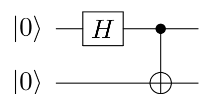
Quantum Computing: An Applied Approach

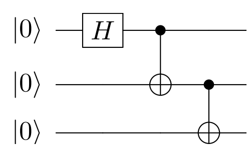
Chapter 3 Problems: Qubits, Operators, and Measurement

1. What is the final state of the following quantum circuits? Express your answer in Dirac notation.

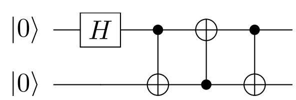
   (a) What is the final state of the following circuit?

   (b) What is the final state of the following circuit?

   (c) What is the final state of the following circuit?

2. Compute the x, y, and z coordinates on the Bloch sphere for each of the following qubit states, and draw the states on the Bloch sphere. Note that states below may be unnormalized.

   (a) $|0\rangle$

   (b) $|1\rangle$

(c) $|0\rangle + |1\rangle$
(d) $|0\rangle + e^{i\phi}|1\rangle$ for $\phi \in \{0, \pi/2, \pi, 3\pi/2\}$
(e) $3/5|0\rangle + 4/5|1\rangle$.

3. Given an initialization of a qubit in state $|0\rangle$ and the following Bloch sphere end states, build a quantum circuit that leads to this state:

   (a) $3/5|0\rangle + 4/5|1\rangle$.
   (b) $|0\rangle + e^{i\phi}|1\rangle$ for $\phi \in \{0, \pi/2, \pi, 3\pi/2\}$.

4. Are the following sets of gates universal? For each, single qubit gates can act on any qubits, and two-qubit gates can act between any pair of qubits.

   (a) $\{H, \mathrm{CNOT}\}$
   (b) $\{H, \mathrm{CNOT}, S\}$
   (c) $\{H, \mathrm{CNOT}, S, T\}$
   (d) $\{H, \mathrm{CNOT}, T\}$
   (e) $\{H, \mathrm{CZ}, S\}$
   (f) $\{H, \mathrm{CZ}, T\}$
   (g) $\{U, \mathrm{CNOT}\}$ where U is an arbitrary single qubit rotation.
   (h) U, CZ where U is an arbitrary single qubit rotation.

5. Let σ be a Pauli operator, e.g., $\sigma \in \{X, Y, Z\}$. Prove that $e^{i\theta\sigma} = \cos\theta I + i\sin\theta\sigma$.

6. Let X, Y, and Z be the usual single qubit Pauli operators. Compute the matrix elements for the single qubit rotation operators $R_x(\theta) := e^{i\theta X/2}$, $R_y(\theta) := e^{i\theta Y/2}$, and $R_z(\theta) := e^{i\theta Z/2}$.

7. Prove that $R_x(\theta_2)R_x(\theta_1) = R_x(\theta_1 + \theta_2)$ and similarly for R_y and R_z.

8. Why is it important that we represent qubits as complex Hilbert spaces? Why would a real-valued vector space not suffice? How do we represent the Hilbert space of a five-qubit system?

9. Consider a qubit in the state $|\psi\rangle = 0.6|0\rangle + 0.8|1\rangle$. What is the probability of measuring the $|0\rangle$ state? What is the probability of measuring the $|1\rangle$ state?

10. Suppose the qubit $|\psi\rangle = 0.6|0\rangle + 0.8|1\rangle$ is measured and the outcome is $|0\rangle$. What is the probability of measuring the $|+\rangle$ state? What is the probability of measuring the $|-\rangle$ state?

11. If we decided to build a quantum computer with qudits that are 4-level systems - let's call these 4-qudits - how many such 4-qudits would we need to represent the same computational space as a 10^6 qubit quantum computer?

12. Prove the following identities:

    (a) $HXH = Z$

    (b) $HZH = X$

    (c) $HYH = -Y$

    (d) $H^2 = I$

    (e) $SWAP_{ij} = CNOT_{ij}, CNOT_{ji}, CNOT_{ij}$

    (f) $R_{z,1}(\theta) CNOT_{1,2} = CNOT_{1,2} R_{z,1}(\theta)$