Homework on Singular Value Decomposition and Least Mean Squares

November 5, 2014

1 Introduction

This homework should be done in autonomy during weeks 45 and 46. The objective is to investigate more deeper SVD and LMS. You can work alone or in group. No formal or written document is asked but I can correct anything that you send to me. This is based on a voluntary basis on your side. A correction will be available on my website at the end of week 46 but I want to highlight that you cannot make progress by only looking at the solutions.

Good Luck and let SVD be with you !
Olivier

2 Left and right eigenvectors, SVD and pseudoinverse of a matrix

Question 1:
Show that if λ is an eigenvalue of a matrix $[\mathbf{A}]$, then it is also an eigenvalue of $[\mathbf{A}]^t$.

Let $[\mathbf{M}]$ be the following (unsymmetric) matrix

$$[\mathbf{M}] = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Question 2:
Find the eigenvalues of matrix $[\mathbf{M}]$.

Question 3:
Find a matrix $[\mathbf{\Phi}]$ of eigenvectors of matrix $[\mathbf{M}]$. These vectors are called right eigenvectors.

Question 4:

Find a matrix $[\mathbf{\Psi}]$ of left eigenvectors of matrix $[\mathbf{M}]$ *i.e.* the eigenvectors of $[\mathbf{M}]^t$. Order the eigenvectors of this matrix so that the the columns of $[\mathbf{\Phi}]$ and $[\mathbf{\Psi}]$ are associated to the same eigenvalue.

Question 5:

Show that $[\mathbf{\Psi}]^t[\mathbf{\Phi}]$ is a diagonal matrix.

Question 6:

Deduce the inverse of matrix $[\mathbf{\Phi}]$.

Question 7:

Find the Singular Value Decomposition of matrix $[\mathbf{M}]$.

Question 8:

Show that for a matrix $[\mathbf{A}] \in \mathcal{M}_{n\times n}(\mathbb{R})$ of rank n (i.e. invertible) whose SVD is $[\mathbf{A}] = [\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^t$ then we have:

$$[\mathbf{A}]^{-1} = [\mathbf{V}][\mathbf{\Sigma}]^{-1}[\mathbf{U}]^t$$

Question 9:

Find the inverse of $[\mathbf{M}]$ through its SVD decomposition.

We will now consider Matrix

$$[\mathbf{M}'] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

This matrix is clearly not invertible. We will now have a look to the concept of pseudoinverse that we will apply to this particular case.

Let $[\mathbf{A}] \in \mathcal{M}_{n\times m}(\mathbb{R})$, the Moore-Penrose pseudoinverse $[\mathbf{A}]^{\#}$ of $[\mathbf{A}]$ is a matrix of $\mathcal{M}_{m\times n}(\mathbb{R})$ fulfilling the following properties:

- $[\mathbf{A}][\mathbf{A}]^{\#}[\mathbf{A}] = [\mathbf{A}]$
- $[\mathbf{A}]^{\#}[\mathbf{A}][\mathbf{A}]^{\#} = [\mathbf{A}]^{\#}$
- $[\mathbf{A}]^{\#}[\mathbf{A}]$ and $[\mathbf{A}][\mathbf{A}]^{\#}$ are symmetric

For a real matrix, the Moore-Penrose pseudoinverse is unique.

Question 10:

Show that for an invertible matrix $[\mathbf{A}] \in \mathcal{M}_{n\times n}(\mathbb{R})$, then then $[\mathbf{A}]^{\#} = [\mathbf{A}]^{-1}$.

Question 11:

Let $[\mathbf{D}] \in \mathcal{M}_{n\times m}(\mathbb{R})$ be a diagonal matrix (i.e. a matrix of the form $D(i,j) = d_i\delta_{ij}$, $d_i \in \mathbb{R}$). Find the Moore-Penrose pseudoinverse $[\mathbf{D}]^{\#}$ of $[\mathbf{D}]$.

Question 12:

Deduce from the SVD of a matrix $[\mathbf{A}]$ the expression of its Moore-Penrose pseudoinverse.

Question 13:

What is the expression of $[\mathbf{M}']^{\#}$?

Question 14:

Find the expressions of the Moore-Penrose pseudoinverses of the following matrices:

$$[\mathbf{M}_1] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, [\mathbf{M}_2] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, [\mathbf{M}_3] = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, [\mathbf{M}_4] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

3 Image decomposition

A prelimianry question:

Question 15:

Let $[\mathbf{A}] \in \mathcal{M}_{n\times m}(\mathbb{R})$ be a matrix of SVD $[\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^t$. Justify the following decomposition

$$[\mathbf{A}] = \sum_{i=1}^{\mathrm{rank}([\mathbf{A}])} \Sigma(i,i)\mathbf{U}(:,i)\mathbf{V}(:,i)^t.$$

In this expression, parentheses are relative to indices of the coefficients of the matrix. The ":" symbol means that we select all the lines/columns in the matrix.

Figure 1: Picture of LAUM

You can find in Figure 1 a jpg picture of your favorite lab. The objective of this exercice is to decompose and reconstruct this image with the help of the Singular Value Decomposition. A script (svd_laum_init.m) is provided with this exercice together with the picture laum.jpg. Highlights of this programme is given here:

```
image=double(imread('laum.jpg'));
[U_R,S_R,V_R]=svd(image(:,:,1));
[U_G,S_G,V_G]=svd(image(:,:,2));
[U_B,S_B,V_B]=svd(image(:,:,3));
image_svd(:,:,1)=U_R*S_R*V_R';
image_svd(:,:,2)=U_G*S_G*V_G';
image_svd(:,:,3)=U_B*S_B*V_B';
imshow(uint8(image_svd))
```

The first line corresponds to the importation of the image into Octave with imread which is a buit-in function. This returns a 3 dimension array. The first two index correspond to the number of the pixel in the horizontal and vertical dimension. The last column is associated to the color (1 for Red, 2 for Green, 3 for Blue). The value in the matrix correspond to the associated level. It is coded in uint8 and should be converted to double to make the SVD. This is done in the next three lines for each basis color. A new variable (image_svd) is then created in which each matrix (R,G,B) is reconstructed through the SVD and displayed with the imshow function.

Question 16:

Modify this program to display the picture obtained with only 3 singular values for each color.

Question 17:

Generalize this program to display several recompositions of the original picture with a loop

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The norm of a square matrix is defined as the largest modulus of its eigenvalues. The null matrix as a norm equal to zero. We can obtain the norm of a matrix with the built-in function **norm**.

Question 18:

Find a criteria based on the norm to estimate the error of the SVD recomposition. Plot this error as a function of the number of kept singular values.

4 Impedance tube measurement fitting

In this exercice, we will consider complex data (pressure signals). We should then first need to answer to a preliminary question to extend the result of the course.

Question 19:
Show that if we want to find $\mathbf{p}$ which minimize $\|[\mathbf{M}\mathbf{p} - \mathbf{y}]\|$, then we have to solve

$$[\mathbf{M}]^*[\mathbf{M}]\mathbf{p} = [\mathbf{M}]^*\mathbf{y}$$

where * denotes the hermitian (complex) conjugation.

We are condidering an impedance tube measurement along the axis of a tube. A sample with a reflection coefficient R to determine is in place at the end of the tube corresponding tho the abscissa $x = 0$. Hence, the pressure profile in the cavity is given by

$$p(x) = A_i e^{-jkx} + RA_i e^{jkx}. \tag{1}$$

Measurements are performed for several positions at $440\,Hz$ and the sound speed is $342\,m/s$. The abscissa of the points are given in the file **position.txt**. A first signal is provided (in the file **P_pure.txt**). This signal correspond to a simulated pressure profile which is free from measurement errors.

Question 20:
Apply the LMS procedure to find the incident amplitude A_i and the reflection coefficient R

We will now consider a second case which corresponds to the same physical problem but for which the signal is truncated at a given level corresponding to background noise. This signal is available in the file **P_background.txt**. While applied to this signal, the previous LMS procedure will lead to a systematic error.

Question 21:
Find one or several strategies to limit this background noise and apply them to the given signal.

The last case correspond to a signal with two sources of errors. The first one is relative to the background noise of the previous case and the second one is associated to pollution of the measurements. It is proposed in the file **P_noisy.txt**

Question 22:
Estimate through a modified LMS procedure the incident amplitude A_i and the reflection coefficient R

A script (**impedance_tube_fit_init.m**) is provided which imports the data and plot the three curves. You can use and modify this script.

```
clear all
close all
clc

f=440;
c=342;
k=2*pi*f/c;

x=load('position.txt');
P_pure=load('P_pure.txt');
P_pure=P_pure(:,1)+i*P_pure(:,2);
P_background=load('P_background.txt');
P_background=P_background(:,1)+i*P_background(:,2);
P_noisy=load('P_noisy.txt');
P_noisy=P_noisy(:,1)+i*P_noisy(:,2);


figure(1)
hold on
plot(x,abs(P_pure),'Linewidth',3)
plot(x,abs(P_background),'k','Linewidth',3)
plot(x,abs(P_noisy),'m','Linewidth',3)
set(gca,'Fontsize',15)
title('Modulus of the three signals')
xlabel('Position (m)')
ylabel('Amplitude (au)')

figure(2)
hold on
plot(x,angle(P_pure),'Linewidth',3)
plot(x,angle(P_background),'m','Linewidth',3)
plot(x,angle(P_noisy),'k','Linewidth',3)
set(gca,'Fontsize',15)
title('Phase of the three signals')
xlabel('Position (m)')
ylabel('Phase (rad)')
```