Bott periodicity theorem and Morse theory

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1 Basic statement and some lemmas

This reading report most follows the last chapter in Milnor's *Morse Theory*. And I would like to talk something about Bott periodicity theorem, i.e. $\pi_{i-1}U \cong \pi_{i+1}U$ for $i \geq 1$.

A simple observation induces the **definition** of $\pi_i U$. For each m, theres exists a fibration

$$U(m) \to U(m+1) \to S^{2m+1}.$$

This is a fibration because $U(m+1)$ acts transitively on $S^{2m+1} = \{v \in \mathbb{C}^{m+1} | \ \|v\|^2 = 1\}$ with the stabalizer $U(m)$. From the homotopy long exact sequence

$$\cdots \to \pi_i S^{2m+1} \to \pi_{i-1}U(m) \to \pi_{i-1}U(m-1) \to \pi_{i-1}S^{2m+1} \ldots$$

of this fibration we could see that

$$\pi_{i-1}U(m) \cong \pi_i U(m), \quad \text{for } i \leq 2m.$$

Since m is arbitrary, we have isomorphisms

$$\pi_{i-1}U(m) \cong \pi_i U(m) \cong \pi_i U(m+1) \cong \cdots, \quad \text{for } i \leq 2m.$$

These isomorphisms are all induced by the inclusion homomorphism. From this **stable phenomenon**, we will let $\pi_{i-1}U \triangleq \pi_{i-1}U(m) \quad \text{for } i \leq 2m$.

Now, we need two **facts**

$$\pi_i SU(m) \cong \pi_i U(m), \quad \text{for } i \neq 1 \tag{1}$$

$$\pi_{i-1}U(m) \cong \pi_i G_m(\mathbb{C}^{2m}), \quad \text{for } 1 \leq i \leq 2m \tag{2}$$

Here, $G_m(\mathbb{C}^{2m})$ is the complex Grassmannian manifold, consisting of all m dimensional vector spaces of $\mathbb{C}^{2m}$. The isomorphism (1) quickly follows from the fibration $SU(m) \to U(m) \to S^1$ and its homotopy long exact sequence. For the isomorphism (2), by $G_m(\mathbb{C}^{2m}) \cong \frac{U(2m)}{U(m)\times U(m)}$(Consider that $U(2m)$ acts on $m-dimensional$ subspaces of $\mathbb{C}^{2m}$ and the stabalizer), we could consider two fibrations $U(m) \to U(2m) \to U(2m)/U(m)$, which will give us $\pi_i(U(2m)/U(m)) = 0$ for $i \leq 2m$ (for $i = 2m$, use the fact that $\pi_{2m}U(m) \to \pi_{2m}U(m+1) \cong \pi_{2m}U(2m)$ is onto), and $U(m) \to U(2m) \to G_m\mathbb{C}^{2m}$, from which we could obtain the isomorphism (2).

From now, by these two facts, our efforts will **focus on** proving

$$\pi_i G_m(\mathbb{C}^{2m}) \cong \pi_{i+1} SU(2m)$$

for $i \leq 2m$. Bott states that the inclusion $G_m(\mathbb{C}^2 m) \hookrightarrow \Omega(SU(2m); I, -I)$ induces this above isomorphism. Here, $\Omega(SU(2m); I, -I)$ represents the piecewise smooth path space of $SU(2m)$ from I to $-I$. We build three lemmas to prove this statement.

Lemma 1 *The space of minimal geodesic from I to $-I$ in $SU(2m)$ is hemeomorphic to the complex Grassmannian manifold $G_m(\mathbb{C}^{2m})$.*

Lemma 2 *Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$.*

Actually, it is naturally to consider **Lemma 1** since the minimal geodesics connecting two particular points might be natural elements to discuss once we get a manifold, even if we do not know anything about Morse theory. Then, we are supposed to prove these two lemmas because of the following **Lemma 3**, in which we will see the real power of Morse theory.

Lemma 3 *Let M be a complete Riemannian manifold, and let $p, q \in M$ be two points with distance $\rho(p,q) = \sqrt{d}$. If the space Ω^d(all piecewise smooth paths with enerygy $\leq d$) of minimal geodesics from p to q is a topological manifold, and if every non-minial geodesic from p to q has index $\geq \lambda_0$, then $\pi_i(\Omega, \Omega^d)$ is zero for $0 \leq i < \lambda_0$.*

It is **hard** to recognize and remember these statements at the first glancing, especially the Lemma 3. But we could consider some more concrete examples to verify and recognize the consequence of **Lemma 3**.

The Freudenthal suspension theorem *The homotopy group $\pi_i(S^n)$ is isomorphic to $\pi_{i+1}(S^{n+1})$ for $i \leq 2n-2$.*

This is a direct consequence of **Lemma 3**. Take p, q be antipodal points(say, the North and South Pole) in S^{n+1}. Then, the space Ω^{π^2}(all piecewise smooth paths with enerygy $\leq \pi^2$) is the space of all minimal geodesics(say, the parallel) from p to q, which could also be identified with the equator $S^n \subset S^{n+1}$. The non-minimal geodesics from $p \to q$ must have idex $\geq 2 \times (n+1-1)$ since a non-minimal geodesic might be in order $p \to q \to p \to q$ along parallel, which winds one and a half times around S^{n+1}. Then, by the homotopy long exact sequence and **Lemma 3**, we have $\pi_i(S^n) \cong \pi_i(\Omega^{\pi^2}) \cong \pi_i(\Omega(S^{n+1}; p, q)) \cong \pi_{i+1}(S^{n+1})$ for $i \leq 2n-2$.
In order to verify the consequence of **Lemma 3** in this case, there is **another explanation** of this theorem. By a result in Morse theory, we could see that the loop space ΩS^{n+1} has the homotopy type of a CW-complex with one cell each in the dimension $0, n, 2n, 3n, \ldots$. By Sard's theorem, we could get that every $h : S^i \to \Omega S^{n+1}, i \leq 2n-2$ is homotopic to a map $h' : S^i \to S^n \subset \Omega S^{n+1}$ and it induces an isomorphism $\pi_i(S^n) \cong \pi_i(\Omega S^{n+1}), i \leq 2n-2$, from which we could also get the **Freudenthal suspension theorem** and verify the consequence of **Lemma 3** in this case by applying homotopy long exact sequence

If we don't choose antipodal points, we could also get a similar result from the process of the above alternative explanation and verify the conclusion of **Lemma 3** in this case.

2 Sketchy proof of lemmas

Now, let me show some basic ideas about these three lemmas.

Sketchy proof of Lemma 1: Consider the set of all geodesics in $SU(2m)$ from I to $-I$, i.e. look for all $A \in \mathrm{T}_I SU(2m)$ such that $exp(A) = -I$. Since $exp(TAT^{-1}) = exp(A) = -I$ for $\forall\ T \in U(2m)$, we could assume that A is in diagonal form first. Then

$$A = \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_n \end{pmatrix},$$

from which we could get that $a_j = k_j i\pi$ for some odd integers $k_1, \ldots, k_{2m}$. In the minimal geodesic case with $trace(A) = 0$, since the length of the geodesic $t \to exp(tA)$ from $t = 0$ to $t = 1$ is $\|A\| = \sqrt{trace(AA^*)}$, k_j must be ± 1 and $a_1 + a_2 + \cdots + a_{2m} = 0$, which means that the number of 1 in $\{k_j\}$ equals to the number of -1 in $\{k_j\}$. Hence, once we assign an arbitrary m dimensional sub-vector-space of $\mathbb{C}^{2m}$ to the eigenspace of A corresponding to eigenvalue $i\pi$, we get an A satisfying the above requirements about minimal geodesics. Thus, we get the Lemma 1.

Sketchy proof of Lemma 2: The Lie algebra

$$\mathfrak{g}' = \mathrm{T}_I SU(2m)$$

consists of all $2m \times 2m$ skew-Hermitian matrices with trace zero. A matrix $A \in \mathfrak{g}'$ corresponds to a geodesic γ from I to $-I$ if and only if the eigenvalues of A have the form $i\pi k_1, \ldots, i\pi k_{2m}$ are odd zeros with sum zero, which is from the idea about Lemma 1. We are supposed to compute the index of any non-minimal geodesic from I to $-I$ on $SU(2m)$ now, for which we need a **Lemma 4** about symmetric space.

Lemma 4 *$\gamma : \mathbb{R} \to M^n$ is a geodesic in a symmetric manifold. $V = \frac{d\gamma}{dt}(0)$ is the velocity vector at $p = \gamma(0)$. Define*

$$K_V : T_pM \to T_pM, W \mapsto R(V, W)V,$$

whose eigenvalues are $e_1, \ldots, e_n$.
Then, the conjugate points to p along γ are the points $\gamma(\pi k/\sqrt{e_i})$ with non-zero integer k and positive eigenvalus e_i of K_V. The multiplicity of $\gamma(t)$ as a conjugate point is equal to the number of e_i such that t is a multiple of $\pi/\sqrt{e_i}$ (that is, t might equal to $\pi k_1/\sqrt{e_1}$ and $\pi k_2/\sqrt{e_2}$).

Proof. Firstly, K_V is self-adjoint from

$$< R(V, W_1)V, W_2 >=< R(V, W_2)V, W_1 >$$

Therefore, at $p = \gamma(0)$, we could choose an orthonormal basis $U_1, \ldots, U_n$ in T_pM so that

$$K_V(U_i) = e_i U_i.$$

Then extend U_i along γ by parallel transportation. We have that $R(V, U_i)V = e_i U_i$ along γ since $R(V, U_i)V$ is also parallel vector fields along γ. Then, any vector field W along γ is in a form

$$W(t) = w_1(t)U_1(t) + \cdots + w_n(t)U_n(t).$$

If W is a Jacobi field, then $\frac{D^2W}{dt^2} + K_V(W) = 0$, from which we could get that

$$\sum(\frac{d^2 w_i}{dt^2} + e_i w_i)U_i = 0.$$

Hence, in $e_i > 0$ and $w_i(0) = 0$ cases,

$$w_i(t) = c_i sin(\sqrt{e_i} t), \text{ for some constant } c_i.$$

The zeros of w_i are the multiples of $\pi/\sqrt{e_i}$.
If $e_i \leq 0$, w_i only vanishes at $t = 0$. □

Thanks to Lemma 4, we only need to analyze the eigenvalues of

$$K_A : \mathfrak{g}' \to \mathfrak{g}'$$

where

$$K_A(W) = R(A, W)A = -\frac{1}{4}[A, [A, W]].$$

The last equation follows from the fact that $< [X, Y], Z >=< X, [Y, Z] >$ in Lie groups with left and right Riemannian metric.
Let $W = (w_{ij}) \in \mathfrak{g}'$. Since $K_{TAT^{-1}}(W) = TK_A(TWT^{-1})T^{-1}$ for $T \in U(2m)$, we could assume that A is a diagonal matrix in favor of computing eigenvalues. Assume

$$A = \begin{pmatrix} i\pi k_1 & & \\ & \ddots & \\ & & i\pi k_{2m} \end{pmatrix},$$

Then $K_A(W) = (\frac{\pi^2}{4}(k_i - k_j)^2 w_{ij})$. Then, by Lemma 4 and be careful to compute each $\frac{\pi^2}{4}(k_i - k_j)^2$(These are exact eigenvalues!), we could get

$$\lambda = \sum_{k_i > k_j} 2(\frac{k_i - k_j}{2} - 1)$$

for the index of the geodesic.
By careful but easy computation in non-minimal geodesic cases, we could get that their index $\lambda \geq 2(m + 1)$. (In case of minimal geodesic, since all k_i must be in $\{\pm 1\}$, this formula gives that $\lambda = 0$, as was to be expected.)

Sketchy proof of Lemma 3: A brief consideration is that there is a *correspondence* between *index* and *cell* according to the *Fundamental theorem of Morse theory*, by which those non-minimal geodesics, whose index $\geq \lambda_0$, correspond to cells with dimension $\geq \lambda_0$. Then, $\pi_i(\Omega, \Omega^d)$ almost equals to 0 for $0 \leq i < \lambda_0$, because the relative homotopy is a kind of *quotient*, which means that only non-minimal geodesics and their correspondent cells *remain* in the space.

For technique details, there are several facts that are supposed to prove. And I give some rough ideas.

(1) *It is sufficient to prove that* $\pi_i(\mathrm{Int}\Omega^c, \Omega^d) = 0$ *for arbitrarily large* c.
This follows from the proof of *Fundamental theorem of Morse theory*, which construct an ascending chain $\Omega^{a_0} \subset \Omega^{a_1} \cdots \subset \Omega$. Then we take the direct limit to get $\pi_i(\Omega, \Omega^d) = 0$.

(2) $\mathrm{Int}\Omega^c$ *contains a smooth manifold* $\mathrm{Int}\Omega^c(t_0, t_1, \ldots, t_k)$ *(piecewise geodesic space) as a deformation retract.*
Broken geodesics are uniquely determined by those *cut points*.

(3) *Let* $f : M \to R$*. It is possible to choose a smooth function* $g : M^{\tilde{c}} \to R$ *for some* $\tilde{c}$ *such that* g *approximates* f *closely and approximates* f*'s first and second derivatives. Also,* g *has no degenerate critical points and all critical points of* g *in a suitable compact subset have index* $\geq \lambda_0$

These three facts are basic consequences about Morse theory. And there is another result following from Sard's theorem

(4) *Attach a* k*-cell to* M *with* $k \geq \lambda_0$ *by a map* σ*. Then, the map* $h : S^r \to M \cup_\sigma e^k$ with $r \leq \lambda - 1$ is homotopic to a map $h' : S^r \to M$.

Now, with the help of these four facts, we could manage to get the Lemma 3. Finally, with all these tough but deserved works, we are able to get the amazing Bott periodicity theorem.

By the way, I do not recognize the real power of Morse theory until I complete this reading report. When I first read the **Lemma 3**, I didn't even try to understand it but just verified it and just regarded it as an ordinary lemma. When I wrote this reading report, I reread the **Lemma 3** and the **Freudenthal suspension theorem** again and tried to understand them. And now, I could see the geometry(or to say, topology) meaning of this lemma and this theorem. But even if I read somethings and wrote somethings about Milnor's book, I think there are much more about Morse theory and other relative theories (like Lie group theory) for me to discover and study.

References

[1] J.Milnor, *Morse Theory*, Princeton University Press, (1973), ISBN 0-691-08008-9

[2] Raoul Bott, Loring W.Tu, *Differential Forms in Algebraic Topology*, Springer, (1982), ISBN 0-387-90613-4