Bounds for Generalized Gamma Distributed Fading Channels

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***Abstract*—In this article, upper bounds for the outage probability and moment generating functions of the fading channels with $\alpha-\mu$ distribution are derived. The proposed upper bound for the moment generating function is used in the performance evaluation of $M-PSK$ communication systems. The derived expressions are simple to use and do not require complex software tools to evaluate. We verified the proposed expressions via numerical computations.**

***Index Terms*—Generalized Gamma distribution, Hölder inequality, bounds,**

I. Introduction

Over the recent years mathematical distribution functions are used to characterize the fading phenomenon occurring in wireless communications. Some of these distributions are Rayleigh, Rice, Nakagami-m, and Weibull [1]. However, measurement results show that none of the distributions seem to fit to the experimental data over a wide range of the fading parameters [3]. For a limited range of fading parameters one of them can be preferred over the others. The generalized gamma distribution was proposed by Stacy in [2]. The work presented in [2] was a purely mathematical model in which statistical properties of the distribution are investigated. The $\alpha-\mu$ distribution which is an alternate name for the generalized gamma distribution is re-invented in [3]-[4] where the parameters of the distribution are associated with the physical parameters of the propagation medium. The $\alpha-\mu$ distribution characterizes the nonlinearity of the propagation medium and clustering of the multipath waves based on the fading parameters α and μ [4]. In addition, this general distribution is flexible, easy to handle mathematically, and measurement results show that it fits to the measured data over a wide range of fading parameters. The $\alpha-\mu$ distribution involves many well known distributions such as gamma, Nakagami-m, exponential, Weibull, one sided Gaussian, and Rayleigh. Second order statistics of the $\alpha-\mu$ distribution such as average fade duration and level crossing rate for equal gain combining and maximal-ratio combining schemes are investigated in [5].

Bit error rate performance of communications systems can be computed using moment generating functions [1]. Due to its general form it is not possible to obtain simple expressions for the moment generating function of the $\alpha-\mu$ distribution. In this letter we evaluate simple upper bounds for the outage probability, and the moment generating function of the $\alpha-\mu$ distribution. And we compute performance bounds for the $M-PSK$ communication systems using the upper bounds derived for moment generating function of the $\alpha-\mu$ distribution. The outline of the letter is as follows. In Section II we review $\alpha-\mu$ distribution and derive upper bound for the outage probability. Section III includes the derivation of upper bounds for the moment generating function and performance of $M-PSK$ communication systems. Finally conclusions are drawn in Section IV.

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II. Bounds for the Outage Probability

The $\alpha-\mu$ distribution is defined in [4] as

$$f_{\tilde{R}}(r)=\frac{\alpha\mu^{\mu}r^{\alpha\mu-1}}{\Gamma(\mu)\bar{r}^{\alpha\mu}}\exp\left(-\mu\left(\frac{r}{\bar{r}}\right)^{\alpha}\right),\ \alpha>0,\ \mu>0 \quad (1)$$

where $\Gamma(\mu)=\int_0^\infty x^{\mu-1}e^{-x}dx$ is the well known Gamma function, α is a fading parameter, μ is the normalized variance of the channel envelope $\tilde{R}$, and $\bar{r}$ is the α-root mean value of the channel envelope. This distribution is a general form for many well known distributions such as Rayleigh ($\alpha=2$, $\mu=1$), Nakagami-m ($\alpha=2$, and μ is fading parameter), Weibull ($\mu=1$, and $\alpha/2$ is fading parameter). Using (1) signal-to-noise ratio (SNR) probability density function (pdf) for $\alpha-\mu$ distributed fading channel can be expressed as [4]

$$f_{\tilde{\gamma}}(\gamma)=\frac{\alpha\mu^{\mu}\gamma^{\alpha\mu/2-1}}{2\Gamma(\mu)\bar{\gamma}^{\alpha\mu/2}}\exp\left(-\mu\left(\frac{\gamma}{\bar{\gamma}}\right)^{\alpha/2}\right) \quad (2)$$

where $\bar{\gamma}$ is the average SNR, i.e, γ_{avg}. The outage probability for $\alpha-\mu$ distributed fading channel is computed using [1] as

$$P_{\text{out}}=P(\tilde{\gamma}\leq\gamma_{thr})=\int_0^{\gamma_{thr}}f_{\tilde{\gamma}}(\gamma)d\gamma \quad (3)$$

which leads to the following equation upon substitution of $\alpha-\mu$ pdf

$$P_{\text{out}}=\frac{\alpha\mu^{\mu}}{2\Gamma(\mu)\bar{\gamma}^{\alpha\mu/2}}\int_0^{\gamma_{thr}}k(\gamma)d\gamma \quad (4)$$

where the function $k(\gamma)$ is given as

$$k(\gamma)=\gamma^{\alpha\mu/2-1}\exp\left(-\mu\left(\frac{\gamma}{\bar{\gamma}}\right)^{\alpha/2}\right). \quad (5)$$

Using Hölder integral inequality [6]

$$\int_\Omega f(x)g(x)dx\leq\left(\int_\Omega f(x)^m dx\right)^{1/m}\left(\int_\Omega g(x)^n dx\right)^{1/n} \quad (6)$$

with $1/m+1/n=1,\ m,\ n>1$ and $f,\ g$ nonnegative integrable function (4) can be written as

$$P_{\text{out}}\leq\frac{\alpha\mu^{\mu}}{2\Gamma(\mu)\bar{\gamma}^{\alpha\mu/2}}\left(\int_0^{\gamma_{thr}}\gamma^{(\alpha\frac{\mu}{2}-1)m}d\gamma\right)^{1/m}$$
$$\times\left(\int_0^{\gamma_{thr}}\exp\left(-n\mu(\frac{\gamma}{\bar{\gamma}})^{\alpha/2}\right)d\gamma\right)^{1/n} \quad (7)$$

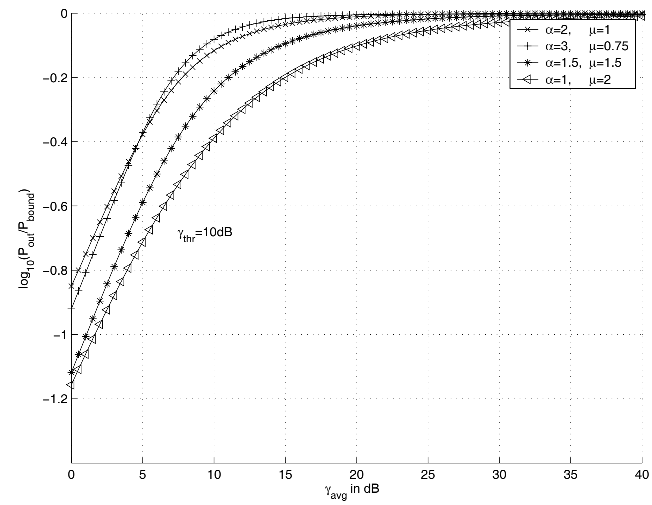

Fig. 1. Outage probability graph for $\alpha-\mu$ distribution, $n=m=2$.

where the first integral can be expressed in closed form whereas the second one cannot be written in terms of the known functions. The integrand function in the second integral expression is a monotonically decreasing exponential function. Using $N_\gamma+1$ values of γ such that $0=\gamma_0\leq\gamma_1\cdots\leq\gamma_{N_\gamma}=\gamma_{thr}$ and replacing second integral expression with its lower Riemann sum we obtain the following upper bound for outage probability

$$P_{\text{out}}\leq\frac{\alpha\mu^\mu}{2\Gamma(\mu)\bar{\gamma}^{\alpha\mu/2}}\left(\frac{\gamma_{thr}{}^{(\alpha\frac{\mu}{2}-1)m+1}}{(\alpha\frac{\mu}{2}-1)m+1}\right)^{1/m}\times\left(\sum_{i=1}^{N_\gamma}\exp\left(-n\mu(\frac{\gamma_{i-1}}{\bar{\gamma}})^{\alpha/2}\right)(\gamma_i-\gamma_{i-1})\right)^{1/n}\quad(8)$$

on which if $N_\gamma=2$ is chosen, with $\gamma_1=\frac{\gamma_{thr}}{2}$ and $\gamma_2=\gamma_{thr}$, the bound expression takes the following form

$$P_{\text{out}}\leq\frac{\alpha\mu^\mu}{2\Gamma(\mu)\bar{\gamma}^{\alpha\mu/2}}\left(\frac{\gamma_{thr}{}^{(\alpha\frac{\mu}{2}-1)m+1}}{(\alpha\frac{\mu}{2}-1)m+1}\right)^{1/m}\times\left(\frac{\gamma_{thr}}{2}+\frac{\gamma_{thr}}{2}\exp(-n\mu(\frac{\gamma_{thr}/2}{\bar{\gamma}})^{\alpha/2})\right)^{1/n}.\quad(9)$$

In Fig. 1 exact outage probability and its bounds are depicted together. It is clear from Fig. 1 that bounds become very tight especially at moderate and high SNR values.

III. Moment Generating Function and Average Error Probability

Moment generating function is a useful mathematical expression used to evaluate the performance of communication systems and it is defined as:

$$M_\gamma(s)=\mathrm{E}\left(e^{-s\gamma}\right).\quad(10)$$

For $\alpha-\mu$ distribution moment generating function takes the following form

$$M_\gamma(s)=\frac{\alpha\mu^\mu}{2\Gamma(\mu)\bar{\gamma}^{\alpha\mu/2}}\int_0^\infty e^{-s\gamma}k(\gamma)d\gamma.\quad(11)$$

Employing Hölder inequality [6] the moment generating function of the $\alpha-\mu$ distribution leads to the following inequality

$$M_\gamma(s)\leq\frac{\alpha\mu^\mu}{2\Gamma(\mu)\bar{\gamma}^{\alpha\mu/2}}\left[\int_0^\infty e^{-ms\gamma}d\gamma\right]^{1/m}\left[\int_0^\infty k^n(\gamma)d\gamma\right]^{1/n}\quad(12)$$

where the integrals can be evaluated resulting in the following inequality

$$M_\gamma(s)\leq s^{-\frac{1}{m}}\left(\frac{1}{m}\right)^{\frac{1}{m}}\frac{\alpha\mu^\mu}{2\Gamma(\mu)}\frac{\bar{\gamma}^{-1+\frac{1}{n}}}{(\mu n)^{\mu-\frac{2}{\alpha}+\frac{2}{\alpha n}}}\times\left[\frac{2}{\alpha}\Gamma\left(\mu n-\frac{2}{\alpha}(n-1)\right)\right]^{\frac{1}{n}}.\quad(13)$$

A. Performance Evaluation for M-PSK Systems

Bit error performance of communication systems employing $M-PSK$ modulation is evaluated using the moment generating function as [1]

$$P_{\text{e}}(\gamma)=\frac{1}{\pi}\int_0^{\pi-\pi/M}M_\gamma\left(\frac{\sin^2(\pi/M)}{\sin^2(\theta)}\right)d\theta.\quad(14)$$

When (13) is used in $M-PSK$ performance expression we get the following bound for the BER performance of communication systems with $\alpha-\mu$ distribution

$$P_{\text{e}}(\gamma)\leq\frac{1}{2\pi}\left(\frac{1}{m}\right)^{\frac{1}{m}}\sin^{-2/m}\left(\frac{\pi}{M}\right)\frac{\alpha\mu^{\frac{2}{\alpha}(1-\frac{1}{n})}}{\Gamma(\mu)}\frac{\bar{\gamma}^{-1+\frac{1}{n}}}{n^{\mu-\frac{2}{\alpha}(1-1\frac{1}{n})}}\times\left[\frac{2}{\alpha}\Gamma(\mu n-\frac{2}{\alpha}(n-1))\right]^{\frac{1}{n}}\int_0^{\pi-\pi/M}\sin^{2/m}\theta d\theta\quad(15)$$

where using $N_\theta+1$ values of θ such that $0=\theta_0\leq\theta_1\cdots\leq\theta_{N_\theta}=\pi-\pi/M$ the integral expression can be bounded by its upper Riemann sum as

$$\int_0^{\pi-\pi/M}\sin^{2/m}\theta d\theta\leq\sum_{i=1}^{N_\theta}(\sin^{2/m}\theta_i)(\theta_i-\theta_{i-1})\quad(16)$$

when used in (15) leads to

$$P_{\text{e}}(\gamma)\leq\frac{1}{2\pi}\left(\frac{1}{m}\right)^{\frac{1}{m}}\sin^{-2/m}\left(\frac{\pi}{M}\right)\frac{\alpha\mu^{\frac{2}{\alpha}(1-\frac{1}{n})}}{\Gamma(\mu)}\frac{\bar{\gamma}^{-1+\frac{1}{n}}}{n^{\mu-\frac{2}{\alpha}(1-1\frac{1}{n})}}\times\left[\frac{2}{\alpha}\Gamma(\mu n-\frac{2}{\alpha}(n-1))\right]^{\frac{1}{n}}\sum_{i=1}^{N_\theta}(\sin^{2/m}\theta_i)(\theta_i-\theta_{i-1}).\quad(17)$$

B. Rayleigh and Nakagami-m Cases

When the fading parameters $\alpha=2,\ \mu=1$ are used in (15) and choosing $n\to\infty,\ m\to1$ the performance bound for the Rayleigh distribution case is found as

$$P_{\text{e}}(\gamma)\leq\frac{1}{2\pi\bar{\gamma}}\sin^{-2}\left(\frac{\pi}{M}\right)\left[\pi-\frac{\pi}{M}+\frac{1}{2}\sin\left(\frac{2\pi}{M}\right)\right].\quad(18)$$

The performance bounds for the Rayleigh case are compared to the exact ones in Fig. 2 where the plots are drawn for $BPSK$ and $8-PSK$ modulation schemes. To check the tightness of the Rayleigh bounds the logarithm of ratio of upper bound and exact curve is depicted in Fig. 3 where

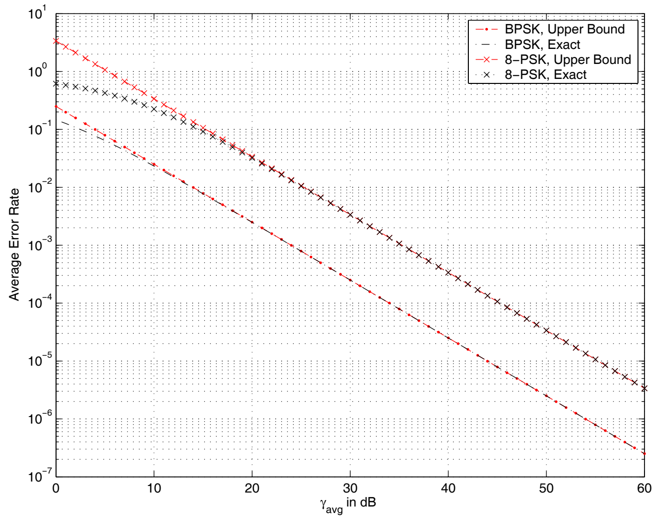

Fig. 2. Performance upper bounds for Rayleigh fading case.

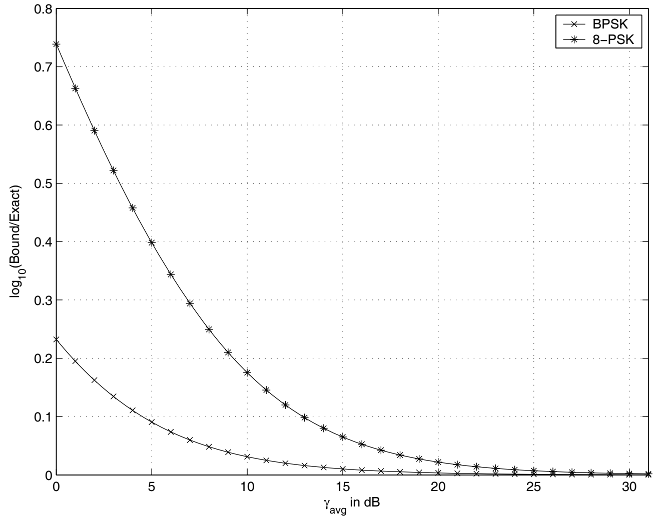

Fig. 3. Difference between upper bounds and exact values.

it is clear that the proposed bound is very tight especially for moderate to high SNR values. For different values of the α and μ parameters the performance curves for $M-PSK$ modulation are depicted in Fig. 4 along with the proposed bounds. Since our proposed bounds cover all α and μ ranges, the tightness of the bound varies depending on α and μ values. In addition, the parameters m and n also have effect on the tightness of the bound along with the α and μ values. While giving values to the m and n, attention should be paid such that no complex terms or negative terms appear on the right hand side of (15). And the large values of n are preferred, for instance, considering the case $\alpha \times \mu = 2$ the parameter values are chosen as $n \to \infty$, $m \to 1$. When practical ranges of interest for the values of α and μ are considered, our proposed bounds give sufficient idea about the performance of the $M-PSK$ communication systems. It is also clear from Fig. 4 that for some parameter values the bound may diverge at very high SNR ranges.

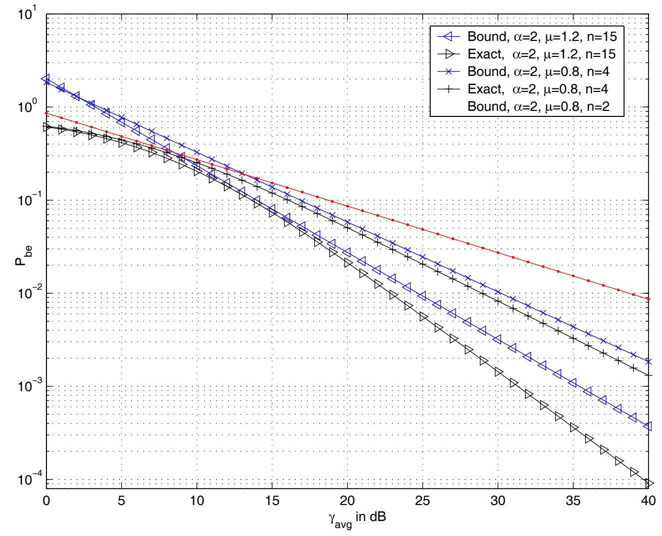

Fig. 4. 8-PSK performance graphs and bounds.

IV. Conclusions

In this manuscript we derived bound expressions for the outage probability and moment generating function of $\alpha-\mu$ distribution. The proposed bounds are very simple in mathematics and can be computed using simple software tools without requiring complex software libraries. Using numerical computation it is shown that the proposed bounds are valid over a wide range of $\alpha-\mu$ parameter values which cover several types of distributions. During the derivation of the bound expressions we made use of the monotonic behavior of some of the functions.

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