The Microstructure of Cointegrated Assets

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Abstract

We generalize the micro price model of Stoikov (2017) [19] to multiple assets. This yields a notion of fair prices, as a function of the observable state of multiple order books. We compute the microprices of two highly cointegrated assets, using Level-1 data collected on Interactive Brokers. We design an execution algorithm based on this two dimentional micro-price and show that it can save half of the bid-ask spread cost.

1 Introduction

Stoikov (2017) [19] proposed a method to compute the fair price of a single asset, as a function of the state of the order book. The main idea is to define the expected mid price after i mid price moves

$$P_i = E\left[M_{\tau_i} \mid X_t\right] \tag{1}$$

where τ_i is the time of the ith mid price move and X_t is the state of the order book. If the data is symmetrized appropriately, the limit

$$P_{micro} = \lim_{i\to\infty} P_i \tag{2}$$

can be shown to exist, and we call it the microprice. In the case of Markov dynamics, the model is flexible enough to fit a range of microstructures, from big tick stocks like BAC to small tick stocks like CVX.

In this paper, we generalize this model to include multiple orderbooks of cointegrated assets. We will focus on one particular example: two leveraged ETFs, with tickers SH and SDS, that are designed to give investors daily returns with a -1 or -2 leverage with respect to the ticker SPY. Since both SH and SDS are in the \$13-\$20 range and the spread is almost always 1 tick, the bid and

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offer sizes have an important role on the next price moves of SH and SDS. This offers a test case where the microstructure and cointegration of the assets both play a role in determining their fair price.

Gould [12] provides an overview of the literature on single order books. Single orders books and their dynamics have been modeled as a Markov chains both as a discrete ([6] [15]) or as a continuous process ([3] [18]). More recently, there has been work on Hawkes process see for example [16]. The market impact, placement of orders in an orderbook have been studied for a long time, for example Silviu Predoiu (2011) [17], Gatheral (2011, 2013) [10][11], or Byrd (2020) [4]. For trading strategies around these dynamics see for example (Ruo (2021) [20],Abergel (2020) [1] or Zare (2021) [23]). The only paper we have come across that consider multiple orderbooks focus on the cross impact Cont (2020) [5]. Additionally, our methodology allows the integration of information that can live on different time scales. Even though prices movements are significantly less frequent than order book changes, our methodology allows us to integrate both and conclude one micro price adjustment.

2 Data

Intraday financial data is typically large, expensive and bound by legal licenses. As a result, empirical research in market microstructure can be challenging to reproduce or falsify. To address this issue, we have included a template for retrieving intraday data from Interactive Brokers on a Github repository.

The IB Python API allows us to collect data on multiple order books. We focus our study on two ETFs that are linked to the performance of the S&P 500. The SH targets the inverse of the daily return of the SP 500. The SDS targets two times the inverse daily return of the S&P 500. The average frequency of quote updates is around 10 seconds, so each day there are 2000+ data points.

	time	bid1	bid_size1	ask1	ask_size1	bid2	bid_size2	ask2	ask_size2
0	2020-12-03 09:49:50.499130-05:00	18.41	4631	18.42	11366	13.01	1576	13.02	2045
1	2020-12-03 09:50:00.518556-05:00	18.41	5562	18.42	11286	13.01	1728	13.02	1879
2	2020-12-03 09:50:10.581464-05:00	18.40	14666	18.41	244	13.01	1820	13.02	1506
3	2020-12-03 09:50:19.612368-05:00	18.41	8151	18.42	11729	13.01	1820	13.02	1506
4	2020-12-03 09:50:19.612368-05:00	18.41	8151	18.42	11729	13.01	2003	13.02	1156
...	...	...	...	...	...	...	...	...	...
2582	2020-12-03 15:49:09.138292-05:00	18.36	11324	18.37	6166	12.95	1992	12.96	2855
2583	2020-12-03 15:49:18.978898-05:00	18.36	10905	18.37	6233	12.95	1557	12.96	3035
2584	2020-12-03 15:49:28.996096-05:00	18.36	11367	18.37	6266	12.95	2087	12.96	2692
2585	2020-12-03 15:49:39.038233-05:00	18.36	11664	18.37	2841	12.95	2583	12.96	2015
2586	2020-12-03 15:49:49.073615-05:00	18.37	7570	18.38	12523	12.95	3269	12.96	795

Table 1: Level-1 order data for (1: SH, 2: SDS) on December 3, 2020

Leveraged ETFs may not be appropriate for buy and hold investors (Avellaneda (2009) [2]). However, at the intraday timescale, the following plot shows that the relationship between SH and SDS midprices is essentially linear.

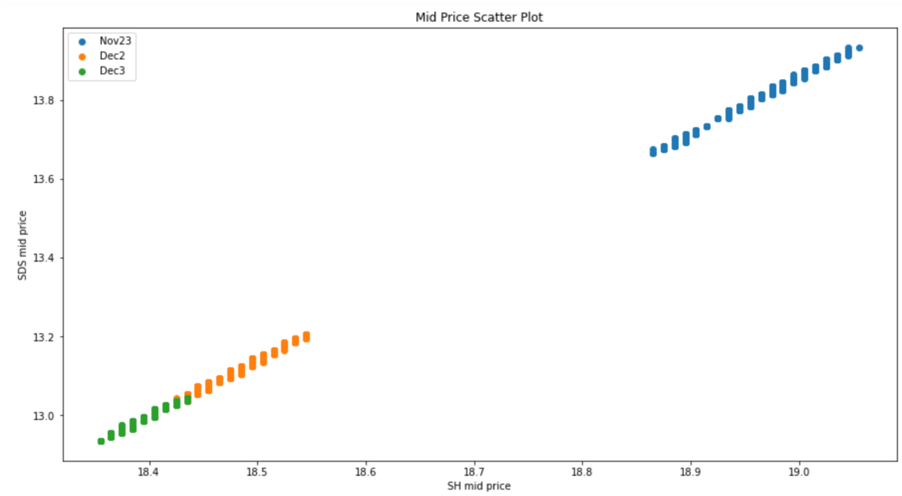

Figure 1: Scatter plot of mid price comparison of SH and SDS on November 23, December 2, and December 3, 2020.

Our modeling relies on two empirical features of the data. First, the regression residual between the two ETFs is a good predictor of the performance of a long/short portfolio in the two assets. Second, the imbalance at the top of each order book is a good predictor of the next price move. The goal of our model is to combine these factors with the mid-price to obtain a generalized formula for the microprice of each ETF. Note that we have used a symmetrization procedure described in Appendix A.

2.1 The relationship between SH and SDS midprices

To reveal the price relationship between SH and SDS, we construct a linear regression model [8] between the midprices of SH and SDS.

$$\boldsymbol{M_t}^{SH} = \alpha + \beta \boldsymbol{M_t}^{SDS} + \epsilon_t$$

The residual of this model gives a view of which price is relatively under or over-priced. For example, a negative residual means that SDS is relatively over-priced and SH is relatively under-priced. Therefore, if we are long one share of SH and short beta shares of SDS , i.e. $\boldsymbol{P_t}^{Combo} = \boldsymbol{M_t}^{SH} - \beta \boldsymbol{M_t}^{SDS}$, we expect the combo price to increase. Figure 2 illustrates the negative relationship between the residual ϵ_t and the forward PNL of the combo

$$F(\epsilon_t) = E[\boldsymbol{P}_{t+1}^{Combo} - \boldsymbol{P}_t^{Combo} | \epsilon_t]$$

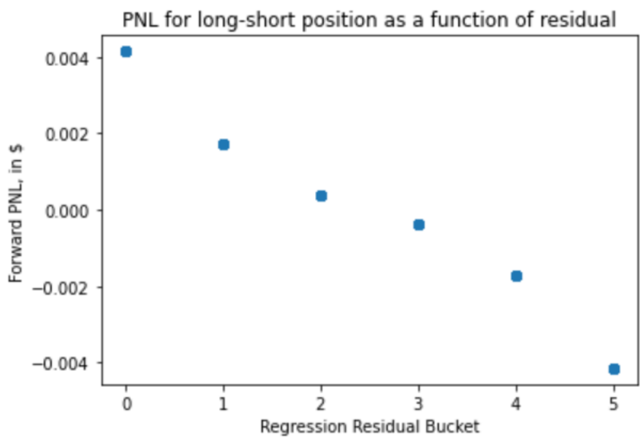

Figure 2: Cointegration residuals versus Forward PNL on November 23, 2020

2.2 The imbalance of Level-1 quotes

The imbalance is defined as

$$\boldsymbol{I_t} = \frac{Q_t^b}{Q_t^b + Q_t^a}$$

where Q_t^b is the best bid size, and Q_t^a is the best ask size. It is a measure of the instantaneous excess demand for the asset and has been shown to have prediction power on the next price movement of the asset. In other words, the function

$$G(I_t) = E[\boldsymbol{M_{t+1}} - \boldsymbol{M_t} | I_t]$$

is increasing in I_t.

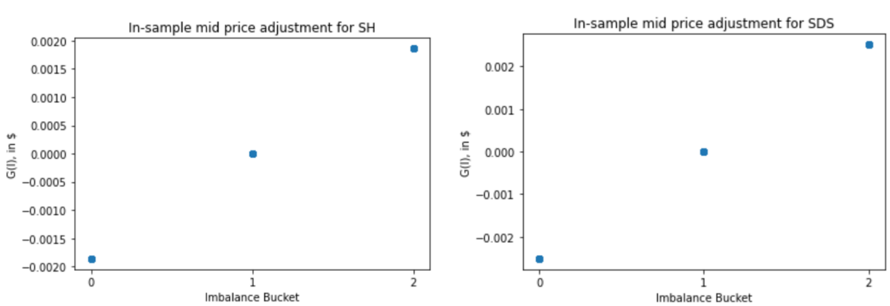

Figure 3: Imbalance v.s. next price movement on Nov 23, 2020

3 Model

We model the dynamic of mid-prices for n different securities and define the $1 \times n$ vector $\boldsymbol{M}_{\tau_0} = [M_1, M_2, \ldots, M_n]_{\tau_0}$ to be the mid-prices at time τ_0. Also we let stopping times $\tau_1, ..., \tau_i$ represent the times when any of the mid-prices $\boldsymbol{M}$ change, i.e.

$$\tau_k = \inf\{u > \tau_{k-1} \mid \boldsymbol{M}_u - \boldsymbol{M}_{u-} \neq \boldsymbol{0}\}, k = 1, ..., i \tag{3}$$

X_t is a finite-state Markov chain which can take m possible values $x_1, x_2, \ldots, x_m$ and can contain variables like the order book imbalance and midprice relationships. For example, if we use the discretized state space described in the previous section, with 3 imbalance buckets for SH, 3 imbalanced buckets for SDS and 6 regression residual buckets, we have $m = 3^2 \times 6 = 54$ states for X_t.

Definition 1. *The **microprice** is the limit*

$$\boldsymbol{P}_{micro} = \lim_{i \to \infty} \boldsymbol{P}_i \tag{4}$$

of the expected mid price after i mid price moves

$$\boldsymbol{P}_i = E\left[\boldsymbol{M}_{\tau_i} \mid \mathcal{F}_{\tau_0}\right] \tag{5}$$

Note that the order book information is given by the filtration generated by the multi-dimensional Markov process: $\mathcal{F}_t = \sigma(\boldsymbol{M}_t, X_t)$

Assumption 1. *The expected mid price increments are independent of the mid price level:*

$$E\left[\boldsymbol{M}_{\tau_i} - \boldsymbol{M}_{\tau_{i-1}} \mid \boldsymbol{M}_t = \boldsymbol{M}, X_t = X\right] = E\left[\boldsymbol{M}_{\tau_i} - \boldsymbol{M}_{\tau_{i-1}} \mid X_t = X\right], t \leq \tau_{i-1} \tag{6}$$

This assumption ensures that the dynamics of the price is the same at each tick. The mid-price predictions can be expressed in terms of the combination of state variables X_t.

Theorem 3.1. *Given Assumptions 1, the prediction of the i-th mid-price at τ_0 can only take m different values which is fully determined by X_{τ_0}, thus we can summarize the predictions using the formula:*

$$\boldsymbol{P}_i = \boldsymbol{P}_0 + \sum_{k=1}^{i} \boldsymbol{G}_k \tag{7}$$

Where $\boldsymbol{P}$ and $\boldsymbol{G}$ are both $m \times n$ matrix, specifically:

$$\begin{aligned} (\boldsymbol{P}_i)_{j\cdot} &= E\left[\boldsymbol{M}_{\tau_i} \mid X_{\tau_0} = x_j\right] \\ (\boldsymbol{G}_i)_{j\cdot} &= E\left[\boldsymbol{M}_{\tau_i} - \boldsymbol{M}_{\tau_{i-1}} \mid X_{\tau_0} = x_j\right] \end{aligned} \tag{8}$$

i.e. the j-th row of matrix $\boldsymbol{P}_i$ and matrix $\boldsymbol{G}_i$ is the expectation of the i-th mid-price and i-th mid-price change respectively, given $X_{\tau_0} = x_j$

Proof. Based on Assumptions 1 and Assumption 2, we have:

$$\begin{aligned} E\left[\boldsymbol{M}_{\tau_i} \mid \mathcal{F}_{\tau_0}\right] &= E\left[\boldsymbol{M}_{\tau_0} \mid \boldsymbol{M}_{\tau_0}, X_{\tau_0}\right] + \sum_{k=1}^{i} E\left[\boldsymbol{M}_{\tau_k} - \boldsymbol{M}_{\tau_{k-1}} \mid \boldsymbol{M}_{\tau_0}, X_{\tau_0}\right] \\ &= \boldsymbol{M}_{\tau_0} + \sum_{k=1}^{i} E\left[\boldsymbol{M}_{\tau_k} - \boldsymbol{M}_{\tau_{k-1}} \mid X_{\tau_0}\right] \end{aligned} \tag{9}$$

As shown in formula(5), the expectation of i-th mid-price at time τ_0 is only determined by X_{τ_0} which by Assumption 1 can only take value from $\Omega = \{x_1, x_2, \ldots, x_m\}$, thus $E\left[\boldsymbol{M}_{\tau_i} \mid \mathcal{F}_{\tau_0}\right]$ has m possible results based on our setting which is completely included in $\boldsymbol{P}_i$. Also given formulae(4),(5) we have:

$$(\boldsymbol{P}_i)_{j\cdot} = (\boldsymbol{P}_0)_{j\cdot} + \sum_{k=1}^{i} (\boldsymbol{G}_k)_{j\cdot} \Longrightarrow \boldsymbol{P}_i = \boldsymbol{P}_0 + \sum_{k=1}^{i} \boldsymbol{G}_k \tag{10}$$

□

Assumption 2. *At each timestamp τ_i when the mid-price vector $\boldsymbol{M}$ changes, there is only one security's mid price change and it can only be up one tick or down one tick. This is reasonable as almost surely no price change of two or more different securities can happen at exact same time, which has been proven true in micro-second level tick data. Also, as the price for most securities is quite stable and can rarely change more than a tick at a time, although our model framework can also take ≥ 2 tick change for a single price jump, we set the magnitude restriction for simplification purpose*

Theorem 3.2. *The price adjustment matrix of the first price change can be computed as:*

$$\boldsymbol{G}_1 = (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{R} \boldsymbol{Z} \tag{11}$$

where $\boldsymbol{I}$ is $m \times m$ Identity matrix, $\boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{Z}$ are $m \times m, m \times 2n, 2n \times n$ matrix respectively, where

$$\begin{aligned} &Q_{ij} := P\left(\boldsymbol{M}_{t+1} - \boldsymbol{M}_t = \boldsymbol{0} \wedge X_{t+1} = x_j \mid X_t = x_i\right) \\ &R_{iz} := P\left(\boldsymbol{M}_{t+1} - \boldsymbol{M}_t = \boldsymbol{m}_z \neq \boldsymbol{0} \mid X_t = x_i\right) \\ &Z_{z\cdot} := \boldsymbol{m}_z \end{aligned} \tag{12}$$

$\boldsymbol{m}_z$ is $1 \times n$ vector with all members to be 0 except for the k^{th} member to be ticksize if $z = 2k-1$, $-$ticksize if $z = 2k$, $k = 1, 2, \ldots, n$

Proof. Using standard techniques for discrete time Markov processes with absorbing states, see Hoel et al. (1972) [14], we have:

$$\begin{aligned} (\boldsymbol{G}_1)_{j\cdot} &= E\left[\boldsymbol{M}_{\tau_1} - \boldsymbol{M}_{\tau_0} \mid X_{\tau_0} = x_j\right] \\ &= \sum_{z \in Z} \boldsymbol{m}_z \cdot P\left(\boldsymbol{M}_{\tau_1} - \boldsymbol{M}_{\tau_0} = \boldsymbol{m}_z \mid X_{\tau_0} = x_j\right) \\ &= \sum_{z \in Z} \sum_{u} \boldsymbol{m}_z \cdot P\left(\boldsymbol{M}_{\tau_1} - \boldsymbol{M}_{\tau_0} = \boldsymbol{m}_z \wedge \tau_1 - \tau_0 = u \mid X_{\tau_0} = x_j\right) \end{aligned} \tag{13}$$

Based on formula (9) and (10) we have

$$\boldsymbol{G}_1 = \left(\sum_s \boldsymbol{Q}^{s-1} \boldsymbol{R} \right) \boldsymbol{Z} = (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{R} \boldsymbol{Z} \tag{14}$$

□

Theorem 3.3. *Given Assumption 2, the expected mid-price Matrix at i-th price change:*

$$\boldsymbol{P}_i = \boldsymbol{P}_0 + \sum_{k=1}^{i} \boldsymbol{G}_k = \boldsymbol{P}_0 + \sum_{k=0}^{i-1} \boldsymbol{B}^k \boldsymbol{G}_1 \tag{15}$$

where

$$\begin{aligned} \boldsymbol{B} &:= (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{T} \\ \boldsymbol{T}_{ij} &:= P\left(\boldsymbol{M}_{t+1} - \boldsymbol{M}_t \neq \boldsymbol{0} \wedge X_{t+1} = x_j \mid X_t = x_i\right) \end{aligned} \tag{16}$$

Please find the detailed proof of theorem 3.3 in Appendix B.

Theorem 3.4. *$\boldsymbol{P}_{micro}$ exists, i.e. the process $\boldsymbol{P}_i$ converges if $\boldsymbol{B}_* \boldsymbol{G}_1 = 0$, where*

$$\boldsymbol{B}_* = \lim_{k \to \infty} \boldsymbol{B}^k \tag{17}$$

Please find the detailed proof of theorem 3.4 in Appendix C. This theorem shows that the micro price limit converges if a certain martingale condition holds. It turns out that symmetrizing the data in the way described in Appendix A, is sufficient to ensure this condition.

4 Results

4.1 Markov Estimation

In all we have the estimation procedure is as follows:

- Symmetrize the data (see Appendix A), so that for every discretized observation $(\boldsymbol{I}_t, PR_t, dM)$ there is a corresponding symmetric observation $((3,3) - \boldsymbol{I}_t, 6 - PR_t, -dM)$.[1]
- Estimate transition probabilities Q[2], T and R. Recall that these matrices are $3^2 * 6 \times 3^2 * 6, 3^2 * 6 \times 3^2 * 6$ and $3^2 * 6 \times 4$ respectively.[3]
- Compute $\boldsymbol{G}_1 = (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{R} \boldsymbol{Z}$

[1] If you use Python, you can think the (3,3) as (2,2) and 6 as 5 because Python range starts from 0, not 1.

[2] See an example of the transient matrix Q in Appendix D.

[3] Since our data has a frequency of 10 seconds, it has some cases of both assets' price moving simultaneously. We will remove these states from the data. In real high-frequency environment, co-movement of prices rarely happens.

- Compute $\boldsymbol{B} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{T}$
- Compute the micro-price adjustment:

$$\boldsymbol{G}_* = \boldsymbol{P}_{micro} - \boldsymbol{M}_{\tau_0} = \boldsymbol{G}_1 + \sum_{i=1}^{\infty} \boldsymbol{B}^i \boldsymbol{G}_1$$

In practice, the microprice adjustments will converge after 5-10 midprice moves.

4.2 Trading Strategies

The four strategies we will compare, will choose between buying and selling one of two securities. In the case of the SH ETF, the buy or sell will be for \$1000, and in the case of the SDS, the size will be for \$500. The actual number of shares will vary based on the price of the underlying. The strategies do not choose to buy or sell, but they do choose the security they buy or sell. As such, these are strategies which are part of family of strategies called execution algo's.

We will trade every 5 minutes and we will assume that the market impact is similar for these strategies and securities.

The four strategies we are considering are:

1. **A random choice between the two securities.**

   Since we buy on the offer side and sell on the bid side, one would expect a cost of half the bid offer. We will buy or sell at the beginning of each period.

2. **A security choice based on weighted mid.**

   In this case, we choose the security where the order book is tilted most favorable. We will still buy on the offer side and sell on the bid side. We will scale the quantity of the transaction by the appropriate dollar price each time we execute on either security. We will calculate the average number of shares \$1000 and \$500 can trade: $\bar{N}^{SH} = 1000/avg(M^{SH})$ and $\bar{N}^{SDS} = 500/avg(M^{SDS})$. We will use these share numbers as multipliers to the weighted mid when deciding which security to trade. We would hope that using the order book information would reduce the cost associated with the strategy.

3. **A choice based on micro price.**

Instead of weighted mid, we use micro price to select the security. This will verify the advantage of micro price over weighted mid under the same micro-structure.

4. **American sniper based on the extended model.**

   In this case, we will buy or sell either security during the period based on the current Markov state's absolute difference of two micro price adjustments multiplied with corresponding shares multiplier ($|\bar{N}^{SH} \times adj^{SH} - \bar{N}^{SDS} \times adj^{SDS}|$) exceeding certain threshold. We will buy or sell the security that first exceeds the threshold. If neither security exceeds the threshold for the period, we will buy or sell at the end of the period. We think that the best opportunities for trading are when the two adjustments have the biggest difference, e.g. SH is $25 \times 0.003 = 0.075$ and SDS is $20 \times -0.002 = -0.04$, then we should definitely buy SH.

4.3 Training and Testing results

The tables below compare the cumulative profit and loss (PNL) of the buys and sells, and sum of buy and sell PNL, both in sample and out of sample. The PNL is the dollar amount of remaining cash and inventories priced at the last mid prices of the day. Initially, we have 0 cash and 0 inventory. We also assume, that we can execute our orders immediately. For the in sample, we will compare the strategies based on the same data we used to calibrate the model. In the testing section, we use data from a different date with the in-sample cointegration parameters, state splitting and corresponding micro price adjustment from the training data.

algo	sum_PNL	buy_PNL	sell_PNL	train_or_test
1. random algo	-54.140	113.767	-167.907	train
2. weighted mid algo	-37.362	121.827	-159.189	train
3. micro algo	-35.712	122.647	-158.359	train
4. American sniper algo_20	-19.993	130.502	-150.495	train
1. random algo	-67.189	355.581	-422.770	test
2. weighted mid algo	-56.297	359.469	-415.765	test
3. micro algo	-49.223	360.868	-410.091	test
4. American sniper algo_20	-43.717	365.522	-409.239	test

Table 2: Train on November 23, 2020 and test on December 2, 2020.

	sum_PNL	buy_PNL	sell_PNL	train_or_test
algo				
1. random algo	-61.606	302.557	-364.163	train
2. weighted mid algo	-51.582	307.148	-358.730	train
3. micro algo	-42.145	311.796	-353.941	train
4. American sniper algo_20	-25.586	296.368	-321.953	train
1. random algo	-67.604	228.278	-295.881	test
2. weighted mid algo	-43.642	240.726	-284.368	test
3. micro algo	-41.012	242.038	-283.049	test
4. American sniper algo_20	-24.656	154.932	-179.588	test

Table 3: Train on December 2, 2020 and test on December 3, 2020.

Now we can summarize the sum_PNL data for the above 2 test periods versus the crossing-spread cost of random choice. We make the random algo cost 100%, as a reference. The other algo's cost are decreasing comparing to the random one. The micro price outperforms the weighted mid and random algo. The American sniper gives further reduction of the cost because of its optional executing time over the period of 5 minutes.

	Cost %
1. random algo	100
2.weighted mid algo	74
3. micro algo	67
4. American sniper algo	51

Table 4: Summarizing the two test days' costs relative to the cost of random choice.

5 Conclusion

In this paper, we have extended our previous micro structure model, that looks at a single orderbook, to multiple order books and we added the price relationship between the securities through co-integration. We compared multiple trading strategies and showed that for the case we considered, this approach outperforms three other trading strategies, inclusive our previous model. In the trading strategy we considered, the execution algo has to choose between two securities. The algo is required to buy/sell a certain amount within a certain timeframe of either security. The resulting execution cost of our algo are about half of a random choice trading strategy.

Our approach can be applied in other execution algorithmic strategies, where one looks for the most cost efficient approach, for the reduction of correlated risk or inside of spreaders where one seeks to purchase one security and sell another highly correlated asset.

We envision that the approach can not only be extended to other highly correlated assets but can also be extended over more securities. In this paper we only explored one specific example of two highly correlated ETF's. Examples of the former in fixed income are the rolls of bond futures. These quarterly expiring contracts on highly correlated assets are both active at the end/beginning of each cycle. Cash securities versus bond futures are another example of the former. Some of these relationships are bound by contractual arbitrage constraints while others are related to statistical arbitrage. Examples of the latter, are Eurodollars, where we have many highly correlated individual contracts. In the example we investigated we only considered two inverse ETF's out of a larger set of inverse ETF's. There are also regular ETF's of the same family, other correlated ETF families and futures.

Besides the application of our model in execution algorithms, it can also be used as part a market making algorithm, or be used as an outlier detection. In a market making algorithm one creates a bid and an offer around a central price, the output of the model can be the input in such an algorithm since it combines information from multiple order books. On the latter point, since the markov chain creates transition probabilities, these probabilities can be compared with the ones observed in the marketplace. We have found that markov chain models are effective tools for this purpose and compare favorably with more complex models for pattern and outlier detection.

And finally, our markov chain approach can easily be extended into a Markov Decision Process, which lends itself to be part of reinforcement learning setup. In Vyetrenko (2021) [22] Q learning was used in a simulated market to learn optimal execution. The limitations of a simulated market were pointed out in Vyetrenko (2019) [21]. These limitations can be removed by having an environment where different trading strategies interact through a central limit order book. Additionally, even though, we described the discrete version of our model in this paper, the continuous differentiable version will give access to additional reinforcement learning algorithms and enable us to implement online learning models.

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Appendix A Symetrizing the data (See page 3)

Since the daily raw data has up or down trend, it will be biased for our later model construction. We eliminate the trend effect by symmetrizing the data using the following steps:

1. Duplicate the bid and ask price series and put negative sign in front of them. To ensure that the ask price is larger than the bid price, we also need to switch the bid and ask label so that $P^{Ask} \to -P^{Bid}$, $P^{Bid} \to -P^{Ask}$.

2. Since we want the prices remains to be a time series, we increase all the negative prices by the difference between the last price of the raw data and the first price of the negative prices in order to close the connecting gap.

3. Switch the bid and ask size to retain the supply and demand flow of each price movement.

So if the price moved from 18 to 19, we would mirror it in the second part by having it move back from 19 to 18 in exact mirroring fasting. The graph below illustrates this. (The time axis has been standardized to UTC timezone.)

Figure 4: Symmetrized mid price of SH on November 23

Appendix B Proof of Theorem 3.3 (See page 7)

Theorem 3.3. *Given Assumption 2, the expected mid-price Matrix at i-th price change:*

$$\boldsymbol{P}_i = \boldsymbol{P}_0 + \sum_{k=1}^{i} \boldsymbol{G}_k = \boldsymbol{P}_0 + \sum_{k=0}^{i-1} \boldsymbol{B}^k \boldsymbol{G}_1 \tag{15}$$

where

$$\begin{aligned} &\boldsymbol{B} := (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{T} \\ &\boldsymbol{T}_{ij} := P\left(\boldsymbol{M}_{t+1} - \boldsymbol{M}_t \neq \boldsymbol{0} \wedge X_{t+1} = x_j \mid X_t = x_i\right) \end{aligned} \tag{16}$$

Proof. Given assumption 3, if mid price changes, $\boldsymbol{M}_{t+1} - \boldsymbol{M}_t$ can only take $2n$ different values for k in $1, 2, \ldots, i$:

$$\begin{aligned} \boldsymbol{T} &= \sum_{z=1}^{2n} \boldsymbol{T}_z, \ \textit{where} \\ (\boldsymbol{T}_z)_{ij} &:= P\left(\boldsymbol{M}_{t+1} - \boldsymbol{M}_t = \boldsymbol{m}_z \neq \boldsymbol{0} \wedge X_{t+1} = x_j \mid X_t = x_i\right) \end{aligned} \tag{18}$$

Thus we have:

$$\boldsymbol{B} = \sum_{z=1}^{2n} (\boldsymbol{I} - \boldsymbol{Q})^{-1} \boldsymbol{T}_z = \sum_{z=1}^{2n} \boldsymbol{B}_z \tag{19}$$

Similar to formula (9) and (10) we can prove

$$(\boldsymbol{B}_z)_{ij} := P\left(\boldsymbol{M}_{\tau_k} - \boldsymbol{M}_{\tau_{k-1}} = \boldsymbol{m}_z \neq \boldsymbol{0} \wedge X_{\tau_k} = x_j \mid X_{\tau_{k-1}} = x_i\right) \tag{20}$$

which is identical for $k = 1, 2, \ldots, i$ based on assumption 2, also we have:

$$\begin{aligned} (\boldsymbol{G}_i)_{j\cdot} &= E\left[\boldsymbol{M}_{\tau_i} - \boldsymbol{M}_{\tau_{i-1}} \mid X_{\tau_0} = x_j\right] \\ &= \sum_{X_{\tau_1}} \cdots \sum_{X_{\tau_{i-1}}} P\left(X_{\tau_1} \cdots X_{\tau_{i-1}} \mid X_{\tau_0} = x_j\right) \cdot E\left[M_{\tau_i} - M_{\tau_{i-1}} \mid X_{\tau_{i-1}}\right] \end{aligned} \tag{21}$$

Let $d\boldsymbol{M}_{\tau_k} = \boldsymbol{M}_{\tau_k} - \boldsymbol{M}_{\tau_{k-1}}$, we have:

$$\begin{aligned} &\sum_{X_{\tau_1}} \cdots \sum_{X_{\tau_{i-1}}} P\left(X_{\tau_1} \cdots X_{\tau_{i-1}} \mid X_{\tau_0} = x_j\right) \\ &= \sum_{dM_{\tau_1}} \cdots \sum_{dM_{\tau_{i-1}}} \sum_{X_{\tau_1}} \cdots \sum_{X_{\tau_{i-1}}} P\left(dM_{\tau_1}, X_{\tau_1} \mid X_{\tau_0} = x_j\right) \cdots P\left(dM_{\tau_{i-1}}, X_{\tau_{i-1}} \mid X_{\tau_{k-2}}\right) \end{aligned} \tag{22}$$

Based on formula(17),(18),(19) we have:

$$\boldsymbol{G}_k = (\boldsymbol{B}_{\boldsymbol{1}} + \cdots + \boldsymbol{B}_{2n})^k \boldsymbol{G}_1 = \boldsymbol{B}^k \boldsymbol{G}_1 \tag{23}$$

which takes all combination of state and price paths into consideration □

Appendix C Proof of Theorem 3.4 (See page 7)

Theorem 3.4. *$\boldsymbol{P}_{micro}$ exists, i.e. the process $\boldsymbol{P}_i$ converges if $\boldsymbol{B}_* \boldsymbol{G}_1 = 0$, where*

$$\boldsymbol{B}_* = \lim_{k \to \infty} \boldsymbol{B}^k \tag{17}$$

Proof. The matrix $\boldsymbol{B}$ is a regular stochastic matrix so it can be decomposed:

$$\boldsymbol{B} = \boldsymbol{B}_* + \sum_{j=2}^{nm} \lambda_j \boldsymbol{B}_j \tag{24}$$

where $\boldsymbol{B}_*$ is the unique stationary distribution and the eigenvalues $|\lambda_j| < 1$ and $\boldsymbol{B}_j = f_j \pi_j$ wheref_j and π_j are the left and right eigen vectors of $\boldsymbol{B}$. This is a spectral representation of matrix $\boldsymbol{B}$ that is known as the Perron Frobenius Theorem. Therefore,

$$\boldsymbol{P}_i = \boldsymbol{P}_0 + \sum_{k=0}^{i} \boldsymbol{B}^k \boldsymbol{G}_1 = \boldsymbol{P}_0 + \boldsymbol{G}_1 + \sum_{k=1}^{i} \left(\boldsymbol{B}^k - \boldsymbol{B}_* \right) \boldsymbol{G}_1 \tag{25}$$

follows from the assumption that $\boldsymbol{B}_* \boldsymbol{G}_1 = 0$. Note that

$$\boldsymbol{B}^k = \boldsymbol{B}_* + \sum_{j=2}^{nm} \lambda_j^k \boldsymbol{B}_j \tag{26}$$

for $k \geq 1$. Therefore

$$\boldsymbol{P}_i = \boldsymbol{P}_0 + \boldsymbol{G}_1 + \sum_{k=1}^{i} \left(\sum_{j=2}^{nm} \lambda_j^k \boldsymbol{B}_j \right) \boldsymbol{G}_1 = \boldsymbol{P}_0 + \boldsymbol{G}_1 + \sum_{j=2}^{nm} \left(\sum_{k=1}^{i} \lambda_j^k \right) \boldsymbol{B}_j \boldsymbol{G}_1 \tag{27}$$

The micro-price can be written explicitly in terms of these matrices:

$$\boldsymbol{P}_{micro} = \lim_{i \to \infty} \boldsymbol{P}_i = \boldsymbol{P}_0 + \boldsymbol{G}_1 + \sum_{j=2}^{nm} \frac{\lambda_j}{1 - \lambda_j} \boldsymbol{B}_j \boldsymbol{G}_1 \tag{28}$$

and therefore converges.

□

Appendix D Example of Transient Matrix (See page 7)

Below is the transient matrix Q of November 23, 2020. The left vertical labels with 3 digits indicate the current states. For example, '502' means that current residual bucket is 5, current SH imbalance bucket is 0, and current SDS imbalance bucket is 2. The horizontal labels with 5 digits indicate the state in the next time step. The last two digits '00' means that SH and SDS mid price remains the same as before.

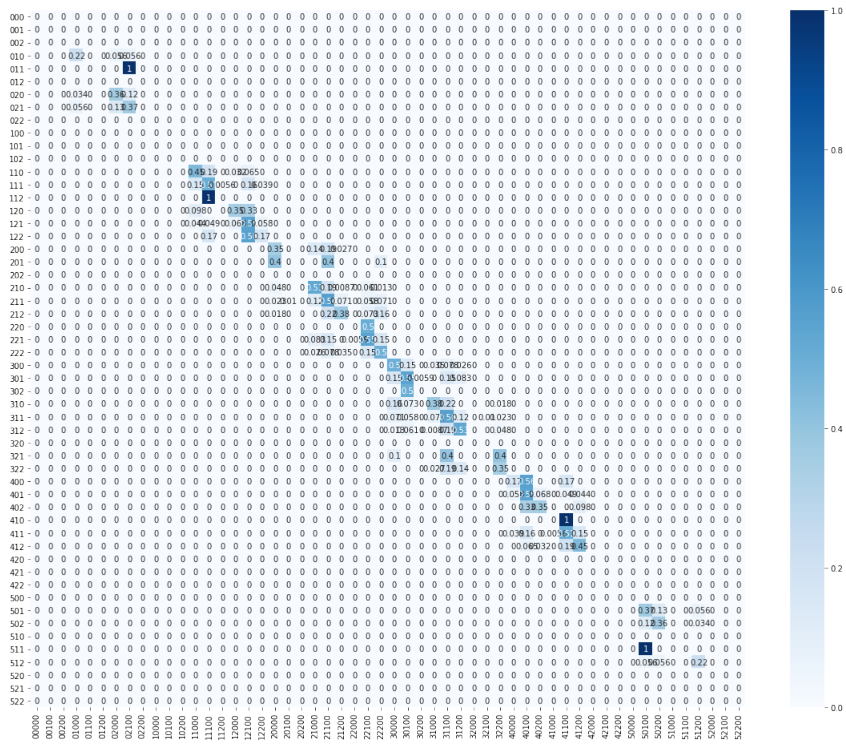

Figure 5: Transient matrix Q with 54×54 dimension