Bit-Error Rate of Binary Digital Modulation Schemes in Generalized Gamma Fading Channels

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***Abstract*— We derive a closed-form expression for the bit-error rate of binary digital modulation schemes in a generalized fading channel that is modeled by the three-parameter generalized gamma distribution. This distribution is very versatile and generalizes or accurately approximates many of the commonly used channel models for multipath, shadow, and composite fading. The result is expressed in terms of Meijer's G-function, which can be easily evaluated numerically.**

***Index Terms*— Generalized gamma fading, Generalized Nakagami fading, Fox's H-Function, Meijer's G-Function.**

I. Introduction

MULTIPATH fading and shadowing severely degrade the performance of wireless communication systems. Many statistical distributions are available in the literature to model multipath and shadow fading in such systems [1]. However, with the ever-increasing demand for personal communication services (PCS) anywhere and at any time, wireless systems are required to operate in increasingly hostile environments and must also operate with less bandwidth, less power and more interference. Therefore, wireless system designers must understand the nature of the operating environment in order to adequately predict the performance of mobile radio systems. A versatile envelope distribution which generalizes many of the commonly used models for multipath and shadow fading is the three-parameter generalized gamma distribution [2], [3], [4], [5]. It has such envelope distributions as the Rayleigh, Nakagami, and Weibull [6] distributions as special cases, and the lognormal distribution as a limiting case. As such, it can be used to characterize both multipath and shadow fading processes. It has been shown that the generalized gamma distribution (like the Suzuki distribution) also provides a good fit to measured data in the presence of both multipath and shadow fading [2]. Average bit error rate (BER) expressions for digital binary phase shift keying (BPSK) and binary frequency shift keying (BFSK) modulation schemes operating over the generalized gamma fading environment were presented in [2] (and references therein) as infinite series. In this letter, we derive a closed-form expression for the average BER for both coherent and noncoherent/differentially coherent binary digital modulations in this generalized fading environment.

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II. Channel Model

We assume that the fading environment is such that the received signal envelope, R, is a three-parameter generalized gamma random variable that has the probability density function (pdf) given by [2], [3], [4], [5]

$$p_R(r) = \frac{2\nu}{(\Omega/m)^m \Gamma(m)} r^{2\nu m-1} \exp\left(-\frac{mr^{2\nu}}{\Omega}\right), \quad r \geq 0 \quad (1)$$

where m is the fading parameter, Ω is the power-scaling parameter, and ν is the shape parameter. Such commonly used channel fading models as the Rayleigh ($m = \nu = 1$), Nakagami-m ($\nu = 1$), Weibull ($m = 1$), and lognormal ($m \to \infty$, $\nu \to 0$) distributions are special or limiting cases of the generalized gamma distribution. In [5], the pdf in (1) was used as a generalization of the Nakagami fading channel in modeling backscattered ultrasonic echo from tissues. The parameter ν was introduced to model more severe fading conditions than are possible with the Nakagami distribution. In an additive white Gaussian noise (AWGN) channel, the pdf of the instantaneous signal-to-noise ratio (SNR), $\gamma = R^2 \frac{E_b}{N_0}$, can be expressed in terms of the average SNR, $\bar{\gamma} = E[R^2]\frac{E_b}{N_0}$, as

$$p_\gamma(\gamma) = \frac{\nu(\beta/\bar{\gamma})^{m\nu}}{\Gamma(m)} \gamma^{\nu m-1} \exp\left\{-\left(\frac{\beta\gamma}{\bar{\gamma}}\right)^\nu\right\}, \quad \gamma \geq 0 \quad (2)$$

where $\beta = \frac{\Gamma(m+1/\nu)}{\Gamma(m)}$. Note that when $\nu = 1$ (Nakagami-m fading), we have $\beta = m$, as expected. The pdf in (2) was first introduced by Stacy [7] as a generalization of the (two-parameter) gamma distribution. It is different from the generalization of the gamma distribution presented in [8], which models the power of Rice-faded envelopes.

III. Calculation of Bit-Error Rate

The conditional BER in an AWGN channel may be written in compact form as [[1], eq. (8.100)]

$$P_E(\gamma) = \frac{\Gamma(b, a\gamma)}{2\Gamma(b)}, \quad (3)$$

where $a = 1$ for BPSK and $1/2$ for BFSK, and $b = 1$ for noncoherent BFSK (NCFSK)/differentially coherent BPSK (DPSK) and $1/2$ for coherent BFSK/BPSK. $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}dt$ is the complementary incomplete gamma function [[9], eq. (8.350.2)]. In a flat-fading environment, the average BER is given by

$$\bar{P}_E = \int_0^\infty P_E(\gamma)p_\gamma(\gamma)d\gamma. \quad (4)$$

Then, substituting (2) and (3) in (4), the average BER in a generalized gamma fading channel becomes

$$\bar{P}_E = \frac{\nu(\beta/\bar{\gamma})^{m\nu}}{2\Gamma(b)\Gamma(m)} \int_0^\infty \gamma^{m\nu-1}\Gamma(b, a\gamma) e^{-\left(\frac{\beta\gamma}{\bar{\gamma}}\right)^\nu} d\gamma. \quad (5)$$

To evaluate the integral in (5), we note from the definition of the gamma function,

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx, \quad \mathrm{Re}\{s\} > 0, \quad (6)$$

that we may obtain the inverse Mellin transform of the exponential function as [10]

$$\exp(-x) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} \Gamma(-s) x^s ds, \quad \sigma = \mathrm{Re}\{s\} < 0, \quad (7)$$

where $j = \sqrt{-1}$. Using (7) in (5) and interchanging the order of integration since the integrand converges absolutely, we have

$$\begin{aligned}\bar{P}_E &= \frac{\nu(\beta/\bar{\gamma})^{m\nu}}{2\Gamma(b)\Gamma(m)} \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} (\beta/\bar{\gamma})^{s\nu}\Gamma(-s) \\ &\quad \times \int_0^\infty \gamma^{\nu(s+m)-1}\Gamma(b, a\gamma) d\gamma ds. \quad (8)\end{aligned}$$

The inner integral in (8) may be evaluated with the help of [[9], eq. (6-455)]. Upon replacing $\nu(m+s)$ with s we obtain

$$\begin{aligned}\bar{P}_E &= \frac{1}{2\Gamma(b)\Gamma(m)} \frac{1}{2\pi j} \\ &\quad \times \int_C \left(\frac{\beta}{a\bar{\gamma}}\right)^s \frac{\Gamma(b+s)\Gamma(s)\Gamma(m-\frac{s}{\nu})}{\Gamma(1+s)} ds, \quad (9)\end{aligned}$$

where the channel fading parameters m and ν have arbitrary real values and the path of the integration is the imaginary axis (in the complex $s-$ plane) which is indented, if necessary, in order to separate the poles of $\Gamma(b+s)$ and $\Gamma(s)$ from the poles of $\Gamma(m - s/\nu)$. The result in (9) may be expressed in terms of Fox's H-function (see [11] for definition and properties):

$$\bar{P}_E = \frac{1}{2\Gamma(b)\Gamma(m)} H_{2,2}^{1,2}\left[\frac{\beta}{a\bar{\gamma}} \,\middle|\, \begin{matrix}(1-b,1),(1,1) \\ (m,1/\nu),(0,1)\end{matrix}\right]. (10)$$

To facilitate the numerical computation of $\bar{P}_E$, we restrict our analysis to rational values of the parameter ν (that is, we let $\nu = l/k$, where l and k are arbitrary integers) and replace s with sl in (9) to obtain

$$\begin{aligned}\bar{P}_E &= \frac{l}{2\Gamma(b)\Gamma(m)} \frac{1}{2\pi j} \\ &\quad \times \int_C \left(\frac{\beta}{a\bar{\gamma}}\right)^{ls} \frac{\Gamma(b+ls)\Gamma(ls)\Gamma(m-ks)}{\Gamma(1+ls)} ds. (11)\end{aligned}$$

Finally, we may use the multiplication theorem for the gamma function [[9], (8.335)]

$$\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{i-1}{n} + x\right) \quad (12)$$

to express (11) as

$$\bar{P}_E = \frac{A(k,l)}{2\pi j}\int_C \left(\frac{l\beta}{k^{k/l} a\bar{\gamma}}\right)^{ls} \chi(s) ds, \quad (13)$$

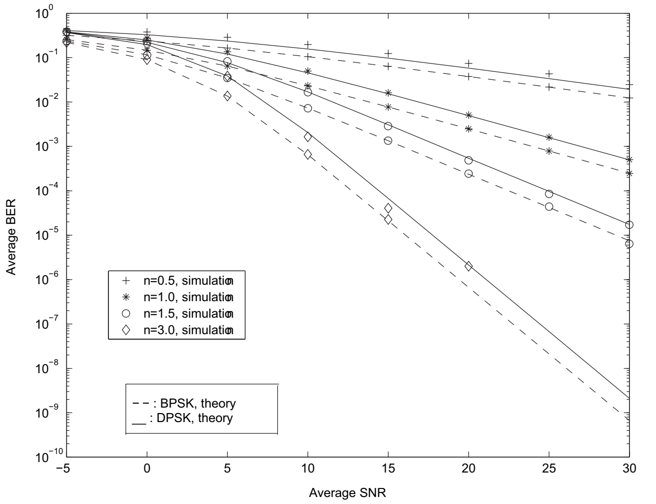

Fig. 1. Average BER versus $\bar{\gamma}$ for coherent and differentially coherent PSK in a generalized gamma fading channel with $m = 1$, for several values of ν.

where

$$A(k,l) = \frac{(l)^{b-\frac{1}{2}}(k)^{m-\frac{1}{2}}(2\pi)^{\frac{2-l-k}{2}}}{2\Gamma(b)\Gamma(m)},$$

$$\chi(s) = \frac{\prod_{t=1}^{k}\Gamma\left(\frac{t+m-1}{k} - s\right)\prod_{i=1}^{l}\Gamma\left(\frac{i+b-1}{l} + s\right)\Gamma\left(\frac{i-1}{l} + s\right)}{\prod_{i=1}^{l}\Gamma\left(\frac{i}{l} + s\right)}.$$

The average BER in (13) is now written in terms of the Meijer's G-function [[9], (9.3)], [11]

$$\bar{P}_E = A(k,l) G_{2l,k+l}^{k,2l}\left[\left(\frac{l\beta}{k^{k/l} a\bar{\gamma}}\right)^l \,\middle|\, \begin{matrix}\alpha_1,\ldots,\alpha_{2l} \\ \beta_1,\ldots,\beta_{k+l}\end{matrix}\right] \quad (14)$$

where

$$\alpha_n = \begin{cases} 1 - \frac{n+b-1}{l} & n = 1,2,\ldots,l \\ 1 - \frac{n-l-1}{l} & n = l+1, l+2, \ldots, 2l \end{cases}$$

and

$$\beta_n = \begin{cases} \frac{n+m-1}{k} & n = 1,2,\ldots,k \\ 1 - \frac{n-k}{l} & n = k+1, k+2, \ldots, k+l \end{cases}.$$

In the special case when $\nu = 1$ ($l = k = 1$), (14) reduces to ([9], [12], [11])

$$\begin{aligned}\bar{P}_E &= \frac{1}{2\Gamma(b)\Gamma(m)} G_{2,2}^{1,2}\left[\frac{m}{a\bar{\gamma}} \,\middle|\, \begin{matrix}1-b,1 \\ m,0\end{matrix}\right] \\ &= \frac{\Gamma(m+b)}{2\Gamma(b)\Gamma(m+1)}\left(\frac{m}{m+a\bar{\gamma}}\right)^m \left(\frac{a\bar{\gamma}}{m+a\bar{\gamma}}\right)^b \\ &\quad \times {}_2F_1\left(1, m+b; m+1; \frac{m}{m+a\bar{\gamma}}\right) \quad (15)\end{aligned}$$

where ${}_2F_1(a,b;c;x)$ is the Gaussian hypergeometric function [9], which is the more familiar expression for the average BER in a flat-fading Nakagami channel (see [[1], (8.105)], for example). Note that in the special case of Rayleigh fading $\nu = m = 1$, it is straight forward to show that (15) reduces to

$$\bar{P}_E = \frac{1}{2}\left[1 - \left(\frac{a\bar{\gamma}}{1+a\bar{\gamma}}\right)^b\right], \quad (16)$$

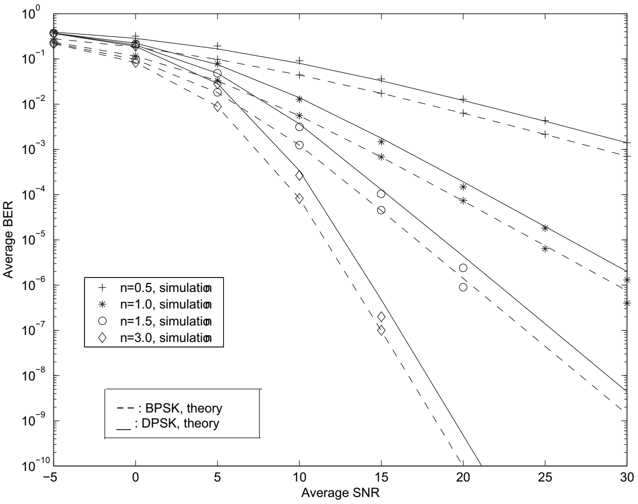

Fig. 2. Average BER versus $\bar{\gamma}$ for coherent and differentially coherent PSK in a generalized gamma fading channel with $m = 2$, for several values of ν.

which agrees with the four results in [[13], pp. 774] as expected. In Figs. 1 and 2, we have plotted the average BER for coherent and differentially coherent binary PSK versus average SNR, $\bar{\gamma}$, for several values of ν and for $m = 1$ and $m = 2$, respectively. We observe that increasing ν decreases the severity of fading. Thus by varying both ν and m, more variety of fading conditions can be modeled than are possible with the Nakagami-m or Weibull fading models.

IV. Conclusion

In this paper we have derived a closed-form expression for the average bit error rate of a digital radio system operating in a flat-fading channel in which the fading envelope is characterized by the three-parameter generalized gamma distribution. The result is expressed in terms of Meijer's G-function, which can be evaluated with the aid of commercially available software like Mathematica and Maple. The results will be useful in predicting the performance of digital mobile radio systems operating in a wide range of fading environments.

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References

[1] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels - A Unified Approach to Performance Analysis*. Wiley-Interscience, 2000.
[2] A. J. Coulson, A. G. Williamson, and R. G. Vaughan, "Improved fading distribution for mobile radio," *IEE Proc. F-Commun.*, vol. 145, pp. 197–202, June 1998.
[3] J. Griffiths and J. P. McGeehan, "Interrelationship between some statistical distributions used in radio-wave propagation," *IEE Proc. F-Commun.*, vol. 129, pp. 411–417, Dec. 1982.
[4] M. D. Yacoub, "The $\alpha - \mu$ distribution: a general fading distribution," in *Proc. IEEE PIMRC*, Sept. 2002, pp. 629–633.
[5] P. M. Shankar, "Ultrasonic tissue characterization using a generalized Nakagami model," *IEEE Trans. Ultrasonics, Ferroelect. Freq. Contr.*, vol. 48, pp. 1716–1720, Nov. 2001.
[6] J. Cheng, C. Tellambura, and N. C. Beaulieu, "Performance of digital linear modulations on Weibull slow-fading channels," *IEEE Trans. Commun.*, vol. 52, pp. 1265–1268, Aug. 2004.
[7] E. W. Stacy, "A generalization of the gamma distribution," *Ann. Math. Stat.*, vol. 33, pp. 1187–1192, Sept. 1962.
[8] J. Cheng and T. Berger, "Performance analysis for MRC and postdetection EGC over generalized gamma fading channels," in *Proc. IEEE WCNC*, Mar. 2003, pp. 120–125.
[9] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*. Academic Press, 1980.
[10] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*. New York: McGraw-Hill, 1954.
[11] A. M. Mathai and R. K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*. New York: John Wiley and Sons, 1978.
[12] ——, *Lecture Notes in Mathematics*. New York: Springer-Verlag, 1973.
[13] J. G. Proakis, *Digital Communications*, 3rd ed. New York: McGraw-Hill, 1995.