1 Question 1

1.1 Part a)

Let Ω be the sample space. Therefore $\mathrm{P}(\{\Omega\}) = 1$. Adding all the joint pmf values must sum to 1:

$$\begin{aligned}
\{\Omega\} &= \bigcup_x \bigcup_y \{\mathrm{X} = x\} \cap \{\mathrm{Y} = y\} \\
\mathrm{P}(\{\Omega\}) &= 1 \\
\implies 1 &= \mathrm{P}((\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) \cup \ldots \cup (\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) + \ldots + \mathrm{P}(\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= (p - \frac{1}{16}) + (\frac{1}{4} - p) + (0) + (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p) \\
1 &= -\frac{1}{16} + \frac{4}{16} + \frac{7}{16} + \frac{1}{16} + \frac{1}{16} + \frac{4}{16} \\
1 &= 1
\end{aligned}$$

Unfortunately, this tells us no information about p. From the definition of probability, $\mathrm{P}(\{c\})$ for $c \in \Omega$ must be greater or equal to 0, $\mathrm{P}(\{c \in \Omega\}) \geq 0$. This can be used to restrict the possible values of p:

$$\begin{aligned}
\mathrm{P}(A \subseteq \Omega) &\geq 0 \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
p - \frac{1}{16} &\geq 0 \\
p &\geq \frac{1}{16} \\
\implies \mathrm{P}(\{\mathrm{X} = 0\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
\frac{1}{4} - p &\geq 0 \\
p &\leq \frac{1}{4} \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = 1\}) &\geq 0 \\
p + \frac{1}{16} &\geq 0 \\
p &\leq \frac{1}{16}
\end{aligned}$$

$$\implies p \in [\frac{1}{16}, \frac{1}{4}] \tag{1}$$

Therefore, $\frac{1}{16} \leq p \leq \frac{1}{4}$, and can be any value within this range.

1.2 Part b)

Aim is to find $\mathrm{P}(\{\mathrm{X} = \mathrm{Y}\})$:

$$
\begin{aligned}
P(\{X = Y\}) &= \sum_a P(\{X = a\} \cap \{Y = a\}) \\
&= P(\{X = -1\} \cap \{Y = -1\}) + P(\{X = 0\} \cap \{Y = 0\}) + P(\{X = 1\} \cap \{Y = 1\}) \\
&= (p - \frac{1}{16}) + (\frac{3}{16}) + (\frac{1}{4} - p) \\
&= \frac{6}{16} = \frac{3}{8}
\end{aligned}
$$

1.3 Part c)

The marginal pdf of X is $f_X(x)$, which is equal to $P(\{X = x\})$ and can be manually evaluated:

$$
\begin{aligned}
P(\{X = -1\}) &= \sum_y P(\{X = -1\} \cap \{Y = y\}) \\
&= (p - \frac{1}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) \\
&= 2p + \frac{1}{8} \\
P(\{X = 0\}) &= \sum_y P(\{X = 0\} \cap \{Y = y\}) \\
&= (\frac{1}{4} - p) + (\frac{3}{16}) + (\frac{1}{16}) \\
&= -p + \frac{1}{2} \\
P(\{X = 1\}) &= \sum_y P(\{X = 1\} \cap \{Y = y\}) \\
&= (0) + (\frac{1}{8}) + (\frac{1}{4} - p) \\
&= -p + \frac{3}{8} \\
\implies f_X(x) = P(\{X = x\}) &= \begin{cases} 2p + \frac{1}{8} & x = -1 \\ -p + \frac{1}{2} & x = 0 \\ -p + \frac{3}{8} & x = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}(\{\mathrm{Y}=-1\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=-1\}) \\
&= (p-\frac{1}{16}) + (\frac{1}{4}-p) + (0) \\
&= \frac{3}{16} \\
\mathrm{P}(\{\mathrm{Y}=0\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=0\}) \\
&= (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) \\
&= \frac{7}{16} \\
\mathrm{P}(\{\mathrm{Y}=1\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=1\}) \\
&= (p+\frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4}-p) \\
&= \frac{6}{18} = \frac{3}{8} \\
\implies f_{\mathrm{Y}}(x) = \mathrm{P}(\{\mathrm{Y}=y\}) &= \begin{cases} \frac{3}{16} & y=-1 \\ \frac{7}{16} & y=0 \\ \frac{3}{8} & y=1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

1.4 Part d)

X and Y are independant if

$$
\mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=y\}) = \mathrm{P}(\{\mathrm{X}=x\})\cdot\mathrm{P}(\{\mathrm{Y}=y\}) = f_{\mathrm{X}}(x)\cdot f_{\mathrm{Y}}(y) \tag{2}
$$

for all possible values x and y. Therefore, this must be true for $x=-1$ and $y=1$:

$$
\begin{aligned}
\mathrm{LHS} &= \mathrm{P}(\{\mathrm{X}=-1\}\cap\{\mathrm{Y}=1\}) \\
&= p + \frac{1}{16}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{RHS} &= \mathrm{P}(\{\mathrm{X}=-1\})\mathrm{P}(\{\mathrm{Y}=1\}) \\
&= (2p+\frac{1}{8})(\frac{3}{8}) \\
&= \frac{3}{4}p + \frac{3}{64}
\end{aligned}
$$

As shown above, LHS and RHS are only equal for zero or one values of p. Letting LHS = RHS, we can find this exact value (or lack thereof):

$$
\begin{aligned}
p + \frac{1}{16} &= \frac{3}{4}p + \frac{3}{64} \\
\frac{1}{4}p &= \frac{3}{64} - \frac{1}{16} \\
p &= -\frac{1}{64}\cdot 4 = -\frac{1}{16}
\end{aligned}
$$

Therefore LHS = RHS only when $p = -\frac{1}{16}$, however from (1) this is not within the potential domain of p. Therefore LHS $\neq$ RHS, showing one counterexample to (2), hence X and Y are not independent.

1.5 Part e)

$$\begin{aligned}
\mathrm{E}(\mathrm{X}) &= \sum_x x\mathrm{P}(\{\mathrm{X} = x\}) \\
&= -1(2p + \frac{1}{8}) + 0(-p + \frac{1}{2}) + 1(-p + \frac{3}{8}) \\
&= -2p - \frac{1}{8} - p + \frac{3}{8} \\
\therefore \mathrm{E}(\mathrm{X}) &= -3p + \frac{1}{4} \\
\mathrm{E}(\mathrm{Y}) &= \sum_y y\mathrm{P}(\{\mathrm{Y} = y\}) \\
&= -1(\frac{3}{16}) + 0(\frac{7}{16}) + 1(\frac{3}{8}) \\
\therefore \mathrm{E}(\mathrm{Y}) &= -\frac{3}{16} + \frac{3}{8} = \frac{3}{16} \\
\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) &= \mathrm{E}[(\mathrm{X} - \mathrm{E}(\mathrm{X})(\mathrm{Y} - \mathrm{E}(\mathrm{Y})] \\
\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) &= \sum_{c\in\Omega} (\mathrm{X}(c) - \mathrm{E}(\mathrm{X}))(\mathrm{Y}(c) - \mathrm{E}(\mathrm{Y}))\mathrm{P}(\{c\}) \\
&= \sum_{x,y} (x - (-3p + \frac{1}{4}))(y - (\frac{3}{16}))\mathrm{P}(\{\mathrm{X} = x\} \cap \{\mathrm{Y} = y\})
\end{aligned}$$

Expanding this sum is tedious and results in nine trinomials. The following sum for the Cov(X, Y) expansion significantly reduces the algebra necessary by computing E(XY) instead:

$$
\begin{aligned}
\mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= \mathrm{E}(\mathrm{XY}) - \mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y}) \\
\mathrm{E}(\mathrm{XY}) &= \sum_{c\in\Omega} \mathrm{X}(c)\mathrm{Y}(c)\mathrm{P}(\{c\}) \\
&= \sum_{x,y} xy\mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=y\}) \\
&= (-1)(-1)(p-\frac{1}{16}) + (-1)(0)(\frac{1}{4}-p) + (-1)(1)(0) \\
&+ (0)(-1)(\frac{1}{8}) + (0)(0)(\frac{3}{16}) + (0)(1)(\frac{1}{8}) \\
&+ (1)(-1)(p+\frac{1}{16}) + (1)(0)(\frac{1}{16}) + (1)(1)(\frac{1}{4}-p) \\
&= (p-\frac{1}{16}) - (p+\frac{1}{16}) + (\frac{1}{4}-p) \\
\therefore \mathrm{E}(\mathrm{XY}) &= -p+\frac{1}{8} \\
\implies \mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= (-p+\frac{1}{8}) - (-3p+\frac{1}{4})(\frac{3}{16}) \\
&= -p+\frac{1}{8}+\frac{9}{16}p-\frac{3}{64} \\
\therefore \mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= -\frac{7}{16}p-\frac{5}{64}
\end{aligned}
$$

2 Question 2

Ω is continuous, which implies X is continuous and Y is continuous. Let D be the set of all $< x, y >$ that is inside (or on the boundary) of the triangle given.

2.1 Part a)

We are told that the joint pdf of X and Y is uniform over D, and assume it is 0 everywhere else. The area of the triangle D on a cartisian plane is $A = \frac{1}{2}bh = 1$, and we know

$$\int_{d \in D} f_{\mathrm{X,Y}}(d)\mathrm{d}d = 1$$

Since $f_{\mathrm{X,Y}}$ is uniform, this integral can be interpreted as the geometric volume of a triangular prism, extruded from D by $f_{\mathrm{X,Y}}$

$$\begin{aligned} A \cdot f_{\mathrm{X,Y}} &= 1 \\ f_{\mathrm{X,Y}} &= 1 \end{aligned}$$

Therefore the joint pdf of (X, Y) is $f_{\mathrm{X,Y}} = 1$.

2.2 Part b)

The set of vectors D_2 containing all vectors $< x, y >$ satisfying $x > y$ in D forms a triangle on a cartisian plane with verticies at $(0, 0)$ $(0.5, 0.5)$ and $(1, 0)$. The area of this triangle is exactly $\frac{1}{4}$ the total area of D, since $A = \frac{1}{2}bh = \frac{1}{2} \cdot 1 \cdot 0.5 = \frac{1}{4}$. Therefore:

$$\begin{aligned} \mathrm{P}(\{\mathrm{X} \geq \mathrm{Y}\}) &= \int_{d \in D_2} f_{\mathrm{X,Y}}(d)\mathrm{d}d \\ &= f_{\mathrm{X,Y}} \cdot A \\ &= \frac{1}{4} \\ \therefore \mathrm{P}(\{\mathrm{X} \geq \mathrm{Y}\}) &= \frac{1}{4} \end{aligned}$$

2.3 Part c)

Since

$$\begin{aligned} F_{\mathrm{X}} &= \int_{-\infty}^{x} f_{\mathrm{X}}(x)\mathrm{d}x \\ f_{\mathrm{X}}(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F_{\mathrm{X}}(x) \end{aligned}$$

Figure 1 illustrates the geometric cases involved with evaluating f_{X}:

We know

$$\begin{aligned} F_{\mathrm{X}}(-1) &= 0 & \qquad F_{\mathrm{X}}(x < -1) &= 0 \\ F_{\mathrm{X}}(1) &= 1 & \qquad F_{\mathrm{X}}(x > 1) &= 1 \end{aligned}$$

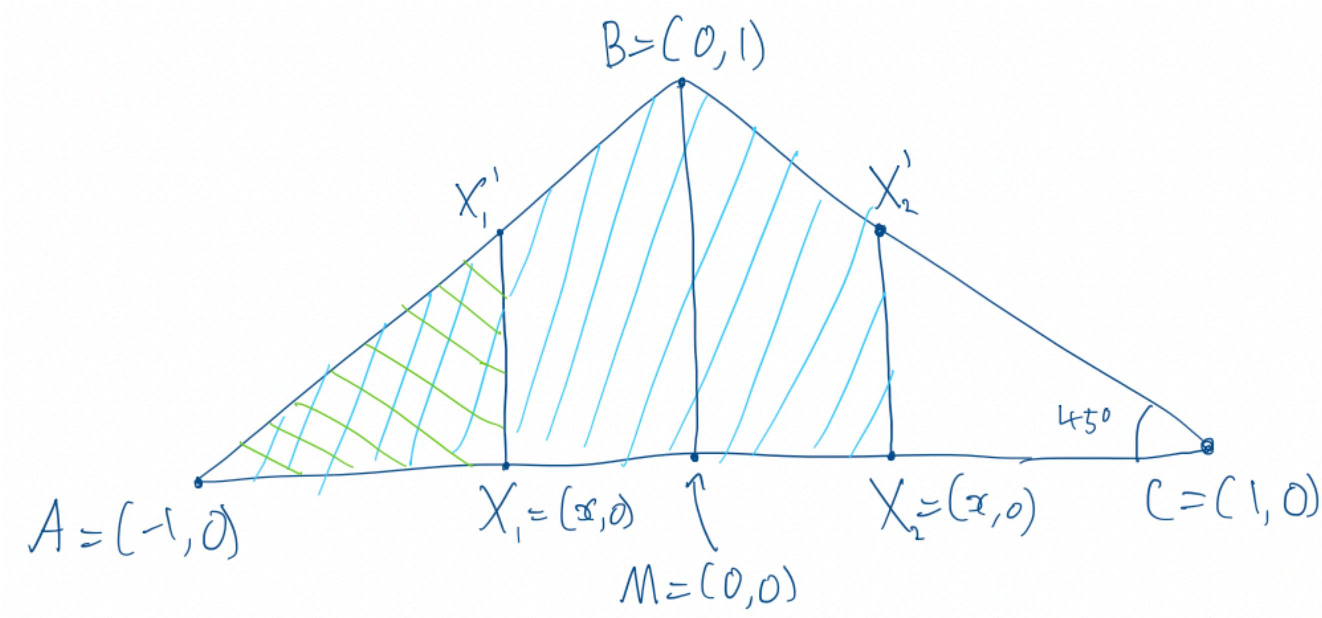

Figure 1: Geometric rendering of D showing the cases of F_{X}

Case 1: $-1 \leq x \leq 0$. This implies F_{X} is the area of $\triangle \mathrm{AX}_1'\mathrm{X}_1$. Let $\mathrm{AX}_1 = x_1 = \mathrm{X}_1\mathrm{X}_1'$, which implies $x_1 = x + 1$:

$$\begin{aligned}\text{Area}(\mathrm{AX}_1'\mathrm{X}_1) &= \frac{1}{2}bh \\ &= \frac{1}{2}x_1^2\end{aligned}$$

Therefore

$$F_{\mathrm{X}}(-1 \leq x \leq 0) = \frac{1}{2}(x+1)^2$$

Case 2: $0 < x \leq 1$. This implies $F_{\mathrm{X}}(x)$ is the area of $\mathrm{ABX}_2'\mathrm{X}_2$. Note $x = MX_2$, and that $X_2X_2' = X_2C$

$$
\begin{aligned}
\text{Area}(\text{ABX}_2'\text{X}_2) &= \text{Area}(\triangle\text{ABM}) + \text{Area}(\text{MBX}_2'\text{X}_2) \\
\text{Area}(\triangle ABM) &= \frac{1}{2} \\
\text{Area}(MBX_2'X_2) &= \text{Area}(MBC) - \text{Area}(X_2X_2'C) \\
\text{Area}(MBC) &= \text{Area}(ABM) = \frac{1}{2} \\
\text{Area}(X_2X_2'C) &= \frac{1}{2}bh \\
&= \frac{1}{2} \cdot (1-x) \cdot (1-x) \\
&= \frac{1}{2}(1-x)^2
\end{aligned}
$$

This is enough information to express F_{X}:

$$
\begin{aligned}
\implies F_{\text{X}}(0 < x \leq 1) &= (\frac{1}{2}) + (\frac{1}{2} - \frac{1}{2}(1-x)^2) \\
&= 1 - \frac{1}{2}(1 - 2x + x^2) \\
&= 1 - \frac{1}{2} + x - \frac{1}{2}x^2 \\
&= -\frac{1}{2}x^2 + x + \frac{1}{2}
\end{aligned}
$$

Combining cases:

$$
\implies F_{\text{X}} = \begin{cases} 0 & : x < -1 \\ \frac{1}{2}(1+x)^2 & : -1 \leq x \leq 0 \\ -\frac{1}{2}x^2 + x + \frac{1}{2} & : 0 < x \leq 1 \\ 1 & : x > 1 \end{cases}
$$

$$
\therefore f_{\text{X}}(x) = \begin{cases} x+1 & : -1 \leq x \leq 0 \\ -x+1 & : 0 < x \leq 1 \\ 0 & : \text{otherwise} \end{cases}
$$

This so happens to be the geometric shape of D on a cartesian plane, ABC.

$$
F_{\text{Y}}(y \leq 0) = 0
$$

$$
F_{\text{Y}}(y \geq 1) = 1
$$

Case $0 < y < 1$

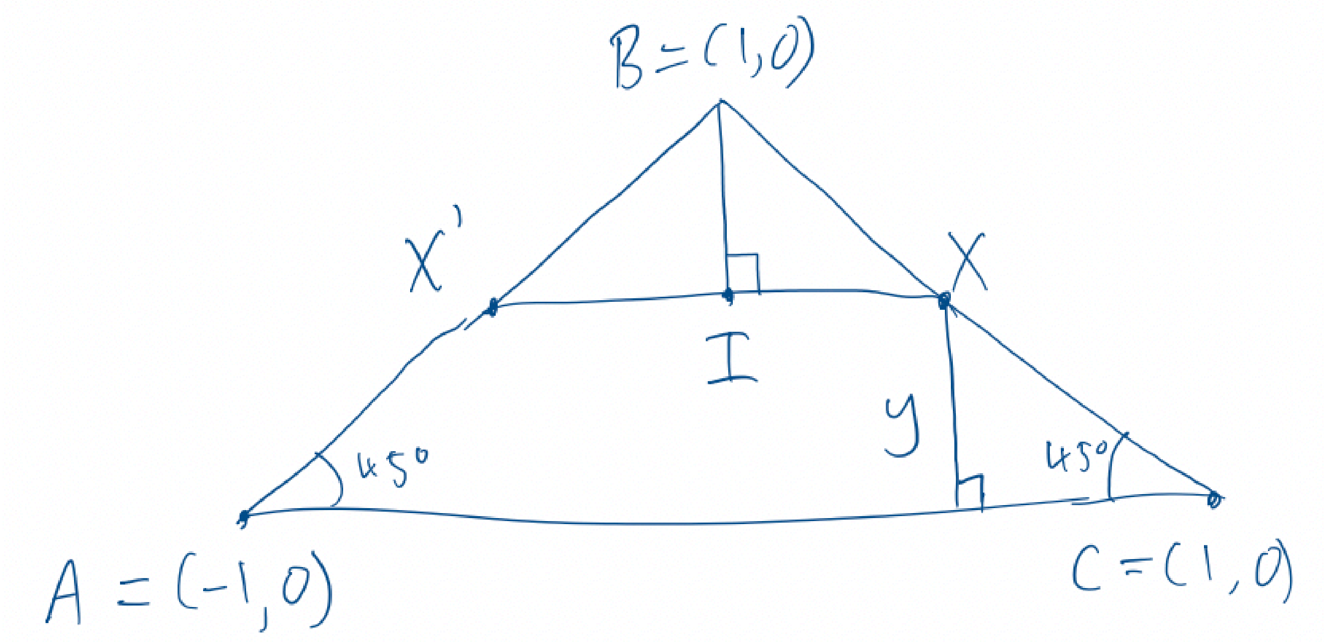

Figure 2: Geometric rendering of D showing the cases of F_{Y}

$$\begin{aligned}
F_{\mathrm{Y}}(y) &= \text{Area}(AX'XC) \\
&= \text{Area}(\triangle ABC) - \text{Area}(\triangle XBX') \\
&= 1 - 2 \cdot \text{Area}(\triangle XIB)
\end{aligned}$$

Note IB $= 1 - y =$ IX

$$\begin{aligned}
\text{Area}(\triangle XIB) &= \frac{1}{2}bh \\
&= \frac{1}{2}(1-y)^2 \\
&= \frac{1}{2}(y^2 - 2y + 1) \implies F_{\mathrm{Y}} \qquad\qquad = 1 - (y^2 - 2y + 1) \\
&= -y^2 + 2y
\end{aligned}$$

Combining cases

$$F_{\mathrm{Y}}(y) = \begin{cases} 1 & : y \geq 1 \\ -y^2 + 2y & : 0 < y < 1 \\ 0 & : y < 0 \end{cases}$$

$$f_{\mathrm{Y}}(y) = \begin{cases} -2y + 2 & : 0 < y < 1 \\ 0 & : \text{otherwise} \end{cases}$$

The geometric interpretation of this is not as intuitive ro realise. Morph A to (0, 2), B to (0, 1) and C to (0, 0), and the initial value of $f_{\mathrm{X}}(0) = AC$ is now placed on the y-axis.

2.4 Part d)

The continuous independance rule can be stated like so:

$$f_{\mathrm{X,Y}}(< x, y >) = f_{\mathrm{X}}(x) \cdot f_{\mathrm{Y}}(y) \quad \forall x, y \in \mathbb{R}$$

Suppose $< x, y > \in D$.

$$\begin{aligned}
\mathrm{LHS} &= f_{\mathrm{X,Y}}(< x, y >) \\
&= 1\mathrm{RHS} & &= f_{\mathrm{Y}}(y) \cdot f_{\mathrm{X}}(x) \\
&= (-2y + 2) \cdot \begin{cases} x + 1 & : -1 \leq x \leq 0 \\ -x + 1 & ; 0 < x \leq 1 \end{cases}
\end{aligned}$$

Suppose further $< x, y > = < 0, 0 >$

$$\begin{aligned}
\mathrm{RHS} &= (-2 \cdot 0 + 2)(0 + 1) \\
&= 2
\end{aligned}$$

Therefore there exists an $< x, y > \in D$ such that the independance rule fails, LHS $\neq$ RHS. Therefore X dn Y are not independant.

2.5 Part e)

It can be immediately noted that, due to the symmetry across the $x = 0$ ”line” and the fact that $\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \int\limits_{x,y} xy\mathrm{P}(\{\mathrm{X} = x\} \cap \{\mathrm{Y} = y\})$ is negative (or zero) for $x < 0$ and positive (or zero) for $x > 0$ demonstrates intuitively that $\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = 0$.

Somehow this seems dissatisfying to me. $\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = \mathrm{E}(\mathrm{XY}) - \mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y})$ is another way to evaluate $\mathrm{Cov}(\mathrm{X}, \mathrm{Y})$. This method requires evaluating a double integral or equivalent.

$$\begin{aligned}
\mathrm{E}(\mathrm{X}) &= \int_{-\infty}^{+\infty} x f_{\mathrm{X}}(x) \, \mathrm{d}x \\
&= \int_{-1}^{0} x(x + 1) \, \mathrm{d}x + \int_{0}^{1} x(-x + 1) \, \mathrm{d}x \\
&= \int_{-1}^{0} x^2 + x \, \mathrm{d}x + \int_{0}^{1} -x^2 + x \, \mathrm{d}x \\
&= \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_{x=-1}^{x=0} + -\frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_{x=0}^{x=1} \\
&= (0 + 0) - (\frac{1}{3}(-1)^3 + \frac{1}{2}(-1)^2) + (-\frac{1}{3}(1)^3 + \frac{1}{2}(1)^2) - (0 + 0) \\
&= +\frac{1}{3} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2} \\
&= 1
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}(\mathrm{Y}) &= \int_{-\infty}^{+\infty} y f_{\mathrm{Y}}(y)\,\mathrm{d}y \\
&= \int_{0}^{1} -2y + 2\,\mathrm{d}y \\
&= \left. -y^2 + 2y \right|_{y=0}^{y=1} \\
&= (-(1)^2 + 2(1)) - (0+0) \\
&= -1 + 2 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}(\mathrm{XY}) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy f_{\mathrm{X}}(x) f_{\mathrm{Y}}(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-\infty}^{+\infty} x \left(\int_{0}^{1} y f_{\mathrm{Y}}(y)\,\mathrm{d}y \right) f_{\mathrm{X}}(x)\,\mathrm{d}x \\
&= \mathrm{E}(\mathrm{Y}) \int_{-1}^{+1} x f_{\mathrm{X}}(x)\,\mathrm{d}x \\
&= \mathrm{E}(\mathrm{Y}) \cdot \mathrm{E}(\mathrm{X}) \\
&= 1
\end{aligned}
$$

Therefore $\mathrm{Cov}(\mathrm{X}, \mathrm{Y}) = 1 - 1 \cdot 1 = 0$