Channel Capacity and Second-Order Statistics in Weibull Fading

Article *in* IEEE Communications Letters · July 2004

DOI: 10.1109/LCOMM.2004.831319 · Source: IEEE Xplore

George K. Karagiannidis
Aristotle University of Thessaloniki

George S Tombras
National and Kapodistrian University of Athens,

Channel Capacity and Second-Order Statistics in Weibull Fading

Nikos C. Sagias, *Student Member, IEEE*, Dimitris A. Zogas, *Student Member, IEEE*, George K. Karagiannidis, *Senior Member, IEEE*, and George S. Tombras, *Senior Member, IEEE*

***Abstract*—The second-order statistics and the channel capacity of the Weibull fading channel are studied. Exact closed-form expressions are derived for the average level crossing rate, the average fade duration, as well as the average Shannon's channel capacity of the Weibull fading process. Numerical results are presented to illustrate the proposed mathematical analysis and to examine the effects of the fading severity on the concerned quantities.**

***Index Terms*—Fade duration, level crossing rate (LCR), Shannon's channel capacity, Weibull fading channels.**

I. Introduction

IN THE PAST, experimental data have shown that the Weibull fading channel model exhibits an excellent fit both for indoor [1] and outdoor environments [2]. Recently, the appropriateness of the Weibull distribution to model fading channels was also reported in [3], where a path-loss model for the Digital Enhanced Cordless Telecommunications (DECT) system at 1.89 GHz, was studied. Previously published works related to the performance analysis of digital communications receivers over Weibull fading channels include the following. In [4], the Weibull fading channel model was considered for the evaluation of the first two moments of the output signal-to-noise power ratio (SNR) in generalized selection combining (GSC) receivers. In [5], the performance of switched and stay diversity receivers in Weibull fading was studied. In [6], dual selection combining (SC) receivers in correlated Weibull fading were considered and in [7], important performance measures such as the outage probability and the average output SNR were studied, for L-branch SC receivers over independent and identically distributed Weibull fading channels. However, to the best of the authors' knowledge, there is not any publication in the open technical literature related to important Weibull fading channel characteristics, such us average level crossing rate (LCR), average fade duration (AFD) [8] and average Shannon's channel capacity [9], [10]. In this letter, exact closed-form expressions for the average LCR, the AFD and the average channel capacity of the Weibull fading channel model are derived. Selected numerical results are presented to outline the proposed mathematical analysis. The effect of the fading severity on the channel's statistics is pointed out and compared to the well-known Rayleigh channel model.

Manuscript received November 11, 2003. The associate editor coordinating the review of this letter and approving it for publication was Dr. Z. Xu.

N. C. Sagias and G. S. Tombras are with the Laboratory of Electronics, Department of Physics, University of Athens, Panepistimiopolis, 15784 Athens, Greece (e-mail: nsagias@space.noa.gr; gtombras@cc.uoa.gr).

G. K. Karagiannidis is with the Institute for Space Applications & Remote Sensing, National Observatory of Athens, Palea Penteli, 15236 Athens, Greece (e-mail: gkarag@space.noa.gr).

D. A. Zogas is with the Electrical & Computer Engineering Department, University of Patras, Rion, 26442 Patras, Greece (e-mail: zogas@space.noa.gr).

Digital Object Identifier 10.1109/LCOMM.2004.831319

II. Second Order Statistics

Let r be the received sampled envelope and $\dot{r}$ its derivative with respect to time, with joined probability density function (pdf) $p_{\dot{r},r}(\dot{r},r)$. The average LCR and AFD are defined as

$$N(r) \triangleq \int_0^\infty \dot{r}\, p_{\dot{r},r}\left(\dot{r},r\right) d\dot{r} \tag{1}$$

$$\tau(r) \triangleq \frac{F_r(r)}{N(r)} \tag{2}$$

respectively, where $F_r(r)$ is the cumulative distribution function (cdf) of r. In the following, the first- and the second-order statistics of r are studied, in order to evaluate (1) and (2) in closed-form expressions. Let $x_1(t)$ and $x_2(t)$ be the in-phase and quadrature components of a narrow-band process at timing instance t, such that

$$x_1(t) = \sqrt{2}\sigma \sum_{k=1}^{K} d_k \cos(2\pi f_k t - \vartheta_k)$$
$$x_2(t) = \sqrt{2}\sigma \sum_{k=1}^{K} d_k \sin(2\pi f_k t - \vartheta_k) \tag{3}$$

where $\sqrt{2}\sigma d_k$ is the fading amplitude of the kth wave, with $\sum_{k=1}^{K} d_k^2 = 1$, ϑ_k is the random phase uniformly distributed in $[0, 2\pi)$, K is the number of the waves and f_k is the Doppler shift, with $f_k = f_d \cos(\theta_k)$, where f_d is the maximum Doppler shift and θ_k is the corresponding angle of wave arrival. Taking into account the central limit theorem, for fixed t and for a large value of K, $x_i(t)$ ($i = 1, 2$) can be considered as zero mean Gaussian process with variance σ^2, i.e., $E\langle x_i \rangle = 0$ and $E\langle x_i^2 \rangle = \sigma^2$, where $E\langle\cdot\rangle$ denotes expectation. It is convenient to alleviate this notation, by omitting the variable t in the equations. It is well-known that a sum of two quadrature Gaussian components is also a Gaussian process, i.e., $z = x_1 + jx_2 = x\exp(j\varphi)$, where $\sqrt{j} = -1$. The random phase $\varphi = \tan^{-1}\left(x_2/x_1\right)$ is uniformly distributed in $[0, 2\pi)$ and the envelope $x = \sqrt{x_1^2 + x_2^2}$ is Rayleigh distributed, with pdf

$$p_x(x) = \frac{2x}{\Omega}\exp\left(-\frac{x^2}{\Omega}\right) \tag{4}$$

where $\Omega = E\langle x^2 \rangle = 2\sigma^2$. Let the received sampled signal be $y = z^{2/\beta} = x^{2/\beta} \exp(j2\varphi/\beta)$, where β is a positive real constant value. Using [11, eq. (5-5)] and (4), the corresponding pdf of the envelope

$$r = x^{2/\beta} \tag{5}$$

of the received signal can be easily obtained as

$$p_r(r) = \frac{\beta}{\Omega} r^{\beta-1} \exp\left(-\frac{r^\beta}{\Omega}\right) \tag{6}$$

with $E\langle r^\beta \rangle = \Omega$. It is easily recognized that the pdf of r follows the Weibull distribution [12, Ch. 17] with fading parameter β, which expresses the severity of fading. As the value of β increases, the severity of the fading decreases, while for the special case of $\beta = 2$, (6) reduces to the well-known Rayleigh pdf. The corresponding cdf of r is given by

$$F_r(r) = 1 - \exp\left(-\frac{r^\beta}{\Omega}\right). \tag{7}$$

Using (5), the derivative $\dot{r}$ of r with respect to time is

$$\dot{r} = \frac{2}{\beta} r^{1-(\beta/2)} \dot{x} \tag{8}$$

where $\dot{x}$ is the time derivative of x. For isotropic scattering, $\dot{x}$ is a Gaussian distributed random variable (rv) with zero mean and variance $\hat{\sigma}^2 = \sigma^2 2\pi^2 f_d^2$ [8], and thus, $\dot{r}$ conditioned on r is also a zero mean Gaussian rv. The standard deviation of $\dot{r}$ conditioned on r can be obtained from (8) as

$$\hat{\sigma}_r = \frac{2}{\beta} r^{1-(\beta/2)} \hat{\sigma} \tag{9}$$

and the corresponding pdf is given by

$$p_{\dot{r}}(\dot{r}|r) = \frac{1}{\sqrt{2\pi}\hat{\sigma}_r} \exp\left(-\frac{\dot{r}^2}{2\hat{\sigma}_r^2}\right). \tag{10}$$

The joined pdf of $\dot{r}$ and r can be obtained replacing (6), (9) and (10) into $p_{\dot{r},r}(\dot{r}, r) = p_{\dot{r}}(\dot{r}|r)\, p_r(r)$ [11, eq. (7-4)], yielding

$$p_{\dot{r},r}(\dot{r}, r) = \frac{\beta^2\, r^{(3\beta/2)-2}}{f_d\, (2\pi\,\Omega)^{3/2}} \exp\left[-\frac{\beta^2\, r^{(\beta-2)}\, \dot{r}^2}{8\pi^2 f_d^2\, \Omega} - \frac{r^\beta}{\Omega}\right]. \tag{11}$$

After replacing (11) into (1) and normalizing the signal level to its root mean square (rms) value, $\rho = r/r_{\mathrm{rms}}$, with $r_{\mathrm{rms}} = \sqrt{E\langle r^2 \rangle} = \Omega^{1/\beta}/\sqrt{a}$ and $a = 1/\Gamma(1 + 2/\beta)$, where $\Gamma(\cdot)$ is the Gamma function [13, eq. (8.310/1)], the average LCR for the Weibull channel can be obtained in simple closed-form as

$$N(\rho) = \sqrt{2\pi} f_d \left(\frac{\rho}{\sqrt{a}}\right)^{\beta/2} \exp\left[-\left(\frac{\rho}{\sqrt{a}}\right)^\beta\right]. \tag{12}$$

The expression for the AFD is obtained by normalizing the signal level to its rms value in (7) and replacing then, together with (12), into (2) resulting in

$$\tau(\rho) = \frac{1 - \exp\left[-\left(\frac{\rho}{\sqrt{a}}\right)^\beta\right]}{\sqrt{2\pi} f_d \left(\frac{\rho}{\sqrt{a}}\right)^{\beta/2} \exp\left[-\left(\frac{\rho}{\sqrt{a}}\right)^\beta\right]}. \tag{13}$$

Note, that when $\beta = 2$, (12) and (13) reduce to previous published expressions for the well-known Rayleigh model [8, eqs. (1.3–35) and (1.3–43)]. The maximum value of the average LCR can be derived solving the equation which is obtained by differentiating (12) with respect to ρ, setting the result equal to zero, i.e., $dN(\rho)/d\rho|_{\rho=\rho_{\max}} = 0$ and then replacing $\rho_{\max}$ into (12). It can be easily shown that the average LCR is maximized at $\rho_{\max} = 2^{(-1/\beta)}\sqrt{a}$ as $N(\rho_{\max}) = f_d\sqrt{\pi/e}$. It is interesting to note that the severity of fading does not affect $N(\rho_{\max})$.

III. Average Channel Capacity

We consider a signal's transmission of bandwidth BW and symbols' energy E_s. Since the nth power of a Weibull rv, with parameters β and Ω, follows also the Weibull distribution, with parameters β/n and Ω, [12, Ch. 17], the pdf of the received SNR per symbol, defined as $\gamma = r^2 E_s/N_0$, with N_0 the double-sided noise power spectral density of the additive white Gaussian noise (AWGN), can be written as [5]

$$p_\gamma(\gamma) = \frac{\beta}{2a\overline{\gamma}} \left(\frac{\gamma}{a\overline{\gamma}}\right)^{(\beta/2)-1} \exp\left[-\left(\frac{\gamma}{a\overline{\gamma}}\right)^{\beta/2}\right] \tag{14}$$

where $\overline{\gamma}$ is the corresponding average SNR per symbol, $\overline{\gamma} = \Gamma(1 + 2/\beta)\Omega^{2/\beta} E_s/N_0$. The average channel capacity, in Shannon's sense, is defined as [9]

$$\overline{C} \triangleq BW \int_0^\infty \log_2(1 + \gamma) p_\gamma(\gamma) d\gamma. \tag{15}$$

By replacing (14) into (15), the average channel capacity for the Weibull fading channel is written as

$$\begin{aligned}\overline{C} = &\frac{BW\beta}{2\,(a\overline{\gamma})^{\beta/2} \ln(2)} \int_0^\infty \gamma^{(\beta/2)-1} \ln(1 + \gamma) \\ &\times \exp\left[-\,(a\overline{\gamma})^{-(\beta/2)}\, \gamma^{\beta/2}\right] d\gamma.\end{aligned} \tag{16}$$

The above integral can be evaluated in closed-form as follows. By expressing the logarithmic and exponential integrands in (16) as Meijer's G-functions [13, eq. (9.301)], i.e., $\ln(1+\gamma) = G_{2,2}^{1,2}[\gamma|_{1,0}^{1,1}]$ and $\exp\{-[\gamma/(a\overline{\gamma})]^{\beta/2}\} = G_{0,1}^{1,0}[[\gamma/(a\overline{\gamma})]^{\beta/2}|_{0}^{-}]$ [14, eq. (11)] and using [14, eq. (21)], the integral in (16) can be solved in closed-form and the average channel capacity can be obtained as

$$\begin{aligned}\overline{C} = &\frac{\beta\,(a\overline{\gamma})^{-(\beta/2)}}{2\ln(2)} \frac{BW\sqrt{k}l^{-1}}{(2\pi)^{(k+2l-3/2)}} \\ &\times G_{2l,k+2l}^{k+2l,l}\left[\frac{(a\overline{\gamma})^{-(\beta k/2)}}{k^k}\middle| \begin{matrix} I\left(l,-\frac{\beta}{2}\right), I\left(l, 1-\frac{\beta}{2}\right) \\ I(k,0), I\left(l,-\frac{\beta}{2}\right), I\left(l,-\frac{\beta}{2}\right)\end{matrix}\right]\end{aligned} \tag{17}$$

where $I(n, \xi) \triangleq \xi/n, (\xi + 1)/n, \ldots, (\xi + n - 1)/n$, with ξ an arbitrary real value and n positive integer. Moreover, $l/k = \beta/2$, where k and l are positive integers. Depending upon the value of β, a set with minimum values of k and l can be properly chosen (e.g., for $\beta = 1.4$ we have to choose $k = 10$ and $l = 7$). Note, that for the special case of β being integer $k = 2$ and $l = \beta$.

IV. Numerical Results

We have numerically evaluated (12), (13) and (17) and the results are depicted in Figs. 1–3, respectively. In Fig. 1, the nor-

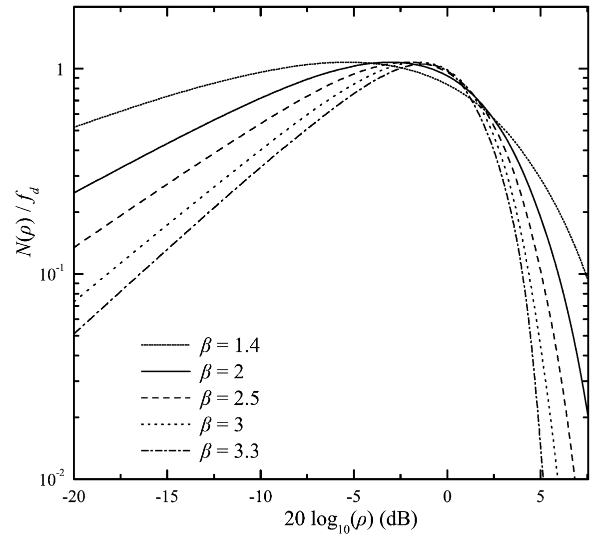

Fig. 1. Normalized average LCR versus normalized envelope level.

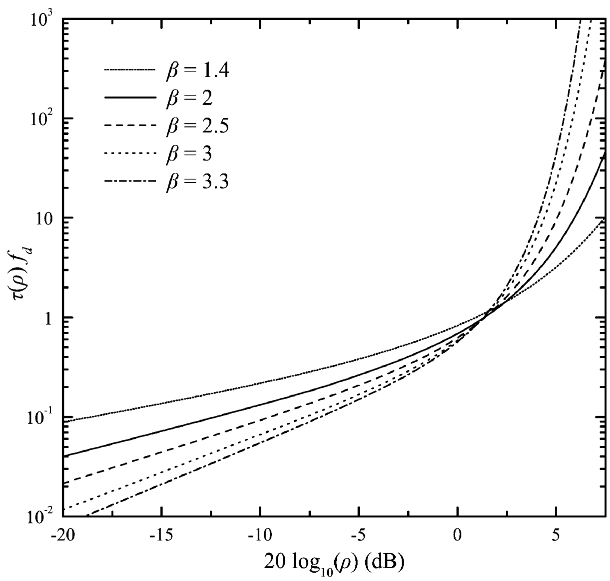

Fig. 2. Normalized AFD versus normalized envelope level.

malized average LCR is plotted as a function of the normalized envelope level ρ for several values of β. As it was expected, when the fading severity increases (i.e., β decreases) the normalized average LCR increases, which means that fades occur more frequently. Moreover, lower signal levels are crossed less frequently, whereas higher signal level are crossed more frequently. In Fig. 2, the normalized AFD is plotted as a function of the normalized envelope level ρ for several values of β. It easily recognized that, the less fading severity, the less time the signal remains in deep fades. In Fig. 3, the normalized average channel capacity (spectral efficiency) is plotted as a function of the average SNR per symbol. For comparison purposes, the normalized channel capacity for the AWGN channel

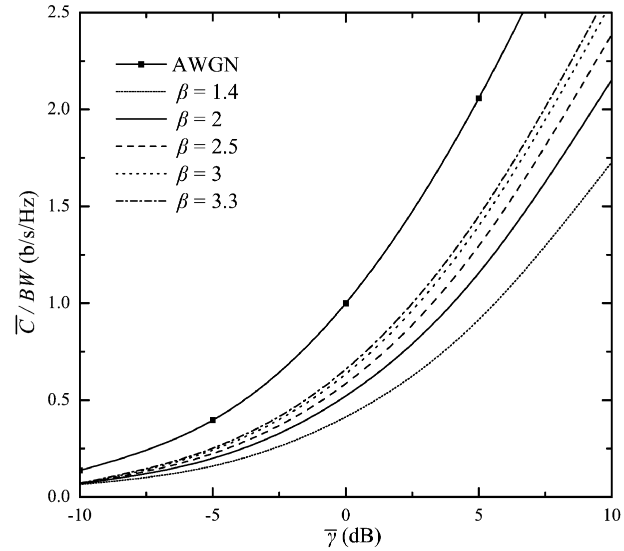

Fig. 3. Normalized average channel capacity versus average SNR per symbol.

$(C/BW = \log_2(1+\gamma))$ is also plotted. As it was expected, the average capacity of the Weibull fading channel is always less than the capacity provided by the AWGN channel.

References

[1] H. Hashemi, "The indoor radio propagation channel," *Proc. IEEE*, vol. 81, pp. 943–968, July 1993.

[2] N. S. Adawi *et al.*, "Coverage prediction for mobile radio systems operating in the 800/900 MHz frequency range," *IEEE Trans. Veh. Technol.*, vol. 37, pp. 3–72, Feb. 1988.

[3] F. Babich and G. Lombardi, "Statistical analysis and characterization of the indoor propagation channel," *IEEE Trans. Commun.*, vol. 48, pp. 455–464, Mar. 2000.

[4] M.-S. Alouini and M. K. Simon, "Performance of generalized selection combining over Weibull fading channels," in *Proc. IEEE Vehicular Technology Conf.*, Rhodes, Greece, May 2001, pp. 1735–1739.

[5] N. C. Sagias, D. A. Zogas, G. K. Karagiannidis, and G. S. Tombras, "Performance analysis of switched diversity receivers in Weibull fading," *Electron. Lett.*, vol. 39, no. 20, pp. 1472–1474, Oct. 2003.

[6] N. C. Sagias, G. K. Karagiannidis, D. A. Zogas, P. T. Mathiopoulos, and G. S. Tombras, "Performance analysis of dual selection diversity in correlated Weibull fading channels," *IEEE Trans. Commun.*, to be published.

[7] N. C. Sagias, P. T. Mathiopoulos, and G. S. Tombras, "Selection diversity receivers in Weibull fading: Outage probability and average signal-to-noise ratio," *Electron. Lett.*, vol. 39, no. 25, pp. 1859–1860, Dec. 2003.

[8] W. C. Jakes, *Microwave Mobile Communications*: John Wiley, 1974.

[9] W. C. Y. Lee, "Estimate of channel capacity in Rayleigh fading environment," *IEEE Trans. Veh. Technol.*, vol. 39, pp. 187–189, 1990.

[10] F. Lazarakis, G. S. Tombras, and K. Dangakis, "Average channel capacity in a mobile radio enviroment with Rician statistics," *IEICE Trans. Commun.*, vol. E77-B, no. 7, pp. 971–977, July 1994.

[11] A. Papoulis, *Probability, Random Variables, and Stochastic Procceses*, 3rd ed: McGraw-Hill.

[12] K. Bury, *Statistical Distributions in Engineering*: Cambridge Univ. Press, 1999.

[13] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. New York: Academic, 1994.

[14] V. S. Adamchik and O. I. Marichev, "The algorithm for calculating integrals of hypergeometric type functions and its realization in REDUCE system," in *Proc. Int. Conf. on Symbolic and Algebraic Computation*, Tokyo, Japan, 1990, pp. 212–224.