

# Math TA Problem Set

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# 1 Problem 1

**Problem** A string of length  $L$  inches is cut into two pieces. One piece is formed into a square, and the other piece is formed into a circle. Where should the string be cut in order to minimize the sums of the areas of the square and circle? USE CALCULUS.

**Solution** Suppose the string of length  $L$  is cut at  $x$ . Then, the two pieces form lengths of  $x$  and  $L - x$ , respectively. Wlog, suppose that the piece of length  $x$  forms the square, and the piece of length  $L - x$  forms the circle. The area of the square will be the square of the length of the side, where the length of a side will be one-fourth that of the total length. Hence, the area of the square will be  $\frac{x^2}{16}$ . Now, the area of the circle will be  $\pi r^2$ , where  $r$  is the radius of the circle. Since the circumference of the circle will have length  $L - x$ , this means that  $2\pi r = L - x$ . Hence,  $r = \frac{L-x}{2\pi}$ , and the area of the circle is

$$\begin{aligned}\pi r^2 &= \pi \left( \frac{L-x}{2\pi} \right)^2 \\ &= \frac{\pi(L-x)^2}{4\pi^2} \\ &= \frac{(L-x)^2}{4\pi}.\end{aligned}$$

Thus, the sum of the areas of the square and the circle will be  $\frac{x^2}{16} + \frac{(L-x)^2}{4\pi}$ . We want this area minimized. Let  $f(x) = \frac{x^2}{16} + \frac{(L-x)^2}{4\pi}$ . Then,

$$\begin{aligned}f'(x) &= \frac{x}{8} + \frac{2(L-x)(-1)}{4\pi} \\ &= \frac{x}{8} + \frac{x}{2\pi} - \frac{L}{2\pi} \\ &= \frac{\pi x + 4x}{8\pi} - \frac{L}{2\pi}.\end{aligned}$$

Setting  $f'(x) = 0$  gives

$$\begin{aligned}\frac{\pi x + 4x}{8\pi} &= \frac{L}{2\pi} \\ \implies \pi x + 4x &= 8\pi \times \frac{L}{2\pi} \\ \implies \pi x + 4x &= 4L \\ \implies (\pi + 4)x &= 4L \\ \implies x &= \frac{4L}{\pi + 4}.\end{aligned}$$

We see that  $f''(x) = \frac{4+\pi}{8\pi}$ , which is positive. This means that  $f(x)$  has a local minimum at  $x = \frac{4L}{\pi+4}$ . Thus, the string should be cut at  $x = \frac{4L}{\pi+4}$  inches to minimize the total area. Plugging in  $\frac{4L}{\pi+4}$  to  $f(x)$ , we get that

$$\begin{aligned}f\left(\frac{4L}{\pi+4}\right) &= \frac{\left(\frac{4L}{\pi+4}\right)^2}{16} + \frac{\left(L - \left(\frac{4L}{\pi+4}\right)\right)^2}{4\pi} \\ &= \frac{16L^2}{16(\pi+4)^2} + \frac{L^2 - \frac{8L}{\pi+4} + \frac{16L^2}{(\pi+4)^2}}{4\pi} \\ &= \frac{16L^2}{16(\pi+4)^2} + \frac{L^2}{4\pi} - \frac{8L}{4\pi(\pi+4)} + \frac{16L^2}{4\pi(\pi+4)^2} \\ &= 16\pi L^2 + 4(\pi+4)L^2 - 32(\pi+4)L + 64L^2 \\ &= (20\pi+4)L^2 + (60-32\pi)L\end{aligned}$$

is the minimal total area with a string of length  $L$ .

## 2 Problem 2

**Problem** Evaluate the integral  $\int \frac{1}{\sqrt{e^{2x}-25}} dx$ . Show work to justify all steps using techniques of integration. Using an integral table is not a sufficient explanation.

**Solution** We can use trigonometric substitution for this problem. Let  $e^x = 5 \sec \theta$ . Then,  $x = \ln(5 \sec \theta)$  for  $\theta \in (-\pi/2, \pi/2)$ , and

$$\begin{aligned} dx &= \frac{1}{5 \sec \theta} \times 5 \sec \theta \tan \theta d\theta \\ &= \tan \theta d\theta. \end{aligned}$$

Next, we have

$$\begin{aligned} \int \frac{1}{\sqrt{e^{2x}-25}} dx &= \int \frac{\tan \theta d\theta}{\sqrt{25 \sec^2 \theta - 25}} \\ &= \int \frac{\tan \theta d\theta}{\sqrt{25(\sec^2 \theta - 1)}} \\ &= \frac{1}{5} \int \frac{\tan \theta d\theta}{\sqrt{\tan^2 \theta}} \\ &= \frac{1}{5} \int \frac{\tan \theta d\theta}{\tan \theta} \\ &= \frac{1}{5} \int d\theta \\ &= \frac{1}{5} \theta + C. \end{aligned}$$

Since  $e^x = 5 \sec \theta$ , it follows that  $\theta = \arccos(5e^{-x})$ . Therefore, the integral evaluates to  $\frac{\arccos(5e^{-x})}{5} + C$ .

### 3 Problem 3

**Problem** Let  $V$  be the set of vectors in  $\mathbb{R}^3$  that have the form  $\begin{bmatrix} a + 2b \\ 2a - b \\ -a + b \end{bmatrix}$ , where  $a, b \in \mathbb{R}$ .

- (a) Prove that  $V$  is a subspace of  $\mathbb{R}^3$ .
- (b) Find a basis for  $V$ .

**Solution**

(a) First, note that  $V$  is a subset of  $\mathbb{R}^3$ , since  $V$  only consists of vectors with three real components. To prove that  $V$  is a subspace of  $\mathbb{R}^3$ , we must show that  $V$  is

- (i) nonempty and contains the zero vector
- (ii) closed under scalar multiplication
- (iii) closed under vector addition.

To show (i), we let  $a = 0, b = 0$ . Then,  $a$  and  $b$  are in  $\mathbb{R}$ , and we see that  $\begin{bmatrix} 0 + 2 \cdot 0 \\ 2 \cdot 0 - 0 \\ -0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  is in  $V$ .

To show (ii), let  $a, b, c \in \mathbb{R}$ . Let  $\vec{v} = \begin{bmatrix} a + 2b \\ 2a - b \\ -a + b \end{bmatrix}$ . Then, clearly  $\vec{v} \in V$ . We want to show that  $c\vec{v} \in V$ . Notice,

$$\begin{aligned} c\vec{v} &= c \begin{bmatrix} a + 2b \\ 2a - b \\ -a + b \end{bmatrix} \\ &= \begin{bmatrix} c(a + 2b) \\ c(2a - b) \\ c(-a + b) \end{bmatrix} \\ &= \begin{bmatrix} ac + 2bc \\ 2ac - bc \\ -ac + bc \end{bmatrix}. \end{aligned}$$

Now, letting  $d = ac, e = bc$ , we get that

$$c\vec{v} = \begin{bmatrix} d + 2e \\ 2d - e \\ -d + e \end{bmatrix}.$$

Since  $d, e \in \mathbb{R}$ , and  $c\vec{v}$  is in the form of a vector in  $V$ , we have that  $c\vec{v} \in V$ .

To show (iii), let  $a, b, d, e \in \mathbb{R}$ . Then, let  $\vec{v}_1 = \begin{bmatrix} a + 2b \\ 2a - b \\ -a + b \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} d + 2e \\ 2d - e \\ -d + e \end{bmatrix}$ . Clearly,  $\vec{v}_1$  and  $\vec{v}_2$  are in  $V$ . We want to show that  $\vec{v}_1 + \vec{v}_2 \in V$ . Notice,

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &= \begin{bmatrix} a + 2b \\ 2a - b \\ -a + b \end{bmatrix} + \begin{bmatrix} d + 2e \\ 2d - e \\ -d + e \end{bmatrix} \\ &= \begin{bmatrix} a + 2b + d + 2e \\ 2a - b + 2d - e \\ -a + b - d + e \end{bmatrix} \\ &= \begin{bmatrix} a + d + 2(b + e) \\ 2(a + d) - (b + e) \\ -(a + d) + b + e \end{bmatrix}. \end{aligned}$$

Let  $f = a + d, g = b + e$ . Then,  $f, g \in \mathbb{R}$ , and

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} f + 2g \\ 2f - g \\ -f + g \end{bmatrix}.$$

This is clearly in the form of a vector in  $V$ . Hence,  $\vec{v}_1 + \vec{v}_2 \in V$ .

Since  $V \subseteq \mathbb{R}^3$ , and since (i), (ii), and (iii) hold, we conclude that  $V$  is a subspace of  $\mathbb{R}^3$ .

(b) To find a basis for  $V$ , we must find a set of linearly independent vectors that span  $V$ . If we let  $a = 1, b = 0$ , then we get the vector  $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ . Now if we let  $a = 0, b = 1$ , we get  $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ . Since these are not scalar multiples of each other, they are linearly independent. Now, notice that any vector in  $V$  is of the form  $\begin{bmatrix} a + 2b \\ 2a - b \\ -a + b \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ -a \end{bmatrix} + \begin{bmatrix} 2b \\ -b \\ b \end{bmatrix}$ , with  $a, b \in \mathbb{R}$ . This is a linear combination of two independent vectors in  $\mathbb{R}^3$ .

Therefore,  $V$  is a plane in  $\mathbb{R}^3$ , and a possible basis for  $V$  is  $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$ .

## 4 Problem 4

**Problem** Use Green's Theorem to evaluate  $\oint_C (3xy + y^2) dx + (2xy + 5x^2) dy$ , where  $C : (x-1)^2 + (y+2)^2 = 1$ . Assume that the curve  $C$  is traversed in a counterclockwise manner.

**Solution** First, we recall Green's Theorem:

**Definition.** (Green's Theorem)

Let  $C$  be a positively oriented, piecewise smooth, simple closed curve in a plane, and let  $D$  be the region bounded by  $C$ . If  $L$  and  $M$  are functions of  $(x, y)$  defined on an open region containing  $D$  and have continuous partial derivatives there, then

$$\oint_C (L dx + M dy) = \iint_D \left( \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy$$

where the path of integration along  $C$  is counterclockwise.

We are given that  $C : (x-1)^2 + (y+2)^2 = 1$ . Notice that this is a unit circle with center  $(1, -2)$  in the Cartesian plane. We see that  $C$  is a piecewise smooth, simple closed curve in the Cartesian plane. Since we are told that the curve  $C$  is traversed in a counterclockwise manner,  $C$  is positively oriented (the interior always faces left, while the exterior always faces right). We see that  $D$  is the unit disk bounded by  $C$ . Let  $L = 3xy + y^2$ , and let  $M = 2xy + 5x^2$ . These functions are defined on all of the Cartesian plane, which is an open region containing  $D$ , and

$$\frac{\partial L}{\partial x} = 3y$$

$$\frac{\partial L}{\partial y} = 3x + 2y$$

$$\frac{\partial M}{\partial x} = 2y + 10x$$

$$\frac{\partial M}{\partial y} = 2x$$

are continuous everywhere in the Cartesian plane (they form planes in  $\mathbb{R}^3$ , which are continuous), and thus on  $D$ . Therefore, by Green's Theorem,

$$\oint_C (3xy + y^2) dx + (2xy + 5x^2) dy = \iint_D (2y + 10x - (3x + 2y)) dx dy.$$

Now, we are to evaluate the right side of the equality. We can simplify the right side integral to  $7 \iint_D x dx dy$ . Recall that  $D : \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + (y+2)^2 \leq 1\}$ . We can convert  $D$  into a region  $D'$  described by polar coordinates. Since for our region  $D$  we have a disk of radius 1 centered at  $(1, -2)$ , we have

$$r = 1$$

$$r^2 = 1$$

$$r^2(\cos^2 \theta + \sin^2 \theta) = 1$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = 1$$

$$(r \cos \theta)^2 + (r \sin \theta)^2 = 1$$

$$((r \cos \theta + 1) - 1)^2 + ((r \sin \theta - 2) + 2)^2 = 1$$

$$(x - 1)^2 + (y + 2)^2 = 1.$$

We can use the formulas  $x(r, \theta) = r \cos \theta + 1$ ,  $y(r, \theta) = r \sin \theta - 2$ . As  $\theta$  goes from 0 to  $2\pi$ ,  $x$  and  $y$  trace out a circle of radius  $r$  centered at  $(1, -2)$ . For example, we see that if  $r = 1$ , then at  $\theta = 0$ ,  $x = 2$  and  $y = -2$ . Evaluating  $(x(1, \theta), (y(1, \theta)))$  from  $\theta = 0$  to  $\theta = 2\pi$  traces out  $C$ . For our region  $D'$ , we see that  $0 \leq r \leq 1$  and  $0 \leq \theta < 2\pi$ . So,  $D' : \{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta < 2\pi\}$ . We recall that we can perform a change of variables for integration using the following formula:

$$\iint_R f(x, y) dA = \iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA'$$

where under the transformation  $x = g(u, v)$ ,  $y = h(u, v)$  the region  $S$  in the  $u - v$  plane becomes the region  $R$  in the  $x - y$  plane, and  $dA'$  is in terms of  $u$  and  $v$ . Note that  $\frac{\partial(x, y)}{\partial(u, v)}$  is the Jacobian of the transformation  $x = g(u, v)$ ,  $y = h(u, v)$  defined by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Let  $dA = dx dy$ . We see that, under the standard polar transformation  $x = r \cos \theta$ ,  $y = r \sin \theta$ , we get

$$\begin{aligned} \frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= (\cos \theta)(r \cos \theta) - (\sin \theta)(-r \sin \theta) \\ &= r \cos^2 \theta - (-r \sin^2 \theta) \\ &= r \cos^2 \theta + r \sin^2 \theta \\ &= r(\cos^2 \theta + \sin^2 \theta) \\ &= r. \end{aligned}$$

Thus,  $dA = dx dy = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr d\theta = |r| dr d\theta = r dr d\theta = dA'$  (since  $|r| = r$ , as  $r$  is taken to be nonnegative). Hence, we see that

$$\iint_D f(x, y) dx dy = \iint_{D'} f(x(r, \theta), y(r, \theta)) r dr d\theta.$$

We are ready to evaluate  $7 \iint_D x dx dy$ . Recall that in our transformation from  $D'$  to  $D$ , we have  $x(r, \theta) = r \cos \theta + 1$ . So, by solving out, we get

$$\begin{aligned} 7 \int_{-3}^{-1} \int_{-\sqrt{1-(y+2)^2+1}}^{\sqrt{1-(y+2)^2+1}} x dx dy &= 7 \int_0^{2\pi} \int_0^1 (r \cos \theta + 1) r dr d\theta \\ &= 7 \int_0^{2\pi} \int_0^1 (r^2 \cos \theta + r) dr d\theta \\ &= 7 \int_0^{2\pi} \left[ \frac{r^3 \cos \theta}{3} + \frac{r^2}{2} \right] \Big|_0^1 d\theta \\ &= 7 \int_0^{2\pi} \left( \frac{\cos \theta}{3} + \frac{1}{2} \right) d\theta \\ &= 7 \left[ \frac{\sin \theta}{3} + \frac{\theta}{2} \right] \Big|_0^{2\pi} \\ &= 7\pi. \end{aligned}$$

Hence, by Green's Theorem,  $\oint_C (3xy + y^2) dx + (2xy + 5x^2) dy = 7\pi$ .

If we wanted to directly calculate  $\oint_C (3xy + y^2) dx + (2xy + 5x^2) dy$ , we would use the fact that the functions  $x(1, \theta) = x(\theta) = \cos \theta + 1$ ,  $y(1, \theta) = y(\theta) = \sin \theta - 2$  trace out  $C$  as  $\theta$  varies from 0 to  $2\pi$  (this follows from

our earlier reasoning). We also see that  $\frac{dx}{d\theta} = -\sin \theta$ ,  $\frac{dy}{d\theta} = \cos \theta$ . Let  $I = \oint_C (3xy + y^2) dx + (2xy + 5x^2) dy$ .

$$\begin{aligned}
 I &= \int_0^{2\pi} (3(\cos \theta + 1)(\sin \theta - 2) + (\sin \theta - 2)^2)(-\sin \theta d\theta) + (2(\cos \theta + 1)(\sin \theta - 2) + 5(\cos \theta + 1)^2)(\cos \theta d\theta) \\
 &= \int_0^{2\pi} (3 \cos \theta \sin \theta - 6 \cos \theta + 3 \sin \theta - 6 + \sin^2 \theta - 4 \sin \theta + 4)(-\sin \theta d\theta) \\
 &\quad + \int_0^{2\pi} (2 \cos \theta \sin \theta - 4 \cos \theta + 2 \sin \theta - 4 + 5 \cos^2 \theta + 10 \cos \theta + 5)(\cos \theta d\theta) \\
 &= \int_0^{2\pi} (-3 \cos \theta \sin^2 \theta + 6 \cos \theta \sin \theta - 3 \sin^2 \theta + 6 \sin \theta - \sin^3 \theta + 4 \sin^2 \theta - 4 \sin \theta) d\theta \\
 &\quad + \int_0^{2\pi} (2 \cos^2 \theta \sin \theta - 4 \cos^2 \theta + 2 \sin \theta \cos \theta - 4 \cos \theta + 5 \cos^3 \theta + 10 \cos^2 \theta + 5 \cos \theta) d\theta \\
 &= \int_0^{2\pi} -3 \cos \theta \sin^2 \theta d\theta + \int_0^{2\pi} 6 \cos \theta \sin \theta d\theta - \int_0^{2\pi} 3 \sin^2 \theta d\theta \\
 &\quad + \int_0^{2\pi} 6 \sin \theta d\theta - \int_0^{2\pi} \sin^3 \theta d\theta + \int_0^{2\pi} 4 \sin^2 \theta d\theta - \int_0^{2\pi} 4 \sin \theta d\theta \\
 &\quad + \int_0^{2\pi} 2 \cos^2 \theta \sin \theta d\theta - \int_0^{2\pi} 4 \cos^2 \theta d\theta + \int_0^{2\pi} 2 \sin \theta \cos \theta d\theta \\
 &\quad - \int_0^{2\pi} 4 \cos \theta d\theta + \int_0^{2\pi} 5 \cos^3 \theta d\theta + \int_0^{2\pi} 10 \cos^2 \theta d\theta + \int_0^{2\pi} 5 \cos \theta d\theta \\
 &= \int_0^{2\pi} -3 \cos \theta \sin^2 \theta d\theta + \int_0^{2\pi} 8 \cos \theta \sin \theta d\theta + \int_0^{2\pi} \sin^2 \theta d\theta + \int_0^{2\pi} 2 \sin \theta d\theta - \int_0^{2\pi} \sin^3 \theta d\theta \\
 &\quad + \int_0^{2\pi} 2 \cos^2 \theta \sin \theta d\theta + \int_0^{2\pi} 6 \cos^2 \theta d\theta + \int_0^{2\pi} \cos \theta d\theta + \int_0^{2\pi} 5 \cos^3 \theta d\theta \\
 &= 0 + 0 + \pi + 0 + 0 + 0 + 6\pi + 0 + 0 \\
 &= 7\pi
 \end{aligned}$$

which is what we obtained from Green's Theorem.