A Comprehensive Report on Game Theory

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Contents

1 Introduction to Game Theory

Game Theory is a powerful framework for analyzing strategic interactions among rational decision-makers. In this mid-term report, we will provide an extensive overview of various topics covered thus far, exploring key concepts and insights gained. We will examine the interplay between these topics, highlighting the fundamental principles and techniques that underpin the field of Game Theory.

2 Assumptions in Game Theory

Before delving into specific topics, it is essential to understand the key assumptions that form the basis of Game Theory. These assumptions provide a framework for modeling strategic interactions and enable the analysis of rational decision-making. The main assumptions in Game Theory include:

2.1 Rationality

Players are assumed to be rational decision-makers who aim to maximize their own utility or payoff. They evaluate the available strategies and make choices that lead to the most favorable outcomes for themselves.

2.2 Common Knowledge

Players have access to the same information and knowledge about the game, including the rules, available strategies, and payoffs. They are aware that other players are rational and possess the same knowledge.

2.3 Sequential and Simultaneous Moves

Games can be classified as either sequential or simultaneous. In sequential games, players take turns making decisions, whereas in simultaneous games, players make decisions simultaneously without knowing the choices of others.

2.4 Complete Information

Players have complete information about the game, including the available strategies, payoffs, and the order of moves. They have a full understanding of the game structure and the implications of different actions.

3 Strategic Analysis: Dominance and Nash Equilibrium

3.1 The Prisoner's Dilemma and Strict Dominance

The Prisoner's Dilemma serves as a fundamental example of strategic interaction. By examining the concept of strict dominance, we can identify and eliminate strategies that are always inferior, simplifying the analysis of games and revealing dominant strategies.

	Confess	Don't Confess
Confess	(-5, -5)	(-1, -10)
Don't Confess	(-10, -1)	(-2, -2)

Table 1: Prisoner's Dilemma

3.2 Iterated Elimination of Strictly Dominated Strategies

Through the process of iteratively eliminating strategies that are dominated at each stage of the game, we can refine our analysis and focus on strategies that are rational and viable.

3.3 Battle of the Sexes

The Battle of the Sexes game illustrates the concept of coordination failure. Players have different preferences over outcomes, leading to multiple Nash equilibria, including one where players coordinate on different strategies.

3.4 Pure Strategy Nash Equilibrium and the Stag Hunt

Pure strategy Nash equilibrium represents a stable outcome in a game where no player has an incentive to unilaterally deviate. The Stag Hunt game exemplifies the concept of coordination and highlights the importance of mutual cooperation for achieving desirable outcomes.

	Stag	Hare
Stag	(4, 4)	(0, 3)
Hare	(3, 0)	(2, 2)

Table 2: Stag Hunt

3.5 Best Responses and Mixed Strategy Nash Equilibrium

Understanding best responses is crucial for identifying Nash equilibria in games with mixed strategies. We explore the concept of mixed strategy Nash equilibrium and the associated matching algorithms that determine the probabilities players assign to different strategies.

	Heads	Tails
Heads	(1, -1)	(-1, 1)
Tails	(-1, 1)	(1, -1)

Table 3: Matching Pennies

To find the mixed strategy Nash equilibrium in the Matching Pennies game, we solve for the probabilities p and $1-p$ that each player assigns to their respective strategies. The best response for Player 1 is given by:

$$\frac{\partial u_1}{\partial p} = \frac{\partial((1,-1)p + (-1,1)(1-p))}{\partial p} = 2p - 1$$

Setting this expression to zero, we find that $p = \frac{1}{2}$ is a best response for Player 1. Similarly, the best response for Player 2 is given by:

$$\frac{\partial u_2}{\partial p} = \frac{\partial((1,-1)p + (-1,1)(1-p))}{\partial p} = 1 - 2p$$

Setting this expression to zero, we find that $p = \frac{1}{2}$ is also a best response for Player 2. Therefore, the mixed strategy Nash equilibrium is $p = \frac{1}{2}$ for both players.

4 Subgame Perfect Equilibrium and Induction

4.1 Backward Induction

Backward induction is a powerful technique that allows us to solve sequential games by analyzing the optimal choices at each stage, working backward from the final decision. By identifying subgame perfect equilibria, we can ascertain optimal strategies in complex games with multiple stages.

4.2 Problems and Limitations of Backward Induction

While backward induction is a valuable tool, it faces challenges in certain situations. We examine the limitations and potential pitfalls of backward induction, including cases where it fails to provide unique or reasonable solutions.

4.3 Forward Induction

In contrast to backward induction, forward induction focuses on predicting player behavior in dynamic games with incomplete information. By considering a player's reasoning about others' future actions, we can make strategic predictions that align with observed outcomes.

5 Strategic Concepts and Applications

5.1 Commitment Problems and Punishment Strategies

Commitment problems arise when players cannot credibly commit to their proposed actions. We explore strategies such as tying hands and burning bridges that enable players to overcome commitment problems and achieve better outcomes.

5.2 The Centipede Game and Time-Dependent Strategies

The Centipede Game challenges traditional backward induction reasoning by introducing time-dependent strategies. We examine the implications of this game and the strategic considerations that arise from players' decision-making over time.

5.3 Probability Distributions and Comparative Statics

Probability distributions play a crucial role in analyzing uncertain outcomes in games. We delve into the concept of comparative statics, which explores how changes in exogenous factors impact equilibrium predictions and players' strategies.

5.4 The Support of Mixed Strategies and Equilibrium Stability

The support of mixed strategies refers to the range of possible actions players choose with positive probability. Understanding the support helps determine the stability of equilibrium outcomes and the likelihood of players deviating from their chosen strategies.

6 Auctions and Expected Utility Theory

6.1 Second Price Auctions

Definition: In a second-price auction, bidders submit sealed bids, and the highest bidder wins the item but pays the second-highest bid price.

Example: Let's consider an auction for a smartphone. Bidder A bids $600, Bidder B bids $800, and Bidder C bids $700. Bidder B wins the smartphone and pays $700 (the second-highest bid).

6.2 Expected Utility Theory

Definition: Expected utility theory combines the probabilities of outcomes with the associated utilities to determine the best course of action.

Mathematical Equation:

$$EU = \sum(Probability\, of\, Outcome_i \times Utility\, of\, Outcome_i)$$

7 Repeated Prisoner's Dilemma and Strategies

7.1 Repeated Prisoner's Dilemma (Finite)

Definition: The repeated prisoner's dilemma involves two players repeatedly facing the classic prisoner's dilemma, forcing players to consider the future impact of their decisions.

Example: In a repeated prisoner's dilemma, Player 1 and Player 2 have the choice to cooperate (C) or betray (D) in each round. The payoff matrix is as follows:

	C (Cooperate)	**D (Betray)**
C	3, 3	0, 5
D	5, 0	1, 1

7.2 Grim Trigger Strategy

Description: Grim trigger is a simple strategy where a player cooperates until the opponent defects. After that, the player defects in all subsequent rounds, leading to a cycle of mutual defection.

Example: In a repeated prisoner's dilemma, if Player 1 and Player 2 both start by cooperating, and Player 2 defects at some point, Player 1 will also defect in all subsequent rounds.

7.3 Tit-for-Tat Strategy

Description: Tit-for-Tat starts with cooperation and then mirrors the opponent's last move. It forgives and returns to cooperation after one mutual defection, promoting a cycle of cooperation.

Example: In a repeated prisoner's dilemma, if both players start by cooperating, and Player 2 defects in round 2, Player 1 will also defect in round 2. However, if Player 2 cooperates in round 3, Player 1 will also cooperate in round 3.

8 Bayesian Nash Equilibrium and Incomplete Information

8.1 Bayesian Nash Equilibrium

Definition: Bayesian Nash equilibrium extends the concept of Nash equilibrium to games with incomplete information, where players have private information and update their beliefs.

Example: Let's consider a sealed-bid auction for a rare coin, where bidders have private information about the coin's authenticity. Each bidder assigns a probability to the coin being authentic or counterfeit based on their private information.

8.2 Incomplete Information Games

Description: Incomplete information games model situations where players are uncertain about the type or strategy of their opponents, leading to strategic challenges.

Example: A seller has two types of used cars, a "good" car with a high value and a "bad" car with a low value. The buyer does not know which type of car the seller is offering. The seller decides whether to sell the "good" or "bad" car to maximize their profit, while the buyer tries to infer the true quality of the car.

9 Equilibrium Concepts in Game Theory

9.1 Perfect Bayesian Equilibrium

Definition: Perfect Bayesian equilibrium is a refinement of Bayesian Nash equilibrium, where players' strategies are consistent with their beliefs, and off-path beliefs are correct.

Example: In an incomplete information game, a perfect Bayesian equilibrium occurs when each player's strategy is optimal given their beliefs, and their beliefs are consistent with Bayes' rule.

9.2 Screening Games and Adverse Selection

Description: Screening games involve one player, the "principal," attempting to learn information about the other player, the "agent." Adverse selection refers to a situation where one party has more information than the other in a transaction, leading to potentially negative consequences for the party with less information.

Example: In the context of insurance, the insurance company (principal) designs contracts to screen customers (agents) based on their risk profiles. High-risk customers are charged higher premiums to protect the company from adverse selection.

10 Signaling Games and Equilibrium Concepts

10.1 Signaling Games

Definition: Signaling games involve one player, the "sender," sending a signal to reveal private information to the other player, the "receiver."

Example: In a job market, candidates with different abilities want to convey their true ability level to potential employers. High-ability candidates may choose to invest in education to signal their competence.

10.2 Separating Equilibrium and Pooling Equilibrium

Description: In signaling games, players can reach either a separating equilibrium, where types reveal their true information, or a pooling equilibrium, where multiple types converge to a single action.

Example: In the job market, a separating equilibrium occurs when high-ability candidates invest in education, and low-ability candidates do not. A pooling equilibrium happens when all candidates choose not to invest in education, making it difficult for employers to distinguish between high and low ability.

11 Risk Attitudes and Decision Making

11.1 Risk Averse, Risk Neutral, and Risk Acceptant

Description: Decision-makers have varying attitudes towards risk. Risk-averse individuals prefer lower-risk options, while risk-neutral individuals are indifferent, and risk-acceptant individuals favor higher-risk choices.

Example: Consider three investors with different risk attitudes. Investor X is risk-averse, Investor Y is risk-neutral, and Investor Z is risk-acceptant.

Investor	Risk Attitude	Option A (Expected Return)	Option B (Expected Return)
X	Risk-Averse	\$5000 (Utility = 400)	\$3000 (Utility = 450)
Y	Risk-Neutral	\$4000	\$4000
Z	Risk-Acceptant	\$3500 (Utility = 300)	\$5000 (Utility = 350)

12 The Winner's Curse

12.1 The Winner's Curse

Description: The winner's curse occurs when the winning bidder in an auction overestimates the value of the won item due to information asymmetry.

Example: In an art auction, multiple bidders compete for a rare painting. The winning bidder may end up overpaying due to an overly optimistic estimate of the painting's value compared to other bidders.

13 Poker and Information Asymmetry

13.1 The Role of Antes in Poker

Description: Antes in poker ensure that there is a minimum pot size, encouraging players to participate actively in the game.

Example: In a poker game, antes are a forced bet that every player must contribute to the pot before the hand starts. This creates an initial pot size and encourages players to remain engaged in the game.

14 The Beer Quiche Game

Definition: The beer quiche game is a classic example in game theory where players' preferences for beer and quiche result in a paradoxical outcome.

Example: Suppose there are two players, Player A and Player B, who have preferences as follows:

	Player A's Preferences	Player B's Preferences
Beer	3, 0	0, 3
Quiche	0, 3	3, 0

If Player A chooses beer, Player B will choose quiche, and vice versa, leading to a paradox where both players are better off by swapping their preferences.

15 Conclusion

In conclusion, game theory provides valuable insights into decision-making in various scenarios, including auctions, repeated interactions, information asymmetry, and strategic interactions. Understanding equilibrium concepts and players' rationality aids in predicting and analyzing outcomes in real-world situations, making game theory a powerful tool for researchers, economists, and policymakers. The use of mathematical equations and examples enhances the understanding of the concepts presented in this report.

16 Future Directions

By combining theoretical analysis, empirical applications, and computational methods, we aim to gain a comprehensive understanding of Game Theory's broad applications and its relevance in real-world scenarios. With a solid foundation in the concepts covered thus far, we are well-equipped to explore and contribute to the expanding field of Game Theory in the remaining duration of the program.