1 Question 1

1.1 Part a)

Let Ω be the sample space. Therefore $P(\{\Omega\}) = 1$. Adding all the joint pmf values must sum to 1:

$$\{\Omega\} = \bigcup_x \bigcup_y \{X = x\} \cap \{Y = y\}$$

$$P(\{\Omega\}) = 1$$

$$\implies 1 = P((\{X = -1\} \cap \{Y = -1\}) \cup \ldots \cup (\{X = 1\} \cap \{Y = 1\}))$$

$$= P(\{X = -1\} \cap \{Y = -1\}) + \ldots + P(\{X = 1\} \cap \{Y = 1\}))$$

$$= (p - \frac{1}{16}) + (\frac{1}{4} - p) + (0) + (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p)$$

$$1 = -\frac{1}{16} + \frac{4}{16} + \frac{7}{16} + \frac{1}{16} + \frac{1}{16} + \frac{4}{16}$$

$$1 = 1$$

Unfortunately, this tells us no information about p. From the definition of probability, $P(\{c\})$ for $c \in \Omega$ must be greater or equal to 0, $P(\{c \in \Omega\}) \geq 0$. This can be used to restrict the possible values of p:

$$P(A \subseteq \Omega) \geq 0$$

$$\implies P(\{X = -1\} \cap \{Y = -1\}) \geq 0$$

$$p - \frac{1}{16} \geq 0$$

$$p \geq \frac{1}{16}$$

$$\implies P(\{X = 0\} \cap \{Y = -1\}) \geq 0$$

$$\frac{1}{4} - p \geq 0$$

$$p \leq \frac{1}{4}$$

$$\implies P(\{X = -1\} \cap \{Y = 1\}) \geq 0$$

$$p + \frac{1}{16} \geq 0$$

$$p \leq \frac{1}{16}$$

$$\implies p \in [\frac{1}{16}, \frac{1}{4}] \tag{1}$$

Therefore, $\frac{1}{16} \leq p \leq \frac{1}{4}$, and can be any value within this range.

1.2 Part b)

Aim is to find $P(\{X = Y\})$:

$$P(\{X = Y\}) = \sum_a P(\{X = a\} \cap \{Y = a\})$$
$$= P(\{X = -1\} \cap \{Y = -1\}) + P(\{X = 0\} \cap \{Y = 0\}) + P(\{X = 1\} \cap \{Y = 1\})$$
$$= (p - \frac{1}{16}) + (\frac{3}{16}) + (\frac{1}{4} - p)$$
$$= \frac{6}{16} = \frac{3}{}$$

1.3   Part c)

The marginal pdf of X is $f_X(x)$, which is equal to $P(\{X = x\})$ and can be manually evaluated:

$$P(\{X = -1\}) = \sum_y P(\{X = -1\} \cap \{Y = y\})$$
$$= (p - \frac{1}{16}) + (\frac{1}{8}) + (p + \frac{1}{16})$$
$$= 2p + \frac{1}{8}$$
$$P(\{X = 0\}) = \sum_y P(\{X = 0\} \cap \{Y = y\})$$
$$= (\frac{1}{4} - p) + (\frac{3}{16}) + (\frac{1}{16})$$
$$= -p + \frac{1}{2}$$
$$P(\{X = 1\}) = \sum_y P(\{X = 1\} \cap \{Y = y\})$$
$$= (0) + (\frac{1}{8}) + (\frac{1}{4} - p)$$
$$= -p + \frac{3}{8}$$
$$\implies f_X(x) = P(\{X = x\}) = \begin{cases} 2p + \frac{1}{8} & x = -1 \\ -p + \frac{1}{2} & x = 0 \\ -p + \frac{3}{8} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P(\{Y = -1\}) = \sum_x P(\{X = x\} \cap \{Y = -1\})$$

$$= (p - \frac{1}{16}) + (\frac{1}{8}) + (p + \frac{1}{16})$$

$$= p + \frac{1}{8}$$

$$P(\{Y = 0\}) = \sum_x P(\{X = x\} \cap \{Y = 0\})$$

$$= (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8})$$

$$= \frac{7}{16}$$

$$P(\{Y = 1\}) = \sum_x P(\{X = x\} \cap \{Y = 1\})$$

$$= (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p)$$

$$= \frac{6}{18} = \frac{3}{8}$$

$$\implies f_Y(x) = P(\{Y = y\}) = \begin{cases} p + \frac{1}{8} & y = -1 \\ \frac{7}{16} & y = 0 \\ \frac{3}{8} & y = 1 \\ 0 & \text{otherwise} \end{cases}$$

1.4 Part d)

X and Y are independant if

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\}) \cdot P(\{Y = y\}) = f_X(x) \cdot f_Y(y) \tag{2}$$

for all possible values x and y. Therefore, this must be true for $x = -1$ and $y = 1$:

$$\text{LHS} = P(\{X = -1\} \cap \{Y = 1\})$$
$$= p - \frac{1}{16}$$

$$\text{RHS} = P(\{X = -1\})P(\{Y = 1\})$$
$$= (2p + \frac{1}{8})(\frac{3}{8})$$
$$= \frac{3}{4}p + \frac{3}{64}$$

As shown above, LHS and RHS are only equal for zero or one values of p. Letting LHS = RHS, we can find this exact value (or lack thereof):

$$p - \frac{1}{16} = \frac{3}{4}p + \frac{3}{64}$$
$$\frac{1}{4}p = \frac{3}{64} + \frac{1}{16}$$
$$p = \frac{7}{64} \cdot 4 = \frac{7}{16}$$

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Therefore LHS = RHS only when $p = \frac{7}{16}$, however from (1) this is not within the potential domain of p. Therefore LHS $\neq$ RHS, showing one counterexample to (2), hence X and Y are not independent.