STAT1301 Advanced Analysis of Scientific Data

Semester 2, 2025, Assignment 2

Caleb Yates s49886350

1 Introduction

Throughout the report, the following syntactical shortcuts and notation will be used.

If X is a random variable of the sample space Ω, an abbreviation of set notation is as follows:

$$\text{Abbreviate } \{d : \forall d \in \Omega \text{ and } \mathrm{X}(d) = x\} \tag{1}$$
$$\text{As } \{\mathrm{X} = x\} \tag{2}$$

Additionally, when thinking in terms of sets becomes obselete,

$$\text{Abbreviate } \mathrm{P}(\{d : \forall d \in \Omega \text{ and } \mathrm{X}(d) = x\}) \tag{3}$$
$$\text{As } \mathrm{P}(\mathrm{X} = x) \tag{4}$$

The abbriviation will be used with inequalities as well.

Given some random variable X, there must exist a function mapping from the sample space Ω to the domain of X, which can be at most $\mathbb{R}$. This function is (intuitively) named X. This function incidentally defines the random variable, which is the motivating reason for using its letter to represent its mapping. The notation Domain[X] will be used throughout this report to indicate the domain of the function mapping X and hence the random variable X itself by definition.

Also, $N(\mu, \sigma)$ indicates that σ is $\sqrt{\mathrm{Var}}$, aka the standard deviation. This is opposed to the syntax of $N(\mu, \sigma^2)$. For clarity, $\sigma =$ will be explicitely written to avoid ambiguity.

2 Question 1

To begin, lets define the sample space

$$\Omega = \{(a, b) \in \{1, 2, 3, 4, 5, 6\}\}$$
$$|\Omega| = 36$$

Notice this is uniform, and hence that a and b are independant

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{36} \tag{5}$$

2.1 Part a)

Let X be a random variable representing the payout of a given dice roll $(a, b) \in \Omega$:

$$\mathrm{X}((a, b) \in \Omega) = a \cdot b$$

Let f_{X} be the PMF of X. Note $f_{\mathrm{X}}(x \in \Omega) = \mathrm{P}(\{\mathrm{X} = x\})$. By cases, the probability distribution of X can be deduced:

$$f_{\mathrm{X}}(1) = \frac{1}{36} \qquad f_{\mathrm{X}}(8) = \frac{2}{36} \qquad f_{\mathrm{X}}(18) = \frac{2}{36}$$
$$f_{\mathrm{X}}(2) = \frac{2}{36} \qquad f_{\mathrm{X}}(9) = \frac{1}{36} \qquad f_{\mathrm{X}}(20) = \frac{2}{36}$$
$$f_{\mathrm{X}}(3) = \frac{2}{36} \qquad f_{\mathrm{X}}(10) = \frac{2}{36} \qquad f_{\mathrm{X}}(24) = \frac{2}{36}$$
$$f_{\mathrm{X}}(4) = \frac{3}{36} \qquad f_{\mathrm{X}}(12) = \frac{4}{36} \qquad f_{\mathrm{X}}(25) = \frac{1}{36}$$
$$f_{\mathrm{X}}(5) = \frac{2}{36} \qquad f_{\mathrm{X}}(15) = \frac{2}{36} \qquad f_{\mathrm{X}}(30) = \frac{2}{36}$$
$$f_{\mathrm{X}}(6) = \frac{4}{36} \qquad f_{\mathrm{X}}(16) = \frac{1}{36} \qquad f_{\mathrm{X}}(36) = \frac{1}{36}$$

For all other values x, $f_{\mathrm{X}}(x) = 0$

2.2 Part b)

This makes determining the expected value of X trivial:

$$\begin{aligned}
\mathrm{E(X)} &= \sum_{c\in\Omega} \mathrm{X}(c)\mathrm{P}(c) && (6)\\
&= \sum_{x\in\mathrm{Domain[X]}} x\mathrm{P}(\{\mathrm{X}=x\}) && (7)\\
&= 1\cdot f_{\mathrm{X}}(1) + 2\cdot f_{\mathrm{X}}(2) + \cdots 30\cdot f_{\mathrm{X}}(30) + 36\cdot f_{\mathrm{X}}(36) && (8)\\
&= \frac{1}{36} + \frac{4}{36} + \cdots \frac{60}{36} + \frac{36}{36} && (9)\\
&= \frac{441}{36} = \frac{49}{4} = 12.25 && (10)
\end{aligned}$$

2.3 Part c)

Evaluating Var(X) is similarly trivial

$$\begin{aligned}
\mathrm{Var(X)} &= \mathrm{E}[(\mathrm{X}-\mathrm{E}(X))^2] && (11)\\
&= \sum_{c\in\Omega} (\mathrm{X}(c) - \frac{49}{4})^2\mathrm{P}(\{c\}) && (12)\\
&= \sum_{x\in\mathrm{Domain[X]}} (x - \frac{49}{4})^2\mathrm{P}(\{\mathrm{X}=x\}) && (13)\\
&= (1-\frac{49}{4})^2\cdot\frac{1}{36} + (2-\frac{49}{4})^2\cdot\frac{2}{36} + \cdots (30-\frac{49}{4})^2\cdot\frac{2}{36} + (36-\frac{49}{4})^2\cdot\frac{1}{36} && (14)\\
&= \frac{11515}{144} \approx 79.97 && (15)\\
\implies \sigma_{\mathrm{X}} &= \sqrt{\mathrm{Var(X)}} = \sqrt{\frac{11515}{144}} \approx 8.942 && (16)
\end{aligned}$$

3 Question 2

Understanding this question in terms of a sample space isn't very fruitful. Ω is completely unspecified, we can only deduce that $|\Omega| \geq (0, 20)$, which implies it is continuous. $\mathrm{P}(A) : \exists A \in \Omega$ is also completely unknown.

3.1 Part a)

Let X be the continuous random variable of algae growth as measured in grams of biomass produced. Note Domain[X] $= (0, 20)$.

Since X is a random variable, its PDF f_{X} must sum to 1:

$$
\begin{aligned}
1 &= \int_{c \in \Omega} \mathrm{P}(\{c\}) && (17)\\
&= \int_{x \in \mathrm{Domain}[X]} \mathrm{P}(\{\mathrm{X} = x\}) && (18)\\
&= \int_0^{20} c(x^2 - 60x + 800)\mathrm{d}x && (19)\\
&= c[\frac{1}{3}x^3 - 30x^2 + 800x]_{x=0}^{x=20} && (20)\\
1/c &= [\frac{1}{3}(20)^3 - 30(20)^2 + 800(20)] - [0 - 0 + 0] && (21)\\
1/c &= \frac{20000}{3} && (22)\\
c &= \frac{3}{20000} && (23)
\end{aligned}
$$

3.2 Part b)

Let F_{X} be the CDF of X:

$$
\begin{aligned}
F_{\mathrm{X}} &= \int_{-\infty}^{x} f_{\mathrm{X}}(x)\mathrm{d}x && (24)\\
&= \int_0^x c(x^2 - 60x + 800)\mathrm{d}x && (25)\\
&= c[\frac{1}{3}x^3 - 30x^2 + 800x]_{x=0}^{x=x} && (26)\\
\frac{F_{\mathrm{X}}}{c} &= [\frac{1}{3}x^3 - 30x^2 + 800x] - [\frac{1}{3}0^3 - 30 \cdot 0^2 + 800 \cdot 0] && (27)\\
\implies F_{\mathrm{X}} &= c(\frac{1}{3}x^3 - 30x^2 + 800x) \text{ for } 0 \leq x \leq 20 && (28)\\
&= \frac{1}{20000}x^3 - \frac{9}{2000}x^2 + \frac{3}{25}x && (29)
\end{aligned}
$$

3.3 Part c)

$$\mathrm{E(X)} = \int_{x \in \mathrm{Domain[X]}} x f_{\mathrm{X}} \mathrm{d}x \tag{30}$$

$$= \int_0^{20} x \cdot c(x^2 - 60x + 800)\mathrm{d}x \tag{31}$$

$$\frac{\mathrm{E(X)}}{c} = \int_0^{20} x^3 - 60x^2 + 800x \mathrm{d}x \tag{32}$$

$$= [\frac{1}{4}x^4 - 20x^3 + 400x^2]_{x=0}^{x=20} \tag{33}$$

$$= [\frac{1}{4}(20)^4 - 20(20)^3 + 400(20)^2] - [0 - 0 + 0] \tag{34}$$

$$= 40000 - 160000 + 160000 \tag{35}$$

$$\mathrm{E(X)} = c \cdot 40000 \tag{36}$$

$$\mathrm{E(X)} = 6 \text{ grams} \tag{37}$$

3.4 Part d)

$$\mathrm{P}(\{\mathrm{X} > 10\} | \{\mathrm{X} > 2\}) \tag{38}$$

$$= \frac{\mathrm{P}(\{\mathrm{X} > 10\} \cap \{\mathrm{X} > 2\})}{\{\mathrm{X} > 2\}} \tag{39}$$

$$= \frac{\mathrm{P}(\{\mathrm{X} > 10\})}{\mathrm{P}(\{\mathrm{X} > 2\})} \tag{40}$$

From the CDF definition of X, $\mathrm{P}(\{\mathrm{X} < x\}) = f_{\mathrm{X}}(x)$

$$\implies \mathrm{P}(\{\mathrm{X} > 10\}) = 1 - \mathrm{P}(\{\mathrm{X} < 10\}) \tag{41}$$

$$= 1 - F_{\mathrm{X}}(10) \tag{42}$$

$$= 1 - \frac{4}{5} \tag{43}$$

$$= \frac{1}{5} \tag{44}$$

$$\implies \mathrm{P}(\{\mathrm{X} > 2\}) = 1 - \mathrm{P}(\{\mathrm{X} < 2\}) \tag{45}$$

$$= 1 - F_{\mathrm{X}}(2) \tag{46}$$

$$= 1 - \frac{139}{625} \tag{47}$$

$$= \frac{486}{625} \tag{48}$$

$$\implies \frac{\mathrm{P}(\{\mathrm{X} > 10\})}{\mathrm{P}(\{\mathrm{X} > 2\})} = \frac{\frac{1}{5}}{\frac{486}{625}} = \frac{125}{486} \approx 0.2572 \tag{49}$$

Therefore, the probability that the biomass exceeds 10 grams, given that it is detectable, is above in (49) $= \frac{125}{486}$.

4 Question 3

Assume that $p = 0.25$ for all the products, not just the 25 that were sampled.

The sample space for this is again completely unspecified, and the P probability function is practically useless for this question. For convenience, the sample space Ω is therefore defined as the domain of X, representing the number of products passing the specific inspection.

$$\Omega = \{1, 2, 3...24, 25\}$$

This makes the definition of X trivial, and its domain incidentally the entire sample space:

$$\mathrm{X}(a \in \Omega) = a$$

4.1 Part a)

Since each product has a $p = 0.25$ probability of passing inspection, and there are 25 products, and it is assumed each inspection and product is independant of each other, X is a binomial distribution:

$$\mathrm{X} \sim \mathrm{Bin}(n = 25, p = 0.25)$$

Notes the following theorems about binomial distributions and X:

$$\mathrm{P}(\{\mathrm{X} = x\}) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{25}{x} 0.25^x \cdot 0.75^{25-x}$$

$$\mathrm{E}(\mathrm{X}) = np = \frac{25}{4}$$

$$\mathrm{Var}(\mathrm{X}) = np(1-p) = \frac{75}{16}$$

4.2 Part b)

Let X_2 be the random variable representing the probability distribution of X with an n parameter such that the probability of finding a defect-free product eexceeds 99%:

$$\mathrm{X}_2 \sim \mathrm{Bin}(n, p = 0.25)$$

$$\mathrm{P}(\{\mathrm{X}_2 \geq 1\}) > 0.99 \tag{50}$$
$$0.99 < \mathrm{P}(\{\mathrm{X}_2 \geq 1\}) \tag{51}$$
$$0.99 < 1 - \mathrm{P}(\{\mathrm{X}_2 = 0\}) \tag{52}$$
$$0.99 - 1 < -\binom{n}{0}(0.25)^0(0.75)^n \tag{53}$$
$$0.01 > 1 \cdot 1 \cdot 0.75^n \tag{54}$$
$$\log_{0.75} 0.01 > n \tag{55}$$
$$\implies n < \log_{0.75} 0.01 \approx 16.008 \tag{56}$$

Therefore the minimum (integer) sample size is $n = 16$.

4.3 Part c)

The random variable Y is dependant on X. Given a possibility $a \in \Omega$ from the sample space, $\mathrm{Y}(a)$ explicitely depends upon $\mathrm{X}(a)$ such that it exactly equals:

$$\mathrm{Y}(a \in \Omega) = 3\mathrm{X}(a) - (25 - \mathrm{X}(a)) \tag{57}$$
$$= 4\mathrm{X}(a) - 25 \tag{58}$$

This allows us to calculate E(X) and Var(X) relatively easily using probability theorems:

$$\mathrm{E}(\mathrm{Y}) = \mathrm{E}(4\mathrm{X} - 25) \tag{59}$$
$$= 4\mathrm{E}(\mathrm{X}) - 25 \tag{60}$$
$$= 4 \cdot \frac{25}{4} - 25 \tag{61}$$
$$= 0 \tag{62}$$

$$\mathrm{Var}(\mathrm{Y}) = \mathrm{Var}(4\mathrm{X} - 25) \tag{63}$$
$$= 4^2\mathrm{Var}(\mathrm{X}) \tag{64}$$
$$= 16 \cdot \frac{75}{16} \tag{65}$$
$$= 75 \tag{66}$$

4.4 Part d)

Since Y is defined in terms of X, this isn't too difficult to evaluate:

$$\mathrm{P}(\{\mathrm{Y} \geq 27\}) = \mathrm{P}(\{4\mathrm{X} - 25 \geq 27\}) \tag{67}$$
$$= \mathrm{P}(\{4\mathrm{X} \geq 52\}) \tag{68}$$
$$= \mathrm{P}(\{\mathrm{X} \geq 13\}) \tag{69}$$
$$\approx 0.00337 \tag{70}$$

This can be calculated by running 1 - pbinom(12, 25, 0.25) in R

5 Question 4

Let $\Omega = (-\infty, +\infty)$ in units ℃, representing the continuous range of possible temperatures in the storeroom. An argument could be made to limit this to $(-\infty, 8)$.

Let X be a random variable for the temperature inside the storeroom.

5.1 Part a)

$$\mathrm{X} \sim N(\mu = 7.5℃, \sigma = 0.3℃)$$

$$\begin{aligned}
\mathrm{P}(\{7.2 < \mathrm{X} < 8\}) &= \mathrm{P}(\{\frac{7.2 - 7.5}{0.3} < \frac{\mathrm{X} - \mu}{\sigma} < \frac{8 - 7.5}{0.3}\}) && (71)\\
&= \mathrm{P}(\{-\frac{2}{3} < \mathrm{Z} < \frac{5}{3}\}) && (72)\\
&= \mathrm{P}(\{\mathrm{Z} < \frac{5}{3}\}) - \mathrm{P}(\{-\frac{2}{3} < \mathrm{Z}\}) && (73)\\
&= \mathrm{P}(\{\mathrm{Z} < \frac{5}{3}\}) - \mathrm{P}(\{\mathrm{Z} > -\frac{2}{3}\}) && (74)\\
&= \mathrm{P}(\{\mathrm{Z} < \frac{5}{3}\}) - (1 - \mathrm{P}(\{\mathrm{Z} < \frac{2}{3}\})) && (75)\\
&= \mathrm{P}(\{\mathrm{Z} < \frac{5}{3}\}) + \mathrm{P}(\{\mathrm{Z} < \frac{2}{3}\}) - 1 && (76)
\end{aligned}$$

Using stats tables this equals $0.9515 + 0.7454 - 1 = 0.6969$. Using R running pnorm($\frac{5}{3}$) - pnorm($-\frac{2}{3}$) $= 0.6997$.

5.2 Part b)

$$\mathrm{X} \sim N(\mu, \sigma = 0.3℃)$$

$$\begin{aligned}
\mathrm{P}(\{\mathrm{X} > 8℃\}) &= 1\% && (77)\\
0.01 &= \mathrm{P}(\{\mathrm{X} > 8\}) && (78)\\
&= 1 - \mathrm{P}(\{\mathrm{X} < 8\}) && (79)\\
0.99 &= \mathrm{P}(\{\mathrm{X} < 8\}) && (80)\\
&= \mathrm{P}(\{\frac{\mathrm{X} - \mu}{\sigma} = \frac{8 - \mu}{\sigma}\}) && (81)\\
0.99 &= \mathrm{P}(\{\mathrm{Z} = \frac{8 - \mu}{3}\}) && (82)\\
& && (83)
\end{aligned}$$

Let z be the value which satisfies $\mathrm{P}(\{\mathrm{Z} < z\}) = 0.99$.

$$\implies \frac{8-\mu}{3} = z \tag{84}$$
$$8-\mu = 3z \tag{85}$$
$$-\mu = 3x - 8 \tag{86}$$
$$\mu = 8 - 3z \tag{87}$$

Using the stats table, $z \approx 2.33$ which implies $\mu \approx 8 - 3 \cdot 2.33 = 1.01$℃. Using R, $z =$ qnorm(0.99) ≈ 2.326348, which implies $\mu \approx 8 - 3 \cdot 2.326348 \approx 1.021$℃.

5.3 Part c)

We are given no information about the parameters of X

$$\mathrm{X} \sim N(\mu, \sigma)$$

$$\mathrm{P}(\{\mu - 1℃ < \mathrm{X} < \mu + 1℃\}) = 95\% \tag{88}$$
$$0.95 = \mathrm{P}(\{\frac{(\mu-1)-\mu}{\sigma} < \frac{\mathrm{X}-\mu}{\sigma} < \frac{(\mu-1)-\mu}{\sigma}\}) \tag{89}$$
$$= \mathrm{P}(\{\frac{-1}{\sigma} < \mathrm{Z} < \frac{+1}{\sigma}\}) \tag{90}$$
$$= 1 - (\mathrm{P}(\{\mathrm{Z} < \frac{-1}{\sigma}\}) + \mathrm{P}(\{\mathrm{Z} > \frac{+1}{\sigma}\})) \tag{91}$$
$$= 1 - 2\mathrm{P}(\{\mathrm{Z} < \frac{-1}{\sigma}\}) \tag{92}$$
$$0.05 = 2\mathrm{P}(\{\mathrm{Z} < \frac{-1}{\sigma}\}) \tag{93}$$
$$0.025 = \mathrm{P}(\{\mathrm{Z} < \frac{-1}{\sigma}\}) \tag{94}$$
$$1 - 0.025 = 1 - \mathrm{P}(\{\mathrm{Z} < \frac{-1}{\sigma}\}) \tag{95}$$
$$0.975 = \mathrm{P}(\{\mathrm{Z} < \frac{+1}{\sigma}\}) \tag{96}$$
$$\text{Let } z \text{ be the solution to } 0.975 = \mathrm{P}(\{\mathrm{Z} < z\}) \tag{97}$$
$$\implies z = \frac{+1}{\sigma} \tag{98}$$
$$\implies \sigma = \frac{1}{z} \tag{99}$$

Using the stats table, $z \approx 1.96$ which implies $\sigma \approx \frac{1}{1.96} \approx 0.510204 \approx 0.51$. Using R $z =$ qnorm(0.975) ≈ 1.959964 which implies $\sigma \approx \frac{1}{1.959964} \approx 0.5102135 \approx 0.51$.