# Construction of the $G_2$ algebra

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This reading report most follows contents in Humphreys's *Introduction to Lie Algebras and Representation Theory*. Our focuses are on the constructions of $G_2$ algebra, whose **Cartan matrix** is:

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

And I filled some details which are not in this book.

## 1 Criterion for semisimplicity

We begin this reading report with a criterion for reductive Lie algebra's semisimplicity, while it is well-known that we could compute the **Killing form** of an arbitrarily given Lie algebra to test its semisimplicity in general. But for **reductive** Lie algebra, which means that Rad $L$, the maximal solvable ideal of $L$, equals to the center $Z(L)$, we have a result that $L = [LL] \oplus Z(L)$ with $[LL]$ semisimple. Hence, once we could show that $Z(L) = 0$, the $L$ is semisimple automatically.

**Proposition.** (a)*Let $L$ be reductive. Then $L = [LL] \oplus Z(L)$, and $[LL]$ is semisimple.*

(b)*Let $L \subset \mathfrak{gl}(V)$($V$ finite dimensional) be a nonzero Lie algebra acting irreducibly on $V$. Then $L$ is reductive, with dim $Z(L) \leq 1$. If in addition $L \subset \mathfrak{sl}(V)$, then $L$ is semisimple.*

*Proof.* (a) If $L$ is abelian, then the conclusion is trivial. Hence, we could suppose that $L$ is reductive but not abelian, so $L' = L/Z(L) = L/\text{Rad } L$ is semisimple by definition. Then ad $L \cong$ ad $L' \cong L'$ acts completely reducibly on $L$ by **Weyl's theorem**, which states that a finite dimensional representation of a semisimple Lie algebra is completely reducible. Hence, $Z(L)$, an $L-$submodule of $L$, has an $L-$submodule complement $M$, which also means that $M$ is an ideal of $L$. Write $L = M \oplus Z(L)$. Then, $[LL] = [MM] \subset M$. But $[LL]$ maps onto $L'$ by the canonical map, so $L = [LL] \oplus Z(L)$ (Or we can use the fact that $[MM] = M$ since $M \cong L/Z(L) \cong L'$ is semisimple, then $[LL] = [MM] = M$).
(b) We briefly denote Rad $L$ as $S$. Because $S$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$, according to **Lie's theorem**, $S$ has a common eigenvector $v$ in $V$, say $s.v = \lambda(s)v$ ($s \in S$) with $\lambda$ linear. We want to show that $S$ acts diagonally on $V$ as the scalar $\lambda(s)$ to get dim $S \leq 1$. Since $L$ acts irreducibly on $V$, all vectors in $V$ are obtained by repeated application of elements in $L$ and their linear combinations. Hence, $V$ has a basis with the form

$$E_0 \cup E_1 \cup E_2 \cdots \cup E_t$$

with $E_0 = \{v\}$ and

$$E_i = \{x_{i1,1}.x_{i1,2}.\dots.x_{i1,i}.v,\ x_{i2,1}.x_{i2,2}.\dots.x_{i2,i}.v,\ \dots\ ,\ x_{is_i,1}.x_{is_i,2}.\dots.x_{is_i,i}.v\}$$

for $i = 1, 2, \dots, t$. We could also require this basis equipping the property that $\forall w \in V$ with a form $w = y_1.y_2....y_l.v, l \leq t$, $w$ could also be expressed as a linear combination of vectors in $E_0 \cup \dots \cup E_l$. Hence, we only need to consider the action of $S$ on this basis. Clearly, since $S$ is an ideal, for any $x \in L$, $[x, s] \in S$ (here, $[\cdot, \cdot]$ is the canonical lie bracket of $\mathfrak{gl}(V)$ ), then

$$s.x.v = \lambda(s)x.v + \lambda([s, x]).v$$

By this formula, we could conclude that the matrices relative to the above basis of all $s \in S$ are triangular, with the only diagonal entry $\lambda(s)$(This could be verified by induction since we have chosen a basis $E_0 \cup \dots \cup E_t$ with above properties, for example, $s.(x.(y.v)) = x.(s.(y.v)) + [sx].(y.v) = \lambda(s)(x.(y.v)) + \lambda([sy])x.v + \lambda([sx])y.v + \lambda([[sx]y])v$ and $x.v, y.v$ are linear combinations of vectors in $E_0 \cup E_1$).Nevertheless, the commutators $[SL] \subset S \subset \mathfrak{gl}(V)$ have trace zero, which forces $\lambda$ to vanish on $[SL]$. Hence, by the above formula, $S \subset \mathfrak{gl}(V)$ acts diagonally on $V$ as the scalar $\lambda(s)$, then dim $S \leq 1$ and $S = Z(L)$ by $Z(L) \subset S \subset Z(\mathfrak{gl}(V)) \cap L \subset Z(L)$. Hence $L$ is reductive.
If $L \subset \mathfrak{sl}(V)$, since the only diagonal matrix in $\mathfrak{sl}(V)$ is 0, then $S = 0$, and $L = [LL]$ is semisimple. □

**Remark:** The technique about **trace** in (b) is also used in proving the **Lie's Theorem** in section 4 of Humphreys's book.

## 2 Explicit construction of algebra $G_2$

In general, once we have a fixed root system $\Phi$, with base $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$, according to **Serre's Theorem**, we could construct a semisimple Lie algebra with corresponding root system $\Phi$ following some relations. In the irreducible root system cases, say $G_2$, this Lie algebra is automatically a simple Lie algebra by the correspondence between the direct sum decomposition of a semisimple Lie algebra and irreducible components of its root system.

In Humphreys's book, there are two constructions of $G_2$. The first is to regard $L$, the simple algebra of type $G_2$, as a subalgebra of $L_0 = \mathfrak{o}(7)$, the simple algebra of type $B_3$. In this way, it just assigns the corresponding root vectors of $L$ to some vectors in $L_0$ and verifies that $L$ is a 14−dimensional Lie algebra. Then, just directly verify that $L$ acts irreducibly on $V = \mathbb{F}^7$ and use the **criterion of semisimplicity** above to show L is semisimple and type $G_2$. The second, which is a more interesting way, is to regard it as the Lie algebra Der $\mathfrak{C}$, where $\mathfrak{C}$ is an 8−dimensional nonassociative algebra and called the octonion algebra. I would like to present the **idea** about the second construction.

Before our construction, we are supposed to describe this $\mathfrak{C}$ first. Let $\{e_1, e_2, e_3\}$ be the usual standard orthonormal basis of $\mathbb{F}^3$ (in most cases of Humphreys's book, we could just regard $\mathbb{F}$ as $\mathbb{C}$) and the usual inner product for $u = u_1e_1 + u_2e_2 + u_3e_3, v = v_1e_1 + v_2e_2 + v_3e_3$ in $\mathbb{F}^3$ is

$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$. It is clear that we could also define a cross product in $\mathbb{F}^3$:

$$u \times v = det \begin{bmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Then, as a vector space, $\mathfrak{C}$ is the sum of two copies of $\mathbb{F}^3$ and two copies of $\mathbb{F}$. For convenience, we could write elements in $\mathfrak{C}$ as $2 \times 2$ matrices in the form:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix}, a, b \in \mathbb{F} \text{ and } v, w \in \mathbb{F}^3$$

We add and multiply scalars just as we would for matrices. However, the product in $\mathfrak{C}$ is given by a little complicate formula:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' - v \cdot w' & av' + b'v + w \times w' \\ a'w + bw' + v \times v' & bb' - w \cdot v' \end{bmatrix}$$

This product is obviously bilinear. Then, fix a basis $\{c_1, \ldots, c_8\}$:

$$c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ c_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ c_{2+i} = \begin{bmatrix} 0 & e_i \\ 0 & 0 \end{bmatrix}, \ c_{5+i} = \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix}$$

By careful computation, we could get the multiplication table of $\mathfrak{C}$: See the table above.

Table 1: Multiplication Table of $\mathfrak{C}$

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ | $c_8$ |
|---|---|---|---|---|---|---|---|---|
| $c_1$ | $c_1$ | $0$ | $c_3$ | $c_4$ | $c_5$ | $0$ | $0$ | $0$ |
| $c_2$ | $0$ | $c_2$ | $0$ | $0$ | $0$ | $c_6$ | $c_7$ | $c_8$ |
| $c_3$ | $0$ | $c_3$ | $0$ | $c_8$ | $-c_7$ | $-c_1$ | $0$ | $0$ |
| $c_4$ | $0$ | $c_4$ | $-c_8$ | $0$ | $c_6$ | $0$ | $-c_1$ | $0$ |
| $c_5$ | $0$ | $c_5$ | $c_7$ | $-c_6$ | $0$ | $0$ | $0$ | $-c_1$ |
| $c_6$ | $c_6$ | $0$ | $-c_2$ | $0$ | $0$ | $0$ | $c_5$ | $-c_4$ |
| $c_7$ | $c_7$ | $0$ | $0$ | $-c_2$ | $0$ | $-c_5$ | $0$ | $c_3$ |
| $c_8$ | $c_8$ | $0$ | $0$ | $0$ | $-c_2$ | $c_4$ | $-c_3$ | $0$ |

Notice that $c_1 + c_2$ acts as identity in this table and check that $\mathfrak{C}_0 = [\mathfrak{C}, \mathfrak{C}]$(here $[c, c'] = cc' - c'c$ for $c, c' \in \mathfrak{C}$) is a subspace with codimension 1 and is spanned by the basis $\{c_1 - c_2, c_3, c_4, \ldots, c_8\}$. By the product rule, any derivation of $\mathfrak{C}$ kills the multiples of $c_1 + c_2$ ($\delta(x) = \delta((c_1 + c_2)x) = \delta(x) + \delta(c_1 + c_2)x$, for all $x \in \mathfrak{C}$). Moreover, a derivation leaves $\mathfrak{C}_0$ invariant since $\delta([c, c']) = \delta(c)c' + c\delta(c') - \delta(c')c - c'\delta(c) = [\delta(c), c'] + [c, \delta(c')] \in \mathfrak{C}_0$.

Now, set $L = \text{Der}\ \mathfrak{C}$. Obviously, $L$ acts faithfully on $\mathfrak{C}_0$ because if $x \in L$ acts trivially on $\mathfrak{C}_0$, we will get that $x$ acts trivially on the whole $\mathfrak{C}$ since any derivation kills $c_1 + c_2$ and $\mathfrak{C}_0$ is spanned by $\{c_1 - c_2, c_3, c_4, \ldots, c_8\}$. Denote by $\phi : L \to \mathfrak{gl}(7, \mathbb{F})$ the associated matrix representation with the basis of $\mathfrak{C}_0$ chosen as above. The idea here is to exhibit some derivations of $\mathfrak{C}$ clearly. As the long roots in the root system of $G_2$ form a system of type $A_2$, $L$ contain a copy of $\mathfrak{sl}(3, \mathbb{F})$. Then, for $x \in \mathfrak{sl}(3, \mathbb{F})$, set:

$$\delta : \mathfrak{sl}(3, \mathbb{F}) \to \text{End}(\mathfrak{C}) \quad x \mapsto \delta(x)$$

with

$$\delta(x)(\begin{bmatrix} a & v \\ w & b \end{bmatrix}) = \begin{bmatrix} 0 & x(v) \\ -x^t(w) & 0 \end{bmatrix}$$

By the fact that $\det(x(v_1), v_2, v_3) + \det(v_1, x(v_2), x_3) + \det(v_1, v_2, x(v_3)) = 0$ for $v_1, v_2, v_3 \in \mathbb{F}^3$ and $x \in \mathfrak{sl}(3, \mathbb{F})$ (this fact is also **true in higher dimension** and the key to prove it is that the trace of $x$ is zero and then verify it by choosing the standard basis of $\mathfrak{sl}(n, \mathbb{F})$ and $\mathbb{F}^n$ for higher $n$), we could prove that these $\delta(x)$ are actually in $\mathrm{Der}\mathfrak{C}$. Also, this representation of $\mathfrak{sl}(3, \mathbb{F})$ is actually nontrivial according to this form.

However, in order to construct the whole $L$ which is semisimple, the above representation about $\mathfrak{sl}(3, \mathbb{F})$ is not enough because we want to use the **criterion of semisimplicity** before,for which we are supposed to show that $\mathrm{Der}\mathfrak{C}$ acts on $\mathfrak{C}_0$ irreducibly while $\mathfrak{sl}(3, \mathbb{F})$ kills $c_1 - c_2$. Hence, we need to locate more derivations to show the irreducibility.
Fortunately, we do not need to locate too much derivations thanks to the representation of $\mathfrak{sl}(3, \mathbb{F})$. For example, if we let $w = w_1e_1 + w_2e_2 + w_3e_3,\ v = v_1e_1 + v_2e_2 + v_3e_3$, then:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \mapsto \begin{bmatrix} 0 & v_1e_1 - v_2e_2 \\ -w_1e_1 + w_2e_2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & (v_1 - v_2)e_1 \\ (w_1 - w_2)e_2 & 0 \end{bmatrix}$$

by setting $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in $\delta(x)$. Hence, the only trouble is that $\mathfrak{sl}(3, \mathbb{F})$ will kills $c_1 - c_2$ while we could choose elements in $\mathfrak{sl}(3, \mathbb{F})$ and get $c_3, \ldots, c_8$ in above way. Now, we need a lemma:

**Lemma.** *In any algebra $\mathfrak{A}$ satisfying the alternative laws, i.e. $x^2y = x(xy),\ yx^2 = (yx)x$, an endomorphism of the following form is actually a derivation: $[L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ ($a, b \in \mathfrak{A}$, $L_a$ = left multiplication by $a$, $R_a$ = right multiplication by $a$)*

*Proof.* Let $x = a + b$, $y = c$, we could get an equivalent statement of the alternative laws:

$$(ab)c + (ba)c = a(bc) + b(ac) \quad \text{and} \quad c(ba) + c(ab) = (cb)a + (ca)b$$

Denote $(xy)z - x(yz)$ as $[x, y, z]$, which is called associator, then we show this lemma by several steps:
**(1)** $[L_a, L_b] = L_{[a,b]} - 2[L_a, R_b]$ and $[R_a, R_b] = -R_{[a,b]} - 2[L_a, R_b]$.
For any $x \in \mathfrak{U}$, $[L_a, L_b](x) = a(bx) - b(ax)$, $L_{[a,b]} - 2[L_a, R_b](x) = ((ab) - (ba))x - 2(a(xb)) + 2((ax)b)$, then

$$\begin{aligned}
&[L_a, L_b](x) = L_{[a,b]} - 2[L_a, R_b](x) \\
\Longleftrightarrow\ & a(bx) - b(ax) = ((ab) - (ba))x - 2(a(xb)) + 2((ax)b) \\
\Longleftrightarrow\ & (ab)x + (ax)b - a(xb) - b(ax) = ((ab) - (ba))x - 2(a(xb)) + 2((ax)b) \\
\Longleftrightarrow\ & -b(ax) = -(ba)x - a(xb) + (ax)b \\
\Longleftrightarrow\ & a(bx) - (ab)x = -a(xb) + (ax)b \\
\Longleftrightarrow\ & a(bx) - (ax)b = (ab)x - a(xb) \\
\Longleftrightarrow\ & a(bx) - (ax)b = -(ax)b + a(xb) + a(bx) - a(xb)
\end{aligned}$$

The $[R_a, R_b] = -R_{[a,b]} - 2[L_a, R_b]$ is similar.
**(2)** $L_a(xy) = ((L_a + R_a)x)y - x(L_a y)$ and $R_a(xy) = -(R_a x)y + x((L_a + R_a)y)$ for any $x, y \in \mathfrak{A}$.

$$\begin{aligned} & L_a(xy) = ((L_a + R_a)x)y - x(L_a y) \\ \Longleftrightarrow & a(xy) = (ax + xa)y - x(ay) \\ \Longleftrightarrow & a(xy) + x(ay) = (ax + xa)y \end{aligned}$$

The $R_a(xy) = -(R_a x)y + x((L_a + R_a)y)$ is similar.
**(3)** $(L_a - R_a)(xy) = ((L_a - R_a)x)y + x((L_a - R_a)y) + [x, 3a, y]$.
According to (2), $(L_a - R_a)(xy) = ((L_a - R_a)x)y + 2(R_a x)y - x(L_a y) + x((L_a - R_a)y) - 2x(L_a y) + (R_a x)y$, and $(R_a x)y - x(L_a y) = [x, a, y]$ by definition.
**(4)** $[L_a, R_b](x) = [a, b, x] = (ab)x - a(bx) = b(ax) - (ba)x = x(ba) - (xb)a$.
This is quickly from $[L_a, R_b](x) = -[a, x, b] = [a, b, x] = -[b, a, x] = -[x, b, a]$ by the alternativity laws.
**(5)** The **left Moufang identity**, i.e. $((xy)x)z = x(y(xz))$.
According to (4), $((xy)x)z - x(y(xz)) = ((xy)x)z - (xy)(xz) + (xy)(xz) - x(y(xz)) = [xy, x, z] + [x, y, xz] = -[x, xy, z] - [x, xz, y] = -(x^2 y)z + x((xy)z) - (x^2 z)y + x((xz)y) = -x^2(yz + zy) + x(x(yz + zy)) = -x^2(yz + zy) + x^2(yz + zy) = 0$.
Moreover, let $x = a + b$, we could get the **linearized left Moufang identity**: $((ay)b)z + ((by)a)z = a(y(bz)) + b(y(az))$.
**(6)** $[L_a, R_b](xy) = ([L_a, R_b]x)y + x([L_a, R_b]y) + [x, [a, b], y]$.
According to (4), $[L_a, R_b](xy) = (ab)(xy) - a(b(xy))$,

$$(ab)(xy) = ((ab)x + x(ab))y - x((ab)y),$$

by the **linearized left Moufang identity**,

$$a(b(xy)) = ((ab)x)y + ((xb)a)y - x(b(ay)),$$

then, according to (4),

$$\begin{aligned} [L_a, R_b](xy) &= (x(ab) - (xb)a)y - x((ab)y - b(ay)) \\ &= (x(ba) - (xb)a)y + (x([a, b]))y + x(b(ay) - (ba)y) - x(([a, b])y) \\ &= ([L_a, R_b]x)y + x([L_a, R_b]y) + [x, [a, b], y] \end{aligned}$$

**(7)** $\delta = [L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ is a derivation.
We could rewrite $\delta$ as $L_{[a,b]} - R_{[a,b]} - 3[L_a, R_b]$ by (1), then, by (2),

$$(L_{[a,b]} - R_{[a,b]})(xy) - ((L_{[a,b]} - R_{[a,b]})x)y - x((L_{[a,b]} - R_{[a,b]})y) = 3[x, [a, b], y],$$

by (6),

$$3[L_a, R_b](xy) - 3([L_a, R_b]x)y - 3x([L_a, R_b]y) = 3[x, [a, b], y],$$

subtracts these two equations, we get $\delta(xy) - \delta(x)y - x\delta(y) = 3[x, [a, b], y] - 3[x, [a, b], y] = 0$ $\square$

Now, we show that $\mathfrak{C}$ satisfies the alternative laws $x^2 y = x(xy)$ and $yx^2 = (yx)x$ for any

$x, y \in \mathfrak{C}$. For $x^2y = x(xy)$,

$$\begin{aligned}
&\begin{bmatrix} a & v \\ w & b \end{bmatrix}^2 \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} \\
&= \begin{bmatrix} (a^2 - v\cdot w)a' - (a+b)v\cdot w' & (a^2 - v\cdot w)v' + b'(a+b)v + (a+b)w \times w' \\ a'(a+b)w + (b^2 - v\cdot w)w' + (a+b)v \times v' & -(a+b)w\cdot v' + b'(b^2 - v\cdot w) \end{bmatrix} \\
&= \begin{bmatrix} a & v \\ w & b \end{bmatrix} \left( \begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} \right)
\end{aligned}$$

And $yx^2 = (yx)x$ is similar.
Hence, by the above lemma, endomorphisms of the form $[L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ for $a, b \in \mathfrak{C}$ are in $\mathrm{Der}\mathfrak{C}$. Now, we could consider the irreducibility. For example, we could consider $\delta_{13} = [L_1, L_3] + [L_1, R_3] + [R_1, R_3]$ (here, for simplicity, we denote $L_{c_i}$ as $L_i$ and $R_{c_i}$ as $R_i$ for $i = 1, 2, \ldots, 8$, $\{c_i\}$ is the basis we chose earlier), and let $w = w_1e_1 + w_2e_2 + w_3e_3,\ v = v_1e_1 + v_2e_2 + v_3e_3$, then:

$$L_3\left(\begin{bmatrix} a & v \\ w & b \end{bmatrix}\right) = \left(\begin{bmatrix} -w_1 & be_1 \\ v_2e_3 - v_3e_2 & 0 \end{bmatrix}\right) \quad , \quad R_3\left(\begin{bmatrix} a & v \\ w & b \end{bmatrix}\right) = \begin{bmatrix} 0 & ae_1 \\ v_3e_2 - v_2e_3 & -w_1 \end{bmatrix}$$

$$L_1\left(\begin{bmatrix} a & v \\ w & b \end{bmatrix}\right) = \begin{bmatrix} a & v \\ 0 & 0 \end{bmatrix} \quad , \quad R_1\left(\begin{bmatrix} a & v \\ w & b \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ w & 0 \end{bmatrix}$$

$$\delta_{13}\left(\begin{bmatrix} a & v \\ w & b \end{bmatrix}\right) = \begin{bmatrix} -w_1 & (b-a)e_1 \\ v_3e_2 - v_2e_3 & w_1 \end{bmatrix} \in \mathfrak{C}_\circ$$

From this example, we could see that the combination of $\delta_{ij}, i = 1, 2, j \geq 3$ and the action of $\mathfrak{sl}(3, \mathbb{F})$ could actually show the irreducibility of the action of $\mathrm{Der}\mathfrak{C}$ on $\mathfrak{C}_\circ$. Thanks to the **criterion of semisimplicity**, $\mathrm{Der}\mathfrak{C}$ is **semisimple**.

Now, back to the faithful representation of $\mathfrak{sl}(3, \mathbb{F})$ we constructed earlier, and $L = \mathrm{Der}\mathfrak{C}$. Denote the image in $L$ of $\mathfrak{sl}(3, \mathbb{F})$ as $M$ and the image of the diagonal subalgebra of $\mathfrak{sl}(3, \mathbb{F})$ as $H$. It could be directly verified that the matrices in $\mathfrak{gl}(8, \mathbb{F})$ commuting with $H$ all have the form $\mathrm{diag}(x, a_3, a_4, a_5, a_6, a_7, a_8)$, where $x \in \mathfrak{gl}(2, \mathbb{F})$, with the basis $c_1, c_2, \ldots c_8$. Notice that such a matrix represents a derivation of $\mathfrak{C}$ only if it is already in $H$ (This fact could be showed by applying the multiplication table of $\mathfrak{C}$ showed earlier. Say, let $\delta = diag(x, a_3, a_4, a_5, a_6, a_7, a_8)$ be a derivation, by $\delta(c_1^2) = \delta(c_1)c_1 + c_1\delta(c_1)$ and $\delta(c_2^2) = \delta(c_2)c_2 + c_2\delta(c_2)$, $x$ must be $0 \in \mathfrak{gl}(2, \mathbb{F})$, and by $c_3, \ldots, c_8$ to get a relation between $a_3, \ldots, a_8$, which shows that $\delta$ must be in $H$).
Hence, $H$ is its own centralizer in $L$. Since the representation of $\mathfrak{sl}(3, \mathbb{F})$ we constructed is faithful, then $Z(M) = Z(\mathfrak{sl}(3, \mathbb{F})) = 0$, $Z(L) = Z(L) \cap H \subset (H \cap Z(M)) = 0$. Hence, $L = \mathrm{Der}\mathfrak{C}$ is **simple** according to the **criterion of semisimplicity**(We could also deduce $Z(L) = 0$ by facts that $L$ does not contain constant matrices other than 0 while $Z(L)$ consists of constant matrices as showed in the proof of the **criterion of semisimplicity**). And $H$, which is two dimensional, is exactly the **Cartan subalgebra** of $L = \mathrm{Der}\mathfrak{C}$. By our construction, $L \supsetneq \mathfrak{sl}(3, \mathbb{F})$, which is a system of type $A_2$, hence, $L$ could not be type $B_2$ and $A_2$, and the only possibility left is $G_2$.

# References


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