,5} = 1 + 0{,}5 + \frac{0{,}5^2}{2} + \frac{0{,}5^3}{6} + \frac{0{,}5^4}{24} + \frac{0{,}5^5}{120}$

$= 1{,}6486\,97917$

$R_5 = 2{,}3 \cdot 10^{-5}$ $\underline{\underline{\text{Ja}}}$ $\underline{\underline{n=5}}$

8.9

$$\arctan 2x = 2x - \frac{8x^3}{3}$$

$$\sin 2x = 2x - \frac{8x^3}{6}$$

$$\arctan 2x - \sin 2x = 2x - \frac{8x^3 .2}{3.2} - 2x + \frac{8x^3}{6}$$

$$= \frac{-8x^3}{6} = \frac{-4x^3}{3}$$

$$\lim_{x\to 0} \frac{\arctan 2x - \sin 2x}{x^3} = \frac{-\frac{4}{3} x^3}{x^3}$$

$$= -4/3$$

8.10

$$f(x) = \ln(1+x)^{\cos x} = \cos x . \ln(1+x)$$

$$\cos x = 1 - \frac{x^2}{2}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$f(x) = \cos x . \ln(1+x)$$

$$= \left(1 - \frac{x^2}{2}\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{2} + \cancel{\frac{x^4}{4}} - \cancel{\frac{x^5}{6}}$$

$$f(x) = \ln(1+x)^{\cos x}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{2}$$

$$= x - \frac{x^2}{2} + \frac{2x^3}{6} - \frac{3x^3}{6}$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6}$$

8.1

$f(x) = \sin\left(x^2 - \frac{\pi}{6}\right)$ $\quad f(0) = -\frac{1}{2}$

$f'(x) = 2x \cdot \cos\left(x^2 - \frac{\pi}{6}\right)$ $\quad f'(0) = 0$

$f''(x) = 2\cos\left(x^2 - \frac{\pi}{6}\right) - 2x \cdot 2x \sin\left(x^2 - \frac{\pi}{6}\right)$ $\quad f''(0) = \frac{\cancel{2}\cdot\sqrt{3}}{\cancel{2}}$

$2\cos\left(x^2 - \frac{\pi}{6}\right) - 4x^2 \sin\left(x^2 - \frac{\pi}{6}\right)$

$f'''(x) = -4x \sin\left(x^2 - \frac{\pi}{6}\right) - 8x \sin\left(x^2 - \frac{\pi}{6}\right) - 8x^3 \cos\left(x^2 - \frac{\pi}{6}\right)$

$f'''(0) = 0 - 0 - 0$

$f'''(0) = 0$

$f^{(4)}(x) = -4\sin\left(x^2 - \frac{\pi}{6}\right) - 8x^2\cos\left(x^2 - \frac{\pi}{6}\right) - 8\sin\left(x^2 - \frac{\pi}{6}\right) - 16x^2\cos\left(x^2 - \frac{\pi}{6}\right) - 24x^2\cos\left(x^2 - \frac{\pi}{6}\right) + 16x^4\sin\left(x^2 - \frac{\pi}{6}\right)$

$f^{(4)}(0) = -4 \cdot \sin\left(-\frac{\pi}{6}\right) - 0 - 8 \cdot \sin\left(-\frac{\pi}{6}\right) + 0 - 0 + 0$

$= -4 \cdot \left(-\frac{1}{2}\right) - 8 \cdot \left(-\frac{1}{2}\right)$

$= 2 + 4 = 6 = f^{(4)}(0)$

$f(x) = f(0) + \cancel{f'(0)x} + f''(0)\frac{x^2}{2} + \cancel{f'''(0)\frac{x^3}{6}} + \frac{f^{(4)}(0)x^4}{24}$

$\sin\left(x^2 - \frac{\pi}{6}\right) = -\frac{1}{2} + \sqrt{3}\,\frac{x^2}{2} + \frac{\cancel{6}\,x^4}{\cancel{24}}$

8.12

$$F(t) = e^{-\frac{\Delta h}{p}\frac{1}{T}}$$

$$F'(t) = -\frac{\Delta h}{p}\left(-\frac{1}{T^2}\right) e^{-\frac{\Delta h}{p}\frac{1}{T}}$$

$$F'(t) = \frac{\Delta h}{p} \cdot \frac{1}{T^2}\, e^{-\frac{\Delta h}{p}\frac{1}{T}}$$

$$= \frac{\Delta h}{p} T^{-2} e^{-\frac{\Delta h}{p}\frac{1}{T}}$$

$$F''(t) = -2\frac{\Delta h}{p}\frac{1}{T^3} e^{-\frac{\Delta h}{p}\frac{1}{T}}$$

$$+ \frac{\Delta h}{p}\frac{1}{T^2}\left(-\frac{\Delta h}{p}\right)\left(-\frac{1}{T^2}\right) e^{-\frac{\Delta h}{p}\frac{1}{T}}$$

$$F''(t) = -\frac{2\Delta h}{pT^3} e^{-\frac{\Delta h}{pT}} + \left(\frac{\Delta h}{pT^2}\right)^2 e^{-\frac{\Delta h}{pT}}$$

i punkten $T = 10$

$$\Rightarrow \quad F(10) = e^{-\frac{\Delta h}{10p}}$$

$$F'(10) = \frac{\Delta h}{100p} e^{-\frac{\Delta h}{10p}}$$

$$F''(10) = \frac{-2\Delta h}{1000p} e^{-\frac{\Delta h}{10p}} + \left(\frac{\Delta h}{100p}\right)^2 e^{-\frac{\Delta h}{10p}}$$

Nu kör vi med Taylor utveckling:

$$F(T) = F(10) + F'(10)(T-10) + \frac{F''(10)}{2}(T-10)^2$$

ersätt $F(10)$, $F'(10)$ och $F''(10)$

med det som vi räknade i sidan innan.

$$F(T) = e^{-\frac{\Delta h}{10p}} + \frac{\Delta h}{100p} e^{-\frac{\Delta h}{10p}} (T-10)$$

$$+ \frac{(T-10)^2}{2} \left[\frac{-2\Delta h}{1000p} e^{-\frac{\Delta h}{10p}} + \left(\frac{\Delta h}{100p}\right)^2 e^{-\frac{\Delta h}{10h}} \right]$$

$\Longrightarrow$ dvs:

$$F(t) = e^{-\frac{\Delta h}{10p}} \left[1 + \frac{\Delta h}{100p}(T-10) + \frac{(T-10)^2}{2} \left(\frac{-2\Delta h}{1000p} + \left(\frac{\Delta h}{100p}\right)^2 \right) \right]$$

$\blacksquare$

8.13 $N(t) = \frac{1500(1+t)}{15+t^2}$

Vi utgår från formeln

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

derivera båda leden

$$\frac{1}{1+x} = 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{4x^3}{4}$$

$$= 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5$$

är en känd formel

$$N(t) = (1+t) \cdot \frac{1500 \;/15}{15+t^2 \;/15}$$

$$N(t) = (1+t) \cdot \frac{100}{1+\frac{t^2}{15}}$$

Vi börjar med:

$$\frac{1}{1+\frac{t^2}{15}} = 1 - \frac{t^2}{15} + \frac{t^4}{15^2} + \cdots$$

multiplicera med 100

$$\frac{100}{1+\frac{t^2}{15}} = 100\left(1 - \frac{t^2}{15} + \cancel{\frac{t^4}{15^2}}\right)$$

och till sist multiplicera med $(1+t)$

$$(1+t)\cdot\frac{100}{1+\frac{t^2}{15}} = (1+t)\cdot 100\left(1-\frac{t^2}{15}+\dots\right)$$

$$\underline{N(t)} = (1+t)\cdot\left(1-\frac{t^2}{15}\right)\cdot 100$$

$$N(t) = \left(1-\frac{t^2}{15}+t-\cancel{\frac{t^3}{15}}\right)100$$

$$N(t) = \left(1+t-\frac{t^2}{15}\right)\cdot 100$$

bara de tre första termerna.

$$N(t) = \left(1+t-\frac{t^2}{15}\right)$$

Skissa (miniräknare eller Geogebra)