Capacity Bounds for kth Best Path Selection over Generalized Fading Channels

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***Abstract*—Exact ergodic capacity calculation for fading wireless channels typically involves time-consuming numerical evaluation of infinite integrals. In this paper, lower and upper bounds on ergodic capacity for kth best path are presented. These bounds have simple analytic expressions which allow their fast evaluation. Numerical results show that the newly proposed bounds closely approximate the exact ergodic capacity for a large variety of system configurations.**

***Index Terms*—Ergodic capacity, convexity, order statistics, Jensen's inequality, generalized Gamma distribution.**

I. Introduction

Diversity combining techniques can combat performance degradation of wireless communication systems due to multi-path fading. Selection combining (SC) is one of the low complexity diversity combining schemes where the diversity path with highest signal-to-noise ratio (SNR) is selected for signal transmission or reception [1], [2]. In practical implementation environment, however, the kth best path, instead of the best path, might be selected. For example in cognitive radio environment with underlay implementation, secondary transmitter may transmit only when the amount of interference introduced to the primary receiver is below a certain level. When antenna selection is employed at the secondary transmitter, it may not select the best path due to excessive interference to the primary receiver [3]. Selection of kth best path may also occur in relay networks when the best path is not available due to scheduling or load-balancing condition [4], [5]. Furthermore, in generalized selection multiuser diversity (GSMuD) scheme, first k best users are selected based on their SNRs for simultaneous channel access [6]. Therefore, it is of great practical interest to study the impact of selecting kth best path on the system performance.

Ergodic capacity is one of the most popular performance measures for wireless communication systems [1], [2]. Unfortunately, ergodic capacity, being expectation of instantaneous capacity over the distribution of received SNR, is often difficult to be expressed in simple closed forms for generalized fading environment especially with diversity combining. The recent attempt based on moment generating function (MGF) results in single integration of Meijer's G and Fox's H functions but typically does not lead to closed form expressions of the ergodic capacity [7], [8]. Therefore, to avoid time consuming numerical integration and subsequent analysis, researchers typically resort to its tight bounds [9]–[12]. The bounds in [9] and [10] are applicable for dual hop relay and MIMO channels under Rayleigh and Nakagami-m fading respectively and cannot be directly extended to analyze selection diversity schemes. Authors in [11] presented bounds on ergodic capacity for dual branch SC in correlated Nakagami-m environment by truncating an infinite sum representation of capacity. However, bounds in [11] cannot be generalized to kth best path selection in generalized fading environment easily. On the other hand, [12] presents simple bounds on the ergodic capacity but these bounds are applicable for only the best path in Rayleigh fading environment. To the best of authors' knowledge, no bounds are available for the ergodic capacity of the kth best path selection over general fading channels. Therefore, in this paper we provide bounds on the ergodic capacity for the kth best path in a generalized Gamma fading environment. Our developed bounds have simple analytic expressions which allow their fast evaluation. These bounds also apply to the best path selection in Rayleigh fading environment but turn out to be slightly tighter than those proposed in [12].

The rest of the paper is organized as follows. First of all system model is described in section II. Lower and upper bounds on the ergodic capacity are presented in section III. Different applications highlighting usefulness of the bounds are illustrated in section IV which are followed by some concluding remarks.

II. System Model

Consider a system where transmitter chooses kth best path out of N independent and identically distributed (IID) paths for transmitting information to the receiver. Let h_i and X_i, respectively, denote the channel gain and the squared Euclidean norm of channel gain of the ith path, $i = 1, 2, \cdots, N$. The received signal can be represented as

$$y = h_{(N-k+1)} s + w, \tag{1}$$

where s is the complex valued symbol transmitted with (normalized) power ρ, w is zero mean complex white Gaussian noise with variance $\mathbb{E}[ww^H] = 1$ and $h_{(N-k+1)}$ is the flat fading channel gain of the kth best channel for $k = 1, 2, \cdots, N$. Therefore, the instantaneous SNR at the receiver side is $\rho X_{(N-k+1)}$ where $X_{(N-k+1)} = |h_{(N-k+1)}|^2$ and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(N)}$.

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It is well known that the probability density function (PDF) of $X_{(N-k+1)}$ can be written in terms of PDF, $f(x)$, and cumulative distribution function (CDF), $F(x)$, of X_i as [13]

$$f_{X_{(N-k+1)}}(x) = k \binom{N}{k} f(x) F(x)^{N-k} (1 - F(x))^{k-1}. \quad (2)$$

The ergodic capacity of the chosen path, $\overline{C_k}(\rho)$, can then be calculated as

$$\overline{C_k}(\rho) = \int_0^\infty f_{X_{(N-k+1)}}(x) \log_2(1 + \rho x)\, dx. \quad (3)$$

In general, it is difficult to obtain closed form expressions for $\overline{C_k}(\rho)$ over generalized fading channels. Simple yet tight bounds on ergodic capacity can help the designer to quickly predict system performance.

For a variety of fading channels, X_i can be modeled as a random variable (RV) with generalized Gamma distribution. The PDF, $f(x)$, and CDF, $F(x)$, of generalized Gamma distributed random variable are given by [14]

$$f(x) = \frac{\lambda x^{\lambda c - 1}}{\beta^c \Gamma(c)} e^{-\frac{x^\lambda}{\beta}} u(x), \quad (4)$$

and

$$F(x) = \left(1 - \frac{\Gamma(c, x^\lambda/\beta)}{\Gamma(c)}\right) u(x), \quad (5)$$

respectively, where the parameters λ, c and β are all positive reals, $\Gamma(c) = \int_0^\infty t^{c-1} e^{-t} dt$ is the Euler's Gamma function, $\Gamma(c, x) = \int_x^\infty t^{c-1} e^{-t} dt$ is the incomplete Gamma function, and $u(x)$ is the unit step function. The generalized Gamma distribution specializes to channel gain distribution for Rayleigh, Weibull, and Nakagami-m fading channels for different values of parameters c and λ [14].

III. Bounds on Ergodic Capacity

The following theorem defines bounds on ergodic capacity of the kth best channel when the statistics of X_i have certain properties.

Theorem 1. *If the survival function of X_i, $\overline{F}(x) = 1 - F(x)$, is log-concave and $1/F\left(\frac{2^x - 1}{\rho}\right)$ is convex, then the ergodic capacity for the kth best channel amongst N IID channels can be bounded as*

$$l_k(\rho) \le \overline{C_k}(\rho) \le u_k(\rho), \quad (6)$$

where $l_k(\rho) = \log_2\left(1 + \rho F^{-1}\left(1 - \frac{k}{N}\right)\right)$, $u_k(\rho) = \log_2\left(1 + \rho F^{-1}\left(1 - e^{H(k-1) - H(N)}\right)\right)$ and $H(k)$ is the Harmonic number defined as

$$H(k) = \begin{cases} \sum_{j=1}^{k} \frac{1}{j} & \text{if } k \in \mathbb{Z}_+ \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

where $\mathbb{Z}_+$ denotes set of positive integers.

Proof is given in Appendix A.

The following proposition states the condition on parameters of generalized Gamma distribution for which the bounds in (6) hold.

Proposition 1. *For channels with generalized Gamma distributed SNR, the bounds on ergodic capacity in* (6) *hold if $\lambda \ge 1$ and $c \ge 1/\lambda$.*

Proof is given in Appendix B.

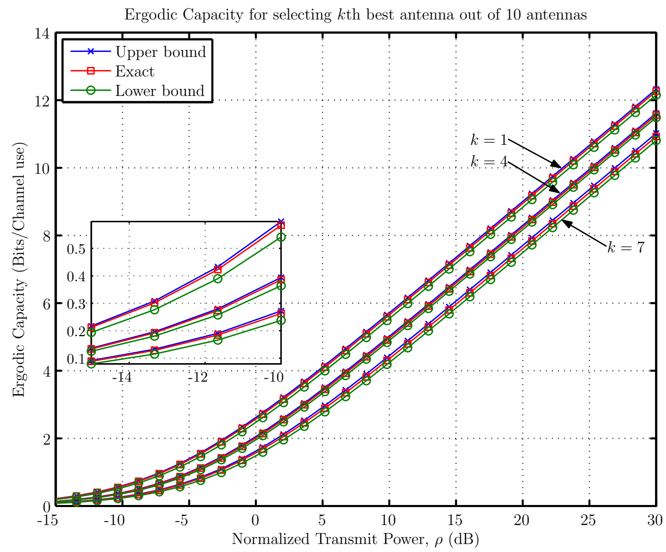

Fig. 1. Ergodic capacity and bounds versus normalized transmit power, ρ, for $k = 1, 4, 7$.

Remark: For generalized Gamma distribution $F^{-1}(p)$ can be calculated using MATLAB's in-built function `gaminv()`.

IV. Examples and Applications

A. Selection Combining over Weibull Fading Channels

Weibull distribution is used to characterize both channel amplitude and SNR for the mobile radio systems operating in 800/900 MHz band [1]. The PDF of the received SNR for Weibull fading with shape parameter η and scale parameter α can be obtained by setting $c = 1$, $\lambda = \eta$ and $\beta = \alpha^\eta$ in (4) [1], [14]. The inverse CDF, $F^{-1}(p)$, of Weibull RV can be calculated as $F^{-1}(p) = \alpha\left(-\ln(1-p)\right)^{1/\eta}$. If the Weibull fading parameter, $\eta \ge 1$, then the bounds in (6) hold.

Fig. 1 shows the ergodic capacity for the kth best out of 10 IID Weibull fading channels along with its upper and lower bounds for $\eta = 2$ and $\alpha = 3$. Here the exact ergodic capacity is calculated by numerical integration of (3) which takes a significant amount of time as compared with the evaluation of upper or lower bound. It is clear from the figure that bounds are tight for a wide range of system parameters.

B. Transmit Antenna Selection with MRC

Consider a Rayleigh fading channel in which transmitter selects the kth best transmit antenna based on combined SNR when maximal ratio combining (MRC) diversity scheme is employed at the receiver. Note that the combined SNR is a chi-squared RV with $2n$ degrees of freedom where n is the number of MRC branches. Chi-squared distribution can be represented as generalized Gamma distribution with $\lambda = 1$ and $c = n$. Since $n \ge 1$, the bounds in (6) hold. The quantile function, $F^{-1}(p)$, of chi-square RV can easily be evaluated using `chi2inv()` in MATLAB.

Fig. 2 displays the numerically computed ergodic capacity along with the lower and upper bounds given in (6) for different values of k. For the best channel selection ($k = 1$), the upper bound of [12] is also plotted in the figure. It is easy to observe that the upper bound of [12] is slightly looser

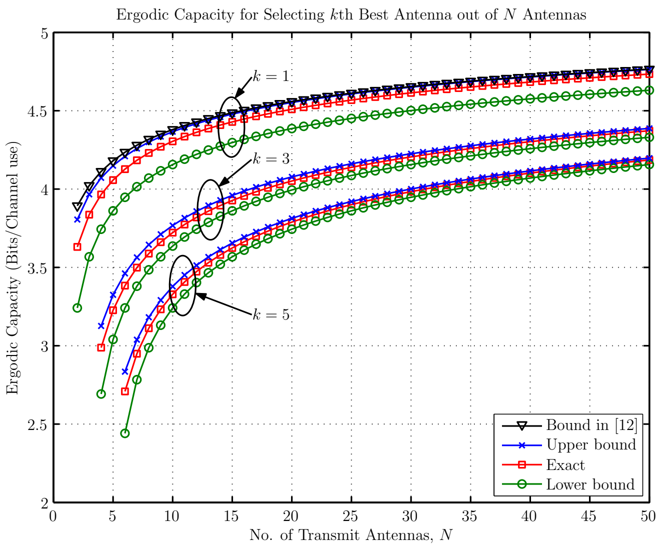

Fig. 2. Ergodic capacity and bounds versus N for $k = 1, 3, 5$, $\beta = 1$ and $n = 3$ at $\rho = 5$ dB.

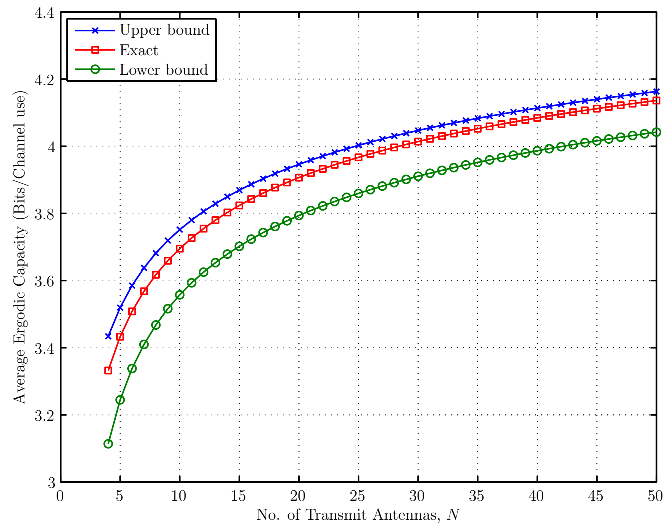

Fig. 3. Ergodic capacity and bounds versus N for Nakagami-m fading, $m = 3$, $\Omega = 2$ and $\rho = 5$ dB.

than our proposed upper bound, $u_k(\rho)$, though the difference between the upper bound in [12] and the proposed bound, $u_k(\rho)$, is reduced for large number of transmit antennas, N. The lower bound proposed in [12] is exactly the same as the proposed $l_k(\rho)$ for $k = 1$. Furthermore, it is interesting to observe that our proposed bounds get tighter with increasing k.

C. Non Ideal Channel Selection

Consider a case where a selection diversity scheme is used over Nakagami-m fading environment. The PDF of RV X_i in Nakagam-m fading has a Gamma PDF with mean Ω which can be modeled as generalized Gamma distribution in (4) with $\lambda = 1$, $c = m$ and $\beta = \Omega/m$ [1], [14]. For $m \geq 1$, bounds in (6) hold.

Furthermore, consider a case in which transmitter, due to imperfect channel estimation or de-correlation of channel response in TDD systems or due to feedback delay, occasionally selects kth best antenna instead of the best one. Let p_k be the probability of selecting kth best antenna, the average ergodic capacity would then be

$$\overline{C}(\rho) = \sum_{k=1}^{N} p_k \overline{C_k}(\rho). \tag{8}$$

Using (6), average ergodic capacity of such diversity schemes can be bounded as

$$\sum_{k=1}^{N} p_k l_k(\rho) \leq \overline{C}(\rho) \leq \sum_{k=1}^{N} p_k u_k(\rho). \tag{9}$$

Fig. 3 shows the average ergodic capacity, $\overline{C}(\rho)$ versus number of antennas for Nakagami-m fading channel. Here the probability of selecting the kth antennas are $p_1 = 0.85$, $p_2 = 0.1$, $p_3 = 0.05$ and $p_i = 0$ for $i \geq 4$. We can observe from this figure that the upper bound provides a good approximation to the actual average ergodic capacity.

V. Conclusion

Simple yet tight bounds on the ergodic capacity for the kth best channel were presented in this letter. Numerical results for variety of application scenarios were presented to demonstrate the tightness of the bounds to the actual ergodic capacity.

Appendix A
Proof of Theorem 1

The following lemma is used to derive bounds on the ergodic capacity.

Lemma 1. *Let X and Y be two RVs with CDF $F_X(x)$ and $F_Y(y)$, respectively. If $F_Y^{-1}(F_X(x))$ is convex then*

$$F_X(\mathbb{E}[X_{(N-k+1)}]) \leq F_Y(\mathbb{E}[Y_{(N-k+1)}]), \tag{10}$$

where $X_{(N-k+1)}$ (or $Y_{(N-k+1)}$) denotes kth maximum amongst N IID random variables with CDF $F_X(x)$ (or $F_Y(y)$). Inequalities are reversed if $F_Y^{-1}(F_X(x))$ is a concave function.

Proof: Inequality (10) can be proved by considering new variate $Z = F_Y^{-1}(F_X(X_{(N-k+1)}))$. It is easy to observe that

$$f_Z(z) = f_Y(z) \frac{f_{X_{(N-k+1)}}(F_X^{-1}(F_Y(z)))}{f_X(F_X^{-1}(F_Y(z)))}. \tag{11}$$

By using (2) one can easily observe that $Z = Y_{(N-k+1)}$. Result of (10) immediately follows from the following Jensen's inequality

$$F_Y^{-1}(F_X(\mathbb{E}[X_{(N-k+1)}])) \leq \mathbb{E}[F_Y^{-1}(F_X(X_{(N-k+1)}))]. \tag{12}$$

Inequality sign in (12) is reversed if $F_Y^{-1}(F_X(x))$ is concave. ■

For proving the capacity bounds, we follow a similar approach of [15, Ch. 4]. First of all, the upper bound is proved by considering $F_Y(y) = 1 - e^{-y}$ for $0 \leq y < \infty$. If $F_Y^{-1}(F(x)) = -\log(1 - F(x))$ is convex then by (10), we have

$$\mathbb{E}[X_{(N-k+1)}] \leq F^{-1}\left(1 - e^{-\mathbb{E}[Y_{(N-k+1)}]}\right). \tag{13}$$

Using the fact that $\mathbb{E}[Y_{(N-k+1)}] = \sum_{j=k}^{N} 1/j = H(N) - H(k-1)$, upper bound in (6) can easily be derived. Since $\log_2(x)$ is a monotonically increasing concave function, we can write

$$\overline{C_k}(\rho) \leq \log_2(1+\rho\mathbb{E}[X_{(N-k+1)}]) \leq u_k(\rho). \quad (14)$$

For the lower bound, consider $F_Y(y) = -1/y$ for $-\infty < y \leq -1$. Let us define a new variate $R = \log_2(1+\rho X)$. Now, if $F_Y^{-1}(F_R(r)) = -1/F_R(r)$ is concave then

$$F_R\left(\mathbb{E}[R_{(N-k+1)}]\right) \geq F_Y\left(\mathbb{E}[Y_{(N-k+1)}]\right). \quad (15)$$

Since $F_Y\left[\mathbb{E}[Y_{(N-k+1)}]\right] = 1 - k/N$ and $F_R(r) = F(\frac{2^r-1}{\rho})$, (15) immediately results in

$$\overline{C_k}(\rho) = \mathbb{E}[R_{(N-k+1)}] \geq \log_2\left(1+\rho F^{-1}\left(1-k/N\right)\right). \quad (16)$$

□

Appendix B
Proof of Proposition 1

For generalized Gamma distribution

$$\frac{d^2}{dx^2}\left(\log f(x)\right) = -\frac{\lambda c - 1}{x^2} - \frac{\lambda(\lambda-1)x^{\lambda-2}}{\beta}. \quad (17)$$

Since for $\lambda \geq 1$ and $c \geq 1/\lambda$, $\frac{d^2}{dx^2}\left(\log f(x)\right) \leq 0$. Therefore the survival function, $\overline{F}(x)$, is log-concave [16]. Hence upper bound on capacity holds. Furthermore, log-concavity of $f(x)$ implies log-concavity of $F(x)$ [16]. Therefore we have [17]

$$f(x)^2 - f'(x)F(x) \geq 0. \quad (18)$$

For the lower bound, one has to show that $1/F\left(\frac{2^x-1}{\rho}\right)$ is convex. Simple manipulations show that $\frac{d^2}{dx^2}\left(F\left(\frac{2^x-1}{\rho}\right)\right) \geq 0$ is equivalent to

$$\left(y+\frac{1}{\rho}\right)\left(2f(y)^2 - f'(y)F(y)\right) \geq f(y)F(y)\ \forall\ y \geq 0. \quad (19)$$

Using (18), it is easy to see that if

$$\left(y+\frac{1}{\rho}\right)\left(2f(y)^2 - f'(y)F(y)\right) \geq y\left(2f(y)^2 - f'(y)F(y)\right). \quad (20)$$

As such, if

$$y\left(2f(y)^2 - f'(y)F(y)\right) \geq f(y)F(y)\ \forall\ y \geq 0. \quad (21)$$

holds, then so does (19). Inequality (21) obviously holds for $y = 0$. For $y > 0$, (21) can be shown to be equivalent to

$$-c + \frac{y^\lambda}{\beta} + \frac{2\beta^{-c}e^{-y^\lambda/\beta}y^{\lambda c}}{\Gamma(c) - \Gamma(c, y^\lambda/\beta)} \geq 0. \quad (22)$$

By letting $z = \frac{y^\lambda}{\beta}$ and using the fact that $c(\Gamma(c) - \Gamma(c,z)) = z^c e^{-z} + \Gamma(c+1) - \Gamma(c+1,z)$, (22) simplifies to

$$e^{-z} \geq \int_0^z \left(\left(\frac{t}{z}\right)^c - \left(\frac{t}{z}\right)^{c-1}\right) e^{-t} dt. \quad (23)$$

Since for $0 \leq t \leq z$, $\left(\frac{t}{z}\right)^c - \left(\frac{t}{z}\right)^{c-1} \leq 0$, we have indeed

$$\int_0^z \left(\left(\frac{t}{z}\right)^c - \left(\frac{t}{z}\right)^{c-1}\right) e^{-t} dt \leq 0 \leq e^{-z}, \quad (24)$$

which concludes the proof. □

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