Generation of Nakagami-*m* Statistics by Monte Carlo Method for Fading Communication Channels

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***Abstract*—Nakagami-*m* distributions for varying fading parameter *m* are widely accepted as best fit to model multipath propagation from severe to light fading. Although simulation results based on approximation of inverse cumulative distribution function are reported recently, very few results are there based on Monte Carlo generation of such random variates. In this paper, statistics for Nakagami-*m* distribution are generated from Monte Carlo simulation, and results are corroborated with theoretical values of expectations and autocorrelation function.**

***Index Terms*—Fading, Nakagami-*m*, Monte Carlo, histogram, autocorrelation.**

I. Introduction

Radio signals in urban areas experience reflection, diffraction, and scattering of waves, collectively referred to as multipath (or short-term) fading [1]. Urban wireless communication channels are particularly affected by the interference of faded signals with the direct path because of constructive or destructive phases in the received signal.

Several probability density functions (pdf) such as Rayleigh, Rician, log-normal distributions are proposed to model the fading random variable (rv). Nakagami-*m* distribution is widely accepted to be the best fit for multipath fading in communication channels because of its flexibility to model a variety of fading conditions those are less or more severe than the cases modeled by the Rayleigh pdf [2],[3]. Moreover, the statistics generated through Nakagami-*m* distributions fits more accurately with experimental data for many physical propagation channels than other distributions [4]. Although several results of modeling Nakagami-*m* distribution are reported recently for arbitrary values of *m* few attempts are reported based on Monte Carlo simulation [5],[6]. Monte Carlo simulation is well accepted for robust generation of relative frequency of occurrence of a rv over large numbers of sample events.

In this paper, we report results of generating random variates with Nakagami-*m* pdf by Monte Carlo simulations. We show Nakagami-*m* pdf for both integer and fractional values of *m* from the first principles of Monte Carlo simulation representing light to severe conditions of fading in communication channels. The statistics generated from simulation are validated with theoretical expectations for Gamma, chi-square, and Nakagami-*m* distributions. Robustness of simulation is also proved from calculated values of autocorrelation function (acf).

II. Statistics of a Fading Random Variable

The pdf of Nakagami-*m* distribution for a rv $R > 0$ is

$$f_R(r) = \frac{2m^m r^{2m-1}}{\Gamma(m)\Omega^m} \exp(-\frac{m}{\Omega} r^2) \tag{1}$$

where $\Omega = E[R^2]$ is the instantaneous power, and $E[.]$ is the expectation operator. In (1) $\Gamma(.)$ is the Gamma function, and *m* is the parameter that determines depth of fading.

$$m = \frac{\Omega^2}{E[(R^2 - \Omega)^2]}, m \geq \frac{1}{2}. \tag{2}$$

For values of $m < 1$, fading is severe than Rayleigh model which is a special case in (1) for $m = 1$. Expectations of Nakagami-*m* pdf are

$$E[R^k] = \frac{\Gamma(m + \frac{k}{2})}{\Gamma(m)} (\frac{\Omega}{m})^{\frac{k}{2}}. \tag{3}$$

The pdf in (1) is closely related to a Gamma distributed rv $\gamma = R^2$, whose pdf is

$$f_G(\gamma) = \frac{1}{\Gamma(m)} (\frac{m}{\Omega})^m \gamma^{m-1} \exp(-\frac{m}{\Omega}\gamma). \tag{4}$$

We note that chi-square distribution is a special case of Gamma distribution for $m = \frac{n}{2}$, *n* being a positive integer, and $\Omega = 2m$. An rv *X* with chi-square distribution and *n* degrees of freedom is denoted as χ_n^2, and its pdf is

$$f_{\chi^2}(x) = \frac{1}{2^m \Gamma(m)} x^{m-1} \exp(-\frac{x}{2}). \tag{5}$$

Such χ_n^2 distribution is generated as $\sum_{i=1}^{N} X_i^2$ where $X_i, i = 1, 2..N$ is the standard normal rv $\aleph[0, 1)$ that has zero mean and variance equal to one. The pdf in $(1), (4), (5)$ represent a family of long-tailed distributions. The extent of the tail of distributions depend on values of fading parameter *m*. A brute force generation of Nakagami-*m* distributions is

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obtained by taking square-root of χ_n^2 rv for integer values of *n*. Family of Nakagami-*m* pdf are shown in Fig. 1 after polynomial curve-fitting of relative frequency data generated by Monte Carlo simulation of normal distributions.

III. Generation of Rayleigh, Gamma, and Chi-Square Distributions

Monte Carlo simulation of random variates starts with generation of uniformly distributed pseudorandom variates, and apply inverse transform of cumulative distribution functions (cdf) to generate other distributions. We choose Wichmann-Hill algorithm to generate rv with uniform distribution as it is robust to pass standard statistical tests to generate samples with very long periods [7] . Essentially we use three component generators with slightly different periods in iterations for sample data generation.

$$\begin{aligned} x_{i+1} &= (171x_i)mod(30269) \\ y_{i+1} &= (170y_{i)})mod(30307) \\ z_{i+1} &= (172z_i)mod(30323). \end{aligned} \tag{6}$$

The three component generators are combined to produce the output

$$u_i = [\frac{x_i}{30269} + \frac{y_i}{30307} + \frac{z_i}{30323}z_i]mod(1). \tag{7}$$

The period of repetition for uniform rv U is approximately 7.0×10^{12} that is large enough to utilize this generator for producing subsequent distributions. Normal distributions those are independent, and identically distributed (iid) are generated from independent uniform variates as

$$\begin{aligned} X &= \sqrt{-2\sigma \ln(U_1)\cos(2\pi U_2)} \\ Y &= \sqrt{-2\sigma \ln(U_2)\sin(2\pi U_1)}. \end{aligned} \tag{8}$$

U_1 and U_2 are generated as in (7). Rayleigh distribution is given by $R = \sqrt{X^2 + Y^2}$ that is a Nakagami distribution as shown in Fig. 1.

Exponential distribution is a special case of Gamma distribution in (4) for $m = 1$. A Gamma distribution of *m*-degrees freedom is generated from summation of *m* iid exponential distributions. An exponential rv X is derived from inverse transformation of cdf as $X = F_X^{-1}(u)$,where $F_X(x)$ is the cdf of an exponential pdf.

$$F_X(x) = 1 - \exp(-\Omega x). \tag{9}$$

Equating the cdf of X to the uniform rv U gives $\exp(-\Omega x) = 1 - u$. The rv U has same pdf as $(1 - U)$, and X after inverse transformation of cdf is

$$X = -\frac{1}{\Omega}\ln(U). \tag{10}$$

Relative frequency distribution for Gamma variates with $m = 3, 4, 6$ degrees of freedom generated by Monte Carlo simulation are shown in Fig. 2 (a). Corresponding histograms for χ_n^2 distributions with $n = 2m$ degrees freedom are in Fig.

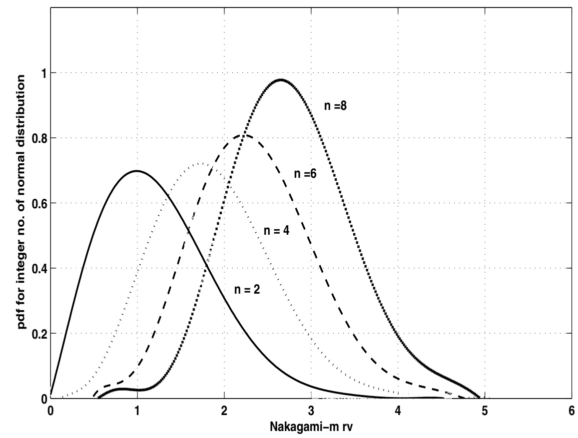

Fig. 1. Nakagami-*m* pdf from addition of integer number of normal distributions.

2(b). Expectations of *k*-th order of the random variates are obtained as

$$\hat{E}[z^k] = E[\sum_{i=1}^{n} z_i^k]. \tag{11}$$

Values of expectations for the distributions are compared in Table 1 with theoretical moments derived from expressions in Section II .

IV. Generation of Statistics for Nakagami-*m* Distribution

There is no closed form expression for the inverse cdf of Nakagami-*m* variates, and it is difficult to apply such direct inverse transformation technique here. Inverse cdf for Nakagami-*m* rv is derived here from the transformation of cdf of a Rayleigh rv as Rayleigh distributed fading is a special case for Nakagami-*m* pdf. Specifically, cdf of a Rayleigh rv is

$$F_{Ray}(r) = 1 - \exp(-\frac{r^2}{2\sigma^2}), r > 0 \tag{12}$$

where σ^2 is the variance of underlying normal rv $\aleph[0, \sigma]$. Setting $F_{Ray}(R) = U$ yields

$$u = 1 - \exp(-\frac{r^2}{2\sigma^2}), U\epsilon[0, 1) \tag{13}$$

as the transformation of R into a uniform rv U. Such uniform rv is generated here from the Rayleigh rv for $\sigma = 1$. Inverse cdf for Nakagami-*m* distribution is the substitution $F_R^{-1}(u)$ providing the desired rv. Proceeding further, approximate inverse cdf for the Nakagami-*m* variate is generated through an ancillary variable [6]

$$\eta = (\sqrt{\ln\frac{1}{1-u}})^{\frac{1}{m}}. \tag{14}$$

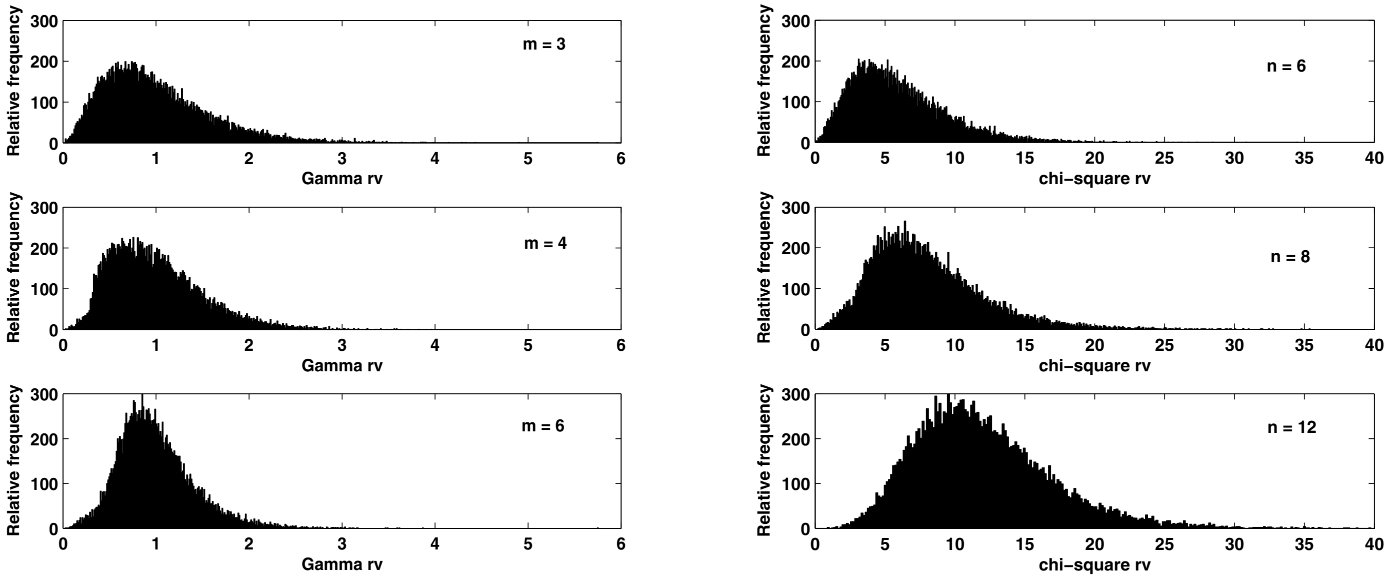

Fig. 2. (a) Relative frequency of Gamma distribution, top: m = 3, middle: m = 4, bottom : m = 6; (b) Relative frequency of χ_n^2 distribution, top: n = 6, middle: n = 8, bottom : n = 12.

Approximation in inverse cdf is

$$F_R^{-1}(u) \approx G(\eta) = \eta + \frac{a_1\eta + a_2\eta^2 + a_3\eta^3}{1 + b_1\eta + b_2\eta^2}. \qquad (15)$$

The coefficients a_1, a_2, a_3, b_1, b_2 are chosen to minimize approximation error in $F_R^{-1}(u)$ that stands for the rv having Nakagami-*m* distribution. Understandably, there is no one set of coefficients for all values of *m*, and a table of values of the coefficients are in [6] for $m < 1$, and integer values. For $m = 1$, $G(\eta)$ reduces to the expression of inverse Rayleigh cdf. Relative frequency of Nakagami-*m* distribution for integer values of *m* are shown in histograms of Fig. 3. The approximate pdf of the rv for values of $m < 1$, and other values are shown in Fig. 4. Theoretical density functions are derived from (1) utilizing the values of *m* and Ω from the sample data generated. It is seen in Fig. 4 that approximation in (15) is fairly close to the theoretical pdf of Nakagami-*m* rv for both fractional and integer values of *m*. Expectations of such distributions closely match the theoretical estimates given in (3), and are produced in Table 2.

Autocorrelation function(acf) of the amplitude is an important statistical indicator for the extent of fading. An empirical estimate of normalized acf is obtained as normalization of the inner product of *i*-th sample R_i with a shifted sample $R_{i+\delta}$, where δ is the discrete relative time-shift between the samples.

$$\overline{f_{ACF}(\delta)} = \frac{\sum_{i=0}^{n-\delta-1} R_i R_{i+\delta}}{\sum_{i=0}^{n-\delta-1} R_i^2}. \qquad (16)$$

Such acf is computed for 2×10^4 sample values of Nakagami-*m* distributions for values of *m* ranging from 0.65, 1.5, 4.0, and 8.0. Variation of the normalized acf with time-shift parameter δ are shown in Fig. 5.

It is seen from the density functions in Fig. 4 that values of $m < 1$ indicate severe fading. This corresponds to the acf plots in Fig. 5 where for values $m < 1$, the acf function falls very sharply after $\delta = 0$. This shows that for severe fading the samples are largely uncorrelated. On the other hand, for larger values of *m* the acf function reaches towards a constant borne by the fact that as *m* increases Nakagami-*m* pdf approaches an impulse. This fact is verified in the lower pane of Fig. 3, and pdf of distributions with larger values of *m*.

TABLE I
SIMULATION AND THEORETICAL RESULTS OF EXPECTATIONS FOR HISTOGRAMS IN FIG. 2.

Order (k)	$f_{Ray}(r)$ $(\sigma = 1.0)$ Theo	Prac	$f_G(\gamma)(m = 4, \Omega = 1)$ Theo	Prac	$f_{\chi^2(x)}$ $(m = 8)$ Theo	Prac
$E[z]$	1.2533	1.2545	1.0	0.9992	8.0	7.9826
$E[z^2]$	2.0	2.0036	1.0625	1.2496	16	16.2049
$variance$	0.4292	0.4297	0.25	0.2512	80	79.9261

TABLE II
SIMULATION AND THEORETICAL RESULTS OF EXPECTATIONS FOR NAKAGAMI-*m* DISTRIBUTIONS.

Order (k)	$m = 0.75, \Omega = 0.12$ Theo	Prac	$m = 1.5, \Omega = 0.375$ Theo	Prac	$m = 4.0, \Omega = 1.5$ Theo	Prac
$E[z]$	0.7395	0.7394	1.128	1.1283	1.939	1.9389
$E[z^2]$	0.7491	0.7491	1.4995	1.4995	4.007	4.007
$E[z^3]$	0.9213	0.9216	2.2545	2.2545	8.7284	8.7249
$E[z^4]$	1.3062	1.3037	3.7763	3.741	20.014	19.9985

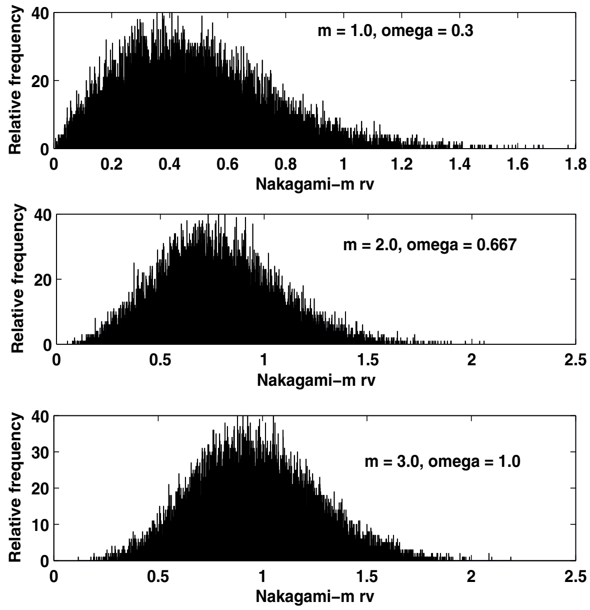

Fig. 3. Histogram of Nakagami-m pdf for integer values of m. Top: m = 1.0, middle: m = 2.0, bottom: m = 3.0 .

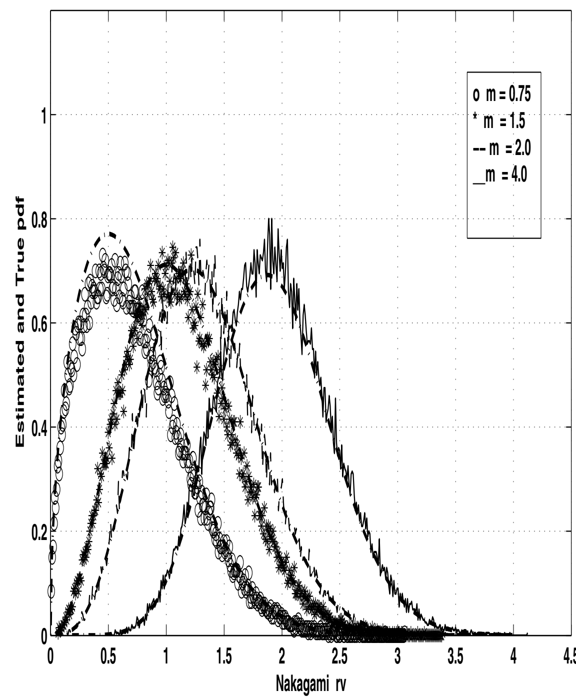

Fig. 4. Monte Carlo estimate and theoretical pdf for Nakagami-m rv. Dotted lines are for theoretical pdf.

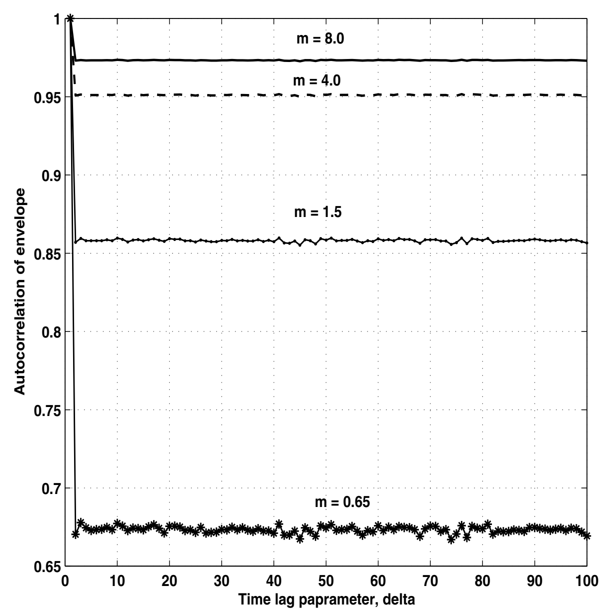

Fig. 5. Normalized autocorrelation of Nakagami-m random varaites for different values of m.

V. Conclusion

Monte Carlo simulation for Nakagami-m distributions with different values of fading parameter m varying from less than one to integer values are reported in the paper. Extensive simulation are done for Rayleigh, Gamma, chi-square, and Nakagami-m distributions, and histograms of relative frequency for each distribution are shown in the paper. Results of simulation for Nakagami-m pdf closely match the theoretical values of expectations from the distribution. Empirical results of autocorrelation among the samples of Nakagami-m distribution show strong corroboration of the effectiveness of simulation results shown in the paper.

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