Probability Distributions of SIR in a Rayleigh Fading Channel

Murad Ridwan
School of Electrical and Computer Engineering
Addis Ababa Institute of Technology
Addis Ababa University, Ethiopia
Email: murad.ridwan@aait.edu.et

Mohammad Abdo
School of Electrical and Computer Engineering
Addis Ababa Institute of Technology
Addis Ababa University, Ethiopia
Email: abdotuko@yahoo.com

Eduard Jorswieck
Communication Laboratory
Dresden University of Technology, TUD
Dresden, Germany
Email: jorswieck@ifn.et.tu-dresden.de

***Abstract*—In fading channel when the interfere signal is a random variable, the resulting signal-to-interference-plus-noise ratio (SINR) or signal-to-interference ratio (SIR) is also random and its distribution has to be known for a correct description of the receiver. In this paper, we derive in closed form expression the probability distributions of SIR when more than one Rayleigh distributed interfere signals are present. SINR or SIR involves the sum and ratio of random variables. When the interfere signals are more than one, it becomes very difficult, if not impossible, to obtain a tractable closed form expression of the probability density function (PDF) or cumulative density function (CDF) of SINR or SIR. The SIR is studied in noise limited systems, i.e. where the interfere signals dominate over noise. If the noise is to be accounted for, which is typically AWGN, the SINR may be studied. In our case, adding AWGN has no effect on the overall probability distribution except for a change in its moments. So to keep the mathematical complexity at bay, only the SIR is considered.**

***Index Terms*– SIR, SINR, PDF, CDF, Rayleigh fading, Ratio of random variables.**

I. Introduction

The distribution of the ratio of two random variables, X/Y, has been extensively investigated by many authors especially when X and Y are independent and identically distributed (i.i.d.) random variables [1]-[10]. It is of interest in many areas of science and engineering such as in fading wireless communication, Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology and inventory ratios in economics [2].

In wireless communication, the Rayleigh fading is typically encountered in land mobile channels in densely populated urban areas where there are many obstacles which make line-of-sight paths rare.

In a Rayleigh fading channel, the signal power is distributed exponentially with PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

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where λ is the rate parameter and is related to the average signal power Ω by

$$\Omega = \mathbf{E}(X) = \frac{1}{\lambda} \tag{2}$$

where $\mathbf{E}(\cdot)$ is the expectation operator.

The SIR is defined as

$$SIR = \frac{X_0}{X_1 + X_2 + \ldots + X_N} \tag{3}$$

where X_0 is power of the desired signal while X_i $(i = 1, \ldots, N)$ are the powers of the interferer signals. In a Rayleigh fading channel, the X_i's follow the exponential distribution. The problem is to derive the probability distributions of SIR.

The paper is arranged as follows. In Section II, a closed form expression for the sum of N exponential random variables is derived. Section III presents the probability distributions of SIR in Rayleigh fading channel.

II. Sum of N Exponential Random Variables

The SIR (3) involves ratios of random variables. The denominator contains sum of N exponentially distributed random variables. In this section we derive the PDF of the sum of N i.i.d. exponential random variables.

Consider a random variable Z defined as the sum of two random variables X_1 and X_2,

$$Z = X_1 + X_2 \tag{4}$$

From elementary probability theory, it is to be recalled that the CDF of Z is

$$F_z(z) = Pr(X_1 + X_2 \leq z) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-x_1} f_{X_1X_2}(x_1, x_2) dx_2 \right] dx_1$$

Then, the PDF of Z becomes

$$f_z(z) = \frac{d}{dz} F_z(z) = \int_{-\infty}^{\infty} f_{X_1X_2}(x_1, z - x_1) dx_1 \tag{5}$$

If X_1 and X_2 are independent, (5) reduces to

$$f_z(z) = \frac{d}{dz}F_z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1)f_{X_2}(z-x_1)dx_1 \tag{6}$$

which is the convolution integral.

Similarly consider the quotient Z of two random variables, i.e.,

$$Z = X_1/X_2 \tag{7}$$

Set $W = X_2$. The joint PDF of Z and W is given by

$$f_{ZW}(z,w) = f_{X_1X_2}(x_1,x_2)|J(z,w)| \tag{8}$$

where $J(z,w)$ is the Jacobian of the transformation

$$J(z,w) = \begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial z} & \frac{\partial x_2}{\partial w} \end{vmatrix}$$

The PDF of Z is the marginal PDF

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)dw \tag{9}$$

Thus using (8) and (9), the PDF of (7) for independent X_1 and X_2 becomes

$$f_Z(z) = \int_{-\infty}^{\infty} |w| f_{X_1}(zw)f_{X_2}(w)dw \tag{10}$$

To find the expression for the sum of N independent exponential random variables, first the expressions for $N=2$ and $N=3$ are obtained. Then the results are extended to the general case using mathematical induction.

Theorem 1: The PDF of $Z = X_1 + X_2$, where X_1 and X_2 are independent exponential random variables is

$$f_Z(z) = \begin{cases} \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 z} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 z}, & z \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Proof: Using (6),

$$\begin{aligned} f_Z(z) &= \int_0^z f_{X_1}(x_1)f_{X_2}(z-x_1)\,dx_1 \\ &= \lambda_1\lambda_2 e^{-\lambda_2 z}\int_0^z e^{x_1(\lambda_2-\lambda_1)}\,dx_1 \\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[e^{-\lambda_1 z} - e^{-\lambda_2 z}\right] \\ &= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}e^{-\lambda_1 z} + \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}e^{-\lambda_2 z} \end{aligned}$$

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Corollary 1: The PDF of $Z = X_1+X_2+X_3$, where X_1, X_2 and X_3 are independent exponential random variables is

$$f_Z(z) = \begin{cases} \frac{\lambda_1\lambda_2\lambda_3 e^{-\lambda_1 z}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{\lambda_1\lambda_2\lambda_3 e^{-\lambda_2 z}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} \\ +\frac{\lambda_1\lambda_2\lambda_3 e^{-\lambda_3 z}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}, & z \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Proof: Set $Y = X_1 + X_2$, $Z = Y + X_3$. The PDF of Y is (11). Then

$$\begin{aligned} f_Z(z) &= \int_0^z f_{X_3}(x_3)f_Y(z-x_3)\,dx_3 \\ &= \int_0^z \lambda_3 e^{-\lambda_3 x_3}\left[\frac{\lambda_1\lambda_2 e^{-\lambda_1(z-x_3)}}{\lambda_2-\lambda_1} + \frac{\lambda_1\lambda_2 e^{-\lambda_2(z-x_3)}}{\lambda_1-\lambda_2}\right]\,dx_3 \\ &= \frac{\lambda_1\lambda_2\lambda_3}{\lambda_2-\lambda_1}e^{-\lambda_1 z}\int_0^z e^{(\lambda_1-\lambda_3)x_3}dx_3 \\ &\quad + \frac{\lambda_1\lambda_2\lambda_3}{\lambda_1-\lambda_2}e^{-\lambda_2 z}\int_0^z e^{(\lambda_2-\lambda_3)x_3}dx_3 \\ &= \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_1-\lambda_3)}e^{-\lambda_1 z}\left[e^{(\lambda_1-\lambda_3)z}-1\right] \\ &\quad + \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}e^{-\lambda_2 z}\left[e^{(\lambda_2-\lambda_3)z}-1\right] \\ &= \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}e^{-\lambda_1 z} + \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}e^{-\lambda_2 z} \\ &\quad + \frac{\lambda_1\lambda_2\lambda_3}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}e^{-\lambda_3 z} \end{aligned}$$

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Inferring from the patters emerging from (11) and (12), one can state the corollary for the sum of N independent exponential random variables.

Corollary 2: The PDF of $Z = X_1 + X_2 + \ldots + X_N$, where X_i $(i = 1, \ldots, N)$ are independent exponential random variables is

$$f_Z(z) = \begin{cases} \prod_{k=1}^{N}\lambda_k \sum_{i=1}^{N} \frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\ j\neq i}}^{N}(\lambda_j-\lambda_i)}, & z \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Proof: The corollary will be proved using mathematical induction.
The equation is true for $i=2$, viz.,

$$\begin{aligned} f_Z(z) &= \prod_{k=1}^{2}\lambda_k \sum_{i=1}^{2} \frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\ j\neq i}}^{2}(\lambda_j-\lambda_i)} \\ &= \lambda_1\lambda_2\left[\frac{1}{\lambda_2-\lambda_1}e^{-\lambda_1 z} + \frac{1}{\lambda_1-\lambda_2}e^{-\lambda_2 z}\right] \end{aligned}$$

Assume (13) is true for $i = n$. One should show that it is true for $i = n+1$.
Let $Z = Y + X_{n+1}$ where Y's PDF is (13). Then,

$$\begin{aligned} f_Z(z) &= \int_0^z f_{X_{n+1}}(x_{n+1})f_Y(z-x_{n+1})\,dx_{n+1} \\ &= \int_0^z \lambda_{n+1}e^{-\lambda_{n+1}x_{n+1}}\left[\prod_{k=1}^{n}\lambda_k \sum_{i=1}^{n}\frac{e^{-\lambda_i(z-x_{n+1})}}{\prod_{\substack{j=1\\ j\neq i}}^{n}(\lambda_j-\lambda_i)}\right]dx_{n+1} \end{aligned}$$

$$= \left(\lambda_{n+1}\prod_{k=1}^{n}\lambda_k\right)\sum_{i=1}^{n}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)} \times \int_0^z e^{-\lambda_{n+1}x_{n+1}}e^{-\lambda_i(z-x_{n+1})}dx_{n+1}$$

$$= \left(\prod_{k=1}^{n+1}\lambda_k\right)\sum_{i=1}^{n}\frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)} \times \int_0^z e^{(\lambda_i-\lambda_{n+1})x_{n+1}}dx_{n+1}$$

$$= \left(\prod_{k=1}^{n+1}\lambda_k\right)\sum_{i=1}^{n}\frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)}\left[\frac{e^{(\lambda_i-\lambda_{n+1})z}-1}{\lambda_i-\lambda_{n+1}}\right]$$

$$= \left(\prod_{k=1}^{n+1}\lambda_k\right)\sum_{i=1}^{n}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)}\left[\frac{e^{-\lambda_{n+1}z}-e^{-\lambda_i z}}{\lambda_i-\lambda_{n+1}}\right]$$

$$= \left(\prod_{k=1}^{n+1}\lambda_k\right)\left[\sum_{i=1}^{n}\frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)(\lambda_{n+1}-\lambda_i)} + \sum_{i=1}^{n}\frac{e^{-\lambda_{n+1}z}}{\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)(\lambda_i-\lambda_{n+1})}\right]$$

Consider the first and the second summations inside the bracket of the last expression. The expression in the denominator of the first summation can be simplified as

$$\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)(\lambda_{n+1}-\lambda_i) = (\lambda_1-\lambda_i)\underbrace{\ldots}_{j\neq i}(\lambda_n-\lambda_i)(\lambda_{n+1}-\lambda_i) = \prod_{\substack{j=1\\j\neq i}}^{n+1}(\lambda_j-\lambda_i)$$

The second summation can further be simplified as

$$\sum_{i=1}^{n}\frac{e^{-\lambda_{n+1}z}}{\prod_{\substack{j=1\\j\neq i}}^{n}(\lambda_j-\lambda_i)(\lambda_i-\lambda_{n+1})}$$
$$= e^{-\lambda_{n+1}z}\left\{\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\cdots(\lambda_n-\lambda_1)(\lambda_1-\lambda_{n+1})} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\cdots(\lambda_n-\lambda_2)(\lambda_2-\lambda_{n+1})}+\ldots \ldots + \frac{1}{(\lambda_n-\lambda_1)(\lambda_n-\lambda_1)\cdots(\lambda_n-\lambda_{n-1})(\lambda_n-\lambda_{n+1})}\right\}$$
$$= e^{-\lambda_{n+1}z}\left\{\frac{1}{(\lambda_{n+1}-\lambda_1)(\lambda_{n+1}-\lambda_2)\cdots(\lambda_{n+1}-\lambda_n)}\right\}$$
$$= e^{-\lambda_{n+1}z}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{n+1}(\lambda_j-\lambda_i)}$$

Thus, putting the results

$$f_Z(z) = \left(\prod_{k=1}^{n+1}\lambda_k\right)\left[\sum_{i=1}^{n}\frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\j\neq i}}^{n+1}(\lambda_j-\lambda_i)} + \frac{e^{-\lambda_{n+1}z}}{\prod_{\substack{j=1\\j\neq i}}^{n+1}(\lambda_j-\lambda_i)}\right]$$

$$= \left(\prod_{k=1}^{n+1}\lambda_k\right)\sum_{i=1}^{n+1}\frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\j\neq i}}^{n+1}(\lambda_j-\lambda_i)}$$

which proves the corollary.

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III. Probability Distributions of SIR

Theorem 2: The PDF of SIR (3) is given by

$$f_Z(z) = \begin{cases} \prod_{k=0}^{N}\lambda_k\sum_{i=1}^{N}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{N}(\lambda_j-\lambda_i)(\lambda_i+\lambda_0 z)^2}, & z\geq 0;\\ 0, & \text{otherwise.}\end{cases} \quad (14)$$

Proof: Let

$$W = X_1 + X_2 + \ldots + X_N$$

W's PDF is given by (13). Then the PDF of $SIR \equiv Z = X_0/W$ using (10) is

$$f_Z(z) = \int_{-\infty}^{\infty}|w|f_{X_0}(zw)f_Y(w)\,dw$$
$$= \int_0^{\infty} w f_{X_0}(zw)f_Y(w)\,dw$$
$$= \int_0^{\infty} w\lambda_0 e^{-\lambda_0 zw}\left[\prod_{k=1}^{N}\lambda_k\sum_{i=1}^{N}\frac{e^{-\lambda_i z}}{\prod_{\substack{j=1\\j\neq i}}^{N}(\lambda_j-\lambda_i)}\right]dw$$
$$= \lambda_0\prod_{k=1}^{N}\lambda_k\sum_{i=1}^{N}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{N}(\lambda_j-\lambda_i)}\int_0^{\infty} we^{-(\lambda_0 z+\lambda_i)w}\,dw$$
$$= \prod_{k=0}^{N}\lambda_k\sum_{i=1}^{N}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{N}(\lambda_j-\lambda_i)}\frac{1}{(\lambda_i+\lambda_0 z)^2}$$

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In terms of the average signal power $\Omega_i = 1/\lambda_i$ ($i = 0,\ldots,N$), the PDF of SIR (14) can be expressed as,

$$f_Z(z) = \begin{cases} \prod_{k=0}^{N}\frac{1}{\Omega_k}\sum_{i=1}^{N}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{N}\left(\frac{1}{\Omega_j}-\frac{1}{\Omega_i}\right)\left(\frac{1}{\Omega_i}+\frac{1}{\Omega_0}z\right)^2}, & z\geq 0;\\ 0, & \text{otherwise.}\end{cases} \quad (15)$$

Theorem 3: The CDF of SIR is

$$F_Z(z) = \begin{cases} \prod_{k=0}^{N}\frac{1}{\Omega_k}\sum_{i=1}^{N}\frac{1}{\prod_{\substack{j=1\\j\neq i}}^{N}\left(\frac{1}{\Omega_j}-\frac{1}{\Omega_i}\right)}\frac{z}{\frac{1}{\Omega_i^2}+\frac{1}{\Omega_0\Omega_i}z}, & z\geq 0;\\ 0, & \text{otherwise.}\end{cases} \quad (16)$$

Proof: The CDF is

$$
\begin{aligned}
F_Z(z) &= Pr(Z \le z) = \int_{-\infty}^{z} f_Z(z)\, dz \\
&= \prod_{k=0}^{N} \frac{1}{\Omega_k} \sum_{i=1}^{N} \frac{1}{\prod\limits_{\substack{j=1\\ j\neq i}}^{N} \left(\frac{1}{\Omega_j} - \frac{1}{\Omega_i}\right)} \int_0^z \frac{dz}{\left(\frac{1}{\Omega_i} + \frac{1}{\Omega_0} z\right)^2} \\
&= \prod_{k=0}^{N} \frac{1}{\Omega_k} \sum_{i=1}^{N} \frac{1}{\prod\limits_{\substack{j=1\\ j\neq i}}^{N} \left(\frac{1}{\Omega_j} - \frac{1}{\Omega_i}\right)} \frac{z}{\frac{1}{\Omega_i^2} + \frac{1}{\Omega_0 \Omega_i} z}
\end{aligned}
$$

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An outage occurs if the instantaneous SIR falls below some acceptable $SIR_{\min}$. Thus the *outage probability* can be given as

$$
\begin{aligned}
p_{\text{out}} &= F_Z(SIR_{\min}) \\
&= \prod_{k=0}^{N} \frac{1}{\Omega_k} \sum_{i=1}^{N} \frac{1}{\prod\limits_{\substack{j=1\\ j\neq i}}^{N} \left(\frac{1}{\Omega_j} - \frac{1}{\Omega_i}\right)} \frac{SIR_{\min}}{\frac{1}{\Omega_i^2} + \frac{1}{\Omega_0 \Omega_i} SIR_{\min}}
\end{aligned} \tag{17}
$$

IV. Conclusion

In fading channels the $SINR$ and SIR are random variables and their distributions have to be known for a correct description of the receiver. In this paper, the distributions of SIR in Rayleigh fading channel have been derived in a closed form expression. Results for the probability density function, cumulative density function and outage probabilities have been shown. Other fading channels can be similarly studied.

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