Bestämda integraler.

Def $\int f(x)dx = F(x) + C$

där $F(x)$ är en primitiv funktion till $f(x)$ och $C \in \mathbb{R}$ godt.

Ex $\int x\,dx = \frac{x^2}{2} + C$

Denna kallas för obestämd integral

Def $\int_a^b f(x)dx = F(b) - F(a)$

Kallas för bestämd integral.

Ex $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \left[\frac{4}{2}\right] - \left[\frac{1}{2}\right] = 3/2$

Anm talet C behövs inte i bestämda integralen

$\int_1^2 x\,dx = \left[\frac{x^2}{2} + C\right]_1^2 = \left[\frac{4}{2} + \cancel{C}\right] - \left[\frac{1}{2} + \cancel{C}\right]$

$\frac{4}{2} - 1/2 = 3/2$

$$\int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2}$$

$$= \left[-\cos \tfrac{\pi}{2}\right] - \left[-\cos 0\right] = -0 + 1 = 1$$

Viktig

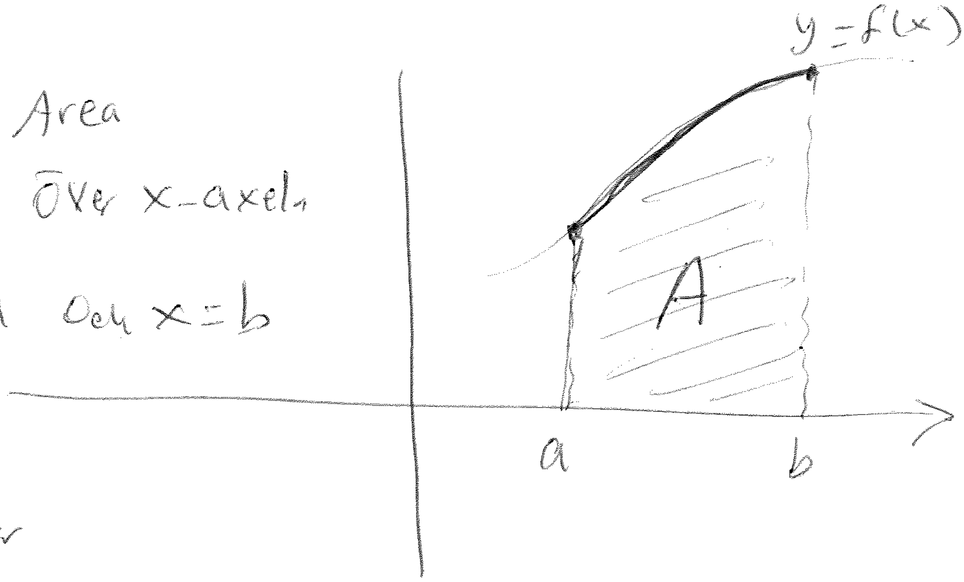
Sambandet mellan Area och Bestämd integral.

Låt $f(x)$ vara en funktion som är

Kontinuerlig mellan $x = a$ och $x = b$

och ligger ovanför x-axeln

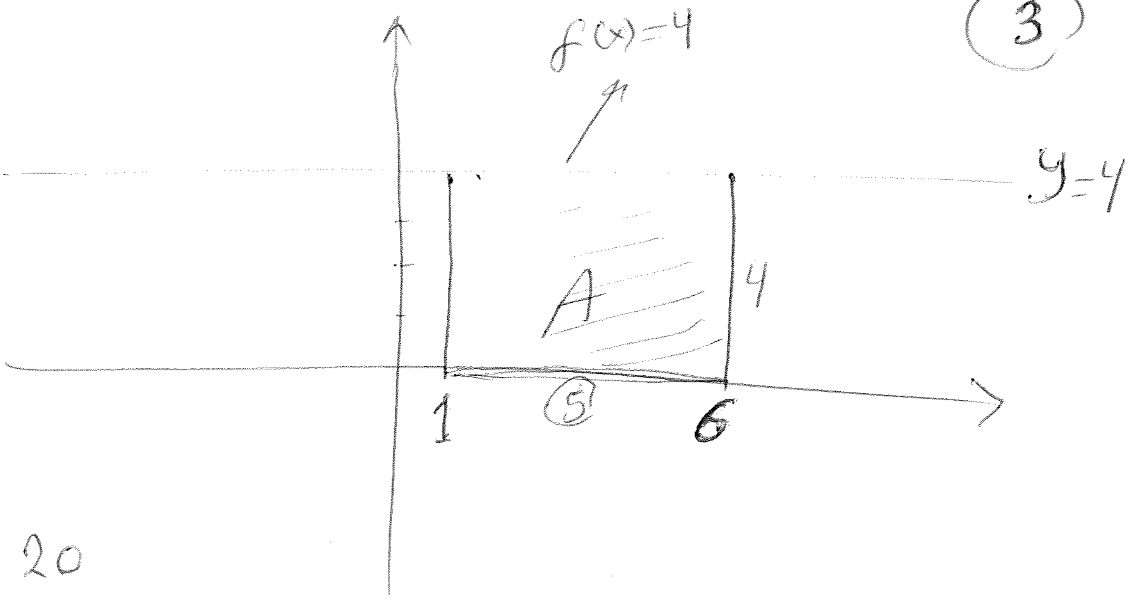
Det finns en Area under $f(x)$, över x-axeln mellan $x = a$ och $x = b$



Då gäller

$$A = \text{Arean} = \int_a^b f(x)\,dx$$

Ex

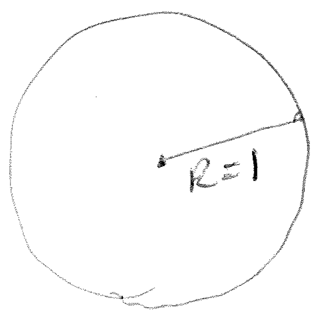


Area = 20

$$\int_1^6 4\,dx = \left[4x\right]_1^6 = 24 - 4 = 20$$

Beräkning av Cirkel area med integral

$A = \pi R^2 = \pi$

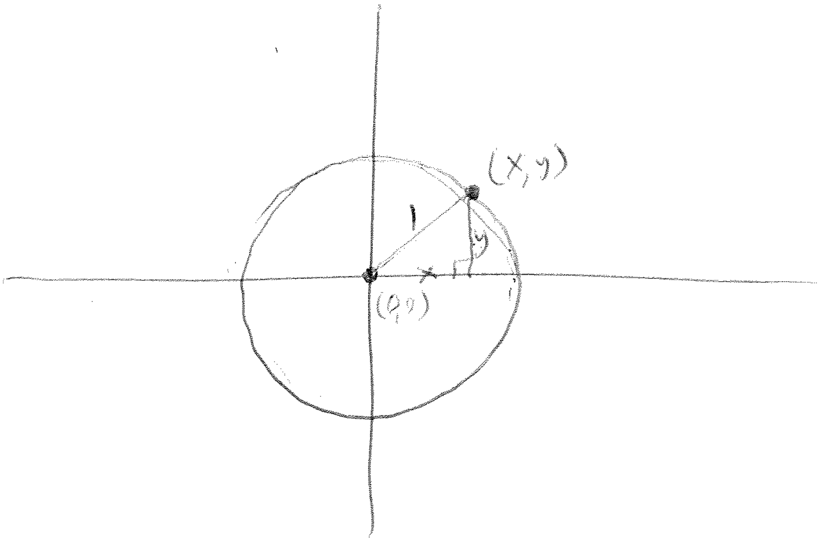


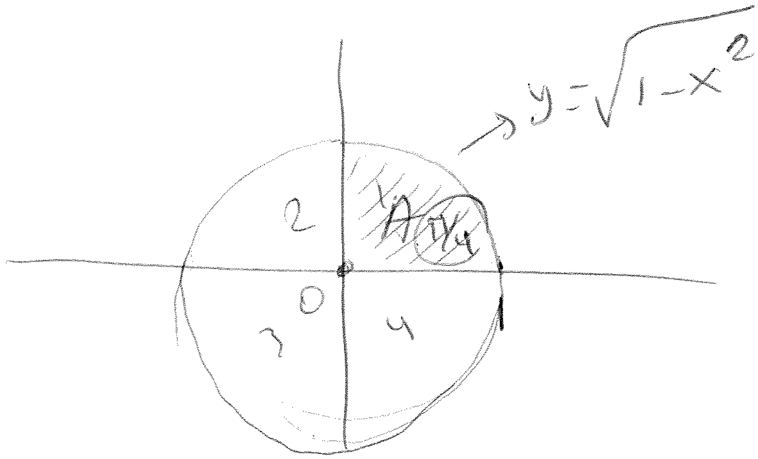
$x^2 + y^2 = 1$

$\Downarrow$

$y^2 = 1 - x^2$

$y = \sqrt{1 - x^2}$





$$A = \int_0^1 \sqrt{1 - x^2}\, dx$$

$$\left(\begin{array}{l} x = \sin t \\ dx = \cos t \, dt \\ x = 0 \Rightarrow \sin t = 0 \rightarrow t = 0 \\ x = 1 \Rightarrow \sin t = 1 \rightarrow t = \pi/2 \end{array} \right)$$

$$A = \int_0^{\pi/2} \sqrt{1 - \sin^2 t} \cdot \cos t \, dt$$

$$A = \int_0^{\pi/2} \sqrt{\cos^2 t} \cdot \cos t \, dt$$

$$= \int_0^{\pi/2} \cos t \cdot \cos t \, dt$$

$$= \int_0^{\pi/2} \cos^2 t \, dt = \int_0^{\pi/2} \frac{1 + \cos 2t}{2} \, dt$$

$$= \int_0^{\pi/2} \left(\tfrac{1}{2} + \tfrac{1}{2} \cos 2t \right) dt$$

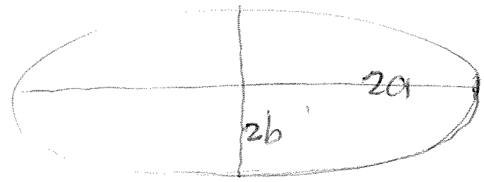
$$\left[\frac{1}{2} t + \frac{1}{2} \cdot \frac{1}{2} \sin 2t \right]_0^{\pi/2}$$

$$\left[\frac{1}{2} t + \frac{1}{4} \sin 2t \right]_0^{\pi/2}$$

$$\left[\pi/4 + \tfrac{1}{4} \cancel{\sin \pi} \right] - \left[0 + \tfrac{1}{4} \cancel{\sin 0} \right] = \pi/4$$

Cirkelns Area $= 4 \times \pi/4 = \pi$

Arean av en ellips



Arean $= \pi \cdot a \cdot b$

ellipsens ekvation

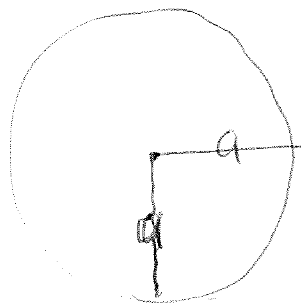
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Downarrow$$

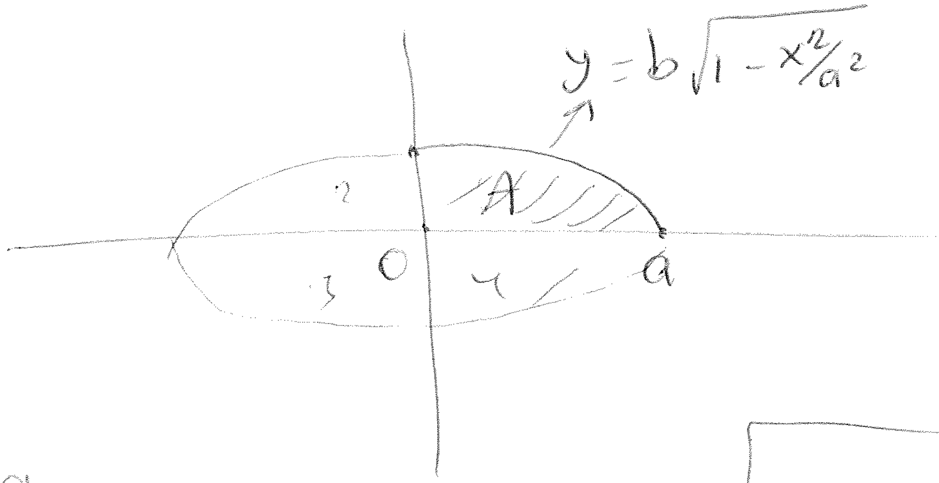
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\boxed{y = b\sqrt{1 - \frac{x^2}{a^2}}}$$



$\pi \cdot a \cdot a = \pi a^2$



$$A = \int_0^a b\sqrt{1 - x^2/a^2}\, dx$$

$$1 - \frac{x^2}{a^2} = 1 - \frac{\cancel{a^2}\sin^2 t}{\cancel{a^2}}$$

$$1 - \sin^2 t$$

$$x = a\sin t$$

$$dx = a\cos t\, dt$$

$$x = 0 \rightarrow a\sin t = 0 \rightarrow t = 0$$

$$x = a \rightarrow \cancel{a}\sin t = \cancel{a} \rightarrow t = \pi/2$$

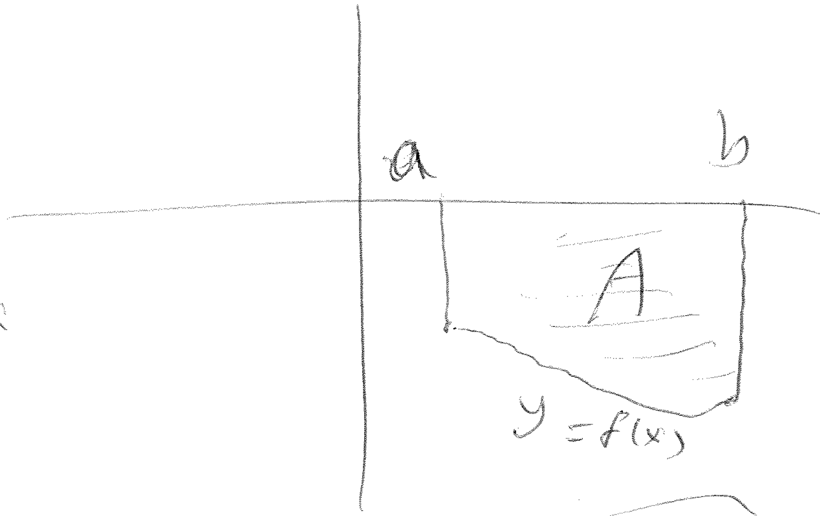
$$A = \int_0^{\pi/2} b\sqrt{1 - \sin^2 t} \cdot a\cos t\, dt$$

$$= ab \int_0^{\pi/2} \cos^2 t\, dt = ab \cdot \pi/4$$

$$\text{ellips} = \cancel{4} \times ab \times \pi/\cancel{4} = a.b.\pi$$

Anm

Om $f(x)$ ligger under x-axeln



$$A = -\int_a^b f(x)\,dx$$

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx$$

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\,dx \qquad \int \sin x \cdot \left(\frac{1}{1+\cos x}\right) dx$$

$$\left(\begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \\ x=0 \rightarrow t=\cos 0 = 1 \\ x=\pi/2 \rightarrow t = \cos \pi/2 = 0 \end{array}\right)$$

(9)

$$= \int_1^0 \frac{\cancel{\sin x}}{1+t} \cdot \frac{dt}{\cancel{-\sin x}}$$

$$= -\int_1^0 \frac{1}{1+t}\, dt$$

$$\boxed{\int_b^a = -\int_a^b}$$

$$\int_0^1 \frac{1}{1+t}\, dt = \Big[\ln|1+t|\Big]_0^1$$

$$= \ln 2 - \cancel{\ln 1} = \ln 2.$$

AnM

Det finns ett annat alternativ för gräns.

10

$$\int_0^1 x\sqrt{1-x^2}\,dx$$

derivatan framför

$\boxed{\text{lägg undan } \int_0^1}$

$$\left(\begin{array}{c} 1-x^2=t \\ -2x\,dx=dt \end{array}\right)$$

$$\int x\sqrt{1-x^2}\,dx$$

$$=\int \cancel{x}\sqrt{t}\cdot\frac{dt}{-2\cancel{x}} = -\tfrac{1}{2}\int\sqrt{t}\,dt$$

$$=-\tfrac{1}{2}\int t^{1/2}\,dt = -\tfrac{1}{2}\,\frac{1}{1/2+1}\,t^{1/2+1}$$

$$= -\tfrac{1}{3}\,t^{3/2} = \left[-\tfrac{1}{3}(1-x^2)^{3/2}\right]_0^1$$

$$\left(-\tfrac{1}{3}\cancel{(0)^{3/2}}\right)-\left[-\tfrac{1}{3}(1-0)^{3/2}\right] = 1/3$$

Def udda och jämna funktion

$f(x)$ kallas för udda om

$$f(-x) = -f(x).$$

Ex $f(x) = x^3$

$$f(-x) = (-x)^3 = (-1)^3 x^3 = -x^3$$

$$\boxed{f(-x) = -f(x)}$$

$f(x) = x$

x^5

x^7

x^9

$\vdots$

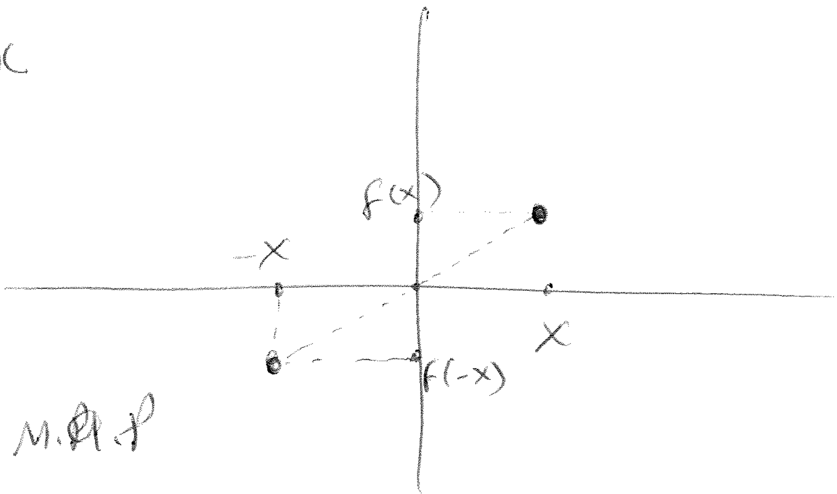
$f(x) = \sin x$

$f(-x) = \sin(-x)$

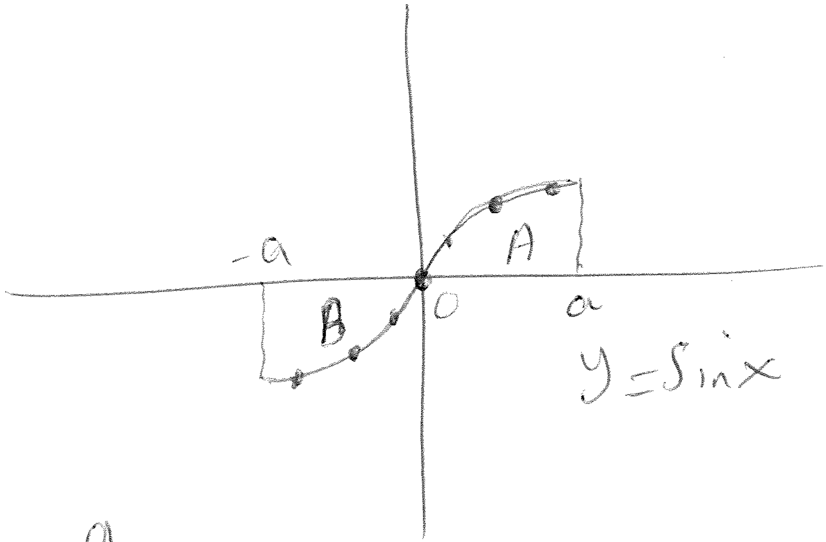
$= -\sin x$

$= -f(x)$

Geometrisk



Symmetrisk m.a.p.
origo

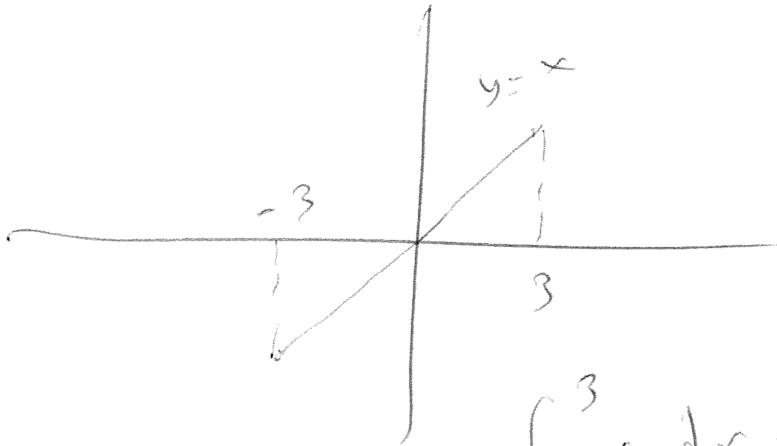


viktig $\int_{-a}^{a} \text{udda} = 0$

$$A = \int_{0}^{a} f(x)\,dx$$

$$B = -\int_{-a}^{0} f(x)\,dx$$

$$\int_{-a}^{a} = A + B = 0$$



$$\int_{-3}^{3} x\,dx = 0$$

$$\int_{-3}^{3} x\,dx = \left[\frac{x^2}{2}\right]_{-3}^{3}$$

$$\left[\frac{9}{2}\right] - \left[\frac{(-3)^2}{2}\right] = \frac{9}{2} - 9/2 = 0$$

Ex

$$\int_{-5}^{5} \frac{\sin x}{x^2 + x^{100} + 5}\,dx = 0$$

$$f(x) = \frac{\sin x}{x^{100} + x^2 + 5}$$

$$f(-x) = \frac{\sin(-x)}{(-x)^{100} + (-x)^2 + 5} = \frac{-\sin x}{x^{100} + x^2 + 5}$$

$$= -f(x)$$

$$\boxed{f(-x) = -f(x)}$$

Def Jämn funktion

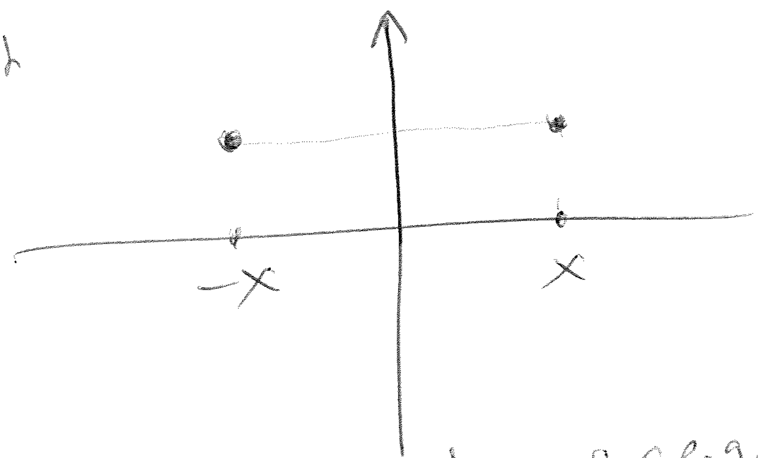
$f(x)$ kallas jämn om

$$f(-x) = f(x)$$

$$f(x) = x^2$$
$$x^4$$
$$x^6$$
$$x^8$$
$$\vdots$$

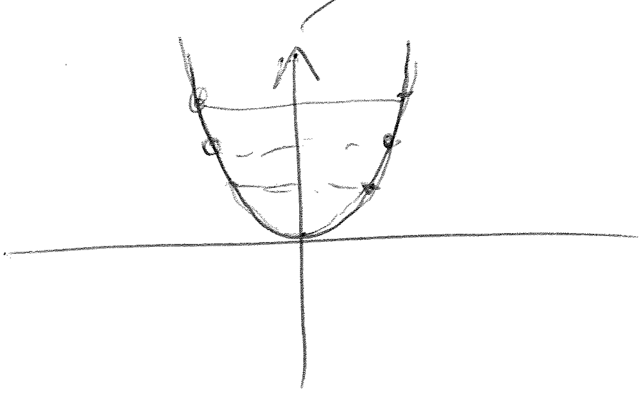
$$f(x) = x^2 \qquad f(-x) = (-x)^2 = x^2$$

Geomet

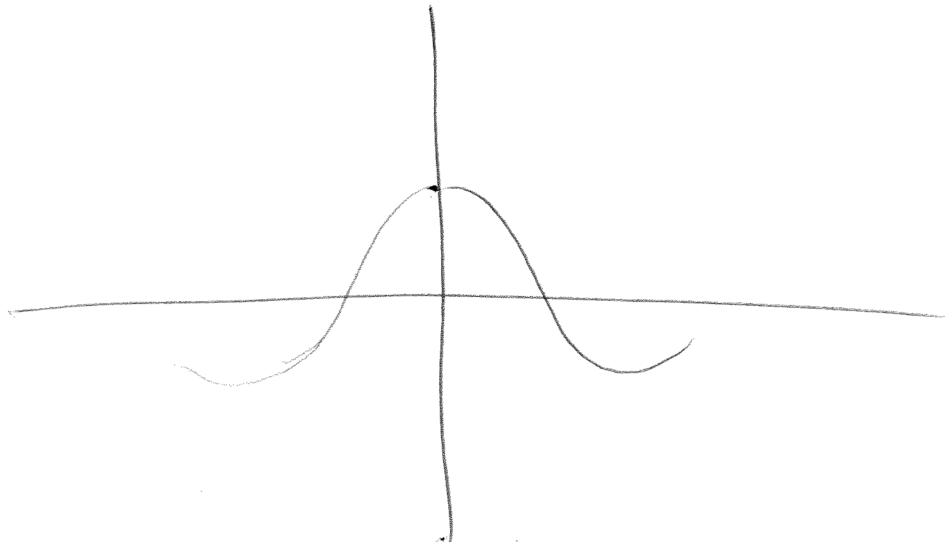


Spegeln fe spegeln

$y = x^2$

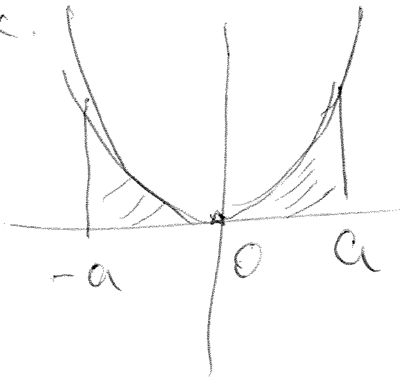


$f(x) = \cos x$ är jämn



Spegel.

$f(-x) = \cos(-x) = \cos x$.



$$\int_{-a}^{a} f(x)\,dx =$$

$$2\int_{0}^{a} f(x)\,dx.$$

$$\int_{-3}^{3} x^2\,dx = 2\int_{0}^{3} x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^3$$

$$= 2 \cdot 9 = 18$$