2. 分段线性插值的误差估计

由前述余项定理可知，n次Lagrange插值多项式的余项为：

$$R_n(x) = f(x) - L_n(x) \quad = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$$

则分段线性插值$L_1(x)$的余项为

$$R_1(x) = f(x) - L_1(x) \quad = f(x) - L_1^{(k)}(x)$$

$$= \frac{f''(\xi)}{2}(x - x_k)(x - x_{k+1}) \quad \xi, x \in [x_k, x_{k+1}], \text{且}\xi\text{与}x\text{有关}$$

$$|R_1(x)| \le \frac{1}{2} \cdot \max_{a \le x \le b} |f''(x)| \cdot \max_{\substack{a \le x \le b \\ k}} |(x - x_k)(x - x_{k+1})|$$

$$\le \frac{1}{2} \cdot M_2 \cdot \frac{1}{4}h^2 \quad = \frac{1}{8}M_2 h^2$$

二、分段二次Lagrange插值

1. 分段二次插值的构造

设插值节点为x_i，函数值为y_i，i=0,1,2,……,n

$h_i=x_{i+1}-x_i$，i=0,1,2,……,n–1，

任取三个相邻的节点x_{k-1}，x_k，x_{k+1}，以 [x_{k-1}，x_{k+1}]为插值区间构造二次Langrange插值多项式：

$$L_2^{(k)}(x)=y_{k-1}l_{k-1}(x)+y_kl_k(x)+y_{k+1}l_{k+1}(x)$$

$$L_2^{(k)}(x)=y_{k-1}\frac{(x-x_k)(x-x_{k+1})}{(x_{k-1}-x_k)(x_{k-1}-x_{k+1})}+y_k\frac{(x-x_{k-1})(x-x_{k+1})}{(x_k-x_{k-1})(x_k-x_{k+1})}$$

$$+y_{k+1}\frac{(x-x_{k-1})(x-x_k)}{(x_{k+1}-x_{k-1})(x_{k+1}-x_k)}\qquad k=1,2,\cdots,n-1$$

2. 分段二次插值的误差估计

由于 $$R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$$

那么分段二次插值$L_2(x)$的余项为：

$$R_2(x)=f(x)-L_2(x)=f(x)-L_2^{(k)}(x)$$

$$=\frac{f'''(\xi)}{6}(x-x_{k-1})(x-x_k)(x-x_{k+1})$$

$\xi, x\in[x_{k-1}, x_{k+1}]$,
且ξ与x有关

$$|R_2(x)|\le\frac{1}{6}\cdot\max_{a\le x\le b}|f'''(x)|\cdot\max_{\substack{x_{k-1}\le x\le x_{k+1}\\ k}}|(x-x_{k-1})(x-x_k)(x-x_{k+1})|$$

$$\le\frac{1}{6}\cdot M_3\cdot\frac{2\sqrt{3}}{9}h^3=\frac{\sqrt{3}}{27}M_3h^3$$

例： 设 $f(x)$ 在各节点处的数据为

i	0	1	2	3	4	5
x_i	0.30	0.40	0.55	0.65	0.80	1.05
y_i	0.30163	0.41075	0.57815	0.69675	0.87335	1.18885

求 $f(x)$ 在 $x=0.36, 0.42, 0.75, 0.98, 1.1$ 处的近似值.

（ 用分段线性、二次插值 ）

解： (1) 分段线性Lagrange插值的公式为

$$L_1^{(k)}(x) = y_k \frac{x - x_{k+1}}{x_k - x_{k+1}} + y_{k+1} \frac{x - x_k}{x_{k+1} - x_k}$$

$$k = 0, 1, \cdots, n-1$$

$$\boldsymbol{f(0.36)\approx} \; L_1^{(0)}(0.36) = 0.30163\times\frac{0.36-0.4}{0.3-0.4} + 0.41075\times\frac{0.36-0.3}{0.4-0.3}$$

$$= 0.36711$$

$$f(0.42)\approx \; L_1^{(1)}(0.42) = 0.41075\times\frac{0.42-0.55}{0.4-0.55} + 0.57815\times\frac{0.42-0.4}{0.55-0.4}$$

$$= 0.43307$$

同理 $f(0.75)\approx L_1^{(3)}(0.75) = 0.81448$

$$f(0.98)\approx L_1^{(4)}(0.98) = 1.10051$$

$$f(1.1)\approx \; L_1^{(4)}(1.1) = 0.87335\times\frac{1.1-1.05}{0.8-1.05} + 1.18885\times\frac{1.1-0.8}{1.05-0.8}$$

$$= 1.25195$$

(2) 分段二次Lagrange插值的公式为

$$L_2^{(k)}(x) = y_{k-1}\frac{(x-x_k)(x-x_{k+1})}{(x_{k-1}-x_k)(x_{k-1}-x_{k+1})} + y_k\frac{(x-x_{k-1})(x-x_{k+1})}{(x_k-x_{k-1})(x_k-x_{k+1})}$$

$$+y_{k+1}\frac{(x-x_{k-1})(x-x_k)}{(x_{k+1}-x_{k-1})(x_{k+1}-x_k)} \qquad k=1,2,\cdots,n-1$$

$$f(0.36) \approx L_2^{(1)}(0.36) = y_0\frac{(0.36-x_1)(0.36-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$+y_1\frac{(0.36-x_0)(0.36-x_2)}{(x_1-x_0)(x_1-x_2)} \qquad +y_2\frac{(0.36-x_0)(0.36-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$=0.36686$$

$$f(0.42) \approx$$

$$L_2^{(1)}(0.42) = y_0 \frac{(0.42 - x_1)(0.42 - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(0.42 - x_0)(0.42 - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$+ y_2 \frac{(0.42 - x_0)(0.42 - x_1)}{(x_2 - x_0)(x_2 - x_1)} = 0.43281$$

$$f(0.75) \approx$$

$$L_2^{(4)}(0.75) = y_3 \frac{(0.75 - x_4)(0.75 - x_5)}{(x_3 - x_4)(x_3 - x_5)} + y_4 \frac{(0.75 - x_3)(0.75 - x_5)}{(x_4 - x_5)(x_4 - x_5)}$$

$$+ y_5 \frac{(0.75 - x_3)(0.75 - x_4)}{(x_5 - x_3)(x_5 - x_4)} = 0.81343$$

$$f(0.98) \approx L_2^{(4)}(0.98) = 1.09784 \qquad f(1.1) \approx L_2^{(4)}(1.1) = 1.25513$$

5.4 均差与Newton插值

一、均差及其性质

*Lagrange*插值多项式理论上较方便，但当节点增加时，全部基函数$l_k(x)$都要变，在实际运算中并不方便.

可将插值多项式表示为如下形式：

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots$$
$$+ a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

其中$a_0,a_1,\ldots\ldots,a_n$待定,可由$P_n(x_i)=f_i$ $(i=0,1,\ldots\ldots,n)$确定. f_i为节点处的函数值.

当$x=x_0$时，

$$P(x_0)=f_0=a_0 \qquad a_0=f_0$$

当$x=x_1$时，

$$P(x_1)=f_1=a_0+a_1(x_1-x_0) \qquad a_1=\frac{f_1-f_0}{x_1-x_0}$$

当$x=x_2$时，

$$P(x_2)=f_2=a_0+a_1(x_2-x_0)+a_2(x_2-x_0)(x_2-x_1)$$

$$P(x_2)=f_2=a_0+a_1(x_2-x_0)+a_2(x_2-x_0)(x_2-x_1)$$

$$a_2=\frac{f_2-a_0-a_1(x_2-x_0)}{(x_2-x_0)(x_2-x_1)}=\frac{f_2-f_0-\dfrac{f_1-f_0}{x_1-x_0}(x_2-x_0)}{(x_2-x_0)(x_2-x_1)}$$

$$=\frac{\dfrac{f_2-f_0}{x_2-x_0}-\dfrac{f_1-f_0}{x_1-x_0}}{x_2-x_1}=\frac{\dfrac{f_2-f_1}{x_2-x_1}-\dfrac{f_1-f_0}{x_1-x_0}}{x_2-x_0}$$

再继续下去待定系数的形式将更复杂，为此引入均差的概念：

定义：

设$f(x)$在互异节点x_i处的函数值为f_i，i=0,1,…,n，称

$$f[x_i, x_j] = \frac{f_j - f_i}{x_j - x_i} \quad (i \neq j)$$

为$f(x)$关于节点x_i，x_j的一阶均差.

两个一阶均差的均差

$$f[x_i, x_j, x_k] = \frac{f[x_j, x_k] - f[x_i, x_j]}{x_k - x_i} \quad (i \neq j \neq k)$$

称为$f(x)$关于节点x_i，x_j，x_k的二阶均差.

一般地，两个$n-1$阶均差的均差

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0}$$

称为n阶均差（也称差商）.

均差的性质：

（1）$f(x)$的n阶均差可表示为函数值$f(x_0)$,$f(x_1)$, ……, $f(x_n)$的线性组合，即

$$f[x_0,x_1,\cdots,x_n]=\sum_{i=0}^{n}\frac{f(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

（2） 均差具有对称性，即任意调换节点的次序，均差的值不变.

如 $$f[x_0,x_1,x_2]=f[x_0,x_2,x_1]=f[x_2,x_1,x_0]$$

（3）设$f(x)$在$[a,b]$上具有n阶导数，且$x_0,x_1,\ldots,x_n \in[a,b]$，则$n$阶均差与导数的关系如下：

$$f[x_0,x_1,\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!} \qquad \xi\in[a,b]$$

均差的计算方法(表格法): **均差表**

x_k	$f(x_k)$	一阶均差	二阶均差	三阶均差	四阶均差
x_0	$f(x_0)$				
		$f[x_0,x_1]$			
x_1	$f(x_1)$		$f[x_0,x_1,x_2]$		
		$f[x_1,x_2]$		$f[x_0,x_1,x_2,x_3]$	
x_2	$f(x_2)$		$f[x_1,x_2,x_3]$		$f[x_0,x_1,\cdots,x_4]$
		$f[x_2,x_3]$		$f[x_1,x_2,x_3,x_4]$	
x_3	$f(x_3)$		$f[x_2,x_3,x_4]$		
		$f[x_3,x_4]$			
x_4	$f(x_4)$				

规定函数值为零阶均差

例： 已知函数$f(x)$的函数值列表如下：

x	–2	–1	0	1	3
y	–56	–16	–2	–2	4

列出一至三阶的均差表.

解：

x	$f(x)$	一阶均差	二阶均差	三阶均差
–2	–56			
–1	–16	40		
0	–2	14	–13	
1	–2	0	–7	2
3	4	3	1	2

二、Newton插值公式

据均差定义，把$x \neq x_i$看成$[a,b]$上一点，则

$$f[x_0,x_1,\cdots,x_k,x]=\frac{f[x_0,x_1,\cdots,x_k]-f[x_0,x_1,\cdots,x_{k-1},x]}{x_k-x}$$

即 $f[x_0,x_1,\cdots,x_{k-1},x]$

$$=f[x_0,x_1,\cdots,x_k]+f[x_0,x_1,\cdots,x_k,x](x-x_k)$$

因此可得

$$f(x)=f(x_0)+f[x,x_0](x-x_0)$$

$$f[x,x_0]=f[x_0,x_1]+f[x,x_0,x_1](x-x_1)$$

……

$$f[x,x_0,\cdots,x_{n-1}]=f[x_0,x_1,\cdots,x_n]+f[x,x_0,\cdots,x_n](x-x_n)$$

将后一式代入前一式，得

$$
\begin{aligned}
f(x) = f(x_0) &+ f[x_0, x_1](x - x_0) \\
&+ f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots\cdots \\
&+ f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1}) \\
&+ f[x, x_0, \cdots, x_n]\omega_{n+1}(x)
\end{aligned}
$$

$$R_n(x) = f(x) - N_n(x) = f[x, x_0, \cdots, x_n]\omega_{n+1}(x)$$

$$= N_n(x) + R_n(x) \qquad (\omega_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n))$$

其中
$$
\begin{aligned}
N_n(x) = f(x_0) &+ f[x_0, x_1](x - x_0) + \cdots\cdots \\
&+ f[x_0, x_1, \cdots, x_n](x - x_0)\cdots(x - x_{n-1})
\end{aligned}
$$

称$N_n(x)$为**Newton均差插值多项式**.

注：

（1）Newton插值多项式的系数为均差表中各阶均差的第一个数据；

（2）Newton插值多项式的基函数为$\omega_i(x)$, $i=0,1,\ldots\ldots,n$；

（3）Newton插值多项式的插值余项为$R_n(x)$.

例： 已知$f(x)$的函数表，求4次牛顿插值多项式，并由此计算$f(0.596)$的近似值.

x_k	$f(x_k)$	一阶均差	二阶均差	三阶均差	四阶均差	五阶均差
0.40	0.41075					
0.55	0.57815	1.11600				
0.65	0.69675	1.18600	0.28000			
0.80	0.88811	1.27573	0.35893	0.19733		
0.90	1.02652	1.38410	0.43348	0.21300	0.03134	
1.05	1.25382	1.51533	0.52493	0.22863	0.03126	–0.00012

从表中可以看到4阶均差几乎为常数，故取4次插值多项式即可，于是：

$$N_4(x)=0.41075+1.166(x-0.4)+0.28(x-0.4)(x-0.55)$$
$$+0.19733(x-0.4)(x-0.55)(x-0.65)$$
$$+0.03134(x-0.4)(x-0.55)(x-0.65)(x-0.8)$$

可得 $$f(0.596)\approx N_4(0.596)=0.63192$$

截断误差为：

$$\left|R_4(x)\right|\approx\left|f[x_0,x_1,x_2,x_3,x_4,x_5]\omega_5(0.596)\right|\leq 3.63\times 10^{-9}$$

这说明截断误差很小.

截断误差的估计：

此例中，五阶均差$f[x,x_0,x_1,\ldots\ldots,x_4]$是用$f[x_0,x_1,\ldots\ldots,x_5]$来近似的.

另一种方法是取x=0.596，由$f(0.596)\approx 0.61392$求得$f[x,x_0,x_1,\ldots\ldots,x_4]$的近似值，进而计算$|R_4(x)|$.

5.5 埃尔米特插值（Hermite）

Newton插值和Lagrange插值虽然构造比较简单，但都存在插值曲线在节点处有尖点，不光滑，插值多项式在节点处不可导等缺点.

埃尔米特插值的基本思想为：

已知节点处函数值及对应节点导数值，求使其函数值及导数值均相等的插值多项式.

设$a \le x_0 < x_1 < \dots\dots < x_n \le b$上，

$y_j = f(x_j), m_j = f'(x_j)$ $(j=0,1,2\dots,n)$

求$H(x)$，使 $H(x_j) = y_j, H'(x_j) = m_j$ $(j=0,1,2\dots,n)$

共有$2n+2$个条件，可唯一确定一次数$\leq 2n+1$的多项式$H_{2n+1}(x)=H(x)$.

形式：

$$H_{2n+1}(x)=a_0+a_1x+\cdots+a_{2n+1}x^{2n+1}$$

一般来说，Hermite插值多项式的次数如果太高会影响收敛性和稳定性，因此$2n+1$不宜太大，仍用分段插值.

故仅考虑$n=1$的情况，即三次Hermite插值.

一、三次Hermite插值公式

考虑只有两个节点的插值问题：

设$f(x)$在节点x_0，x_1处的函数值为y_0，y_1；在节点x_0，x_1处的一阶导数值为y'_0，y'_1.

两个节点最高可以用3次Hermite多项式$H_3(x)$作为插值函数.

$H_3(x)$应满足条件：

$$H_3(x_0) = y_0 \qquad H_3(x_1) = y_1$$

$$H_3'(x_0) = y_0' \qquad H_3'(x_1) = y_1'$$

采用基函数方法构造.

$H_3(x)$应用四个插值基函数表示.

设$H_3(x)$的插值基函数为$\alpha_0(x)$， $\alpha_1(x)$，$\beta_0(x)$， $\beta_1(x)$，则

$$H_3(x)=y_0\alpha_0(x)+y_1\alpha_1(x)+y_0'\beta_0(x)+y_1'\beta_1(x)$$

$$H_3'(x)=y_0\alpha_0'(x)+y_1\alpha_1'(x)+y_0'\beta_0'(x)+y_1'\beta_1'(x)$$

其中

$\boxed{\alpha_0(x_0)=1}$	$\boxed{\alpha_0(x_1)=0}$	$\boxed{\alpha_0'(x_0)=0}$	$\boxed{\alpha_0'(x_1)=0}$
$\alpha_1(x_0)=0$	$\alpha_1(x_1)=1$	$\alpha_1'(x_0)=0$	$\alpha_1'(x_1)=0$
$\beta_0(x_0)=0$	$\beta_0(x_1)=0$	$\beta_0'(x_0)=1$	$\beta_0'(x_1)=0$
$\beta_1(x_0)=0$	$\beta_1(x_1)=0$	$\beta_1'(x_0)=0$	$\beta_1'(x_1)=1$

可知x_1是$\alpha_0(x)$的二重零点，即可假设

$$\alpha_0(x)=(x-x_1)^2(ax+b)$$

由

$$\alpha_0(x_0)=1 \qquad \alpha_0'(x_0)=0$$

可得

$$a=-\frac{2}{(x_0-x_1)^3} \qquad b=\frac{1}{(x_0-x_1)^2}+\frac{2x_0}{(x_0-x_1)^3}$$

$$\begin{aligned}\alpha_0(x)&=(x-x_1)^2(ax+b)\\&=(x-x_1)^2\left(-\frac{2x}{(x_0-x_1)^3}+\frac{1}{(x_0-x_1)^2}+\frac{2x_0}{(x_0-x_1)^3}\right)\\&=\frac{(x-x_1)^2}{(x_0-x_1)^2}\left(1+\frac{2x_0}{x_0-x_1}-\frac{2x}{x_0-x_1}\right)\end{aligned}$$

$$\alpha_0(x)=\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2 \quad =(1+2l_1(x))\cdot l_0^2(x)$$

Lagrange
插值基函数

同理可得

$$\alpha_1(x)=(1+2l_0(x))\cdot l_1^2(x) \quad =\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2$$

$$\beta_0(x)=(x-x_0)\cdot l_0^2(x) \quad =(x-x_0)\left(\frac{x-x_1}{x_0-x_1}\right)^2$$

$$\beta_1(x)=(x-x_1)\cdot l_1^2(x) \quad =(x-x_1)\left(\frac{x-x_0}{x_1-x_0}\right)^2$$

将以上结果代入

$$H_3(x)=y_0\alpha_0(x)+y_1\alpha_1(x)+y_0'\beta_0(x)+y_1'\beta_1(x)$$

得两个节点的三次Hermite插值公式：

$$H_3(x)=y_0\alpha_0(x)+y_1\alpha_1(x)+y_0'\beta_0(x)+y_1'\beta_1(x)$$

$$=y_0(1+2l_1(x))\cdot l_0^2(x)+y_1(1+2l_0(x))\cdot l_1^2(x)$$

$$+y_0'(x-x_0)\cdot l_0^2(x)\ \ +y_1'(x-x_1)\cdot l_1^2(x)$$

$$=y_0\left(1+2\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2+y_1\left(1+2\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2$$

$$+y_0'\left(x-x_0\right)\left(\frac{x-x_1}{x_0-x_1}\right)^2+y_1'\left(x-x_1\right)\left(\frac{x-x_0}{x_1-x_0}\right)^2$$