

Construction of the G_2 algebra

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This reading report most follows contents in Humphreys's *Introduction to Lie Algebras and Representation Theory*. Our focuses are on the constructions of G_2 algebra, whose **Cartan matrices** are:

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

And I filled some details which are not in this book.

1 Criterion for semisimplicity

We begin this reading report with a criterion for reductive Lie algebra's semisimplicity, while it is well-known that we could compute the **Killing form** of an arbitrarily given Lie algebra to test its semisimplicity in general. But for **reductive** Lie algebra, which means that $\text{Rad } L$, the maximal solvable ideal of L , equals to the center $Z(L)$, we have a result that $L = [LL] \oplus Z(L)$ with $[LL]$ semisimple. Hence, once we could show that $Z(L) = 0$, the L is semisimple automatically.

Proposition. (a) Let L be reductive. Then $L = [LL] \oplus Z(L)$, and $[LL]$ is semisimple.

(b) Let $L \subset \mathfrak{gl}(V)$ (V finite dimensional) be a nonzero Lie algebra acting irreducibly on V . Then L is reductive, with $\dim Z(L) \leq 1$. If in addition $L \subset \mathfrak{sl}(V)$, then L is semisimple.

Proof. (a) If L is abelian, then the conclusion is trivial. Hence, we could suppose that L is reductive but not abelian, so $L' = L/Z(L) = L/\text{Rad } L$ is semisimple by definition. Then $\text{ad } L \cong \text{ad } L' \cong L'$ acts completely reducible on L by **Weyl's theorem**, which states that a finite dimensional representation of a semisimple Lie algebra is completely reducible. Hence, $Z(L)$, an L -submodule of L , has an L -submodule complement M , which also means that M is an ideal of L . Write $L = M \oplus Z(L)$. Then, $[LL] = [MM] \subset M$. But $[LL]$ maps onto L' by the canonical map, so $L = [LL] \oplus Z(L)$ (Or we can use the fact that $[MM] = M$ since $M \cong L/Z(L) \cong L'$ is semisimple, then $[LL] = [MM] = M$).

(b) We briefly denote $\text{Rad } L$ as S . Because S is a solvable Lie subalgebra of $\mathfrak{gl}(V)$, according to **Lie's theorem**, S has a common eigenvector v in V , say $s.v = \lambda(s)v$ ($s \in S$) with λ linear. We want to show that S acts diagonally on V as the scalar $\lambda(s)$ to get $\dim S \leq 1$. Since L acts irreducibly on V , V has a basis with the form $\{v, x_1.v, \dots, x_n.v\}$. Hence, we only need to consider the action of S on this basis. Clearly, since S is an ideal, for any $x \in L$, $[x, s] \in S$ (here, $[\cdot, \cdot]$ is the canonical lie bracket of $\mathfrak{gl}(V)$), then

$$s.x.v = \lambda(s)x.v + \lambda([s, x]).v$$

By this formula, we could conclude that the matrices relative to the above basis of all $s \in S$ are triangular, with the only diagonal entry $\lambda(s)$. Nevertheless, the commutators $[SL] \subset S \subset \mathfrak{gl}(V)$ have trace zero, which forces λ to vanish on $[SL]$. Hence, by the above formula, $S \subset \mathfrak{gl}(V)$ acts diagonally on V as the scalar $\lambda(s)$, then $\dim S \leq 1$ and $S = Z(L)$ by $Z(L) \subset S \subset Z(\mathfrak{gl}(V)) \cap L \subset Z(L)$. Hence L is reductive.

If $L \subset \mathfrak{sl}(V)$, since the only diagonal matrix in $\mathfrak{sl}(V)$ is 0, then $S = 0$, and $L = [LL]$ is semisimple. \square

Remark: The technique about **trace** in (b) is also used in proving the **Lie's Theorem** in section 4 of Humphreys's book.

2 Explicit construction of algebra G_2

In general, once we have a fixed root system Φ , with base $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$, according to **Serre's Theorem**, we could construct a semisimple Lie algebra with corresponding root system Φ following some relations. In the irreducible root system cases, say G_2 , this Lie algebra is automatically a simple Lie algebra by the correspondence between the direct sum decomposition of a semisimple Lie algebra and irreducible components of its root system.

In Humphreys's book, there are two constructions of G_2 . The first is to regard L , the simple algebra of type G_2 , as a subalgebra of $L_0 = \mathfrak{o}(7)$, the simple algebra of type B_3 . In this way, it just assigns the corresponding root vectors of L to some vectors in L_0 and verifies that L is a 14-dimensional Lie algebra. Then, just directly verify that L acts irreducibly on $V = \mathbb{F}^7$ and use the **criterion of semisimplicity** above to show L is semisimple and type G_2 . The second, which is a more interesting way, is to regard it as the Lie algebra $\text{Der } \mathfrak{C}$, where \mathfrak{C} is an 8-dimensional nonassociative algebra and called the octonion algebra. I would like to present the **idea** about the second construction.

Before our construction, we are supposed to describe this \mathfrak{C} first. Let $\{e_1, e_2, e_3\}$ be the usual standard orthonormal basis of \mathbb{F}^3 (in most cases of Humphreys's book, we could just regard \mathbb{F} as \mathbb{C}) and the usual inner product for $u = u_1e_1 + u_2e_2 + u_3e_3, v = v_1e_1 + v_2e_2 + v_3e_3$ in \mathbb{F}^3 is $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$. It is clear that we could also define a cross product in \mathbb{F}^3 :

$$u \times v = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

Then, as a vector space, \mathfrak{C} is the sum of two copies of \mathbb{F}^3 and two copies of \mathbb{F} . For convenience, we could write elements in \mathfrak{C} as 2×2 matrices in the form:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix}, a, b \in \mathbb{F} \text{ and } v, w \in \mathbb{F}^3$$

We add and multiply scalars just as we would for matrices. However, the product in \mathfrak{C} is given by a little complicate formula:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} = \begin{bmatrix} aa' - v \cdot w' & av' + b'v + w \times w' \\ a'w + bw' + v \times v' & bb' - w \cdot v' \end{bmatrix}$$

This product is obviously bilinear. Then, fix a basis $\{c_1, \dots, c_8\}$:

$$c_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad c_{2+i} = \begin{bmatrix} 0 & e_i \\ 0 & 0 \end{bmatrix}, \quad c_{5+i} = \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix}$$

By careful computation, we could get the multiplication table of \mathfrak{C} :

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
c_1	c_1	0	c_3	c_4	c_5	0	0	0
c_2	0	c_1	0	0	0	c_6	c_7	c_8
c_3	0	c_3	0	c_8	$-c_7$	$-c_1$	0	0
c_4	0	c_4	$-c_8$	0	c_6	0	$-c_1$	0
c_5	0	c_5	c_7	$-c_6$	0	0	0	$-c_1$
c_6	c_6	0	$-c_2$	0	0	0	c_5	$-c_4$
c_7	c_7	0	0	$-c_2$	0	$-c_5$	0	c_3
c_8	c_8	0	0	0	$-c_2$	c_4	$-c_3$	0

Notice that $c_1 + c_2$ acts as identity in this table and check that $\mathfrak{C}_0 = [\mathfrak{C}, \mathfrak{C}]$ (here $[c, c'] = cc' - c'c$ for $c, c' \in \mathfrak{C}$) is a subspace with codimension 1 and is spanned by the basis $\{e_1 - e_2, e_3, e_4, \dots, e_8\}$. By the product rule, any derivation of \mathfrak{C} kills the multiples of $c_1 + c_2$ ($\delta(x) = \delta((c_1 + c_2)x) = \delta(x) + \delta(c_1 + c_2)x$, for all $x \in \mathfrak{C}$). Moreover, a derivation leaves \mathfrak{C}_0 invariant since $\delta([c, c']) = \delta(c)c' + c\delta(c') - \delta(c')c - c'\delta(c) = [\delta(c), c'] + [c, \delta(c')] \in \mathfrak{C}_0$.

Now, set $L = \text{Der } \mathfrak{C}$. Obviously, L acts faithfully on \mathfrak{C}_0 because if $x \in L$ acts trivially on \mathfrak{C}_0 , we will get that x acts trivially on the whole \mathfrak{C} since any derivation kills $c_1 + c_2$ and \mathfrak{C}_0 is spanned by $\{c_1 - c_2, c_3, c_4, \dots, c_8\}$. Denote by $\phi : L \rightarrow \mathfrak{gl}(7, \mathbb{F})$ the associated matrix representation with the basis of \mathfrak{C}_0 chosen as above. The idea here is to exhibit some derivations of \mathfrak{C} clearly. As the long roots in the root system of G_2 form a system of type A_2 , L contain a copy of $\mathfrak{sl}(3, \mathbb{F})$. Then, for $x \in \mathfrak{sl}(3, \mathbb{F})$, set:

$$\delta : \mathfrak{sl}(3, \mathbb{F}) \rightarrow \text{End}(\mathfrak{C}) \quad x \mapsto \delta(x)$$

with

$$\delta(x) \left(\begin{bmatrix} a & v \\ w & b \end{bmatrix} \right) = \begin{bmatrix} 0 & x(v) \\ -x^t(w) & 0 \end{bmatrix}$$

By the fact that $\det(x(v_1), v_2, v_3) + \det(v_1, x(v_2), v_3) + \det(v_1, v_2, x(v_3)) = 0$ for $v_1, v_2, v_3 \in \mathbb{F}^3$ and $x \in \mathfrak{sl}(3, \mathbb{F})$ (this fact is also **true in higher dimension** and the key to prove it is that the trace of x is zero and then verify it by choosing the standard basis of $\mathfrak{sl}(n, \mathbb{F})$ and \mathbb{F}^n for higher n), we could prove that these $\delta(x)$ are actually in $\text{Der } \mathfrak{C}$. Also, this representation of $\mathfrak{sl}(3, \mathbb{F})$ is actually nontrivial according to this form.

However, in order to construct the whole L which is semisimple, the above representation about $\mathfrak{sl}(3, \mathbb{F})$ is not enough because we want to use the **criterion of semisimplicity** before, for which we are supposed to show that $\text{Der } \mathfrak{C}$ acts on \mathfrak{C}_0 irreducibly while $\mathfrak{sl}(3, \mathbb{F})$ kills $c_1 - c_2$. Hence, we need to locate more derivations to show the irreducibility.

Fortunately, we do not need to locate too much derivations thanks to the representation of $\mathfrak{sl}(3, \mathbb{F})$. For example, if we let $w = w_1e_1 + w_2e_2 + w_3e_3$, $v = v_1e_1 + v_2e_2 + v_3e_3$, then:

$$\begin{bmatrix} a & v \\ w & b \end{bmatrix} \mapsto \begin{bmatrix} 0 & v_1e_1 - v_2e_2 \\ -w_1e_1 + w_2e_2 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & (v_1 - v_2)e_1 \\ (w_1 - w_2)e_2 & 0 \end{bmatrix}$$

by setting $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in $\delta(x)$. Hence, the only trouble is that $\mathfrak{sl}(3, \mathbb{F})$ will kills $c_1 - c_2$ while we could choose elements in $\mathfrak{sl}(3, \mathbb{F})$ and get c_3, \dots, c_8 in above way. Now, we need a lemma:

Lemma. *In any algebra \mathfrak{A} satisfying the alternative laws, i.e. $x^2y = x(xy)$, $yx^2 = (yx)x$, an endomorphism of the following form is actually a derivation: $[L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ ($a, b \in \mathfrak{A}$, $L_a =$ left multiplication by a , $R_a =$ right multiplication by a)*

Proof. Let $x = a + b$, $y = c$, we could get an equivalent statement of the alternative laws:

$$(ab)c + (ba)c = a(bc) + b(ac) \quad \text{and} \quad c(ba) + c(ab) = (cb)a + (ca)b$$

Denote $(xy)z - x(yz)$ as $[x, y, z]$, which is called associator, then we show this lemma by several steps:

(1) $[L_a, L_b] = L_{[a, b]} - 2[L_a, R_b]$ and $[R_a, R_b] = -R_{[a, b]} - 2[L_a, R_b]$.

For any $x \in \mathfrak{A}$, $[L_a, L_b](x) = a(bx) - b(ax)$, $L_{[a, b]} - 2[L_a, R_b](x) = ((ab) - (ba))x - 2(a(xb)) + 2((ax)b)$, then

$$\begin{aligned} [L_a, L_b](x) &= L_{[a, b]} - 2[L_a, R_b](x) \\ \iff a(bx) - b(ax) &= ((ab) - (ba))x - 2(a(xb)) + 2((ax)b) \\ \iff (ab)x + (ax)b - a(xb) - b(ax) &= ((ab) - (ba))x - 2(a(xb)) + 2((ax)b) \\ \iff -b(ax) &= -(ba)x - a(xb) + (ax)b \\ \iff a(bx) - (ab)x &= -a(xb) + (ax)b \\ \iff a(bx) - (ax)b &= (ab)x - a(xb) \\ \iff a(bx) - (ax)b &= -(ax)b + a(xb) + a(bx) - a(xb) \end{aligned}$$

The $[R_a, R_b] = -R_{[a, b]} - 2[L_a, R_b]$ is similar.

(2) $L_a(xy) = ((L_a + R_a)x)y - x(L_a y)$ and $R_a(xy) = -(R_a x)y + x((L_a + R_a)y)$ for any $x, y \in \mathfrak{A}$.

$$\begin{aligned} L_a(xy) &= ((L_a + R_a)x)y - x(L_a y) \\ \iff a(xy) &= (ax + xa)y - x(ay) \\ \iff a(xy) + x(ay) &= (ax + xa)y \end{aligned}$$

The $R_a(xy) = -(R_a x)y + x((L_a + R_a)y)$ is similar.

(3) $(L_a - R_a)(xy) = ((L_a - R_a)x)y + x((L_a - R_a)y) + [x, 3a, y]$.

According to (2), $(L_a - R_a)(xy) = ((L_a - R_a)x)y + 2(R_a x)y - x(L_a y) + x((L_a - R_a)y) - 2x(L_a y) + (R_a x)y$, and $(R_a x)y - x(L_a y) = [x, a, y]$ by definition.

(4) $[L_a, R_b](x) = [a, b, x] = (ab)x - a(bx) = b(ax) - (ba)x = x(ba) - (xb)a$.

This is quickly from $[L_a, R_b](x) = -[a, x, b] = [a, b, x] = -[b, a, x] = -[x, b, a]$ by the alternative laws.

(5) The **left Moufang identity**, i.e. $((xy)x)z = x(y(xz))$.

According to (4), $((xy)x)z - x(y(xz)) = ((xy)x)z - (xy)(xz) + (xy)(xz) - x(y(xz)) = [xy, x, z] + [x, y, xz] = -[x, xy, z] - [x, xz, y] = -(x^2y)z + x((xy)z) - (x^2z)y + x((xz)y) = -x^2(yz + zy) + x(x(yz + zy)) = -x^2(yz + zy) + x^2(yz + zy) = 0$.

Moreover, let $x = a + b$, we could get the **linearized left Moufang identity**:

$$((ay)b)z + ((by)a)z = a(y(bz)) + b(y(az)).$$

$$(6) [L_a, R_b](xy) = ([L_a, R_b]x)y + x([L_a, R_b]y) + [x, [a, b], y].$$

According to (4), $[L_a, R_b](xy) = (ab)(xy) - a(b(xy))$,

$$(ab)(xy) = ((ab)x + x(ab))y - x((ab)y),$$

by the **linearized left Moufang identity**,

$$a(b(xy)) = ((ab)x)y + ((xb)a)y - x(b(ay)),$$

then, according to (4),

$$\begin{aligned} [L_a, R_b](xy) &= (x(ab) - (xb)a)y - x((ab)y - b(ay)) \\ &= (x(ba) - (xb)a)y + (x([a, b]))y + x(b(ay) - (ba)y) - x([a, b]y) \\ &= ([L_a, R_b]x)y + x([L_a, R_b]y) + [x, [a, b], y] \end{aligned}$$

(7) $\delta = [L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ is a derivation.

We could rewrite δ as $L_{[a,b]} - R_{[a,b]} - 3[L_a, R_b]$ by (1), then, by (2),

$$(L_{[a,b]} - R_{[a,b]})(xy) - ((L_{[a,b]} - R_{[a,b]})x)y - x((L_{[a,b]} - R_{[a,b]})y) = 3[x, [a, b], y],$$

by (6),

$$3[L_a, R_b](xy) - 3([L_a, R_b]x)y - 3x([L_a, R_b]y) = 3[x, [a, b], y],$$

subtracts these two equations, we get $\delta(xy) - \delta(x)y - x\delta(y) = 3[x, [a, b], y] - 3[x, [a, b], y] = 0$ \square

Now, we show that \mathfrak{C} satisfies the alternative laws $x^2y = x(xy)$ and $yx^2 = (yx)x$ for any $x, y \in \mathfrak{C}$. For $x^2y = x(xy)$,

$$\begin{aligned} &\begin{bmatrix} a & v \\ w & b \end{bmatrix}^2 \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} \\ &= \begin{bmatrix} (a^2 - v \cdot w)a' - (a+b)v \cdot w' & (a^2 - v \cdot w)v' + b'(a+b)v + (a+b)w \times w' \\ a'(a+b)w + (b^2 - v \cdot w)w' + (a+b)v \times v' & -(a+b)w \cdot v' + b'(b^2 - v \cdot w) \end{bmatrix} \\ &= \begin{bmatrix} a & v \\ w & b \end{bmatrix} \left(\begin{bmatrix} a & v \\ w & b \end{bmatrix} \begin{bmatrix} a' & v' \\ w' & b' \end{bmatrix} \right) \end{aligned}$$

And $yx^2 = (yx)x$ is similar.

Hence, by the above lemma, endomorphisms of the form $[L_a, L_b] + [L_a, R_b] + [R_a, R_b]$ for $a, b \in \mathfrak{C}$ are in $Der\mathfrak{C}$. Now, we could consider the irreducibility. For example, we could consider $\delta_{13} = [L_1, L_3] + [L_1, R_3] + [R_1, R_3]$ (here, for simplicity, we denote L_{c_i} as L_i and R_{c_i} as R_i for $i = 1, 2, \dots, 8$, $\{c_i\}$ is the basis we chose earlier), and let $w = w_1e_1 + w_2e_2 + w_3e_3$, $v = v_1e_1 + v_2e_2 + v_3e_3$, then:

$$\begin{aligned} L_3 \left(\begin{bmatrix} a & v \\ w & b \end{bmatrix} \right) &= \left(\begin{bmatrix} -w_1 & be_1 \\ v_2e_3 - v_3e_2 & 0 \end{bmatrix} \right) \quad , \quad R_3 \left(\begin{bmatrix} a & v \\ w & b \end{bmatrix} \right) = \begin{bmatrix} 0 & ae_1 \\ v_3e_2 - v_2e_3 & -w_1 \end{bmatrix} \\ L_1 \left(\begin{bmatrix} a & v \\ w & b \end{bmatrix} \right) &= \begin{bmatrix} a & v \\ 0 & 0 \end{bmatrix} \quad , \quad R_1 \left(\begin{bmatrix} a & v \\ w & b \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ w & 0 \end{bmatrix} \end{aligned}$$

$$\delta_{13} \left(\begin{bmatrix} a & v \\ w & b \end{bmatrix} \right) = \begin{bmatrix} -w_1 & (b-a)e_1 \\ v_3e_2 - v_2e_3 & w_1 \end{bmatrix} \in \mathfrak{C}_o$$

From this example, we could see that the combination of $\delta_{ij}, i = 1, 2, j \geq 3$ and the action of $\mathfrak{sl}(3, \mathbb{F})$ could actually show the irreducibility of the action of $Der\mathfrak{C}$ on \mathfrak{C}_o . Thanks to the **criterion of semisimplicity**, $Der\mathfrak{C}$ is **semisimple**.

Now, back to the faithful representation of $\mathfrak{sl}(3, \mathbb{F})$ we constructed earlier, and $L = Der\mathfrak{C}$. Denote the image in L of $\mathfrak{sl}(3, \mathbb{F})$ as M and the image of the diagonal subalgebra of $\mathfrak{sl}(3, \mathbb{F})$ as H . It could be directly verified that the matrices in $\mathfrak{gl}(8, \mathbb{F})$ commuting with H all have the form $diag(x, a_3, a_4, a_5, a_6, a_7, a_8)$, where $x \in \mathfrak{gl}(2, \mathbb{F})$, with the basis c_1, c_2, \dots, c_8 . Notice that such a matrix represents a derivation of \mathfrak{C} only if it is already in H (This fact could be showed by applying the multiplication table of \mathfrak{C} showed earlier. Say, let $\delta = diag(x, a_3, a_4, a_5, a_6, a_7, a_8)$ be a derivation, by $\delta(c_1^2) = \delta(c_1)c_1 + c_1\delta(c_1)$ and $\delta(c_2^2) = \delta(c_2)c_2 + c_2\delta(c_2)$, x must be 0 in $\mathfrak{gl}(2, \mathbb{F})$, and by c_3, \dots, c_8 to get a relation between a_3, \dots, a_8 , which shows that δ must be in H). Hence, H is its own centralizer in L . Since the representation of $\mathfrak{sl}(3, \mathbb{F})$ we constructed is faithful, then $Z(M) = Z(\mathfrak{sl}(3, \mathbb{F})) = 0$, $Z(L) = Z(L) \cap H \subset (H \cap Z(M)) = 0$. Hence, $L = Der\mathfrak{C}$ is **simple** according to the **criterion of semisimplicity** (We could also deduce $Z(L) = 0$ by facts that L does not contain constant matrices other than 0 while $Z(L)$ consists of constant matrices as showed in the proof of the **criterion of semisimplicity**). And H , which is two dimensional, is exactly the **Cartan subalgebra** of $L = Der\mathfrak{C}$. By our construction, $L \not\supseteq \mathfrak{sl}(3, \mathbb{F})$, which is a system of type A_2 , hence, L could not be type B_2 and A_2 , and the only possibility left is G_2 .

References

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