Assignment 1

STAT1301 Advanced Analysis of Scientific Data

Semester 2, 2025

Student Number: 49886351

DUE DATE: 11 / 9 / 2025

Question 1

Part a)

Assume each team must play exactly once
Assume the listed order of the two teams in any one match doesn't matter

$\therefore$ The first match has ${}^{10}\mathbf{C}_2$ possibilities
$\therefore$ The second match has ${}^{8}\mathbf{C}_2$ possibilities
$\therefore$ The third match has ${}^{6}\mathbf{C}_2$ possibilities
$\therefore$ The fourth match has ${}^{4}\mathbf{C}_2$ possibilities
$\therefore$ The fifth match has ${}^{2}\mathbf{C}_2$ possibilities

Therefore the total number of possibilities for team assignments is:

(1) $$\#N = {}^{10}\mathbf{C}_2 \cdot {}^{8}\mathbf{C}_2 \cdot {}^{6}\mathbf{C}_2 \cdot {}^{4}\mathbf{C}_2 \cdot {}^{2}\mathbf{C}_2$$

Assume the order of matches in the round doesn't matter
⇒ The total number of possibilities in (1) double-counts many team allocations

Given any one team allocation for a round of 5 matches, by rearranging the match order there is a total of $5!$ permutations, of which only one should be counted as unique since it was assumed the order of matches in a round doesn't matter

Therefore the total number of ways the teams can be assigned to the matches is:

(2) $$\therefore \#N = \frac{{}^{10}\mathbf{C}_2 \cdot {}^{8}\mathbf{C}_2 \cdot {}^{6}\mathbf{C}_2 \cdot {}^{4}\mathbf{C}_2 \cdot {}^{2}\mathbf{C}_2}{5!}$$

Evaluating (2):

(3) $$\implies \#N = \frac{45 \cdot 28 \cdot 15 \cdot 6 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

(4) $$\implies \#N = \frac{113400}{120} = 945$$

Therefore there are **945** ways the teams can be assigned to the matches

Part b)

Generalizing from (2) above noting symmetry:

(5) $$\#N = \frac{{}^{2n}\mathbf{C}_2 \cdot {}^{2n-2}\mathbf{C}_2 \cdot {}^{2n-4}\mathbf{C}_2 \cdots {}^{4}\mathbf{C}_2 \cdot {}^{2}\mathbf{C}_2}{n!}$$

Question 2

Let a be the hours of the second employee
Let b be the hours of the third employee

(1) $$\implies 10 + a + b = 180$$

(2) $$\implies a + b = 170$$

(3) $$\Omega = \{(a, b) : \forall a + b = 170\}$$

Let X be a random variable for a
Let $f_X(x)$ be the PDF of X
Let $F_X(x)$ be the CDF of X

(4) $$F_X(170) = \int_0^{170} f_X(x)\,\mathrm{d}x = 1$$

Assume that "randomly assigned" implies a uniform distribution

(5) $$f_X(x) = f_X(y) : \forall x, y \in [0,170]$$

(6) $$\implies f_X(x) = c : \exists c \in \mathbb{R} \cap \{\frac{\mathrm{d}c}{\mathrm{d}x} = 0\}$$ (c is a constant)

(7) $$\implies 1 = \int_0^{170} f_X(x)\,\mathrm{d}x$$

(8) $$1 = \int_0^{170} c\ \mathrm{d}x$$

(9) $$1 = 170c - 0$$

(10) $$c = \frac{1}{170}$$

(11) $$\implies f_X(x) = \frac{1}{170}$$

(12) $$\implies F_X(x) = \frac{1}{170}x : x \in [0,170]$$

Now knowing the probability distribution of X, the desired probability can be computed from the CDF using the definition $\mathrm{P}(X < a) = F_X(a) : a \in [0,170]$:

(13) $$P(X > 90) = 1 - P(X < 90)$$

(14) $$P(X > 90) = 1 - F_X(90)$$

(15) $$P(X > 90) = 1 - \frac{1}{170} \cdot 90$$

(16) $$P(X > 90) = \frac{17}{17} - \frac{9}{17}$$

$$P(X > 90) = \frac{8}{17} \tag{17}$$

This is the probability that the second employee works for over 90 hours, from symmetry the third employee will also have the same probability, hence the total probability is twice (17):

$$2 \cdot P(X > 90) = 2 \cdot \frac{8}{17} \tag{18}$$

$$\therefore = \frac{16}{17} \approx 0.941 \approx 94.1\,\% \tag{19}$$

$\therefore$ The probability that either the second or third employee is randomly assigned more than 90 hours is $\frac{16}{17}$ which is **≈94.1%**

Question 3

Part a)

Each student can pick floor 2, 3, 4, 5, 6, 7 or 8

(1) $$\Omega = \{(x_1, x_2, x_3, x_4, x_5) : \forall x_1, x_2, x_3, x_4, x_5 \in [2,8]\}$$

Calculating the number of possibilities of 5 students each with 7 independent possible choices:

(2) $$\implies |\Omega| = 7^5$$

Part b)

Assume "randomly" implies a uniform distribution

(3) $$\implies \mathrm{P}(A) = \frac{|A|}{|\Omega|} : \forall A \subseteq \Omega$$

Let A be the event that each student exits on a different floor

(4) $$\implies A = \{(x_1, x_2, x_3, x_4, x_5) : x_i \neq x_j \forall 1 \leq i < j \leq 5\} \text{ where } A \subset \Omega$$

To calculate the cardinality of A, notice that of the seven possible floors exactly five must be exited from. The number of possible combinations of 5 floors from a pool of 7 is ${}^7\mathrm{C}_5$. For each combination of floor choices, each of the five students may exit in any permutation, giving $5!$ permutations, hence

(5) $$|A| = {}^7\mathrm{C}_5 \cdot 5!$$

From this $\mathrm{P}(A)$ may be calculated

(6) $$\mathrm{P}(A) = \frac{{}^7\mathrm{C}_5 \cdot 5!}{7^5}$$

(7) $$\implies \mathrm{P}(A) = \frac{21 \cdot 120}{16807}$$

(8) $$\implies \mathrm{P}(A) = \frac{21 \cdot 120}{16807} = \frac{360}{2401} \approx 0.1499 \approx 15\,\%$$

∴ The probability that all five students exit on different floors is approximately **≈15%.**

Part c)

Let A be the event that all students will exit on different floors

(9) $$\text{Let } A = \{(x_1, x_2, x_3, x_4, x_5) : x_i \neq x_j \forall 1 \leq i < j \leq 5\} \text{where } A \subset \Omega$$

This set can be (exhaustively) partitioned into two sets B and C, defined below under Case 1 and Case 2

Case 1: No student exits from floor six

Let B be the partition of A where none of x_1, x_2, x_3, x_4, or x_5 is 6

(10) $$B = \{(x_1, x_2, x_3, x_4, x_5) : \forall x_1, x_2, x_3, x_4, x_5 \in \{2,3,4,5,7,8\}\} \text{ where } B \subset A$$

The probability p of a student exiting from one the remaining 6 floors $n \in \{2,3,4,5,7,8\}$ is

Let $p(n) = \frac{1-0.5}{6} = \frac{1}{12}$ where $n \in \{2,3,4,5,7,8\}$

Calculating P for a single combination of student choices is therefore a constant with respect to the specific floors each student exits from:

(11) $$\mathrm{P}(\{(x_1,x_2,x_3,x_4,x_5)\}) = p(x_1)p(x_2)p(x_3)p(x_4)p(x_5)\ \forall x_1,x_2,x_3,x_4,x_5 \in [\{2,3,4,5,7,8\}$$

(12) $$\mathrm{P}(\{(x_1,x_2,x_3,x_4,x_5)\}) = \frac{1}{12}\cdot\frac{1}{12}\cdot\frac{1}{12}\cdot\frac{1}{12}\cdot\frac{1}{12}\ \forall x_1,x_2,x_3,x_4,x_5 \in \{2,3,4,5,7,8\}$$

(13) $$\mathrm{P}(\{(x_1,x_2,x_3,x_4,x_5)\}) = \frac{1}{12^5}\ \forall x_1,x_2,x_3,x_4,x_5 \in \{2,3,4,5,7,8\}$$

From the infinite disjoint union rule of probability, we can generalize (13) to any sized event set B by adding all the disjoint probabilities, which becomes simple multiplication as the probability of every permutation of student exits are all equal

(14) $$\mathrm{P}(B) = \frac{|B|}{12^5}$$

Now the problem is simplified to calculating the cardinality of B. This is the same as the number of permutations of 5 students each with 6 mutually exclusive choices, or rearranging is the same as the permutations of 6 choices assigned each to one of 5 students

(15) $$|B| = {}^6\mathrm{P}_5$$

(16) $$\implies \mathrm{P}(B) = \frac{{}^6\mathrm{P}_5}{12^5}$$

(17) $$\implies \mathrm{P}(B) = \frac{720}{248832} = \frac{5}{1728}$$

Case 2: A student exits from floor six

Let C be the partition of A where one of x_1, x_2, x_3, x_4, or x_5 is 6

(18) $$C = \{(x_1,x_2,x_3,x_4,x_5) : 6 \in \{x_1,x_2,x_3,x_4,x_5\}\} \text{ where } C \subset A$$

To help express the probability function $\mathrm{P}(B)$, the piecewise probability p that a student exits on floor n was defined:

(19) $$\text{Let } p(n) = \begin{cases} \frac{1}{12} & n \in \{2,3,4,5,7,8\} \\ \frac{1}{2} & n = 6 \end{cases}$$

Like Case 1, fortunately, $\mathrm{P}(c \in C)$ turns out to be a constant no matter the permutation of student choices as exactly one $p(x_i)$ must be $p(6) = 0.5$ and multiplication is commutative

(20) $$\mathrm{P}(\{(x_1,x_2,x_3,x_4,x_5)\}) = p(x_1)p(x_2)p(x_3)p(x_4)p(x_5)\ \forall (x_1,x_2,x_3,x_4,x_5) \in C$$

(21) $$\mathrm{P}(\{(x_1,x_2,x_3,x_4,x_5)\}) = \frac{1}{12}\cdot\frac{1}{12}\cdot\frac{1}{2}\cdot\frac{1}{12}\cdot\frac{1}{12}\ \forall (x_1,x_2,x_3,x_4,x_5) \in C$$

(22) $$\mathrm{P}(\{(x_1,x_2,x_3,x_4,x_5)\}) = \frac{1}{12^4\cdot 2}\ \forall (x_1,x_2,x_3,x_4,x_5) \in C$$

From the infinite disjoint union rule of probability we can generalize (22), similarly to (14)

$$\mathrm{P}(C) = \frac{|C|}{12^4 \cdot 2} \tag{23}$$

To calculate the cardinality of C, consider the four students who picked one of the six remaining floors that were not floor six. The number of permutations of these 4 students each with 6 mutually exclusive choices is ${}^6\mathrm{P}_4$. Taking into account the student that chose floor six, there were five possible students that could have chosen floor six. Hence:

$$|C| = 5 \cdot {}^6\mathrm{P}_4 \tag{24}$$

$$\implies \mathrm{P}(C) = \frac{5 \cdot {}^6\mathrm{P}_4}{12^4 \cdot 2} \tag{25}$$

$$\implies \mathrm{P}(C) = \frac{1800}{41472} = \frac{25}{576} \tag{26}$$

Merging Case 1 and Case 2

Now we have $B \cap C = \varnothing$ and $B \cup C = A$ (from partition), as well as $\mathrm{P}(B)$ and $\mathrm{P}(C)$, we can use the (finite) disjoint union rule of probability to calculate $\mathrm{P}(A)$

$$\mathrm{P}(B \cup C) = P(B) + P(C) \text{ where } B \cap C = \varnothing \tag{27}$$

$$\mathrm{P}(A) = P(B) + P(C) \tag{28}$$

$$\mathrm{P}(A) = \frac{5}{1728} + \frac{25}{576} = \frac{5}{108} \approx 0.0463 \approx 4.6\,\% \tag{29}$$

∴ The probability of all five students exiting on different floors is **≈4.6%.**

Question 4

Let $x, y, z \in \{\mathrm{T}, \mathrm{F}\}$ where T is boolean true and F is boolean false represent the (independent) occurrence of hazard X, Y and Z respectively

(1) $$\Omega = \{(x, y, z) : \forall x, y, z \in \{\mathrm{T}, \mathrm{F}\}\}$$

(2) $$|\Omega| = 2^3 = 8$$

For brevity, $\mathrm{P}(\{x, y, z)\})$ will be written as $\mathrm{P}(x, y, z)$
From the question, (3) (4) and (5) can be deduced about the probability function P

(3) $$\mathrm{P}(\mathrm{T},\mathrm{F},\mathrm{F}) = \mathrm{P}(\mathrm{F},\mathrm{T},\mathrm{F}) = \mathrm{P}(\mathrm{F},\mathrm{F},\mathrm{T}) = 0.1$$

(4) $$\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{F}) = \mathrm{P}(\mathrm{T},\mathrm{F},\mathrm{T}) = \mathrm{P}(\mathrm{F},\mathrm{T},\mathrm{T}) = 0.12$$

(5) $$\mathrm{P}(\{(\mathrm{T},\mathrm{T},\mathrm{T})\} \mid \{(\mathrm{T},\mathrm{T},\mathrm{F}), (\mathrm{T},\mathrm{T},\mathrm{T})\}) = \frac{1}{3}$$

(3) and (4) provide six of the eight base cases of the probability function, from (5) using set and probability rules the final two cases can be deduced:

(6) $$\frac{\mathrm{P}(\{(\mathrm{T},\mathrm{T},\mathrm{T})\} \cap \{(\mathrm{T},\mathrm{T},\mathrm{F}), (\mathrm{T},\mathrm{T},\mathrm{T})\})}{\mathrm{P}(\{(\mathrm{T},\mathrm{T},\mathrm{F}), (\mathrm{T},\mathrm{T},\mathrm{T})\})} = \frac{1}{3}$$

(7) $$\frac{\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T})}{\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{F}) + \mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T})} = \frac{1}{3}$$

(8) $$\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T}) = \frac{1}{3}\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T}) + \frac{1}{3}\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{F})$$

(9) $$\frac{2}{3}\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T}) = \frac{1}{3}\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{F})$$

(10) $$\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T}) = \frac{1}{2}\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{F})$$

Recall from (4) that $\mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{F}) = 0.12$

(11) $$\implies \mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T}) = \frac{1}{2} \cdot 0.12 = 0.06$$

Now only $\mathrm{P}(\mathrm{F},\mathrm{F},\mathrm{F})$ remains unknown, which can be computed by summing all other probabilities and subtracting that from $\mathrm{P}(\Omega) = 1$:

(12) $$\mathrm{P}(\mathrm{F},\mathrm{F},\mathrm{F}) = 1 - (\mathrm{P}(\mathrm{T},\mathrm{F},\mathrm{F}) + \mathrm{P}(\mathrm{F},\mathrm{T},\mathrm{F}) + \mathrm{P}(\mathrm{F},\mathrm{F},\mathrm{T}) + \\ \mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{F}) + \mathrm{P}(\mathrm{T},\mathrm{F},\mathrm{T}) + \mathrm{P}(\mathrm{F},\mathrm{T},\mathrm{T}) + \mathrm{P}(\mathrm{T},\mathrm{T},\mathrm{T}))$$

(13) $$\mathrm{P}(\mathrm{F},\mathrm{F},\mathrm{F}) = 1 - (0.1 + 0.1 + 0.1 + 0.12 + 0.12 + 0.12 + 0.06)$$

(14) $$\mathrm{P}(\mathrm{F},\mathrm{F},\mathrm{F}) = 1 - 0.72$$

(15) $$\mathrm{P}(\mathrm{F},\mathrm{F},\mathrm{F}) = 0.28$$

At this point, enough information is known about P to compute it for any event set $A \subseteq \Omega$ using the finite disjoint union rule of probability
The question then asks to compute (16), which becomes simple set and probability manipulation

$$P(\{F, F, F\} \mid \{(F, y, z) : \forall y, z \in \{T, F\}\}) \tag{16}$$

$$= \frac{P(\{F, F, F\} \cap \{(F, F, F), (F, F, T), (F, T, F), (F, T, T)\})}{P(\{(F, F, F), (F, F, T), (F, T, F), (F, T, T)\})} \tag{17}$$

$$= \frac{P(F, F, F)}{P(F, F, F) + P(F, F, T) + P(F, T, F) + P(F, T, T)} \tag{18}$$

$$= \frac{0.28}{0.28 + 0.1 + 0.1 + 0.12} = \frac{7}{15} \approx 0.4667 \approx 47\,\% \tag{19}$$

∴ The probability that an amphibian is unaffected by any of the three hazards assuming it is not affected by hazard X is **≈47%**.

Question 5

Let P represent the scenario of being given a placebo vaccine, and P' being given the new vaccine
Let I represent the scenario of developing immunity, and I' not developing immunity

(1) $$\Omega = \{(P,I),(P,I'),(P',I),(P',I')\}$$

Knowing the probability of receiving the real vaccine is p can be represented as (2), and the converse as (3):

(2) $$\mathrm{P}(\{(P,x) : \forall x \in \{I,I'\}\}) = p$$

(3) $$\iff \mathrm{P}(\{(P',x) : \forall x \in \{I,I'\}\}) = 1-p$$

Knowing the probability of developing immunity if given the real vaccine equals α is represented in (4):

(4) $$\mathrm{P}(\{(P,I),(P',I)\} \,|\, \{(P',I),(P',I')\}) = \alpha$$

Knowing the probability of developing immunity if given the placebo vaccine equals β is represented in (5):

(5) $$\mathrm{P}(\{(P,I),(P',I)\} \,|\, \{(P,I),(P,I')\}) = \beta$$

Part a)

The probability that a participant in the trial develops immunity is expressed in (6):

(6) $$\mathrm{P}(\{(P,I),(P',I)\})$$

(7) $$= \mathrm{P}(P,I) + \mathrm{P}(P',I)$$

To evaluate $\mathrm{P}(P,I)$ in terms of the constants p, α and β, (5) can be rearranged using set and probability rules

(8) $$\beta = \frac{\mathrm{P}(\{(P,I),(P',I)\}) \cap \{(P,I),(P,I')\}}{\mathrm{P}(\{(P,I),(P,I')\})}$$

(9) $$\beta = \frac{\mathrm{P}(P,I)}{p}$$

(10) $$\mathrm{P}(P,I) = \beta p$$

To evaluate $\mathrm{P}(P',I)$ in terms of the constants p, α and β, (4) can be rearranged using set and probability rules

(11) $$\alpha = \frac{\mathrm{P}(\{(P,I),(P',I)\} \cap \{(P',I),(P',I')\})}{\mathrm{P}(\{(P',I),(P',I')\})}$$

(12) $$\alpha = \frac{\mathrm{P}(P',I)}{1-p}$$

(13) $$\mathrm{P}(P',I) = \alpha(1-p)$$

Substituting (10) and (13) into (7):

(14) $$\mathrm{P}(P,I) + \mathrm{P}(P',I) = \beta p + \alpha(1-p)$$

$\therefore$ The probability that a participant in the trial develops immunity is $\beta p + \alpha(1-p)$

Part b)

The probability that a participant receives the new vaccine given they develop immunity is expressed in (15):

(15) $$\mathrm{P}(\{(P', I), (P', I')\} \mid \{(P, I), (P', I)\})$$

(16) $$= \frac{\mathrm{P}(P', I)}{\mathrm{P}(\{(P, I), (P', I)\})}$$

(17) $$= \frac{\alpha(1-p)}{\beta p + \alpha(1-p)}$$

∴ The probability that a participant received the new vaccine given they developed immunity is stated in (17) above

Part c)

Let Ω_2 be a second sample space, representing both independent participants

(18) $$\Omega_2 = \{(\Omega_x, \Omega_y) : \forall \Omega_x, \Omega_y \subseteq \Omega\}$$

As each participant is independent from each other, a second probability function P_2 can be defined as the simple multiplication between each participants probabilities:

(19) $$\mathrm{P}_2(\{(\Omega_x, \Omega_y)\}) = \mathrm{P}(\Omega_x) \cdot \mathrm{P}(\Omega_y) \ \ \forall \Omega_x, \Omega_y \in \Omega$$

Given both both participants develop immunity, the probability that either only the first participant or the second participant was given the real vaccine is expressed in (20):

(20) $$\mathrm{P}_2(\{((P, i_1), (P', i_2)), ((P', i_3), (P, i_4)) \forall i_1, i_2, i_3, i_4 \in \{I, I'\}\} \mid \{((x_1, I), (x_2, I)) : \forall x_1, x_2 \in \{P, P'\}\})$$

Using set and probability rules, this can be expressed in terms of the constants p, α and β:

(21) $$= \frac{\mathrm{P}_2(\{((P, i_1), (P', i_2)), ((P', i_3), (P, i_4)) : \forall i_1, i_2, i_3, i_4 \in \{I, I'\}\} \cap \{((x_1, I), (x_2, I)) : \forall x_1, x_2 \in \{P, P'\}\})}{\mathrm{P}_2(\{((x_1, I), (x_2, I)) : \forall x_1, x_2 \in \{P, P'\}\})}$$

(22) $$= \frac{\mathrm{P}_2(\{((P, I), (P', I)), ((P', I), (P, I))\})}{\mathrm{P}_2(\{((P, I), (P, I)), ((P', I), (P, I)), ((P, I), (P', I)), ((P', I), (P', I))\})}$$

Using (19) and the disjoint union rule to reduce into P territory:

(23) $$= \frac{\mathrm{P}(P, I) \cdot \mathrm{P}(P', I) + \mathrm{P}(P', I) \cdot \mathrm{P}(P, I)}{\mathrm{P}(P, I)^2 + 2 \times \mathrm{P}(P, I) \cdot \mathrm{P}(P', I) + \mathrm{P}(P', I)^2}$$

Substituting (10) and (13):

(24) $$= \frac{2 \times \beta p \cdot \alpha(1-p)}{(\beta p)^2 + 2 \times \beta p \cdot \alpha(1-p) + (\alpha(1-p))^2}$$

(25) $$= \frac{2\alpha\beta p(1-p)}{\alpha^2(1-p)^2 + \beta^2 p^2 + 2\alpha\beta p(1-p)}$$

∴ The probability of exactly one of the two participants receiving the new vaccine given they both develop immunity is stated in (25) above