Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

An Algorithmic Solution to the Blotto Game Using Multimarginal Couplings

Vianney Perchet, Philippe Rigollet, Thibaut Le Gouic

informs
https://pubsonline.informs.org/journal/opre
OPERATIONS RESEARCH
Articles in Advance, pp. 1–15
ISSN 0030-364X (print), ISSN 1526-5463 (online)

Crosscutting Areas

An Algorithmic Solution to the Blotto Game Using Multimarginal Couplings

Vianney Perchet,[a,b,*] Philippe Rigollet,[c] Thibaut Le Gouic[d]

[a] Centre de Recherche en Économie et Statistique (CREST), ENSAE, 91120 Palaiseau, France; [b] Criteo AI Laboratory, Team Fairplay, 75009 Paris, France; [c] Mathematics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139l; [d] Institut de Mathématiques de Marseille, Ecole Centrale de Marseille, 13453 Marseille, France

*Corresponding author

Contact: vianney.perchet@normalesup.org, https://orcid.org/0000-0002-9333-264X (VP); rigollet@math.mit.edu (PR); thibaut.le_gouic@math.cnrs.fr (TLG)

Received: February 3, 2023
Revised: June 2, 2023
Accepted: February 12, 2024
Published Online in Articles in Advance: March 22, 2024

Area of Review: Machine Learning and Data Science

https://doi.org/10.1287/opre.2023.0049

Abstract. We describe an efficient algorithm to compute solutions for the general two-player Blotto game on n battlefields with heterogeneous values. Whereas explicit constructions for such solutions have been limited to specific, largely symmetric or homogeneous setups, this algorithmic resolution covers the most general situation to date: a value-asymmetric game with an asymmetric budget with sufficient symmetry and homogeneity. The proposed algorithm rests on recent theoretical advances regarding Sinkhorn iterations for matrix and tensor scaling. An important case that had been out of reach of previous attempts is that of heterogeneous but symmetric battlefield values with asymmetric budgets. In this case, the Blotto game is constant-sum, so optimal solutions exist, and our algorithm samples from an ε-optimal solution in time $\tilde{\mathcal{O}}(n^2 + \varepsilon^{-4})$, independent of budgets and battlefield values, up to some natural normalization. In the case of asymmetric values where optimal solutions need not exist but Nash equilibria do, our algorithm samples from an ε-Nash equilibrium with similar complexity but where implicit constants depend on various parameters of the game such as battlefield values.

Funding: V. Perchet acknowledges support from the French National Research Agency (ANR) [Grant ANR-19-CE23-0026] as well as the support grant, and Investissements d'Avenir [Grant LabEx Ecodec/ANR-11-LABX-0047]. P. Rigollet is supported by the NSF [Grants IIS-1838071, DMS-2022448, and CCF-2106377].
Supplemental Material: The online appendix is available at https://doi.org/10.1287/opre.2023.0049.

Keywords: Blotto • equilibria computations • optimal transport

1. Introduction

A century ago, Emile Borel published his seminal paper, "The Theory of Play and Integral Equations with Skew-Symmetric Kernels" (Borel 1921; see also von Neumann and Fréchet 1959, p. 157). Although perhaps not as conspicuous, it predates von Neumann's monumental work, "On the Theory of Games of Strategy" (von Neumann 1928) by several years. In this work, Borel describes what is now called the Blotto game and introduces the notions of strategy and mixed strategies and even foresees the fruitful interactions between game theory and economics that are to be observed throughout the century. As such, the Blotto game is considered to be the genesis of modern game theory (Fréchet 1953, Nakayama 2006). Despite its prestigious pedigree, equilibrium strategies for this game are only known in special cases.

Blotto is a resource-allocation game in which two players compete over n different battlefields simultaneously, allocating resources to each battlefield. The following two additional characteristics are perhaps the most salient features of the Blotto game:

1. *Winner-takes-all*: For each battlefield, the player allocating the most resources to a given battlefield wins the battlefield.

2. *Fixed budget*: each player is subject to a fixed—and deterministic—budget that mixed strategies should satisfy almost surely.

Despite its apparent simplicity, the Blotto game captures a variety of practical situations that extend far beyond the context of the military terminology described. These include political strategy (Myerson 1993, Merolla et al. 2005), network security (Labib et al. 2015, Ferdowsi et al. 2021), and various forms of practical auction markets (Masucci and Silva 2015, Hajimirsaadeghi and Mandayam 2017). For instance, in the bipartisan plurality voting game (Laslier and Picard 2002), two parties compete in each of the 50 states for some election. They both allocate part of their overall budget campaigns to each state, and the winner of a specific state is the party that allocated the highest budget to it—this simplification is for illustration purposes, but in practice, many other factors are at play, fortunately for democracy. Winning a state secures a number of senators/electoral votes, and the objective of the plurality game is to maximize the cumulative number of votes over all states.

The goal of this paper is to efficiently construct a Nash equilibrium for this game or, when they exist, an optimal strategy.

1.1. Prior Work

Despite its century-long existence, Nash equilibria for the Blotto game are only known under various restrictions on the main parameters of the problem: the budget of each player and the value given to each battlefield.

- *Budget.* A large fraction of the literature considers the case where the players have *symmetric budgets*, starting with the original problem of Borel (Borel 1921) and in most of the main contributions throughout the 20th century (Borel 1921, Borel and Ville 1938, Gross 1950, Gross and Wagner 1950, Laslier 2002, Thomas 2018). The case of symmetric budgets is well understood except in the setup where players may disagree on the value of a battlefield that was recently introduced (Kovenock and Roberson 2021).
- *Battlefields.* When the two players have different budgets, the situation becomes more complex, as the poorest will have to forfeit some battlefields. In this case, only partial results are known. To understand what "partial" means, recall that full generality of the battlefield values occurs when (i) players may assign a different value to a given battlefield—we say that the values are *asymmetric*—and (ii) these values may vary across battlefield—we say that the battlefields are *heterogeneous*. Partial results are known for symmetric values. Even under this simplifying assumption, the case of heterogeneous battlefields remains poorly understood except in the case of two battlefields (Macdonell and Mastronardi 2015). In the case of more than two battlefields, Nash equilibria are known for homogeneous battlefields (Roberson 2006) or under stringent assumptions on the battlefield values (Schwartz et al. 2014) that essentially reduce to the homogeneous case.

We refer the reader to Table 1 for a survey of recent advances. Whereas we tackle the most general setup to date, we stress that an important case was not covered by prior literature: the case of asymmetric budget and heterogeneous and symmetric values. Indeed, in this case, the game is constant-sum, and optimal strategies exist. Our results also cover the case of asymmetric values introduced very recently (Kovenock and Roberson 2021), but this setup leads to only Nash equilibria rather than optimal strategies. Whenever possible, we conflate the two setups and simply refer to a *solution* to the Blotto.

A discrete version of the Blotto game where both budgets and allocations are required to be integral was also introduced in Borel's original paper (Borel 1921). Explicit optimal solutions were provided (Hart 2008) for the homogeneous and symmetric version of the discrete game; see also Hortala-Vallve and Llorente-

Table 1. Variants of the Continuous Blotto Game and Their Solutions

	Asymmetrical budget	Heterogeneous values	Asymmetrical values	More than three battlefields	Complete results
Borel and Ville (1938)	✗	✗	✗	✗	✓
Gross and Wagner (1950)	✗	✗	✗	✓	✓
Weinstein (2012)	✗	✗	✗	✓	✓
Gross (1950)	✗	✓	✗	✓	✓
Laslier (2002)	✗	✓	✗	✓	✓
Thomas (2018)	✗	✓	✗	✓	✓
Gross and Wagner (1950)	✓	✓	✗	✗	✓
Roberson (2006)	✓	✗	✗	✓	✓
Macdonell and Mastronardi (2015)	✓	✓	✗	✗	✓
Schwartz et al. (2014)	✓	✓	✗	✓	✗
Kovenock and Roberson (2021)	✓	✓	✓	✓	✗
This paper	✓	✓	✓	✓	✓

Note. The last column, "Complete results," indicates whether results obtained hold with possibly strong assumptions on the different values (for instance, there always exist more than three battlefields with the exact same value).

Saguer (2012) for partial solutions in the asymmetric-value case. More recently, this discrete version has seen significant computational advances (Behnezhad et al. 2017, Ahmadinejad et al. 2019, Beaglehole et al. 2022). Conceptually, this line of work is close to the present paper in the sense that it provides an algorithm to sample from (approximate) solutions. Moreover, the discrete Blotto game can be seen as a discretization of the continuous version of interest here and that could be quantified using the arguments of Section 4. However carrying out this analysis, for instance, based on Beaglehole et al. (2022, theorem 4.2), leads to worse dependence on n and ε compared with Theorem 2 here. More strikingly, the complexity bound of Theorem 2 does not depend on budgets or battlefield values. On the other hand, solving the discrete symmetric Blotto can be reduced to solving some linear program with a polynomial number (in budget and battlefields) constraints and variables, as first indicated in Ahmadinejad et al. (2019), later improved in Behnezhad et al. (2017). The two lines of work also differ in more profound ways. First and foremost, the approach employed here is fundamentally different: it aims at mixing known solutions for the related Lotto game, whereas solutions to the discrete Blotto games are more agnostic, so it is unclear what the marginals of the resulting strategy are. In particular, the present approach allows us to sample from ε-Nash equilibria in the asymmetric-value case, whereas this setup is currently out of reach for solutions to the discrete Blotto game.

Finally, note that our approach also yields new (existential) results for the discrete Blotto game. Because they are not the focus of our contribution, they are relegated to the electronic companion.

1.2. Our Contributions

All of the solutions given for two-player games have consisted of constructing explicit solutions. Because of the budget constraints, these strategies can be decomposed into two parts: *marginal distributions* that indicate which (random) strategy to play on each battlefield and a *coupling* that correlates the marginal strategies in such a way to ensure that the budget constraint is satisfied almost surely.

The first question may be studied independently of the second by considering what is known as the (general) *Lotto* game (Bell and Cover 1980). In this game, the budget constraint need only be enforced in *expectation* with respect to the randomization of the mixed strategies. Whereas this setup lacks a defining characteristic of the Blotto game (fixed budget), it has the advantage of lending itself to more amenable computations. Indeed, unlike the Blotto game, a complete solution to the Lotto game was recently proposed in Kovenock and Roberson (2021), where the authors describe an explicit Nash equilibrium in the most general case: asymmetric budget and asymmetric and heterogeneous values.

In light of this progress, a natural question is whether the marginal solutions discovered in Kovenock and Roberson (2021) can be *coupled* in such a way that the budget constraint is satisfied almost surely. We provide a positive answer to this question by appealing to an existing result from the theory of *joint mixability* (Wang and Wang 2016). Mixability asks the following question: can n random variables $X_1, \dots, X_n$ with prescribed marginal distributions $X_i \sim P_i$ be coupled in such a way that $\mathrm{var}(X_1 + \cdots + X_n) = 0$? Joint mixability is precisely the step required to go from a Lotto solution to a Blotto one by coupling the marginals of the Lotto solution in such a way that the budget constraint is satisfied. The idea of coupling random variables to satisfy almost certain constraints also appears in the design of a quite specific type of auction (Border 1991); we mention here that, interestingly, there are connections between the Lotto solutions and those of a so-called "all-pay auction" (Kovenock and Roberson 2021). The arguments are fundamentally different, as they rely on some separating hyperplane theorem in an infinite dimension. Unfortunately, even in this somewhat specific problem, this theorem does not provide constructive proof, merely an existence result.

In this paper, we exploit a simple and new connection between joint mixability and the theory of *multimarginal couplings* that has recently received a regain of interest in the context of optimal transport (Agueh and Carlier 2011, Di Marino et al. 2017, Altschuler and Boix-Adsera 2020). In multimarginal optimal transport, the goal is to optimize a cost over the space of couplings with given marginals. Unlike the case of two marginals that arise in traditional optimal transport, this question raises significant computational challenges and often leads to NP-hardness (Altschuler and Boix-Adsera 2021). In the language of optimization, joint mixability merely asks if the set of constraints is nonempty. We propose an algorithmic solution to the Blotto problem by efficiently constructing a coupling that satisfies the budget constraint almost surely and can be easily sampled from. Our construction relies on three key steps: first, we reduce the problem to a small number of marginals to bypass the inherent NP-hardness of multimarginal problems; second, we discretize the marginals; and finally, we employ a multimarginal version of the Sinkhorn algorithm (Sinkhorn 1964, Sinkhorn and Knopp 1967) to construct a coupling of the discretized marginals. After a simple smoothing step, we produce a sampling with continuous marginals that are close to the ones prescribed by the Lotto solutions and from which it is straightforward to sample. Furthermore, we quantify the combined effect of discretization error and of the Sinkhorn algorithm on the value of the game, effectively leading to an *approximate* Nash equilibrium and even to an approximately optimal solution in the case of symmetric values. We mention here that the level of approximation is a parameter of the algorithm; that is, it can be chosen arbitrarily small for any number of battlefields and values (under mild balancedness conditions described later). This is in contrast with other approximation approaches (Vu et al. 2018), where the approximation factor decreases as a function of the number of battlefields.

We exhibit tight—or near tight in the asymmetric values case—conditions for the mixability of *specific* Lotto solutions into Blotto solutions; see Corollaries 1 and 2. Whereas these conditions are reasonable and cover most cases, some heavily skewed games, either in terms of budget asymmetry or values inhomogeneity, are not covered by our results. More specifically, even in the simple cases with extremely skewed parameters where marginals of the Blotto solution have been computed (Roberson 2006), they are quite different from the aforementioned Lotto solutions (they are mixtures of Dirac masses). We even prove, on the contrary, that the latter cannot be the marginals of any Blotto solutions. We leave it as an open question to exhibit Lotto strategies that can be mixed into Blotto ones even for such games.

The rest of this paper is organized as follows. In the next section, we recall the solution for the Lotto game and show that it can be turned into solutions for the Blotto game. This existential result simply appeals to existing results of joint mixability. We move from an existential to an algorithmic result in Section 3 by proceeding in three steps: first, we reduce the problem to the case $n=4$, then we discretize the problem, and finally, we apply the Sinkhorn algorithm to couple the resulting marginals in an appropriate fashion. The main product of Section 3 is Algorithm 8, which shows how to sample from an approximate solution to the Blotto game. Finally, we provide a detailed complexity analysis for this algorithm in Section 4, showing, in particular, that it runs in time polynomial in the parameters of the Blotto game and the approximation error ε. Finally, our techniques also yield new results for the discrete Blotto game largely studied by Hart (2008) and Hortala-Vallve and Llorente-Saguer (2012) that are of independent interest. We postpone them to the electronic companion.

Notation. For any integer n, define $[n]=\{1,\dots,n\}$. We use $\mathbf{1}$ to denote an all-ones vector or tensor. Note that the dimension of this vector will be clear from the context but may vary across occurrences. For any two vectors x, y, we denote their entry-wise (Hadamard) product $x \odot y$ and their entry-wise division $x \oslash y$ whenever y has only nonzero entries. For any two real numbers a and b, we denote by $a \vee b$ their maximum and by $a \wedge b$ their minimum.

2. Solutions for Blotto and Lotto Games

The goal of this section is to describe the Blotto game and its connection to the Lotto game for which explicit solutions are known. We first recall a solution for the Lotto game derived in Kovenock and Roberson (2021) and show that it can be readily turned into a Blotto solution using the theory of joint mixability.

2.1. The Blotto Game

The classical two-player *Blotto* game is formalized as follows. Two players, respectively denoted by A and B, are competing over $n \geq 2$ battlefields denoted by $i \in [n]$. Because we focus on two-player games where both players obey the same rules, it will be convenient when describing the game to denote by $P \in \{A,B\}$ either player and by $\overline{P}$ the other player so that $(P,\overline{P}) \in \{(A,B),(B,A)\}$.

The *datum* of a Blotto game is as follows. Player $P \in \{A,B\}$ has a total budget of $T_P > 0$ to allocate across the n battlefields. Moreover, the player valuates battlefield $i \in [n]$ to $v_{P,i} > 0$, which may differ from $v_{\overline{P},i}$. Without loss of generality, we assume that $T_A \geq T_B$ in order to break symmetry and that

$$\sum_{i\in[n]} v_{A,i} = \sum_{i\in[n]} v_{B,i} = 1. \tag{1}$$

Indeed, multiplying the value of all battlefields by the same constant has no impact on the players' strategies. We mention here that this normalization naturally impacts the quality of approximated solutions in the following sense. Consider a Blotto game where Equation (1) is not satisfied for player A; that is, $\sum_{i\in[n]} v_{A,i} = \overline{v}_A \neq 1$. Any ε-Nash equilibrium of the normalized game (i.e., with respect to the battlefield values $v'_{A,i} = v_{A,i}/\overline{v}_A$ and $v'_{B,i} = v_{B,i}/\overline{v}_B$) induces a $\overline{v}\varepsilon$-Nash equilibrium of the original game, where $\overline{v} = \overline{v}_A \vee \overline{v}_B$.

The *rules* of the Blotto game are as follows. A pure strategy for player P is an allocation vector $x_P = (x_{P,1}, \dots, x_{P,n})$, where $x_{P,i} > 0$ is the amount allocated to battlefield $i \in [n]$. A mixed strategy for player P is a probability distribution over pure strategies. A salient feature of the Blotto game is that a player P is constrained to playing strategies that satisfy the budget constraint: $x_{P,1} + \dots x_{P,n} \leq T_P$. In turn, admissible mixed strategies for the Blotto games are random vectors $X_P \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} X_{P,i} \leq T_P \quad \text{almost surely}. \tag{2}$$

Given two pure strategies x_P and $x_{\overline{P}}$ for players P and $\overline{P}$, respectively, player P wins battlefield $i \in [n]$ if $x_{P,i} > x_{\overline{P},i}$ and receives a reward $v_{P,i} > 0$. Ties $x_{P,i} = x_{\overline{P},i}$ are broken arbitrarily, as they are irrelevant to our analysis.

The existence of Nash equilibria is a consequence of standard game-theoretic arguments (Reny 1999). Unfortunately, these general results say little about the structure of equilibrium strategies. At the end of this section, we make partial progress toward this question by describing the marginals of such equilibrium strategies. However, these remain existential results in essence.

This is in stark contrast with the associated Lotto game, described in the following section, where the hard budget constraint is dropped in favor of a constraint in expectation and whose explicit solutions have been computed.

2.2. The Associated Lotto Game

A *Lotto* game has the same data and rules as its associated Blotto game except for the almost sure Budget Constraint (2), which is relaxed to the following *expected* budget constraint:

$$\sum_{i=1}^{n} \mathbb{E}[X_{P,i}] \leq T_P. \tag{3}$$

This relaxation greatly simplifies the game. In fact, Kovenock and Roberson (2021) have recently elicited an explicit characterization of a nontrivial Nash equilibrium for the most general version of the Lotto game to date; see Table 1. In the rest of Section 2.2, we describe their solution in detail because it is the basis for ours.

Finding an optimal strategy for the Lotto game amounts to finding a stationary point for an optimization problem subject to constraints of Form (3). Because of the linearity of expectation, the associated Lagrangian is decomposed as the sum of n terms, one per battlefield, that are each mathematically equivalent to an "all-pay" auction whose solutions are well-known.

More explicitly, Nash equilibria of the Lotto problem depend on two parameters, $\gamma \geq 0$ and $\lambda \geq 0$, that are set later on. First, given any $\gamma \geq 0$, consider the subsets of battlefields $\mathcal{N}(\gamma)$ that are at least γ-times more valuable to A than to B:

$$\mathcal{N}(\gamma) = \left\{ i \in [n], \frac{v_{A,i}}{v_{B,i}} \geq \gamma \right\}.$$

Given a scaling parameter $\lambda \geq 0$ to be defined later, the mixed strategy of player A at equilibrium prescribes allocating a (random) budget of $X_{A,i}$ to battlefield i with distribution given by

$$X_{A,i} \sim \begin{cases} \mathsf{Unif}\left[0, \dfrac{\gamma v_{B,i}}{\lambda}\right] & \text{if } i \in \mathcal{N}(\gamma) \\ \left(1 - \dfrac{v_{A,i}}{\gamma v_{B,i}}\right)\delta_0 + \dfrac{v_{A,i}}{\gamma v_{B,i}} \mathsf{Unif}\left[0, \dfrac{v_{A,i}}{\lambda}\right] & \text{if } i \notin \mathcal{N}(\gamma), \end{cases}$$

where δ_0 denotes the Dirac point mass at zero.

The strategy of player B is given by

$$X_{B,i} \sim \begin{cases} \left(1 - \dfrac{\gamma v_{B,i}}{v_{A,i}}\right)\delta_0 + \dfrac{\gamma v_{B,i}}{v_{A,i}} \mathsf{Unif}\left[0, \dfrac{\gamma v_{B,i}}{\lambda}\right] & \text{if } i \in \mathcal{N}(\gamma), \\ \mathsf{Unif}\left[0, \dfrac{v_{A,i}}{\lambda}\right] & \text{if } i \notin \mathcal{N}(\gamma). \end{cases}$$

Note that the strategies of A and B are the same except that the roles of $v_{A,i}$ and $\gamma v_{B,i}$ are switched. In that sense, γ plays the role of an "exchange" rate that accounts for discrepancies between budgets and valuations across the two players.

It remains to find the parameters γ and λ using the budget constraints. For this set of strategies, saturating the total budget constraint (2) readily yields the following two equations:

$$\lambda T_A = \sum_{i \in \mathcal{N}(\gamma)} \frac{\gamma v_{B,i}}{2} + \sum_{i \notin \mathcal{N}(\gamma)} \frac{(v_{A,i})^2}{2\gamma v_{B,i}} = \frac{1}{2}\sum_{i=1}^{n} (\gamma v_{B,i}) \wedge \frac{(v_{A,i})^2}{\gamma v_{B,i}}, \tag{4}$$

$$\lambda T_B = \sum_{i \in \mathcal{N}(\gamma)} \frac{(\gamma v_{B,i})^2}{2 v_{A,i}} + \sum_{i \notin \mathcal{N}(\gamma)} \frac{v_{A,i}}{2} = \frac{1}{2}\sum_{i=1}^{n} \frac{(\gamma v_{B,i})^2}{v_{A,i}} \wedge v_{A,i}. \tag{5}$$

Any pair (γ, λ) solving this system of two equations yields a Nash equilibrium. It remains to show that such solutions may be computed efficiently. Observe that eliminating λ from the equations yields the following nonlinear equation in γ:

$$\begin{aligned} f(\gamma) &:= \gamma^3 \left(T_A \sum_{i \in \mathcal{N}(\gamma)} \frac{v_{B,i}^2}{v_{A,i}} \right) - \gamma^2 T_B \sum_{i \in \mathcal{N}(\gamma)} v_{B,i} \\ &\quad + \gamma T_A \sum_{i \notin \mathcal{N}(\gamma)} v_{A,i} - T_B \sum_{i \notin \mathcal{N}(\gamma)} \frac{v_{A,i}^2}{v_{B,i}} \\ &= \gamma T_A \sum_{i=1}^{n} \left(\gamma^2 \frac{v_{B,i}^2}{v_{A,i}} \wedge v_{A,i} \right) \\ &\quad - T_B \sum_{i=1}^{n} \frac{v_{A,i}}{v_{B,i}} \left(\gamma^2 \frac{v_{B,i}^2}{v_{A,i}} \wedge v_{A,i} \right) = 0. \end{aligned} \tag{6}$$

Any solution γ^* to this equation readily yields a unique λ^* by plugging it into either (4) or (5); both equations will yield the same solution by (6). In turn, the existence and efficient computation of solutions γ^* to (6) are ensured by the following proposition. The bounds on γ^* presented in the following proposition depend on the distance between the vectors of battlefield values $v_A := (v_{A,1}, \ldots, v_{A,n})$ and $v_B := (v_{B,1}, \ldots, v_{B,n})$. Interestingly, the natural measure of distance that emerges is the χ^2-divergence that commonly arises in information theory and statistics; see, for example, Polyanskiy and Wu (2022). The χ^2-divergence $\chi^2(u\|v)$ between two probability vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ is defined by

$$\chi^2(u\|v) = \sum_{i=1}^{n} \frac{u_i^2}{v_i} - 1 = \sum_{i=1}^{n} \left(\frac{u_i}{v_i} - 1\right)^2 v_i.$$

It is clear that $\chi^2(u\|v) \geq 0$ with equality if and only if $u = v$.

Proposition 1. *Equation (6) has the following properties:*

1. *It always has at least one and at most $3n + 3$ solutions γ^*.*
2. *Any solution γ^* satisfies*

$$\frac{T_B}{T_A} \frac{1}{1 + \chi^2(v_B\|v_A)} \leq \gamma^* \leq \frac{T_B}{T_A}(1 + \chi^2(v_A\|v_B)).$$

3. *Computing all solutions can be done in $\mathcal{O}(n \log n)$ operations.*

The proof is based on standard computations, hence postponed to Electronic Companion EC.2, with the associated Algorithm 2.

Remark 1. With symmetric values, that is, when $v_{A,i} = v_{B,i} = v_i$, the game is constant-sum, and each player has a then unique[1] optimal strategy given by a unique pair (γ^*, λ^*) (Kovenock and Roberson 2021). In fact, in that case, the unique (γ^*, λ^*) can be computed analytically as $\gamma^* = T_B/T_A$ and $\lambda^* = T_B/(2T_A^2)$; this can be easily seen from Proposition 1 (point 2) because $\chi^2(v_A\|v_B) = \chi^2(v_B\|v_A) = 0$. With these parameters, the optimal strategy of player A is to choose $X_{A,i}$ uniformly at random on $[0, 2T_A v_i]$, and that of player B is to forfeit each battlefield with probability $1 - \frac{T_B}{T_A}$ and to choose $X_{B,i}$ uniformly at random on $[0, 2T_A v_i]$ on battlefield i if not forfeited.

2.3. From Lotto to Blotto

In the previous section, we described how to compute solutions of a Lotto game. To turn a strategy for the Lotto game into a strategy for the Blotto game, one can *couple* the marginal strategies of a Lotto game, effectively turning Constraint (3) on the expected budget into the almost sure Budget Constraint (2).

Stated otherwise, a solution to the associated Lotto game induces a solution to the original Blotto game if the random variables $\{X_{A,i}\}_{i\in[n]}$ (and similarly, $\{X_{B,i}\}_i$) are *jointly mixable* (Wang and Wang 2016).

Definition 1. A family of k random variables $Z_1, \ldots, Z_k$ with finite expectations is *jointly mixable* if there exists a coupling π such that if $(Z_1, \ldots, Z_k) \sim \pi$,

$$\sum_{i=1}^{k} Z_i = \sum_{i=1}^{k} \mathbb{E}[Z_i], \quad \text{almost surely}.$$

In that case, the coupling π is called a *joint mix*.

Obviously, not all random k-tuples variables are jointly mixable. Take, for example, Z_1, Z_2, and Z_3 to be Bernoulli with parameter $1/2$. Then, $\mathbb{E}[Z_1] + \mathbb{E}[Z_2] + \mathbb{E}[Z_3] = 3/2$, whereas there is no coupling of the Z_is such that their sum equals a fractional number.

Whereas the full characterization of jointly mixable distribution is a complex question, some conditions, either sufficient or necessary, for joint mixability have been derived. The following proposition is a simple extension of a result of Wang and Wang (2016) (see also Zimin (2020) on the mixability of distributions with monotone densities.

Proposition 2. *For $i = 1, \ldots, k$, let $p_i \in [0,1]$, $b_i > 0$ be fixed parameters, and let Z_i be a random variable with distribution given by the following mixture:*

$$Z_i \sim (1 - p_i)\delta_0 + p_i \, \mathsf{Unif}[0, b_i]. \tag{7}$$

Then, $Z_1, \ldots, Z_k$ are jointly mixable if and only if

$$\max_{1\leq i\leq k} b_i \leq \frac{1}{2}\sum_{i=1}^{k} p_i b_i. \tag{8}$$

This proposition is a consequence of a few standard computations; its proof is delayed to Electronic Companion EC.3.

We are now in a position to state the main result of this section: the marginal distributions of the Lotto game described are jointly mixable into a solution to the Blotto game. To that end, we instantiate Proposition 2 to the parameters of the marginal distributions described in Section 2.2.

Theorem 1. *Let γ^*, λ^* be the parameters of Nash equilibrium of the Lotto game described in Section 2.2. Then, the marginal distributions can be coupled into a Nash equilibrium for the corresponding Blotto game if and only if*

$$\max_{i\in[n]}(\gamma^* v_{B,i} \wedge v_{A,i}) \leq \lambda^* T_B. \tag{9}$$

Inequality (9) is simply an instantiation of (8); hence, details are postponed to Electronic Companion EC.3.

Condition (9) of the previous theorem relies on the values γ^*, λ^* that define the solution of the Lotto game. In light of the bounds obtained in Proposition 1, these parameters may be eliminated to produce a sufficient condition for the existence of said solution for Blotto games with symmetric values. Recall that in this case, the game is constant-sum, so a solution is, in fact, an optimal strategy. This result is captured in the following corollary, which is a straightforward consequence of Theorem 1.

Corollary 1. *Assume symmetric values: $v_A = v_B = v$. Then, the marginal distributions of the optimal Lotto strategy described in Section 2.2 with $\gamma^* = T_B/T_A$ and $\lambda^* = T_B/(2T_A^2)$*

can be coupled into an optimal strategy for the corresponding Blotto game if and only if

$$\max_{i\in[n]} v_i \le \frac{T_B}{2T_A}.$$

In fact, a sufficient condition may be derived in the case of nonsymmetric values.

Corollary 2. *Assume that battlefields are balanced in the sense that there exists* $r \in (0,1)$ *such that*

$$\chi^2(v_A \| v_B) \vee \chi^2(v_B \| v_A) \le r^2.$$

Then, the marginal distributions of the optimal Lotto strategy described in Section 2.2 can be coupled into an optimal strategy for the corresponding Blotto game as long as

$$\max_{i\in[n]} v_{B,i} \le \frac{T_B}{2T_A}(1-r).$$

The proof of this result is based solely on computations; it is postponed to Electronic Companion EC.4. If battlefields have relatively similar values in the sense that, for some $\kappa > 1$, $\frac{v_{B,i}}{v_{B,j}} \le \kappa$ for any $i,j \in [n]$, then $\max_{i\in[n]} v_{B,i} \le \frac{\kappa}{n}$; hence Corollary 2 holds as soon as $T_A \le nT_B \frac{1-r}{2\kappa}$. Stated otherwise, the larger budget cannot be of the order of n times the smaller budget, which is rather mild. Note that the result of Corollary 2 is tight in the sense that if $r \to 0$, it recovers the result of Corollary 1. It is unclear whether the dependence in r is sharp in our result, and it is an interesting question to address in future work.

Under rather general conditions, Corollaries 1 and 2 show the *existence* of solutions with marginal distributions of the Lotto game derived in Kovenock and Roberson (2021). It remains to show that such a coupling may be realized efficiently. This is done in the next section.

3. An Efficient Algorithm to Compute Solutions

Deriving solutions, either optimal strategies in the constant-sum setting or Nash equilibria, remains one of the major open problems surrounding the Blotto game. Previous attempts at this task have focused on deriving an *explicit* coupling between marginals. This is possible in specific cases. For example, several explicit couplings between n random variables $X_i \sim \mathsf{Unif}[0,1], i = 1, \ldots, n$ are known (Rüschendorf and Uckelmann 2002, Knott and Smith 2006). In particular, this provides a solution to some Blotto problems with sufficient symmetry. However, this explicit approach fails for more general problems, and, in particular, in the important case of asymmetric budget such as the one covered in Corollary 1. In this paper, we take another route by describing the efficient algorithm Lotto2Blotto, whose pseudocode is postponed to the Electronic Companion EC.1, that computes an ε-approximate solution with time complexity that is polynomial in n and $1/\varepsilon$.

In light of the previous section, our goal is to find an algorithm that efficiently computes a coupling between the marginal Lotto strategies described earlier. This task faces two major hurdles.

On the one hand, the continuous nature of the marginals described does not lend itself to efficient algorithms that typically work with discrete quantities. Instead, we propose to simply discretize the marginals at a scale of order $\varepsilon > 0$. In particular, this prevents us from replicating exactly the marginals of the Lotto game, but we can show that the error employed in said discretization remains of the same order once propagated to the utility of a given player.

On the other hand, the mere description of a coupling between n discrete marginals on $\mathcal{O}(1/\varepsilon)$ atoms is an object of size $\mathcal{O}(1/\varepsilon^n)$, which is exponential in the number n of battlefields. To overcome this limitation, we develop a careful scheme that allows us to reduce the problem to the case of four marginals instead of n.

Finally, we employ recent developments in computational optimal transport to couple our four marginals using a variant of Sinkhorn iterations (Sinkhorn 1964, Sinkhorn and Knopp 1967, Cuturi 2013).

3.1. Reductions

The typical size of a coupling with n marginals is exponential in n. Whereas this issue is, in general, hopeless to overcome, we can exploit some of the structure of the problem at hand. Indeed, a similar principle has been recently employed in multimarginal optimal transport to devise polynomial-time algorithms under additional structure (Altschuler and Boix-Adsera 2020). More specifically, we reduce our problem to the case where there are only four marginals that remain mixable if the original marginals are mixable.

This reduction is done in two steps. Recall that the marginals for the Lotto game described in Section 2.2 are either uniform distributions or mixtures of a uniform distribution with a Dirac point mass at zero. Our first step reduces to the case where $n-1$ marginals are uniform and only one is a mixture as earlier. In our second step, we further reduce to the case where there are three uniform marginals and one mixture.

Throughout this section, we focus on player A for brevity. Reductions for player B are analogous.

3.1.1. Step 1. Reduction to a Single Mixture. The marginal distributions described in Section 2.2 consist of $|\mathcal{N}(\gamma^*)|$ uniform distributions and $n - |\mathcal{N}(\gamma^*)|$ mixtures of a uniform distribution and a Dirac point mass at zero, and our goal is to efficiently couple them into a joint mix coupling π that has these marginals and satisfies the Blotto budget constraint. For clarity, we also regard uniform distributions as mixture distributions albeit with zero weight on the point mass. Otherwise, we say that a distribution is a *strict* mixture. The goal of this first step is to reduce this coupling problem to the case where there are $n-1$ uniform distributions and one single strict mixture. To that end, we show that such a coupling π may be obtained as a mixture of n joint mixes $\pi_k, k = 1, \ldots, n$:

$$\pi = \sum_{k=1}^{n} q_k \pi_k, \quad q_k \geq 0, \sum_k q_k = 1,$$

where the marginal distributions of π_k consist of, at most, one strict mixture, the rest being uniform distributions. Moreover, this decomposition can be computed efficiently as the solution of a simple greedy procedure.

Lemma 1. *Let γ^*, λ^* be the parameter of a solution for the Lotto game, and assume that Mixability Condition (9) holds. Then, there exists a family $\pi_1, \ldots, \pi_n$ of couplings and a set of nonnegative weights $q_k \geq 0, \sum_k q_k = 1$ such that*

1. *The marginal distributions of $(X^k_{A,1}, \ldots, X^k_{A,n}) \sim \pi_k$ are given by*

$$X^k_{A,i} \sim (1 - p_i^{(k)})\delta_0 + p_i^{(k)} \mathsf{Unif}\left[0, \frac{v_{A,i} \wedge (\gamma^* v_{B,i})}{\lambda^*}\right]$$

for some $p_i^{(k)} \in [0,1], i, k \in [n]$ with, at most, one $p_i^{(k)}$ in $(0,1)$ for each k.

2. *Each coupling $\pi_k, k = 1, \ldots, n$ is a joint mix.*
3. *The mixture of couplings*

$$\pi = \sum_{k=1}^{n} q_k \pi_k. \tag{10}$$

is a solution for the Blotto game.

4. *The total complexity of computing the weights $q_k, p_i^{(k)}$, $i, k \in [n]$ scales as $\mathcal{O}(n^2 \log n)$.*

Note that the mixture of couplings π in (10) is necessarily a joint mix as a mixture of joint mixes. To sample from it, player A simply samples π_k with probability q_k and plays according to the strategy prescribed by it.

The geometric proof of this lemma is delayed to Electronic Companion EC.5, along with the pseudocode of associated algorithm Decomp.

3.1.2. Step 2. Reduction to Four Random Variables.

The previous step reduces the joint mixability problem of n general mixtures to a simpler one where, at most, one strict mixture is involved. Still, computing—in fact, even describing—a coupling of n variables requires generically exponential (in n) time and memory. To overcome this limitation, we reduce the number of random variables from n to a constant number.

The following lemma states that each π_k can be realized as the coupling of three new uniform random variables and a strict mixture, thus reducing the mixability question from n to only four random variables. A careful inspection of the proof of Lemma 2 indicates that the reduction may lead to three marginals rather than four. In that case, two marginals are uniform and one is a strict mixture. To handle this case, some adjustments are needed; in particular—and obviously—with the size of the resulting coupling. However, extensions from four to three marginals are straightforward, and we omit this case for clarity.

Lemma 2. *Fix $n \geq 4, k \in [n]$, and assume without loss of generality that the last marginal of the coupling π_k from Lemma 1 is a strict mixture. Then, $(X_{A,1}, \ldots, X_{A,n}) \sim \pi_k$ may be constructed from three uniform random variables Y_1, Y_2, Y_3 and a partition $\mathcal{I}_0 \sqcup \mathcal{I}_1 \sqcup \mathcal{I}_2 \sqcup \mathcal{I}_3 = [n-1]$ as follows. Set $X_{A,i} = 0$ for all $i \in \mathcal{I}_0$, and*

$$X_{A,i} = \theta_i Y_j, \qquad i \in \mathcal{I}_j, \quad j \in \{1,2,3\},$$

where $\theta_1, \ldots, \theta_{n-1} \in [0,1]$ are such that

$$\sum_{i \in \mathcal{I}_j} \theta_i = 1, \quad j = 1,2,3.$$

In particular, it holds that

$$\sum_{i=1}^{n-1} X_{A,i} = Y_1 + Y_2 + Y_3 \quad \textit{almost surely},$$

and $(Y_1, Y_2, Y_3, X_{A,n})$ are jointly mixable. The support of Y_j is $[0, b_j^*]$ where

$$b_j^* = \sum_{i \in \mathcal{I}_j} \frac{\gamma^* v_{B,i}}{\lambda^*} \wedge \frac{v_{A,i}}{\lambda^*}.$$

Moreover, the θ_i's, the sets $\mathcal{I}_j$, and the parameters of the distributions of Y_1, Y_2, Y_3 can each be computed in constant time.

The proof of this lemma, based on standard mixability arguments, is postponed to Electronic Companion EC.7, with the pseudocode of the corresponding algorithm Reduc.

Note that any joint mix of $(Y_1, Y_2, Y_3, X_{A,n})$ readily yields a joint mix of $(X_{A,1}, \ldots, X_{A,n})$ by defining $X_{A,i} = \theta_i Y_{j(i)}$, where $j(i) \in [3]$ is the unique integer such that $i \in \mathcal{I}_{j(i)}$.

3.2. Discretization

The problem of finding a solution for the Blotto game has been reduced to the construction of a coupling of (at most) four random variables, three of them being uniform over some intervals and the fourth one being a mixture between a Dirac mass at zero and some uniform distribution. Throughout this section, we denote these random variables as $Y_1, \ldots, Y_4$ for simplicity; in the notation of the previous section, they correspond to Y_1, Y_2, Y_3, and $X_{A,n}$ respectively.

Unfortunately, even in this simple case, finding explicit, closed-form couplings appears to be possible only under stringent additional conditions that limit the scope of the Blotto game. To overcome this limitation, we take an algorithmic approach, describing an efficient way to find an approximate solution. To that end, we obviously need to work with discrete random variables and describe here a coupling between these discretized random variables.

Let $(Y_1, Y_2, Y_3, Y_4) \sim \varpi$ be jointly mixed so that

$$Y_1 + Y_2 + Y_3 + Y_4 = T_A. \tag{11}$$

Moreover, let $h > 0$ be some (small) discretization parameter. Define the quantized random variables $\tilde{Y}_i$ by

$$\tilde{Y}_i = \left\lfloor \frac{Y_i}{h} \right\rfloor, i = 1,2,3, \qquad \tilde{Y}_4 = \frac{T_A}{h} - \left\lfloor \frac{T_A - Y_4}{h} \right\rfloor. \tag{12}$$

Our goal is to compute any of the joint distributions $\mathsf{D}(\varpi)$ of the vector $\tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4)$ when ϖ ranges over joint mixes.

As a first step toward this goal, note that these discretized random variables need not be jointly mixable. Indeed, in general, we have $\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3 \le \lfloor (Y_1 + Y_2 + Y_3)/h \rfloor$, but equality may fail to hold because of discretization errors. To account for these, let $\varepsilon \in \{0,1,2\}$ be defined as

$$\varepsilon = (T_A/h - \tilde{Y}_4) - (\tilde{Y}_1 + \tilde{Y}_2 + \tilde{Y}_3), \tag{13}$$

and consider the augmented random vector $\tilde{Y}_+ = (\tilde{X}, \varepsilon)$. In light of (13), $\tilde{Y}_+ \in \mathbb{R}^5$ lives almost surely on a four dimensional subspace. As such, its distribution may be represented by a four-tensor (Γ_{ijke}) with entries given by

$$\Gamma_{ijke} = \mathbb{P}(\tilde{Y}_1 = i, \tilde{Y}_2 = j, \tilde{Y}_3 = k, \varepsilon = e).$$

In particular, note that $e \in \{0,1,2\}$, whereas i, j, and k each range in a set of integers of size $\Theta(1/h)$. Using (13), we can read off the distribution of $\tilde{Y}_4$ from this tensor.

This tensor is subject to four sets of linear constraints, one for each of the marginal constraints given in (12). They are given by

$$\sum_{jke} \Gamma_{ijke} = \mathbb{P}(\tilde{Y}_1 = i) \;\; \forall i, \quad \sum_{ike} \Gamma_{ijke} = \mathbb{P}(\tilde{Y}_2 = j) \;\; \forall j,$$
$$\sum_{ije} \Gamma_{ijke} = \mathbb{P}(\tilde{Y}_3 = k) \;\; \forall k,$$

and, in light of (13), by

$$\sum_{i+j+k+e=\ell} \Gamma_{ijke} = \mathbb{P}(T_A/h - \tilde{Y}_4 = \ell) \;\; \forall \ell.$$

Note that, indeed, any draw from a distribution that satisfies the constraints shown here yields a random vector $(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \varepsilon)$. Defining $\tilde{Y}_4$ by solving (13) yields a vector $\tilde{Y} \sim \mathsf{D}(\varpi)$ for some joint mix ϖ defined as earlier. In other words, $\tilde{Y}$ is indeed the discretization of random variables drawn from a joint mix (though it need not be jointly mixable itself).

Because the random variables Y_j, for $j \in [3]$, constructed at the previous step, have a support equal to $[0, b_j^*]$ where $b_j^* = \sum_{i \in \mathcal{I}_j} \frac{\gamma^* v_{B,i}}{\lambda^*} \wedge \frac{v_{A,i}}{\lambda^*}$, the reduced (to four random variables) and discretized problem reduces to finding some tensor (Γ_{ijke}) with $3 \cdot \lfloor \frac{b_1^*}{h} + 1 \rfloor \cdot \lfloor \frac{b_2^*}{h} + 1 \rfloor \cdot \lfloor \frac{b_3^*}{h} + 1 \rfloor$ entries satisfying, at most, $4 \cdot \lfloor \frac{\max b_j^*}{h} + 1 \rfloor$ linear constraints. Although this can be done simply via linear programming (hence polynomially in h^{-1}, more precisely in $\tilde{\mathcal{O}}(1/h^{8,5})$ with Vaidya's algorithm), a quite efficient and more popular way is to use a variant of Sinkhorn-Knopp algorithm that quickly finds approximated solutions. This is more relevant, as this linear program is already some approximation of the original problem; hence, there is no point in solving it exactly.

The pseudocode of Algorithm Discretize can be found in Section EC.8 in the electronic companion.

3.3. Tensor Scaling Using Sinkhorn Iterations

In light of the previous sections, we have reduced our problem to that of finding coupling in the form of a four-tensor $(\Gamma_{ijke}, i \in [d_1], j \in [d_2], k \in [d_3], e \in \{0,1,2\})$ with nonnegative entries subject to marginal constraints. We approach this problem from a computational perspective and propose an algorithm that converges rapidly to a feasible solution. To describe this algorithm, recall that its input is four probability vectors $\tilde{\mu}_j \in \mathbb{R}^{d_j}, j = 1, \dots, 4$, with $d_4 = d_1 + d_2 + d_3 + 2$ that represent the probability mass functions of the discretized random variable $\tilde{Y}_1$, $\tilde{Y}_2, \tilde{Y}_3, T_A/h - \tilde{Y}_4$ defined in the previous section: $\tilde{\mu}_{j,\cdot} = \mathbb{P}(\tilde{Y}_j = \cdot), j = 1, \dots, 3$, $\tilde{\mu}_{4,\cdot} = \mathbb{P}(T_A/h - \tilde{Y}_j = \cdot)$.

The linear constraints take the form

$$\bar{\Gamma}_i^{(1)} := \sum_{jke} \Gamma_{ijke} = \tilde{\mu}_{1,i} \qquad \forall i \in [d_1], \tag{14}$$

$$\bar{\Gamma}_j^{(2)} := \sum_{ike} \Gamma_{ijke} = \tilde{\mu}_{2,j} \qquad \forall j \in [d_2], \tag{15}$$

$$\bar{\Gamma}_k^{(3)} := \sum_{ije} \Gamma_{ijke} = \tilde{\mu}_{3,k} \qquad \forall k \in [d_3], \tag{16}$$

$$\bar{\Gamma}_\ell^{(4)} := \sum_{i+j+k+e=\ell} \Gamma_{ijke} = \tilde{\mu}_{4,\ell} \qquad \forall \ell \in [d_4]. \tag{17}$$

Denote by $\mathcal{G}$ the set of tensors $\Gamma = (\Gamma_{ijke})$ that satisfy these constraints.

To solve this problem, we propose to project the all-ones tensor $\mathbf{1}$ onto $\mathcal{G}$ using the Kullback-Leibler (KL) divergence. Recall that the KL divergence between two nonnegative tensors Γ, Γ' is given by

$$\mathsf{KL}(\Gamma \| \Gamma') = \sum_{ijke} \Gamma_{ijke} \log\left(\frac{\Gamma_{ijke}}{\Gamma'_{ijke}}\right).$$

In particular, $\mathsf{KL}(\Gamma \| \mathbf{1})$ is simply the (negative) entropy $\mathsf{H}(\Gamma)$ of Γ, and we aim to solve the convex optimization problem

$$\min_{\Gamma\in\mathcal{G}} \mathsf{H}(\Gamma) = \sum_{ijke} \Gamma_{ijke}\log(\Gamma_{ijke}).$$

Whereas many algorithms are available to solve this problem (Bubeck 2015), its specific structure can be exploited efficiently. Indeed, first-order optimality conditions imply that any optimal Γ must be of the form

$$\Gamma_{ijke} = \xi_{1,i}\cdot\xi_{2,j}\cdot\xi_{3,k}\cdot\xi_{4,i+j+k+e}, \tag{18}$$

for some scaling vectors $\xi_j \in (0,\infty)^{d_j}, j = 1,\dots,4$. This representation readily calls for an iterative tensor scaling algorithm similar to the Sinkhorn algorithm (Sinkhorn 1964, Sinkhorn and Knopp 1967, Cuturi 2013). Tensor scaling has been investigated in more classical setups (Altschuler and Boix-Adsera 2020, Lin et al. 2022) that slightly differ from the present setup because the fourth marginal constraint takes a special form. Nevertheless, the implementation of the algorithm Sinkhorn remains straightforward and is presented in Section EC.9 in the electronic companion. Its analysis is also a straightforward extension of that for the traditional matrix case (Altschuler et al. 2017). More specifically, following the exact same lines as the one of theorem 4.3 in Lin et al. 2022), we readily get the following result.

Proposition 3. *Define*

$$\tilde{\mu}_{\min} = \min_{i\in[4],j,\tilde{\mu}_{i,j}\neq 0} \tilde{\mu}_{i,j}.$$

Algorithm Sinkhorn terminates and returns a tensor Γ such that $\sum_{i=1}^4 \|\overline{\Gamma}^{(i)} - \tilde{\mu}_i\|_1 \le \eta$ after, at most, $32\eta^{-1}(1-\log\tilde{\mu}_{\min})$ iterations. Moreover, each marginal $\overline{\Gamma}^{(i)}$ of Γ has positive entries that sum to one and, hence, is a probability vector.

What have we accomplished so far? Through several reductions and a tensor scaling algorithm, given the datum of a Blotto game, we are able to compute a joint distribution that corresponds to an approximate solution. In Section 4, we evaluate the accuracy of this approximation in terms of the value of the game by showing that the various approximations (discretization and numerical precision of the algorithm) do not blow up when propagated back into the reductions. Before that, we investigate an important operational question: how to sample a strategy from the resulting coupling Γ.

3.4. From Coupling to Sampling

Finding an efficient construction of (approximate) equilibria or optimal strategies is only relevant if it can be associated with some efficient sampling method so that a player may query a sampler and receive the allocation $(X_{A,1},\dots,X_{A,n})$ that they should play on each battlefield. In light of the various reduction steps employed earlier, it is sufficient to sample a four-tuple

$$(\tilde{Y}_1,\tilde{Y}_2,\tilde{Y}_3,\varepsilon) \in [0,b_1^*/h]\times[0,b_2^*/h]\times[0,b_3^*/h]\times\{0,1,2\}$$

from the output Γ of the algorithm Sinkhorn. Indeed, from $(\tilde{Y}_1,\tilde{Y}_2,\tilde{Y}_3,\varepsilon)$, we obtain the random variables $Y_i, i = 1,\dots,4$ that are approximately distributed from the joint mix ϖ as follows.

To ensure that the marginal distributions are continuous, let $U \sim \mathsf{Unif}[0,1]$ and define

$$Y_1' = \left(\tilde{Y}_1 + \frac{\varepsilon}{3} + \frac{U}{3}\right)h \wedge b_1^*, \quad Y_2' = \left(\tilde{Y}_2 + \frac{\varepsilon}{3} + \frac{U}{3}\right)h \wedge b_2^*,$$
$$Y_3' = \left(\tilde{Y}_3 + \frac{\varepsilon}{3} + \frac{U}{3}\right)h \wedge b_3^*.$$

To correct for potential boundary effects, define $S = Y_1' + Y_2' + Y_3'$ and

$$\zeta = \mathbb{1}\{S > T_A\}\frac{T_A - S}{3} + \mathbb{1}\{S < T_A - b_4^*\}\frac{T_A - b_4^* - S}{3}.$$

Then, take $\overline{Y}_1 = Y_1' + \zeta$, $\overline{Y}_2 = Y_2' + \zeta$, $\overline{Y}_3 = Y_3' + \zeta$, and $\overline{Y}_4 = T_A - (Y_1 + Y_2 + Y_3)$.

We call this procedure the *smoothing* procedure. Finally, as mentioned earlier, just define $X_{A,i} = \theta_i \overline{Y}_{j(i)}$, where $j(i) \in [3]$ is the unique integer such that $i \in \mathcal{I}_{j(i)}$.

Note that the random variable $U \sim \mathsf{Unif}[0,1]$ is superfluous and theoretical results would follow by taking $U = 0$. Its role is simply to ensure, for cosmetic reasons, that the random marginal distributions are continuous apart from the potential point mass at zero.

It remains to sample $(\tilde{Y}_1,\tilde{Y}_2,\tilde{Y}_3,\varepsilon)$ from the output Γ of Algorithm Sinkhorn. This is quite straightforward in light of the factored form of Γ. Indeed, recall that the coupling output by Algorithm Sinkhorn has Form (18).

$$\Gamma_{ijke} = \xi_{1,i}\cdot\xi_{2,j}\cdot\xi_{3,k}\cdot\xi_{4,i+j+k+e}, \quad \forall i \in [d_1], j \in d_2, k \in [d_3], e \in \{0,1,2\}.$$

As a consequence, we can draw from Γ as follows:

1. Set $\tilde{Y}_1 = i \in [d_1]$ with probability proportional to $\xi_{1,i}$.
2. Set $\tilde{Y}_2 = j \in [d_2]$ with probability proportional to $\xi_{2,j}$.
3. Set $\tilde{Y}_3 = k \in [d_3]$ with probability proportional to $\xi_{3,k}$.
4. Conditionally on $(\tilde{Y}_1,\tilde{Y}_2,\tilde{Y}_3)$, set $\varepsilon = e \in \{0,1,2\}$ with probability proportional to $\xi_{4,\tilde{Y}_1+\tilde{Y}_2+\tilde{Y}_3+e}$.

The pseudocode of Algorithm Sample can be found in Section EC.10 of the electronic companion.

4. Approximation Errors and Computational Complexity

The construction of the previous section relies on various approximations, each of them inducing some error that can be mitigated at the cost of additional computational complexity by tuning the discretization parameter h of Section 3.2 and the tolerance parameter η in Algorithm Sinkhorn. In this section, we study the computational complexity required to reach an ε-approximate solution.

4.1. From Approximate Strategies to Approximate Solutions

Note that the very notion of "approximate solution" strongly depends on whether the problem is value symmetric ($v_{A,i} = v_{B,i}$ for all i) or asymmetric ($v_{A,i} \neq v_{B,i}$ for some i). Indeed, in the former case, the game is constant-sum, and optimal strategies do exist. This is no longer true in the latter case where only Nash equilibria are considered. As a consequence, we can consider approximations of a single optimal strategy in value-symmetric games, whereas we will have to consider approximations of a pair of equilibrium strategies in value-asymmetric ones. In the following two sections, we consider each case separately. In the remaining, we shall focus the analysis on player A, but it is almost identical for player B; hence, we do not repeat it for the sake of clarity.

4.1.1. The Value-Symmetric Case. A value-symmetric Blotto game, where $v_{A,i} = v_{B,i} = v_i$ for all I, is constant-sum and optimal strategies exist for each player. In particular, this allows us to provide strong approximation guarantees by controlling how suboptimal the expected utility of a player is.

To check this well-known fact on our specific instance, consider the utility of player A. Set two equilibrium parameters $\gamma^* = T_B/T_A \le 1$ and $\lambda^* = T_B/(2T_A^2)$ (see Corollary 1) defining an optimal strategy and observe that $\mathcal{N}(\gamma^*) = [n]$ because $\gamma^* \le 1$. For $i \in [n]$, let $X_{A,i} \sim \mathsf{Unif}[0, 2T_A v_i]$ denote the amount allocated by player A to battlefield i according to this optimal strategy, and denote by $F_{A,i}$ its cumulative distribution function (cdf). The expected utility (a.k.a. reward) of player A if player B chooses allocation $x_B = (x_{B,i})_i$ depends only on the sequence $F_A = (F_{A,1}, \ldots, F_{A,n})$ of marginal cdfs rather than the whole coupling. It is given by

$$\mathcal{U}_A(F_A, x_B) = \sum_{i=1}^n v_{A,i}\mathbb{P}(X_{A,i} > x_{B,i}) = \sum_{i=1}^n v_i(1 - F_{A,i}(x_{B,i}))$$
$$= 1 - \sum_{i=1}^n v_i \wedge \frac{x_{B,i}}{2T_A} \ge 1 - \frac{T_B}{2T_A}.$$

where we used the fact that the v_i's sum to one and the $x_{B,i}$'s sum to, at most, T_B. Moreover, if B employs the mixed strategy $X_{B,1}, \ldots, X_{B,n}$ described in Corollary 1, the utility of player A, denoted $\mathcal{U}_A(F_A, F_B)$, changes as follows. Let $U_{A,i}, U_{B,i} \sim \mathsf{Unif}[0, 2T_A v_i]$ be a sequence of uniform random variables such that $U_{A,i}$ is independent of $U_{B,i}$. In particular, $\mathbb{P}(U_{A,i} > U_{B,i}) = .5$ and

$$\mathcal{U}_A(F_A, F_B) = \left(1 - \frac{T_B}{T_A}\right)\sum_{i=1}^n v_i(1 - F_{A,i}(0))$$
$$+ \frac{T_B}{T_A}\sum_{i=1}^n v_i\, \mathbb{P}(U_{A,i} > U_{B,i})$$
$$= \left(1 - \frac{T_B}{T_A}\right) + \frac{T_B}{2T_A} = 1 - \frac{T_B}{2T_A}.$$

Similar inequalities can be obtained by reversing the roles of players A and B. In particular, this implies that the value of the game is $1 - \frac{T_B}{2T_A}$, and thus, the strategy of player A given in Corollary 1 is optimal, and its optimal utility is given by $1 - \frac{T_B}{2T_A}$.

To estimate the cost of the various approximations incurred by player A, let $X_{A,i}^{h,\eta} \sim P_{A,i}^{h,\eta}$ denote the strategy on battlefield $i \in [n]$. The notation is meant to emphasize that the approximation error stems from two sources: the precision level η of Algorithm Sinkhorn and the grid size h of the discretization procedure in Section 3.2. In particular, we write $P_{A,i}^{0,0} := P_{A,i}$. Cognizant of this approximation error, player A may take advantage of the suboptimality of the strategy of player A and respond with the best-response strategy denoted $X_B^{h,\eta} = (X_{B,1}^{h,\eta}, \ldots, X_{B,n}^{h,\eta})$. As a result, the suboptimality gap of player A's expected utility is controlled as follows:

$$\mathcal{U}_A(F_A, F_B) - \mathcal{U}_A(F_A^{h,\eta}, F_B^{h,\eta})$$
$$= \sum_{i=1}^n v_i\mathbb{P}(X_{A,i} > X_{B,i}) - \sum_{i=1}^n v_i\mathbb{P}(X_{A,i}^{h,\eta} > X_{B,i}^{h,\eta})$$
$$\le \sum_{i=1}^n v_i\mathbb{P}(X_{A,i} > X_{B,i}^{h,\eta}) - \sum_{i=1}^n v_i\mathbb{P}(X_{A,i}^{h,\eta} > X_{B,i}^{h,\eta})$$
$$\le \sum_{i=1}^n v_i \sup_{x>0}[\mathbb{P}(X_{A,i} > x) - \mathbb{P}(X_{A,i}^{h,\eta} > x)],$$

where, in the first inequality, we use the fact that X_B is an optimal response for B when A plays X_A.

It will be convenient in the sequel to further bound this quantity using the ∞-Wasserstein distance—see Santambrogio (2015, section 5.5.1)—between $P_{A,i}$ and $P_{A,i}^{h,\eta}$, denoted $W_\infty(P_{A,i}, P_{A,i}^{h,\eta})$. Indeed, to show that $W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) \le \omega$ for some $\omega \ge 0$, it is sufficient to exhibit a coupling of $X_{A,i}, X_{A,i}^{h,\eta}$ such that $|X_{A,i} - X_{A,i}^{h,\eta}| < \omega$ almost surely. Later, we often do so implicitly, as such couplings are, in all instances, trivial.

Fix $\omega > 0$ and assume $W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) \le \omega$. Then, for any $\omega \ge 0$, we have

$$\mathbb{P}(X_{A,i} > x) - \mathbb{P}(X_{A,i}^{h,\eta} > x) \le \mathbb{P}(x < X_{A,i} < x + \omega) \le \frac{\lambda^* \omega}{\gamma^* v_i},$$

where we used the fact that $X_{A,i} \sim \mathsf{Unif}[0, \gamma^* v_i / \lambda^*]$. These two displays together yield that the suboptimality gap for player A is controlled as

$$\mathcal{U}_A(F_A, F_B) - \mathcal{U}_A(F_A^{h,\eta}, F_B^{h,\eta}) \le \frac{\lambda^*}{\gamma^*} \sum_{i=1}^n W_\infty(P_{A,i}, P_{A,i}^{h,\eta}).$$

Recall that step 1 in the reduction consists of decomposing $P_{A,i}$ as a mixture of (at most) n other distributions; that is, $P_{A,i} = \sum_k q_k P_{A,i}^{(k)}$. Accordingly, we also have constructed $P_{A,i}^{h,\eta}$ as a mixture $P_{A,i}^{h,\eta} = \sum_k q_k P_{A,i}^{h,\eta,(k)}$. It follows readily from the definition of W_∞ that

$$W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) \le \sum_k q_k W_\infty(P_{A,i}^{(k)}, P_{A,i}^{h,\eta,(k)}).$$

In particular, controlling each term on the right-hand side uniformly in k results in the same control on the desired error. Therefore, without loss of generality, we may assume that $P_{A,i}^{(k)} = P_{A,i}$ and $P_{A,i}^{h,\eta,(k)} = P_{A,i}^{h,\eta}$ so as to keep the notation light. Moreover, as earlier, we assume, without loss of generality, the last marginal $P_{A,n}$ is the only strict mixture.

4.1.2. The Value-Asymmetric Case. When values are asymmetric, the game is no longer constant-sum, and we shift our focus from optimal strategies to Nash equilibria. In this context, the notion of approximation is more subtle and has to be carried out jointly for both players.

For any set of marginal cdfs $G_A = (G_{A,1}, \dots, G_{A,n})$ and $G_B = (G_{B,1}, \dots, G_{B,n})$, denote by $\mathcal{U}_A(G_A, G_B)$ the expected utility of player A if player A plays according to strategy G_A, whereas player B plays according to strategy B.

A pair (F_A, F_B) is a Nash equilibrium if

$$\mathcal{U}_A(F_A, F_B) \ge \mathcal{U}_A(G_A, F_B), \text{ for all admissible } G_A.$$

Writing

$$\begin{aligned}\mathcal{U}_A(F_A, F_B) &= \mathcal{U}_A(F_A, F_B) - \mathcal{U}_A(F_A^{h,\eta}, F_B) + \mathcal{U}_A(F_A^{h,\eta}, F_B) \\ &\quad - \mathcal{U}_A(F_A^{h,\eta}, F_B^{h,\eta}) + \mathcal{U}_A(F_A^{h,\eta}, F_B^{h,\eta}), \\ \mathcal{U}_A(G_A, F_B) &= \mathcal{U}_A(G_A, F_B) - \mathcal{U}_A(G_A, F_B^{h,\eta}) + \mathcal{U}_A(G_A, F_B^{h,\eta})\end{aligned}$$

it readily follows from these two displays that

$$\mathcal{U}_A(F_A^{h,\eta}, F_B^{h,\eta}) + \varepsilon \ge \mathcal{U}_A(G_A, F_B^{h,\eta}), \text{ for all admissible } G_A,$$

where, using similar computations as earlier,

$$\varepsilon \le \sum_{i=1}^n v_{A,i} \left[\frac{\lambda^*}{\gamma^* v_{B,i}} W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) + \frac{2\lambda^*}{v_{A,i}} W_\infty(P_{B,i}, P_{B,i}^{h,\eta}) \right].$$

4.2. Control of the Errors

In both cases, symmetric or asymmetric values, a control of the approximation error follows from controlling $W_\infty(P_{A,i}, P_{A,i}^{h,\eta})$. In the rest of this section, we sightly abuse notation and write $W_\infty(X, Y)$ when the distributions of the random variables X and Y are clear from the context.

Recall that for any $i \in [n]$ because $X_{A,i} = \theta_i Y_{k(i)}$ and similarly for the approximate versions, for some fixed $\theta_i \in [0,1]$, we have for any $h, \eta \ge 0$ that

$$W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) = \theta_i W_\infty(Y_{j(i)}, \overline{Y}_{j(i)}),$$

where Y_j is the result of the reductions and is defined in Lemma 2, whereas $\overline{Y}_j$ is the output the smoothing procedure and is defined in Section 3.4.

Recall that the discrepancy between $\overline{Y}_j$ and the target Y_j stems from three approximations: discretization error ($h > 0$), numerical error ($\eta > 0$), and the error because of the smoothing step. The error coming from the smoothing step is easy to control: for any $j \in [4]$, we have $W_\infty(Y_j, \overline{Y}_j) \le W_\infty(Y_j, h\tilde{Y}_j) + h$. We have proved that

$$W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) \le \theta_i W_\infty(Y_{j(i)}, h\tilde{Y}_{j(i)}) + \theta_i h. \tag{19}$$

As a result, it is sufficient to control the discretization error and the numerical error at the level of the variables $\tilde{Y}_j$. To emphasize the presence of these errors, we employ the same notation as for $P_{A,i}^{h,\eta}$ and write $\tilde{Y}_j = \tilde{Y}_j^{h,\eta}$ for $h > 0, \eta \ge 0$. By the triangle inequality, we have

$$W_\infty(Y_j, h\tilde{Y}_j) \le \underbrace{W_\infty(Y_j, h\tilde{Y}_j^{h,0})}_{\text{discretization error}} + \underbrace{W_\infty(h\tilde{Y}_j^{h,0}, h\tilde{Y}_j^{h,\eta})}_{\text{numerical error}}.$$

The discretization error is trivial to control. Indeed, in light of the coupling provided by (12), we get that

$$W_\infty(Y_j, h\tilde{Y}_j^{h,0}) \le h. \tag{20}$$

Finally, to control the numerical error $W_\infty(h\tilde{Y}_j^{h,0}, h\tilde{Y}_j^{h,\eta})$, recall that the tolerance $\eta > 0$ in Algorithm Sinkhorn controls the ℓ_1 error between the current marginals and the targets. Hence, we need to bound the ∞-Wasserstein distance by the ℓ_1 distance. This is quite straightforward because $\tilde{Y}_j^{h,\eta}$ has bounded support for $\eta \ge 0$. Indeed, recall from Lemma 2 that for any $j \in [4]$, we have that $\tilde{Y}_j^{h,\eta} \in [0, b_j^*/h]$, where

$$b_j^* = \frac{1}{\lambda^*} \sum_{i \in \mathcal{I}_j} (\gamma^* v_{B,i}) \wedge v_{A,i}.$$

Hence, $W_\infty(h\tilde{Y}_j^{h,0}, h\tilde{Y}_j^{h,\eta}) \le b_j^* \eta$, and we have proved that

$$W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) \le 2\theta_i h + \theta_i b_{k(i)}^* \eta.$$

In particular, for the value-symmetric case, because

$$b_{k(i)} = \frac{\gamma^*}{\lambda^*} \sum_{l \in \mathcal{I}_{k(i)}} v_l \le \frac{\gamma^*}{\lambda^*},$$

we get the following simple bound

$$W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) \le 2\theta_i h + \frac{\theta_i \gamma^* \eta}{\lambda^*}.$$

Because $\sum_i \theta_i = 4$, the suboptimality gap for player A is controlled as

$$\mathcal{U}_A(F_A) - \mathcal{U}_A(F_A^{h,\eta}) \le \frac{\lambda^*}{\gamma^*} \sum_{i=1}^n W_\infty(P_{A,i}, P_{A,i}^{h,\eta}) \le 8\frac{\lambda^*}{\gamma^*} h + 4\eta. \tag{21}$$

In the value-asymmetric case, we get a suboptimality for player A smaller than

$$\varepsilon_A = \left(16 + \frac{8}{\gamma^*} \max_i \frac{v_{A,i}}{v_{B,i}}\right)\lambda^* h + \left(8\gamma^* + 4 \max_i \frac{v_{A,i}}{v_{B,i}}\right)\eta \tag{22}$$

and, with symmetric arguments, a suboptimality for player B smaller than

$$\varepsilon_B = \left(16 + 8 \max_i \frac{v_{B,i}}{v_{A,i}}\right)\frac{\lambda^* h}{\gamma^*} + \left(8 + 4 \max_i \frac{v_{B,i}}{v_{A,i}}\right)\eta.$$

4.3. Computational Complexity

In this section, we tally the complexity required to achieve either ε-suboptimality gap in the value-symmetric case or an ε-Nash equilibrium in the value-asymmetric case.

Before making this distinction recall the various steps that were employed, together with their computational complexity.

Lotto Step. Computing one (and, actually, all) pair of parameters (γ, λ) requires $\mathcal{O}(n \log n)$ operations; see Proposition 1.

Step 1: Computing all couplings π_k and their associated convex weights q_k requires $\mathcal{O}(n^2)$ operations; see Lemma 1.

Step 2: Given some coupling π_k computed at step 1, the reduction from n to only four random variables requires $\mathcal{O}(n^2)$ operations; see Lemma 2.

Step 3: The discretization step is computationally costless.

Step 4: The Sinkhorn algorithm requires $\mathcal{O}\big(\frac{\log(1/\tilde{\mu}_{\min})}{\eta}\big)$ operations; see Proposition 3. Because the marginal distributions μ_i are h-discretizations of either uniform on an interval of size at most $\sum \frac{\gamma^* v_{B,i}}{\lambda^*} \le \frac{\gamma^*}{\lambda^*}$ or uniform on interval of length $\frac{v_{A,i}}{\lambda^*}$ with weight $\frac{v_{A,i}}{\gamma^* v_{B,i}}$, it holds that $\tilde{\mu}_{\min} \ge \frac{h\lambda^*}{\gamma^*}$. At each iteration of Sinkhorn, all the components of the tensor are computed, hence a complexity, per iteration, in $\mathcal{O}\big(\big(\frac{\gamma^*}{h\lambda^*}\big)\big)^3$.

Step 5: The sampling cost comes from the reconstruction of the budget allocation from the four random variables, constructed at step 2 and sampled from the coupling computed at step 4. The sampling step has a linear cost with respect to the discretization size $\mathcal{O}\big(\frac{\gamma^*}{h\lambda^*}\big)$, whereas the reconstruction complexity scales linearly with respect to the number of battlefields $\mathcal{O}(n)$.

We are now in a position to state our main theorems. We begin with the symmetric-value case for player A. The result for player B is completely analogous and therefore omitted.

Theorem 2. *Consider the two-player Blotto game on n battlefields with symmetric values $v_1, \dots, v_n$, where player A has a budget of T_A and player B has a budget of $T_B \le T_A$, and assume that these data satisfy the conditions of Corollary 1. Fix $\varepsilon > 0$ and let $\eta = \varepsilon/4$ and $h = \varepsilon T_A/8$. Then, the procedure described in Algorithm Sample samples from an ε-suboptimal strategy for player A in time*

$$\mathcal{O}\left(n^2 + \frac{\log(1/\varepsilon)}{\varepsilon^4}\right).$$

Proof. Note first that the preprocessing cost associated with steps 1 through 4 is $\mathcal{O}(n^2)$.

To compute the cost of Sinkhorn iterations, observe that the parameters η and h are chosen in such a way that each term on the right-hand side of (21) is equal to $\varepsilon/2$:

$$8\frac{\lambda^*}{\gamma^*} h = 4\eta = \frac{\varepsilon}{2}.$$

Hence,

$$\tilde{\mu}_{\min} \ge \frac{h\lambda^*}{\gamma^*} = \frac{\varepsilon}{16}, \quad \text{and} \quad \left(\frac{\gamma^*}{h\lambda^*}\right)^3 = \left(\frac{16}{\varepsilon}\right)^3.$$

Moreover, because $\eta = \varepsilon/8$, we get that the total complexity of Sinkhorn iterations is

$$\mathcal{O}\left(\left(\frac{\gamma^*}{h\lambda^*}\right)^3 \frac{\log(1/\tilde{\mu}_{\min})}{\eta}\right) = \mathcal{O}\left(\varepsilon^{-3}\frac{\log(1/\varepsilon)}{\varepsilon}\right) = \mathcal{O}(\varepsilon^{-4}\log(1/\varepsilon)).$$

Finally, the last step has a total cost of $\mathcal{O}(n + \varepsilon)$, which is negligible with respect to the combination of the previous steps. □

It is worth noting that in the value-symmetric case, the computational complexity of our procedure is *independent* of the datum of the problem (budgets and values) under Normalization (1). Note that this normalization merely scales the utility and should, of course, affect the desired accuracy parameter ε.

We now move to the asymmetric-value case and characterize the complexity of our procedure to compute an ε-Nash equilibrium for the Blotto game. As earlier, we focus on player A only.

Theorem 3. *Consider the Blotto game on n battlefields with asymmetric values $v_{P,1},\dots,v_{P,n}$, $P\in\{A,B\}$, where player $P\in\{A,B\}$ has budget T_P with $T_B\le T_A$ and assume that these data satisfy the conditions of Corollary 1. Define*

$$\mathfrak{m}=\max_i \frac{v_{A,i}}{v_{B,i}}\vee\frac{v_{B,i}}{v_{A,i}}.$$

Fix $\varepsilon>0$ and let

$$\eta=\frac{\varepsilon}{24\mathfrak{m}},\qquad h=\frac{\gamma^*}{\lambda^*}\frac{\varepsilon}{48\mathfrak{m}}.$$

Then, Algorithm Lotto2Blotto samples from an ε-Nash equilibrium in time

$$\mathcal{O}\left(n^2+\left(\frac{\mathfrak{m}}{\varepsilon}\right)^4\log\left(\frac{\mathfrak{m}}{\varepsilon}\right)\right).$$

Note first that the preprocessing cost associated with steps 1 through 4 is $\mathcal{O}(n^2)$.

To compute the cost of Sinkhorn iterations, observe that the parameters η and h are chosen in such a way that each term on the right-hand side of (22) is smaller than $\varepsilon/2$:

$$\left(16+\frac{8}{\gamma^*}\mathfrak{m}\right)\lambda^*h,(4\gamma^*+2\mathfrak{m})\eta\le\frac{\varepsilon}{2}.$$

Hence,

$$\tilde{\mu}_{\min}\ge\frac{h\lambda^*}{\gamma^*}=\frac{\varepsilon}{48\mathfrak{m}},\quad\text{and}\quad\left(\frac{\gamma^*}{h\lambda^*}\right)^3=\left(\frac{48\mathfrak{m}}{\varepsilon}\right)^3.$$

Together with the prescribed value of η and because $\gamma^*\le\mathfrak{m}$ because of Proposition 1, we get that the total complexity of Sinkhorn iterations is

$$\mathcal{O}\left(\left(\frac{\gamma^*}{h\lambda^*}\right)^3\frac{\log(1/\tilde{\mu}_{\min})}{\eta}\right)=\mathcal{O}\left(\left(\frac{\mathfrak{m}}{\varepsilon}\right)^4\log\left(\frac{\mathfrak{m}}{\varepsilon}\right)\right).$$

Finally, the last step has a total cost of $\mathcal{O}(n+\varepsilon)$, which is negligible with respect to the combination of the previous steps.

Acknowledgments

An extended abstract of this paper has been published at the conference EC'2022.

Endnote

[1] In the case of the Lotto game, it is natural to call a strategy an equivalence class of strategies with the same marginals.

References

Agueh M, Carlier G (2011) Barycenters in the Wasserstein space. *SIAM J. Math. Anal.* 43(2):904–924.

Ahmadinejad A, Dehghani S, Hajiaghayi M, Lucier B, Mahini H, Seddighin S (2019) From duels to battlefields: Computing equilibria of blotto and other games. *Math. Oper. Res.* 44(4):1304–1325.

Altschuler JM, Boix-Adsera E (2020) Polynomial-time algorithms for multimarginal optimal transport problems with structure. Preprint, submitted August 7, https://arxiv.org/abs/2008.03006.

Altschuler J, Boix-Adsera E (2021) Hardness results for multimarginal optimal transport problems. *Discrete Optim.* 42:100669.

Altschuler J, Weed J, Rigollet P (2017) Near-linear time approximation algorithms for optimal transport via Sinkhorn iteration. *Advances Neural Inform. Processing Systems 30 (NIPS 2017)* (Curran Associates Inc., Red Hook, NY), 1961–1971.

Beaglehole D, Hopkins M, Kane D, Liu S, Lovett S (2022) An efficient approximation algorithm for the Colonel Blotto game. https://www.researchgate.net/publication/358142876_An_Efficient_Approximation_Algorithm_for_the_Colonel_Blotto_Game/fulltext/61f2f483c5e3103375c4e85f/An-Efficient-Approximation-Algorithm-for-the-Colonel-Blotto-Game.pdf?_tp=eyJjb250ZXh0Ijp7ImZpcnN0UGFnZSI6InB1YmxpY2F0aW9uIiwicGFnZSI6InB1YmxpY2F0aW9uIn19.

Behnezhad S, Dehghani S, Derakhshan M, HajiAghayi M, Seddighin S (2017) Faster and simpler algorithm for optimal strategies of Blotto game. *Proc. AAAI Conf. Artificial Intelligence*, vol. 31 (Association for the Advancement of Artificial Intelligence, Washington, DC), 369–375.

Bell RM, Cover TM (1980) Competitive optimality of logarithmic investment. *Math. Oper. Res.* 5(2):161–166.

Border KC (1991) Implementation of reduced form auctions: A geometric approach. *Econometrica* 59(4):1175–1187.

Borel E (1921) La théorie du jeu et les équations intégrales à noyau symétrique. *Comptes rendus de l'Académie des Sciences* 173:1304–1308.

Borel E, Ville J (1938) *Applications de la Théorie des Probabilités aux Jeux de Hasard: Cours Professé à la Faculté des Sciences de Paris* (Gauthier-Villars, Paris).

Bubeck S (2015) Convex optimization: Algorithms and complexity. *Foundations Trends Machine Learning* 8(3–4):231–357.

Cuturi M (2013) Sinkhorn distances: Lightspeed computation of optimal transport. Burges CJC, Bottou L, Welling M, Ghahramani Z, Weinberger KQ, eds. *Advances Neural Inform. Processing Systems 26 (NIPS 2013)* (Curran Associates, Inc., Red Hook, NY), 2292–2300.

Di Marino S, Gerolin A, Nenna L (2017) Optimal transportation theory with repulsive costs. Bergounioux M, Oudet É, Rumpf M, Carlier G, Champion T, Santambrogio F, eds. *Topological Optimization and Optimal Transport*, vol. 17 (De Gruyter, Berlin, Boston), 204–256.

Ferdowsi A, Saad W, Mandayam NB (2021) Colonel Blotto game for sensor protection in interdependent critical infrastructure. *IEEE Internet Things J.* 8(4):2857–2874.

Fréchet M (1953) Emile Borel, initiator of the theory of psychological games and its application. *Econometrica* 21:95–96.

Gross O (1950) The symmetric Blotto game. Technical report, Rand Corporation, Santa Monica, CA.

Gross O, Wagner R (1950) A continuous Colonel Blotto game. Technical report, Rand Corporation, Santa Monica, CA.

Hajimirsaadeghi M, Mandayam NB (2017) A dynamic Colonel Blotto game model for spectrum sharing in wireless networks. *2017 55th Annual Allerton Conf. Comm. Control Comput. (Allerton)* (IEEE, Piscataway, NJ), 287–294.

Hart S (2008) Discrete Colonel Blotto and general Lotto games. *Internat. J. Game Theory* 36(3):441–460.

Hortala-Vallve R, Llorente-Saguer A (2012) Pure strategy Nash equilibria in non-zero sum Colonel Blotto games. *Internat. J. Game Theory* 41(2):331–343.

Knott M, Smith C (2006) Choosing joint distributions so that the variance of the sum is small. *J. Multivariate Anal.* 97(8):1757–1765.

Kovenock D, Roberson B (2021) Generalizations of the general Lotto and Colonel Blotto games. *Econom. Theory* 71(3):997–1032.

Labib M, Ha S, Saad W, Reed JH (2015) A Colonel Blotto game for anti-jamming in the internet of things. *2015 IEEE Global Comm. Conf. (GLOBECOM)* (IEEE, Piscataway, NJ), 1–6.

Laslier JF (2002) How two-party competition treats minorities. *Rev. Econom. Design* 7(3):297–307.

Laslier JF, Picard N (2002) Distributive politics and electoral competition. *J. Econom. Theory* 103(1):106–130.

Lin T, Ho N, Cuturi M, Jordan MI (2022) On the complexity of approximating multimarginal optimal transport. *J. Machine Learn. Res.* 23(65):1–43.

Macdonell ST, Mastronardi N (2015) Waging simple wars: A complete characterization of two-battlefield Blotto equilibria. *Econom. Theory* 58(1):183–216.

Masucci AM, Silva A (2015) Defensive resource allocation in social networks. *2015 54th IEEE Conf. Decision Control (CDC)* (IEEE, Piscataway, NJ), 2927–2932.

Merolla J, Munger M, Tofias M (2005) In play: A commentary on strategies in the 2004 U.S. presidential election. *Public Choice* 123(1/2):19–37.

Myerson RB (1993) Incentives to cultivate favored minorities under alternative electoral systems. *Amer. Political Sci. Rev.* 87(4):856–869.

Nakayama M (2006) The dawn of modern theory of games. Kusuoka S, Yamazaki A, eds. *Advances in Mathematical Economics*, vol. 9 (Springer, Tokyo), 73–97.

Polyanskiy Y, Wu Y (2022) *Information Theory Information Theory: From Coding to Learning* (Cambridge University Press, Cambridge, UK).

Reny PJ (1999) On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica* 67(5):1029–1056.

Roberson B (2006) The Colonel Blotto game. *Econom. Theory* 29(1): 1–24.

Rüschendorf L, Uckelmann L (2002) Variance minimization and random variables with constant sum. Cuadras CM, Fortiana J, Rodriguez-Lallena JA, eds. *Distributions with Given Marginals and Statistical Modelling* (Springer, Dordrecht, Netherlands), 211–222.

Santambrogio F (2015) *Optimal Transport for Applied Mathematicians.* Progress in Nonlinear Differential Equations and Their Applications, vol. 87 (Birkhäuser/Springer, Cham, Switzerland).

Schwartz G, Loiseau P, Sastry SS (2014) The heterogeneous Colonel Blotto game. *2014 7th Internat. Conf. NETwork Games COntrol OPtim. (NetGCoop)* (IEEE, Piscataway, NJ), 232–238.

Sinkhorn R (1964) A relationship between arbitrary positive matrices and doubly stochastic matrices. *Ann. Math. Statist.* 35(2):876–879.

Sinkhorn R, Knopp P (1967) Concerning nonnegative matrices and doubly stochastic matrices. *Pacific J. Math.* 21(2):343–348.

Thomas C (2018) N-dimensional Blotto game with heterogeneous battlefield values. *Econom. Theory* 65(3):509–544.

von Neumann J (1928) Zur Theorie der Gesellschaftsspiele. *Math. Ann.* 100:295–320.

von Neumann J, Fréchet M (1959) Le rôle d'émile borel dans la théorie des jeux. *Rev. Econom. Polit.* 69(2):139–167.

Vu DQ, Loiseau P, Silva A (2018) Efficient computation of approximate equilibria in discrete colonel blotto games. *Proc. 27th Internat. Joint Conf. Artificial Intelligence* (AAAI Press, Washington, DC), 519–526.

Wang B, Wang R (2016) Joint mixability. *Math. Oper. Res.* 41(3):808–826.

Weinstein J (2012) Two notes on the Blotto game. *B. E. J. Theoretical Econom.* 12(1):1–13.

Zimin AP (2020) On existence of measure with given marginals supported on a hyperplane. Preprint, submitted October 14, https://arxiv.org/abs/2010.07263.

Vianney Perchet is a professor at the Centre de recherche en économie et statistique (CREST) at the ENSAE since October 2019. Mainly focusing on the interplay between machine learning and game theory, his themes of research are at the junction of mathematics, computer science, and economics. The spectrum of his interest ranges from pure theory (say, optimal rates of convergence of algorithms) to pure applications (modeling user behavior, optimization of recommender systems, etc.) He is also part-time principal researcher in the Criteo AI Laboratory in Paris, working on efficient exploration in recommender systems.

Philippe Rigollet is a professor of mathematics at MIT, where he serves as the chair of applied mathematics. He works at the intersection of statistics, machine learning, and optimization, focusing primarily on the design and analysis of efficient statistical methods. His current research is on statistical optimal transport and the mathematical theory behind transformers. His research has been recognized by the CAREER award from the Nation Science Foundation and a best paper award at the Conference on Learning Theory in 2013 for his pioneering work on statistical-to-computational trade-offs. He is also a recognized speaker and has been selected to present his work on statistical optimal transport during a Medallion lecture by the Institute for Mathematical Statistics as well as the 2019 Saint-Flour Lectures in Probability and Statistics.

Thibaut Le Gouic has been a maître de conférences (associate professor) at the Institut de Mathématiques de Marseille and the Ecole Centrale de Marseille since 2014. He was a visiting researcher at the Faculty of Computer Science at the Higher School of Economics (HSE) in 2018–2019 and the Department of Mathematics at MIT in 2019–2021. His research focuses on probability theory and statistics. His contributions include work on empirical barycenters, optimal transport, sampling algorithms, and statistical fairness.