Proximity-based approximation algorithms for pessimistic bilevel programs

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Abstract

To be filled

1 Introduction

Bilevel programs model sequential decision-making between players in a strategic game, where the first player (called the leader) makes decisions with knowledge about how the second player (called the follower) will respond. Both players aim to maximize their own objectives subject to their own constraints. The follower has the benefit of *observing* and knowing the decision made by the leader and responding to that; thus, the follower solves a traditional optimization problem. In contrast, the leader has to *anticipate* the follower's response and make a decision to maximize their own objective function, given the follower's response. Even the simplest version of the problem where both the leader and the follower solve a linear program is known to be NP-complete. Vatsalya does there need to be a citation here?

Bilevel programs are of interest in many business applications, where firms act strategically and influence each other. A large body of literature in supply chain management Vatsalya Sentence was incomplete; adding a basic extension to complete it - to be changed later examines the use of bilevel programming to optimize strategic decisions across supply chains.

When integer constraints are involved with all other constraints and the objective functions being linear, and the leader's variables appearing in the follower's constraints, the problem falls in a complexity class called Σ_2^p, and is actually *complete* for this class of decision problems. In other words, it is conjectured that given oracle access to solve NP-hard problems, one would take exponential time to solve these problems.

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Sriram To cite: Kleinert et al. (2021)

Sriram Finding John's ellipsoid: https://math.stackexchange.com/questions/3495898/constrained-convex-optimization/3496110#3496110

Sriram To cite Boyd and Vandenberghe, Convex optimization Eqn 8.15, Section 8.4.2. Lowner John's ellipsoid of a polyhedron is found using a convex-optimization problem. Sriram Useful url: https://see.stanford.edu/materials/lsocoee364a/08GeometricalProbs.pdf

Vatsalya Paper flow: First we define the general case of the bilevel stackelberg game. We discuss how it is difficult to solve it, and propose approximations for two specific cases (Skl-Obj) and (Skl-Cons), while making clear the difference between both cases. Then we continue with definition, algorithm, theorem, proof of both cases.

We define two types of Stackelberg games - optimistic and pessimistic. We use generic convex functions f_x, f_y, g_x and g_y, which will be sharpened in later sections.

2 The model

First, we define a Stackelberg game, which is also referred to as *bilevel programs* in the literature.

Definition 1 (Integer Convex Stackelberg Game)**.** An Optimistic Integer Convex Stackelberg Game (Stackelberg) is a problem of the form

$$\begin{aligned} \min_{x\in\mathbb{Z}^{n_x}, y\in\mathbb{Z}^{n_y}} \quad &: \quad f_x(x,y) && \text{s.t.} \\ g_x(x,y) \quad &\leq \quad 0 && (1) \\ y \quad &\in \quad \arg\min_{y\in\mathbb{Z}^{n_y}} \{f_y(x,y) \,:\, g_y(x,y) \leq 0\} && \text{(Skl-Gen-O)} \end{aligned}$$

–eq:Sklbrg-Gen-O˝

A Pessimistic Integer Convex Stackelberg Game (Stackelberg) is a problem of the form

$$\begin{aligned} \min_{x\in\mathbb{Z}^{n_x}} \max_{y\in\mathbb{Z}^{n_y}} \quad &: \quad f_x(x,y) && \text{s.t.} \\ g_x(x,y) \quad &\leq \quad 0 && \\ y \quad &\in \quad \arg\min_{y\in\mathbb{Z}^{n_y}} \{f_y(x,y) \,:\, g_y(x,y) \leq 0\} && \text{(Skl-Gen-P)} \end{aligned}$$

–eq:Sklbrg-Gen-P˝

Here f_x, f_y, g_x and g_y are any convex functions of (x, y). We call the player deciding x as the leader and the player deciding y as the follower.

In a Stackelberg game, using the notation as above, a player referred to as the leader chooses $x \in \mathbb{Z}^{n_x}$ first. Observing the leader's choice, a player referred to as the follower solves an optimisation problem parameterised in x, to decide their variables y. In turn, the leader's objective function and feasible set themselves are parameterised in y, and they anticipate y's behaviour to choose x.

Now, given a leader's decision x, it is possible that the follower has multiple optimal solutions. If the leader can control or influence the follower to choose a decision that is to their liking, among the multiple optimal solutions that the follower has, it is said to be the optimistic version of the problem. In other words, in the optimistic version, among the multiple optima, the follower chooses y which is the most favourable to the leader. On the other hand, if the leader has little to no influence on the follower, then the leader should possibly plan for the worst alternative that the follower might pick, giving rise to the pessimistic version of the problem. So, in the pessimistic version, we assume that the follower chooses the solution which is the least favourable to the leader. The example below highlights the difference.

Example 1. *Consider the following problem*

$$\begin{aligned} \min_{x\in\mathbb{Z}} \max_{y\in\mathbb{Z}} \quad &: \quad 100(x-1)^2 + y && \textit{s.t.} \\ y \quad &\in \quad \arg\min_{y\in\mathbb{Z}} \left\{y^2 - xy\right\} \end{aligned}$$

and

$$\begin{aligned} \min_{x\in\mathbb{Z}} \min_{y\in\mathbb{Z}} \quad &: \quad 100(x-1)^2 + y && \textit{s.t.} \\ y \quad &\in \quad \arg\min_{y\in\mathbb{Z}} \left\{y^2 - xy\right\} \end{aligned}$$

The former is the pessimistic version of the problem while the latter is the optimistic version of the problem. Observe that if the leader chooses an x that is different from 1, then their objective value will grow fast. Thus, the leader always chooses $x = 1$. Given $x = 1$, the follower has two optimal solutions: $y = 0$, and $y = 1$. While the follower is indifferent between these choices, the leader prefers $y = 0$. Thus the solution to the optimistic version is $(1, 0)$ with the leader's objective value of 0 and the solution to the pessimistic version is $(1, 1)$ with the leader's objective value being 1.

In this paper, while we are interested in Stackelberg game where the follower has integer constraints in the problem, a useful subproblem that is the one where the integer constraints of the follower are relaxed. It is well known that this neither need to be a relaxation or a restriction of the original Stackelberg game.

Definition 2 (Follower relaxed version (FRV))**.** Given a pessimistic or optimistic version of the Stackelberg game, the follower relaxed version of the problem is the one where the integer constraints of the follower's problem are relaxed. In FRV, we retain the pessimistic or optimistic assumptions as in the original Stackelberg game.

For example, the FRV of (Skl-Gen-O) is

$$\begin{aligned} \min_{x\in\mathbb{Z}^{n_x}, y\in\mathbb{R}^{n_y}} \quad &: \quad f_x(x,y) && \text{s.t.} \\ g_x(x,y) \quad &\le \quad 0 \\ y \quad &\in \quad \arg\min_{y\in\mathbb{R}^{n_y}} \left\{f_y(x,y) \,:\, g_y(x,y) \le 0\right\} \end{aligned}$$

Solving the (Skl-Gen-O) or (Skl-Gen-P) problem is generally known to be computationally hard. Vatsalya Here, would it be good to include a few lines describing how having an integer constraint on both variables makes the problem much more difficult to solve than if only x is an integer? Maybe we would need to cite some paper for this assertion. From the perspective of someone new to this topic, it is not obvious why the problem is complex and why relaxing a constraint would make it less complex. Sriram Yes. The complexity details are not known for quadratic, for linear, you can say that if only x is integer, it is NP-complete, while if both x and y are integer, it is Σ_2^p hard. Look at the paragraph below eqn (22) in https://doi.org/10.1007/s10107-017-1189-5 - and the papers cited there. That's what you might have to cite. However, solving their respective FRVs could be relatively easier. For example, we note that if f_x and f_y are linear and the regions defined by $g_x(x, y) \le 0$ and $g_y(x, y) \le 0$ are polyhedra, then the FRVs are already NP-complete. However, the original (Skl-Gen-P) and (Skl-Gen-O) are Σ_2^p-complete. Moreover, in the linear versions, there are integer-programming reformulations based on big-M, or reformulations into complementarity problems that can be solved readily by solvers. However, analogous solvers for the problems *with* integer constraints for the followers are yet to mature.

In this paper we discuss two variations of (Skl-Gen-P) and (Skl-Gen-O) categorised based on how the leader and follower interact. We construct feasible solutions to specialised variations by temporarily relaxing the integer constraint on the follower (i.e., solving the FRV), and then prove that such solutions approximate the actual solutions in various settings. Sriram What are those two variants etc. - Discuss

3 Follower with a convex quadratic objective

In this section, we consider settings where the follower solves a convex quadratic problem with no constraints other than the integer constraint. Interaction between the leader and follower occurs exclusively through the objective function - the leader's decision x appears in the follower's objective function.

Definition 3 (Integer Convex Quadratic Stackelberg Game)**.** An Integer Convex Quadratic Stackelberg Game is a variation of (Skl-Gen-P) or (Skl-Gen-O) where the leader's objective $f_x(x, y) = h_x(x) + d_x(y)$ is a linear function; the leader's feasible set is independent of the follower's variables, i.e., $g_x(x, y) = g_x(x)$; the follower's objective $f_y(x, y) = \frac{1}{2}y^\top Q_y y + (C_y x + d_y)^\top y$ is a (strictly) convex quadratic function with the leader's variable appearing only in the linear term, and the follower has no constraints, i.e., $g_y(x, y) = -1$. Moreover, h_x, d_x and g_x are convex functions of x, and Q_y is a symmetric positive definite matrix.

The Pessimistic Integer Convex Quadratic Stackelberg Game is of the form:

$$\begin{aligned} \min_{x\in\mathbb{Z}^{n_x}} \max_{y\in\mathbb{Z}^{n_y}} \quad &: \quad h_x(x) + d_x(y) \qquad \text{s.t.} \\ g_x(x) \quad &\le \quad 0 \\ y \quad &\in \quad \arg\min_{y\in\mathbb{Z}^{n_y}} \left\{ \frac{1}{2} y^\top Q_y y + (C_y x)^\top y + {d_y}^\top y \right\} \end{aligned} \tag{Skl-Quad-Obj-P}$$

eq:Sklbrg-Quad-P

And the Optimistic Integer Convex Quadratic Stackelberg Game is of the form:

$$\begin{aligned} \min_{x\in\mathbb{Z}^{n_x}, y\in\mathbb{Z}^{n_y}} \quad &: \quad h_x(x) + d_x(y) \qquad \text{s.t.} \\ g_x(x) \quad &\le \quad 0 \end{aligned}$$

Algorithm 1 The Relaxed Foresight Algorithm

alg:apx

Input: An instance of (Skl-Quad-Obj-O) or (Skl-Quad-Obj-P) defined by $h_x, d_x, Q_y, C_y, d_y, g_x$

Output: $x^\dagger \in \mathbb{Z}^{n_x}$ and $y^\dagger \in \mathbb{Z}^{n_y}$

1: Solve the FRV of (Skl-Quad-Obj-O) or (Skl-Quad-Obj-P) to get the optimal solution $(x^\dagger, \widehat{y}^\dagger)$.

2: Solve the convex quadratic integer program given by $\min_{y \in \mathbb{Z}^{n_y}} \frac{1}{2} y^\top Q_y y + (C_y x^\dagger + d_y)^\top y$ to obtain the optimal solution $y^\dagger$. If there are multiple optimal y-s, choose the one that has the largest value of ${d_x}^\top y$ in case of (Skl-Quad-Obj-P), and the smallest value in case of (Skl-Quad-Obj-O).

3: **return** $x^\dagger, y^\dagger$.

$$y \quad \in \quad \arg\min_{y \in \mathbb{Z}^{n_y}} \left\{ \frac{1}{2} y^\top Q_y y + (C_y x)^\top y + {d_y}^\top y \right\} \tag{Skl-Quad-Obj-O}$$

eq:Sklbrg-Quad-O

Observe that in (Skl-Quad-Obj-P) and (Skl-Quad-Obj-O), if one relaxes the follower's constraints that $y \in \mathbb{Z}^{n_y}$ and allows $y \in \mathbb{R}^{n_y}$, then the problem is easier to solve. It also does not matter whether we consider the optimistic version or pessimistic version in the FRV, as the follower always has a unique optimal solution. This is because any solution to the set of equations given by $Q_y y = -(C_y x + d_y)$ models the optimal follower response. Since Q_y is positive definite, it is nonsingular and hence there is always a unique solution to the system of equations. Thus, the game with the *relaxed* follower problem can be rewritten with the same objective, and the follower's problem replaced by a system of linear equality constraints $Q_y y + C_y x = -d_y$. This convex quadratic integer program with linear inequality constraints can be solved using many off-the-shelf commercial solvers including Gurobi Sriram cite, CPLEX Sriram cite, SCIP Sriram cite etc. Now, we present a description of Algorithm 1.

The relaxed foresight algorithm for (Skl-Quad-Obj-O) and (Skl-Quad-Obj-P). In (Skl-Quad-Obj-O) and (Skl-Quad-Obj-P), we model the leader's behaviour as follows. The leader makes a decision, under the assumption that the follower does not have to restrict herself to $\mathbb{Z}^{n_y}$. As discussed above, this is computationally easier. Following that, the follower responds to the leader's decision optimally. Examples can be created where this is suboptimal. Vatsalya To add However, the theory developed to prove results for the Nash version of the problem Sriram To cite, helps provide us bounds on the Stackelberg version as well. We prove that Algorithm 1 serves as an approximation algorithm to solve (Skl-Quad-Obj-O) and (Skl-Quad-Obj-P).

In particular, we want to prove that the solution $(x^\dagger, y^\dagger)$ obtained from Algorithm 1 is approximate to the true optimal solution $(x^\star, y^\star)$ for (Skl-Quad-Obj-O) or (Skl-Quad-Obj-P). We do so by finding an upper bound on the difference between the true optimal objective value $f^\star = f_x(x^\star, y^\star)$, and the objective value obtained from Algorithm 1, $f^\dagger = f_x(x^\dagger, y^\dagger)$.

We first define the proximity measure for an instance of (Skl-Quad-Obj-O) or (Skl-Quad-Obj-P), which gives us a measure for the difference between the continuous and integer optima to the follower's problem.

def:prox

$$
\begin{array}{ccc}
f_x(x^\star, y^\star) & \leq & f_x(x^\dagger, y^\dagger) \\
\approx & & \approx \\
f_x(x^\star, \widehat{y}^\star) & \geq & f_x(x^\dagger, \widehat{y}^\dagger)
\end{array}
$$

Figure 1: Visual representation of proof of (5a): The true solution $(x^\star, y^\star)$ is optimal to (Skl-Quad-Obj-P) and (Skl-Quad-Obj-O), whereas the solution $(x^\dagger, y^\dagger)$ obtained from the algorithm is feasible for the same. This leads to the top inequality. Similarly, the bottom inequality arises from the fact that $(x^\dagger, \widehat{y}^\dagger)$ is optimal to the FRV of (Skl-Quad-Obj-P) and (Skl-Quad-Obj-O) while $(x^\star, \widehat{y}^\star)$ is feasible for it. The approximate equalities holds due to Theorem 5 and the fact that f_x, f_y are Lipschitz continuous. The approximations and the inequality in the bottom, together indicate that $f_x(x^\dagger, y^\dagger) \leq f_x(x^\star, y^\star) + \varepsilon$ where ε is bounded.

fig:quad-ex-ante

Definition 4. Let Q be a given positive definite matrix. The proximity bound of Q with respect to the ℓ_p vector norm is denoted by $\overline{\pi_p}(Q)$ and is the optimal objective value of the problem

$$
\max_{d \in \mathbb{R}^n} \max_{\substack{u \in \mathbb{R}^n \\ v \in \mathbb{Z}^n}} \quad : \quad \|u - v\|_p \qquad \text{s.t.} \tag{4a}
$$

$$
u \quad \in \quad \arg\min \left\{ \frac{1}{2} y^\top Q y + d^\top y : y \in \mathbb{R}^n \right\} \tag{4b}
$$

$$
v \quad \in \quad \arg\min \left\{ \frac{1}{2} y^\top Q y + d^\top y : y \in \mathbb{Z}^n \right\} \tag{4c}
$$

eq:prox:obj eq:prox:constrC eq:prox:constr eq:prox

If p is not explicitly mentioned, then $p = 2$ is used.

We use Vatsalya cite theorem to obtain a bound for the proximity measure.

thm:proxQuad

Theorem 5 (Sriram To cite: Citation not available yet)**.** Given a positive definite matrix Q of dimension $n \times n$, $\overline{\pi}(Q) \leq \frac{\phi_n}{4}\sqrt{\frac{\lambda_1}{\lambda_n}}$ where λ_1 and λ_n are the largest and the smallest singular values of Q and ϕ_n is a constant dependent only on the dimension n and $\phi_n \leq n^{5/2}$.

The bound $\phi_n \leq n^{5/2}$ comes from Barvinok (2002) but tighter bounds are conjectured.

Proposition 6. If an $n \times n$ matrix Q is a *diagonal* positive definite matrix, then $\pi(Q) = \frac{\sqrt{n}}{2}$.

Sriram To prove?

Definition 7. A function $f : D \to \mathbb{R}$ is said to be ℓ_p-Lipschitz continuous with a Lipschitz constant L_p over some $D \subseteq \mathbb{R}^n$, if $\|f(x) - f(y)\|_p \leq L_p \|x - y\|_p$ for every $x, y \in D$.

thm:eapGty

Theorem 8 (Ex-ante and ex-post guarantees in ℓ_2 norm)**.** Suppose $d_x(\cdot)$ is ℓ_2-Lipschitz continuous over $\mathbb{R}^{n_y}$ with a Lipschitz constant L_2. Let the optimal objective obtained by the *leader* in (Skl-Quad-Obj-O) or (Skl-Quad-Obj-P) be $f^\star$. Then, the following bounds hold for the objective $f^\dagger$ obtained using

Algorithm 1:

$$
\begin{aligned}
f^{\dagger} &\leq f^{\star} + 2L_2\overline{\pi}\left(Q_y\right) \leq f^{\star} + 2L_2\frac{\lambda_1\phi_{n_y}}{\lambda_n} && \text{(5a)}\\
f^{\dagger} &\leq f^{\star} + L_2(\overline{\pi}\left(Q_y\right) + \left\|y^{\dagger} - \widehat{y}^{\dagger}\right\|_2) && \text{(5b)}
\end{aligned}
$$

Here, (5a) is the ex-ante guarantee and (5b) is the ex-post guarantee in the ℓ_2 norm. Further, if Q_y is a diagonal matrix, then in (5a), $f^{\dagger}$ is at most $f^{\star} + L_2\sqrt{n_y}$.

By providing a finite bound for the difference between the objective value from the true solution and the one obtained from Algorithm 1, this theorem establishes that the solution obtained from Algorithm 1 approximates the true optimal solution. Now we proceed to prove Theorem 8. An intuition behind the proof of the theorem is provided in Figure 1.

Proof of Theorem 8. Suppose $(x^{\star}, y^{\star})$ is the (unknown) optimal solution to (Skl-Quad-Obj-O) or (Skl-Quad-Obj-P). Let $\widehat{y}^{\star}$ be the continuous minimiser of the lower-level problem, given the upper-level decision $x^{\star}$. Given a leader's decision, the continuous relaxation of the follower's problem has a unique solution since Q_y is assumed to be positive definite. Since there is a unique continuous minimiser, the FRV is same for both the pessimistic and the optimistic version of the problem.

From Theorem 5, we know that $\|\widehat{y}^{\star} - y^{\star}\|_2 \leq \overline{\pi_2}\left(Q_y\right)$. This inequality holds irrespective of whether $y^{\star}$ is chosen pessimistically or optimistically as Theorem 5 holds for every choice of integer and continuous minimisers.

Furthermore, since $d_x(\cdot)$ is Lipschitz continuous with constant L_2, we know that $|d_x(\widehat{y}^{\star}) - d_x(y^{\star})| \leq L_2\left\|\widehat{y}^{\star} - y^{\star}\right\|_2$, implying that $d_x(\widehat{y}^{\star}) \leq d_x(y^{\star}) + L_2\overline{\pi_2}\left(Q_y\right)$.

Now, since $(x^{\dagger}, \widehat{y}^{\dagger})$ is an optimal solution to the FRV while $(x^{\star}, \widehat{y}^{\star})$ is feasible to the FRV, we have

$$
\begin{aligned}
h_x(x^{\dagger}) + d_x(\widehat{y}^{\dagger}) &\leq h_x(x^{\star}) + d_x(\widehat{y}^{\star}) && \text{(6a)}\\
&\leq h_x(x^{\star}) + d_x(y^{\star}) + L_2\left\|\widehat{y}^{\star} - y^{\star}\right\|_2 && \text{(6b)}\\
&\leq f^{\star} + L_2\overline{\pi_2}\left(Q_y\right) && \text{(6c)}
\end{aligned}
$$

where $f^{\star} = h_x(x^{\star}) + {d_x}^{\top}y^{\star}$ is the optimum objective value Sriram Fix this sentence. However, $(x^{\dagger}, \widehat{y}^{\dagger})$ is not necessarily feasible to (Skl-Quad-Obj-P) or (Skl-Quad-Obj-O). $y^{\dagger}$ is an integer optimum to the follower given $x^{\dagger}$, hence $(x^{\dagger}, y^{\dagger})$ is feasible.

Given $(x^{\dagger},\ \widehat{y}^{\dagger})$ is the continuous optimum to the lower-level problem and $y^{\dagger}$ is an integer optimum to the lower-level problem, keeping the optimistic/pessimistic assumptions in mind, we know from Theorem 5 that $\left\|y^{\dagger} - \widehat{y}^{\dagger}\right\| \leq \overline{\pi}\left(Q_y\right)$. We also know that $d_x(y^{\dagger}) \leq d_x(\widehat{y}^{\dagger}) + L\overline{\pi}\left(Q_y\right)$. We use these results to provide a bound for $f_x(x^{\dagger}, y^{\dagger}) - f_x(x^{\dagger}, \widehat{y}^{\dagger})$, the latter of which has been proved to approximate $f_x(x^{\star}, y^{\star})$ in (6). We note that

$$
\begin{aligned}
h_x(x^{\dagger}) + d_x(y^{\dagger}) &\leq h_x(x^{\dagger}) + d_x(\widehat{y}^{\dagger}) + L_2\left\|y^{\dagger} - \widehat{y}^{\dagger}\right\|_2 && \text{(7a)}\\
&\leq f^{\star} + 2L_2\overline{\pi_2}\left(Q_y\right) && \text{(7b)}
\end{aligned}
$$

eq:t1-2
where the first inequality is from $d_x(\cdot)$ being Lipschitz continuous, and the last inequality follows from (6). The ex-ante guarantee (5a) is proved by (6) and (7) together.

If $y^\dagger$ and $\widehat{y}^\dagger$ from the algorithm are known, then from (6),

$$
\begin{aligned}
h_x(x^\dagger) + d_x(y^\dagger) \quad &\leq \quad h_x(x^\dagger) + d_x(\widehat{y}^\dagger) + L_2 \left\|y^\dagger - \widehat{y}^\dagger\right\|_2 && \text{(8a)}\\
&\leq \quad f^\star + L_2 \overline{\pi_2}\left(Q_y\right) + L_2 \left\|y^\dagger - \widehat{y}^\dagger\right\|_2 && \text{(8b)}
\end{aligned}
$$

eq:t1-3
where the last inequality follows from (6).

The ex-post guarantee (5b) is proved by (6) and (8) together. This is tighter bound for $f^\star - f^\dagger$, given that the approximate solution has already been obtained from Algorithm 1. ■

Theorem 8 shows that the solution obtained from Algorithm 1 approximates the true solution, since the objective value from both solutions differs by a finite amount.

If the follower has a linear objective function in (Skl-Quad-Obj-P), we can use a known norm in place of the Lipschitz constant L_∞ in Theorem 8.

cor:eapGty

Corollary 9 (Corollary to Theorem 8)**.** Suppose the function $d_x(y) := {d_x}^\top y$. Then, the following bounds hold for the objective $f^\dagger$ obtained using Algorithm 1:

$$
\begin{aligned}
f^\dagger \quad &\leq \quad f^\star + \sqrt{{d_x}^\top {Q_y}^{-1} d_x}\,\frac{\phi_{n_y}}{2} \leq f^\star + \|d_x\| \frac{1}{\sqrt{\lambda_n}} \frac{\phi_{n_y}}{2} && \text{(9a)}\\
f^\dagger \quad &\leq \quad f^\star + \|d_x\|_2 \,\overline{\pi_2}\left(Q_y\right) + \|d_x\|_2 \left\|y^\dagger - \widehat{y}^\dagger\right\|_2 && \text{(9b)}
\end{aligned}
$$

where $f^\star$ is the true optimal objective value.

Vatsalya Might need to change (9b), after working out the complete proof for (9a).

Proof. If $f(x) = c^\top x$, i.e., a linear function, then $f(x)$ is L_2-Lipschitz continuous with a Lipschitz constant $\|c\|_2$. With $d_x(y) := {d_x}^\top y$, we can replace the Lipschitz constant L used in (6), (16) and (17) with $\|d_x\|_2$. Therefore,

$$
\begin{aligned}
f^\dagger \quad &\leq \quad f^\star + \|d_x\|_2 \,\overline{\pi_2}\left(Q_y\right) && \text{(10a)}\\
f^\dagger \quad &\leq \quad f^\star + 2\,\|d_x\|_2 \,\overline{\pi_2}\left(Q_y\right) && \text{(10b)}\\
f^\dagger \quad &\leq \quad f^\star + \|d_x\|_2 \,\overline{\pi_2}\left(Q_y\right) + \|d_x\|_2 \left\|y^\dagger - \widehat{y}^\dagger\right\|_2 && \text{(10c)}
\end{aligned}
$$

where (10a) and (10b) prove (9a), while (10a) and (10c) prove (9b). **Vatsalya** to add: proof of how (10a) and (10b) result in the theorem statement. ■

The above theorems are specific to the case of the problem considered here. In contrast, it is well known that relaxing the integrality constraints in the follower's problem gives very little information about the primal, except for the case when the follower minimises the same function that the leader maximises (in which case, relaxing the follower's problem is a restriction to the leader). Also, an analogous algorithm for linear games provides no approximation, and the difference in the objectives could be arbitrarily bad. The proofs rely on the fact that Q_y is positive definite (and not just positive semidefinite), and that there are no other constraints on the follower beside the integrality constraints.

Algorithm 2 The Modified Relaxed Foresight Algorithm

alg:apx-l

Input: An instance of (Skl-Lin-Cons-O) and (Skl-Lin-Cons-P) defined by h_x, d_x, g_x, d_y, D_y

Output: $x^\dagger \in \mathbb{Z}^{n_x}$ and $y^\dagger \in \mathbb{Z}^{n_y}$

1: Solve the FRV of (Skl-Lin-Cons-P) to get the optimal solution $(x^\dagger, \hat{y}^\dagger)$.

2: Solve the convex integer program given by

$$\min_{y\in\mathbb{Z}^{n_y}} \quad : \quad {d_y}^\top y \qquad \text{s.t.} \tag{11a}$$

$$g_y(x^\dagger) + D_y y \quad \leq \quad 0 \tag{11b}$$

alg:apx-l:p to obtain the optimum solution $y^\dagger$. If there are multiple optimal y-s, choose the one that has the largest value of $d_x(y)$.

3: Solve the convex integer program given by (11) to obtain the optimum solution $y_o^\dagger$. If there are multiple optimal y-s, choose the one that has the smallest value of $d_x(y)$.

4: **return** $x^\dagger, y^\dagger, x_o^\dagger, y_o^\dagger$.

4 With linear constraints on the follower

sec:linear

In this section, we consider settings where the follower solves a linear problem with a constraint that depends on the leader's decision x. Interaction between the leader and follower occurs exclusively through the constraint on the follower.

Definition 10 (Integer Convex Linear Stackelberg Game)**.** A Integer Convex Linear Stackelberg Game (IL-Stackelberg) is a variation of (Skl-Gen-O) or (Skl-Gen-P) where:

$$\begin{aligned}
f_x(x,y) &= h_x(x) + d_x(y)\\
g_x(x,y) &= g_x(x)\\
f_y(x,y) &= {d_y}^\top y\\
g_y(x,y) &= g_y(x) + D_y y \leq 0
\end{aligned}$$

Here h_x and g_x are convex functions of x, d_x is a convex function on y, and d_y is a linear transformation. D_y is a matrix with integral values.

Note the following caveats in the above decision. The leader's feasible set is not affected by the follower's decisions. The leader's variables appear only in the follower's constraints.

The Pessimistic Integer Convex Linear Stackelberg Game is of the form:

$$\begin{aligned}
\min_{x\in\mathbb{Z}^{n_x}} \max_{y\in\mathbb{Z}^{n_y}} \quad &: \quad h_x(x) + d_x(y) \qquad \text{s.t.}\\
g_x(x) \quad &\leq \quad 0\\
y \quad &\in \quad \arg\min_{y\in\mathbb{Z}^{n_y}} \left\{ {d_y}^\top y \,:\, g_y(x) + D_y y \leq 0 \right\}
\end{aligned} \tag{Skl-Lin-Cons-P}$$

eq:Sklbrg-Lin-P

And the Optimistic Integer Convex Linear Stackelberg Game is of the form:

$$\min_{x\in\mathbb{Z}^{n_x},y\in\mathbb{Z}^{n_y}} \quad : \quad h_x(x) + d_x(y) \qquad \text{s.t.}$$

$$g_x(x) \leq 0$$

$$y \in \arg\min_{y\in\mathbb{Z}^{n_y}} \left\{ {d_y}^\top y \,:\, g_y(x) + D_y y \leq 0 \right\} \tag{Skl-Lin-Cons-O}$$

The modified relaxed foresight algorithm for (Skl-Lin-Cons-O) and (Skl-Lin-Cons-P). In (Skl-Lin-Cons-O) and (Skl-Lin-Cons-O), we model the leader's behaviour as follows. The leader makes a decision, under the assumption that the follower does not have to restrict herself to $\mathbb{Z}^{n_y}$. Following that, the follower responds to the leader's decision optimally. We prove that Algorithm 2 serves as an approximation algorithm to solve (Skl-Lin-Cons-P).

In particular, we want to prove that the solution $(x^\dagger, y^\dagger)$ obtained from Algorithm 2 is approximate to the true optimal solution $(x^\star, y^\star)$ for (Skl-Lin-Cons-P). We do so by finding an upper bound on the difference between the true optimal objective value $f^\star = f_x(x^\star, y^\star)$, and the objective value obtained from Algorithm 2, $f^\dagger = f_x(x^\dagger, y^\dagger)$.

We first look into a theorem defining the proximity measure for an instance of (Skl-Lin-Cons-P), which gives us a measure for the difference between the continuous and integer optima to the follower's problem.

Theorem 11 (Proximity theorem (Cook et al., 1986))**.** Let D_y be an integral matrix such that each subdeterminant is at most $\Delta(D_y)$ in absolute value, d_y and b be vectors such that $D_y y \leq b$ has an integral solution and $\min\{{d_y}^\top y : D_y y \leq b\}$ exists. Suppose that both

$$\min\left\{{d_y}^\top y \mid D_y y \leq b\right\} \tag{12a}$$

$$\min\left\{{d_y}^\top y \mid D_y y \leq b; y \in \mathbb{Z}^n\right\} \tag{12b}$$

are finite. Then,

$$\forall\, \widehat{y} \text{ optimal to (12a) } \exists\, \widetilde{y} \text{ optimal to (12b) s.t. } \|\widehat{y} - \widetilde{y}\|_\infty \leq n\Delta(D_y) \quad \text{and} \tag{13a}$$

$$\forall\, \widetilde{y} \text{ optimal to (12b) } \exists\, \widehat{y} \text{ optimal to (12a) s.t. } \|y - \widetilde{y}\|_\infty \leq n\Delta(D_y). \tag{13b}$$

Notice that Theorem 11 discusses the existence of a follower's integer optimum $\widetilde{y}$ that differs from the continuous optimum by the given bound. There may still be other integral optima that are farther away from $\widehat{y}$. To ascertain whether the follower's optimum obtained from Algorithm 2 $y^\dagger$ can approximate the true solution $y^\star$, we must therefore have a measure of the largest distance between any two integral follower solutions.

Definition 12 (Pessimistic-Optimistic Difference)**.** Vatsalya Think of a better name for the theorem.

Given an Integer Stackelberg game, we define the *pessimistic-optimistic difference* with respect to ℓ_p norm, denoted by $\overline{\omega_p}$ as the largest distance between any pair of follower optimal solutions. Formally, $\overline{\omega_p} := \max\{\omega_p(x) : x \in D\}$, where $\omega_p(x) := \max\left\{\left\|y^1 - y^2\right\|_p : y^1, y^2 \text{ are optimal to follower given } x\right\}$.

Definition 12 sets the bound on how far could the followers' solution could be, for any given a leader's solution. For example, if it is known that the follower has a unique optimal solution for every feasible leader's decision x, then $\overline{\omega_p} = 0$ for any p. Sriram Hints on how to compute $\overline{\omega_\infty}$. As another example, if it is known that the follower's feasible set is contained in an ℓ_p ball of radius R, then $\overline{\omega_p} \le 2R$.

4.1 Guarantees for when there are few variables and many constraints

When the number of variables n in (Skl-Lin-Cons-P) is less than the number of constraints m, we can use Theorem 11 to prove that the solution $(x^\dagger, y^\dagger)$ to (Skl-Lin-Cons-P) obtained from Algorithm 2 approximates the true solution $(x^\star, y^\star)$ by providing an upper bound for the difference between the objective function values $f^\dagger$ and $f^\star$ obtained using both solutions respectively.

The existence of an integral solution $\widetilde{y}$ to the follower's problem in (Skl-Lin-Cons-P) that is proximal to the solution of the FRV $\widehat{y}$ lets us develop a theorem to prove that the solution $(x^\dagger, y^\dagger)$ obtained from Algorithm 2 approximates the true solution $(x^\star, y^\star)$.

thm:eapGty-L

Theorem 13 (Ex-ante and ex-post guarantees in L_∞ norm)**.** Let the optimal objective obtained by the *leader* in (Skl-Lin-Cons-P) be $f^\star$. Let $d_x : \mathbb{R}^{n_y} \to \mathbb{R}$ be ℓ_∞ Lipschitz continuous with a Lipschitz constant L_∞. Then, the following bounds hold for the objective $f^\dagger$ obtained using Algorithm 2:

$$
\begin{aligned}
f^\dagger &\le f^\star + 2L_\infty n\Delta(D_y) + L_\infty \overline{\omega_\infty} \qquad \text{and} && \text{–thm:eaGty-L–} \quad (14a)\\
f^\dagger &\le f^\star + 2L_\infty n\Delta(D_y) + d_x(y^\dagger) - d_x(y_o^\dagger) && \text{–thm:epGty-L–} \quad (14b)
\end{aligned}
$$

Here, (14a) is the ex-ante guarantee and (14b) is the ex-post guarantee in the ℓ_∞ norm.

Proof of Theorem 13. Let $(x^\star, y^\star)$ be a true optimal solution for (Skl-Lin-Cons-P). Let $\widehat{y}^\star$ be the continuous minimiser of the follower's problem, given the leader's decision $x^\star$. We know from Theorem 11 that there exists $\widetilde{y}^\star$ feasible and optimal to the (integer-constrained) follower problem given the leader's decision $x^\star$, such that $\|\widetilde{y}^\star - \widehat{y}^\star\|_\infty \le n\Delta(D_y)$. We note that, while $\widetilde{y}^\star$ is optimal to the follower's problem, it is only *an* optimal solution, and it need not be a pessimistic or optimistic solution to the follower's problem. In other words, the following series of inequalities could possibly hold: $d_x(y^\star) > d_x(\widetilde{y}^\star) > d_x(y_o^\star)$ where $y^\star$ and $y_o^\star$ are respectively the pessimistic and the optimistic solutions of the follower's problems given the leader's decision $x^\star$.

Now, by the fact that $(x^\dagger, \widehat{y}^\dagger)$ are obtained as optimal solutions to the corresponding FRV, we have

$$
\begin{aligned}
h_x(x^\dagger) + d_x(\widehat{y}^\dagger) &\le h_x(x^\star) + d_x(\widehat{y}^\star) && (15a)\\
&\le h_x(x^\star) + d_x(\widetilde{y}^\star) + L_\infty \|\widehat{y}^\star - \widetilde{y}^\star\|_\infty && (15b)\\
&\le h_x(x^\star) + d_x(\widetilde{y}^\star) + L_\infty n\Delta(D_y) && (15c)\\
&\le h_x(x^\star) + d_x(y^\star) + L_\infty n\Delta(D_y) && (15d)\\
&= f^\star + L_\infty n\Delta(D_y) && (15e)
\end{aligned}
$$

where $f^\star = h_x(x^\star) + d_x(y^\star)$ is the optimum objective value of (Skl-Lin-Cons-P). The first inequality in (15) is due to the fact that $(x^\dagger, \widehat{y}^\dagger)$ is feasible and optimal to the FRV, while $(x^\star, \widehat{y}^\star)$ is feasible to the FRV. The second inequality follows from the Lipschitz continuity of d_x. i.e., $|d_x(\widehat{y}^\star) - d_x(\widetilde{y}^\star)| \le L_\infty \left\| \widehat{y}^\star - \widetilde{y}^\star \right\|_\infty$ implying that $d_x(\widehat{y}^\star) \le d_x(\widetilde{y}^\star) + L_\infty \left\| \widehat{y}^\star - \widetilde{y}^\star \right\|_\infty$. The third inequality follows from Theorem 11. The last inequality is due to the fact that $(x^\dagger, \widetilde{y}^\dagger)$ is an integer feasible and optimal solution to the lower level, and also a pessimistic optimum. i.e., an optimum that has the largest value of $d_x(y)$.

However, $(x^\dagger, \widehat{y}^\dagger)$ is not necessarily feasible to (Skl-Lin-Cons-P) since $\widehat{y}^\dagger$ may not satisfy the integrality requirements. The corresponding feasible point is $(x^\dagger, y^\dagger)$.

Given $x^\dagger$, $\widehat{y}^\dagger$ is the continuous optimum to the lower-level problem, we know from Theorem 11 that there exists $\widetilde{y}^\dagger$, an integer optimum to the lower-level problem such that $\left\| \widehat{y}^\dagger - \widetilde{y}^\dagger \right\|_\infty \le n\Delta(D_y)$

Thus,

$$
\begin{aligned}
h_x(x^\dagger) + d_x(y^\dagger) &= h_x(x^\dagger) + d_x(\widehat{y}^\dagger) + d_x(y^\dagger) - d_x(\widehat{y}^\dagger) && \text{(16a)}\\
&\le f^\star + L_\infty n\Delta(D_y) + d_x(y^\dagger) - d_x(\widehat{y}^\dagger) + d_x(\widetilde{y}^\dagger) - d_x(\widetilde{y}^\dagger) && \text{(16b)}\\
&\le f^\star + L_\infty n\Delta(D_y) + d_x(y^\dagger) - d_x(\widetilde{y}^\dagger) + L_\infty \left\| \widetilde{y}^\dagger - \widehat{y}^\dagger \right\|_\infty && \text{(16c)}\\
&\le f^\star + L_\infty n\Delta(D_y) + d_x(y^\dagger) - d_x(\widetilde{y}^\dagger) + L_\infty n\Delta(D_y) && \text{(16d)}\\
&\le f^\star + 2L_\infty n\Delta(D_y) + d_x(y^\dagger) - d_x(y_o^\dagger) && \text{(16e)}
\end{aligned}
$$

where the first inequality follows from (15), the second inequality is because d_x is Lipschitz continuous, the third inequality follows from Theorem 11, and the last inequality holds because $\widetilde{y}^\dagger$ is an integer solution to a problem that has $y^\dagger$ as the pessimistic and $y_o^\dagger$ as the optimistic solutions, meaning that $d_x(y^\dagger) \ge d_x(\widetilde{y}^\dagger) \ge d_x(y_o^\dagger)$. This concludes the proof for (14b). Using the pessimistic-optimistic difference $\overline{\omega_\infty}$ from Definition 12 can give us an ex-ante guarantee as stated in (14a), the proof for which continues after (15) and (16) as

$$
\begin{aligned}
h_x(x^\dagger) + d_x(y^\dagger) &\le f^\star + 2L_\infty n\Delta(D_y) + d_x(y^\dagger) - d_x(y_o^\dagger) && \text{(17a)}\\
&\le f^\star + 2L_\infty n\Delta(D_y) + L_\infty \left\| y^\dagger - y_o^\dagger \right\|_\infty && \text{(17b)}\\
&\le f^\star + 2L_\infty n\Delta(D_y) + L_\infty \omega_\infty(x^\dagger) && \text{(17c)}\\
&\le f^\star + 2L_\infty n\Delta(D_y) + L_\infty \overline{\omega_\infty} && \text{(17d)}
\end{aligned}
$$

where the first inequality follows from (16), the second from the fact that d_x is Lipschitz continuous, and the last two from the definitions of $\omega_\infty(x^\dagger)$ and $\overline{\omega_\infty}$ respectively, from Definition 12. ■

Theorem 13 shows that the solution obtained from Algorithm 2 approximates the true solution, since the objective value from both solutions differs by a finite amount.

If the follower has a linear objective function in (Skl-Lin-Cons-P), we can use a known norm in place of the Lipschitz constant L_∞ in Theorem 13.

cor:eapGty-L

Corollary 14 (Corollary to Theorem 13)**.** Suppose the function $d_x(y) := {d_x}^\top y$, and $f^\star$ be the solution to (Skl-Lin-Cons-P). Then, the following bounds hold for the objective $f^\dagger$ obtained using Algorithm 2:

$$
\begin{aligned}
f^\dagger &\le f^\star + 2\left\|d_x\right\|_1 n\Delta(D_y) + \left\|d_x\right\|_1 \overline{\omega_\infty} \qquad \text{and} && \text{(18a)}\\
f^\dagger &\le f^\star + 2\left\|d_x\right\|_1 n\Delta(D_y) + {d_x}^\top(y^\dagger - y_o^\dagger) && \text{(18b)}
\end{aligned}
$$

Here, (18a) is a corollary to (14a), and (18b) is a corollary to (14b).

Proof of Corollary 14. For a Lipschitz continuous function $d_x(\cdot)$, we know that $|d_x(y_1) - d_x(y_2)| \le L_\infty \left\|y_1 - y_2\right\|_\infty$.

When $d_x(y) := {d_x}^\top y$ however, we observe that ${d_x}^\top y_1 - {d_x}^\top y_2 = {d_x}^\top(y_1 - y_2) \le \left\|d_x\right\|_1 \left\|y_1 - y_2\right\|_\infty$, from the Cauchy-Schwartz inequality.

Therefore,

$$
\begin{aligned}
h_x(x^\dagger) + {d_x}^\top \widehat{y}^\dagger &\le f^\star + \left\|d_x\right\|_1 n\Delta(D_y) && \text{(19a)}\\
h_x(x^\dagger) + {d_x}^\top y^\dagger &\le f^\star + 2\left\|d_x\right\|_1 n\Delta(D_y) + {d_x}^\top(y^\dagger - y_o^\dagger) && \text{(19b)}\\
h_x(x^\dagger) + {d_x}^\top y^\dagger &\le f^\star + 2\left\|d_x\right\|_1 n\Delta(D_y) + \left\|d_x\right\|_1 \overline{\omega_\infty} && \text{(19c)}
\end{aligned}
$$

where (19a) and (19b) prove (18b), and (19a) to (19c) together prove (18a). ∎

Specifically in this case, we can use the following extension to Theorem 11 which allows for the follower to have a separable convex objective function.

thm:W-M

Theorem 15 (ℓ_∞ proximity between relaxed and integral solutions (Werman and Magagnosc, 1991))**.** Let D_y be an integral matrix such that each subdeterminant is at most $\Delta(D_y)$ in absolute value, b a vector and $f_{y_i} : \mathbb{R} \to \mathbb{R}$ be convex functions. Suppose that both

$$
\min\left\{\sum_i f_i(y_i) \mid D_y y \le b\right\} \tag{20a}
$$

$$
\min\left\{\sum_i f_i(y_i) \mid D_y y \le b; y \in \mathbb{Z}^n\right\} \tag{20b}
$$

are finite. Then,

$$
\begin{aligned}
&\forall\ \widehat{y} \text{ optimal to (20a) } \exists\ \widetilde{y} \text{ optimal to (20b) s.t. } \left\|\widehat{y} - \widetilde{y}\right\|_\infty \le n\Delta(D_y) \qquad \text{and} && \text{(21a)}\\
&\forall\ \widetilde{y} \text{ optimal to (20b) } \exists\ \widehat{y} \text{ optimal to (20a) s.t. } \left\|y - \widetilde{y}\right\|_\infty \le n\Delta(D_y). && \text{(21b)}
\end{aligned}
$$

This is an extension of Theorem 11 where each f_{y_i} is a linear function.

Remark 1. Theorem 13 and Corollary 14 can be extended to the case of (Skl-Lin-Cons-P) where the follower's objective function is a sum of one-dimensional convex functions. i.e., $f_y(y) := \sum_i f_i(y_i)$, where $f_{y_i} : \mathbb{R} \to \mathbb{R}$ are convex functions. The authors in Werman and Magagnosc (1991) call such functions as *separable convex functions*. As stated in Theorem 15, the authors prove a result analogous to Theorem 11. Using similar steps as in the proofs of Theorem 13 and Corollary 14, this extension can be obtained.

4.2 Guarantees for when there are many variables and few constraints

Theorem 16 (ℓ_1 norm proximity between relaxed and integer solutions (Eisenbrand and Weismantel, 2018))**.** Let D_y and b be integral matrices such that $\delta(D_y)$ is the upper bound on the absolute values of entries in D_y. Suppose that both

$$\min \left\{ {p_y}^\top y \mid D_y y \le b \right\} \tag{22a}$$

$$\min \left\{ {p_y}^\top y \mid D_y y \le b; y \in \mathbb{Z}^n \right\} \tag{22b}$$

are finite with optimal solutions $y^\dagger$ and $y^\star$ respectively. Then,

$$\forall\ \widehat{y} \text{ optimal to (22a) } \exists\ \widetilde{y} \text{ optimal to (22b) s.t. } \|\widehat{y} - \widetilde{y}\|_1 \le m(2m\delta(D_y) + 1)^m \tag{23a}$$

$$\forall\ \widetilde{y} \text{ optimal to (22b) } \exists\ \widehat{y} \text{ optimal to (22a) s.t. } \|\widehat{y} - \widetilde{y}\|_1 \le m(2m\delta(D_y) + 1)^m \tag{23b}$$

where m is the number of rows in matrix D_y, i.e., the number of constraints in the problem.

In Eisenbrand and Weismantel (2018), Theorem 16 is presented for programs in the standard form, i.e., with constraints $Ax = b$ and $x \ge 0$. The same can be converted to the form used in (22) by adding variables x^+, x^- and s such that $A(x^+ - x^-) + s = b$ and $x^+, x^-, s \ge 0$, so that we have $A(x^+ - x^-) = D_y y$. Thus, we have $2n + m$ variables in the problem where n is the number of elements in y, while the number of constraints m remains the same.

While Theorem 13 shows that Algorithm 2 provides an approximation for the true solution of (Skl-Lin-Cons-P), there might be cases where the bound $n\Delta(D_y)$ is large and therefore not very helpful in proving that the objective $f^\dagger$ obtained from Algorithm 2 is proximal to the true objective $f^\star$. This can happen when n, the number of variables in the (Skl-Lin-Cons-P), is large. Theorem 16 can be useful in such cases because the proximity bound defined here depends on the number of constraints m rather than the number of variables.

The following theorem utilises Theorem 16 to prove that the solution $(x^\dagger, y^\dagger)$ obtained from Algorithm 2 approximates the true solution $(x^\star, y^\star)$.

Theorem 17 (Ex-ante and ex-post guarantees in ℓ_1 norm)**.** Let the optimal objective obtained by the *leader* in (Skl-Lin-Cons-P) be $f^\star$ and $g_x(x)$ be an integral vector. Let $d_x : \mathbb{R}^{n_y} \to \mathbb{R}$ be ℓ_1 Lipschitz continuous with a Lipschitz constant L_1. Then, the following bounds hold for the objective $f^\dagger$ obtained using Algorithm 2:

$$f^\dagger \quad \le \quad f^\star + 2L_1 m(2m\delta(D_y) + 1)^m + L_1 \overline{\omega_1} \qquad \text{and} \tag{24a}$$

$$f^\dagger \quad \le \quad f^\star + 2L_1 m(2m\delta(D_y) + 1)^m + d_x(y^\dagger) - d_x(y_o^\dagger) \tag{24b}$$

where m is the number of rows in matrix D_y. (24a) is the ex-ante guarantee and (24b) is the ex-post guarantee in the ℓ_1 norm.

Proof of Theorem 17. Let $\widehat{y}^\star$ be the continuous minimiser of the lower-level problem, given the upper-level decision $x^\star$. We know that from Theorem 16 that there exists $\|\widetilde{y}^\star - \widehat{y}^\star\|_1 \le m(2m\delta(D_y) + 1)^m$, where $\widetilde{y}^\star$ is feasible and optimal to the unrelaxed follower's problem. We note that, while $\widetilde{y}^\star$ is optimal to the follower's problem, it is only *an* optimal solution, and it need not match with the pessimistic or optimistic solution under the Stackelberg construct.

Now, by the fact that the approximate solutions are obtained as optimal solutions to Algorithm 2, we have

$$
\begin{aligned}
h_x(x^\dagger) + d_x(\widehat{y}^\dagger) &\leq h_x(x^\star) + d_x(\widehat{y}^\star) && \text{(25a)}\\
&\leq h_x(x^\star) + d_x(\widetilde{y}^\star) + L_1 \left\|\widehat{y}^\star - \widetilde{y}^\star\right\|_1 && \text{(25b)}\\
&\leq h_x(x^\star) + d_x(\widetilde{y}^\star) + L_1 m(2m\delta(D_y)+1)^m && \text{(25c)}\\
&\leq h_x(x^\star) + d_x(y^\star) + L_1 m(2m\delta(D_y)+1)^m && \text{(25d)}\\
&= f^\star + L_1 m(2m\delta(D_y)+1)^m && \text{(25e)}
\end{aligned}
$$

eq:t2-1m where $f^\star = h_x(x^\star) + d_x(y^\star)$ is the optimum objective value. The first inequality in (25) is due to the fact that $(x^\dagger, \widehat{y}^\dagger)$ is feasible and optimal to the FRV, while $(x^\star, \widehat{y}^\star)$ is a feasible point to the FRV. The second inequality follows from the Lipschitz continuity of the function $d_x(y)$, and the third inequality is due to Theorem 16. The last inequality is due to the fact that $(x^\dagger, \widetilde{y}^\dagger)$ is an integer feasible and optimal solution to the lower level, and also a pessimistic optimum. i.e., an optimum that has the largest value of $d_x(y)$.

However, $(x^\dagger, \widehat{y}^\dagger)$ is not feasible to (Skl-Lin-Cons-P). The corresponding feasible point is $(x^\dagger, y^\dagger)$. Given $x^\dagger$, $\widehat{y}^\dagger$ is the continuous optimum to the follower's problem, we know from Theorem 16 that there exists $\widetilde{y}^\dagger$, an integer optimum to the follower's problem such that $\left\|\widehat{y}^\dagger - \widetilde{y}^\dagger\right\|_1 \leq m(2m\delta(D_y)+1)^m$.

Thus,

$$
\begin{aligned}
h_x(x^\dagger) + d_x(y^\dagger) &= h_x(x^\dagger) + d_x(\widehat{y}^\dagger) + d_x(y^\dagger) - d_x(\widehat{y}^\dagger) && \text{(26a)}\\
&\leq f^\star + L_1 m(2m\delta(D_y)+1)^m + d_x(y^\dagger) - d_x(\widetilde{y}^\dagger) + d_x(\widetilde{y}^\dagger) - d_x(\widehat{y}^\dagger) && \text{(26b)}\\
&\leq f^\star + L_1 m(2m\delta(D_y)+1)^m + d_x(y^\dagger) - d_x(\widetilde{y}^\dagger) + L_1 \left\|\widetilde{y}^\dagger - \widehat{y}^\dagger\right\|_1 && \text{(26c)}\\
&\leq f^\star + L_1 m(2m\delta(D_y)+1)^m + d_x(y^\dagger) - d_x(\widetilde{y}^\dagger) + L_1 m(2m\delta(D_y)+1)^m && \text{(26d)}\\
&\leq f^\star + 2L_1 m(2m\delta(D_y)+1)^m + d_x(y^\dagger) - d_x(y_o^\dagger) && \text{(26e)}
\end{aligned}
$$

eq:t2-2m where the first inequality follows from (25), the second inequality is because d_x is Lipschitz continuous, the third inequality follows from Theorem 16, and the last inequality holds because $\widetilde{y}^\dagger$ is an integer solution to a problem that has $y^\dagger$ as the pessimistic and $y_o^\dagger$ as the optimistic solutions, meaning that $d_x(y^\dagger) \geq d_x(\widetilde{y}^\dagger) \geq d_x(y_o^\dagger)$.

This concludes the proof for (24b), since we solve Algorithm 2 to obtain $y^\dagger$ and $y_o^\dagger$. Using the pessimistic-optimistic difference $\overline{\omega_1}$ from Definition 12 can give us an ex-ante guarantee as stated in (24a), the proof for which continues after (25) and (26) as

$$
\begin{aligned}
h_x(x^\dagger) + d_x(y^\dagger) &\leq f^\star + 2L_1 m(2m\delta(D_y)+1)^m + d_x(y^\dagger) - d_x(y_o^\dagger) && \text{(27a)}\\
&\leq f^\star + 2L_1 m(2m\delta(D_y)+1)^m + L_1 \left\|y^\dagger - y_o^\dagger\right\|_1 && \text{(27b)}
\end{aligned}
$$

$$
\begin{aligned}
&\le \quad f^\star + 2L_1 m(2m\delta(D_y)+1)^m + L_1\omega_1(x^\dagger) && \text{(27c)}\\
&\le \quad f^\star + 2L_1 m(2m\delta(D_y)+1)^m + L_1\overline{\omega_1} && \text{(27d)}
\end{aligned}
$$

where the first inequality follows from (26), the second from the fact that d_x is Lipschitz continuous, and the last two from the definitions of $\omega_1(x^\dagger)$ and $\overline{\omega_1}$ respectively, from Definition 12. ■

If the follower has a linear objective function in (Skl-Lin-Cons-P), we can use a known norm in place of the Lipschitz constant L_1 in Theorem 17.

Corollary 18 (Corollary to Theorem 17)**.** Suppose the function $d_x(y) := {d_x}^\top y$, and $f^\star$ be the solution to (Skl-Lin-Cons-P). Then, the following bounds hold for the objective $f^\dagger$ obtained using Algorithm 2:

$$
\begin{aligned}
f^\dagger &\le f^\star + 2\left\|d_x\right\|_\infty m(2m\delta(D_y)+1)^m + \left\|d_x\right\|_\infty \overline{\omega_1} \qquad \text{and} && \text{(28a)}\\
f^\dagger &\le f^\star + 2\left\|d_x\right\|_\infty m(2m\delta(D_y)+1)^m + {d_x}^\top (y^\dagger - y_o^\dagger) && \text{(28b)}
\end{aligned}
$$

here, (28a) is the corollary to (24a), and (28b) is the corollary to (24b).

Proof of Corollary 18. For a Lipschitz continuous function $d_x(\cdot)$, we know that $|d_x(y_1) - d_x(y_2)| \le L_1 \left\|y_2 - y_1\right\|_1$.

When $d_x(y) := {d_x}^\top y$ however, we observe that ${d_x}^\top y_1 - {d_x}^\top y_2 = {d_x}^\top(y_1 - y_2) \le \left\|d_x\right\|_\infty \left\|y_1 - y_2\right\|_1$, from the Cauchy-Schwartz inequality. So we can replace L_1 in (25) to (27) with $\left\|d_x\right\|_\infty$.

Therefore,

$$
\begin{aligned}
h_x(x^\dagger) + {d_x}^\top \widehat{y}^\dagger &\le f^\star + \left\|d_x\right\|_\infty m(2m\delta(D_y)+1)^m && \text{(29a)}\\
h_x(x^\dagger) + {d_x}^\top y^\dagger &\le f^\star + 2\left\|d_x\right\|_\infty m(2m\delta(D_y)+1)^m + {d_x}^\top(y^\dagger - y_o^\dagger) && \text{(29b)}\\
h_x(x^\dagger) + {d_x}^\top y^\dagger &\le f^\star + 2\left\|d_x\right\|_\infty m(2m\delta(D_y)+1)^m + \left\|d_x\right\|_\infty \overline{\omega_1} && \text{(29c)}
\end{aligned}
$$

where (29a) and (29b) prove (28b), and (29a) to (29c) together prove (28a). ■

Algorithm to find an upper bound on $\overline{\omega_\infty}$ and $\overline{\omega_1}$. Given a follower's optimisation problem, $\overline{\omega_\infty}$ and $\overline{\omega_1}$ can be calculated by solving a problem that maximises $\left\|y^1 - y^2\right\|_\infty$ while restricting the follower's decision y to the feasibility constraints in (Skl-Lin-Cons-P) and ensuring that y^1 and y^2 yield the same objective function value in (Skl-Lin-Cons-P). Consider the problem of maximising $\left\|y^1 - y^2\right\|_p$ subject to the following constraints.

$$
\begin{aligned}
g_x(x) &\le 0 && \text{(30a)}\\
g_y(x) + D_y y^1 &\le 0 && \text{(30b)}\\
g_y(x) + D_y y^2 &\le 0 && \text{(30c)}\\
{d_y}^\top y^1 &= {d_y}^\top y^2 && \text{(30d)}\\
x &\in \mathbb{Z}^{n_x} && \text{(30e)}\\
y^1, y^2 &\in \mathbb{Z}^{n_y} && \text{(30f)}
\end{aligned}
$$

lp:no-diff-best

This will provide an upper bound on $\overline{\omega_p}$.

Now, we are interested in the case $p = \infty$. This means, we want to identify the component of $y^1 - y^2$ which has the largest absolute value. Let $z_i = \left|y_i^1 - y_i^2\right|$. We want to represent this using linear and integer constraints. This can be done by adding the following constraints for each i.

$$
\begin{aligned}
z_i &\geq y_i^1 - y_i^2 && \text{(31a)}\\
z_i &\geq y_i^2 - y_i^1 && \text{(31b)}\\
z_i &\leq y_i^1 - y_i^2 + Mu_i && \text{(31c)}\\
z_i &\leq y_i^2 - y_i^1 + M(1 - u_i) && \text{(31d)}\\
u_i &\in \{0, 1\} && \text{(31e)}
\end{aligned}
$$

eq:no:abs

where M is a large number.

We wish to find the maximum z_i across all i from 1 to n, where n is the number of elements in y^1 and y^2. Let $t = \max z_i \ \forall\ i$. The LP that helps us find t is

$$
\begin{aligned}
\max \quad &: \quad t \qquad\qquad \text{s.t.} && \text{(32a)}\\
t &\geq z_i && \text{(32b)}\\
t &\leq z_i + M(1 - b_i) && \text{(32c)}\\
b_i &\in \{0, 1\} && \text{(32d)}\\
\sum_i b_i &= 1 && \text{(32e)}\\
& && \text{(32f)}
\end{aligned}
$$

eq:infNormOmega

This gives us the upper bound on $\overline{\omega_\infty}$.

To find an upper bound on $\overline{\omega_1}$, we can maximize $\sum_i z_i$ subject to (30) and (31).

5 Complexity

First, we quantify the computational complexity of the pessimistic version of the Integer Convex Quadratic Stackelberg Game. We prove in this section, that deciding whether the optimal objective value of the problem is something or not to be Σ_2^p-hard. This means, the problem is in the second level in the polynomial hierarchy of complexity classes. This further indicates, unless $NP = \Sigma_2^p$, which is a very unlikely situation in complexity theory, there are no algorithms that can run asymptotically faster than $O(2^{2^n})$ time where n is the size of binary encoding of the problem. In other words, unless very unlikely results in complexity theory hold. one might have to solve an exponential large NP-hard problem, to solve a Σ_2^p-hard problem. We formally state the result below.

thm:QuadIntHard

Theorem 19. The pessimistic version of the *Integer Convex Quadratic Stackelberg Game as stated in* (Skl-Quad-Obj-P) *is* Σ_2^p*-hard.*

Complexity proofs are typically provided by reducing a different (hard) problem to the given problem. This way, if the given problem is solved, then so is the original hard problem, indicating the hardness of the given problem. Here, we show that a problem called Subset-Sum-Interval, that is known to be Σ_2^p-hard can be reduced into the pessimistic version of the Integer Convex Quadratic Stackelberg Game. To this end, we define Subset-Sum-Interval and formally state its hardness below.

Definition 20 (Subset-Sum-Interval). Subset-Sum-Interval is the following decision problem:

> Given a sequence $q_1, \ldots q_k$ of positive integers, two more positive integers R and r, such that $r \le k$, decide if there exists S such that $R \le S < R + 2^r$ such that there does not exist a subset $I \subseteq \{1, 2, \ldots, k\}$, $\sum_{i \in I} q_i = S$.

One can observe that if such an S is given, checking that $\sum_{i \in I} q_i \ne S$ for every subset I is already NP-hard, as it is an instance of the Subset-Sum problem. Eggermont and Woeginger (2013) showed that Subset-Sum-Interval belongs to the complexity class Σ_2^p, and it is indeed complete for this class. We formally state this below.

Theorem 21 ((Eggermont and Woeginger, 2013)). Subset-Sum-Interval is Σ_2^p-complete.

Now, we are in a position to prove Theorem 19. The proof of this theorem is motivated by the proof in Caprara et al. (2014, Sec 3.3) where a certain version (DNeg) of bilevel knapsack problem is Σ_2^p-hard. While significant book-keeping is required for the proof, we use notations involving superscripts p, d and o to hint at the analogous forms of *padding items*, *dummy items* and *ordinary items* used in Caprara et al. (2014, Sec 3.3). We also note that the proof in this paper does not trivially follow from the complexity results in Caprara et al. (2014, Sec 3.3) as the follower is not allowed to have any constraints besides the integer constraints in our context.

Proof of Theorem 19. To prove Theorem 19, we show that an instance of Subset-Sum-Interval can be rewritten as an instance of (Skl-Quad-Obj-P) of size bounded by a polynomial in the size of the Subset-Sum-Interval instance. Thus, if (Skl-Quad-Obj-P) can be solved fast, this reformulation can be used to solve Subset-Sum-Interval fast, which is unlikely. Moreover, we exclude Subset-Sum-Interval instances with the trivial case where R cannot be represented as a subset sum of q_1 to q_k. This can be checked with an oracle to the Subset-sum problem (which is NP-complete), and does not alter the complexity of Subset-Sum-Interval.

Given an instance of Subset-Sum-Interval, we construct an instance of (Skl-Quad-Obj-P), where some of the follower variables are restricted to be binary. Suppose a variable y_i should be made to be binary, this can be achieved by including the term $M^2(y_i^2 - y_i)$ for a sufficiently large value of M. For $y_i = 0$ or 1, this expression evaluates to 0, and for any other integer value of y_i, the expression evaluates to a large number which is at least M^2. Thus, by choosing large values of M, the follower could be forced to choose binary values for y_i. Secondly, adding the said term $M^2(y_i^2 - y_i)$ never introduces any non-convexity in the problem, as $M(y_i^2 - y_i)$ is a convex quadratic function of y_i. Thus for the rest of the proof, we arbitrarily allow any follower variable to be binary.

Now, given an instance of SUBSET-SUM-INTERVAL with $q_1, \dots, q_k$, we define the constants $Q := \sum_{i=1}^{k} q_i$, and $B := R + 2^r - 1 + rQ$.

Further, we define $4r+2k+1$ variables controlloed by the follower. The first set of variables notated as $y^p \in \{0,1\}^r$, $y^d \in \{0,1\}^r$, $y^o \in \{0,1\}^k$, and $y_s \in \{0,1\}$ account for $2r+k+1$ variables. The remaining $2r+k$ variables are notated as $z^p \in \{0,1\}^r, z^d \in \{0,1\}^r$ and $z^o \in \{0,1\}^k$. For notational convenience, the variables y^p, z^p, y^d and z^d are indexed from 0 to $r-1$. y^o and z^o are indexed from 1 to k. The leader has $2r+k$ variables notated as $x^p \in \{0,1\}^r$ and $x^d \in \{0,1\}^r$ The objective function of the follower is

$$\left(\sum_{i=0}^{r-1} \left((Q+2^i) y_i^p + Q y_i^d \right) + \sum_{i=1}^{k} q_i y_i^o + y_s - B \right)^2 + \\ M \sum_{i=0}^{r-1} \left(x_i^p + y_i^p - z_i^p \right)^2 + M \sum_{i=0}^{r-1} \left(x_i^d + y_i^d - z_i^d \right)^2 . \tag{33a}$$

We note that in the above expression, we have not included the terms that enforce the binary constraints on the follower's variables. As needed in (Skl-Quad-Obj-P), all the decision variables of the follower are reqeuired to be integers.

The leader's constraints are the following.

$$\sum_{i=0}^{r-1} x_i^p + \sum_{i=0}^{r-1} x_i^d \quad \le \quad r \tag{33b}$$

$$x_i^p, x_i^d \quad \in \quad \{0,1\} \tag{33c}$$

The leader's objective is to minimise

$$\sum_{i=0}^{r-1} (Q+2^i) y_i^p + \sum_{i=0}^{r-1} Q y_i^d + \sum_{i=1}^{k} q_i y_i^o \tag{33d}$$

eq:QuadHardProb

Having defined the equivalent Integer Convex Quadratic Stackelberg Game in (33), we first analyse the follower's objective function. The last term, as mentioned in the objective function ensures that $x_i^d + y_i^d - z_i^d = 0$. If not, the objective increases by M, which the follower does not want. Since z_i^d is non-negative (binary in particular), and it does not appear anywhere, we can simplify this to indicate a follower constraint $x_i^d + y_i^d \le 1$ for each i from 0 to $r-1$,

Analogously, by looking at the penultimate term, we can think of it to be equivalent to $x_i^p + y_i^p \le 1$ constraint added to the follower's problem.

Finally, the first term is minimised when

$$\sum_{i=0}^{r-1} (Q+2^i) y_i^p + \sum_{i=0}^{r-1} Q y_i^d + \sum_{i=1}^{k} q_i y_i^o \quad = \quad B - y_s$$

The LHS there is exactly the leader's objective. y_s in the RHS can be chosen by the follower to be 0 or 1. Thus, this term is minimised if the leader's objective is either B or $B-1$. Moreover, the pessimistic assumption ensures that, given both the choices, the follower will try to ensure that the leader's objective is B and not $B-1$.

Next, we state that achieving an optimal objective value strictly better than B indicates we have a YES instance of SUBSET-SUM-INTERVAL.

Claim 1. If the answer to the SUBSET-SUM-INTERVAL instance is YES, then the leader's optimal objective value for the problem in (33) is $B-1$ or smaller.

Proof of claim 1. Let S be the smallest integer between R and $R+2^r$ that cannot be written as a subset sum of q_1 to q_k. This means S cannot be written as a subset sum of q_1 to q_k but $S-1$ can be. The constraints on the leader restricts them to choose at most r components of the x^p and x^d variables to be 1. We build a feasible solution to the leader that has objective value of at most $B-1$ now. Since $S-R$ is a positive integer less than 2^r, it can be written as a sum of first r powers of 2. i.e., $S-R=\sum_{i\in I}2^i$ where $I\subseteq\{0,\dots,r-1\}$. Now, let the leader choose $x^p_i=1, x^d_i=0$ if $i\in I$ and $x^p_i=0, x^d_i=1$ if $i\notin I$. Clearly, the leader has picked exactly r components of x^p, x^d to be 1.

As analysed earlier, the follower essentially has constraints of the form $x^p_i+y^p_i\le 1$ and $x^d_i+y^d_i\le 1$. Observe that among the $2r$ components in x^p and x^d, the leader has already set r of them to 1. This means, at most the remaining r of the components in y^p and y^d could be set to 1 by the follower. If the follower sets anything fewer than r components to 1 (let's say $\ell<r$ components), then the largest value that the leader's objective could possibly make is at most $\ell Q+(2^r-1)-(S-R)+Q$. The last $+Q$ is due to setting all of $y^o_i s$ to 1. However, this objective is strictly less than B as needed. So, if this is optimal to follower, then we are already done for the claim. However, this need not be optimal to the follower. Alternatively, if the follower sets all r components to 1, then the first two terms, i.e., $\sum_{i=0}^{r-1}(Q+2^i)y^p_i+\sum_{i=0}^{r-1}Qy^d_i$ evaluates to exactly $rQ+(2^r-1)+(R-S)=B-S$. Note that we said S cannot be written as subset sum, but $S-1$ can be written as a subset sum of q_1 to q_k. This means, the following. Suppose $\sum_{i\in I}q_i=S-1$ for some $I\subseteq\{1,\dots,k\}$, then setting $y^o_i=1$ for $i\in I$ and the remaining $y^o_i=0$ makes $\sum_{i=0}^{r-1}(Q+2^i)y^p_i+\sum_{i=0}^{r-1}Qy^d_i+\sum_{i=1}^{k}q_iy^o_i=B-1$. Now, the follower can choose $y_s=1$, which will be optimal for the follower. However, this means that the leader has an objective of $B-1$ as required.

Claim 2. If the answer to the SUBSET-SUM-INTERVAL instance is NO, then the leader's optimal objective value for the problem in (33) is B.

Proof of claim 2. The leader must choose at most r components of x^p and x^d to be 1. Among the components that the leader did not set to 1, the follower can arbitrarily choose r components of y^p and y^d, and set them to 1. Suppose $T:=\sum_{i=0}^{r-1}(Q+2^i)y^p_i+\sum_{i=0}^{r-1}Qy^d_i$, we can observe that T will be between rQ (if all of y^ds are 1s) and $rQ+2^r-1$ (if all of y^ps are 1s). Now, since it is a NO instance of SUBSET-SUM-INTERVAL, both $B-T$ and $B-T-1$ can be written as subset sum of q_1 to q_k. But among the two, because of the pessimistic assumption, the follower will make a choice that hurts the leader the most. So, let $I\subseteq\{1,\dots,k\}$ such that $\sum_{i\in I}q_k=B-T$, the follower will set $y^o_i=1$ for $i\in I$ and set

$y_s = 0$. This makes the leader's objective equal to $T + (B - T) = B$, completing the proof. ■

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