Math TA Problem Set

Athanasios Grivakis

Spring 2024
Last update: September 5, 2024

Contents

1 Problem 1

Problem A string of length L inches is cut into two pieces. One piece is formed into a square, and the other piece is formed into a circle. Where should the string be cut in order to minimize the sums of the areas of the square and circle? USE CALCULUS.

Solution Suppose the string of length L is cut at x. Then, the two pieces form lengths of x and $L - x$, respectively. Wlog, suppose that the piece of length x forms the square, and the piece of length $L - x$ forms the circle. The area of the square will be the square of the length of the side, where the length of a side will be one-fourth that of the total length. Hence, the area of the square will be $\frac{x^2}{16}$. Now, the area of the circle will be πr^2, where r is the radius of the circle. Since the circumference of the circle will have length $L - x$, this means that $2\pi r = L - x$. Hence, $r = \frac{L-x}{2\pi}$, and the area of the circle is

$$\begin{aligned}
\pi r^2 &= \pi\left(\frac{L-x}{2\pi}\right)^2 \\
&= \frac{\pi(L-x)^2}{4\pi^2} \\
&= \frac{(L-x)^2}{4\pi}.
\end{aligned}$$

Thus, the sum of the areas of the square and the circle will be $\frac{x^2}{16} + \frac{(L-x)^2}{4\pi}$. We want this area minimized. Let $f(x) = \frac{x^2}{16} + \frac{(L-x)^2}{4\pi}$. Then,

$$\begin{aligned}
f'(x) &= \frac{x}{8} + \frac{2(L-x)(-1)}{4\pi} \\
&= \frac{x}{8} + \frac{x}{2\pi} - \frac{L}{2\pi} \\
&= \frac{\pi x + 4x}{8\pi} - \frac{L}{2\pi}.
\end{aligned}$$

Setting $f'(x) = 0$ gives

$$\begin{aligned}
&\frac{\pi x + 4x}{8\pi} = \frac{L}{2\pi} \\
\implies &\pi x + 4x = 8\pi \times \frac{L}{2\pi} \\
\implies &\pi x + 4x = 4L \\
\implies &(\pi + 4)x = 4L \\
\implies &x = \frac{4L}{\pi + 4}.
\end{aligned}$$

We see that $f''(x) = \frac{4+\pi}{8\pi}$, which is positive. This means that $f(x)$ has a local minimum at $x = \frac{4L}{\pi+4}$. Thus, the string should be cut at $x = \frac{4L}{\pi+4}$ inches to minimize the total area. Plugging in $\frac{4L}{\pi+4}$ to $f(x)$, we get that

$$\begin{aligned}
f\left(\frac{4L}{\pi+4}\right) &= \frac{\left(\frac{4L}{\pi+4}\right)^2}{16} + \frac{\left(L - \left(\frac{4L}{\pi+4}\right)\right)^2}{4\pi} \\
&= \frac{16L^2}{16(\pi+4)^2} + \frac{L^2 - \frac{8L}{\pi+4} + \frac{16L^2}{(\pi+4)^2}}{4\pi} \\
&= \frac{16L^2}{16(\pi+4)^2} + \frac{L^2}{4\pi} - \frac{8L}{4\pi(\pi+4)} + \frac{16L^2}{4\pi(\pi+4)^2} \\
&= 16\pi L^2 + 4(\pi+4)L^2 - 32(\pi+4)L + 64L^2 \\
&= (20\pi + 4)L^2 + (60 - 32\pi)L
\end{aligned}$$

is the minimal total area with a string of length L.

2 Problem 2

Problem Evaluate the integral $\int \frac{1}{\sqrt{e^{2x}-25}}\,dx$. Show work to justify all steps using techniques of integration. Using an integral table is not a sufficient explanation.

Solution We can use trigonometric substitution for this problem. Let $e^x = 5\sec\theta$. Then, $x = \ln(5\sec\theta)$ for $\theta \in (-\pi/2, \pi/2)$, and

$$\begin{aligned} dx &= \frac{1}{5\sec\theta} \times 5\sec\theta\tan\theta\,d\theta \\ &= \tan\theta\,d\theta. \end{aligned}$$

Next, we have

$$\begin{aligned} \int \frac{1}{\sqrt{e^{2x}-25}}\,dx &= \int \frac{\tan\theta\,d\theta}{\sqrt{25\sec^2\theta - 25}} \\ &= \int \frac{\tan\theta\,d\theta}{\sqrt{25(\sec^2\theta - 1)}} \\ &= \frac{1}{5}\int \frac{\tan\theta\,d\theta}{\sqrt{\tan^2\theta}} \\ &= \frac{1}{5}\int \frac{\tan\theta\,d\theta}{|\tan\theta|} \\ &= \frac{1}{5}\int \pm 1 d\theta \\ &= \frac{1}{5}\theta + C. \end{aligned}$$

where ± 1 depends on the value of θ. Notice that since $\tan\theta$ and θ have the same sign over $(\pi/2, \pi/2)$, the θ term we get after integrating eliminates the need for $\pm$. Since $e^x = 5\sec\theta$, it follows that $\theta = \arccos(5e^{-x})$. Therefore, the integral evaluates to $\frac{\arccos(5e^{-x})}{5} + C$.

3 Problem 3

Problem Let V be the set of vectors in $\mathbb{R}^3$ that have the form $\begin{bmatrix} a+2b \\ 2a-b \\ -a+b \end{bmatrix}$, where $a, b \in \mathbb{R}$.

(a) Prove that V is a subspace of $\mathbb{R}^3$.

(b) Find a basis for V.

Solution

(a) First, note that V is a subset of $\mathbb{R}^3$, since V only consists of vectors with three real components. To prove that V is a subspace of $\mathbb{R}^3$, we must show that V is

(i) nonempty and contains the zero vector

(ii) closed under scalar multiplication

(iii) closed under vector addition.

To show (i), we let $a = 0, b = 0$. Then, a and b are in $\mathbb{R}$, and we see that $\begin{bmatrix} 0+20 \\ 20-0 \\ -0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in V.

To show (ii), let $a, b, c \in \mathbb{R}$. Let $\vec{v} = \begin{bmatrix} a+2b \\ 2a-b \\ -a+b \end{bmatrix}$. Then, clearly $\vec{v} \in V$. We want to show that $c\vec{v} \in V$. Notice,

$$\begin{aligned} c\vec{v} &= c \begin{bmatrix} a+2b \\ 2a-b \\ -a+b \end{bmatrix} \\ &= \begin{bmatrix} c(a+2b) \\ c(2a-b) \\ c(-a+b) \end{bmatrix} \\ &= \begin{bmatrix} ac+2bc \\ 2ac-bc \\ -ac+bc \end{bmatrix}. \end{aligned}$$

Now, letting $d = ac, e = bc$, we get that

$$c\vec{v} = \begin{bmatrix} d+2e \\ 2d-e \\ -d+e \end{bmatrix}.$$

Since $d, e \in \mathbb{R}$, and $c\vec{v}$ is in the form of a vector in V, we have that $c\vec{v} \in V$.

To show (iii), let $a, b, d, e \in \mathbb{R}$. Then, let $\vec{v_1} = \begin{bmatrix} a+2b \\ 2a-b \\ -a+b \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} d+2e \\ 2d-e \\ -d+e \end{bmatrix}$. Clearly, $\vec{v_1}$ and $\vec{v_2}$ are in V. We want to show that $\vec{v_1} + \vec{v_2} \in V$. Notice,

$$\begin{aligned} \vec{v_1} + \vec{v_2} &= \begin{bmatrix} a+2b \\ 2a-b \\ -a+b \end{bmatrix} + \begin{bmatrix} d+2e \\ 2d-e \\ -d+e \end{bmatrix} \\ &= \begin{bmatrix} a+2b+d+2e \\ 2a-b+2d-e \\ -a+b-d+e \end{bmatrix} \\ &= \begin{bmatrix} a+d+2(b+e) \\ 2(a+d)-(b+e) \\ -(a+d)+b+e \end{bmatrix}. \end{aligned}$$

Let $f = a + d, g = b + e$. Then, $f, g \in \mathbb{R}$, and

$$\vec{v_1} + \vec{v_2} = \begin{bmatrix} f + 2g \\ 2f - g \\ -f + g \end{bmatrix}.$$

This is clearly in the form of a vector in V. Hence, $\vec{v_1} + \vec{v_2} \in V$.
Since $V \subseteq \mathbb{R}^3$, and since (i), (ii), and (iii) hold, we conclude that V is a subspace of $\mathbb{R}^3$.

(b) To find a basis for V, we must find a set of linearly independent vectors that span V. If we let $a = 1, b = 0$, then we get the vector $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$. Now if we let $a = 0, b = 1$, we get $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$. Since these are not scalar multiples of each other, they are linearly independent. Now, notice that any vector in V is of the form $\begin{bmatrix} a + 2b \\ 2a - b \\ -a + b \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ -a \end{bmatrix} + \begin{bmatrix} 2b \\ b \\ b \end{bmatrix}$, with $a, b \in \mathbb{R}$. This is a linear combination of two independent vectors in $\mathbb{R}^3$.
Therefore, V is a plane in $\mathbb{R}^3$, and a possible basis for V is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

4 Problem 4

Problem Use Green's Theorem to evaluate $\oint_C (3xy+y^2)\,dx+(2xy+5x^2)\,dy$, where $C:(x-1)^2+(y+2)^2=1$. Assume that the curve C is traversed in a counterclockwise manner.

Solution First, we recall Green's Theorem:

> **Definition.** (Green's Theorem)
> Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x,y) defined on an open region containing D and have continuous partial derivatives there, then
>
> $$\oint_C (L\,dx+M\,dy)=\iint_D\left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)\,dx\,dy$$
>
> where the path of integration along C is counterclockwise.

We are given that $C:(x-1)^2+(y+2)^2=1$. Notice that this is a unit circle with center $(1,-2)$ in the Cartesian plane. We see that C is a piecewise smooth, simple closed curve in the Cartesian plane. Since we are told that the curve C is traversed in a counterclockwise manner, C is positively oriented (the interior always faces left, while the exterior always faces right). We see that D is the unit disk bounded by C. Let $L=3xy+y^2$, and let $M=2xy+5x^2$. These functions are defined on all of the Cartesian plane, which is an open region containing D, and

$$\frac{\partial L}{\partial x}=3y \qquad \frac{\partial L}{\partial y}=3x+2y$$
$$\frac{\partial M}{\partial x}=2y+10x \qquad \frac{\partial M}{\partial y}=2x$$

are continuous everywhere in the Cartesian plane (they form planes in $\mathbb{R}^3$, which are continuous), and thus on D. Therefore, by Green's Theorem,

$$\oint_C (3xy+y^2)\,dx+(2xy+5x^2)\,dy=\iint_D (2y+10x-(3x+2y))\,dx\,dy.$$

Now, we are to evaluate the right side of the equality. We can simplify the right side integral to $7\iint_D x\,dx\,dy$. Recall that $D:\{(x,y)\in\mathbb{R}^2 \mid (x-1)^2+(y+2)^2\le 1\}$. We can convert D into a region D' described by polar coordinates. Since for our region D we have a disk of radius 1 centered at $(1,-2)$, we have

$$\begin{aligned}
r&=1\\
r^2&=1\\
r^2(\cos^2\theta+\sin^2\theta)&=1\\
r^2\cos^2\theta+r^2\sin^2\theta&=1\\
(r\cos\theta)^2+(r\sin\theta)^2&=1\\
((r\cos\theta+1)-1)^2+((r\sin\theta-2)+2)^2&=1\\
(x-1)^2+(y+2)^2&=1.
\end{aligned}$$

We can use the formulas $x(r,\theta)=r\cos\theta+1$, $y(r,\theta)=r\sin\theta-2$. As θ goes from 0 to 2π, x and y trace out a circle of radius r centered at $(1,-2)$. For example, we see that if $r=1$, then at $\theta=0$, $x=2$ and $y=-2$. Evaluating $(x(1,\theta),(y(1,\theta)))$ from $\theta=0$ to $\theta=2\pi$ traces out C. For our region D', we see that $0\le r\le 1$ and $0\le\theta<2\pi$. So, $D':\{(r,\theta):0\le r\le 1, 0\le\theta<2\pi\}$. We recall that we can perform a change of variables for integration using the following formula:

$$\iint_R f(x,y)\,dA=\iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA'$$

where under the transformation $x = g(u, v), y = h(u, v)$ the region S in the $u - v$ plane becomes the region R in the $x - y$ plane, and dA' is in terms of u and v. Note that $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian of the transformation $x = g(u, v), y = h(u, v)$ defined by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Let $dA = \, dx\, dy$. We see that, under the standard polar transformation $x = r\cos\theta, y = r\sin\theta$, we get

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta) \\
&= r\cos^2\theta - (-r\sin^2\theta) \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r(\cos^2\theta + \sin^2\theta) \\
&= r.
\end{aligned}$$

Thus, $dA = \, dx\, dy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \, dr\, d\theta = |r| \, dr\, d\theta = r\, dr\, d\theta = dA'$ (since $|r| = r$, as r is taken to be nonnegative). Hence, we see that

$$\iint_D f(x,y)\, dx\, dy = \iint_{D'} f(x(r,\theta), y(r,\theta))r\, dr\, d\theta.$$

We are ready to evaluate $7\iint_D x\, dx\, dy$. Recall that in our transformation from D' to D, we have $x(r,\theta) = r\cos\theta + 1$. So, by solving out, we get

$$\begin{aligned}
7\int_{-3}^{-1}\int_{-\sqrt{1-(y+2)^2}+1}^{\sqrt{1-(y+2)^2}+1} x\, dx\, dy &= 7\int_0^{2\pi}\int_0^1 (r\cos\theta + 1)r\, dr\, d\theta \\
&= 7\int_0^{2\pi}\int_0^1 (r^2\cos\theta + r)\, dr\, d\theta \\
&= 7\int_0^{2\pi}\left[\frac{r^3\cos\theta}{3} + \frac{r^2}{2}\right]\Bigg|_0^1 d\theta \\
&= 7\int_0^{2\pi}\left(\frac{\cos\theta}{3} + \frac{1}{2}\right) d\theta \\
&= 7\left[\frac{\sin\theta}{3} + \frac{\theta}{2}\right]\Bigg|_0^{2\pi} \\
&= 7\pi.
\end{aligned}$$

Hence, by Green's Theorem, $\oint_C (3xy + y^2)\, dx + (2xy + 5x^2)\, dy = 7\pi$.

If we wanted to directly calculate $\oint_C (3xy + y^2)\, dx + (2xy + 5x^2)\, dy$, we would use the fact that the functions $x(1,\theta) = x(\theta) = \cos\theta + 1$, $y(1,\theta) = y(\theta) = \sin\theta - 2$ trace out C as θ varies from 0 to 2π (this follows from our earlier reasoning). We also see that $\frac{dx}{d\theta} = -\sin\theta$, $\frac{dy}{d\theta} = \cos\theta$. Let $I = \oint_C (3xy + y^2)\,dx + (2xy + 5x^2)\,dy$.

$$
\begin{aligned}
I &= \int_0^{2\pi} (3(\cos\theta+1)(\sin\theta-2) + (\sin\theta-2)^2)(-\sin\theta\,d\theta) + (2(\cos\theta+1)(\sin\theta-2) + 5(\cos\theta+1)^2)(\cos\theta\,d\theta) \\
&= \int_0^{2\pi} (3\cos\theta\sin\theta - 6\cos\theta + 3\sin\theta - 6 + \sin^2\theta - 4\sin\theta + 4)(-\sin\theta\,d\theta) \\
&+ \int_0^{2\pi} (2\cos\theta\sin\theta - 4\cos\theta + 2\sin\theta - 4 + 5\cos^2\theta + 10\cos\theta + 5)(\cos\theta\,d\theta) \\
&= \int_0^{2\pi} (-3\cos\theta\sin^2\theta + 6\cos\theta\sin\theta - 3\sin^2\theta + 6\sin\theta - \sin^3\theta + 4\sin^2\theta - 4\sin\theta)\,d\theta \\
&+ \int_0^{2\pi} (2\cos^2\theta\sin\theta - 4\cos^2\theta + 2\sin\theta\cos\theta - 4\cos\theta + 5\cos^3\theta + 10\cos^2\theta + 5\cos\theta)\,d\theta \\
&= \int_0^{2\pi} -3\cos\theta\sin^2\theta\,d\theta + \int_0^{2\pi} 6\cos\theta\sin\theta\,d\theta - \int_0^{2\pi} 3\sin^2\theta\,d\theta \\
&+ \int_0^{2\pi} 6\sin\theta\,d\theta - \int_0^{2\pi} \sin^3\theta\,d\theta + \int_0^{2\pi} 4\sin^2\theta\,d\theta - \int_0^{2\pi} 4\sin\theta\,d\theta \\
&+ \int_0^{2\pi} 2\cos^2\theta\sin\theta\,d\theta - \int_0^{2\pi} 4\cos^2\theta\,d\theta + \int_0^{2\pi} 2\sin\theta\cos\theta\,d\theta \\
&- \int_0^{2\pi} 4\cos\theta\,d\theta + \int_0^{2\pi} 5\cos^3\theta\,d\theta + \int_0^{2\pi} 10\cos^2\theta\,d\theta + \int_0^{2\pi} 5\cos\theta\,d\theta \\
&= \int_0^{2\pi} -3\cos\theta\sin^2\theta\,d\theta + \int_0^{2\pi} 8\cos\theta\sin\theta\,d\theta + \int_0^{2\pi} \sin^2\theta\,d\theta + \int_0^{2\pi} 2\sin\theta\,d\theta - \int_0^{2\pi} \sin^3\theta\,d\theta \\
&+ \int_0^{2\pi} 2\cos^2\theta\sin\theta\,d\theta + \int_0^{2\pi} 6\cos^2\theta\,d\theta + \int_0^{2\pi} \cos\theta\,d\theta + \int_0^{2\pi} 5\cos^3\theta\,d\theta \\
&= 0 + 0 + \pi + 0 + 0 + 0 + 6\pi + 0 + 0 \\
&= 7\pi
\end{aligned}
$$

which is what we obtained from Green's Theorem.