# Probability

## Probability Model

1. Ingredients of a probability model:
   (a) Sample space $\Omega$: set of all possible outcomes.
   (b) Events: subsets $A \subset \Omega$ to which a probability can be assigned.
   (c) Probability: function $\mathbb{P}$ that assigns a number to each possible event $A$, satisfying $0 \le \mathbb{P}(A) \le 1$, $\mathbb{P}(\Omega) = 1$, and the Sum Rule: if $A_1, A_2, \ldots$ cannot occur at the same time, then
   $$\mathbb{P}\left(\bigcup_i A_i\right) = \sum_i \mathbb{P}(A_i).$$
2. $A^c$: $A$ does not occur (*complement* of $A$).
3. $A \cap B$: Both $A$ and $B$ occur (also written $A, B$).
4. $A \cup B$: Either $A$ or $B$ (or both) occur.
5. If $A \cap B = \emptyset$. $A$ and $B$ are *disjoint*.
6. If $A \subset B$ then $\mathbb{P}(A) \le \mathbb{P}(B)$.
7. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ and hence $\mathbb{P}(A^c \mid B) = 1 - \mathbb{P}(A \mid B)$.
8. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

## Probability by Counting

9. When all (finitely many) outcomes in $\Omega$ are *equally likely*, the probability of an event $A \subset \Omega$ is
$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\text{Number of elements in } A}{\text{Number of elements in } \Omega}.$$

10.

| Order | Replacement: Yes | Replacement: No |
|---|---|---|
| Yes | $n^k$ | $\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1)$ |
| No | — | $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}$ |

## Conditional Probability

11. Definition:
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$
12. Events $A$ and $B$ are said to be *independent* if
$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B).$$
13. Product Rule (notation $AB = A \cap B$):
$$\mathbb{P}(A_1 A_2 \ldots A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1) \cdots \mathbb{P}(A_n \mid A_1 \ldots A_{n-1}).$$
14. Product Rule for independent events:
$$\mathbb{P}(A_1 A_2 \ldots A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2) \cdots \mathbb{P}(A_n).$$
15. For an event $B$ and a collection of disjoint events, $E_1, E_2, \ldots, E_k$ with union $\Omega$, the Law of Total Probability:
$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \cdots + P(B \mid E_k)P(E_k). \end{aligned}$$
16. Bayes' Rule:
$$P(A|B) = P(B|A)P(A)/P(B), \text{ provided } P(B) \neq 0.$$

## Random Variables

17. Cumulative distribution function (cdf) of a random variable $X$:
$$F(x) = \mathbb{P}(X \le x), \quad x \in \mathbb{R}.$$
18. Probability mass function (pmf) of a discrete random variable:
$$f(x) = \mathbb{P}(X = x) \quad \text{for all } x.$$
19. Probability density function (pdf) of a continous random variable $X$: positive function $f$ with integral 1 such that:
$$\mathbb{P}(a < X \le b) = F(b) - F(a) = \int_a^b f(u)\,\mathrm{d}u.$$
20. Relations between the cdf and pdf of a continuous random variable:
$$F(x) = \int_{-\infty}^x f(t)\,\mathrm{d}t, \qquad f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x).$$

## Expectation

21. Expectation of a discrete random variable with pmf $f$:
$$\mathbb{E}X = \sum_x x\, f(x).$$
22. Expectation of a continuous random variable with pdf $f$:
$$\mathbb{E}X = \int_{-\infty}^{\infty} x\, f(x)\,\mathrm{d}x.$$
23. Expectation of a function of a discrete random variable $X$ with pmf $f$:
$$\mathbb{E}g(X) = \sum_x g(x)\, f(x)\ .$$
24. Properties of expectation:
   (a) $\mathbb{E}(a\,X + b) = a\,\mathbb{E}(X) + b$,
   (b) $\mathbb{E}(g(X) + h(X)) = \mathbb{E}(g(X)) + \mathbb{E}(h(X))$.
25. Variance: $\mathrm{Var}(X) = \mathbb{E}(X - \mu)^2$, where $\mu = \mathbb{E}X$.
26. Properties of variance:
   (a) $\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$.
   (b) $\mathrm{Var}(a + bX) = b^2\,\mathrm{Var}(X)$.
   (c) $\mathrm{s.d.}(X) = \sqrt{\mathrm{Var}(X)}$.

## Bernoulli Random Variable

27. Notation: $X \sim \mathsf{Ber}(p)$
28. $\mathbb{P}(X = 1) = p$ and $\mathbb{P}(X = 0) = 1 - p$
29. $\mathbb{E}X = p$
30. $\mathrm{Var}(X) = p(1 - p)$

## Binomial Random Variable

31. Notation: $X \sim \mathsf{Bin}(n, p)$
32. $\mathbb{P}(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n.$
33. $\mathbb{E}X = np$
34. $\mathrm{Var}(X) = np(1 - p)$.

## Uniform Random Variable

35. Notation: $X \sim \mathsf{U}[a, b]$
36. Probability density function:
$$f(x) = \frac{1}{b - a}, \quad a \le x \le b \quad \text{(and } f(x) = 0 \text{ otherwise)}.$$
37. $\mathbb{E}X = (a + b)/2$
38. $\mathrm{Var}(X) = (b - a)^2/12$

## Normal Random Variable

39. Notation: $X \sim \mathsf{N}(\mu, \sigma^2)$
40. Probability density function:
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ x \in \mathbb{R}$$
41. $\mathbb{E}X = \mu$
42. $\mathrm{Var}(X) = \sigma^2$
43. Standardization: $\frac{X-\mu}{\sigma} \sim \mathsf{N}(0, 1)$.

## Multiple Random Variables

44. Joint cdf of $X_1, \dots, X_n$:

$$F(x_1, \dots, x_n) = \mathbb{P}(X_1 \le x_1, \dots, X_n \le x_n).$$

45. Joint pmf of discrete $X_1, \dots, X_n$:

$$f(x_1, \dots, x_n) = \mathbb{P}(X_1 = x_1, \dots, X_n = x_n).$$

46. Joint pdf of continuous $X_1, \dots, X_n$ (summarised as $\mathbf{X}$): positive function $f$ with total integral 1 such that

$$\mathbb{P}(\mathbf{X} \in B) = \int_{\mathbf{x} \in B} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \quad \text{for all sets } B.$$

47. $X_1, \dots, X_n$ with joint pmf or pdf $f$ are *independent* if

$$f(x_1, \dots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n) \quad \text{for all } x_1, \dots, x_n.$$

48. Covariance: $\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$, where $\mu_X = \mathbb{E}X$ and $\mu_Y = \mathbb{E}Y$.

49. Correlation coefficient: $\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\, \sigma_Y}$, where $\sigma_X^2 = \mathrm{Var}(X)$ and $\sigma_Y^2 = \mathrm{Var}(Y)$.

50. Properties of Var and Cov:
$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$.
$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$.
$\mathrm{Cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$.
$\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$.
$\mathrm{Cov}(aX + bY, Z) = a\,\mathrm{Cov}(X,Z) + b\,\mathrm{Cov}(Y,Z)$.
$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$.
$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$.
$X$ and $Y$ independent $\Longrightarrow \mathrm{Cov}(X,Y) = 0$.

51. For all random variables $X_1, \dots, X_n$ and constants $a, b_1, \dots, b_n$:

$$\mathbb{E}(a + b_1 X_1 + \cdots + b_n X_n) = a + b_1 \mathbb{E}X_1 + \cdots + b_n \mathbb{E}X_n.$$

52. For *independent* random variables,

$$\mathrm{Var}(a + b_1 X_1 + \cdots + b_n X_n) = b_1^2 \mathrm{Var}(X_1) + \cdots + b_n^2 \mathrm{Var}(X_n).$$

53. For *independent* random variables,

$$\mathbb{E}[X_1 X_2 \cdots X_n] = \mathbb{E}X_1 \mathbb{E}X_2 \cdots \mathbb{E}X_n.$$

54. For independent *normal* random variables $X_1, X_2, \dots, X_n$:

$$Y = a + b_1 X_1 + \cdots + b_n X_n \sim \mathsf{N}\left(a + \sum_{i=1}^n b_i\, \mathbb{E}X_i,\ \sum_{i=1}^n b_i^2\, \mathrm{Var}(X_i)\right).$$

55. Law of large numbers: If $X_1, \dots, X_n$ are iid with finite expectation $\mu$, then for large $n$,

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n} \approx \mu.$$

56. Central limit theorem (CLT) for sums: If $X_1, \dots, X_n$ are iid with finite expectation $\mu$ and variance $\sigma^2 > 0$, then for large $n$:

$$X_1 + \cdots + X_n \overset{\text{approx.}}{\sim} \mathsf{N}(n\mu, n\sigma^2).$$

57. Central limit theorem (CLT) for means (conditions as above):

$$\frac{X_1 + \cdots + X_n}{n} = \bar{X} \overset{\text{approx.}}{\sim} \mathsf{N}\left(\mu, \frac{\sigma^2}{n}\right).$$

58. By the CLT, $X \sim \mathsf{Bin}(n,p)$ has approximately a $\mathsf{N}(np, np(1-p))$ distribution.

59. Continuity correction. Adjustments for approximating a discrete random variable $X$ with a continuous random variable $Y$:
$P(X = x) \approx P(x - 0.5 < Y < x + 0.5)$
$P(X \le x) \approx P(Y \le x + 0.5)$
$P(X < x) \approx P(Y < x - 0.5)$
$P(X \ge x) \approx P(Y \ge x - 0.5)$
$P(X > x) \approx P(Y > x + 0.5)$

## Statistics

60. Sample mean: $\bar{X} = \sum_{i=1}^n X_i / n$.

61. Sample variance:
$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1} \left(\sum_{i=1}^n X_i^2 - n\bar{X}^2\right)$.

### Confidence Intervals

62. 1-sample iid data. Approximate confidence interval for $\mu$:

$$\bar{X} \pm q \frac{S}{\sqrt{n}},$$

where $q$ is the $1 - \alpha/2$ quantile of the $\mathsf{t}_{n-1}$ distribution. This CI is exact when the data is drawn from a normal distribution.

63. 1-sample normal data. Exact confidence interval for $\sigma^2$:

$$\left(\frac{(n-1)S^2}{q_2},\ \frac{(n-1)S^2}{q_1}\right),$$

where $q_1$ and $q_2$ are the $\alpha/2$ and $1 - \alpha/2$ quantiles of the $\chi^2_{n-1}$ distribution.

64. Pooled sample variance:

$$S_p^2 = \frac{(m-1)S_X^2 + (n-1)S_Y^2}{m + n - 2}.$$

65. 2-sample data (assuming equal variances). Approximate confidence interval for $\mu_X - \mu_Y$:

$$\bar{X} - \bar{Y} \pm q\, S_p \sqrt{\frac{1}{m} + \frac{1}{n}},$$

where $q$ is the $1 - \alpha/2$ quantile of the $\mathsf{t}_{m+n-2}$ distribution. This CI is exact when the data for each group is drawn from a normal distribution.

66. 2-sample normal data (unequal variances). Approximate confidence interval for $\mu_X - \mu_Y$:

$$\bar{X} - \bar{Y} \pm q \sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}},$$

where $q$ is the $1 - \alpha/2$ quantile of $t_{\mathrm{df}}$, where df is given by either Welch's approximation or the conservative approximation given below.

67. 1-sample binomial data. Approximate confidence interval for $p$:

$$\hat{P} \pm q \sqrt{\frac{\hat{P}(1 - \hat{P})}{n}},$$

where $\hat{P} = X/n$ and $q$ is the $1 - \alpha/2$ quantile of $\mathsf{N}(0,1)$.

68. 2-sample binomial data. Approximate confidence interval for $p_X - p_Y$:

$$\hat{P}_X - \hat{P}_Y \pm q \sqrt{\frac{\hat{P}_X(1 - \hat{P}_X)}{m} + \frac{\hat{P}_Y(1 - \hat{P}_Y)}{n}},$$

where $\hat{P}_X = X/m$, $\hat{P}_Y = Y/n$ and $q$ is the $1 - \alpha/2$ quantile of $\mathsf{N}(0,1)$.

### Hypothesis Testing

69. The 7 steps for a statistical test: model, hypotheses, test statistic, distribution of the test statistic under $H_0$, outcome of the test statistic, p-value, accept or reject $H_0$.

70. p-value: the probability that under $H_0$ the (random) test statistic takes a value as extreme as or more extreme than the one observed.

71. Type I error: reject a $H_0$ that is actually true.
Type II error: fail to reject a $H_0$ that it is actually false.
Power: probability of correctly rejecting $H_0$ when $H_1$ is true.

72. 1-sample $t$-test. Test statistic:

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}.$$

Under $H_0 : \mu = \mu_0$, we have $T \sim \mathsf{t}_{n-1}$.

73. 1-sample binomial test. Test statistic $X$. Under $H_0 : p = p_0$, we have $X \sim \mathsf{Bin}(n, p_0)$, approximated by $\hat{P} \sim \mathsf{N}(p_0, np_0(1 - p_0))$. Test statistic:

$$Z = \frac{\hat{P} - p_0}{\sqrt{p_0(1 - p_0)/n}} \sim \mathsf{N}(0,1)$$

74. 2-sample $t$-test (assuming equal variances). Test statistic:

$$T = \frac{\bar{X} - \bar{Y}}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}},$$

Under $H_0 : \mu_X = \mu_Y$, we have $T \sim \mathsf{t}_{m+n-2}$.

75. 2-sample $t$-test (unequal variances). Test statistic:

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}}.$$

Under $H_0 : \mu_X = \mu_Y$, we have $T \overset{\text{approx.}}{\sim} \mathsf{t}_{\text{df}}$ distribution with Welch's

$$\text{df} = \frac{\left(\frac{s_X^2}{m} + \frac{s_Y^2}{n}\right)^2}{\frac{s_X^4}{m^2(m-1)} + \frac{s_Y^4}{n^2(n-1)}}.$$

Conservative approximation: $\text{df} = \min(m-1, n-1)$.

76. 2-sample binomial test. Test statistic:

$$Z = \frac{\hat{P}_X - \hat{P}_Y}{\sqrt{\hat{P}(1-\hat{P})\left(\frac{1}{m} + \frac{1}{n}\right)}},$$

where $\hat{P} := \frac{X+Y}{m+n}$. Under $H_0 : p_X = p_Y$, we have $Z \overset{\text{approx.}}{\sim} \mathsf{N}(0,1)$.

77. Chi-squared test:

$$\text{expected} = (\text{row total}) \times (\text{column total})/\text{overall total},$$
$$\chi^2 = \sum (\text{observed - expected})^2/\text{expected},$$
$$\text{df} = (\text{rows} - 1) \times (\text{columns} - 1).$$

## Linear Models

78. 1-factor ANOVA model. Let $Y_{ik}$ be the response for the $k$-th replication at level $i$. Then,

$$Y_{ik} = \mu_i + \varepsilon_{ik}, \quad k = 1, \ldots, n_i, \; i = 1, \ldots, d,$$

where $\{\varepsilon_{ik}\} \overset{\text{iid}}{\sim} \mathsf{N}(0, \sigma^2)$.

79. 1-factor ANOVA. Test statistic $F = \frac{\text{MSF}}{\text{MSE}}$. Under $H_0$ : all $\{\mu_i\}$ are equal, we have $F \sim \mathsf{F}(d-1, n-d)$.

80. 1-factor ANOVA table:

| Source | DF | SS | MS | $F$ | $\mathbb{P}(F > f)$ |
|---|---|---|---|---|---|
| Factor | $d-1$ | SSF | $\frac{\text{SSF}}{d-1}$ | $\frac{\text{MSF}}{\text{MSE}}$ | P-value |
| Error | $n-d$ | SSE | $\frac{\text{SSE}}{n-d}$ | | |
| Total | $n-1$ | SST | | | |

81. 2-factor ANOVA model (factor effects formulation). Let $Y_{ijk}$ be the response for the $k$-th replication at level $(i,j)$. Then,

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk},$$
$$k = 1, \ldots, n_{ij}, \; i = 1, \ldots, d_1, \; j = 1, \ldots, d_2,$$

where $\{\varepsilon_{ijk}\} \sim_{\text{iid}} \mathsf{N}(0, \sigma^2)$.

82. $R^2 = (SST - SSE)/SST$

83. Simple linear regression:

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, \ldots, n,$$

where $\{\varepsilon_i\} \overset{\text{iid}}{\sim} \mathsf{N}(0, \sigma^2)$.

84. Multiple linear regression:

$$Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_d x_{id} + \varepsilon_i, \quad i = 1, \ldots, n,$$

where $\{\varepsilon_i\} \overset{\text{iid}}{\sim} \mathsf{N}(0, \sigma^2)$.

85. Normal linear model:

$$\mathbf{Y} = \mathcal{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where $\boldsymbol{\beta}$ is a vector of parameters and $\boldsymbol{\varepsilon}$ a vector of independent error terms, each $\mathsf{N}(0, \sigma^2)$ distributed.

# Other Mathematical Formulas

86. **Binomial theorem**: $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$.

87. **Geometric sum**: $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ $(a \neq 1)$. If $|a| < 1$ then $1 + a + a^2 + \cdots = \frac{1}{1-a}$.

88. **Logarithms**:

(a) $\log(x\,y) = \log x + \log y$.

(b) $\mathrm{e}^{\log x} = x$.

89. **Exponential**:

(a) $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

(b) $\mathrm{e}^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$.

(c) $\mathrm{e}^{x+y} = \mathrm{e}^x\,\mathrm{e}^y$.

90. **Differentiation**:

(a) $(f+g)' = f' + g'$,

(b) $(fg)' = f'g + fg'$,

(c) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

(d) $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n\,x^{n-1}$.

(e) $\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{e}^x = \mathrm{e}^x$.

(f) $\frac{\mathrm{d}}{\mathrm{d}x} \log(x) = \frac{1}{x}$.

91. **Chain rule**: $(f(g(x)))' = f'(g(x))\,g'(x)$.

92. **Integration**: $\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$, where $F' = f$.

93. **Integration by parts**:
$\int_a^b f(x)\,G(x)\,\mathrm{d}x = [F(x)\,G(x)]_a^b - \int_a^b F(x)\,g(x)\,\mathrm{d}x$. (Here $F' = f$ and $G' = g$.)

# Standard Normal Distribution

This table gives the cumulative distribution function (cdf) $\Phi$ of a $\mathsf{N}(0,1)$-distributed random variable $Z$:

$$\Phi(z) = \mathbb{P}(Z \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-x^2/2}\,\mathrm{d}x.$$

Example: $\Phi(1.65) = \mathbb{P}(Z \le 1.65) = 0.9505$.

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 5000 | 5040 | 5080 | 5120 | 5160 | 5199 | 5239 | 5279 | 5319 | 5359 |
| 0.1 | 5398 | 5438 | 5478 | 5517 | 5557 | 5596 | 5636 | 5675 | 5714 | 5753 |
| 0.2 | 5793 | 5832 | 5871 | 5910 | 5948 | 5987 | 6026 | 6064 | 6103 | 6141 |
| 0.3 | 6179 | 6217 | 6255 | 6293 | 6331 | 6368 | 6406 | 6443 | 6480 | 6517 |
| 0.4 | 6554 | 6591 | 6628 | 6664 | 6700 | 6736 | 6772 | 6808 | 6844 | 6879 |
| 0.5 | 6915 | 6950 | 6985 | 7019 | 7054 | 7088 | 7123 | 7157 | 7190 | 7224 |
| 0.6 | 7257 | 7291 | 7324 | 7357 | 7389 | 7422 | 7454 | 7486 | 7517 | 7549 |
| 0.7 | 7580 | 7611 | 7642 | 7673 | 7704 | 7734 | 7764 | 7794 | 7823 | 7852 |
| 0.8 | 7881 | 7910 | 7939 | 7967 | 7995 | 8023 | 8051 | 8078 | 8106 | 8133 |
| 0.9 | 8159 | 8186 | 8212 | 8238 | 8264 | 8289 | 8315 | 8340 | 8365 | 8389 |
| 1.0 | 8413 | 8438 | 8461 | 8485 | 8508 | 8531 | 8554 | 8577 | 8599 | 8621 |
| 1.1 | 8643 | 8665 | 8686 | 8708 | 8729 | 8749 | 8770 | 8790 | 8810 | 8830 |
| 1.2 | 8849 | 8869 | 8888 | 8907 | 8925 | 8944 | 8962 | 8980 | 8997 | 9015 |
| 1.3 | 9032 | 9049 | 9066 | 9082 | 9099 | 9115 | 9131 | 9147 | 9162 | 9177 |
| 1.4 | 9192 | 9207 | 9222 | 9236 | 9251 | 9265 | 9279 | 9292 | 9306 | 9319 |
| 1.5 | 9332 | 9345 | 9357 | 9370 | 9382 | 9394 | 9406 | 9418 | 9429 | 9441 |
| 1.6 | 9452 | 9463 | 9474 | 9484 | 9495 | 9505 | 9515 | 9525 | 9535 | 9545 |
| 1.7 | 9554 | 9564 | 9573 | 9582 | 9591 | 9599 | 9608 | 9616 | 9625 | 9633 |
| 1.8 | 9641 | 9649 | 9656 | 9664 | 9671 | 9678 | 9686 | 9693 | 9699 | 9706 |
| 1.9 | 9713 | 9719 | 9726 | 9732 | 9738 | 9744 | 9750 | 9756 | 9761 | 9767 |
| 2.0 | 9772 | 9778 | 9783 | 9788 | 9793 | 9798 | 9803 | 9808 | 9812 | 9817 |
| 2.1 | 9821 | 9826 | 9830 | 9834 | 9838 | 9842 | 9846 | 9850 | 9854 | 9857 |
| 2.2 | 9861 | 9864 | 9868 | 9871 | 9875 | 9878 | 9881 | 9884 | 9887 | 9890 |
| 2.3 | 9893 | 9896 | 9898 | 9901 | 9904 | 9906 | 9909 | 9911 | 9913 | 9916 |
| 2.4 | 9918 | 9920 | 9922 | 9925 | 9927 | 9929 | 9931 | 9932 | 9934 | 9936 |
| 2.5 | 9938 | 9940 | 9941 | 9943 | 9945 | 9946 | 9948 | 9949 | 9951 | 9952 |
| 2.6 | 9953 | 9955 | 9956 | 9957 | 9959 | 9960 | 9961 | 9962 | 9963 | 9964 |
| 2.7 | 9965 | 9966 | 9967 | 9968 | 9969 | 9970 | 9971 | 9972 | 9973 | 9974 |
| 2.8 | 9974 | 9975 | 9976 | 9977 | 9977 | 9978 | 9979 | 9979 | 9980 | 9981 |
| 2.9 | 9981 | 9982 | 9982 | 9983 | 9984 | 9984 | 9985 | 9985 | 9986 | 9986 |
| 3.0 | 9987 | 9987 | 9987 | 9988 | 9988 | 9989 | 9989 | 9989 | 9990 | 9990 |
| 3.1 | 9990 | 9991 | 9991 | 9991 | 9992 | 9992 | 9992 | 9992 | 9993 | 9993 |
| 3.2 | 9993 | 9993 | 9994 | 9994 | 9994 | 9994 | 9994 | 9995 | 9995 | 9995 |
| 3.3 | 9995 | 9995 | 9995 | 9996 | 9996 | 9996 | 9996 | 9996 | 9996 | 9997 |
| 3.4 | 9997 | 9997 | 9997 | 9997 | 9997 | 9997 | 9997 | 9997 | 9997 | 9998 |
| 3.5 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 | 9998 |
| 3.6 | 9998 | 9998 | 9999 | 9999 | 9999 | 9999 | 9999 | 9999 | 9999 | 9999 |

## Student (or $t$-distribution)

In this table we list the $\gamma$-quantiles values of the $t_{\mathrm{df}}$-distribution for various values of df and $\gamma$.

**Examples:** The 0.95-quantile for the $t_5$ distribution is 2.02.
Let $T \sim t_5$. Then, $\mathbb{P}(-2.02 < T \leq 2.02) = \mathbb{P}(T_5 \leq 2.02) - \mathbb{P}(T_5 \leq -2.02) = 0.95 - \mathbb{P}(T > 2.02) = 0.95 - 1 + \mathbb{P}(T \leq 2.02) = 0.95 - 1 + 0.95 = 0.90.$

| df \ $\gamma$ | 0.90 | 0.95 | 0.975 | 0.99 | 0.995 | 0.9975 | 0.999 | 0.9995 |
|---|---|---|---|---|---|---|---|---|
| 1 | 3.08 | 6.31 | 12.71 | 31.82 | 63.66 | 127.32 | 318.33 | 636.67 |
| 2 | 1.89 | 2.92 | 4.30 | 6.96 | 9.92 | 14.09 | 22.33 | 31.60 |
| 3 | 1.64 | 2.35 | 3.18 | 4.54 | 5.84 | 7.45 | 10.21 | 12.92 |
| 4 | 1.53 | 2.13 | 2.78 | 3.75 | 4.60 | 5.60 | 7.17 | 8.61 |
| 5 | 1.48 | 2.02 | 2.57 | 3.37 | 4.03 | 4.77 | 5.89 | 6.87 |
| 6 | 1.44 | 1.94 | 2.45 | 3.14 | 3.71 | 4.32 | 5.21 | 5.96 |
| 7 | 1.41 | 1.89 | 2.36 | 3.00 | 3.50 | 4.03 | 4.79 | 5.41 |
| 8 | 1.40 | 1.86 | 2.31 | 2.90 | 3.36 | 3.83 | 4.50 | 5.04 |
| 9 | 1.38 | 1.83 | 2.26 | 2.82 | 3.25 | 3.69 | 4.30 | 4.78 |
| 10 | 1.37 | 1.81 | 2.23 | 2.76 | 3.17 | 3.58 | 4.14 | 4.59 |
| 11 | 1.36 | 1.80 | 2.20 | 2.72 | 3.11 | 3.50 | 4.02 | 4.44 |
| 12 | 1.36 | 1.78 | 2.18 | 2.68 | 3.05 | 3.43 | 3.93 | 4.32 |
| 13 | 1.35 | 1.77 | 2.16 | 2.65 | 3.01 | 3.37 | 3.85 | 4.22 |
| 14 | 1.35 | 1.76 | 2.14 | 2.62 | 2.98 | 3.33 | 3.79 | 4.14 |
| 15 | 1.34 | 1.75 | 2.13 | 2.60 | 2.95 | 3.29 | 3.73 | 4.07 |
| 16 | 1.34 | 1.75 | 2.12 | 2.58 | 2.92 | 3.25 | 3.69 | 4.02 |
| 17 | 1.33 | 1.74 | 2.11 | 2.57 | 2.90 | 3.22 | 3.65 | 3.97 |
| 18 | 1.33 | 1.73 | 2.10 | 2.55 | 2.88 | 3.20 | 3.61 | 3.92 |
| 19 | 1.33 | 1.73 | 2.09 | 2.54 | 2.86 | 3.17 | 3.58 | 3.88 |
| 20 | 1.33 | 1.72 | 2.09 | 2.53 | 2.85 | 3.15 | 3.55 | 3.85 |
| 21 | 1.32 | 1.72 | 2.08 | 2.52 | 2.83 | 3.14 | 3.53 | 3.82 |
| 22 | 1.32 | 1.72 | 2.07 | 2.51 | 2.82 | 3.12 | 3.51 | 3.79 |
| 23 | 1.32 | 1.71 | 2.07 | 2.50 | 2.81 | 3.10 | 3.48 | 3.77 |
| 24 | 1.32 | 1.71 | 2.06 | 2.49 | 2.80 | 3.09 | 3.47 | 3.75 |
| 25 | 1.32 | 1.71 | 2.06 | 2.49 | 2.79 | 3.08 | 3.45 | 3.73 |
| 26 | 1.31 | 1.71 | 2.06 | 2.48 | 2.78 | 3.07 | 3.44 | 3.71 |
| 27 | 1.31 | 1.70 | 2.05 | 2.47 | 2.77 | 3.06 | 3.42 | 3.69 |
| 28 | 1.31 | 1.70 | 2.05 | 2.47 | 2.76 | 3.05 | 3.41 | 3.67 |
| 29 | 1.31 | 1.70 | 2.05 | 2.46 | 2.76 | 3.04 | 3.40 | 3.66 |
| 30 | 1.31 | 1.70 | 2.04 | 2.46 | 2.75 | 3.03 | 3.39 | 3.65 |
| 40 | 1.30 | 1.68 | 2.02 | 2.42 | 2.70 | 2.97 | 3.31 | 3.55 |
| 50 | 1.30 | 1.68 | 2.01 | 2.40 | 2.68 | 2.94 | 3.26 | 3.50 |
| 100 | 1.29 | 1.66 | 1.98 | 2.36 | 2.63 | 2.87 | 3.17 | 3.39 |
| 200 | 1.29 | 1.65 | 1.97 | 2.35 | 2.60 | 2.84 | 3.13 | 3.34 |
| $\infty$ | 1.28 | 1.64 | 1.96 | 2.33 | 2.58 | 2.81 | 3.09 | 3.29 |

## Chi-square-distribution

In this table we list the $\gamma$ quantiles of the $\chi^2_{\text{df}}$ distribution, for various values of df and $\gamma$.

| df \ $\gamma$ | .005 | .010 | .025 | .050 | .100 | .250 | .750 | .900 | .950 | .975 | .990 | .995 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | .000 | .000 | .001 | .004 | .016 | .102 | 1.32 | 2.71 | 3.84 | 5.02 | 6.63 | 7.88 |
| 2 | .010 | .020 | .051 | .103 | .211 | .575 | 2.77 | 4.61 | 5.99 | 7.38 | 9.21 | 10.6 |
| 3 | .072 | .115 | .216 | .352 | .584 | 1.21 | 4.11 | 6.25 | 7.81 | 9.35 | 11.3 | 12.8 |
| 4 | .207 | .297 | .484 | .711 | 1.06 | 1.92 | 5.39 | 7.78 | 9.49 | 11.1 | 13.3 | 14.9 |
| 5 | .412 | .554 | .831 | 1.15 | 1.61 | 2.67 | 6.63 | 9.24 | 11.1 | 12.8 | 15.1 | 16.7 |
| 6 | .676 | .872 | 1.24 | 1.64 | 2.20 | 3.45 | 7.84 | 10.6 | 12.6 | 14.4 | 16.8 | 18.5 |
| 7 | .989 | 1.24 | 1.69 | 2.17 | 2.83 | 4.25 | 9.04 | 12.0 | 14.0 | 16.0 | 18.5 | 20.3 |
| 8 | 1.34 | 1.65 | 2.18 | 2.73 | 3.49 | 5.07 | 10.2 | 13.4 | 15.5 | 17.5 | 20.1 | 22.0 |
| 9 | 1.73 | 2.09 | 2.70 | 3.33 | 4.17 | 5.90 | 11.4 | 14.7 | 16.9 | 19.0 | 21.7 | 23.6 |
| 10 | 2.16 | 2.56 | 3.25 | 3.94 | 4.87 | 6.74 | 12.5 | 16.0 | 18.3 | 20.5 | 23.2 | 25.2 |
| 11 | 2.60 | 3.05 | 3.82 | 4.57 | 5.58 | 7.58 | 13.7 | 17.3 | 19.7 | 21.9 | 24.7 | 26.8 |
| 12 | 3.07 | 3.57 | 4.40 | 5.23 | 6.30 | 8.44 | 14.8 | 18.5 | 21.0 | 23.3 | 26.2 | 28.3 |
| 13 | 3.57 | 4.11 | 5.01 | 5.89 | 7.04 | 9.30 | 16.0 | 19.8 | 22.4 | 24.7 | 27.7 | 29.8 |
| 14 | 4.07 | 4.66 | 5.63 | 6.57 | 7.79 | 10.2 | 17.1 | 21.1 | 23.7 | 26.1 | 29.1 | 31.3 |
| 15 | 4.60 | 5.23 | 6.26 | 7.26 | 8.55 | 11.0 | 18.2 | 22.3 | 25.0 | 27.5 | 30.6 | 32.8 |
| 16 | 5.14 | 5.81 | 6.91 | 7.96 | 9.31 | 11.9 | 19.4 | 23.5 | 26.3 | 28.8 | 32.0 | 34.3 |
| 17 | 5.70 | 6.41 | 7.56 | 8.67 | 10.1 | 12.8 | 20.5 | 24.8 | 27.6 | 30.2 | 33.4 | 35.7 |
| 18 | 6.26 | 7.01 | 8.23 | 9.39 | 10.9 | 13.7 | 21.6 | 26.0 | 28.9 | 31.5 | 34.8 | 37.2 |
| 19 | 6.84 | 7.63 | 8.91 | 10.1 | 11.7 | 14.6 | 22.7 | 27.2 | 30.1 | 32.9 | 36.2 | 38.6 |
| 20 | 7.43 | 8.26 | 9.59 | 10.9 | 12.4 | 15.5 | 23.8 | 28.4 | 31.4 | 34.1 | 37.6 | 40.0 |
| 21 | 8.03 | 8.90 | 10.3 | 11.6 | 13.2 | 16.3 | 24.9 | 29.6 | 32.7 | 35.5 | 38.9 | 41.4 |
| 22 | 8.64 | 9.50 | 11.0 | 12.3 | 14.0 | 17.2 | 26.0 | 30.8 | 33.9 | 36.8 | 40.3 | 42.8 |
| 23 | 9.26 | 10.2 | 11.7 | 13.1 | 14.8 | 18.1 | 27.1 | 32.0 | 35.2 | 38.1 | 41.6 | 44.2 |
| 24 | 9.89 | 10.9 | 12.4 | 13.8 | 15.7 | 19.0 | 28.2 | 33.2 | 36.4 | 39.4 | 43.0 | 45.6 |
| 25 | 10.5 | 11.5 | 13.1 | 14.6 | 16.5 | 19.9 | 29.3 | 34.4 | 37.7 | 40.6 | 44.3 | 46.9 |
| 26 | 11.2 | 12.2 | 13.8 | 15.4 | 17.3 | 20.8 | 30.4 | 35.6 | 38.9 | 41.9 | 45.6 | 48.3 |
| 27 | 11.8 | 12.9 | 14.6 | 16.2 | 18.1 | 21.7 | 31.5 | 36.7 | 40.1 | 43.2 | 47.0 | 49.6 |
| 28 | 12.5 | 13.6 | 15.3 | 16.9 | 18.9 | 22.7 | 32.6 | 37.9 | 41.3 | 44.5 | 48.3 | 51.0 |
| 29 | 13.1 | 14.3 | 16.0 | 17.7 | 19.8 | 23.6 | 33.7 | 39.1 | 42.6 | 45.7 | 49.6 | 52.3 |
| 30 | 13.8 | 15.0 | 16.8 | 18.5 | 20.6 | 24.5 | 34.8 | 40.3 | 43.8 | 47.0 | 50.9 | 53.7 |
| 40 | 20.7 | 22.2 | 24.4 | 26.5 | 29.1 | 33.7 | 45.6 | 51.8 | 55.8 | 59.3 | 63.7 | 66.8 |
| 50 | 28.0 | 29.7 | 32.4 | 34.8 | 37.7 | 42.9 | 56.3 | 63.2 | 67.5 | 71.4 | 76.2 | 79.5 |
| 60 | 35.5 | 37.5 | 40.5 | 43.2 | 46.5 | 52.3 | 67.0 | 74.4 | 79.1 | 83.3 | 88.4 | 92.0 |
| 70 | 43.3 | 45.4 | 48.8 | 51.7 | 55.3 | 61.7 | 77.6 | 85.5 | 90.5 | 95.0 | 100.4 | 104.2 |
| 80 | 51.2 | 53.5 | 57.2 | 60.4 | 64.3 | 71.1 | 88.1 | 96.6 | 101.9 | 106.6 | 112.3 | 116.3 |
| 90 | 59.2 | 61.8 | 65.6 | 69.1 | 73.3 | 80.6 | 98.6 | 107.6 | 113.1 | 118.1 | 124.1 | 128.3 |
| 100 | 67.3 | 70.1 | 74.2 | 77.9 | 82.4 | 90.1 | 109.1 | 118.5 | 124.3 | 129.6 | 135.8 | 140.2 |

Examples: The 0.01-quantile of the $\chi^2_5$ distribution is 0.554. Let $X \sim \chi^2_5$. Then,
$\mathbb{P}(0.412 < X \leq 9.24) = \mathbb{P}(X \leq 9.24) - \mathbb{P}(X \leq 0.412) = 0.90 - 0.005 = 0.895$

**F-distribution**

In the next three pages we list the $\gamma$-quantiles of the $F_{m,n}$ distribution, for $\gamma = 0.99, 0.975$ and $0.95$ and various value of $m$ and $n$.

$\gamma = 0.99$

| | $m=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $n$=1 | 4052 | 5000 | 5403 | 5625 | 5764 | 5859 | 5928 | 5982 | 6023 |
| 2 | 98.5 | 99.0 | 99.2 | 99.2 | 99.3 | 99.3 | 99.4 | 99.4 | 99.4 |
| 3 | 34.1 | 30.8 | 29.5 | 28.7 | 28.2 | 27.9 | 27.7 | 27.5 | 27.3 |
| 4 | 21.2 | 18.0 | 16.7 | 16.0 | 15.5 | 15.2 | 15.0 | 14.8 | 14.7 |
| 5 | 16.3 | 13.3 | 12.1 | 11.4 | 11.0 | 10.7 | 10.5 | 10.3 | 10.2 |
| 6 | 13.7 | 10.9 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 |
| 7 | 12.2 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 |
| 8 | 11.3 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 |
| 9 | 10.6 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 |
| 10 | 10.0 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 |
| 12 | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 |
| 15 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 |
| 20 | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 |
| 30 | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.30 | 3.17 | 3.07 |
| 40 | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 3.12 | 2.99 | 2.89 |
| 60 | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.95 | 2.82 | 2.72 |
| 120 | 6.85 | 4.79 | 3.95 | 3.48 | 3.17 | 2.96 | 2.79 | 2.66 | 2.56 |
| $\infty$ | 6.63 | 4.61 | 3.78 | 3.32 | 3.02 | 2.80 | 2.64 | 2.51 | 2.41 |

| | $m=10$ | 12 | 15 | 20 | 30 | 40 | 60 | 120 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $n$=1 | 6056 | 6106 | 6157 | 6209 | 6261 | 6287 | 6313 | 6339 | 6366 |
| 2 | 99.4 | 99.4 | 99.4 | 99.4 | 99.5 | 99.5 | 99.5 | 99.5 | 99.5 |
| 3 | 27.2 | 27.1 | 26.9 | 26.7 | 26.5 | 26.4 | 26.3 | 26.2 | 26.1 |
| 4 | 14.5 | 14.4 | 14.2 | 14.0 | 13.8 | 13.7 | 13.7 | 13.6 | 13.5 |
| 5 | 10.1 | 9.89 | 9.72 | 9.55 | 9.38 | 9.29 | 9.20 | 9.11 | 9.02 |
| 6 | 7.87 | 7.72 | 7.56 | 7.40 | 7.23 | 7.14 | 7.06 | 6.97 | 6.88 |
| 7 | 6.62 | 6.47 | 6.31 | 6.16 | 5.99 | 5.91 | 5.82 | 5.74 | 5.65 |
| 8 | 5.81 | 5.67 | 5.52 | 5.36 | 5.20 | 5.12 | 5.03 | 4.95 | 4.86 |
| 9 | 5.26 | 5.11 | 4.96 | 4.81 | 4.65 | 4.57 | 4.48 | 4.40 | 4.31 |
| 10 | 4.85 | 4.71 | 4.56 | 4.41 | 4.25 | 4.17 | 4.08 | 4.00 | 3.91 |
| 12 | 4.30 | 4.16 | 4.01 | 3.86 | 3.70 | 3.62 | 3.54 | 3.45 | 3.36 |
| 15 | 3.80 | 3.67 | 3.52 | 3.37 | 3.21 | 3.13 | 3.05 | 2.96 | 2.87 |
| 20 | 3.37 | 3.23 | 3.09 | 2.94 | 2.78 | 2.69 | 2.61 | 2.52 | 2.42 |
| 30 | 2.98 | 2.84 | 2.70 | 2.55 | 2.39 | 2.30 | 2.21 | 2.11 | 2.01 |
| 40 | 2.80 | 2.66 | 2.52 | 2.37 | 2.20 | 2.11 | 2.02 | 1.92 | 1.80 |
| 60 | 2.63 | 2.50 | 2.35 | 2.20 | 2.03 | 1.94 | 1.84 | 1.73 | 1.60 |
| 120 | 2.47 | 2.34 | 2.19 | 2.03 | 1.86 | 1.76 | 1.66 | 1.53 | 1.38 |
| $\infty$ | 2.32 | 2.18 | 2.04 | 1.88 | 1.70 | 1.59 | 1.47 | 1.32 | 1.00 |

Example: If $X \sim F_{7,10}$, then $\mathbb{P}(X \leq 5.20) = 0.99$ .

$\gamma = 0.975$

| | $m=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $n$=1 | 648 | 800 | 864 | 900 | 922 | 937 | 948 | 957 | 963 |
| 2 | 38.5 | 39.0 | 39.2 | 39.2 | 39.3 | 39.3 | 39.4 | 39.4 | 39.4 |
| 3 | 17.4 | 16.0 | 15.4 | 15.1 | 14.9 | 14.7 | 14.6 | 14.5 | 14.5 |
| 4 | 12.2 | 10.6 | 9.98 | 9.60 | 9.36 | 9.20 | 9.07 | 8.98 | 8.90 |
| 5 | 10.0 | 8.43 | 7.76 | 7.39 | 7.15 | 6.98 | 6.85 | 6.76 | 6.68 |
| 6 | 8.81 | 7.26 | 5.60 | 6.23 | 5.99 | 5.82 | 5.70 | 5.60 | 5.52 |
| 7 | 8.07 | 6.54 | 5.89 | 5.52 | 5.29 | 5.12 | 4.99 | 4.90 | 4.82 |
| 8 | 7.57 | 6.05 | 5.42 | 5.05 | 4.82 | 4.65 | 4.53 | 4.43 | 4.36 |
| 9 | 7.21 | 5.71 | 5.08 | 4.72 | 4.48 | 4.32 | 4.20 | 4.10 | 4.03 |
| 10 | 6.94 | 5.46 | 4.83 | 4.47 | 4.24 | 4.07 | 3.95 | 3.85 | 3.78 |
| 12 | 6.55 | 5.10 | 4.47 | 4.12 | 3.89 | 3.73 | 3.61 | 3.51 | 3.44 |
| 15 | 6.20 | 4.77 | 4.15 | 3.80 | 3.58 | 3.41 | 3.29 | 3.20 | 3.12 |
| 20 | 5.87 | 4.46 | 3.86 | 3.51 | 3.29 | 3.13 | 3.01 | 2.91 | 2.84 |
| 30 | 5.57 | 4.18 | 3.59 | 3.25 | 3.03 | 2.87 | 2.75 | 2.65 | 2.57 |
| 40 | 5.42 | 4.05 | 3.46 | 3.13 | 2.90 | 2.74 | 2.62 | 2.53 | 2.45 |
| 60 | 5.29 | 3.93 | 3.34 | 3.01 | 2.79 | 2.63 | 2.51 | 2.41 | 2.33 |
| 120 | 5.15 | 3.80 | 3.33 | 2.89 | 2.67 | 2.52 | 2.39 | 2.30 | 2.22 |
| $\infty$ | 5.02 | 3.69 | 3.12 | 2.79 | 2.67 | 2.41 | 2.29 | 2.19 | 2.11 |

| | $m=10$ | 12 | 15 | 20 | 30 | 40 | 60 | 120 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $n$=1 | 969 | 977 | 985 | 993 | 1001 | 1006 | 1010 | 1014 | 1018 |
| 2 | 39.4 | 39.4 | 39.4 | 39.4 | 39.5 | 39.5 | 39.5 | 39.5 | 39.5 |
| 3 | 14.4 | 14.3 | 14.3 | 14.2 | 14.1 | 14.0 | 14.0 | 13.9 | 13.9 |
| 4 | 8.84 | 8.75 | 8.66 | 8.56 | 8.46 | 8.41 | 8.36 | 8.31 | 8.26 |
| 5 | 6.62 | 6.52 | 6.43 | 6.33 | 6.23 | 6.18 | 6.12 | 6.07 | 6.02 |
| 6 | 5.46 | 5.37 | 5.27 | 5.17 | 5.07 | 5.01 | 4.96 | 4.90 | 4.85 |
| 7 | 4.76 | 4.67 | 4.57 | 4.47 | 4.36 | 4.31 | 4.25 | 4.20 | 4.14 |
| 8 | 4.30 | 4.20 | 4.10 | 4.00 | 3.89 | 3.84 | 3.78 | 3.73 | 3.67 |
| 9 | 3.96 | 3.87 | 3.77 | 3.67 | 3.56 | 3.51 | 3.45 | 3.39 | 3.33 |
| 10 | 3.72 | 3.62 | 3.52 | 3.42 | 3.31 | 3.26 | 3.20 | 3.14 | 3.08 |
| 12 | 3.37 | 3.28 | 3.18 | 3.07 | 2.96 | 2.91 | 2.85 | 2.79 | 2.72 |
| 15 | 3.06 | 2.96 | 2.86 | 2.76 | 2.64 | 2.59 | 2.52 | 2.46 | 2.40 |
| 20 | 2.77 | 2.68 | 2.57 | 2.46 | 2.35 | 2.29 | 2.22 | 2.16 | 2.09 |
| 30 | 2.51 | 2.41 | 2.31 | 2.20 | 2.07 | 2.01 | 1.94 | 1.87 | 1.79 |
| 40 | 2.39 | 2.29 | 2.18 | 2.07 | 1.94 | 1.88 | 1.80 | 1.72 | 1.64 |
| 60 | 2.27 | 2.17 | 2.06 | 1.94 | 1.82 | 1.74 | 1.67 | 1.58 | 1.48 |
| 120 | 2.16 | 2.05 | 1.94 | 1.82 | 1.69 | 1.61 | 1.53 | 1.43 | 1.31 |
| $\infty$ | 2.05 | 1.94 | 1.83 | 1.71 | 1.57 | 1.48 | 1.39 | 1.27 | 1.00 |

Example: if $X \sim F_{7,10}$. Then, $\mathbb{P}(X \leq 3.95) = 0.975$.

$\gamma = 0.95$

| | $m=1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $n$=1 | 161 | 200 | 216 | 225 | 230 | 234 | 237 | 239 | 241 |
| 2 | 18.5 | 19.0 | 19.2 | 19.2 | 19.3 | 19.3 | 19.4 | 19.4 | 19.4 |
| 3 | 10.1 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 |
| $\infty$ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 |

| | $m=10$ | 12 | 15 | 20 | 30 | 40 | 60 | 120 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $n$=1 | 242 | 244 | 246 | 248 | 250 | 251 | 252 | 253 | 254 |
| 2 | 19.4 | 19.4 | 19.4 | 19.4 | 19.5 | 19.5 | 19.5 | 19.5 | 19.5 |
| 3 | 8.79 | 8.74 | 8.70 | 8.66 | 8.62 | 8.59 | 8.57 | 8.55 | 8.53 |
| 4 | 5.96 | 5.91 | 5.86 | 5.80 | 5.75 | 5.72 | 5.69 | 5.66 | 5.63 |
| 5 | 4.74 | 4.68 | 4.62 | 4.56 | 4.50 | 4.46 | 4.43 | 4.40 | 4.37 |
| 6 | 4.06 | 4.00 | 3.94 | 3.87 | 3.81 | 3.77 | 3.74 | 3.70 | 3.67 |
| 7 | 3.64 | 3.57 | 3.51 | 3.44 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
| 8 | 3.35 | 3.28 | 3.22 | 3.15 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
| 9 | 3.14 | 3.07 | 3.01 | 2.94 | 2.86 | 2.83 | 2.79 | 2.75 | 2.71 |
| 10 | 2.98 | 2.91 | 2.85 | 2.77 | 2.70 | 2.66 | 2.62 | 2.58 | 2.54 |
| 12 | 2.75 | 2.69 | 2.62 | 2.54 | 2.47 | 2.43 | 2.38 | 2.34 | 2.30 |
| 15 | 2.54 | 2.48 | 2.40 | 2.33 | 2.25 | 2.20 | 2.16 | 2.11 | 2.07 |
| 20 | 2.35 | 2.28 | 2.20 | 2.12 | 2.04 | 1.99 | 1.95 | 1.90 | 1.84 |
| 30 | 2.16 | 2.09 | 2.01 | 1.93 | 1.84 | 1.79 | 1.74 | 1.68 | 1.62 |
| 40 | 2.08 | 2.00 | 1.92 | 1.84 | 1.74 | 1.69 | 1.64 | 1.58 | 1.51 |
| 60 | 1.99 | 1.92 | 1.84 | 1.75 | 1.65 | 1.59 | 1.53 | 1.47 | 1.39 |
| 120 | 1.91 | 1.83 | 1.75 | 1.66 | 1.55 | 1.50 | 1.43 | 1.35 | 1.25 |
| $\infty$ | 1.83 | 1.75 | 1.67 | 1.57 | 1.46 | 1.39 | 1.32 | 1.22 | 1.00 |

Example: if $X \sim F_{7,10}$. Then, $\mathbb{P}(X \leq 3.14) = 0.95$.