03/07/2018 Modern Mathematics

§1 Functional inequality

In $\mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) \mid \exists R > 0, \text{ s.t. } |f|(x) \equiv 0, \text{ for } |x| \geq R \}$

- Sobolev ineql.

$$\left( \int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq c \int_{\mathbb{R}^n} |\nabla f|(x) dx .$$

here. $|\nabla f|(x) = \sqrt{\sum_{i=1}^n |\partial x_i f|^2 (x)}$

$x = (x_1, \cdots x_n)$.

- Nash ineql.

$$\left( \int_{\mathbb{R}^n} |f|^2(x) dx \right)^{\frac{1}{2} + \frac{1}{n}} \leq c \left( \int_{\mathbb{R}^n} |\nabla f|^2(x) dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} |f|(x) dx \right)^{\frac{2}{n}}$$

- Log-Sobler ineql.

$\mu$ is a measure over $\mathbb{R}^n$.

$$\int_{\mathbb{R}^n} f^2(x) (\ln f)^2(x) d\mu \leq c \int_{\mathbb{R}^n} |\nabla f|^2(x) d\mu$$

- Poincare ineql.

$$\int_{\mathbb{R}^n} f^2(x) d\mu \leq c \int_{\mathbb{R}^n} |\nabla f|^2(x) dx .$$

§2 Gromov – Hausdorff metric (distance)

- $(X, d_X)$, $(Y, d_Y)$ metric spaces

$d_{GH}(X, Y) = \inf \cdots$

Let $\mathcal{M} = \{$All compact metric spaces$\}$

Thm. $(\mathcal{M}, d_{GH})$ is a metric space

- Riemannian mfld is a metric space
- Limits ~~of~~ under $d_{GH}$

Metric space with measure $\mu$ $(X, d, \mu)$.

① $(X, d)$ is a metric space

$$\begin{cases} d(x,y) = d(y,x) \geq 0 \\ d(x,y) = 0 \Leftrightarrow x = y \\ d(x,y) \leq d(x,z) + d(y,z) \end{cases} \quad \forall x, y, z \in X$$

② $\mu$, Borel outer measure, ~~normal~~ regular

1) outer measure
$$\begin{cases} \mu : 2^X \to [0, +\infty) \\ \mu(\phi) = 0 \\ \mu(A) \leq \mu(B) \text{ if } A \subseteq B \\ \mu\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty \mu(A_i) \end{cases}$$

2) Borel sets: $\sigma$-algebra generated by $B_r(x) \triangleq \{ y \in X \mid d(x,y) < r \}$ $(r > 0)$. and $B_r^c(x) = X \setminus B_r(x)$

- Borel sets are measurable: $A$ is a Borel set, then $\mu(A) = \mu(A \cap D) + \mu(A \cap D^c)$ for $\forall D \subset X$

3) regular, $\forall B \subset X$, $\exists A$, a Borel set s.t. $\mu(A) = \mu(B)$ and $B \subset A$

4) $\mu(B_r(x)) < +\infty$, $\forall x \in X$, $r > 0$

5) measurable functions: $f : X \to \mathbb{R}$, $\forall a \in \mathbb{R}$, $\{ x : f(x) \leq a \}$ is measurable

Lebesgue integral:

$f$ is measurable, def: $\forall t>0$

$$f^*(t) = \mu(\{x\in X \mid f(x)>t\})$$

if $f\geq 0$, then $\int_X f\,d\mu \triangleq \int_0^{+\infty} f^*(t)\,dt$

if $f = f^+ - f^-$, $f^+ = \max\{f, 0\}$

$f^- = \max\{-f, 0\}$

then $\int_X f\,d\mu \triangleq \int_X f^+\,d\mu - \int_X f^-\,d\mu$

- $L^p$ - functions: $f$ is measurable, $p>0$

$\int_X |f|^p\,d\mu < +\infty$, denoted as $L^p(X)$.

- Hölder ineql.

$f\in L^p(X)$, $g\in L^q(X)$, $p, q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int_X |fg|\,d\mu \leq \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q d\mu\right)^{\frac{1}{q}}$$

Proof: By Young equality:

$\forall a, b>0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

$$\forall \varepsilon,\ ab = \frac{a}{\varepsilon}\cdot \varepsilon b \leq \frac{a^p}{\varepsilon^p p} + \frac{\varepsilon^q b^q}{q}$$

$$\int_X |fg|\,d\mu \leq \frac{1}{\varepsilon^p p}\int_X |f|^p d\mu + \frac{\varepsilon^q}{q}\int_X |g|^q d\mu.$$

Let $\varepsilon = \|f\|_{L^p}^{\frac{1}{q}}\ \|g\|_{L^q}^{-\frac{1}{p}}$

$\square$

Cor: $f_i\in L^{p_i}$, $i=1,\cdots,k$, $p_i>1$ and $\sum_{i=1}^k \frac{1}{p_i}=1$, then

$$\int_X |f_1\cdots f_k|\,d\mu \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}}$$

Proof: When $k=2$, by Hölder.

When $k=n$, ineql hold.

Then, when $k=n+1$.

$$\int_X |f_1\cdots f_n|\cdot|f_{n+1}|\,d\mu$$

$$\leq \left(\int_X |f_1\cdots f_n|^{q_{n+1}} d\mu\right)^{\frac{1}{q_{n+1}}} \left(\int_X |f_{n+1}|^{p_{n+1}} d\mu\right)^{\frac{1}{p_{n+1}}}$$

$\left(q_{n+1} = \frac{p_{n+1}}{p_{n+1}-1}\right)$

By induction. $\sum_{i=1}^{n+1}\frac{1}{p_i}=1$

$\Rightarrow \sum_{i=1}^{n}\frac{q_{n+1}}{p_i} = 1$, then.

$$\int_X |f_1\cdots f_n|^{q_{n+1}} d\mu$$

$$\leq \prod_{i=1}^n \left(\int_X \left(|f_i|^{q_{n+1}}\right)^{\frac{p_i}{q_{n+1}}} d\mu\right)^{\frac{q_{n+1}}{p_i}}$$

So, $\int_X |f_1\cdots f_{n+1}|\,d\mu$

$$\leq \prod_{i=1}^{n+1}\left(\int_X |f_i|^{p_i} d\mu\right)^{\frac{1}{p_i}}$$

$\square$

Thm. (Sobolev ineql)

Suppose $n\geq 2$, $1\leq p<n$, $q = \frac{np}{n-p}$

Then, $\exists\, C = C(n,p)$, s.t.

$\forall f\in C_0^\infty(\mathbb{R}^n)$, $\|f\|_{L^q} \leq C\|\nabla f\|_{L^p}$

Proof: (Gagliardo – Nirenberg)

Suppose $p=1$, since $f(x)=0$ when $x\to+\infty$

$$f(x) = \int_{-\infty}^{x_i} \partial_{x_i} f(x_1,\cdots x_{i-1}, t, \cdots x_n)\,dt$$

$x=(x_1,\cdots,x_n)$

$$\Rightarrow |f(x)| \le \int_{-\infty}^{\infty} |\partial_{x_i} f|(x)\,dx_i$$

$$\Rightarrow |f(x)|^n \le \prod_{i=1}^n \int_{-\infty}^{\infty} |\partial_{x_i} f|(x)\,dx_i .$$

Since $|\partial_{x_i} f| \le |\nabla f|$.

$$|f(x)|^{\frac{n}{n-1}} \le \left(\prod_{i=1}^n \int_{-\infty}^{\infty} |\nabla f|(x)\,dx_i\right)^{\frac{1}{n-1}}$$

Let $D_i = \{-\infty < x_i < +\infty\}$

no $x_1$ variable!

$$\int_{D_1} |f(x)|^{\frac{n}{n-1}} dx_1 \le \left(\int_{D_1} |\nabla f|\,dx_1\right)^{\frac{1}{n-1}} \cdot \int_{D_1} \prod_{i=2}^n \left(\int_{D_i} |\nabla f|\,dx_i\right)^{\frac{1}{n-1}} dx_1$$

$$\le \left(\int_{D_1} |\nabla f|\,dx_1\right)^{\frac{1}{n-1}} \prod_{i=2}^n \Big(\underbrace{\int_{D_1}\int_{D_i} |\nabla f|\,dx_i\,dx_1}_{=\int_{D_1 D_i} |\nabla f|\,dx_i dx_1}\Big)^{\frac{1}{n-1}}$$

$$\Rightarrow \int_{D_1 D_2} |f(x)|^{\frac{n}{n-1}} dx_1 dx_2$$

$$\le \left(\int_{D_1 D_2} |\nabla f|\,dx_1 dx_2\right)^{\frac{1}{n-1}}$$

$$\cdot \int_{D_2} \left(\int_{D_1} |\nabla f|\,dx_1\right)^{\frac{1}{n-1}} \left(\prod_{i=3}^n \int_{D_1 D_i} |\nabla f|\,dx_i dx_1\right)^{\frac{1}{n-1}} dx_2$$

$$\Rightarrow \int_{\mathbb{R}^n} |f|^{\frac{n}{n-1}} dx$$

$$\le \left(\int_{\mathbb{R}^n} |\nabla f|\,dx\right)^{\frac{n}{n-1}} \quad (\checkmark \text{ for } p=1)$$

(Proof only stable on $\mathbb{R}^n$)

For $1<p<n$, let $g=f^{\alpha}$, $\alpha>1$.

Apply $p=1$ case for $g$.

$$\left(\int_{\mathbb{R}^n} g^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} |\nabla g|\,dx$$

$$|\nabla g| = \sqrt{\sum_{i=1}^n |\partial_{x_i} g|^2}, \quad \partial_{x_i} g = \alpha f^{\alpha-1} \partial_{x_i} f$$

$$= \alpha |f|^{\alpha-1} |\nabla f| .$$

$$\Rightarrow \left(\int_{\mathbb{R}^n} f^{\alpha\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le \alpha \int_{\mathbb{R}^n} |f|^{\alpha-1} |\nabla f|\,dx$$

$$\le \alpha \left(\int_{\mathbb{R}^n} |f|^{(\alpha-1)p'} dx\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} |\nabla f|^p dx\right)^{\frac{1}{p}}$$

$\left(p' = \frac{p}{p-1}\right.$, let $\alpha \frac{n}{n-1} = (\alpha-1)\frac{p}{p-1}$.

then $\alpha = \dfrac{(n-1)p}{n-p} \Rightarrow \dfrac{n\alpha}{n-1} = \dfrac{np}{n-p} = q$

$$\Rightarrow \left(\int_{\mathbb{R}^n} |f|^q dx\right)^{\frac{n-1}{n}} \le \alpha \left(\int_{\mathbb{R}^n} |f|^q dx\right)^{\frac{p-1}{p}} \cdot \|\nabla f\|_{L^p}$$

The logic is, we get $\alpha$ first, then choose $q = \frac{n\alpha}{n-1} = \frac{np}{n-p}$!

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Sobolev inequality

$\forall f \in C_0^\infty(\mathbb{R}^n)$, $n > p \geq 1$, $q = \frac{np}{n-p}$,

$\exists\, c = c(n,p)$, s.t.

$\|f\|_{L^q} \leq c \|\nabla f\|_{L^p}$.

New Proof: $\cdot\ x \in \mathbb{R}^n$, polar words $(r, \theta_1, \cdots, \theta_{n-1})$

$y_1 - x_1 = r\cos\theta_1$

$y_2 - x_2 = r\sin\theta_1 \cos\theta_2$ , $r = |y-x| = \sqrt{\sum_{i=1}^n (y_i - x_i)^2}$

$\vdots$

$y_n - x_n = r\sin\theta_1 \cdots \sin\theta_{n-1}$

$\Rightarrow \frac{\partial y_i}{\partial r} = \frac{y_i - x_i}{r}$

$\Rightarrow \partial_r f(r,\theta) = \sum_{i=1}^n \frac{\partial f}{\partial y_i} \cdot \frac{\partial y_i}{\partial r} = \sum_{i=1}^n \frac{\partial f}{\partial y_i} \cdot \frac{y_i - x_i}{r}$

$\Rightarrow f(x) = -\int_0^\infty \partial_r f(r,\theta)\, dr$, $\forall \theta$

$\Rightarrow f(x) = -\frac{1}{w_{n-1}} \int_{S^{n-1}} \int_0^\infty \partial_r f(r,\theta)\, dr\, d\theta$

Key!

where $w_{n-1} = \mathrm{Vol}(S^{n-1}) = \frac{n\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$

$\Rightarrow f(x) = -\frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{\partial_r f(r,\theta)}{r^{n-1}}\, dy$

$= -\frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{\langle \nabla f, y-x \rangle}{|y-x|^n}\, dy$

Lemma 1: $\forall f \in C_0^\infty(\mathbb{R}^n)$, $\forall x \in \mathbb{R}^n$,

$f(x) = -\frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{\langle \nabla f, y-x \rangle}{|y-x|^n}\, dy$

Remark: $-\Delta_y G(x,y) = \delta_x(y)$,

$G(x,y)$ is the Green-function,

$G(x,y) = \frac{1}{(n-2)\, w_{n-1}} |x-y|^{2-n}$

Here, $\Delta_y = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}$,

$\cdot\ \forall f \in C_0^\infty(\mathbb{R}^n)$

$\int_{\mathbb{R}^n} -f\,(\Delta_y(G(x,y)))\, dy = f(x)$

$\Leftrightarrow \int_{\mathbb{R}^n} -\Delta f(y)\, G(x,y)\, dy = f(x)$

$\Leftrightarrow \int_{\mathbb{R}^n} \langle \nabla f(y), \nabla G(x,y) \rangle dy = f(x)$

$= \frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{\langle \nabla f, y-x \rangle}{|y-x|^n}\, dy$

Lemma 2: (Vitali Covering)

$(X, d)$ is a metric space

If $\mathcal{C} = \{B_i = B_{r_i}(x_i) \subset X,\ r_i \leq R\}$ is a set of open balls, then there exists a $\mathcal{C}' \subset \mathcal{C}$ s.t.

① $B' \cap B'' = \phi$, $\forall B', B'' \in \mathcal{C}'$

② $\bigcup_{B \in \mathcal{C}} B \subseteq \bigcup_{B' \in \mathcal{C}'} \widehat{B'}$

where $\widehat{B'} = B_{5r_i}(x_i)$, $B' = B_{r_i}(x_i)$

Proof of lemma 2:

$\mathcal{C}_i = \{B \in \mathcal{C} :$ B with radius r, s.t. $2^{-i}R < r \leq 2^{-i+1}R\}$

Key!

Now we construct $\mathcal{C}'$ by induction.

Step 1: Choose a max'l subset $\mathcal{C}_1'$ of $\mathcal{C}_1$, s.t. $B' \cap B'' = \phi$, $\forall B', B'' \in \mathcal{C}_1'$ (Applying Zorn's lemma)

Step 2: Suppose we already have $\mathcal{C}_1', \mathcal{C}_2', \cdots, \mathcal{C}_k'$, then $\mathcal{C}_{k+1}'$ is a max'l subset of $\mathcal{C}_{k+1}$ s.t. ① $B' \cap B'' = \phi$, $\forall B', B'' \in \mathcal{C}_{k+1}'$

② $B' \cap B_i = \phi$, $\forall B' \in \mathcal{C}'_{i+1}$,
$B_i \in \mathcal{C}'_i$, $1 \le i \le k$

Now, let $\mathcal{C}' = \bigcup_{i=1}^{\infty} \mathcal{C}'$.

Easy to check the first part of lemma 2.

For the second part, $\forall B \in \mathcal{C}$, $B \in \mathcal{C}_k \setminus \mathcal{C}'$
$\Rightarrow \exists B' \in \bigcup_{i=1}^{k} \mathcal{C}_i$, s.t. $B' \cap B \neq \phi$.

Obviously, $B \subseteq \widehat{B'}$, since if $B = B_r(x)$,
$B' = B_{r'}(x')$, and $2^{-k}R < r', r \le 2^{-k+1}R$,
then $B_{5r'}(x') \supseteq B_r(x)$ □

Def: (Maximal function) $f \in L^1(X)$,
$(X, d, \mu)$ with metric $d$, measure $\mu$.
Now, $\forall x \in X$, the maximal function

$$M(f)(x) \triangleq \sup_{0<r\le 1} \fint_{B_r(x)} |f| d\mu,$$

or $\sup_{B \ni x} \fint_B |f|$

note $\fint_{B_r(x)} |f| d\mu \to 0$ as $r \to \infty$, since $f \in L^1$!

where $\fint_{B_r(x)} = \frac{1}{\mu(B_r(x))} \int_{B_r(x)}$.

Lemma 3: In $(X, d, \mu)$, s.t.

$$\mu(B_r(x)) \ge C_0 \mu(B_{2r}(x))$$

$\forall x \in X$, $0 < r \le 1$. Then, $\forall t > 0$, $f \in L^1(X)$,
we have $t\mu(\{x \mid M(f)(x) > t\}) \le C\|f\|_{L^1}$,
where $C$ is indept to $f$. ($C = C(t, C_0)$)

Proof of lemma 3: By the def of $M(f)$,
let $\Omega = \{x \in X \mid M(f)(x) > t\}$, $\forall x \in \Omega$,
$\exists\ 0 < r_x \le 1$, s.t. $\fint_{B_{r_x}(x)} |f| d\mu \ge \frac{t}{2}$. $(*)$

$\Omega \subseteq \bigcup_{x \in \Omega} B_{r_x}(x)$, $0 < r_x \le 1$.

By Vitali covering (lemma 2)
$\Rightarrow \exists \{B_{r_i}(x_i), x_i \in \Omega\}$, s.t.

① $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \phi$

② $\Omega \subseteq \bigcup_{x \in \Omega} B_{r_x}(x) \subseteq \bigcup_i B_{5r_i}(x_i)$

$$\mu(\Omega) \le \sum_i \mu(B_{5r_i}(x_i)) \le C_0^{-3} \sum_i \mu(B'_{r_i}(x_i)).$$

By $(*)$, $\int_{B_{r_{x_i}}(x_i)} |f| d\mu \ge \frac{t}{2} \mu(B_{r_i}(x_i))$

$$\Rightarrow \mu(\Omega) \le \frac{2}{t} C_0^{-3} \sum_i \int_{B'_{r_i}(x_i)} |f| d\mu \le \frac{2}{t} C_0^{-3} \int_X |f| d\mu$$ □

Now, $\forall f \in C_0^{\infty}(\mathbb{R}^n)$

$$\Rightarrow f(x) = -\frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{\langle \nabla f, y - x \rangle}{|y - x|^n} dy$$

$$\Rightarrow |f(x)| \le \frac{1}{w_{n-1}} \int_{\mathbb{R}^n} \frac{|\nabla f|}{|y - x|^{n-1}} dy \triangleq \frac{1}{w_{n-1}} I_1(|\nabla f|)(x)$$

Def:
$\forall u \in L^1(X)$, $\alpha > 0$, in $(X, d, \mu)$

$$I_\alpha(u)(x) = \int_X \frac{|u|(y)}{(d(x,y))^{n-\alpha}} d\mu(y),$$

which is called Riesz potential.

Lemma 4: In $(X, d, \mu)$, $u \in L^1(X)$,
if $C_1 r^n \le \mu(B_r(x)) \le C_0 r^n$, $\forall x \in X$, $0 < r \le 1$
then $I_1(u)(x) \le C(M(u)(x))^{1-\frac{1}{n}} \|u\|_{L^1}^{\frac{1}{n}}$

what if $M(u) = 0$?

Proof of lemma 4:

$$I_1(u)(x) = \int_X \frac{|u|(y)}{(d(x,y))^{n-1}} d\mu(y)$$

$$= \left(\int_{B_\varepsilon(x)} + \int_{X\setminus B_\varepsilon(x)}\right), \quad \forall\, 0<\varepsilon<1$$

$$= \sum_{i=1}^{\infty} \int_{B_{2^{-i+1}\varepsilon}\setminus \bar{B}_{2^{-i}\varepsilon}} + \int_{X\setminus B_\varepsilon(x)}$$

$$\le \sum_{i=1}^{\infty} \int_{B_{2^{-i+1}\varepsilon}\setminus \bar{B}_{2^{-i}\varepsilon}} 2^{i(n-\overset{\alpha}{1})}\, \varepsilon^{\overset{\alpha}{1}-n}\, |u|(y)\, dy + \varepsilon^{\overset{\alpha}{1}-n}\|u\|_{L^1}$$

$$\le \sum_{i=1}^{\infty} 2^{i(n-\overset{\alpha}{1})}\, \varepsilon^{\overset{\alpha}{1}-n} \int_{B_{2^{-i+1}\varepsilon(x)}} |u|(y)\, d\mu(y)$$

$$+ \varepsilon^{\overset{\alpha}{1}-n}\|u\|_{L^1}$$

$$\le \sum_{i=1}^{\infty} 2^{i(n-\overset{\alpha}{1})}\, \varepsilon^{\overset{\alpha}{1}-n}\, \mu(B_{2^{-i+1}\varepsilon(x)}) ⨍_{B_{2^{-i+1}\varepsilon(x)}} |u|\, d\mu$$

$$+ \varepsilon^{\overset{\alpha}{1}-n}\|u\|_{L^1}$$

$$\le \sum_{i=1}^{\infty} C_0\cdot 2^n\cdot 2^{-i\alpha}\cdot \varepsilon^{\alpha} M(u)(x) + \varepsilon^{\overset{\alpha}{1}-n}\|u\|_{L^1}$$

$$\le C\varepsilon^{\alpha} M(u)(x) + \varepsilon^{\overset{\alpha}{1}-n}\|u\|_{L^1}$$

Let $\varepsilon = \left(\dfrac{M(u)(x)}{\|u\|_{L^1}}\right)^{-\frac{1}{n}}$ $\square$

$\varepsilon$ take the same get $M(u)^{1-\frac{\alpha}{n}}\cdot(\|u\|_{L^1})^{\frac{\alpha}{n}}$

Lemma 5: $h\in L^p(X)$, $p\ge 1$, then

① $\int_X |h|^p\, d\mu = \int_0^\infty \frac{\cancel{1}}{p}\, t^{p-1}\mu(\{x \mid |h|(x)>t\})\,dt \ge c(p)\sum_{i=-\infty}^{\infty} 2^{p(i-1)}\mu(\{x \mid |h|(x)>2^i\})$

② $\int_X |h|^p\, d\mu \le c(p)\sum_{i=-\infty}^{\infty} 2^{pi}\mu(\{x \mid |h|(x)>2^i\})$.

Proof of lemma 5:

$$\int_X |h|^p\, d\mu = \int_X \int_0^\infty \frac{\cancel{1}}{p}\, t^{p-1}\chi_{\{|h|(x)>t\}}\, dt\, d\mu$$

$$= \int_0^\infty \frac{t^{p-1}}{p}\int_X \chi_{\{|h|(x)>t\}}\, dt\, d\mu$$

$$= \int_0^\infty \frac{t^{p-1}}{p}\mu(\{x \mid |h|(x)>t\})\,dt$$

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Want to prove:

$$\left(\int_{\mathbb{R}^n}|f|^{\frac{n}{n-1}}(x)\,dx\right)^{\frac{n-1}{n}} \le C(n)\int_{\mathbb{R}^n}|\nabla f|(x)\,dx$$

$\forall f\in C_0^\infty(\mathbb{R}^n)$

Last week: → Holds on general Riemannian mfld

① $|f(x)| \le \frac{1}{W_{n-1}}\int_{\mathbb{R}^n}\frac{|\nabla f|(y)}{|x-y|^{n-1}}\,dy \triangleq \frac{1}{W_{n-1}} I_1(|\nabla f|)$

② $\forall t>0$, $t\mu(\{x \mid M(u)(x)>t\})\le C\|u\|_{L^1}$

$u\in C_0^\infty(\mathbb{R}^n)$

where $M(u)(x) = \sup_{0<r\le 1} \fint_{B_r(x)}|u|\,dy$

③ $I_1(u)(x) \le C(n)\left(M(u)(x)\right)^{1-\frac{1}{n}}\|u\|_{L^1}^{\frac{1}{n}}$ ($\alpha$)

④ $h\in L^p(\mathbb{R}^n)$, then $\forall p\ge 1$

$$\int_{\mathbb{R}^n}|h|^p\,d\mu \le C(n,p)\sum_{i=-\infty}^{\infty} 2^{p\cdot i}\mu(\{x \mid |h|(x)>2^i\})$$

Today:

Lemma 1: $f\in C_0^\infty(\mathbb{R}^n)$, then $\forall \alpha>0$,

$$\alpha^{\frac{n}{n-1}}\mu(\{x \mid |f(x)|>\alpha\}) \le C(n)\|\nabla f\|_{L^1}^{\frac{n}{n-1}}$$

where $\mu(\Omega)=\int_\Omega dx = \mathrm{Vol}(\Omega)$

Proof: let $u(x)=\nabla f(x)$.

By ①, $\{x \mid |f(x)|>\alpha\}\subseteq\{x \mid I_1(u)(x)>W_{n-1}\alpha\}$

By ③, $\subseteq \{x \mid C(n)(M(u)(x))^{1-\frac{1}{n}}\|u\|_{L^1}^{\frac{1}{n}}>W_{n-1}\alpha\}$

$=\{x \mid M(u)(x)\ge C'(n)\,\alpha^{\frac{n}{n-1}}\|u\|_{L^1}^{-\frac{1}{n-1}}\}$

By ②, $\mu(\{x \mid |f(x)|>\alpha\})$ $\overset{\frac{1}{n-1}+1}{}$ (Holds for Lipschitz)

$\le C''(n)\,\alpha^{-\frac{n}{n-1}}\|u\|_{L^1}^{\frac{n}{n-1}}$ (*) □

Now, we turn to prove Sobolev ineq'l.

$\forall f\in C_0^\infty(\mathbb{R}^n)$, $\forall k\in\mathbb{Z}$, let

$$f_k(x)=\min\{(|f|-2^k)_+,\ 2^k\}$$

where $g_+(x)=\max\{g,0\}$

Note that $f_k$ is Lipschitz.

By (*), $(2^k)^{\frac{n}{n-1}}\mu(\{x \mid |f_k|\ge 2^k\}) \le C(n)\|\nabla f_k\|_{L^1}^{\frac{n}{n-1}}$ (**)

$\{x \mid |f_k(x)|\ge 2^k\} \supseteq \{x \mid |f(x)|\ge 2^{k+1}\}$

$$|\nabla f_k|(x)\le \begin{cases} 0 & |f(x)|>2^{k+1} \\ 0 & |f|(x)<2^k \\ |\nabla f|, & |f|(x)\in[2^k,2^{k+1}] \end{cases}$$

Kato ineq'l.

$|\nabla|f||\le|\nabla f|$, when $f\ne 0$.

By (**), $(2^k)^{\frac{n}{n-1}}\mu(\{x \mid |f|\ge 2^{k+1}\})$

$$\le C(n)\left(\int_{2^k\le|f|\le 2^{k+1}}|\nabla f|\right)^{\frac{n}{n-1}}$$

$$\Rightarrow \int_{\mathbb{R}^n}|f|^{\frac{n}{n-1}}\,d\mu \le C(n)\sum_{i=-\infty}^{\infty} 2^{i\frac{n}{n-1}}\mu(\{x \mid |f|\ge 2^i\})$$

$$\le C'(n)\sum_{k=-\infty}^{\infty}\left(\int_{|f|\in[2^k,2^{k+1}]}|\nabla f|\right)^{\frac{n}{n-1}}$$

$$\le C'(n)\left(\int_{\mathbb{R}^n}|\nabla f|\,d\mu\right)^{\frac{n}{n-1}}$$

$a_1^p+\cdots+ta_n^p\le(a_1+\cdots+ta_n)^p$, $a_n=\min_{1\le i\le n} a_i$

Note: $(a_1^p+\cdots+a_n^p)\le(a_1+\cdots+a_n)^p$, $\forall p\ge 1$, $a_i\ge 0$

by induction ? $\because$ $a_1(a_1+\cdots+a_n)^{p-1}+\cdots+a_n(a_1+\cdots+a_n)^{p-1}\ge a_1^p+\cdots+a_n^p$ □

Thm 1. $\forall\, 0<r\le 1$, $1\le q<n$, $\exists\, C(n,q)>0$, s.t. ① Dirichlet Poincare ineq'l : weak

$\forall f\in C_0^\infty(B_r(0))$, $B_r(0)\subseteq\mathbb{R}^n$,

we have $\fint_{B_r(0)}|f|^q\,d\mu \le C(n,q)\,r^q\fint_{B_r(0)}|\nabla f|^q\,d\mu$

② Neumann – Poincare ineq'l weak

$\forall f \in C^\infty(B_r(0))$, we have

$$\fint_{B_r(0)} \left| f - \fint_{B_r(0)} f\, d\mu \right|^q d\mu \le c(n,q)\, r^q \fint_{B_r(0)} |\nabla f|^q\, d\mu$$

③ Dirichlet – Sobolev ineq'l strong

$\forall f \in C_0^\infty(B_r(0))$, we have

$$\left( \fint_{B_r(0)} f^{q'} d\mu \right)^{\frac{1}{q'}} \le c(n,q)\, r \left( \fint_{B_r(0)} |\nabla f|^q d\mu \right)^{\frac{1}{q}}$$

where $q' = \frac{nq}{n-q}$

④ $f \in C^\infty(B_r(0))$, we have, $\left(q' = \frac{nq}{n-q} > q\right)$ strong

$$\left( \fint_{B_r(0)} \left| f - \fint_{B_r(0)} f\, d\mu \right|^{q'} d\mu \right)^{\frac{1}{q'}} \le c(n,q)\, r \left( \fint_{B_r(0)} |\nabla f|^q d\mu \right)^{\frac{1}{q}}$$

Remark: (i) Riemannian mfld, $Ric \ge -(n-1)$, ①, ②, ③, ④ true.

(ii) ④ $\Rightarrow$ ②

(iii) ④ + ① $\Rightarrow$ ③

Verifying (ii). By Hölder ineq'l,

$$\left( \fint_X f^p d\mu \right)^{\frac{1}{p}} \le \left( \fint_X f^q d\mu \right)^{\frac{1}{q}}, \quad q > p$$

□

Verifying (iii), we need a lemma 2 (Minkowski ineq'l). In $(X,\mu)$, $f, g \in L^p(X,\mu)$, $p \ge 1$, then

$$\|f+g\|_{L^p} \le \|f\|_{L^p} + \|g\|_{L^p}.$$

Proof of lemma: $p=1$ (✓)

$p>1$: $\int_X |f+g|^p d\mu \le \int_X (|f|+|g|)(|f+g|^{p-1}) d\mu$

$$= \int_X |f|\,|f+g|^{p-1} d\mu + \int_X |g|\,|f+g|^{p-1} d\mu$$

$$\le \left(\|g\|_{L^p} + \|f\|_{L^p}\right) \left( \int_X |f+g|^p d\mu \right)^{\frac{1}{q}}, \quad q = \frac{p}{-1+p}$$

□

Now, if ④, ①, then

$$\left( \fint_{B_r(0)} f^{q'} d\mu \right)^{\frac{1}{q'}} \le \left( \fint_{B_r(0)} \left| f - \fint_{B_r(0)} f\, d\mu \right|^{q'} d\mu \right)^{\frac{1}{q'}} + \left| \fint_{B_r(0)} f\, d\mu \right| \quad \text{(by lemma 2)}$$

By ①, $\left| \fint_{B_r(0)} f\, d\mu \right| \le c(n)\, r \fint_{B_r(0)} |\nabla f|\, d\mu$

$$\le c(n)\, r \left( \fint_{B_r(0)} |\nabla f|^q d\mu \right)^{\frac{1}{q}}$$

and By ④, we have ③.

□

Segment ineq'l:

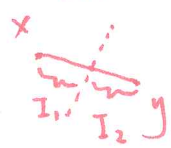
$h: \mathbb{R}^n \to [0,+\infty)$ and $h \in C^0(\mathbb{R}^n)$

def: $\mathcal{F}_h : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,+\infty)$ integral alone the line $x$—$y$

$$\mathcal{F}_h(x,y) \triangleq \int_0^{d(x,y)=|y-x|} h\left(x + t\,\frac{y-x}{|y-x|}\right) dt$$

Then, $\forall x_0, x_1 \in B_{3r}(0) \subseteq \mathbb{R}^n$, $\exists\, c(n)$,

s.t. $$\fint_{B_r(x_0)} \fint_{B_r(x_1)} \mathcal{F}_h(x,y)\, dx\, dy \le c(n)\, r \fint_{B_{10r}(0)} h(x)\, dx$$


Proof: (Cheeger – Colding)

$\forall x \in B_r(x_0)$

$$\int_{B_r(x_1)} \mathcal{F}_h(x,y)\, dy = \int_{B_r(x_1)} \int_0^{|x-y|} h\left(x + t\,\frac{y-x}{|y-x|}\right) dt\, dy$$

$$= \int_{B_r(x_1)} \left( \int_0^{\frac{|x-y|}{2}} + \int_{\frac{|x-y|}{2}}^{|x-y|} \right) dt\, dy \triangleq I_1 + I_2$$

Question: In $(X, d, \mu)$

When Segment ineq'l holds?

$$I_2 = \int_{B_r(x_1)} \int_{\frac{|x-y|}{2}}^{|x-y|} h\left(x + t\frac{y-x}{|y-x|}\right) dt\, dy$$

center at x

$$\le \int_{B_{7r}(x)} \int_{\frac{|x-y|}{2}}^{|x-y|} h\left(x + t\frac{y-x}{|y-x|}\right) dt\, dy$$

$$\le \int_{S^{n-1}} \int_0^{7r} \int_{\frac{s}{2}}^{s=|x-y|} h(x+t\theta)\, dt\, s^{n-1} ds\, d\theta$$

integral of $h(y)$

$$\cancel{\le} \int_{S^{n-1}} \int_0^{7r} \int_{\frac{s}{2}}^{s} t^{n-1} h(x+t\theta) \frac{s^{n-1}}{t^{n-1}}\, dt\, ds\, d\theta$$

Key purpose

$$\le 2^{n-1} \int_{S^{n-1}} \int_0^{7r} \int_{\frac{s}{2}}^{s} h(x+t\theta)\, t^{n-1}\, dt\, ds\, d\theta$$

$$\le 2^{n-1} \int_{S^{n-1}} \int_0^{7r} \int_0^{7r} h(x+t\theta)\, t^{n-1}\, dt\, ds\, d\theta$$

$$= 7r 2^{n-1} \int_{S^{n-1}} \int_0^{7r} h(x+t\theta)\, t^{n-1}\, dt\, d\theta$$

$$= 7r 2^{n-1} \int_{B_{7r}(x)} h(y)\, dy$$

$$\le 7r \cdot 2^{n-1} \int_{B_{10r}(0)} h(y)\, dy$$

$$\Rightarrow \int_{B_r(x_0)} I_2\, dx \le 7r \cdot 2^{n-1} \mathrm{Vol}(B_r(x_0)) \int_{B_{10r}(0)} h(y)\, dy$$

$$\int_{B_r(x_0)} I_1\, dx = \int_{B_r(x_0)} \int_{B_r(x_1)} \int_0^{\frac{|x-y|}{2}} h\left(x + t\frac{y-x}{|y-x|}\right) dt\, dy\, dx$$

$u = |y-x| - t.$

$$= \int_{B_r(x_1)} \int_{B_r(x_0)} \int_0^{\frac{|x-y|}{2}} h\left(x + t\frac{y-x}{|y-x|}\right) dt\, dx\, dy$$

$$= \int_{B_r(x_1)} \int_{B_r(x_0)} \int_{\frac{|x-y|}{2}}^{|x-y|} h\left(y + t\frac{x-y}{|x-y|}\right) dt\, dx\, dy$$

$$\le 7r \cdot 2^{n-1} \mathrm{Vol}(B_r(x_0)) \int_{B_{10r}(0)} h(z)\, dz$$

$$\Rightarrow \int_{B_r(x_0)} \int_{B_r(x_1)} \mathcal{F}_h(x,y)\, dx\, dy$$

$$\le 7 \cdot 2^{n-1} \cdot r \left( \mathrm{Vor}(B_r(x_0)) + \mathrm{Vor}(B_r(x_1)) \right) \int_{B_{10r}(0)} h(z)\, dz$$

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Last time, still need to prove:

① $\fint_{B_r(0)} f^p dx \le C(n,p) r^p \fint_{B_r(0)} |\nabla f|^p dx$

$\forall f \in C_0^\infty(B_r(0))$, $n > p \ge 1$

② $\left( \fint_{B_r(0)} \left| f(x) - \fint_{B_r(0)} f(y) dy \right|^{q'} dx \right)^{\frac{1}{q'}}$

$\le C(n,p) r \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{p}}$.

$q' = \frac{np}{n-p}$, $\forall 1 \le p < n$, $f \in C^\infty(B_r(0))$

What we have:

Segment ineq'l: $h: \mathbb{R}^n \to \mathbb{R}_+$

$$\mathcal{F}_h(x,y) = \int_0^{|x-y|} h\left(x + t\frac{y-x}{|y-x|}\right) dt$$

Then $\forall x_0, x_1 \in B_{3r}(0)$,

$$\fint_{B_r(x_0)} \fint_{B_r(x_1)} \mathcal{F}_h(x,y) dx dy \le C(n) r \fint_{B_{10r}(0)} h(z) dz$$

We need a lemma 1:

$f \in C^2(\mathbb{R}^n)$, then $\forall x, y \in \mathbb{R}^n$

$$|f(x) - f(y)| \le \int_0^{|x-y|} |\nabla f|\left(x + t\frac{y-x}{|y-x|}\right) dt$$

Proof: $|f(x) - f(y)| = \left| \int_0^{|x-y|} \frac{d}{dt} f\left(x + t\frac{y-x}{|y-x|}\right) dt \right|$

$= \left| \int_0^{|x-y|} \nabla f \cdot \frac{y-x}{|y-x|} dt \right| \le \int_0^{|x-y|} |\nabla f|\left(x + t\frac{y-x}{|y-x|}\right) dt$

□

Now we turn back to prove ①:

Proof of ①: Let $x_0 = 0$, $x_1 = (2r, 0, \cdots, 0)$ key!

Let $h(x) = |\nabla f|(x)$. By Segment ineq'l + lemma 1,

$$\fint_{B_r(0)} \fint_{B_r(x_1)} |f^p(x) - f^p(y)| dx dy$$

$$\le \fint_{B_r(0)} \fint_{B_r(x_1)} \mathcal{F}_{|\nabla f^p|}(x,y) dx dy$$

$$\le C(n) r \fint_{B_{10r}(0)} |\nabla f^p|(z) dz$$

Since $\forall y \in B_r(x_1) \Rightarrow d(0,y) \ge r$.

$\Rightarrow f(y) = 0$ $f \in C_0^\infty(B_r(0))$! key!

$$\Rightarrow \fint_{B_r(0)} |f^p(x)| dx \le \frac{C(n) r \int_{B_r(0)} |\nabla f^p| dz}{Vol(B_{10r}(0))}$$

$$\le C(n) r \fint_{B_r(0)} |\nabla f^p| dz$$

$$\le C(n) r p \fint_{B_r(0)} |\nabla f| f^{p-1} dz$$

$$\le C(n,p) r \left( \fint_{B_r(0)} |\nabla f|^p dz \right)^{\frac{1}{p}} \left( \fint_{B_r(0)} f^p dz \right)^{1-\frac{1}{p}}$$

□

lemma 2: (Weak Neumann Poincare ineq'l)

$\forall f \in C^\infty(B_{10r}(0))$, $0 < r \le 1$, $1 \le q < n$

$\exists C = C(q,n)$, s.t.

$$\fint_{B_r(0)} \left| f(y) - \fint_{B_r(0)} f(x) dx \right| dy \le C(n,p) \left( r^q \fint_{B_{10r}(0)} |\nabla f|^q dx \right)^{\frac{1}{q}}$$

Proof: By lemma 1, $\forall x, y \in B_{10r}(0)$

$\Rightarrow |f(x) - f(y)| \le \mathcal{F}_{|\nabla f|}(x,y) = \int_0^{|x-y|} |\nabla f|(\gamma(t)) dt$,

where $\gamma(t) = x + t\frac{y-x}{|y-x|}$

Hence: $|f(x) - f(y)|^q \le \left( \int_0^{|x-y|} |\nabla f|(\gamma(t)) dt \right)^q$

$$\le \int_0^{|x-y|} |\nabla f|^q(\gamma(t)) dt \cdot |x-y|^{q-1}$$

$$\le (20r)^{q-1} \mathcal{F}_{|\nabla f|^q}(x,y)$$

Let $x_0 = x_1 = 0$, by Segment ineq'l

$$\fint_{B_r(0)}\fint_{B_r(0)} |f(x)-f(y)|^q\,dx\,dy$$

$$\le c(n,q)\, r^q \fint_{B_{10r}(0)} |\nabla f|^q(z)\,dz \cdots (*)$$

$r^{q-1}\cdot r$

Back to:

$$\fint_{B_r(0)} \left| f(y) - \fint_{B_r(0)} f(x)dx \right|^q dy$$

$$\le \fint_{B_r(0)} \left| \fint_{B_r(0)} f(y) - f(x)\,dx \right|^q dy$$

$$\le \fint_{B_r(0)} \left( \fint_{B_r(0)} |f(y)-f(x)|dx \right)^q dy$$

Hölder

$$\le \fint_{B_r(0)} \left( \fint_{B_r(0)} |f(y)-f(x)|^q dx \right) dy$$

$$\le c(n,q)\, r^q \fint_{B_{10r}(0)} |\nabla f|^q(z)\,dz, \text{ by } (*)$$

With:

$$\fint_{B_r(0)} \left| f - \fint_{B_r(0)} f(x)dx \right| dy$$

$$\le \left( \fint_{B_r(0)} \left| f - \fint_{B_r(0)} f(x)dx \right|^q dy \right)^{\frac{1}{q}}$$

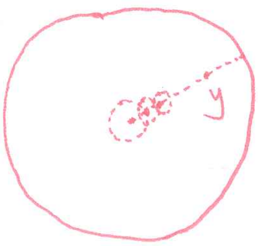
We finish the proof. □

(Chain Ball condition)

lemma 3: $\forall y \in B_r(0) \subseteq \mathbb{R}^n$, $\exists K = k(n) > 0$, and $\{B_0, B_1, \cdots\}$, where $B_i = B_{r_i}(x_i)$ and $r_i = \frac{r}{10}\left(\frac{9}{10}\right)^i$, and:

(1) $B_0 = B_{\frac{r}{10}}(0)$

(2) $10B_i \subseteq B_r(0)$, $\forall i \ge 0$




(3) $B_i \subseteq B_{Kr_i}(y)$

(4) $\exists\, \widetilde{B_i}$ s.t. $\widetilde{B_i} \subset (B_i \cap B_{i+1})$ and $B_i \cup B_{i+1} \subseteq K\widetilde{B_i}$

where, for $B = B_r(x)$, $\lambda B \triangleq B_{\lambda r}(x)$.

Proof: construct $B_i$:

Let $B_0 = B_{\frac{r}{10}}(0)$, $\forall y \in B_r(0)$,

let $\gamma(t) = ty \Rightarrow \gamma(0) = 0$, $\gamma(1) = y$

Suppose $r\left(\frac{9}{10}\right)^{\alpha+1} \le d(y, \partial B_r(0)) < r\left(\frac{9}{10}\right)^{\alpha}$, $\alpha \in \mathbb{N}$

For any $s \in [0,1]$, $R_{\gamma(s)}$ is the maximal $R$ s.t. $B_{10R}(\gamma(s)) \subseteq B_r(0)$

then $R_s \le R_t$ when $s \ge t$

let $x_1 = \partial B_0 \cap \{\gamma[0,1]\}$, $B_1 = B_{R_{x_1}}(x_1)$

$x_2 = \partial B_1 \cap \{\gamma[0,1]\}$, $B_2 = B_{R_{x_2}}(x_2)$

and $|x_2| > |x_1|$

By induction: we already have $B_i$, $1 \le i \le \alpha$.

let $B_j = B_{r_j}(y)$, when $j \ge \alpha + 1$, where $r_j = \frac{r}{10}\left(\frac{9}{10}\right)^j$

We need to check $\{B_i\}$.

Radius: $x_1 = \partial B_0 \cap \{\gamma[0,1]\}$

$\Rightarrow |x_1| = \frac{r}{10}$, $\Rightarrow R_{x_1} = \frac{r}{10}\cdot\frac{9}{10}$

and $|x_2| = \frac{r}{10} + \frac{r}{10}\cdot\frac{9}{10} = r\left(1 - \left(\frac{9}{10}\right)^2\right)$

$\Rightarrow r - |x_2| = r\left(\frac{9}{10}\right)^2$, $R_{x_2} = \frac{r}{10}\left(\frac{9}{10}\right)^2$

$\forall i \le \alpha$, $|x_i| = \sum_{k=0}^{i-1} \frac{r}{10}\left(\frac{9}{10}\right)^i = r\left(1 - \left(\frac{9}{10}\right)^i\right)$

$\Rightarrow r - |x_i| = r\left(\frac{9}{10}\right)^i \Rightarrow R_{x_i} = \frac{r}{10}\left(\frac{9}{10}\right)^i$

(V)

(1), (2) are true

(3): $d(x_i, \partial B_r(0)) = r - |x_i| = 10 r_i$

line connecting $r_i$ and $\bar{y}$

Suppose $\bar{y} = \frac{ry}{|y|}$, then $y \in \overline{r_i \bar{y}}$,

$\Rightarrow B_{20r_i}(\bar{y}) \supseteq B_i$, $B_{40r_i}(y) \supseteq B_{20r_i}(\bar{y})$

Let $K = 40$, (3) is true.

(4) $|x_i| = r\left(1-\left(\frac{9}{10}\right)^i\right)$

$|x_{i+1}| = r\left(1-\left(\frac{9}{10}\right)^{i+1}\right)$

Let $R_i = \frac{r_{i+1}}{2}$, $\tilde{x}_i \in \overrightarrow{x_{i+1}x_i}$, s.t.

$d(\tilde{x}_i, x_{i+1}) = R_i$

Let $\tilde{B}_i = B_{R_i}(\tilde{x}_i)$,

$\Rightarrow B_i \cap B_{i+1} \supset \tilde{B}_i$

$\Rightarrow \lambda B_{R_i}(\tilde{x}_i) \supset B_i \cup B_{i+1}$,

where $\lambda = \dfrac{2r_i - \frac{1}{2}r_{i+1}}{\frac{1}{2}r_{i+1}} = \dfrac{2\times\frac{10}{9}-\frac{1}{2}}{\frac{1}{2}} = \frac{31}{9}$

$\square$

lemma 4: $\forall y \in B_r(0)$, $f \in C^\infty(B_r(0))$ and weak Neuman ineq'l holds in lemma2, then $\left|f(y) - \fint_{B_{\frac{r}{10}}(0)} f(y)\,dy\right|$

$\leq C(n,p)\, r\, M(|\nabla f|^p)(y)^{\frac{n-p}{np}} \cdot \left(\fint_{B_r(0)} |\nabla f|^p\right)^{\frac{1}{p}}$

where $1 \leq p < n$, $M(|\nabla f|^p)(y)$ is maximal function.
Denote $B_{\frac{r}{10}}(0)$ as $B_0$

Proof: $\forall y \in B_r(0)$, by lemma 3, $\exists \{B_0, \cdots\}$

s.t. $B_i = B_{r_i}(x_i)$, $r_i = \frac{r}{10}\left(\frac{9}{10}\right)^i$

$\left|f(y) - \fint_{B_0} f(y)\,dy\right| = \lim\limits_{i\to\infty} \left|f_{B_i} - f_{B_0}\right|$

$\leq \sum\limits_{i=0}^{\infty} \left|f_{B_i} - f_{B_{i+1}}\right|$

$\leq \sum\limits_{i=0}^{\infty} \left(|f_{B_i} - f_{\tilde{B}_i}| + |f_{B_{i+1}} - f_{\tilde{B}_i}|\right)$

$\leq \sum\limits_{i=0}^{\infty} \fint_{\tilde{B}_i} \left|f(z) - \fint_{B_i} f(x)\,dx\right| dz$

$+ \sum\limits_{i=0}^{\infty} \fint_{\tilde{B}_i} \left|f(z) - \fint_{B_{i+1}} f(x)\,dx\right| dz$

$\leq C(n) \sum\limits_{i=0}^{\infty} \fint_{B_i} \left|f(z) - \fint_{B_i} f(x)\,dx\right| dz$

By lemma 2:

$\left|f(y) - \fint_{B_0} f(y)\,dy\right|$

$\leq \sum\limits_{i=0}^{\infty} C(n,p)\, r_i \left(\fint_{B_{10r_i}(x_i)} |\nabla f|^p dz\right)^{\frac{1}{p}}$,

for $0 < s \leq 1$, $= \left(\sum\limits_{r_i \leq r_s} + \sum\limits_{r_i > r_s}\right)$,

The first part: $\sum\limits_{r_i \leq r_s} C(n,p)\, r_i \left(\fint_{B_{10r_i}(x_i)} |\nabla f|^p\right)^{\frac{1}{p}}$

We consider maximal function.

To be continued.

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We want to show:

$\forall f \in C^\infty(B_r(0))$, $1 \le p < n$

- $\left( \fint_{B_r(0)} \left| f - \fint_{B_r(0)} f \right|^{\frac{np}{n-p}} dx \right)^{\frac{n-p}{np}} \le c(n,p)\, r \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{p}}$

- Weak Poincare ineq'l: $\forall f \in C^\infty(B_{10r}(0))$

$$\fint_{B_r(0)} \left| f - \fint_{B_r(0)} f \right| dx \le c(n,p)\, r \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{p}}$$

$B_{10r}(0)$

(we have proved)

- $\forall y \in B_r(0)$, $\exists \{B_0, B_1, \cdots, B_i, \cdots\}$

s.t. ① $B_0 = B_{\frac{r}{10}}(0)$, for $i$ large enough, $x_i = y$

② $B_i = B_{r_i}(x_i)$, $r_i = \frac{r}{10}\left(\frac{9}{10}\right)^i$

③ $B_i \subseteq B_{K(n) r_i}(y)$, $K(n) > 0$

④ $\exists\, \widetilde{B}_i$, s.t. $B_i \cap B_{i+1} \supset \widetilde{B}_i$ and $K(n)\widetilde{B}_i \supset B_i \cup B_{i+1}$

⑤ $10 B_i \subseteq B_r(0)$

(we have proved) — lemma3

- $\forall y \in B_r(0)$, $f \in C_0^\infty(B_r(0))$, Then

$$\left| f(y) - \fint_{B_0} f dx \right| \le c(n,p)\, r\, M(g^p)^{\frac{n-p}{np}}(y) \cdot \left( \fint_{B_r(0)} |\nabla f|^p \right)^{\frac{1}{n}}$$

$p < n$

where $g(y) = \begin{cases} |\nabla f|(y), & y \in B_r(0) \\ 0, & y \bar{\in} B_r(0) \end{cases}$

— lemma4

Proof: by lemma 3, $\exists \{B_0, B_1, \cdots\}$

s.t. $\lim_{i \to \infty} \fint_{B_i} f dx = f(y)$

and $\left| \fint_{B_i} f dx - \fint_{B_0} f dx \right| \le \sum_{k=1}^{i} \left| \fint_{B_k} - \fint_{B_{k-1}} \right|$

$$\le \sum_{k=1}^{i} \left| \fint_{B_k} - \fint_{\widetilde{B}_{k-1}} \right| + \left| \fint_{\widetilde{B}_{k-1}} - \fint_{B_{k-1}} \right|$$

$$\le \sum_{k=1}^{i} \fint_{\widetilde{B}_{k-1}} \left| f(z) - \fint_{B_k} f dx \right| dz + \fint_{\widetilde{B}_{k-1}} \left| f(z) - \fint_{B_{k-1}} f dx \right| dz$$

$$\le \sum_{k=1}^{i} \frac{Vol(B_k)}{Vol(\widetilde{B}_{k-1})} \fint_{B_k} \left| f(z) - \fint_{B_k} f dx \right| dz + \frac{Vol(B_{k-1})}{Vol(\widetilde{B}_{k-1})} \fint_{B_{k-1}} \left| f(z) - \fint_{B_{k-1}} f dx \right| dz$$

$$\le c(n) \sum_{k=0}^{i} \fint_{B_k} \left| f(z) - \fint_{B_k} f(x) dx \right| dz$$

weak Poincare

$$\le c(n,p) \sum_{k=0}^{i} r_k \left( \fint_{10B_k} |\nabla f|^p dz \right)^{\frac{1}{p}}.$$

let $i \to \infty$.

$$\left| f(y) - \fint_{B_0} f dx \right| \le \sum_{k=0}^{\infty} c(n,p)\, r_k \left( \fint_{10B_k} |\nabla f|^p dz \right)^{\frac{1}{p}}$$

key! For $\forall\, 0 < s < 1$, $r_k = \frac{r}{10}\left(\frac{9}{10}\right)^k$

$$\sum_{r_k \le sr} c(n,p)\, r_k \left( \fint_{10B_k} |\nabla f|^p dz \right)^{\frac{1}{p}}$$

$$\le \sum_{r_k \le sr} c(n,p)\, r_k \left( \fint_{B_{10K(n)r_k}(y)} |\nabla f|^p dz \right)^{\frac{1}{p}}$$

since $B_k \subseteq B_{K(n)r_k}(y)$

$$= \sum_{r_k \le sr} c(n,p)\, r_k \left( \fint_{B_{10K(n)r_k}(y)} g^p dz \right)^{\frac{1}{p}}$$

$$\le \sum_{r_k \le sr} c(n,p)\, r_k\, M(g^p)^{\frac{1}{p}}(y)$$

$\sum_{r_k \le sr} r_k \sim sr$

$$\le c(n,p)\, sr\, M(g^p)^{\frac{1}{p}}(y).$$

For $\sum_{r_k > sr} c(n,p)\, r_k \left( \fint_{10B_k} |\nabla f|^p dz \right)^{\frac{1}{p}}$

$$\le \sum_{r_k > sr} c(n,p)\, r_k \cdot r_k^{-\frac{n}{p}} \left( \int_{B_r(0)} |\nabla f|^p dz \right)^{\frac{1}{p}}$$

$\rightarrow \left(\frac{9}{10}\right)^k > 10s$

$$\le c(n,p)\, (sr)^{1-\frac{n}{p}} \left( \int_{B_r(0)} |\nabla f|^p dz \right)^{\frac{1}{p}}$$

$\sum_{r_k > sr} = r\left(1 - \frac{9}{10}\left(\frac{9}{10}\right)^k\right) < r(1 - 9s)$

$$\le c(n,p)\, r\, s^{1-\frac{n}{p}} \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{p}}$$

$$\Rightarrow \left| f(y) - \fint_{B_0} f dx \right|$$

$$\le c(n,p)\, r\, s^{1-\frac{n}{p}} \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{p}}$$

$$+ c(n,p)\, s\, r\, M(g^p)^{\frac{1}{p}}(y)$$

Choose $s$ s.t. $r s^{1-\frac{n}{p}} \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{p}}$

similar method $= s\, r\, M(g^p)^{\frac{1}{p}}(y)$

$$\Rightarrow s = \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{n}} M(g^p)^{-\frac{1}{n}}(y)$$

$$\Rightarrow \left| f(y) - \fint_{B_0} f dx \right| \le c(n,p)\, r \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{n}} \cdot M(g^p)^{\frac{n-p}{np}}(y)$$

$\square$

Lemma 5: $\forall t > 0$

① $t^{\frac{np}{n-p}} \mu\left( \left\{ y : \left| f(y) - \fint_{B_0} f dx \right| \ge t \right\} \right)$

$$\le c\, \mu(B_r(0)) \left( r^p \fint_{B_r(0)} |\nabla f|^p \right)^{\frac{n}{n-p}}$$

② $t^{\frac{np}{n-p}} \mu\left( \{ y : |f|(y) \ge t \} \right)$ || similar.

$$\le c\, \mu(B_r(0)) \left( r^p \fint_{B_r(0)} |\nabla f|^p \right)^{\frac{n}{n-p}}$$

$$+ c \left| \fint_{B_0} f dx \right|^{\frac{np}{n-p}} \mu(B_r(0))$$

Proof: ①: Let $\Omega_t = \left\{ y : \left| f(y) - \fint_{B_0} f dx \right| \ge t \right\}$

lemma 4

$$\Rightarrow \Omega_t \subseteq \left\{ y : M(g^p) \ge c\, t^{\frac{np}{n-p}} \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{-\frac{p}{n-p}} \cdot r^{-\frac{np}{n-p}} \right\} \triangleq \tilde{\Omega}_t$$

Key

$$\Rightarrow \mu(\Omega_t) \le \mu(\tilde{\Omega}_t)$$

By the property of max'l function:

$t\, \mu(\{ x \mid M(u)(x) > t \}) \le C \|u\|_{L^1}$

$$c\, t^{\frac{np}{n-p}} r^{-\frac{np}{n-p}} \left( \fint_{B_r(0)} |\nabla f|^p dx \right)^{-\frac{p}{n-p}}$$

$$\cdot \mu(\Omega_t) \le C(n) \int_{B_{50K(n)r}(0)} g^p(y) dy$$

$$\le C(n) \int_{B_r(0)} |\nabla f|^p dy$$

$$\Rightarrow t^{\frac{np}{n-p}} \mu(\Omega_t)$$

$$\le c\, r^{\frac{np}{n-p}} \mu(B_r(0)) \left( \fint_{B_r(0)} |\nabla f|^p dy \right)^{\frac{n}{n-p}} \quad (*)$$

② $\{ y : |f|(y) \ge t \}$ $f \in C_0^\infty(B_r(0))$!

Key

$$\le \left\{ y : \left| f - \fint_{B_0} f dx \right| \ge \frac{t}{2} \right\} \cup \left\{ y : \left| \fint_{B_0} f dx \right| \ge \frac{t}{2} \right\}$$

$$\Rightarrow t^{\frac{np}{n-p}} \mu\left( \{ y : |f|(y) \ge t \} \cap B_r(0) \right)$$

$= \int_A 1 \le \int_A \frac{\left| \fint_{B_0} f \right|^{\frac{np}{n-p}}}{\left| \frac{t}{2} \right|^{\frac{np}{n-p}}} \le \int_{B_r(0)} \cdots$

$$\le (*) + t^{\frac{np}{n-p}} \mu\left( \left\{ y : \left| \fint_{B_0} f dx \right| \ge \frac{t}{2} \right\} \cap B_r(0) \right)$$

$$\le (*) + C \left| \fint_{B_0} f dx \right|^{\frac{np}{n-p}} \mu(B_r(0)).$$

$\square$

Lemma 6: $\forall\, 1 \le q < \frac{np}{n-p}$

We have: $\left( \fint_{B_r(0)} \left| f(y) - \fint_{B_0} f dx \right|^q dy \right)^{\frac{1}{q}}$

$$\le c \left( r^p \fint_{B_r(0)} |\nabla f|^p dx \right)^{\frac{1}{p}}$$

Proof: Let $h = \left| f(y) - \fint_{B_0} f dx \right|$,

then $\int_{B_r(0)} h^q(x)\, dx$

$$= \int_{B_r(0)} \int_0^{h(x)} q\, t^{q-1} dt\, dx$$

$$= \int_{B_r(0)} \int_0^\infty q\, t^{q-1} \chi_{\{h(x) \ge t\}} dt\, dx$$

$$= \int_0^\infty \int_{B_r(0)} q\, t^{q-1} \chi_{\{h(x) \ge t\}} dx\, dt.$$

$$= \int_s^\infty q t^{q-1} \mu(\{y: h(y) \geq t\} \cap B_r(0)) dt$$

$$+ \int_0^s q t^{q-1} \mu(\{y: h(y) \geq t\} \cap B_r(0)) dt, \quad \forall s>0.$$

$$\leq C(n,p,q) \underline{\int_s^\infty} t^{q-1-\frac{np}{n-p}} dt\, \mu(B_r(0)) \cdot \left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{n}{n-p}}$$

$$+ C(q,n) \underline{\int_0^s} t^{q-1} dt\, \mu(\{B_r(0)\})$$ Key!

$$\leq C(n,p,q) s^{q-\frac{np}{n-p}} \mu(B_r(0)) \left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{n}{n-p}}$$

$$+ C(q,n) s^q \mu(B_r(0))$$

let $s = \left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{1}{p}}$

$$\Rightarrow \int_{B_r(0)} h^q(x) dx \leq C(n,p,q) \left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{q}{p}} \cdot \mu(B_r(0))$$

~~where~~

□

lemma 7: $f \in C^\infty(B_r(0))$

we have: $\left(\fint_{B_r(0)} f^{\frac{np}{n-p}} dx\right)^{\frac{n-p}{np}}$

$$\leq C\left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{1}{p}}$$

$$+ C \fint_{B_r(0)} |f| dx$$

[let $f_i = \min\{2^i, \max\{|f|-2^i, 0\}\}$ ]

Now, assume lemma 7 is true.

we prove what we want finally.

$$\left(\fint_{B_r(0)} \left|f(y) - \fint_{B_r(0)} f dx\right|^{\frac{np}{n-p}} dy\right)^{\frac{n-p}{np}}$$

$$\leq C\left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{1}{p}}$$

$$+ C \fint_{B_r(0)} \left|f - \fint_{B_r(0)} f dx\right| dy$$

But $\fint_{B_r(0)} \left|f - \fint_{B_r(0)} f dx\right| dy$

Key! $\leq \fint_{B_r(0)} \left(\left|f - \fint_{B_0} f dx\right| + \underline{\left|\fint_{B_r(0)} f - \fint_{B_0} f dx\right|}\right) dy$

Then, by lemma 6:

$$\leq C\left(\left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{1}{p}} + \underline{\fint_{B_r(0)} \left|f - \fint_{B_0} f dx\right| dy}\right)$$

put one in another.

again, by lemma 6:

$$\left(\fint_{B_r(0)} \left|f - \fint_{B_r(0)} f dx\right|^{\frac{np}{n-p}} dy\right)^{\frac{n-p}{np}}$$

$$\leq C\left(r^p \fint_{B_r(0)} |\nabla f|^p dx\right)^{\frac{1}{p}}$$

□

Jesen ineq'l:

$h: \mathbb{R} \to \mathbb{R}$, convex function, $h'' \geq 0$.

$\forall f: (X, d, \mu) \to \mathbb{R}$ integrable and $\mu(X) = 1$.

then $\int_X h(f) d\mu \geq h\left(\int_X f d\mu\right)$

Proof: Assume $\int_X f d\mu = a$

since h is convex, $\exists A, B$, s.t.

$h(t) \geq At + B$, $\forall t$ and $\underline{h(a) = Aa + B}$

$$\Rightarrow \int_X h(f) d\mu \geq \int_X (Af + B) d\mu$$

$$\geq Aa + B = h\left(\int_X f d\mu\right).$$

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**Lemma 7:** Assume $f\in C^\infty(B_r(0))$, then

$$\left(\fint_{B_r(0)} f^{\frac{np}{n-p}}\,dx\right)^{\frac{n-p}{np}} \le C(n,p)\left(r^p\fint_{B_r(0)}|\nabla f|^p dx\right)^{\frac{1}{p}} + c\fint_{B_r(0)}|f(x)|\,dx$$

For $\forall\ 1\le p<n$, measure estimate:

$\forall t>0$, $t^{\frac{np}{n-p}}\mu(\{x:|f|>t\}\cap B_r(0))$

$$\le c\mu(B_r(0))\left(r^p\fint_{B_r(0)}|\nabla f|^p\right)^{\frac{n}{n-p}} + c\left(\fint_{B_0} f\right)^{\frac{np}{n-p}}\mu(B_r(0))$$

**Proof of lemma 7:**

Let $f_i(x)=\min\{2^i,\max\{|f|-2^i,0\}\}$

For $i\in\mathbb{Z}$, by measure estimate: $t=2^i$

$$2^{i\frac{np}{n-p}}\mu(\{x:|f_i|\ge 2^i\}\cap B_r(0))$$
$$\le c\mu(B_r(0))\left(r^p\fint_{B_r(0)}|\nabla f_i|^p\right)^{\frac{n}{n-p}} + c\left(\fint_{B_0} f_i\right)^{\frac{np}{n-p}}\mu(B_r(0))$$

$$|\nabla f_i|\le\begin{cases}0 & |f|<2^i,\ |f|>2^{i+1}\\ |\nabla f|, & 2^i\le|f|\le 2^{i+1}\end{cases}$$

Where $|\nabla|f||\le|\nabla f|$.

On the other side, $\{x:|f_i|\ge 2^i\}\supseteq\{x:|f|\ge 2^{i+1}\}$.

$$\fint_{B_0} f_i=\frac{1}{\mu(B_0)}\int_{B_0} f_i(x)\,dx$$
$$\le\frac{1}{\mu(B_0)}\left(\int_{2^i\le|f|\le 2^{i+1}}|f|-2^i\,dx+\int_{|f|>2^{i+1}}2^i\,dx\right)$$
$$\le\frac{1}{\mu(B_0)}\left(\int_{2^i\le|f|\le 2^{i+1}}|f|\,dx+2^i\mu(\{x:|f|\ge 2^{i+1}\})\right)$$

$$\Rightarrow\int_{B_r(0)} f^{\frac{np}{n-p}}dx\le C(n,p)\sum_{i\in\mathbb{Z}}2^{i\frac{np}{n-p}}\mu(\{x:|f|\ge 2^{i+1}\})$$
$$\le c\mu(B_r(0))\sum_{i\in\mathbb{Z}}\left(\frac{r^p}{\mu(B_r(0))}\int_{2^i\le|f|\le 2^{i+1}}|\nabla f|^p dx\right)^{\frac{n}{n-p}}$$
$$+\sum_{i\in\mathbb{Z}}c\left(\frac{1}{\mu(B_0)}\int_{2^i\le|f|\le 2^{i+1}}|f|\,dx+2^i\mu(\{x:|f|\ge 2^{i+1}\})\right)^{\frac{np}{n-p}}\cdot\mu(B_r(0)).$$

Since $(a_1^p+\cdots a_n^p)\le(a_1+\cdots a_n)^p$, $p\ge 1$

$$\Rightarrow\fint_{B_r(0)} f^{\frac{np}{n-p}}dx$$
$$\le c\left(\sum_{i\in\mathbb{Z}}\frac{r^p}{\mu(B_r(0))}\int_{2^i\le|f|\le 2^{i+1}}|\nabla f|^p dx\right)^{\frac{n}{n-p}}$$
$$+c\left(\frac{1}{\mu(B_0)}\sum_{i\in\mathbb{Z}}\left[\int_{2^i\le|f|\le 2^{i+1}}|f|\,dx+2^i\mu(\{x:|f|\ge 2^{i+1}\})\right]\right)^{\frac{np}{n-p}}$$

$\left(a+b\right)^\alpha\le a^\alpha+b^\alpha$, $\alpha<1$

$(a+tb)^\alpha\le a^\alpha+(tb)^\alpha$

$\Leftarrow((a+tb)^\alpha)'\le((tb)^\alpha)'$

$\alpha=\frac{n-p}{np}$

$$\le c\left(\frac{r^p}{\mu(B_r(0))}\int_{B_r(0)}|\nabla f|^p dx\right)^{\frac{n}{n-p}}$$
$$+c\left(\frac{1}{\mu(B_0)}\left[\int_{B_r(0)}|f|\,dx+\int_{B_r(0)}|f|\,dx\right]\right)^{\frac{np}{n-p}}$$

Since $\mu(B_0)=\mu(B_{\frac{r}{10}}(0))=\mu(B_r(0))10^{-n}$

We prove the lemma. $\square$

Log - Soblev ineq'l:

Thm1: $\forall f \in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f^2 dx = 1$, then $\int_{\mathbb{R}^n} f^2 \ln f^2 dx \leq \frac{n}{2} \ln\left(c(n) \int_{\mathbb{R}^n} |\nabla f|^2 dx\right)$

Proof: Since $\left(\int_{\mathbb{R}^n} f^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}$

$\leq c(n) \int_{\mathbb{R}^n} |\nabla f|^2 dx$, (sobler ineq'L)

assume $d\mu = f^2 dx \Rightarrow \int_{\mathbb{R}^n} d\mu = 1$

$\Rightarrow \int_{\mathbb{R}^n} f^{\frac{2n}{n-2}} dx = \int_{\mathbb{R}^n} f^{\frac{4}{n-2}} d\mu$

Let $h(x) = \ln(x)$, by Jensen ineq'l,

$\ln\left(\int_{\mathbb{R}^n} f^{\frac{4}{n-2}} d\mu\right) \geq \int_{\mathbb{R}^n} \ln f^{\frac{4}{n-2}} d\mu$

$= \frac{2}{n-2} \int_{\mathbb{R}^n} \ln f^2 d\mu = \frac{2}{n-2} \int_{\mathbb{R}^n} f^2 \ln f^2 dx$

$\square$

[Remark 1: the best $c(n) = \frac{2}{n\pi e}$ ]

[Remark 2: Riemannian mfld, $Ric \geq 0$, if in log-Soblev, $c(n) = \frac{2}{n\pi e}$ holds, then this mfld isometric to $\mathbb{R}^n$ ]

Thm2: Assume $f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$, assume $d\mu(x) = e^{-\frac{|x|^2}{2}} \frac{dx}{(2\pi)^{\frac{n}{2}}}$

Then $\int_{\mathbb{R}^n} f \ln f \, d\mu - \left(\int_{\mathbb{R}^n} f d\mu\right) \ln\left(\int_{\mathbb{R}^n} f d\mu\right)$

$\leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\mu$

(or write as: $\int_{\mathbb{R}^n} f^2 \ln f^2 d\mu$

$- \int_{\mathbb{R}^n} f^2 d\mu \ln\left(\int_{\mathbb{R}^n} f^2 d\mu\right) \leq 2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$.

Def: (Heat semi-group): $\forall f \in C_0^\infty(\mathbb{R}^n), t > 0$,

$(P_t f)(x) \triangleq \int_{\mathbb{R}^n} f(y) e^{-\frac{|x-y|^2}{4t}} \frac{dy}{(4\pi t)^{\frac{n}{2}}}$

$\sigma^2 = 2t$

denote: $\rho_t(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$

Lemma1: $\forall t, s > 0$.

$\rho_{t+s}(x,y) = \int_{\mathbb{R}^n} \rho_t(x,z) \rho_s(z,y) dz$

Proof: We need a formula:

$\int_{\mathbb{R}^n} e^{-x^T A x + v^T x} dx = \sqrt{\frac{\pi^n}{\det A}} e^{\frac{1}{4} v^T A^{-1} v}$

( $A$ is positive definite, $v$ is const n-dim vector)

Since $\rho_t(x,z) \rho_s(z,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \cdot \frac{1}{(4\pi s)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t} - \frac{|y|^2}{4s}}$

$\cdot e^{-\left(\frac{1}{4t} + \frac{1}{4s}\right)|z|^2 + \langle \frac{x}{2t} + \frac{y}{2s}, z \rangle}$

$\Rightarrow A = \left(\frac{1}{4t} + \frac{1}{4s}\right) E$, $V = \frac{x}{2t} + \frac{y}{2s}$

$\Rightarrow \int_{\mathbb{R}^n} \rho_t(x,z) \rho_s(z,y) dz = \frac{1}{(4\pi t)^{\frac{n}{2}} (4\pi s)^{\frac{n}{2}}} \cdot e^{-\frac{|x|^2}{4t} - \frac{|y|^2}{4s}}$

$\cdot \frac{\pi^{\frac{n}{2}}}{\left(\frac{1}{4t} + \frac{1}{4s}\right)^{\frac{n}{2}}} e^{\frac{ts}{t+s}\left(\frac{|x|^2}{4t^2} + \frac{|y|^2}{4s^2} + \frac{\langle x, y \rangle}{2ts}\right)}$

$= \rho_{t+s}(x,y)$

$\square$

Lemma 2:

① $\lim_{t\to 0^+} (P_t f)(x) = f(x)$

② $(P_t(P_s f))(x) = (P_{t+s}(f))(x),\ \forall t,s>0$

③ $|\nabla(P_t f)(x)| \le \int_{\mathbb{R}^n} |\nabla f|(y)\, \rho_t(x,y)\,dy$

Proof: ① Consider $|P_t f(x) - f(x)|$

$$= \left| \int_{\mathbb{R}^n} (f(x)-f(y))\, \rho_t(x,y)\,dy \right|$$

since $\int_{\mathbb{R}^n} \rho_t(x,y)\,dy = 1$,

then $\le \int_{\mathbb{R}^n} |f(x)-f(y)|\, \rho_t(x,y)\,dy$

$\forall \varepsilon>0,\ \exists \delta$. s.t. when $|x-y|<\delta$,

$|f(y)-f(x)| \le \varepsilon$ $\quad f\in C_0^\infty(\mathbb{R}^n)$

$$\Rightarrow |P_t f - f| \le \varepsilon + \int_{\mathbb{R}^n \setminus B_\delta(x)} |f(y)-f(x)|\, \rho_t(x,y)\,dy$$

Since $|f|\le M$, $\Rightarrow |P_t f - f| \le \varepsilon + M\int_{\mathbb{R}^n\setminus B_\delta(x)} \rho_t(x,y)\,dy$

Consider polar coords: $y = x + r\theta$,

$$\int_{\mathbb{R}^n\setminus B_\delta(x)} \rho_t(x,y)\,dy$$

$$= \int_{S^{n-1}} \int_\delta^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{r^2}{4t}} r^{n-1}\,dr\,d\theta$$

$$\overset{u=\frac{r^2}{4t}}{=} \int_{S^{n-1}} \int_{\frac{\delta^2}{4t}}^\infty \frac{1}{2}\frac{1}{\pi^{\frac{n}{2}}} e^{-s}\cdot s^{\frac{n-2}{2}}\,ds\,d\theta$$

$\to 0$ as $t\to 0^+$ $\quad (\frac{\delta^2}{4t}\to\infty)$

② $(P_t(P_s f))(x) = \int_{\mathbb{R}^n} (P_s f)(y)\, \rho_t(x,y)\,dy$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(z)\rho_s(y,z)\rho_t(x,y)\,dz\,dy$$

$$= \int_{\mathbb{R}^n} f(z)\left(\int_{\mathbb{R}^n} \rho_s(y,z)\rho_t(x,y)\,dy\right)dz$$

$$= \int_{\mathbb{R}^n} f(z)\, \rho_{t+s}(x,z)\,dz$$

$$= (P_{t+s} f)(x)$$

[ like $e^t\cdot e^s = e^{t+s}$ ]

③ $|\nabla(P_t f)(x)|$,

consider $\frac{\partial P_t f}{\partial x_i} = \int_{\mathbb{R}^n} f(y) \frac{\partial \rho_t(x,y)}{\partial x_i}\,dy$

but $\frac{\partial \rho_t(x,y)}{\partial x_i} = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} \left(-\frac{x_i-y_i}{2t}\right)$

$$= -\frac{\partial \rho_t(x,y)}{\partial y_i}.$$

$$\Rightarrow \frac{\partial P_t f}{\partial x_i} = -\int_{\mathbb{R}^n} f(y)\frac{\partial \rho_t(x,y)}{\partial y_i}\,dy$$

$$= \int_{\mathbb{R}^n} \frac{\partial f(y)}{\partial y_i}\rho_t(x,y)\,dy$$

By partial integral.

$$\Rightarrow \underline{\nabla(P_t f)(x) = \int_{\mathbb{R}^n} \nabla f(y)\, \rho_t(x,y)\,dy}.$$

Key!

$$\Rightarrow |\nabla(P_t f)| \le \int_{\mathbb{R}^n} |\nabla f|(y)\, \rho_t(x,y)\,dy$$

Remark $|\int \vec{\alpha}| \le \int |\vec{\alpha}|$.

let $\vec{\beta} = \frac{\int\vec{\alpha}}{|\int\vec{\alpha}|}$, $|\langle \vec{\beta}, \int\vec{\alpha}\rangle| = |\int\langle\vec{\alpha},\vec{\beta}\rangle| \le \int|\langle\vec{\alpha},\vec{\beta}\rangle| \le \int|\vec{\alpha}|$

Def: Laplacian: $\Delta \triangleq \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2}$

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots \frac{\partial^2 f}{\partial x_n^2}.$$

Property: $(\partial_t - \Delta_x)\, \rho_t(x,y) = 0$

Proof: $\partial_t \rho_t(x,y) = \rho_t(x,y)\left(-\frac{n}{2t} + \frac{|x-y|^2}{4t^2}\right)$

and $\frac{\partial}{\partial x_i} \rho_t(x,y) = \rho_t(x,y)\left(-\frac{x_i - y_i}{2t}\right)$

$\frac{\partial^2}{\partial x_i^2} \rho_t(x,y) = \rho_t(x,y)\left(-\frac{1}{2t} + \frac{(x_i-y_i)^2}{2t}\right).$

$\square$

Cor: $(\partial_t - \Delta)(P_t f) = 0.$

Proof: $\Delta(P_t f)(x)$

$= \int_{\mathbb{R}^n} f(y)\, \Delta_x \rho_t(x,y)\, dy$

$= \int_{\mathbb{R}^n} f(y)\, \partial_t \rho_t(x,y)\, dy$

$= \partial_t (P_t f)(x)$

$\square$

Cor: $\Delta(P_t f)(x) = (P_t(\Delta f))(x)$

Proof: $\Delta P_t f(x) = \int_{\mathbb{R}^n} f(y)\, \Delta_x \rho_t(x,y)\, dy$

$= \int_{\mathbb{R}^n} f(y)\, \partial_t \rho_t(x,y)\, dy$

$= \int_{\mathbb{R}^n} f(y)\, \Delta_y \rho_t(x,y)\, dy$ $\quad \int_\Omega u\Delta v - v\Delta u\, dy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, d\sigma(y)$

divergence Thm.

$= \int_{\mathbb{R}^n} \Delta_y f(y)\, \rho_t(x,y)\, dy$ since $f \in C_0^\infty$.

$= P_t(\Delta f)(x).$

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Last time:

Log-Soblev: $d\mu = e^{-\frac{|x|^2}{2}} \cdot (2\pi)^{-\frac{n}{2}} dx$

$\forall f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$, we have

$$\int_{\mathbb{R}^n} f \ln f \, d\mu - \int_{\mathbb{R}^n} f d\mu \ \ln\left(\int_{\mathbb{R}^n} f d\mu\right) \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} d\mu$$

- $P_t(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$, $t>0$, $x, y \in \mathbb{R}^n$
- $(P_t f)(x) \triangleq \int_{\mathbb{R}^n} f(y) P_t(x,y)\, dy$
- $\Delta \triangleq \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2}$

Properties: ① $(\partial_t - \Delta_x) P_t(x,y) = 0$

② $\lim_{t\to 0^+} (P_t f)(x) = f(x)$

③ $(P_t(P_s f))(x) = (P_{t+s} f)(x)$, $\forall t, s > 0$

④ $|\nabla(P_t f)|(x) \leq \int_{\mathbb{R}^n} |\nabla f|(y) P_t(x,y)\, dy$

We first proved $\nabla(P_t f)(x) = \int_{\mathbb{R}^n} (\nabla f)(y) P_t(x,y)\, dy$ ⋯ (*)

⑤ $|\nabla(P_t f)(x)|^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2(y) P_t(x,y)\, dy$

Proof of ⑤:

Way 1: Apply Cauchy ineq'l to (*)

Way 2: Let $F_s(x) = P_s(|\nabla(P_{t-s} f)|^2)(x)$, for $0<s<t$, Key!

$\lim_{s\to 0^+} F_s(x) = |\nabla(P_t f)|^2(x)$

$\lim_{s\to t^-} F_s(x) = P_t(|\nabla f|^2)(x)$

with $\frac{d}{ds} F_s(x) = \frac{d}{ds} P_s(|\nabla(P_{t-s} f)|^2)(x)$

$+ P_s\left(\frac{d}{ds} |\nabla(P_{t-s} f)|^2\right)(x)$

By ① and the last Cor last time, "$\Delta(P_t f)(x) = (P_t(\Delta f))(x)$",

$= P_s(\Delta |\nabla P_{t-s} f|^2)(x)$

$+ P_s\left(2 \langle \nabla(P_{t-s} f), \nabla(\frac{d}{ds} P_{t-s} f)\rangle\right)$

$= P_s(\Delta |\nabla P_{t-s} f|^2)(x)$

$- 2 P_s\left(\langle \nabla(\Delta P_{t-s} f), \nabla(P_{t-s} f)\rangle\right)$.

Now we need a Bochner Formula:

In $\mathbb{R}^n$, $\Delta |\nabla f|^2 = 2 \underbrace{\sum_{i,j=1}^n \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2}_{|\nabla^2 f|^2} + 2\langle \nabla\Delta f, \nabla f\rangle$

Proof of Bochner Formula:

$|\nabla f|^2 = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2$,

$\Delta |\nabla f|^2 = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2$

$= \sum_{j=1}^n \sum_{i=1}^n \frac{\partial}{\partial x_j}\left(2 \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot \frac{\partial f}{\partial x_i}\right)$

$= \sum_{j=1}^n \sum_{i=1}^n \left(2 \frac{\partial^3 f}{\partial x_i (\partial x_j)^2} \cdot \frac{\partial f}{\partial x_i} + 2\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2\right)$

$= 2|\nabla^2 f|^2 + 2 \sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial^2 f}{(\partial x_j)^2}\right) \cdot \frac{\partial f}{\partial x_i}$

$= 2|\nabla^2 f|^2 + 2 \sum_{i=1}^n \frac{\partial}{\partial x_i}(\Delta f) \cdot \frac{\partial f}{\partial x_i}$

$= 2|\nabla^2 f|^2 + 2\langle \nabla\Delta f, \nabla f\rangle$

Now, by Bochner Formula,

$\frac{d}{ds} F_s(x) = P_s\left(2 |\nabla^2 (P_{t-s} f)|^2\right) \geq 0$

$\Rightarrow \lim_{s\to 0^+} F_s(x) \leq \lim_{s\to t^-} F_s(x)$, which is ⑤ $\square$

⑥ $P_t(f\ln f) - P_t f \ln P_t f$

$$= \int_0^t P_s\left(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\right)(x)\, ds, \quad \forall t>0$$

Proof of ⑥: Consider $G_s(x) = P_s(P_{t-s} f \ln P_{t-s} f)(x)$

then $\lim_{s\to 0^+} G_s(x) = P_t f \ln P_t f$ Proved in Homework.

$\lim_{s\to t^-} G_s(x) = P_t(f\ln f)(x)$ using $\lim_{s\to t^-} (P_t - P_s)(f)(x) = 0$ uniformly, only $\|f\|_\infty$ matters

$$\frac{d}{ds} G_s(x) = \left(\frac{d}{ds} P_s\right)\left(P_{t-s} f \ln P_{t-s} f\right) + P_s\left(\frac{d}{ds}\left(P_{t-s} f \ln P_{t-s} f\right)\right)$$

$$= P_s\left(\Delta\left(P_{t-s} f \ln P_{t-s} f\right)\right) + P_s\left(-\Delta(P_{t-s} f)\cdot(\ln P_{t-s} f) - \Delta(P_{t-s} f)\right)$$

And $\Delta\left(P_{t-s} f \ln P_{t-s} f\right)$

$$= \Delta P_{t-s} f\,(\ln P_{t-s} f + 1) + \frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}$$

because $\Delta(g\cdot \ln g) = \mathrm{div}(\nabla(g\cdot\ln g))$

$= \mathrm{div}(\nabla g\cdot \ln g + \nabla g) = \mathrm{div}(\nabla g\,(\ln g + 1))$

$= \Delta g\,(\ln g + 1) + \nabla g \cdot \nabla(\ln g + 1) = \Delta g(\ln g + 1) + \frac{|\nabla g|^2}{g}$

$$\Rightarrow \frac{d}{ds} G_s(x) = P_s\left(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\right)(x)$$

$\square$

⑦ $$\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f} \le P_{t-s}\left(\frac{|\nabla f|^2}{f}\right), \quad (f \ge 0)$$

Proof of ⑦: By ④

$$|\nabla P_{t-s} f| \le \int_{\mathbb{R}^n} |\nabla f|(y)\, \rho_{t-s}(x,y)\, dy$$

$$\le \left(\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f} \rho_{t-s}(x,y)\, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} f \rho_{t-s}(x,y)\, dy\right)^{\frac{1}{2}}$$

Another way:

$$H_u(x) = P_u\left(\frac{|\nabla P_{t-s-u} f|^2}{P_{t-s-u} f}\right)$$

compute $\frac{d}{du} H_u \ge 0$

$\square$

By ⑥, $P_t(f\ln f) - P_t f \ln P_t f$

$$= \int_0^t P_s\left(\frac{|\nabla P_{t-s} f|^2}{P_{t-s} f}\right)(x)\, ds$$

$$\le \int_0^t P_s\cdot\left(P_{t-s}\left(\frac{|\nabla f|^2}{f}\right)\right)(x)\, ds$$

$$= P_t\left(\frac{|\nabla f|^2}{f}\right)\cdot t$$

Let $t = \frac{1}{2}$, $\rho_{\frac{1}{2}}(x,0) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}}$,

when $t = \frac{1}{2}$,

$$\int_{\mathbb{R}^n} f\ln f\, d\mu - \int_{\mathbb{R}^n} f\, d\mu \ln\left(\int_{\mathbb{R}^n} f\, d\mu\right) \le \frac{1}{2}\int_{\mathbb{R}^n} \frac{|\nabla f|^2}{f}\, d\mu, \quad \cdots (***)$$

which is the Log-Soblev ineq'l !

$\square$

Thm. (the most optimized Log-Soblev const.)

$\forall f \in C_0^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} f^2 dx = 1$

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \le \frac{n}{2}\ln\left(\frac{2}{n\pi e}\int_{\mathbb{R}^n} |\nabla f|^2 dx\right)$$

Proof: Let $d\mu = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx$,

let $h(x) = (2\pi)^{\frac{n}{4}} e^{\frac{|x|^2}{4}} f(x)$, then

$$\int_{\mathbb{R}^n} h^2(x)\, d\mu = 1$$

And $\nabla h(x) = (2\pi)^{\frac{n}{4}} e^{\frac{|x|^2}{4}} \left(\nabla f(x) + \frac{x}{2}\cdot f(x)\right)$

$|\nabla h|^2(x) = (2\pi)^{\frac{n}{2}} e^{\frac{|x|^2}{2}} \left( |\nabla f|^2 + \frac{|x|^2}{4} f^2 + \langle x, \nabla f\rangle f \right)$

By the Log-Sobolev (***),

$$\int_{\mathbb{R}^n} h^2 \ln h^2 d\mu - 0$$

$$\le 2\int_{\mathbb{R}^n} |\nabla h|^2(x)\, d\mu$$

$$\Rightarrow \int_{\mathbb{R}^n} \left( |\nabla f|^2 + \frac{|x|^2}{4}\cdot f^2 + \langle x, \nabla f\rangle f \right) dx$$

And $\int_{\mathbb{R}^n} h^2 \ln h^2 d\mu = \int_{\mathbb{R}^n} f^2 \ln\left((2\pi)^{\frac{n}{2}} e^{\frac{|x|^2}{2}} f^2\right) dx$

$$\Rightarrow \int_{\mathbb{R}^n} f^2 \ln f^2 + \frac{|x|^2}{2}\cdot f^2 + \frac{n}{2}\ln(2\pi) f^2 dx$$

$$\le 2\int_{\mathbb{R}^n} \left( |\nabla f|^2 + \frac{|x|^2}{4} f^2 + \frac{1}{2}\langle x, \nabla f^2 \rangle \right) dx$$

$$\Rightarrow \int_{\mathbb{R}^n} f^2 \ln f^2 dx + \frac{n}{2}\ln(2\pi)$$

$$\le 2\int_{\mathbb{R}^n} |\nabla f|^2 dx - n\int_{\mathbb{R}^n} f^2 dx$$

(Partial integral)

$$= 2\int_{\mathbb{R}^n} |\nabla f|^2 dx - n \quad \cdots (****)$$

For $\forall \delta > 0$, Let $F(x) = \delta^{\frac{n}{2}} f(\delta x)$

Key!

$$\Rightarrow \int_{\mathbb{R}^n} F^2(x) dx = \int_{\mathbb{R}^n} \delta^n f^2(\delta x)\, dx = 1$$

by (****), $\int_{\mathbb{R}^n} \delta^n f^2(\delta x) \ln\left(\delta^n f^2(\delta x)\right) dx$

$$+ \frac{n}{2}\ln(2\pi e^2) \le 2\int_{\mathbb{R}^n} |\nabla F|^2(x) dx$$

since $\nabla F = \delta^{\frac{n}{2}+1} \nabla f(\delta x)$

$$\Rightarrow |\nabla F|^2 = \delta^{n+2} |\nabla f|^2(\delta x)$$

let $y = \delta x$,

$$\int_{\mathbb{R}^n} f^2(y) \ln\left(\delta^n f^2(y)\right) dy + \frac{n}{2}\ln(2\pi e^2)$$

$$\le 2\int_{\mathbb{R}^n} \delta^2 |\nabla f|^2(y)\, dy$$

$$\Rightarrow \int_{\mathbb{R}^n} f^2(y) \ln f^2(y)\, dy$$

$$\le 2\delta^2 \int_{\mathbb{R}^n} |\nabla f|^2(y) dy - \frac{n}{2}\ln(2\pi e^2) - n\ln\delta \qquad \delta^2 = t.$$

Consider $\mu(t) = 2t\int_{\mathbb{R}^n} |\nabla f|^2(y)\, dy - \frac{n}{2}\ln t$

$$\mu'(t) = 2\int_{\mathbb{R}} |\nabla f|^2(y) dy - \frac{n}{2}\cdot\frac{1}{t}$$

take $\delta^2 = t = \frac{n}{4}\cdot\left(\int_{\mathbb{R}^n} |\nabla f|^2 dy\right)^{-1}$

$$\Rightarrow \int_{\mathbb{R}^n} f^2(y) \ln f^2(y)\, dy$$

$$\le \frac{n}{2} - \frac{n}{2}\ln(2\pi e^2) + \frac{n}{2}\ln\left(\frac{4}{n}\int_{\mathbb{R}^n} |\nabla f|^2 dy\right)$$

$$= \frac{n}{2}\ln\left(\frac{2}{n\pi e}\int_{\mathbb{R}^n} |\nabla f|^2(y)\, dy\right)$$

Most optimized: If $\int_{\mathbb{R}^n} f^2 dx = 1$, $\int_{\mathbb{R}^n} f^2 \ln f^2 dx \le \phi\left(\int_{\mathbb{R}^n} |\nabla f|^2 dx\right)$ holds For $\forall f \in C_0^\infty(\mathbb{R}^n)$, then $\phi(t) \ge \psi(t) = \frac{n}{2}\ln\left(\frac{2}{n\pi e} t\right)$, $\forall t > 0$ holds.

Proof: Let $f^2(x) = \lambda^n \varphi_n(\lambda x)$, $\lambda > 0$ where $\varphi_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} \Rightarrow \int_{\mathbb{R}^n} f^2(x) dx = 1$

$$f(x) = \lambda^{\frac{n}{2}}(2\pi)^{-\frac{n}{4}} e^{-\frac{|\lambda x|^2}{4}}$$

$$\Rightarrow \nabla f = \lambda^{\frac{n}{2}}(2\pi)^{-\frac{n}{4}} e^{-\frac{|\lambda x|^2}{4}}\left(-\frac{\lambda^2}{2}x\right)$$

$$\Rightarrow |\nabla f|^2 = \lambda^{n}(2\pi)^{-\frac{n}{2}} e^{-\frac{|\lambda x|^2}{2}} \frac{\lambda^4}{4}|x|^2$$

$$\Rightarrow \frac{n}{2}\ln\left(\frac{\lambda^2}{2\pi}\right) - \frac{n}{2} \leq \phi\left(\frac{n\lambda^2}{4}\right) = \phi\left(\int_{\mathbb{R}^n}|\nabla f|^2 dx\right)$$

let $t = \frac{n\lambda^2}{4}$,

$$\phi(t) \geq \frac{n}{2}\ln\left(\frac{2t}{n\pi e}\right)$$

$\square$

$$\frac{\partial}{\partial r} e^{-\frac{r^2}{2}} = -r e^{-\frac{r^2}{2}}$$

$$\|$$

Remark: $\Delta e^{-\frac{|x|^2}{2}} = e^{-\frac{|x|^2}{2}}\left(-n+|x|^2\right)$

$$\Rightarrow 0 = \int_{\mathbb{R}^n} \Delta e^{-\frac{|x|^2}{2}} dx \quad \left|\int_{\Omega} \Delta e^{-\frac{|x|^2}{2}} dx\right| = \left|\int_{\partial\Omega} \langle \nabla e^{-\frac{|x|^2}{2}}, n\rangle d\sigma(x)\right|$$

$$= \int_{\mathbb{R}^n} -n e^{-\frac{|x|^2}{2}} dx + \int_{\mathbb{R}^n} |x|^2 e^{-\frac{|x|^2}{2}} dx \qquad \leq \int_{\partial\Omega} \left|\nabla e^{-\frac{|x|^2}{2}}\right| d\sigma(x)$$

$\to 0$ as $\mathrm{diam}(\Omega) \to +\infty$.

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Poincare ineq'l

let $d\mu_t = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} dx$

$P_t(x,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$

then $d\mu_t(x) = P_t(x,0)dx$, $\int_{\mathbb{R}^n} 1\, d\mu_t(x) = 1$ (or $\mu_t(\mathbb{R}^n)=1$)

Thm. $\forall f \in C_0^\infty(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \left| f(x) - \int_{\mathbb{R}^n} f(z)\, d\mu_t(z) \right|^2 d\mu_t(x)$$

$$\leq 2t \int_{\mathbb{R}^n} |\nabla f|^2 d\mu_t$$

Proof: $LHS = \int_{\mathbb{R}^n} [f^2 + (\int_{\mathbb{R}^n} f(z) d\mu_t(z))^2 - 2f(x)\int_{\mathbb{R}^n} f(z) d\mu_t(z)] d\mu_t(x)$

$= \int_{\mathbb{R}^n} f^2(x)\, d\mu_t(x) - (\int_{\mathbb{R}^n} f(z)\, d\mu_t(z))^2$,

Let $\underline{F_s = P_s((P_{t-s} f)^2)(0)}$, key!

when $s=0$, $F_s|_{s=0} = (P_t f)^2 = (\int_{\mathbb{R}^n} f(x)\, d\mu_t(x))^2$

when $s=t$, $F_s|_{s=t} = \int_{\mathbb{R}^n} f^2(x)\, d\mu_t(x)$

compute $\frac{d}{ds} F_s = P_s(\Delta (P_{t-s} f)^2)$

$\qquad - P_s(2P_{t-s} f\, \Delta P_{t-s} f)$

since $\Delta f^2 = 2f\Delta f + 2|\nabla f|^2$

$\Rightarrow \frac{d}{ds} F_s = 2P_s(|\nabla (P_{t-s} f)|^2)$

$\underline{\frac{d}{ds}(P_s(|\nabla P_{t-s} f|^2))} = P_s(\Delta |\nabla P_{t-s} f|^2$

key! another derivative! $\quad -2\langle \nabla P_{t-s} f, \nabla \Delta P_{t-s} f\rangle)$

$= 2P_s(|\nabla^2 P_{t-s} f|^2)$ by Bochner formula

$\geq 0$

$\Rightarrow 2P_t(|\nabla f|^2) \geq 2P_s(|\nabla (P_{t-s} f)|^2)$, $\forall 0 < s \leq t$

$= \frac{d}{ds} F_s$

$\Rightarrow F_t - F_0 \leq 2t\, P_t(|\nabla f|^2)$ (integral at both side) $\square$

Thm. Assume $\mu$ is a probability measurement, i.e. $\mu(\mathbb{R}^n) = 1$. If for $\forall f \in C_0^\infty(\mathbb{R}^n)$, holds $\int_{\mathbb{R}^n} f^2 \ln f^2 d\mu - \int_{\mathbb{R}^n} f^2 d\mu \ln(\int_{\mathbb{R}^n} f^2 d\mu)$

$\leq \lambda \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$, $(*)$ (Log-Soblev)

then $\int_{\mathbb{R}^n} (f - \underline{\int_{\mathbb{R}^n} f d\mu})^2 d\mu$

$\leq \frac{\lambda}{2} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$, for $\forall f \in C_0^\infty(\mathbb{R}^n)$

Proof: Assume $\int_{\mathbb{R}^n} f d\mu = 0$,

Let $u_t = 1 + tf$, use Log-Sobler ineq'l $(*)$,

$\Rightarrow \int_{\mathbb{R}^n} (1+tf)^2 \ln(1+tf)^2 d\mu$

$- \int_{\mathbb{R}^n} (1+tf)^2 d\mu \ln(\int_{\mathbb{R}^n} (1+tf)^2 d\mu)$

$= \int_{\mathbb{R}^n} (1 + 2tf + t^2 f^2)(\underbrace{2tf - t^2 f^2 + O(t^3)}_{\text{key}}) d\mu$

$- (1 + t^2 \int_{\mathbb{R}^n} f^2 d\mu)(t^2 \int_{\mathbb{R}^n} f^2 + O(t^4) d\mu$

$= \int_{\mathbb{R}^n} (4t^2 f^2 - t^2 f^2 + O(t^3))$

$- t^2 \int_{\mathbb{R}^n} f^2 d\mu + O(t^4)$

$= 2t^2 \int_{\mathbb{R}^n} f^2 d\mu + O(t^3)$.

But $\int_{\mathbb{R}^n} |\nabla u_t|^2 d\mu = t^2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$

$\Rightarrow 2t^2 \int_{\mathbb{R}^n} f^2 d\mu + O(t^3) \leq \lambda t^2 \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$

$\Rightarrow 2\int_{\mathbb{R}^n} f^2 d\mu \leq \lambda \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$ $\square$

Open problem: $\mu$ is a probability measurement

Let $\lambda_L^{-1} = \inf_{f \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla f|^2 d\mu}{\int_{\mathbb{R}^n} f^2 \ln f^2 d\mu - \int_{\mathbb{R}^n} f^2 (\ln \int_{\mathbb{R}^n} f^2 d\mu) d\mu}$

$$\lambda_p^{-1} = \inf_{\substack{f\in C_0^\infty(\mathbb{R}^n)\\ f\neq 0}} \frac{\int_{\mathbb{R}^n} |\nabla f|^2 d\mu}{\int_{\mathbb{R}^n} (f - \int_{\mathbb{R}^n} f d\mu)^2 d\mu}$$

Question, $\lambda_p^{-1} = 2\lambda_L^{-1}$ ??

Remark: by Thm $\Rightarrow \lambda_p^{-1} \geq 2\lambda_L^{-1}$

What we have known:

1) $\mathrm{supp}(\mu) \subseteq [0,1]$, $d\mu = dx$, $n=1$

2) replace $\mathbb{R}^n$ to $S^n = \{x\in\mathbb{R}^{n+1}, |x|=1\}$,

$\mu$ is the probability measurement on $S^n$.

Similar Question: replace $\mathbb{R}^n$ by compact mfld $M$, $\mu$ is $M$'s volume measurement

Question: $\lambda_p = 2\lambda_L$ ??

Thm. On $\mathbb{R}^n$, the following ineq'l are equivalent

(1) Nash ineq'l:

$$\|f\|_2^{2+\frac{4}{n}} \leq c(n) \|f\|_1^{\frac{4}{n}} \int_{\mathbb{R}^n} |\nabla f|^2 dx,$$

for any $f\in C_0^\infty(\mathbb{R}^n)$

(2) Log-Soblev ineq'l I: $\forall f \in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f^2 dx = 1$, we have

$$\int_{\mathbb{R}^n} f^2 \ln f^2 dx \leq \frac{n}{2} \ln\left(c(n) \int_{\mathbb{R}^n} |\nabla f|^2 dx\right)$$

(3) Log-Soblev ineq'l II:

$\forall f\in C_0^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f^2 dx = 1$, $\forall a>0$,

we have $\int_{\mathbb{R}^n} f^2 \ln f^2 dx$

$$\leq a \int_{\mathbb{R}^n} |\nabla f|^2 dx - \frac{n}{2}\ln a + c(n)$$

(4) Soblev ineq'l: $\forall f\in C^\infty(\mathbb{R}^n)$

we have $\left(\int_{\mathbb{R}^n} f^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}} \leq c(n) \int_{\mathbb{R}^n} |\nabla f|^2 dx$

Proof: ④ $\Rightarrow$ ② Jensen ineq'l.

③ $\Leftrightarrow$ ②: $\ln x \leq ax - \ln a - 1$ (② $\Rightarrow$ ③)

$\forall a>0$, $x>0$ holds

③ $\Rightarrow$ ②: for fixed $f$, let $a = \left(\int_{\mathbb{R}^n} |\nabla f|^2 dx\right)^{-1}$

we only need to prove ② $\Rightarrow$ ① and ① $\Rightarrow$ ④

② $\Rightarrow$ ①: Let $g = \lambda f$, $\lambda = \|f\|_2^{-1}$, multiple $\lambda^{2+\frac{4}{n}}$,

the ineq'l in (1) $\Leftrightarrow \|\lambda f\|^{2+\frac{4}{n}}$

$$\leq c(n) \|\lambda f\|_1^{\frac{4}{n}} \int_{\mathbb{R}^n} |\nabla(\lambda f)|^2 dx$$

Note that $\|g\|_2 = \|\lambda f\|_2 = 1$,

we need to prove $1 \leq c(n) \|g\|_1^{\frac{4}{n}} \int_{\mathbb{R}^n} |\nabla g|^2 dx$ (*)

Assume $\int_{\mathbb{R}^n} f^2 dx = 1$ first, let $\phi(u) = \ln \|f\|_{\frac{1}{u}}$

$$\psi(u) = \frac{\phi(u) - \phi(\frac{1}{2})}{u - \frac{1}{2}}$$

$$\phi'(u) = \left(u \ln\left(\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx\right)\right)'$$

$$= \ln\left(\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx\right) + \frac{u\cdot \int_{\mathbb{R}^n} f^{\frac{1}{u}} \ln f (\frac{-1}{u^2}) dx}{\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx}$$

$$= \ln \int_{\mathbb{R}^n} f^{\frac{1}{u}} dx - \frac{1}{u} \frac{\int_{\mathbb{R}^n} f^{\frac{1}{u}} \ln f dx}{\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx},$$

$$\phi''(u) = -\frac{1}{u^2} \frac{\int_{\mathbb{R}^n} f^{\frac{1}{u}} \ln f dx}{\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx} + \frac{1}{u^2} \frac{\int_{\mathbb{R}^n} f^{\frac{1}{u}} \ln f dx}{\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx}$$

$$- \frac{1}{u} \frac{\frac{1}{u^2}\left(\int_{\mathbb{R}^n} f^{\frac{1}{u}} \ln f dx\right)}{\left(\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx\right)^2}$$

$$+ \frac{1}{u^3} \frac{\int_{\mathbb{R}^n} f^{\frac{1}{u}} (\ln f)^2 dx}{\int_{\mathbb{R}^n} f^{\frac{1}{u}} dx}$$

$$= \frac{1}{u^3}\frac{1}{\left(\int_{\mathbb{R}^n} f^{\frac{1}{u}}dx\right)^2}\left(\int_{\mathbb{R}^n} f^{\frac{1}{u}}(\ln f)^2dx \cdot \int_{\mathbb{R}^n} f^{\frac{1}{u}}dx\right.$$

$$\left. - \left(\int_{\mathbb{R}^n} f^{\frac{1}{u}} \ln f dx\right)^2\right) \geq 0,\ \phi \text{ is convex}$$

$$\Rightarrow -\psi(1) \leq -\psi\left(\tfrac{1}{2}\right) = -\phi'\left(\tfrac{1}{2}\right)$$

$$= 2\frac{\int_{\mathbb{R}^n} f^2 \ln f dx}{\int_{\mathbb{R}^n} f^2 dx} - \ln \int_{\mathbb{R}^n} f^2 dx$$

since $\int_{\mathbb{R}^n} f^2 dx = 1$, $\Rightarrow$

$$2\int_{\mathbb{R}^n} f^2 \ln f\, dx \geq -\psi(1) = 2\ln\frac{\|f\|_2}{\|f\|_1}$$

$$\Rightarrow -2\ln\|f\|_1 \leq \int_{\mathbb{R}^n} f^2 \ln f^2 dx$$

$$\leq \frac{n}{2}\ln\left(c(n)\int_{\mathbb{R}^n} |\nabla f|^2 dx\right)$$

$$\Rightarrow \ln\left(c(n)\int_{\mathbb{R}^n} |\nabla f|^2 dx \cdot \|f\|_1^{\frac{4}{n}}\right) \geq 0$$

$$\Rightarrow c(n)\int_{\mathbb{R}^n} |\nabla f|^2 dx\, \|f\|_1^{\frac{4}{n}} \geq 1$$

□

① $\Rightarrow$ ④. If Nash ineq'l holds for compact supp smooth functions, i.e. $C_0^\infty(\mathbb{R}^n)$, then $\forall$ compact supp Lipshitz functions holds

$\forall k \in \mathbb{Z}$, assume $f \geq 0$.

Let $f_k = \min\{(f-2^k), 2^k\}$, then

$$|\nabla f_k| \leq \begin{cases} 0, & \mathbb{R}^n \setminus B_k \\ |\nabla f|, & B_k \end{cases},\quad B_k = \{2^k \leq f(x) \leq 2^{k+1}\},$$

Use Nash on $f_k$,

$$\|f_k\|_2^{2+\frac{4}{n}} \leq c(n)\|f_k\|_1^{\frac{4}{n}} \int_{\mathbb{R}^n} |\nabla f_k|^2 dx,$$

$$\Rightarrow \left(2^{2k}\mu(\{x: f(x) \geq 2^{k+1}\})\right)^{1+\frac{2}{n}}$$

$$\leq c(n)\left(2^k \mu(\{x: f(x) \geq 2^k\})\right)^{\frac{4}{n}} \int_{B_k} |\nabla f| dx,$$

Let $a_k = 2^{\frac{2n}{n-2}\cdot k}\mu(\{x: f(x) \geq 2^k\})$,

then,

$$a_{k+1}^{1+\frac{2}{n}} \leq c(n)\int_{B_k} |\nabla f|^2 \cdot a_k^{\frac{4}{n}}$$

$$\Rightarrow a_{k+1} \leq c(n)\left(\int_{B_k} |\nabla f|^2 dx\right)^{\frac{n}{n+2}} a_k^{\frac{4}{n+2}}$$

$$\Rightarrow \sum_{k\in\mathbb{Z}} a_k = \sum_{k\in\mathbb{Z}} a_{k+1}$$

$$\leq c(n)\sum_{k\in\mathbb{Z}}\left(\int_{B_k} |\nabla f|^2 dx\right)^{\frac{n}{n+2}} a_k^{\frac{4}{n+2}}$$

$$\leq c(n)\left(\sum_{k\in\mathbb{Z}}\left(\int_{B_k} |\nabla f|^2 dx\right)^{\frac{n}{n-2}}\right)^{\frac{n-2}{n+2}}$$

$$\left(\sum_{k\in\mathbb{Z}} a_k\right)^{\frac{4}{n+2}} \quad \text{(Hölder)}$$

$$\Rightarrow \sum_{k\in\mathbb{Z}} a_k \leq c(n)\sum_{k\in\mathbb{Z}}\left(\int_{B_k} |\nabla f|^2 dx\right)^{\frac{n}{n-2}}$$

$$\leq c(n)\left(\sum_{k\in\mathbb{Z}}\int_{B_k} |\nabla f|^2 dx\right)^{\frac{n}{n-2}}$$

$$= c(n)\left(\int_{\mathbb{R}^n} |\nabla f|^2 dx\right)^{\frac{n}{n-2}}$$

$$\Rightarrow \int_{\mathbb{R}^n} f^{\frac{2n}{n-2}} dx \leq \sum_{k\in\mathbb{Z}} a_k \cdot c(n)$$

$$\leq c(n)\left(\int_{\mathbb{R}^n} |\nabla f|^2 dx\right)^{\frac{n}{n-2}}$$

□

05/09/2018 Modern Mathematics

"Lipschitz functions are differentiable"

Def: ( L-Lipschitz ) $f: \mathbb{R}^n \to \mathbb{R}$

If $\forall x, y \in \mathbb{R}^n$, $|f(x) - f(y)| \leq L|x-y|$ holds, then $f$ is an $L$-Lip function.

Def: ( Differentiable )

$f: \mathbb{R}^n \to \mathbb{R}$ is diff at $x_0 \in \mathbb{R}^n$ if $\exists$ $A_{1\times n}$, s.t. $\lim\limits_{y \to x_0} \dfrac{|f(y) - f(x_0) - A(y-x_0)|}{|y-x_0|} = 0$

If $f$ is diff at $x_0$, let $\nabla f(x_0) = A$, which is unique.

Remark: ① $f, g$ are diff at $x_0$, then $f+g$ is diff at $x_0$ with $\nabla(f+g)(x_0) = \nabla f + \nabla g$

② If $f$ is $L$-Lip and diff at $x_0$, then $|\nabla f|(x_0) \leq L$.

Proof of ②: Let $v = \dfrac{\nabla f(x_0)}{|\nabla f(x_0)|}$

$\Rightarrow \lim\limits_{t \to 0} \dfrac{|f(x_0+tv) - f(x_0) - \nabla f \cdot (tv)|}{|tv|} = 0$,

with

$\Rightarrow |\nabla f|(x_0) \leq \dfrac{|f(x_0+tv) - f(x_0)|}{|tv|} + \dfrac{|f(x_0+tv) - f(x_0) - \nabla f \cdot (tv)|}{|tv|}$

and $|f(x_0+tv) - f(x_0)| \leq L|tv|$,

$\Rightarrow |\nabla f| \leq L$ if we let $t \to 0$

Thm. $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then $f$ is diff on $\mathbb{R}^n$ a.e., i.e. $\exists\, \Omega$, with $\mu(\Omega) = 0$, s.t. $\forall x \in \mathbb{R}^n \setminus \Omega$, $f$ is diff.
Here $\mu$ is Lebesgue measurement.

Lemma: $f: \mathbb{R} \to \mathbb{R}$ monotomic, then $f$ is a.e. diff on $\mathbb{R}$. ( Lebesgue's )

Proof of Thm.

Case $n=1$: let $g(x) = f(x) + Lx$

$\Rightarrow g(x) - g(y) = f(x) - f(y) + L(x-y)$

$\Rightarrow g(x) - g(y) \geq 0$, when $x \geq y$.

$\Rightarrow g$ is ~~ab~~ monotomic

$\Rightarrow g$ is a.e. diff $\Rightarrow$ so is $f$.

Case $n \geq 1$:

① If is diff at $x_0$, then $\nabla f = \left( \dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n} \right)$

Proof of ①: Let $v \in \mathbb{R}^n$, $|v| = 1$, if $f$ is diff at $x_0$, then $g(t) = f(x_0 + tv)$ is diff at $t=0$, since:

$0 = \lim\limits_{t \to 0} \dfrac{|f(x_0+tv) - f(x_0) - \nabla f(x_0) \cdot (tv)|}{|tv|}$

$= \lim\limits_{t \to 0} \dfrac{|g(t) - g(0) - (\nabla f(x_0) \cdot v)t|}{|t|}$

$\Rightarrow g'(0) = \langle \nabla f(x_0), v \rangle$,

特别地，取 $v = e_i = (0, \dots, \overset{i\text{th}}{1}, 0 \dots 0)$

$\Rightarrow g_i'(0) = \langle \nabla f(x_0), e_i \rangle$, where $g_i = f(x_0 + te_i)$

Let $\dfrac{\partial f}{\partial x_i}(x_0) = \langle \nabla f(x_0), e_i \rangle$

$\Rightarrow \nabla f(x_0) = \left( \dfrac{\partial f}{\partial x_1}(x_0), \dots, \dfrac{\partial f}{\partial x_n}(x_0) \right)$ $\square$

$\square$

② $\exists\, \Omega$ with $\mu(\Omega)=0$, s.t. $\forall x \in \mathbb{R}^n \setminus \Omega$, $\frac{\partial f}{\partial x_i}(x)$ exists, $i=1,2,\cdots,n$.

Proof of ②: only need to show: for $\forall i=1,\cdots,n$, $\exists\, \Omega_i$, $\mu(\Omega_i)=0$, s.t. $\frac{\partial f}{\partial x_i}(x)$ exists for $\forall x \in \mathbb{R}^n \setminus \Omega_i$. Then, take $\Omega = \bigcup_{i=1}^n \Omega_i$.

Assume $i=1$, for $\forall (x_2,\cdots,x_n) \in \mathbb{R}^{n-1}$, consider $g: \mathbb{R} \to \mathbb{R}$, $g(x) = f(x, x_2, \cdots, x_n)$.

By Case $n=1$, $g$ is a.e. diff, i.e., $\exists\, I_{x_2\cdots x_n} \subset \mathbb{R}$, $\mu(I_{x_2\cdots x_n}) = 0$, s.t. $g$ is diff on $\mathbb{R} \setminus I_{x_2\cdots x_n}$

In particular, $\frac{\partial f}{\partial x_1}(x, x_2, \cdots, x_n) = g'(x)$.

Let $\Omega_1 = \bigcup_{(x_2,\cdots,x_n)\in \mathbb{R}^{n-1}} I_{x_2\cdots x_n}$.

By Fubini's theorem:

$$\int_{\Omega_1} dx = \int_{\mathbb{R}^{n-1}} dx_2 \cdots dx_n \int_{I_{x_2\cdots x_n}} dx_1 = 0$$

$\square$

Remark: $\forall v \in \mathbb{R}^n$, $|v|=1$, $\exists\, \Omega_v$ with $\mu(\Omega_v)=0$, s.t. $\forall x \in \mathbb{R}^n \setminus \Omega_v$, $f(x+tv)$ is diff at $t=0$. Denote its derivative as $D_v f(x)$

③ $\forall v \in \mathbb{R}^n$, $|v|=1$, $\exists\, \Omega_v$ with $\mu(\Omega_v)=0$, s.t. $\forall x \in \mathbb{R}^n \setminus \Omega_v$, $D_v f(x)$ exists and $D_v f(x) = \langle Df(x), v\rangle$, where $Df(x) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)(x)$

Proof of ③: Let $\tilde{\Omega} = \bigcup_{i=1}^n \Omega_i \cup \Omega_v$, then $\forall x \in \mathbb{R}^n \setminus \tilde{\Omega}$, $D_v f$ exists, $Df$ exists.

Consider the integral: $\forall \eta \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} D_v f(x) \cdot \eta(x)\, dx$$

$$= \int_{\mathbb{R}^n} \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \eta(x)\, dx$$

$$= \lim_{t\to 0} \int_{\mathbb{R}^n} \frac{f(x+tv)-f(x)}{t} \eta(x)\, dx$$

(Dominated convergence theorem, dominated by $2|D_v f(x)|$)

$$= \lim_{t\to 0} \int_{\mathbb{R}^n} \frac{f(y)}{t}\eta(y-tv) - \frac{f(y)}{t}\eta(y)\, dy$$

$$= \lim_{t\to 0} \int_{\mathbb{R}^n} f(y) \frac{\eta(y-tv)-\eta(y)}{t}\, dy$$

$$= \int_{\mathbb{R}^n} f(y) \langle \nabla\eta(y), -v\rangle\, dy$$

$$= -\sum_{i=1}^n v_i \int_{\mathbb{R}^n} f(y) \frac{\partial \eta}{\partial y_i}\, dy, \quad v=(v_1,\cdots,v_n)$$

$$= -\sum_{i=1}^n v_i \int_{\mathbb{R}^n} f(y) \lim_{t\to 0} \frac{\eta(y+te_i)-\eta(y)}{t}\, dy$$

$$= \sum_{i=1}^n v_i \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x) \cdot \eta(x)\, dx$$

(By similar process or integral by parts)

$$\Rightarrow \int_{\mathbb{R}^n} (D_v f - \langle Df, v\rangle)\, \eta(x)\, dx = 0$$

$$\Rightarrow D_v f = \langle Df, v\rangle \text{ a.e.}$$

$\square$

④ Proof of Case $n>1$: take a subset of $S^{n-1} = \{x \in \mathbb{R}^n : |x|=1\}$, $\{u_1, \cdots u_i, \cdots\}$, ~~which~~ is denumerable and dense in $S^{n-1}$

By ③, $D_{u_i} f = \langle Df, u_i \rangle$, $\forall \mathbb{R}^n \setminus \tilde{\Omega}_{u_i}$.

Let $\Omega = \bigcup_{i=1}^{\infty} \tilde{\Omega}_i$, which $\mu(\Omega)$ holds.

We show that $\forall x \in \mathbb{R}^n \setminus \Omega$, $f$ is diff at $x$, i.e. $\forall x \in \mathbb{R}^n \setminus \Omega$, $\forall \varepsilon > 0, \exists \delta > 0$, s.t. when $|x-y| < \delta$,

$$\frac{|f(y) - f(x) - \langle Df(x), y-x \rangle|}{|y-x|} < \varepsilon, \ (*)$$

Only need to show: $\forall v \in S^{n-1}$, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon, n, f) > 0$, s.t. when $|t| < \delta$, $\left| \frac{f(x+tv) - f(x)}{t} - D_v f \right| < \varepsilon$

and $D_v f = \langle Df, v \rangle$ $\quad (**)$

Indeed, when $(**)$ holds, ~~then~~ let $v = \frac{y-x}{|y-x|}$, when $|y-x| < \delta$, $(**) \Rightarrow (*)$.

Now we prove $(**)$:

Since $\{u_1, u_2, \cdots u_n, \cdots\}$ is denumerable and dense in $S^{n-1}$. For $\forall \varepsilon > 0$, $\exists$ a subset $\{v_1, \cdots, v_N\} \subset \{u_1, \cdots u_n \cdots\}$, s.t. $\forall v \in S^{n-1}$, $\exists v_i$, s.t. $|v - v_i| < \varepsilon$.

Then $\forall x \in \mathbb{R}^n \setminus \Omega$, since $D_{v_i} f$ exists, $\Rightarrow \exists \delta$, s.t. $\forall i = 1, 2, \cdots, N$, when $|t| < \delta$

$$\left| \frac{f(x+tv_i) - f(x)}{t} - D_{v_i} f \right| < \varepsilon$$

$$\Rightarrow \left| \frac{f(x+tv) - f(x)}{t} - \langle Df, v \rangle \right|$$

$$\leq \left| \frac{f(x+tv_i) - f(x+tv)}{t} \right| + \left| \frac{f(x+tv_i) - f(x)}{t} - \langle Df, v_i \rangle \right| + |\langle Df, v - v_i \rangle|$$

$$\leq L \cdot |v - v_i| + \varepsilon + |Df| \cdot |v - v_i|$$

take the $v_i$ s.t. $|v - v_i| < \varepsilon$,

so, ~~it turns~~ $\leq (L + 1 + |Df|) \cdot \varepsilon$

$\leq (L + 1 + n \cdot L) \cdot \varepsilon$

$\square$

Pointwise Lipschitz const.

$f: \mathbb{R}^n \to \mathbb{R}$, def: $\forall x \in \mathbb{R}^n$,

$$\operatorname{Lip} f(x) \triangleq \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y-x|}$$

Properties: ① If $f$ is $L$-Lipschitz, then $\operatorname{Lip} f(x) \leq L$

② If $f$ is diff at $x$, then $\operatorname{Lip} f(x) = |\nabla f|(x)$.

③ $\operatorname{Lip}(f \pm g)(x) \leq \operatorname{Lip} f(x) + \operatorname{Lip} g(x)$

Proof of ②:

$$\frac{|f(y) - f(x)|}{|y-x|} = \frac{|f(y) - f(x) - \langle \nabla f, y-x \rangle + \langle \nabla f, y-x \rangle|}{|y-x|}$$

$$\leq |\nabla f| + \frac{|f(y) - f(x) - \langle \nabla f(x), y-x \rangle|}{|y-x|}$$

$\Rightarrow \operatorname{Lip} f(x) \leq |\nabla f|(x)$

then, let $v = \frac{\nabla f(x)}{|\nabla f|(x)}$

$$\Rightarrow \frac{|f(x+tv) - f(x)|}{|t|}$$

$$\geq \frac{|\langle \nabla f, tv \rangle|}{|t|} - \frac{|f(x+tv) - f(x) - \langle \nabla f, tv \rangle|}{|t|}$$

$$\Rightarrow \cancel{\lim} \ \mathrm{Lip} f(x) \geq |\nabla f|(x)$$

$\square$

05/16/2018 Modern Mathematics

Smooth Approximation

Let $\eta(t) \triangleq \begin{cases} e^{\frac{1}{t^2-1}}, & |t|<1 \\ 0, & |t| \geq 1 \end{cases}$

then $\eta \in C_0^\infty(\mathbb{R})$, for $\forall x \in \mathbb{R}^n$, let $\psi(x) \triangleq \eta(|x|)$

For $\forall \varepsilon > 0$, let $\psi_\varepsilon(x) \triangleq \eta\left(\frac{|x|}{\varepsilon}\right)$

$$\varphi(x) \triangleq \frac{\psi(x)}{\int_{\mathbb{R}^n} \psi(x)dx}, \quad \varphi_\varepsilon(x) \triangleq \frac{\psi_\varepsilon(x)}{\int_{\mathbb{R}^n} \psi_\varepsilon(x)dx} = \varepsilon^{-n}\varphi\left(\frac{x}{\varepsilon}\right)$$

$\Rightarrow \operatorname{supp}\varphi_\varepsilon \subseteq B_\varepsilon(0)$.

Def: for $\forall f \in L^p(\mathbb{R}^n, \mu)$, $\mu$ is Borel measurement

$$f_\varepsilon(x) \triangleq \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x-y)dy.$$

Thm. $\forall f \in L^p$, then $f_\varepsilon \xrightarrow[L_p]{\varepsilon\to 0} f$ under $L^p$ metric, $1 \leq p < +\infty$

Proof: ① Continuous functions are dense in $L^p$

$\Rightarrow$ continuous functions with compact supp is dense in $L^p$.

Assume $f \in L^p$. $h_i \to f$ in $L^p$ and $h_i \in C_0(\mathbb{R}^n)$

② for $h \in C_0(\mathbb{R}^n)$, then $h_\varepsilon(x) = \int_{\mathbb{R}^n} h(y)\varphi_\varepsilon(y-x)dy$

$$= \int_{\mathbb{R}^n} h(y)\varepsilon^{-n}\varphi\left(\frac{y-x}{\varepsilon}\right)dy = \int_{\mathbb{R}^n} h(x+\varepsilon z)\varphi(z)dz$$

$$\Rightarrow h_\varepsilon(x) - h(x) = \int_{\mathbb{R}^n}(h(x+\varepsilon z) - h(x))\varphi(z)dz.$$

Since $h \in C_0(\mathbb{R}^n) \Rightarrow h$ is uniformly continuous.

$\Rightarrow \forall \varepsilon' > 0$, when $0 < \varepsilon < \varepsilon(\varepsilon', n')$,

$|h(x+\varepsilon z) - h(x)| \leq \varepsilon'$, when $|z| \leq 1$,

note that $\operatorname{supp}\varphi \subseteq B_1(0)$

$\Rightarrow |h_\varepsilon(x) - h(x)| \leq \varepsilon'$.

Since $h_\varepsilon(x)$, $h(x)$ has compact support.

$$\Rightarrow \int_{\mathbb{R}^n} |h_\varepsilon(x) - h(x)|^p d\mu \to 0$$

$$\|f_\varepsilon - f\|_{L^p} \leq \|h_{i,\varepsilon} - h_i\|_{L^p} + \|h_{i,\varepsilon} - f_\varepsilon\|_{L^p} + \|h_i - f\|_{L^p} \quad \cdots (*)$$

Since $(h_{i,\varepsilon} - f_\varepsilon)(x) = \int_{\mathbb{R}^n}(h_i(y) - f(y))\varphi_\varepsilon(y-x)dy$

$$\Rightarrow \int_{\mathbb{R}^n} |h_{i,\varepsilon} - f_\varepsilon|^p d\mu(x)$$

$$\leq \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|h_i(y) - f(y)|\varphi_\varepsilon(y-x)dy\right)^p d\mu(x)$$

$$\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|h_i(y) - f(y)|^p\varphi_\varepsilon(y-x)dy\,d\mu(x)$$

$$\underline{\underline{\text{assume } d\mu = dx}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|h_i(y) - f(y)|^p\varphi_\varepsilon(y-x)dx\,dy$$

$$= \int_{\mathbb{R}^n}|h_i(y) - f(y)|^p dy$$

In $(*)$, let $\varepsilon \to 0$ first $\Rightarrow \lim\limits_{\varepsilon\to 0}\|f_\varepsilon - f\|_{L^p}$

$\leq 2\|h_i - f\|_{L^p}$, then let $i \to +\infty$

$f_\varepsilon \xrightarrow[L_p]{} f$ in $L_p$.

$\square$

Remark: $f_\varepsilon \in C^\infty(\mathbb{R}^n)$,

hence, $C^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, dx)$.

Thm. If $f$ is $L$-Lipschitz with compact support, then $f_\varepsilon \to f$ in $L^\infty$ and $\nabla f_\varepsilon \to \nabla f$ in $L^p$, $\forall 1 \leq p < +\infty$.

Proof: Since $|f_\varepsilon(x) - f(x)| = \left|\int_{\mathbb{R}^n}(f(y) - f(x))\varphi_\varepsilon(y-x)dy\right|$

$$\leq \int_{\mathbb{R}^n}|f(x+\varepsilon z) - f(x)|\varphi(z)dz$$

$$\leq \varepsilon L\int_{\mathbb{R}^n}\varphi(z)dz = \varepsilon L \to 0 \text{ as } \varepsilon \to 0.$$

While $\nabla f_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\, \nabla_x \varphi_\varepsilon(y-x)\,dy$

$= \int_{\mathbb{R}^n} f(y)\,(-\nabla_y \varphi_\varepsilon(y-x))\,dy$

$= \int_{\mathbb{R}^n} \nabla_y f(y) \cdot \varphi_\varepsilon(y-x)\,dy$

$\Rightarrow \nabla f_\varepsilon(x) = (\nabla f)_\varepsilon(x)$

$\Rightarrow \nabla f_\varepsilon \to \nabla f$ in $L^p$ by our previous theorem

$\square$

Recall: for $\forall f \in C_0^\infty(\mathbb{R}^n)$,

$\Rightarrow |f(x)| \leq C(n) \int_{\mathbb{R}^n} \frac{|\nabla f|(y)}{|x-y|^{n-1}}\,dy$

$\Rightarrow \forall \alpha > 0,\ \alpha^{\frac{n}{n-1}} \mu\{x : |f(x)| \geq \alpha\}$

$\leq C(n)\, \|\nabla f\|_{L^1}^{\frac{n}{n-1}}$, $(**)$ where $\mu$ is Lebesgue measurement.

Claim: $(**)$ holds for Lipschitz functions with compact support.

Proof of claim: $f$ is Lipschitz with compact support. Consider

$f_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\, \varphi_\varepsilon(y-x)\,dy$

$\Rightarrow |f_\varepsilon(x) - f(x)| \leq \varepsilon L,\ \forall x \in \mathbb{R}^n$

and $\int_{\mathbb{R}^n} |\nabla(f_\varepsilon - f)(x)|\,dx \to 0$,

in particular, $\forall \varepsilon' > 0$, when $\varepsilon \leq \varepsilon(\varepsilon', n)$,

$\|\nabla f_\varepsilon\|_{L^1} \leq \|\nabla f\|_{L^1} + \varepsilon'$

Apply $(**)$ to $f_\varepsilon$,

$\alpha^{\frac{n}{n-1}} \mu\{x : |f_\varepsilon(x)| \geq \alpha\} \leq C(n)\, \|\nabla f_\varepsilon\|_{L^1}^{\frac{n}{n-1}}$

$\Rightarrow \alpha^{\frac{n}{n-1}} \mu\{x : |f_\varepsilon(x)| \geq \alpha\}$

$\leq C(n)\,(\|\nabla f\|^{\frac{n}{n-1}} + \varepsilon')$

But $\{x : |f(x)| \geq \alpha + \varepsilon L\}$

$\subseteq \{x : |f_\varepsilon(x)| \geq \alpha\}$

$\Rightarrow \alpha^{\frac{n}{n-1}} \mu(\{x : |f|(x) \geq \alpha + \varepsilon L\})$

$\leq C(n)\,(\|\nabla f\|_{L^1}^{\frac{n}{n-1}} + \varepsilon')$

$\Rightarrow (t - \varepsilon L)^{\frac{n}{n-1}} \mu(\{x : |f|(x) \geq t\})$

$\leq C(n)\,(\|\nabla f\|_{L^1}^{\frac{n}{n-1}} + \varepsilon')$

let $\varepsilon \to 0$ and let $\varepsilon' \to 0$

$\Rightarrow \forall t > 0$,

$t^{\frac{n}{n-1}} \mu(\{x : |f|(x) \geq t\})$

$\leq C(n)\, \|\nabla f\|_{L^1}^{\frac{n}{n-1}}$

$\square$

Thm. (Estimate of Volume's lower bound)

$(\mathbb{R}^n, \mu)$ is a measurable space. $\mu$ satisfies:

① $0 < \mu(B_r(x)) < +\infty,\ \forall x \in \mathbb{R}^n, r > 0$

② $\forall x \in \mathbb{R}^n,\ \exists C_x, r_x > 0$, s.t. for $\forall 0 < r < r_x$,

$\mu(B_r(x)) \geq \mu(B_{2r}(x))\, C_x$

Assume $\forall f \in \{\text{compact, Lipschitz}\}$.

we have $\left(\int_{\mathbb{R}^n} |f|^{pk}(x)\,d\mu\right)^{\frac{1}{pk}} \leq C_0 \left(\int_{\mathbb{R}^n} |\nabla f|^p(x)\,d\mu\right)^{\frac{1}{p}}$

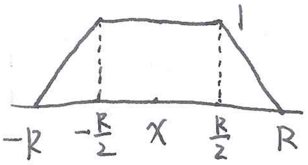
Where $p \geq 1$, $k > 1$, then $\forall x \in \mathbb{R}^n$,

$\mu(B_1(x)) \geq C(n, p, k, C_0) > 0$.

Proof: for $\forall R>0$,

def: $f_R(y)=\begin{cases} 1, & y\in B_{\frac{R}{2}}(x) \\ \frac{2}{R}(R-|y-x|), & y\in B_R(x)\setminus B_{\frac{R}{2}}(x) \\ 0, & y\in \mathbb{R}^n\setminus B_R(x)\end{cases}$

then $f_R$ is Lipschitz.



$, \quad |\nabla f_R(y)|=\begin{cases}\frac{2}{R}, & y\in \overline{B_R(x)}\setminus B_{\frac{R}{2}}(x) \\ 0, & \text{others}\end{cases}$

for $y, y'\in \overline{B_R(x)}\setminus B_{\frac{R}{2}}(x)$

$\Rightarrow |f_R(y)-f_R(y')|\leq \frac{2}{R}|y-y'|$

By Sobolev ineq'l

$$\left(\int_{B_R(x)} f_R^{pk}(y)\,dy\right)^{\frac{1}{pk}}\leq C_0\left(\int_{B_R\setminus B_{\frac{R}{2}}(x)}|\nabla f_R|^p d\mu\right)^{\frac{1}{p}}$$

$$\Rightarrow \mu\left(B_{\frac{R}{2}}(x)\right)^{\frac{1}{pk}}\leq C_0\cdot\frac{2}{R}\cdot\mu(B_R(x))^{\frac{1}{p}}.$$

$$\Rightarrow \mu(B_R(x))\geq \mu(B_{\frac{R}{2}}(x))^{\frac{1}{k}}(2C_0)^{-p}R^p$$

$$\mu(B_{\frac{R}{2}}(x))\geq \mu(B_{\frac{R}{4}}(x))^{\frac{1}{k}}(2C_0)^{-p}\left(\frac{R}{2}\right)^p$$

$$\Rightarrow \mu(B_R(x))\geq \mu(B_{\frac{R}{4}}(x))^{\frac{1}{k^2}}(2C_0)^{-p-\frac{p}{k}}R^{p+\frac{p}{k}}\cdot 2^{-\frac{p}{k}}$$

...

$$\geq \mu\left(B_{\frac{R}{2^N}}(x)\right)^{\frac{1}{k^N}}(2C_0)^{-\sum_{i=0}^{N-1}\frac{p}{k^i}}R^{p\sum_{i=0}^{N-1}\frac{1}{k^i}}\cdot 2^{-p\sum_{i=0}^{N-1}\frac{i}{k^i}},$$

since $k>1$, the last three terms are finite.

$$\Rightarrow \mu(B_R(x))\geq C(C_0,p,k,n)\cdot R^{p\sum_{i=0}^{N-1}\frac{1}{k^i}}\,\mu\left(B_{\frac{R}{2^N}}(x)\right)^{\frac{1}{k^N}}$$

Assume $N_0>1$, s.t. $\frac{r_x}{2}\leq\frac{R}{2^{N_0}}<r_x$

then for any $N>N_0$,

$\mu\left(B_{\frac{R}{2^N}}(x)\right)$

$\geq C_x^{N-N_0}\,\mu\left(B_{\frac{R}{2^{N_0}}}(x)\right)$ by our assumption,

$\geq C_x^{N-N_0}\,\mu\left(B_{\frac{r_x}{2}}(x)\right)$

$$\Rightarrow \mu(B_R(x))\geq C(C_0,p,k,n)R^{p\cdot\sum_{i=0}^{N-1}\frac{1}{k^i}}\cdot C_x^{\frac{N-N_0}{k^N}}\cdot\mu\left(B_{\frac{r_x}{2}}(x)\right)^{\frac{1}{k^N}}$$

$\Rightarrow$ when $N\to\infty$

$$\mu(B_R(x))\geq C(C_0,p,k,n)\,R^{p\cdot\frac{k}{k-1}}$$

□

Remark: for $\mathbb{R}^n$, in standard Sobolev ineq'l, $p>1$, $pk=\frac{np}{n-p}\Rightarrow k=\frac{n}{n-p}$.

$\Rightarrow p\frac{k}{k-1}=n$

$\Rightarrow \mu(B_R(x))\geq C(n)R^n$

□

Cor: for $d\mu=\frac{dx}{(1+|x|^2)^\alpha}$, $\alpha>0$,

$$\left(\int_{\mathbb{R}^n}|f|^{pk}d\mu\right)^{\frac{1}{pk}}\leq C\left(\int_{\mathbb{R}^n}|\nabla f|^p d\mu\right)^{\frac{1}{p}}$$

doesn't hold for all $f\in C_0^\infty(\mathbb{R}^n)$, $p\geq 1$, $k>1$.

Proof: $\forall x\in\mathbb{R}^n$, let $r_x=1$, $C_x=C(n)$.

$\Rightarrow \mu(B_r(x))\geq C_x\,\mu(B_{2r}(x))$, $\forall 0<r\leq r_x$.

But $\mu(B_1(x))\approx\frac{C(n)}{(1+|x|^2)^\alpha}\to 0$, as $|x|\to+\infty$

$\Rightarrow$ Sobolev ineq'l doesn't hold.

□

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# Gromov-Hausdorff metric

- Hausdorff metric

Def: (Metric Space): $(Z, d)$ is a metric space if ① $d(z_1, z_2) = d(z_2, z_1) \geq 0$

② $d(z_1, z_2) = 0 \Leftrightarrow z_1 = z_2$

③ $d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3)$ $\forall z_1, z_2, z_3 \in Z$.

Notations: $B_r(z) \triangleq \{y \in Z: d(y, z) < r\}$

$\bigcup_{x \in A} B_r(x) = B_r(A) \triangleq \{y \in Z: d(y, A) < r\}$

where $A \subseteq Z$, $d(y, A) \triangleq \inf_{z \in A} d(z, y)$

Only consider compact metric space $(Z, d)$, i.e. $\forall \{z_i\} \subset Z$, exists a convergence subseq. Hence $Z$ is complete

Def: (Hausdorff metric). Assume $(Z, d)$ is compact, $\forall X, Y \subseteq Z$,

$$d_H(X, Y) \triangleq \inf\{\varepsilon : X \subset B_\varepsilon(Y), Y \subset B_\varepsilon(X)\}$$

Example 1: $Z = \mathbb{R}$, $X = [0, 1]$, $Y = [2, 3]$

$d_H(X, Y) = 2$ $B_2(X) \not\supseteq Y$

Example 2: $Z = \mathbb{R}$, $X = [0, 1]$, $Y = [0, 1)$

$d_H(X, Y) = 0$

Properties: ① $d_H(X, Y) = \max\{\sup_{y \in Y} d(y, X), \sup_{x \in X} d(x, Y)\}$

② $\forall X_1, X_2, X_3 \subset Z$

$d_H(X_1, X_3) \leq d_H(X_1, X_2) + d_H(X_2, X_3)$

③ If $X, Y \subset Z$ both ~~are~~ closed and $d_H(X, Y) = 0$ $\Rightarrow X = Y$

Proof: ① Let $\max\{\sup_{y \in Y} d(y, x), \sup_{x \in X} d(x, Y)\} = R$, $d_H(X, Y) = r$.

$\forall \varepsilon > 0$, since $d_H(X, Y) = r$

$\Rightarrow X \subseteq B_{r+\varepsilon}(Y)$, $Y \subseteq B_{r+\varepsilon}(X)$.

$\Rightarrow \forall x \in X$, $d(x, Y) \leq r + \varepsilon$.

$\forall y \in Y$, $d(y, X) \leq r + \varepsilon$.

$\Rightarrow R \leq r + \varepsilon \Rightarrow R \leq r$ as $\varepsilon \to 0$.

On the other hand, $\forall \varepsilon > 0$.

$\Rightarrow \sup_{y \in Y} d(y, x) \leq R \Rightarrow Y \subseteq B_{R+\varepsilon}(X)$.

$\sup_{x \in X} d(x, Y) \leq R \Rightarrow X \subseteq B_{R+\varepsilon}(Y)$

$\Rightarrow r \leq R + \varepsilon \Rightarrow r \leq R$ as $\varepsilon \to 0$. $\square$

Remark: $X \subset Y \not\Rightarrow d_H(X, Y) = 0$

② Let $r_{ij} = d_H(X_i, X_j)$

$\Rightarrow \forall \varepsilon > 0$, $X_1 \subseteq B_{r_{12}+\varepsilon}(X_2)$.

$X_2 \subseteq B_{r_{23}+\varepsilon}(X_3)$

$\Rightarrow X_1 \subseteq B_{r_{12}+r_{23}+2\varepsilon}(X_3)$.

Similarly, $X_3 \subseteq B_{r_{12}+r_{23}+2\varepsilon}(X_1)$

$\Rightarrow d_H(X_1, X_3) \leq r_{12} + r_{23} + 2\varepsilon$ and let $\varepsilon \to 0$ $\square$

③ If $d_H(X, Y) = 0$, $X, Y$ closed

Assume $X \setminus Y \neq \phi$, take $x \in X \setminus Y$.

Since $Y$ is closed $\Rightarrow \exists r > 0$, s.t. $B_r(x) \cap Y = \phi \Rightarrow d(x, Y) > 0$, which is absurd by Property ①.

$\Rightarrow X = Y$

$\square$

Consider $\tilde{Z} = \{X \subset Z : X \text{ is closed}\}$

Thm. $(\tilde{Z}, d_H)$ is a complete metric space

Proof: By ②③ $\Rightarrow d_H$ is a metric over $\tilde{Z}$.

Now we show that $\tilde{Z}$ is complete with $d_H$.

Assume $\{X_i \in \tilde{Z}\}$ is a Cauchy seq.,

i.e. $d_H(X_i, X_j) \to 0$ as $i, j \to +\infty$

Assume $d_H(X_i, X_j) \le \varepsilon_i$, $\forall j \ge i$ and $\varepsilon_i \to 0$.

cannot take $\cap X_i$ directly in case of $\varepsilon[\,]$

Let $X_\infty = \bigcap_{i=1}^{\infty} B_{2\varepsilon_i}(X_i)$ ~~may be~~ not $\emptyset$

Then $X_\infty$ is closed, $X_\infty \in \tilde{Z}$.

And $X_\infty \subseteq B_{3\varepsilon_i}(X_i)$, $\forall i \ge 1$.

since $d_H(X_i, X_j) \le \varepsilon_i$, when $j \ge i$,

$\Rightarrow \forall j \ge i$, $X_j \subseteq B_{2\varepsilon_i}(X_i)$

$\Rightarrow \bigcup_{j=i}^{\infty} X_j \subseteq B_{2\varepsilon_i}(X_i)$

$\Rightarrow \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} X_j \subseteq X_\infty$.

Now $\forall i$, $\forall k$,

$d_H(X_i, X_l) \le \varepsilon_i$

$B_{2\varepsilon_i}\left(\bigcup_{j=k}^{\infty} X_j\right) \supseteq B_{2\varepsilon_i}(X_l)$ key

$\supseteq X_i$ if $l > i$. any

$\Rightarrow B_{3\varepsilon_i}(X_\infty) \supseteq X_i$

$\Rightarrow d_H(X_i, X_\infty) \le 3\varepsilon_i \to 0$ as $i \to +\infty$

$\square$

## Gromov – Hausdorff metric

Def: (isometric) For $(X, d_X)$, $(Y, d_Y)$,

if $\exists$ $f: X \to Y$, $h: Y \to X$, s.t.

① $f \circ h = Id_Y$, $h \circ f = Id_X$

② $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$, $\forall x_1, x_2 \in X$

$d_X(h(y_1), h(y_2)) = d_Y(y_1, y_2)$, $\forall y_1, y_2 \in Y$

then we say $(X, d_X)$ is isometric to $(Y, d_Y)$

denote as $X \overset{\text{isometric}}{\simeq} Y$

Def: (isometric embedding) For $(X, d_X)$, $(Y, d_Y)$,

if $\exists$ $f: X \to Y$.

$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$, $\forall x_1, x_2 \in X$,

then we say $f: X \to Y$ is an isometric embedding, denote as $X \xhookrightarrow{\text{isometric}} Y$.

Remark: $f(X) \subset Y$, and

$f(X) \overset{\text{isom.}}{\simeq} X$, $X$ is regarded as a subset of $Y$.

Def: (Gromov – Hausdorff metric)

$(X, d_X)$, $(Y, d_Y)$, compact metric spaces,

$d_{GH}((X, d_X), (Y, d_Y)) \triangleq$

$$\inf_{(Z, d^Z)} \left\{ d_H^Z(X, Y) : X \xhookrightarrow{\text{isom.}} Z,\ Y \xhookrightarrow{\text{isom.}} Z \right\}$$

Def: (Admitted metric) Assume $Z = X \amalg Y$,

if $(Z, d^Z)$ is a metric space s.t.

$d^Z|_X = d_X$, i.e. $\forall x_1, x_2 \in X$, $d^Z(x_1, x_2) = d_X(x_1, x_2)$

$d^Z|_Y = d_Y$, then $d^Z$ is called an admitted metric.

**Lemma:** If we denote

$d'_{GH}((X,d_X),(Y,d_Y)) = \inf\{d_H(X,Y) : d$ is an admitted metric over $X \amalg Y\}$ ($d$ is admitted),

then $d_{GH}((X,d_X),(Y,d_Y)) = d'_{GH}((X,d_X),(Y,d_Y))$

**Example:** $(X,d_X)$, $(Y,d_Y)$ compact, and $diam(X,d_X) \triangleq \max\{d(x_1,x_2), x_1,x_2 \in X\}$

$diam(Y,d_Y) \triangleq \cdots$

s.t. $diam(X,d_X), diam(Y,d_Y) < D$.

Let $Z = X \amalg Y$. $d: Z \times Z \to \mathbb{R}$, symmetric,

s.t. ① $d(x_1,x_2) \triangleq d_X(x_1,x_2)$, $\forall x_1,x_2 \in X$

② $d(y_1,y_2) \triangleq d_Y(y_1,y_2)$, $\forall y_1,y_2 \in Y$

③ $d(x,y) \triangleq \frac{D}{2}$, $\forall x \in X, y \in Y$

Then $d$ is an admitted metric

Proof of Example: Only need to check triangle ineq'l.

$\forall x_1, x_2, y_1, y_2$.

$d(x_1,x_2) \le d(x_1,y_1) + d(x_2,y_1)$

$d(x_1,y_1) \le d(x_1,x_2) + d(x_2,y_1)$,

which is obvious

□

Proof of lemma:

By def, $d_{GH}(X,Y) \le d'_{GH}(X,Y)$

On the other hand.

$\forall \varepsilon > 0$, $\exists (Z, d^Z)$ compact, s.t. $X \xhookrightarrow{isom.} Z$, $Y \xhookrightarrow{isom.} Z$

and $d^Z_H(X,Y) \le \varepsilon + d_{GH}(X,Y)$

Consider $Z \times [0,\varepsilon]$ with metric $d_\varepsilon$ s.t. $\forall (z_1,t_1),(z_2,t_2) \in Z \times [0,\varepsilon]$.

$d_\varepsilon((z_1,t_1),(z_2,t_2))$

$= (d(z_1,z_2)^2 + |t_1-t_2|^2)^{\frac{1}{2}}$

And $X \xhookrightarrow{isom.} Z \times \{0\}$

$Y \xhookrightarrow{isom.} Z \times \{\varepsilon\}$,

then $X \times \{0\} \cup Y \times \{\varepsilon\}$ is a disjoint union, denoted as $X \amalg Y$. then $d_\varepsilon|_{X \amalg Y}$ is an admitted metric, denoted as $d^{X \amalg Y}$, then $d'_{GH}((X,d_X),(Y,d_Y))$

$\le d_H^{X\amalg Y}(X,Y) = d_H^{Z\times[0,\varepsilon]}(X\times\{0\}, Y\times\{\varepsilon\})$

$\le d_H^{Z\times[0,\varepsilon]}(X\times\{0\}, X\times\{\varepsilon\})$

$+ d_H^{Z\times[0,\varepsilon]}(X\times\{\varepsilon\}, Y\times\{\varepsilon\})$

$= \varepsilon + d_H^Z(X,Y)$

$\le 2\varepsilon + d_{GH}(X,Y)$, then let $\varepsilon \to 0$

□

Def: ~~(Quasi-metric)~~ (Pseudometric)

$d_{ps}: Z \times Z \to \mathbb{R}$, s.t.

① $d_{ps}(z_1,z_2) = d_{ps}(z_2,z_1)$

② $d_{ps}(z_1,z_2) \ge 0$

③ $d_{ps}(z_1,z_2) + d_{ps}(z_2,z_3) \ge d(z_1,z_3)$ $\forall z_1,z_2,z_3 \in Z$.

Lemma: $\mathcal{M} \triangleq \{(X,d) ; (X,d) \text{ compact}\}$, then $d_{GH}$ is a pseudometric over $\mathcal{M}$ and $d_{GH}(X,Y) = 0 \Leftrightarrow X \overset{\text{isom.}}{\simeq} Y$.

Proof: ~~As to~~ In the first part, we only need to prove the triangle ineq'l.

i.e. $d_{GH}(X,Y) + d_{GH}(Y,Z) \geq d_{GH}(X,Z) \quad \forall X, Y, Z$.

$\forall \varepsilon > 0$, assume admitted metrics $d_{XY}, d_{YZ}$ s.t. $d_{YZ,H}(Y,Z) \leq d_{GH}(Y,Z) + \varepsilon$,

$d_{XY,H}(X,Y) \leq d_{GH}(X,Y) + \varepsilon$.

def $d$, an admitted metric over $X \amalg Z$

s.t. $d_{XZ}(x,z) \triangleq \inf\limits_{y \in Y} \{ d_{XY}(x,y) + d_{YZ}(y,z) \}$.

Only need to show:

① $d_{XZ}(x_1, z_1) \leq d_{XZ}(x_1, x_2) + d_{XZ}(x_2, z_1)$

② $d_{XZ}(x_1, x_2) \leq d_{XZ}(x_1, z_1) + d_{XZ}(x_2, z_1)$.

(remained to be checked)

then, over $X \amalg Y \amalg Z$, $d_{XY}, d_{YZ}$; $d_{ZX}$ can define an admitted metric

( ... to be continued )

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**Lemma:** $d_{GH}$ is a pseudometric and

$d_{GH}(X,Y)=0 \Leftrightarrow X \overset{\text{iso.m.}}{\simeq} Y$

**Proof:** Only need to prove the triangle ineq'l, i.e. $\forall X, Y, Z$ compact,

$$d_{GH}(X,Y)+d_{GH}(Y,Z) \geq d_{GH}(X,Z).$$

$\forall \varepsilon>0$, $\exists\, d_{XY}, d_{YZ}$ ~~is~~ are admitted metrics over $X \amalg Y$, $Y \amalg Z$, s.t. $d_H^{XY}(X,Y) \leq d_{GH}(X,Y)+\varepsilon$

$d_H^{YZ}(Y,Z) \leq d_{GH}(Y,Z)+\varepsilon$

Now we construct an admitted metric over $X \amalg Y \amalg Z$, $d$: s.t.

$d(x,y)=d_{XY}(x,y)$, $d(y,z)=d_{YZ}(y,z)$

$$d(x,z)=\min_{y\in Y}\{d_{XY}(x,y)+d_{YZ}(y,z)\}$$

Now we verify that $d$ is a metric. Only need to check the triangle ineq'l.

$d(x_1,x_2) \leq d(x_1,z_1)+d(z_1,x_2)$ (*)

$d(x_1,z_1) \leq d(x_1,x_2)+d(x_2,z_1)$. (**)

For (*), $\exists\, y_1$ (since $Y$ is compact), s.t.

$d(x_1,z_1)=d_{XY}(x_1,y_1)+d_{YZ}(y_1,z_1)$, $\exists\, y_2$,

$d(z_1,x_2)=d_{XY}(x_2,y_2)+d_{YZ}(y_2,z_1)$.

$\Rightarrow$ RHS of (*) $\geq d_{XY}(x_1,y_1)+d_{XY}(x_2,y_2)+d_{YZ}(y_1,y_2)$

$= d_{XY}(x_1,y_1)+d_{XY}(x_2,y_2)+\underbrace{d_Y(y_1,y_2)}_{=d_{XY}(y_1,y_2)}$

$\geq d_{XY}(x_1,x_2)=d(x_1,x_2)$.

For (**), $\exists\, y_2$, s.t. $d(x_2,z_1)$

$= d_{XY}(x_2,y_2)+d_{YZ}(y_2,z_1)$

RHS of (**) $\geq d_{XY}(x_1,y_2)+d_{YZ}(y_2,z_1)$

$\geq d(x_1,z_1)$

$\Rightarrow d_{GH}(X,Z) \leq d_H(X,Z)$ on $X \amalg Y \amalg Z$

$\leq d_H(X,Y)+d_H(Y,Z)$

$\leq d_{GH}(X,Y)+d_{GH}(Y,Z)+2\varepsilon$, let $\varepsilon \to 0$

$\Rightarrow d_{GH}$ satisfies triangle ineq'l

$\Rightarrow d_{GH}$ is a pseudometric.

Then we prove $d_{GH}(X,Y)=0 \Leftrightarrow X \overset{\text{iso.m.}}{\simeq} Y$

Choose a subset $A=\{x_i\} \subseteq X$, denumerable and dense. Since $d_{GH}(X,Y)=0$

$\Rightarrow \exists$ an admitted metric $d_i$ over $X \amalg Y$, s.t. $d_{H,i}(X,Y) \leq i^{-1} \to 0$.

For $x_1$, $\exists\, y_{1,i}$ s.t. $d_i(x_1,y_{1,i}) \leq \frac{2}{i}$

Since $Y$ is compact, $\exists$ a subseq $\{y_{1,i_1}\}$ s.t. $y_{1,i_1} \to y_1$, and $d_{i_1}(x_1,y_1)$

$\leq d_{i_1}(x_1,y_{1,i_1})+d_{i_1}(y_{1,i_1},y_1)$

$\leq \frac{2}{i_1}+d_{i_1}(y_{1,i_1},y_1) \to 0$ as $i_1 \to +\infty$.

$\Rightarrow$ Choose $\{d_{i_1}\}$'s subseq $\{d_{i_2}\}$ and $\{y_{2,i_2}\}$ s.t. $y_{2,i_2} \to y_2$ and $d_{i_2}(x_2,y_2) \to 0$ as $i_2 \to +\infty$.

$\Rightarrow \exists\, \{d_\ell\} \subseteq \{d_i\}$ (diagonal of $d_{i_1}, d_{i_2} \cdots$) and $\{y_1,y_2,\cdots\} \subset Y$ s.t. $d_\ell(x_i,y_i) \to 0$ as $\ell \to +\infty$.

Now we define $f: A \to Y$, s.t. $f(x_i)=y_i$

then $d_Y(f(x_i),f(x_j))=d_X(x_i,x_j)$ (***)

by the fact that

$d_Y(\overbrace{f(x_i)}^{=y_i}, \overbrace{f(x_j)}^{=y_j}) = d_\ell(y_i, y_j)$

$\le d_\ell(y_i, x_i) + d_\ell(y_j, x_j) + d_\ell(x_j, x_i)$

$= d_\ell(y_i, x_i) + d_\ell(y_j, x_j) + d_X(x_j, x_i)$

$= d_X(x_j, x_i)$ as $\ell \to +\infty$.

similarly, $d_X(x_i, x_j) \le d_Y(y_i, y_j)$

$\Rightarrow$ (***) holds.

Now we extend $f$ to $X$, $f: X \to Y$,

$\forall x \in X$, $\exists \{x_\alpha\} \subset A$, s.t. $x_\alpha \to x$,

$\{x_\alpha\}$ is a Cauchy seq in $X$ $\Rightarrow$

$\{f(x_\alpha)\}$ is a Cauchy seq in $Y$, then define $f(x) = \lim_{\alpha \to +\infty} f(x_\alpha)$ since $Y$ is compact. This definition is well-defined since if $x'_\alpha \to x$, $\{x'_\alpha\} \subseteq A$.

then $d_X(x'_\alpha, x_\alpha) \le d_X(x'_\alpha, x) + d_X(x, x_\alpha) \to 0$ as $\alpha \to +\infty$

$\Rightarrow d_X(\overbrace{y'_\alpha}^{f(x'_\alpha)}, \overbrace{y_\alpha}^{f(x_\alpha)}) \to 0$ as $\alpha \to +\infty$

$\Rightarrow \lim_{\alpha \to +\infty} f(x_\alpha) = \lim_{\alpha \to +\infty} f(x'_\alpha) = f(x)$

then $\forall x, x' \in X$, choose $\{x_\alpha\}, \{x'_\alpha\} \subset X$, $x_\alpha \to x$, $x'_\alpha \to x'$.

$\Rightarrow d_Y(f(x), f(x')) = \lim_{\alpha \to +\infty} d_Y(f(x_\alpha), f(x'_\alpha))$

$= d_X(x, x')$

Claim: $\lim_{\ell \to +\infty} d_\ell(f(x), x) = 0$.

Proof of claim: choose $\{x_\alpha\} \subset A$, $x_\alpha \to x$.

then $d_\ell(f(x), x) \le d_\ell(f(x), f(x_\alpha))$

$+ d_\ell(f(x_\alpha), x_\alpha) + d_\ell(x_\alpha, x)$

~~$\Rightarrow \lim_{\ell \to +\infty} (f(x), x)$~~

Note that $d_\ell(x_\alpha, x) = d_X(x_\alpha, x)$

$d_\ell(f(x), f(x_\alpha)) = d_Y(f(x), f(x_\alpha))$.

let $\ell \to +\infty$, by the choice of $d_\ell$, $y_\alpha$, $d_\ell(f(x_\alpha), x_\alpha) \to 0$. then we let $\alpha \to +\infty$

$\Rightarrow \lim_{\ell \to +\infty} d_\ell(f(x), x) = 0$.

Now, for $\{d_\ell\}$, choose $Y$'s denumerable dense subset $B = \{z_1, z_2, \dots\}$, then $\exists \{d_{\ell'}\} \subseteq \{d_\ell\}$ and $\{w_1, w_2, \dots\} \subset X$, s.t. $d_{\ell'}(z_i, w_i) \to 0$ as $\ell' \to +\infty$.

we also construct $h: Y \to X$, s.t. $h(z_i) = w_i$, $d_Y(y, y') = d_X(h(y), h(y'))$ and $\lim_{\ell \to +\infty} d_{\ell'}(h(y), y) \to 0$.

Claim: $f \circ h = Id_Y$, $h \circ f = Id_X$.

$d_Y(f \circ h(y), y) = d_{\ell'}(f \circ h(y), y)$

$\le d_{\ell'}(f \circ h(y), h(y)) + d_{\ell'}(h(y), y)$

and let $\ell' \to 0$

$\Rightarrow d_Y(f \circ h(y), y) = 0 \Rightarrow f \circ h = Id_Y$

$\Rightarrow X \overset{\text{iso.m.}}{\simeq} Y$

$\square$

Now, let $\mathcal{M} = \{(X, d) : X \text{ is compact metric space}\} / \{X \overset{\text{iso.m.}}{\simeq} Y\}$

Thm. $(\mathcal{M}, d_{GH})$ is a ~~compact~~ complete metric space.

Proof: By lemma, $d_{GH}$ is a metric over $\mathcal{M}$. Then we need to show that $\mathcal{M}$ is complete

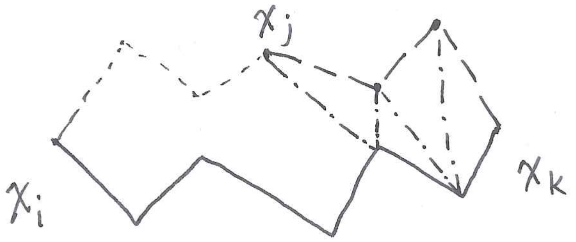
Assume $\{(X_i, d_i) \in \mathcal{M}\}$ is a Cauchy seq, i.e. $d_{GH}(X_i, X_j) \to 0$ as $i, j \to +\infty$.

Then we construct an $X$ s.t. $(X, d) \in \mathcal{M}$ $d_{GH}(X_i, X) \to 0$ as $i \to +\infty$.

Choose $\{x_i\}$'s subseq (again denoted as $\{X_i\}$) s.t. $d_{GH}(X_i, X_{i+1}) < 2^{-i-1}$ and $d_{i,i+1}$ is an admitted metric over $X_i \amalg X_{i+1}$ s.t. $d_H^{i,i+1}(X_i, X_{i+1}) \le 2^{-i}$.

consider $Y = \coprod_{i=1}^{\infty} X_i$, with admitted metric $d$: s.t.

$$d(x_i, x_j) \triangleq \min_{\{x_k \in X_k,\ i+1 \le k \le j-1\}} \sum_{k=i}^{j-1} d_{k,k+1}(x_k, x_{k+1})$$




$d(x_i, x_k) + d(x_j, x_k) \ge d(x_i, x_j)$

hence $d$ satisfies triangle ineq'l.

$\Rightarrow d$ is a metric

Now, we construct $X$. Let $\hat{X} = \{\{x_i\} : x_i \in X_i \text{ and } d(x_i, x_j) \to 0 \text{ as } i, j \to \infty\}$

def $\hat{d}(\{x_i\}, \{y_i\}) = \lim_{i \to +\infty} d(x_i, y_i)$

This is well-defined since

$|d(x_i, y_i) - d(x_j, y_j)| \le d(x_i, x_j) + d(y_i, y_j) \to 0$ as $i, j \to +\infty$

$\Rightarrow \{d(x_i, y_i)\}$ is a Cauchy seq over $\mathbb{R}$

$\Rightarrow \lim_{i \to +\infty} d(x_i, y_i)$ is well-defined.

and $\hat{d}$ is a pseudometric over $\hat{X}$, define equivalence ~~class~~ $\{x_i\} \sim \{y_i\}$

$\iff \hat{d}(\{x_i\}, \{y_i\}) = 0$

Let $X \triangleq \hat{X}/\sim$ and $(X, \hat{d})$ is a metric space.

Define the "metric" over $X \amalg Y$,

$$\tilde{d}(\overset{\in Y}{x_k}, \overset{\in X}{\{y_i\}}) \triangleq \lim_{i \to \infty} d(x_k, y_i).$$

Then $\tilde{d}$ is a pseudometric over $X \amalg Y$.

Consider equivalent class $x_k \sim \{y_i\}$

$\iff \tilde{d}(x_k, \{y_i\}) = 0$

Let $Z = X \amalg Y/\sim$

$\forall x_k \in X_k$, construct $\{y_i\} \in X$ (a point), s.t.

$\tilde{d}(x_k, \{y_i\}) \le 2^{-k+2}$

~~then~~ $\Rightarrow B_{2^{-k+2}}(X) \supset X_k$

Proof of this: Since $d_H(X_k, X_{k+1}) \le 2^{-k}$

$\Rightarrow \exists\, y_{k+1} \in X_{k+1}$, s.t. $\tilde{d}(y_{k+1}, x_k) \le 2^{-k+1}$

Because $d_H(X_{k+1}, X_{k+2}) \le 2^{-k-1}$,

$\Rightarrow \exists\, y_{k+2} \in X_{k+2}$ s.t. $\tilde{d}(y_{k+2}, y_{k+1}) \le 2^{-k}$

$\cdots \Rightarrow \exists\, \{y_i\}$ s.t. $\tilde{d}(y_j, y_{j+1}) \le 2^{-j+1}$

$\Rightarrow \{y_j\} \in X$ and $\tilde{d}(x_k, \{y_i\}) = \lim_{i \to +\infty} d(x_k, y_i) \le 2^{-k+2}$,

where $d(x_k, y_i) \le d(x_k, y_{k+1}) + d(y_{k+1}, y_{k+2}) \cdots d(y_{i-1}, y_i) \le 2^{-k+2}$

$\square$

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Let $\mathcal{M} = \{(X,d) : X \text{ is compact metric space}\} / \{X \overset{\text{iso.m.}}{\simeq} Y\}$

Thm. $(\mathcal{M}, d_{GH})$ is a complete metric space.

Proof: What we have got:

$(X_i, d_i) \in \mathcal{M}$, Cauchy seq.

Assume $d_{GH}(X_i, X_{i+1}) \le 2^{-i}$

Then we constructed $Y = \coprod_{i=1}^{\infty} X_i$ with metric $d$.

$\hat{X} \triangleq \{\{x_i\} : x_i \in X_i,\ d(x_i, x_j) \to 0 \text{ as } i,j \to +\infty\}$

$\hat{d}(\{x_i\}, \{y_i\}) = \lim_{i\to\infty} d(x_i, y_i)$.

Define $\{x_i\} \sim \{y_i\}$ iff $\hat{d}(\{x_i\}, \{y_i\}) = 0$.

$X \triangleq \hat{X}/\sim \Rightarrow (X, \hat{d})$ is a metric space.

Now we want to prove $(X, \hat{d})$ is $X_i$'s limit, and $X$ is compact. hence $X \in \mathcal{M}$.

Consider $Z = X \amalg Y$, with metric (admitted)

$d(\{x_i\}, y_k) = \lim_{i\to\infty} d(y_k, x_i)$, $\{x_i\} \in X_i, y_k \in Y$

Let $\tilde{Z} = Z/\sim$, where $\{x_i\} \sim y_k \Leftrightarrow d(\{x_i\}, y_k) = 0$.

Claim: ① $X_i \subseteq B_{2^{-i+3}}(X)$

② $X \subseteq B_{2^{-i+3}}(X_i)$

Proof of ①: $\forall y_i \in X_i$, find $\{x_k\} \in X$, and $d(\{x_k\}, y_i) < 2^{-i+3}$

Since $d_H(X_i, X_{i+1}) \le 2^{-i+1}$, $\exists x_{i+1} \in X_{i+1}$, s.t. $d(x_{i+1}, y_i) < 2^{-i+2}$

Since $d_H(X_{i+1}, X_{i+2}) \le 2^{-i}$, $\Rightarrow \exists x_{i+2} \in X_{i+2}$ s.t. $d(x_{i+2}, x_{i+1}) < 2^{-i+1}$

...

$\forall j > i$, $\exists x_j \in X_j$, $d(x_j, x_{j-1}) < 2^{-j+3}$

$\Rightarrow \{x_k\} \in X$, and $d(\{x_k\}, y_i)$

$= \lim_{k\to\infty} d(x_k, y_i) \le \left(\sum_{j=i+2}^{\infty} d(x_j, x_{j-1})\right) + d(x_{i+1}, y_i)$

$\le 2^{-i+2} + 2^{-i+2} < 2^{-i+3} \Rightarrow$ ① □

Proof of ②: We ~~only~~ need to show:

$\forall \{y_i\} \in X$, we can find $\{x_i\} \in X$, s.t. $d(x_i, x_{i+1}) \le 2^{-i+2}$ and $d(x_i, y_i) \to 0$ as $i \to \infty$

i.e. $\hat{d}(\{x_i\}, \{y_i\}) = 0$.

Now, $\forall i$, $\exists \alpha_i > i$, s.t. if $j, k \ge \alpha_i$, $d(y_j, y_k) \le 2^{-i-2}$, ~~$\alpha_i > i$~~.

Since $d(X_i, X_{\alpha_i}) \le 2^{-i+1}$

$\Rightarrow \exists x_i \in X_i$, s.t. $d(x_i, y_{\alpha_i}) \le 2^{-i+1}$.

Then $d(x_i, x_{i+1}) \le d(x_i, y_{\alpha_i}) + d(y_{\alpha_i}, y_{\alpha_{i+1}}) + d(x_{i+1}, y_{\alpha_{i+1}})$

$\le 2^{-i+1} + 2^{-i} + 2^{-i-2} \le 2^{-i+2}$.

And $d(x_i, y_i) \le d(x_i, y_{\alpha_i}) + d(y_i, y_{\alpha_i})$

$\le 2^{-i+1} + d(y_i, y_{\alpha_i})$

$\to 0$ as $i \to +\infty$.

Now, return to ②. For $\forall \{y_i\} \in X$, $\exists \{x_i\} \in X$, s.t. $\{x_i\} \sim \{y_i\}$ and $d(x_i, x_{i+1}) \le 2^{-i+2}$

$\Rightarrow d(x_k, \{y_i\}) = d(x_k, \{x_i\})$

$= \lim_{i\to\infty} d(x_k, x_i) \le \lim_{i\to\infty} \sum_{j=k}^{i-1} d(x_j, x_{j+1})$

$\leq \lim_{i\to\infty} \sum_{j=k}^{i-1} 2^{-j+2} \leq 2^{-i+3}$

$\Rightarrow X \subset B_{2^{-k+3}}(X_k)$

$\square$

Now, by claim ① ②, $d_{GH}(X, X_k)$

$\leq 2^{-k+i} \xrightarrow{k\to\infty} 0$

Now we prove $X$ is compact, i.e.

$\forall$ $X$'s seq has a convergent subseq

$\to$ as in claim②, i.e. $d(x_i^\alpha, x_{i+1}^\alpha) \leq 2^{-i+2}$

For $\{\{x_i^\alpha\} \in X, \alpha = 1, 2, \cdots\}$, fix $i$,

since $X_i$ is compact,

$\Rightarrow \{x_i^1, x_i^2, \cdots\} \subseteq X_i$, $\exists$ subseq

$\{x_i^{\alpha'}\} \subseteq X_i$, s.t. $\lim_{\alpha'\to\infty} x_i^{\alpha'} \triangleq x_i^\infty$.

Now, by diagonal principle, $\exists \alpha_1, \alpha_2, \alpha_3 \cdots$,

s.t. $\forall i$, $\lim_{\alpha_i\to\infty} x_i^{\alpha_i} = x_i^\infty$.

We want to prove $\{x_i^\infty\} \in X$ and

$\hat{d}(\{x_i^{\alpha_k}\}, \{x_i^\infty\}) \to 0$ as $k\to+\infty$

since $d(x_i^\infty, x_j^\infty) \leq d(x_i^{\alpha_k}, x_i^{\alpha_\infty})$

$+ d(x_i^{\alpha_k}, x_j^{\alpha_k}) + d(x_j^{\alpha_k}, x_j^\infty)$

$\leq d(x_i^{\alpha_k}, x_i^\infty) + d(x_j^{\alpha_k}, x_j^\infty)$

$+ \sum_{l=i}^{j-1} 2^{-l+2}$ $(j>i)$, and let $\alpha_k \to +\infty$ $(k\to+\infty)$

$d(x_i^\infty, x_j^\infty) \leq \sum_{l=i}^{j-1} 2^{-l+2} \leq 2^{-i+3}$, $j>i$.

$\Rightarrow \{x_i^\infty\} \in X$.

$\hat{d}(\{x_i^{\alpha_k}\}, \{x_i^\infty\}) = \lim_{i\to\infty} d(x_i^{\alpha_k}, x_i^\infty)$

$\leq \lim_{i\to\infty} d(x_j^{\alpha_k}, x_i^{\alpha_k}) + d(x_i^\infty, x_j^\infty)$

$+ d(x_j^{\alpha_k}, x_j^\infty)$, $\forall j<i$ fixed.

$\leq 2^{-j+2} + 2^{-j+3} + d(x_j^{\alpha_k}, x_j^\infty)$, $\forall j$ fixed.

$\Rightarrow \overline{\lim_{k\to+\infty}} \hat{d}(\{x_i^{\alpha_k}\}, \{x_i^\infty\})$

$\leq 2^{-j+2} + 2^{-j+3}$, $\forall j$ fixed

and let $j\to\infty$, we have

$\lim_{k\to\infty} \hat{d}(\{x_i^{\alpha_k}\}, \{x_i^\infty\}) = 0$, i.e.

$\{x_i^{\alpha_k}\} \xrightarrow[k\to\infty]{\hat{d}} \{x_i^\infty\}$

$\square$

Lemma: If $A = \{x_1, x_2, \cdots, x_k\}$, $B = \{y_1, y_2, \cdots, y_k\}$ discrete compact metric spaces

s.t. $|d(x_i, x_j) - d(y_i, y_j)| < \varepsilon$, $\forall i, j$.

then $d_{GH}(A, B) \leq \varepsilon$

Proof: Construct a $d$ over $A \amalg B$, s.t.

$$\begin{cases} d(x_i, y_i) = \varepsilon, \\ d(x_i, y_j) = \min_l \{d(x_i, x_l) + \varepsilon + d(y_j, y_l)\} \end{cases}$$

If $d$ is an admitted metric $\Rightarrow d_H(A, B) = \varepsilon$

$\Rightarrow d_{GH}(A, B) \leq \varepsilon$

triangle ineq'l: ① $d(x_i, x_j) \leq d(x_i, y_l) + d(y_l, x_j)$

② $d(x_i, y_l) \leq d(x_i, x_j) + d(x_j, y_l)$

For ②: $d(x_j, y_l) = d(x_j, x_p) + \varepsilon + d(y_p, y_l)$, for $p$.

$\Rightarrow d(x_i, x_j) + d(x_j, y_l)$

$\geq d(x_i, x_p) + \varepsilon + d(y_p, y_l) \geq d(x_i, y_l)$

For ①: $d(x_i, y_l) = d(x_i, x_p) + \varepsilon + d(y_p, y_l)$

$d(x_j, y_l) = d(x_j, x_q) + \varepsilon + d(y_q, y_l)$

$\Rightarrow d(x_i, y_l) + d(x_j, y_l)$

$\geq d(y_p, y_q) + 2\varepsilon + d(x_i, x_p) + d(x_j, x_q)$

by assumption

$\geq d(x_p, x_q) + \varepsilon + d(x_i, x_p) + d(x_j, x_q)$

$\geq d(x_i, x_j) + \varepsilon > d(x_i, x_j)$

$\square$

Cor: If $\{x_1, x_2, \cdots x_k\} \subseteq X$

$\{y_1, \cdots y_k\} \subseteq Y$.

and $X \subset \bigcup_{i=1}^{k} B_{\varepsilon}(x_i)$, $Y \in \bigcup_{i=1}^{k} B_{\varepsilon}(y_i)$,

and $|d(x_i, x_j) - d(y_i, y_j)| \leq \varepsilon$

$\Rightarrow d_{GH}(X, Y) \leq 3\varepsilon$

Proof: Let $A = \{x_1, \cdots x_k\}$, $B = \{y_1 \cdots y_k\}$,

by the lemma: $d_{GH}(A, B) \leq \varepsilon$.

Since it is a cover, $d_H(A, X) \leq \varepsilon$,

$d_H(B, Y) \leq \varepsilon \Rightarrow d_{GH}(A, X) \leq \varepsilon$.

$d_{GH}(B, Y) \leq \varepsilon$

$\Rightarrow d_{GH}(X, Y) \leq 3\varepsilon$ by triangle ineq'l

$\square$

Def: (Capacity)

Assume $X$ is a compact metric space, and $\forall \varepsilon > 0$. $Cap(\varepsilon) \triangleq Cap_X(\varepsilon)$

$\triangleq$ ~~max~~ $\underset{K}{\arg\max} \{ \bigcup_{i=1}^{K} B_{\frac{\varepsilon}{2}}(x_i) \subseteq X$,

$x_i \in X$, $B_{\frac{\varepsilon}{2}}(x_i) \cap B_{\frac{\varepsilon}{2}}(x_j) = \phi, i \neq j \}$.

all open

Def: (Covering)

$Cov(\varepsilon) = Cov_X(\varepsilon)$

$\triangleq \underset{K}{\arg\min} \{ \bigcup_{i=1}^{K} B_{\varepsilon}(x_i) \supseteq X$,

$x_i \in X \}$.

Properties: ① $Cov_X(\varepsilon) \leq Cap_X(\varepsilon)$

② If $d_{GH}(X, Y) < \delta$, then

1) $Cov_X(\varepsilon + 2\delta) \leq Cov_Y(\varepsilon)$

2) $Cap_X(\varepsilon + 2\delta) \leq Cap_Y(\varepsilon)$

Proof: ① assume $Cap_X(\varepsilon) = N$

$\Rightarrow \exists \{x_1, \cdots, x_N\}$ s.t.

$B_{\frac{\varepsilon}{2}}(x_i) \cap B_{\frac{\varepsilon}{2}}(x_j) = \phi$, $\forall i, j = 1 \cdots, N$.

then $X \subseteq \bigcup_{i=1}^{N} B_{\varepsilon}(x_i)$

Otherwise, $\exists x \in X \setminus \bigcup_{i=1}^{N} B_{\varepsilon}(x_i)$,

s.t. $d(x, x_i) \geq \varepsilon \Rightarrow B_{\frac{\varepsilon}{2}}(x) \cap B_{\frac{\varepsilon}{2}} = \phi, \forall i$.

contradiction!

②2): Assume $Cap_X(\varepsilon + 2\delta) = N$. $\Rightarrow$

$\exists \{x_1, \cdots, x_N\}$ s.t. $d(x_i, x_j) \geq \varepsilon + 2\delta$

then choose $\{y_1, \cdots, y_N\} \subset Y$.

s.t. $d_{X \amalg Y}(x_i, y_i) < \delta$

$$\Rightarrow d(y_i, y_j) \geq d(x_i, y_j) - d_{X \amalg Y}(x_i, y_i) - d(x_j, y_j) > \varepsilon, \ \forall i, j$$

$$\Rightarrow B_{\frac{\varepsilon}{2}}(y_i) \cap B_{\frac{\varepsilon}{2}}(y_j) = \phi$$

$$\Rightarrow \mathrm{Cap}_Y(\varepsilon) \geq N = \mathrm{Cap}_X(\varepsilon + 2\delta).$$

1): Exercise.

$\square$

Thm. (Gromov's precompact theorem).

Assume $\mathcal{C} \subseteq (\mathcal{M}, d_{GH})$, a subset, the TFAE:

① $\mathcal{C}$ is precompact, i.e. $\forall$ seq $\subseteq \mathcal{C}$, has a convergent subseq.

② $\exists$ a function $N_1 : (0,1) \to \mathbb{Z}_{\geq 0}$,
s.t. $\forall X \in \mathcal{C}$, $\mathrm{Cap}_X(\varepsilon) \leq N_1(\varepsilon)$,
$\forall \varepsilon \in (0,1)$

③ $\exists$ a function $N_2 : (0,1) \to \mathbb{Z}_{\geq 0}$,
s.t. $\forall X \in \mathcal{C}$, $\mathrm{Cov}_X(\varepsilon) \leq N_2(\varepsilon)$, $\forall \varepsilon \in (0,1)$

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What we have: Lemma: $A=\{x_1,x_2,\cdots,x_k\}$, $B=\{y_1,y_2,\cdots,y_k\}$ metric spaces. If $\forall\, i,j=1,\cdots,k$,

$|d(x_i,x_j)-d(y_i,y_j)|<\varepsilon$, then $d_{GH}(A,B)\le\varepsilon$.

Capacity: $Cap(\varepsilon)=Cap_X(\varepsilon)\triangleq \underset{k}{\arg\max}\left\{\bigcup_{i=1}^{k}B_{\frac{\varepsilon}{2}}(x_i)\subseteq X,\ x_i\in X,\ B_{\frac{\varepsilon}{2}}(x_i)\cap B_{\frac{\varepsilon}{2}}(x_j)=\phi,\ i\ne j\right\}$

open for all $i$

Covering: $Cov(\varepsilon)=Cov_X(\varepsilon)=\underset{k}{\arg\min}\left\{\bigcup_{i=1}^{k}B_\varepsilon(x_i)\supseteq X,\ x_i\in X\right\}$

Properties: ① $Cov_X(\varepsilon)\le Cap_X(\varepsilon)$

② If $d_{GH}(X,Y)<\delta$, then $Cap_X(\varepsilon+2\delta)\le Cap_Y(\varepsilon)$

$Cov_X(\varepsilon+2\delta)\le Cov_Y(\varepsilon)$.

Thm. (Gromov)

Assume $\mathcal{C}\subset(\mathcal{M},d_{GH})$ is a subset, then

TFAE:

1) $\mathcal{C}$ is precompact

2) $\exists\, N_1:(0,1)\to\mathbb{Z}_{\ge0}$, s.t. $\forall X\in\mathcal{C}$, $Cap_X(\varepsilon)\le N_1(\varepsilon)$, $\forall\varepsilon\in(0,1)$

3) $\exists\, N_2:(0,1)\to\mathbb{Z}_{\ge0}$, s.t. $\forall X\in\mathcal{C}$, $Cov_X(\varepsilon)\le N_2(\varepsilon)$, $\forall\varepsilon\in(0,1)$.

Remark: $A$ is precompact $\Leftrightarrow$ $\forall$ seq. has a convergent subseq. (or Cauchy subseq.), limit may not in $A$

$\Leftrightarrow$ "whole bounded", i.e. $\forall\varepsilon>0$, $Cov_A(\varepsilon)<+\infty$

Proof of Remark: If "whole bounded", then $\forall\{x_i\}\subset A$. $\forall\varepsilon>0$, $Cov_A(\varepsilon)<+\infty$.

$\Rightarrow \exists$ subseq. $\{x_i'\}$, s.t. $d(x_i',x_j')\le 2\varepsilon\ldots$

$\Rightarrow$ precompact

If precompact, $\cdots\Rightarrow$ whole bounded $\square$

Proof of theorem: 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1)

1) $\Rightarrow$ 2): $\forall\varepsilon>0$, since $\mathcal{C}$ is precompact,

$\Rightarrow$ $\mathcal{C}$ is whole bounded

$\Rightarrow \exists$ finite subset $\{X_1,\cdots,X_k\}\subset\mathcal{C}$,

s.t. $\forall X\in\mathcal{C}$, $\exists X_i$, s.t. $d_{GH}(X,X_i)<\frac{\varepsilon}{4}$.

By property ②, $Cap_X(\varepsilon)\le Cap_{X_i}(\frac{\varepsilon}{2})$.

Let $N_1(\varepsilon)=\max\limits_{1\le i\le k}Cap_{X_i}(\frac{\varepsilon}{2})<+\infty$

$\Rightarrow \forall X\in\mathcal{C}$, $Cap_X(\varepsilon)\le N_1(\varepsilon)$

2) $\Rightarrow$ 3): By property ①, choose $N_2=N_1$

3) $\Rightarrow$ 1): Only need to prove $\mathcal{C}$ is whole bounded, i.e. $\forall\varepsilon>0$, $\exists X_1,\cdots X_k\in\mathcal{C}$, s.t. $\forall X\in\mathcal{C}$, $\exists X_i$, s.t. $d_{GH}(X,X_i)<\varepsilon$

Since $Cov_X(\frac{\varepsilon}{2})\le \overset{<+\infty}{N_2(\frac{\varepsilon}{2})}\Rightarrow$

① $diam(X)\le\varepsilon N_2(\frac{\varepsilon}{2})$ (Assume $X$ is connected)

② exists at most $N_2(\frac{\varepsilon}{2})$ points $\{y_\alpha\}$, s.t. $B_{\frac{\varepsilon}{2}}(\{y_\alpha\})\supset X$

i.e. $d_{GH}(\{y_\alpha\},X)\le\frac{\varepsilon}{2}$, where $\{y_\alpha\}\subset X$.

expand $\{y_\alpha\}$ s.t. exact $N_2(\frac{\varepsilon}{2})$ points,

s.t. $d_{GH}(\{y_\alpha\},X)\le\frac{\varepsilon}{2}$

then $\forall\alpha,\beta=1,2,\cdots,N_2(\frac{\varepsilon}{2})$.

Let $d^X_{\alpha\beta}=d_X(y_\alpha,y_\beta)$. then

$d^X_{\alpha\beta}\le diam(X)\le\varepsilon N_2(\frac{\varepsilon}{2})$.

$\Rightarrow (d^X_{\alpha\beta}) \in \mathbb{R}^{N_2(\frac{\varepsilon}{2})\times N_2(\frac{\varepsilon}{2})}$ (matrix)

$\Rightarrow \{ (d^X_{\alpha\beta}) \in \mathbb{R}^{N_2(\frac{\varepsilon}{2})^2} \mid X \in \mathcal{C} \}$

is a bounded set, i.e. precompact set.

$\Rightarrow \exists\ (d^{X_1}_{\alpha\beta}), \cdots (d^{X_k}_{\alpha\beta})$

s.t. $\forall (d^X_{\alpha\beta})$, $\exists (d^{X_i}_{\alpha\beta})$, s.t.

$|(d^X_{\alpha\beta}) - (d^{X_i}_{\alpha\beta})| \leq \frac{\varepsilon}{10}$,

i.e. $\forall \alpha, \beta = 1, \cdots, N_2(\frac{\varepsilon}{10})$, $|d^X_{\alpha\beta} - d^{X_i}_{\alpha\beta}| \leq \frac{\varepsilon}{10}$

By the lemma. $d_{GH}(\{y^X_\alpha\}, \{y^{X_i}_\alpha\}) \leq \frac{\varepsilon}{10}$.

On the other hand, $d_{GH}(\{y^X_\alpha\}, X) \leq \frac{\varepsilon}{2}$,

$d_{GH}(\{y^{X_i}_\alpha\}, X_i) \leq \frac{\varepsilon}{2}$

$\Rightarrow d_{GH}(X, X_i) \leq d_{GH}(\{y^X_\alpha\}, X)$

$+ d_{GH}(\{y^{X_i}_\alpha\}, X_i) + d_{GH}(\{y^X_\alpha\}, \{y^{X_i}_\alpha\})$

$\leq \frac{11\varepsilon}{10}$

i.e. $\exists \{X_1, \cdots X_k\} \subset \mathcal{C}$, s.t.

$B_{2\varepsilon}(\{X_1, \cdots X_k\}) \supset \mathcal{C}$

$\Rightarrow \mathcal{C}$ precompact $\square$

Remark: Gromov theorem. Assume the metric spaces in $\mathcal{C}$ are uniformly bounded.

or assume $\forall x, y \in X \in \mathcal{C}$, $\exists \gamma [0,1] \to X$, $\gamma$ is continuous, s.t. $\gamma(0) = x$, $\gamma(1) = y$ and $d(\gamma(t), \gamma(s)) = |t-s| d(x,y)$, $\forall t, s \in [0,1]$.

Estimate $Cap_X(\varepsilon)$:

- metric space $(X, d, \mu)$ with measurement $\mu$, a Borel measurement.

Proposition: If $(X, d)$ is compact metric space, $X$ has a Borel measurement $\mu$, s.t. $\forall x \in X$, $0 < r < 1$, $\mu(B_r(x)) \geq V_0 \cdot r^n$ and $\mu(X) \leq V$, then $Cap_X(r) \leq \frac{V \cdot 2^n}{V_0 \cdot r^n}$, $\forall r \in (0,1)$

Proof: Assume $Cap_X(r) = N$,

$\Rightarrow \exists \{x_1, \cdots x_N\} \subset X$ s.t. $B_{\frac{r}{2}}(x_i) \cap B_{\frac{r}{2}}(x_j) = \emptyset$

$\Rightarrow V \geq \mu(X) \geq \sum_{i=1}^N \mu(B_{\frac{r}{2}}(x_i))$

$\geq N \cdot V_0 \cdot \frac{r^n}{2^n}$

$\square$

(Doubling measure): $\forall x \in X$, $0 < r \leq \operatorname{diam} X$,

$\mu(B_r(x)) \geq c\mu(B_{2r}(x))$

Proposition: $(X, d)$ has a doubling measure, s.t. 1) $\forall x \in X$, $0 < r \leq \operatorname{diam}(X)$,

$\mu(B_r(x)) \geq c\mu(B_{2r}(x))$

2) $\operatorname{diam}(X) \leq D$,

then $Cap_X(r) \leq \frac{1}{c} (\frac{D}{r})^{-\frac{\log c}{\log 2}}$, $\forall 0 < r \leq 1$

Proof: $\forall r > 0$, $\forall x \in X$,

$\mu(B_{\frac{r}{2}}(x)) \geq c\mu(B_r(x))$

$\geq c^2 \mu(B_{2r}(x)) \cdots \geq c^k \mu(B_{2^{k-1}r}(x))$

Let $2^{k-1} r = D \Rightarrow 2^{k-1} = \frac{D}{r}$

$\Rightarrow k-1 = \frac{\log \frac{D}{r}}{\log 2}$

$\Rightarrow c^k = c^{\frac{\log \frac{D}{r}}{\log 2}+1} = \left(\frac{D}{r}\right)^{\frac{\log c}{\log 2}} \cdot c$

$\Rightarrow \mu(B_{\frac{r}{2}}(x)) \geq \left(\frac{D}{r}\right)^{\frac{\log c}{\log 2}} \cdot c\mu(X)$

Assume $Cap_X(r) = N$, then $\exists \{x_1, \cdots, x_N\} \subset X$, s.t. $B_{\frac{r}{2}}(x_i) \cap B_{\frac{r}{2}}(x_j) = \phi$.

$$\Rightarrow \mu(X) \geq \sum_{i=1}^{N} \mu(B_{\frac{r}{2}}(x_i)) \geq N \cdot \left(\frac{D}{r}\right)^{\frac{\log c}{\log 2}} \cdot c\mu(X)$$

□

Corollary: Assume $\mathcal{C} \subset (\mathcal{M}, d_{GH})$, a subset. If $\forall X \in \mathcal{C}$, ① $\operatorname{diam} X \leq D$

② $\exists$ doubling measure $\mu$, s.t. $\mu(B_r(x)) \geq c\mu(B_{2r}(x))$, $\forall x \in X$, $r \leq D$

then $\mathcal{C}$ is precompact □

Cor2: Let $\mathcal{M}(n, D, \lambda) = \{(M, g) : (M, g)$ n-dim'l complete Riemannian mfld, and $\operatorname{diam}(M,g) \leq D$, $Ric_{(M,g)} \geq -(n-1)\lambda\}$, then $\mathcal{M}(n, D, \lambda) \subseteq (\mathcal{M}, d_{GH})$ is ~~prepc~~ precompact by Volume comparison theorem and Cor 1. □

Noncompact cases:

Pointed metric space: $(X, d, x)$, $x \in X$.

Assume $(X, d_X, x)$, $(Y, d_Y, y)$ are pointed compact metric spaces. ↳ bounded diameter

Def: Pointed Gromov-Hausdorff metric.

$$d^P_{GH}((X, d_X, x), (Y, d_Y, y)) \triangleq \inf_{(Z,d)} \{ d^Z_H(X, Y) + d_Z(x, y) \}$$

inf is taken in all $X \xhookrightarrow{iso.m.} Z$, $Y \xhookrightarrow{iso.m.} Z$.

Remark: Only consider the admitted metrics over $X \amalg Y$.

Eg. $(X, d, x) = ([0,1], d, 0) \subseteq \mathbb{R}$

$(Y, d, y) = ([0,1], d, \frac{1}{2}) \subseteq \mathbb{R}$.

$\Rightarrow d_{GH}((X, d, x), (Y, d, y)) = \frac{1}{2}$

For noncompact complete metric, $(X, d_X, x)$, assume, $\forall y \in X$, $r > 0$, $\varepsilon > 0$,

$B_{r+\varepsilon}(y) = B_\varepsilon(B_r(y))$

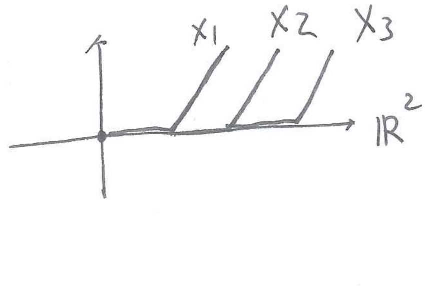
Def. (Convergence of pointed metric spaces)

We say: $(X_i, d_i, x_i) \xrightarrow{d^P_{GH}} (X, d, x)$, if for $\forall R > 0$, we have

$$d^P_{GH}((\overline{B_R(x_i)}, d_i, x_i), (\overline{B_R}(x), d, x)) \to 0$$

as $i \to +\infty$.

E.g.



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Pointed - GH metric

$(X, d_X, x), (Y, d_Y, y)$ compact

Def: $d^P_{GH}((X, d_X, x), (Y, d_Y, y))$

$$\triangleq \inf_{(Z,d)} \{ d^Z_H(X, Y) + d_Z(x, y) \}$$

$X \xhookrightarrow{iso.m.} Z$, $Y \xhookrightarrow{iso.m.} Z$, $Z$ is a compact metric space.

For complete noncompact metric, $(X, d_X, x)$

Assume: 1) bounded, closed subset is compact

2) $\forall \varepsilon > 0$, $r > 0$, $y \in X$,

$$B_\varepsilon(B_r(y)) = B_{\varepsilon + r}(y)$$

Def (PGH - convergence):

$$(X_i, d_i, x_i) \xrightarrow{d^P_{GH}} (X, d, x)$$

if $\forall R > 0$,

$$d^P_{GH}((\overline{B_R(x_i)}, x_i), (\overline{B_R(x)}, x)) \to 0$$

as $i \to \infty$

Lemma: If $d^P_{GH}((\overline{B_R(x_i)}, x_i), (\overline{B_R(x)}, x)) \xrightarrow[i\to\infty]{} 0$,

then $\forall 0 < r \leq R$,

$$d^P_{GH}((\overline{B_r(x_i)}, x_i), (\overline{B_r(x)}, x)) \xrightarrow{i\to\infty} 0$$

Proof: Assume $d_i$ is an admitted metric over $\overline{B_R(x_i)} \amalg \overline{B_R(x)}$, s.t.

$$d_{GH, i}(\overline{B_R(x_i)}, \overline{B_R(x)}) + d_i(x_i, x) < \varepsilon_i \to 0$$

For $\forall z_i \in \overline{B_r(x_i)}$, $\Rightarrow \exists z \in \overline{B_R(x)}$, s.t. $d_i(z_i, z) < \varepsilon_i$

$$\Rightarrow d_i(z, x) \leq d_i(z, z_i) + d_i(z_i, x_i) + d_i(x_i, x)$$
$$\leq r + 2\varepsilon_i$$

$\Rightarrow z \in B_{r+3\varepsilon_i}(x)$

$$\Rightarrow \overline{B_r(x_i)} \subseteq B_{\varepsilon_i}(B_{r+3\varepsilon_i}(x))$$
$$\subseteq B_{r+4\varepsilon_i}(x) \underset{\text{by assumption}}{=} B_{4\varepsilon_i}(B_r(x))$$

Similarly, $\overline{B_r(x)} \subseteq B_{4\varepsilon_i}(B_r(x_i))$

$$\Rightarrow d^P_{GH}((\overline{B_r(x)}, x), (\overline{B_r(x_i)}, x_i))$$
$$\leq 4\varepsilon_i + \varepsilon_i \to 0 \qquad \square$$

Eg1: Assume $X_i = \{0\} \cup [1 + \frac{1}{i}, 1 + \frac{2}{i}] \subseteq \mathbb{R}$

$$X = \{0\} \cup \{1\}$$

then $d^P_{GH}((\overline{B^i_2(0)}, 0), (\overline{B_2(0)}, 0)) \xrightarrow[i\to\infty]{} 0$

But $d^P_{GH}((\overline{B^i_1(0)}, 0), (\overline{B_1(0)}, 0)) \underset{i\to\infty}{\not\to} 0$

where $\overline{B_R(x_i)} = \{x : d(x, x_i) \leq R\}$

Eg2: $X_i = \{0\} \cup [1 - \frac{2}{i}, 1 - \frac{1}{i}] \subseteq \mathbb{R}$

$$X = \{0\} \cup \{1\}$$

then $X_i \xrightarrow{d^P_{GH}} X$, but $(\overline{B^i_1(0)}, (0)) \not\to (\overline{B_1(0)}, 0)$

$P \to$ may not have $P$

where $\overline{B^i_1(0)} = \{x : d(x, 0) < 1\}$

Compactness Theorem:

Assume $(X_i, d_i, x_i)$ are complete metrices. If $\forall R>0$, $0<\varepsilon<1$, $\exists N_R(\varepsilon)\in \mathbb{Z}_{\geq 0}$, s.t. $Cap_{\overline{B_R(x_i)}}(\varepsilon) \leq N_R(\varepsilon)$, $\forall i$, then $\exists$ $x_i$'s subseq, convergent under PGH-metric.

Proof: By Gromov's compact theorem, $\forall R>0$, $\exists$ subseq of $\overline{B_R(x_{i,R})}$, s.t.

$$\overline{B_R(x_{i,R})} \to X_R$$
$$x_{i,R} \to x \in X_R$$

$\Rightarrow X_R = \overline{B_R(x)}$,

By diagonal principle, $\exists$ sub seq of $\overline{B_R(x_{i,R})}$ convergent

$\square$

Tangent cone: $(X,d)$ is a complete metric, $x\in X$, $(Y, d_Y, y)$ is complete

1) If $\exists$ $r_\alpha \to 0$ $(r_\alpha>0)$, s.t.

$$\left(X, \frac{d}{r_\alpha}, x\right) \xrightarrow{\alpha\to\infty} (Y, d_Y, y),$$

then we say $(Y, d_Y, y)$ is a tangent cone at $x$.

2) If $\exists$ $r_\alpha \to +\infty$, s.t.

$$\left(X, \frac{d}{r_\alpha}, x\right) \xrightarrow{\alpha\to\infty} (Y, d_Y, y),$$

then we say $(Y, d_Y, y)$ is a tangent cone of $X$ at $\infty$,

where $\left(X, \frac{d}{r_\alpha}, x\right)$ satisfies:

$$\frac{d}{r_\alpha}(z,w) = \frac{1}{r_\alpha} d(z,w),\ \forall z,w\in X$$

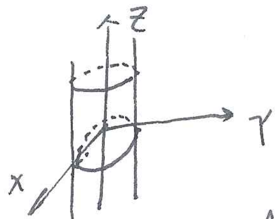
Eg1: $X=\{(x,y): x^2+y^2=1\} \subseteq \mathbb{R}^2$

At $(1,0)$, the tangent cone $\overset{\text{iso.m.}}{\simeq} (\mathbb{R}, d, 0)$

all real numbers.


At $\infty$, the tangent cone is a point

Eg2: $X=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2=1\}\subseteq \mathbb{R}^3$




At $(0,1,0)$, the tangent cone $\overset{\text{iso.m.}}{\simeq} (\mathbb{R}^2, d, 0^2)$

At $\infty$, the tangent cone $\overset{\text{iso.m.}}{\simeq}$ $\mathbb{R}$. (real-axis).

Remark: 1) the tangent cone of $(X,d)$ is indept to the choice of $x$, the basis point.

2) tangent cone is dept to $\{r_\alpha\}$

3) the tangent cone may not exist

4) when the tangent cone exists, it may not be unique

Thm. Assume $(X_i, d_i, x_i) \xrightarrow{d_{GH}^P} (X, d, x)$, and assume $\forall i$, $\exists$ a measurement $\mu_i$, s.t. $\mu_i(B_{2r}(y)) \leq C(R)\mu_i(B_r(y))$, $\forall y\in B_R(x_i)$, $r\leq R$.

then, $\forall z \in X$, the tangent cone exists at $z$. always

Proof: Let $\{r_\alpha\} \to 0$ arbitrarily,

consider $(X, \frac{d}{r_\alpha}, z) \triangleq (\tilde{X}_\alpha, \tilde{d}_\alpha, \tilde{z}_\alpha)$,

we only need to show $\exists$ subseq convergent.

$\Leftarrow \forall R>0, \forall 0<\varepsilon<1, \exists N_R(\varepsilon) \in \mathbb{Z}_{\geq 0}$,

s.t. $\mathrm{Cap}_{\overline{B_R(\tilde{z}_\alpha)}}(\varepsilon) \leq N_R(\varepsilon), \forall \alpha$.

Assume $z_i \in X_i$, s.t. $z_i \to z$,

assume $d(z, x) = R_0$, then $z_i \in B_{2R_0}(x_0)$

$\Rightarrow \mathrm{Cap}_{\overline{B_1(z_i)}}(\varepsilon) \leq C_{R_0, C(R_0)}, \left(\frac{1}{\varepsilon}\right)^{C_{R_0, C(R_0)}}$

Since $\overline{B_1(z_i)} \to \overline{B_1(z)}$

$\Rightarrow d_{GH}(\overline{B_1(z_i)}, \overline{B_1(z)}) \leq \varepsilon_i \to 0$

$\Rightarrow \mathrm{Cap}_{\overline{B_1(z)}}(\varepsilon)$

$\leq \mathrm{Cap}_{\overline{B_1(z_i)}}(\varepsilon - 2\varepsilon_i)$

$\leq \mathrm{Cap}_{\overline{B_1(z_i)}}\left(\frac{\varepsilon}{2}\right)$, for $i$ large enough

$\leq C_{R_0, C(R_0)} \left(\frac{1}{\varepsilon}\right)^{C_{R_0, C(R_0)}}$.

Similarly, $\forall 0<r<1$,

$\mathrm{Cap}_{\overline{B_r(z)}}(\varepsilon) \leq C_{R_0, C(R_0)} \left(\frac{r}{\varepsilon}\right)^{C_{R_0, C(R_0)}}$,

For $\tilde{d}_\alpha$, $\overline{B_r(z)} = \overline{B_{r \cdot r_\alpha^{-1}}(\tilde{z}_\alpha)}$

$\Rightarrow \mathrm{Cap}_{\overline{B_{r\cdot r_\alpha^{-1}}(\tilde{z}_\alpha)}}(\varepsilon \cdot r_\alpha^{-1}) = \mathrm{Cap}_{\overline{B_r(z)}}(\varepsilon)$

$\leq C_{R_0, C(R_0)} \left(\frac{r}{\varepsilon}\right)^{C_{R_0, C(R_0)}}$

For $\forall R>0$, for $\alpha$ large enough,

assume $r \cdot r_\alpha^{-1} = R$, $\tilde{\varepsilon} = \varepsilon \cdot r_\alpha^{-1}$

$\Rightarrow \mathrm{Cap}_{B_R(\tilde{z}_\alpha)}(\tilde{\varepsilon}) \leq C_{R_0, C(R_0)} \left(\frac{r}{\varepsilon}\right)^{C_{R_0, C(R_0)}}$

$= C_{R_0, C(R_0)} \left(\frac{R}{\tilde{\varepsilon}}\right)^{C_{R_0, C(R_0)}}$,

where $\tilde{\varepsilon} \in (0,1) \Rightarrow$ satisfies the compactness theorem

$\Rightarrow$ has a subseq convergent

$\square$

Eg 1: Assume $X = \{(x,y) \in \mathbb{R}^2 : y = x^2\} \subseteq \mathbb{R}^2$,

where $d((x_1,y_1),(x_2,y_2)) = \sqrt{|x_1-x_2|^2 + |y_1-y_2|^2}$.

Q: what is $X$'s tangent cone at $\infty$? And prove it.

Eg 2: Assume $X$ is a point, $Y = S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$,

compute $d_{GH}(X, Y)$. $S^n$ is with Euclidean metric

Eg 3: Assume $u \in C_0^\infty(\mathbb{R}^n)$, if

$\sum_{i=0}^{\infty} \fint_{B_{2^{-i}}(x)} |\nabla u| < \varepsilon$, and $\fint_{B_{\frac{1}{2}}(0)} u = 1$,

$x \in B_{\frac{1}{2}}(0)$, then $|u(x) - 1| \leq C(n)\varepsilon$

Eg 4: $(X, d)$ is a compact metric space, $d_H$ is the Hausdorff metric over $X$.

then $\mathcal{Z} = \{A \subset X, A \text{ is closed}\}$.

try to show that $(\mathcal{Z}, d_H)$ is a compact metric space. (Only need to show that it is compact)