Kapitel Sju

7.1

a) $y = x^3 - 6x^2 + 9x - 4$

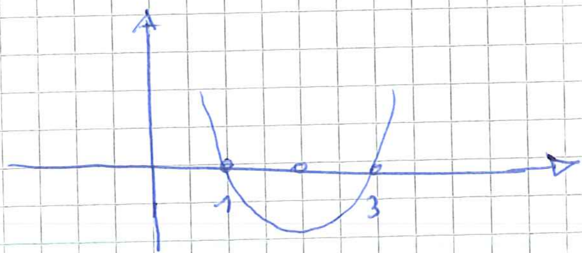
$y' = 3x^2 - 12x + 9$

$y' = 0$ om $x^2 - 4x + 3 = 0$

$1 - 4 + 3 = 0$ (Koefficienterna)

dvs $x = 1$

$x = 3$ ($\frac{3}{1}$ eller $\frac{c}{a}$)



$x = 1$ $y' = 0$

$x = 3$ $y' = 0$

och koefficienten bredvid x^2 är positiv

dvs $\cup$

$y' = 0$ när $x = 1$

$y' = 0$ när $x = 3$

$y' < 0$ när $1 < x < 3$

$y' > 0$ när $x > 1$ och $x > 3$

x		1		3	
y'	+	0	−	0	+
y	$\nearrow$	max	$\searrow$	min	$\nearrow$

ett annat sätt:

$y' = 3x^2 - 12x + 9$ med $x_1 = 1$

$x_2 = \frac{9}{3} = 3$

$y' = 3(x^2 - 4x + 3)$

$y' = 3(x-1)(x-3)$

x		1		3	
$(x-1)$	$-$	0	$+$	$+$	$+$
$(x-3)$	$-$	$-$	$-$	0	$+$
y'	$+$	0	$-$	0	$+$
y	$\nearrow$	max	$\searrow$	min	$\nearrow$

(produkte av $(x-1)$ $(x-3)$)

b) $y = (x^2+1)^3$

$y' = 3 \cdot 2x\,(x^2+1)^2 = 6x\,(x^2+1)^2$

$y' = 0$ om $x = 0$ bara $(x^2+1) > 0$

x		0	
$6x$	$-$	0	$+$
$(x^2+1)^2$	$+$	$+$	$+$
y'	$-$	0	$+$
y	$\searrow$	min	$\nearrow$

(produkten

Växande i rosa

Avtagande i grön

c) $y = \dfrac{x+1}{x} = \dfrac{x}{x} + \dfrac{1}{x} = 1 + \dfrac{1}{x}$

$x \neq 0$

$y' = 0 - \dfrac{1}{x^2}$ $y' = 0$ när ...

$\dfrac{1}{x^2} = 0$ Aldrig

y' är alltid negativ

y är alltid avtagande $\forall x \neq 0$

inga stationära punkter

d) $y = \dfrac{2}{x} + x^2$ $x \neq 0$

$y' = -\dfrac{2}{x^2} + 2x$

$y' = 0$ om $-\dfrac{2}{x^2} + 2x = 0$

$-2 + 2x^3 = 0$

$x^3 = 1$ $x = 1$

x	-1	0		1	2
y'	$-$			0	$+$
y	↘		↘	min →	↗

e) $y = 2\ln x - x^2$ $x > 0$

$y' = \dfrac{2}{x} - 2x$ $x \neq 0$

$y' = 0$ om $\dfrac{2}{x} - 2x = 0$

$2 - 2x^2 = 0$ $2 = 2x^2$

$x = \pm 1$

Men $\ln x$ är definierad för $x > 0$ dvs $x = -1$ Nej!

$x = 1$

x	-1	0	$\frac{1}{2}$	1	2
y'			$+$	0	$-$
y			$\nearrow$	max	$\searrow$

f) $y = \dfrac{x^2}{x^2+1}$

$$y' = \frac{2x(x^2+1) - 2x \cdot x^2}{(x^2+1)^2} = \frac{\cancel{2x^3} + 2x \cancel{- 2x^3}}{(x^2+1)^2}$$

$y' = 0$ om $2x = 0$ (nämnaren är alltid > 0)

$x = 0$

x		0	
y'	$-$	0	$+$
y	$\searrow$	$\leftrightarrow$ min	$\nearrow$

g) $y = e^x(x^2 - 2x + 2)$

$$y' = e^x(x^2 - 2x + 2) + e^x(2x - 2)$$

$$= e^x(x^2 \cancel{- 2x} \cancel{+ 2} \cancel{+ 2x} \cancel{- 2})$$

$$= e^x \cdot x^2 \qquad x^2 \geq 0 \quad e^x > 0$$

y' är alltid positiv

y är alltid växande

$\forall x \in \mathbb{R}$

$x = 0$ är en terasspunkt

x		0	
y'	$+$	0	$+$
y	$\nearrow$	$\leftrightarrow$	$\nearrow$

h) $y = e^x(x^2 - 2x + 3)$

$y' = e^x(x^2 - 2x + 3) + e^x(2x - 2)$

$= e^x(x^2 - \cancel{2x} + 3 \cancel{+ 2x} - 2)$

$= e^x(x^2 + 1)$

$e^x > 0 \quad \forall x \in \mathbb{R}$

$x^2 + 1 > 0 \quad \forall x \in \mathbb{R}$

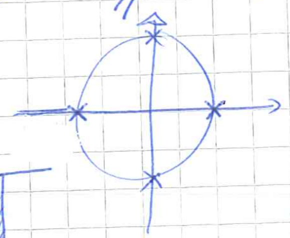
$y' > 0 \quad \forall x \in \mathbb{R}$

y är växande $\forall x \in \mathbb{R}$

saknar stationära punkter

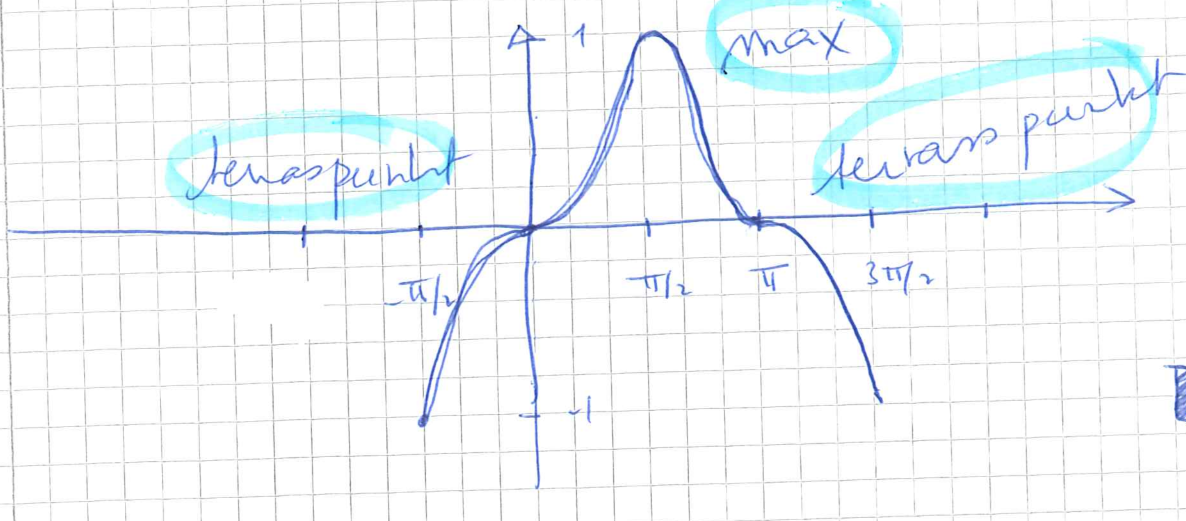
i) $y = \sin^3 x \qquad x \in \left]-\pi/2 \,;\, 3\pi/2\right[= I$

$y' = 3\cos x \sin^2 x \qquad \sin^2 x \geq 0 \quad \forall x \in I$



x	$-\pi/2$		0		$\pi/2$		π		$3\pi/2$
$\sin^2 x$	‖	+	0	+	+	+	0	+	‖
$\cos x$	‖	+	+	+	0	−	−	−	‖
y'	‖	+	0	+	0	−	0	−	‖
y	‖	↗	↔	↗	↔	↘	↔	↘	‖

kurvan kommer att se ut så här:



7.2 Jag råkade göra detta i upp **7.1** $\Rightarrow$ kolla där :-)

7.3

$x \neq 3$

$$y = \frac{x^2}{3-x}$$

$$y' = \frac{2x(3-x) - (-1)(x^2)}{(3-x)^2}$$

$$= \frac{6x - 2x^2 + x^2}{(3-x)^2} = \frac{6x - x^2}{(3-x)^2}$$

$$= \frac{x(6-x)}{(3-x)^2}$$

$y' = 0$ om $x = 0$, $x = 6$

a) $y'(0) = 0$ $\quad x = 0 \quad y = \frac{0}{3-0} = 0$

$y - 0 = 0(x - 0) \quad y = 0$

b) $y'(2) = 8 \quad x = 2 \quad y = \frac{4}{3-4} = -4$

$y + 4 = 8(x - 2) \quad y = 8x - 20$

c) $y'(4) = \frac{4(6-4)}{(3-4)^2} = 8 \quad x = 4 \quad y = \frac{16}{-1} = -16$

$y + 16 = 8(x - 4) \quad y = 8x - 48$

x		0		3		6	
x	$-$	0	$+$	‖	$+$	$+$	$+$
$6-x$	$+$	$+$	$+$	‖	$+$	0	$-$
y'	$-$	0	$+$	‖	$+$	0	$-$
y	$\searrow$	min	$\nearrow$	‖	$\nearrow$	max	$\searrow$

7.4 a) $y = 2x^4 + x - 2$

$y' = 8x^3 + 1$

$y' = 0$ om $8x^3 = -1$ $\quad x^3 = -\frac{1}{8}$

$x = -1/2$

x	-1	$-1/2$		1
y'		0	$+$	$+$
y	$\searrow$	$\rightarrow$ min	$\nearrow$	

$x = -1/2 \quad y = 2\left(-\frac{1}{2}\right)^4 + \left(-\frac{1}{2}\right) - 2$

$= 2 \cdot \frac{1}{16} + -\frac{1}{2} - 2$

$= \frac{1}{8} - \frac{4}{8} - \frac{16}{8}$

$y = -19/8$

$(-1/2 ; -19/8)$ är en minimipunkt

och $y = -19/8$ är minsta värdet ■

b) $y = 1 - x - 2x^2$ $\quad y' = 0$ om $x = -\frac{1}{4}$

$y' = -1 - 4x$

$x = -\frac{1}{4}$ är en maximipunkt

$y = 1 + \frac{1}{4} - 2 \cdot \frac{1}{16} = \frac{8}{8} + \frac{2}{8} - \frac{1}{8} = \frac{9}{8}$ är största värdet

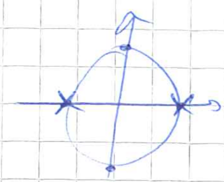
x	-2	$-1/4$	2
y'	$+$	0	$-$
y	$\nearrow$	max	$\searrow$

■

7.5

a) $y = \cos 2x$

$y' = -2\sin 2x$ $\qquad$ $y' = 0$ om



$\sin 2x = 0$

$2x = 0 + 2\pi n$

$2x = \pi + 2\pi n$

dvs $\quad x = \pi n$

$x = \frac{\pi}{2} + \pi n$

eller

$y = \cos 2x$

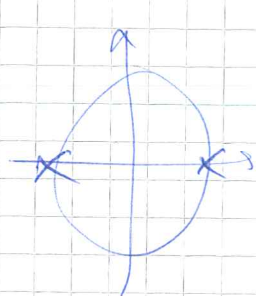
$-1 \leq \cos 2x \leq 1$

$\cos 2x = 1$ $\qquad$ (Max)

$2x = 0 + 2\pi n$

$x = \pi n$

eller $\quad \cos 2x = -1$ $\qquad$ (Min)



$2x = \pi + 2\pi n$

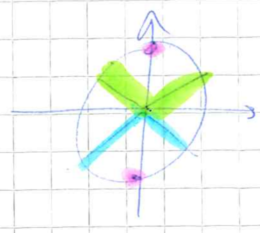
$x = \frac{\pi}{2} + \pi n$

c) $y = \frac{e^x}{x}$ $\qquad$ $y' = \frac{e^x \cdot x - e^x}{x^2}$

$y' = \frac{e^x(x-1)}{x^2}$ $\qquad$ $y' = 0$ om $\quad x = 1$ $\qquad$ $x \neq 0$

x		0		1	
$\frac{e^x}{x^2}$	+		+	+	+
$x-1$	−		−	0	+
y'	−		−	0	
y	$\searrow$		$\searrow$	min	$\nearrow$

7.6 $y = 4\sin^3 x - 3\sin x \qquad x \in [0; 2\pi]$



$$y' = 4\cdot 3\cdot \cos x \cdot \sin^2 x - 3\cos x$$
$$= 3\cos x\,(4\sin^2 x - 1)$$

$y' = 0$ om $\cos x = 0$ eller $\sin^2 x = \frac{1}{4}$

$x_5 = \pi/2$ $\qquad$ $\sin x = \pm 1/2$

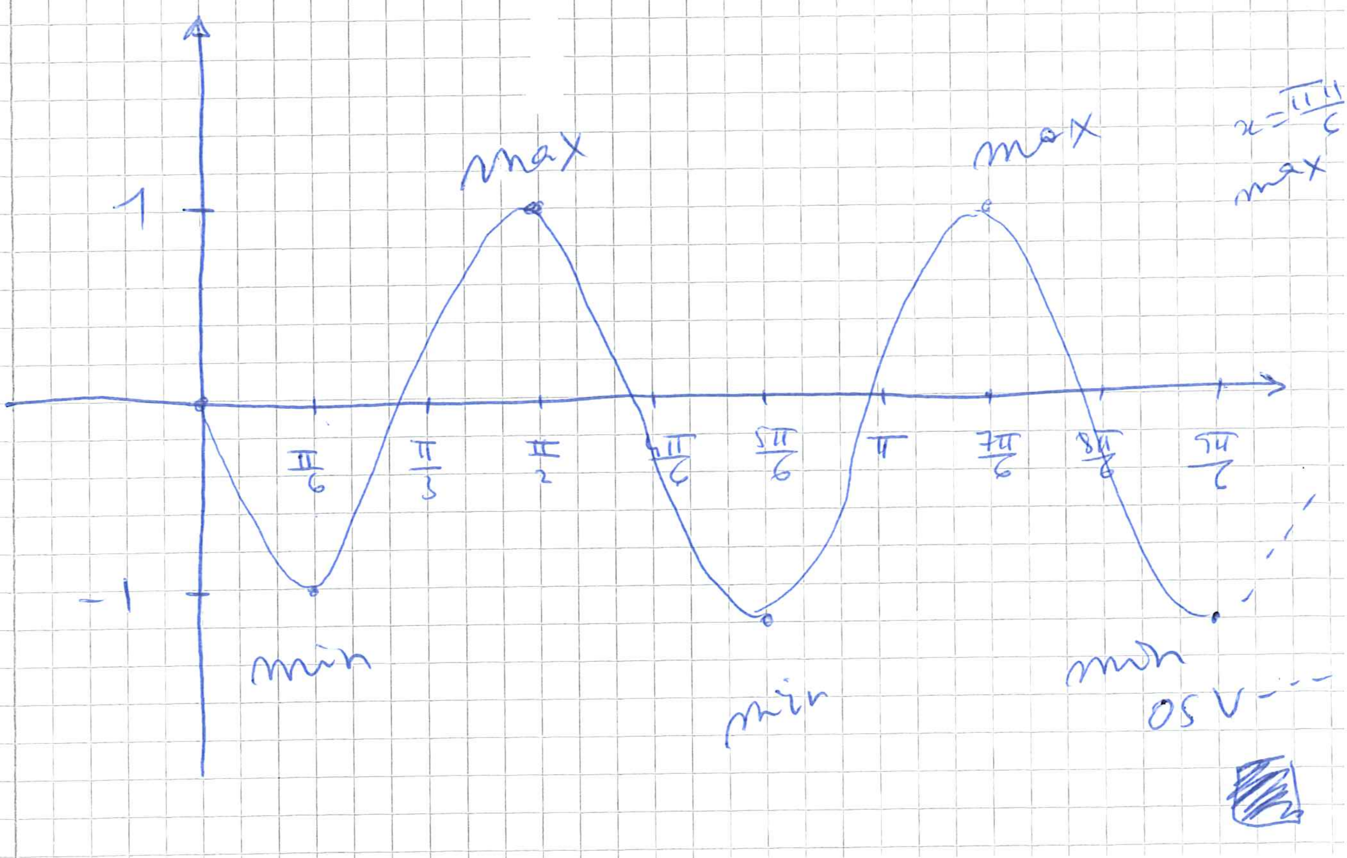
$x_6 = 3\pi/2$

$x_1 = \frac{\pi}{6} \qquad x_2 = \frac{5\pi}{6}$

$x_3 = \pi + \frac{\pi}{6} = 7\pi/6$

$x_4 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

x	0		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$		$\frac{3\pi}{2}$		$\frac{11\pi}{6}$		2π
$3\cos x$	+	+	+	+	0	−	−	−	−	−	0	+	+		+
$4\sin^2 x - 1$	−		0	+	+	+	0	−	0	+	+	+	0		−
y'	−		0	+	0	−	0	+	0	−	0	+	0		−
y	↘		↔ −1	↗	↔ 1	↘	↔	↗	↔	↘	↔ −1	↗	↔		↘



7.7

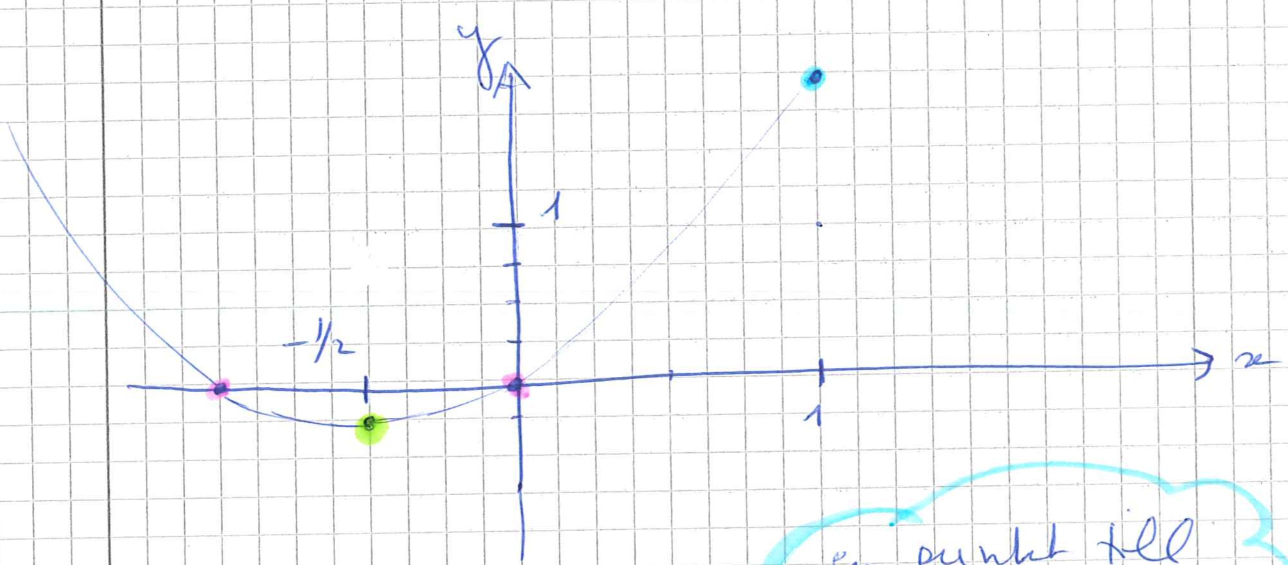
a) $y = x^2 + x = x(x+1)$

$y' = 2x+1$ $\quad y' = 0$ om $x = -1/2$

x		$-1/2$	
y'	$-$	0	$+$
y	$\searrow$	$\leftrightarrow$	$\nearrow$

$x = -1/2$ $\quad y = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4}$

$(-1/2 ; -1/4)$ är en minimipunkt



en punkt till
$x = 1$
$y = 2$ (1^2+1)

$x = 0$
$y = 0$ $(0 ; 0)$

$x = -1$ $\quad y = 0$

jag hittade nollställen dessutom genom $y = x(x+1)$

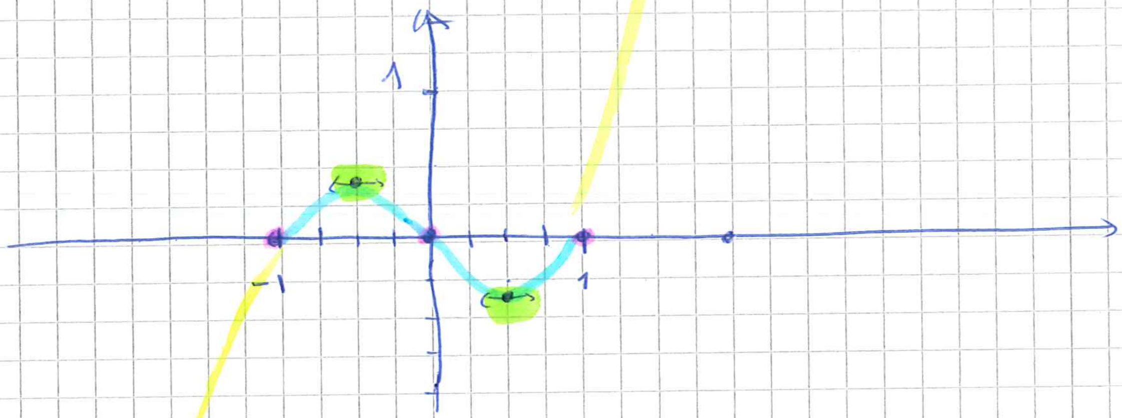
$y = 0$ om $x = 0$
$x = -1$

b) $y = x^3 - x = x(x^2-1)$

Nollställen är i $x=0$, $y=0$ $\quad x=1$, $y=0$ $\quad x=-1$, $y=0$

$y' = 3x^2 - 1 \qquad y' = 0$ när $x^2 = 1/3 \qquad x = \pm\frac{1}{\sqrt{3}}$

x	-1		$-1/\sqrt{3}$		0		$1/\sqrt{3}$		1
y'		$+$	0		$-$		0	$+$	
y		$\nearrow$	Max		$\searrow$		Min	$\nearrow$	



$x = \frac{1}{\sqrt{3}} \qquad y = \frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{1}{\sqrt{3}}\left(\frac{-2}{3}\right) = \frac{-2}{3\sqrt{3}}$

$x = -\frac{1}{\sqrt{3}} \qquad y = -\frac{1}{\sqrt{3}}\left(\frac{1}{3} - 1\right) = \frac{-1}{\sqrt{3}}\left(-\frac{2}{3}\right) = \frac{2}{3\sqrt{3}}$

Maximipunkt $\left(-\frac{1}{\sqrt{3}} ; \frac{2}{3\sqrt{3}}\right)$

Minimipunkt $\left(1/\sqrt{3} ; -2/3\sqrt{3}\right)$

Nollställen/punkter $(0;0)\ (1;0)\ (-1;0)$

Man kan välja mer punkter

t. ex $(2;6)$ och $(-2;-10)$

c) $y = \frac{1}{x+2}$ $\quad x \neq -2$

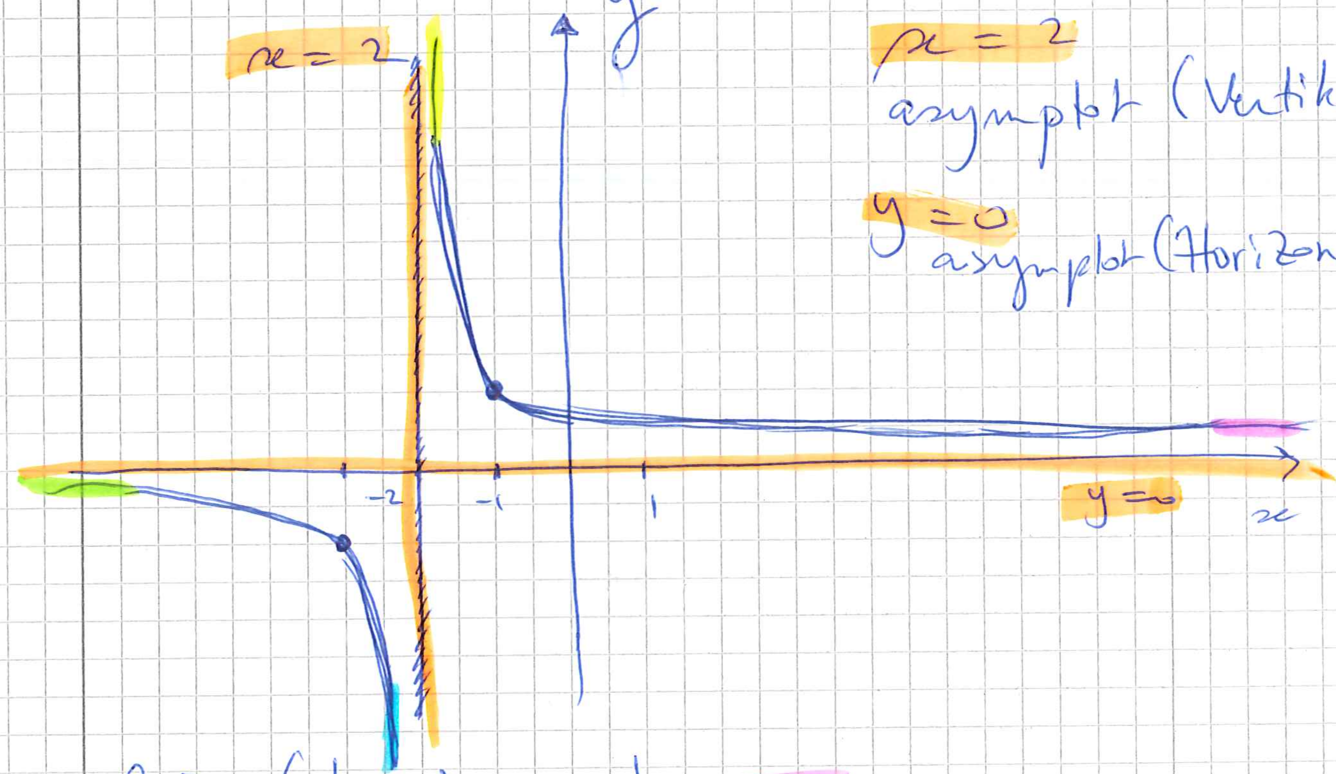
Inga nollställen !

$y' = \frac{-1}{(x+2)^2} < 0$ Alltid avtagande

Inga Max / Min

$\lim_{x \to -2^+} \frac{1}{x+2} \longrightarrow \frac{1}{0+} \longrightarrow +\infty$

$\lim_{x \to -2^-} \frac{1}{x+2} = \frac{1}{0-} \longrightarrow -\infty$



$x = 2$ asymptot (Vertikal)

$y = 0$ asymptot (Horizontell)

$\lim_{x \to +\infty} \left(\frac{1}{x+2}\right) \longrightarrow \frac{1}{\infty +} \Rightarrow 0^+$

$\lim_{x \to -\infty} \left(\frac{1}{x+2}\right) \longrightarrow \frac{1}{\infty -} \longrightarrow 0^-$

Extra! Dessutom $\quad x = -1 \quad y = 1 \qquad x = -3 \quad y = -1$

g) $y = \dfrac{2x-5}{4-x^2} = f(x)$

$4 - x^2 \neq 0$ dvs $x^2 \neq 4$

$x \neq \pm 2$

$x = 2$ vertikal asymptot

$x = -2$ vertikal asymptot

$2x - 5 = 0$ ger mig nollställer

$x = 5/2$ $y = 0$ $(5/2 ; 0)$ nollställe

hur beter sig kurvan i närheten av

$x = 2$, $x = -2$, $x = \pm\infty$

(*) $\lim_{x \to 2^+} f(x) = \dfrac{4-5}{0-} = \dfrac{-1}{0-} \longrightarrow +\infty$

$\lim_{x \to 2^-} f(x) = \dfrac{4-5}{0+} = \dfrac{-1}{0+} \longrightarrow -\infty$

(*) $\lim_{x \to -2^+} f(x) = \dfrac{-4-5}{0+} = \dfrac{-9}{0+} \longrightarrow -\infty$

$\lim_{x \to -2^-} f(x) = \dfrac{-4-5}{0-} = \dfrac{-9}{0-} \longrightarrow +\infty$

(*) $\lim_{x \to +\infty} f(x) = \dfrac{\cancel{x}\left(2 - \frac{5}{x}\right)}{x^{\cancel{2}}\left(\cancel{\frac{4}{x^2}} - 1\right)} \longrightarrow 0^-$

(*) $\lim_{x \to -\infty} f(x) = \dfrac{\cancel{x}\left(2 - \cancel{5/x}\right)}{x^2\left(\cancel{\frac{4}{x^2}} - 1\right)} = \dfrac{1}{x(-1)} \longrightarrow 0^+$

$y = 0$ Horizontell asymptot

$$y' = \frac{2(4-x^2)-(-2x)(2x-5)}{(4-x^2)^2}$$

Teckenstudie för y': $y'=0$ om

$$8-2x^2+4x^2-10x=0$$

$$2x^2-10x+8=0$$

$2-10+8=0$ dvs $x_1=1$

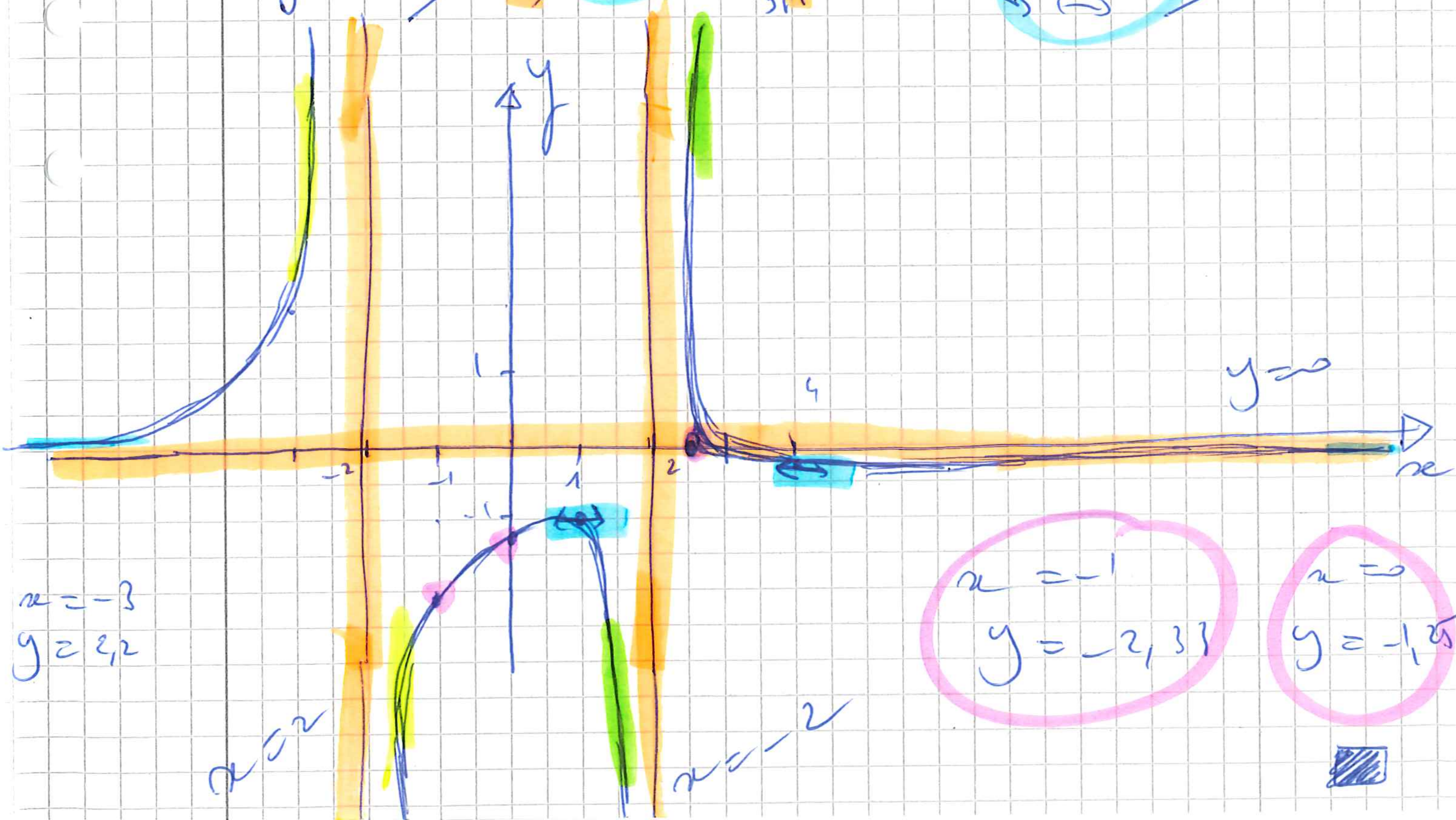
och $x_2 = \frac{c}{a} = \frac{8}{2} = 4$ $\quad x_2=4$

$$y' = 2(x^2-5x+4)$$
$$= 2(x-1)(x-4)$$

$x=1 \quad y=\frac{-3}{+3}$

$x=4 \quad y=-\frac{1}{4}$

x		-2		1		2		$5/2$	4	
$2(x-1)$	$-$		$-$	0	$+$		$+$	$+$	$+$	$+$
$(x-4)$	$-$		$-\ -$	$-$	$-$		$-$	$-$	0	$+$
y'	$+$		$+$	0	$-$		$-$	$-$	0	$+$
y	↗		↗	max	↘		↘		min	↗



$x=-3$
$y=2,2$

$x=-1$
$y=-2,33$

$x=0$
$y=-1,25$

7.8

$$v = kr^2(R-r) \qquad k, R > 0$$
$$v = kr^2R - kr^3 \qquad 0 \leq r \leq R$$

v är störst när $v' = 0$ (Optimering uppgifter)

$v' = 2krR - 3kr^2$ (om k och R är konstanter så är r variabeln)

$v' = 0$ om

$$2krR - 3kr^2 = 0$$

$$kr(2R - 3r) = 0$$

antingen $r = 0$ eller $2R = 3r$

$$r = \frac{2}{3}R$$

om $r = 0$ $\quad v = 0$ (minimum)

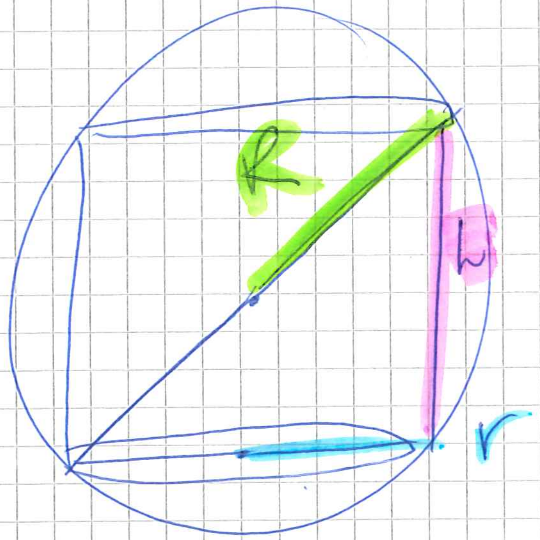
om $r = \frac{2}{3}R$ $\quad v = k\left(\frac{2}{3}R\right)^2\left(R - \frac{2}{3}R\right)$

$$v = k\frac{4}{9}R^2\left(\frac{3R}{3} - \frac{2R}{3}\right)$$

$$v = k\frac{4}{9}R^2\left(\frac{R}{3}\right)$$

$$v = k\frac{4R^3}{27} = \frac{4k}{27}R^3$$

7.9



Vilken radie r ger max volym?

$$h^2 + (2r)^2 = (2R)^2$$

$$h^2 = 4R^2 - 4r^2$$

$$h = +\sqrt{4R^2 - 4r^2}$$

$(h > 0)$

$$V = \pi r^2 h$$

$$V = \pi r^2 \sqrt{4R^2 - 4r^2}$$

$$V = \pi r^2 (4R^2 - 4r^2)^{1/2}$$

$V' = 0$ om volymen är max

$$V' = 2\pi r (4R^2 - 4r^2)^{1/2} + \pi r^2 \cdot \frac{1}{2} (-8r)(4R^2 - 4r^2)^{-1/2}$$

$$0 = 2\pi r \sqrt{4R^2 - 4r^2} - \frac{4\pi r^3}{\sqrt{4R^2 - 4r^2}}$$

dvs $$2\pi r \sqrt{4R^2 - 4r^2} = \frac{4\pi r^3}{\sqrt{4R^2 - 4r^2}}$$

$$2\pi r (4R^2 - 4r^2) = 4\pi r^3$$

$$2\pi r (4R^2 - 4r^2 - 2r^2) = 0$$

$r = 0$ eller $4R^2 = 6r^2$

$r^2 = \frac{4}{6} R^2$ $\quad r = \sqrt{\frac{2}{3}}\, R$ $\quad$ (eftersom $r > 0$)

$$r = \sqrt{\frac{2}{3}}\, R$$

$$h = \sqrt{4R^2 - 4r^2}$$

$$= \sqrt{4R^2 - 4\left(\frac{2}{3}R^2\right)}$$

$$= \sqrt{\frac{12R^2 - 8R^2}{3}}$$

$$= \sqrt{\frac{4R^2}{3}} = \frac{2R}{\sqrt{3}}$$

$$V = \pi r^2 h$$

$$= \pi \cdot \frac{2}{3} R^2 \cdot \frac{2R}{\sqrt{3}}$$

$$= \frac{\pi\ 2R^2 \cdot 2R}{3\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}} \begin{matrix} \cdot\sqrt{3} \\ \cdot\sqrt{3} \end{matrix}$$

$$V_{max} = \frac{4\pi\sqrt{3}R^3}{9}$$

7.10 $h = 20t - 4{,}9t^2$

$h' = 20 - 9{,}8t$ $\quad h' = 0$ om h max

$20 - 9{,}8t = 0 \quad t = \frac{20}{9{,}8} = 2$ s.

$$h(2) = h_{max} = 20 \cdot 2 - 4{,}9 \cdot 4$$

$$= 40 - 19{,}6$$

$$= 20{,}4 \text{ m}$$

7.11

a) $\lim_{x\to 1} \frac{x-1}{x^4-1} = \frac{0}{0}$ (Hopital fungerar här)

$= \lim_{x\to 1} \frac{1}{4x^3} = \frac{1}{4}$ ■

b) $\lim_{x\to 0} \frac{\tan 2x}{\sin x} = \frac{0}{0} \Rightarrow$ Hopital

$= \lim_{x\to 0} \frac{(1+\tan^2 2x)\cdot 2}{\cos x} = \frac{1\cdot 2}{1} = \frac{1\cdot 2}{1} = 2$ ■

c) $\lim_{x\to 0} \frac{\sqrt{x}-\sqrt{1-x^2}}{\sqrt{x^2+1}} = \frac{\sqrt{0}-\sqrt{1}}{\sqrt{1}} = \frac{-1}{1} = -1$ ■

d) $\lim_{x\to 0} \frac{\arctan x}{x} = \frac{0}{0} \Rightarrow$ Hopital

$= \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = \frac{\frac{1}{1}}{1} = 1$ ■

e) $\lim_{x\to 0} \frac{(\arctan x)^2}{x^2} = \frac{0}{0} \Rightarrow$ Hopital

$= \lim_{x\to 0} \frac{\cancel{2}\cdot\frac{1}{1+x^2}\arctan x}{\cancel{2}x} = \lim_{x\to 0} \frac{\arctan x}{x(1+x^2)}$

Hopital igen

$= \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{(1+x^2)+x\cdot 2x} = \lim_{x\to 0} \frac{1}{(1+x^2)(1+x^2+2x^2)}$

$= \frac{1}{1\cdot 1} = 1/1 = 1$ ■

f) $\lim_{x\to \frac{\pi}{2}} \frac{(\cos(x-\pi))}{x-\pi/2} = \frac{0}{0} \Rightarrow$ Hopital

$= \lim_{x\to \frac{\pi}{2}} \frac{-\sin(x-\pi)}{1} = -\frac{\sin(-\frac{\pi}{2})}{1} = -(-1) = 1$ ■

7.12

$$\lim_{x\to 1} \frac{\ln|x-1|}{\frac{1}{x-1}} = \frac{\ln|(1-1)|}{\frac{1}{1-1}} = \frac{\infty}{\infty}$$

$\Rightarrow$ Hopital regeln

$$\lim_{x\to 1} \frac{\ln|x-1|}{\frac{1}{x-1}} = \lim_{x\to 1} \frac{\frac{1}{x-1}}{\frac{-1}{(x-1)^2}}$$

$$= \lim_{x\to 1} \frac{1}{x-1}\cdot\frac{(x-1)^2}{-1} = \lim_{x\to 1} \frac{(x-1)}{-1} = 0$$

7.13

$$\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) =$$

$$\lim_{x\to 1} \left(\frac{x\cdot \ln x - x + 1}{(x-1)\ln x}\right) = \frac{0}{0} \text{ Hopital}$$

$$= \lim_{x\to 1} \frac{\ln x + x\cdot\frac{1}{x} - 1 + 0}{1\cdot \ln x + (x-1)\frac{1}{x}}$$

$$= \lim_{x\to 1} \frac{\ln x \cancel{+1} \cancel{-1}}{\ln x + (x-1)\frac{1}{x}} = \frac{0}{0+0} \text{ Hopital}$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x} + (x-1)\left(-\frac{1}{x^2}\right)} = \frac{1}{1+1+0} = \frac{1}{2}$$

7.14

$P(I) = UI - RI^2$ (I är variabeln)

$P' = U - 2RI$

$P' = 0$ ger P_{max} dvs $U = 2RI$

$$I = \frac{U}{2R}$$

ger största effekten