1 Question 1

1.1 Part a)

Let Ω be the sample space. Therefore $\mathrm{P}(\{\Omega\}) = 1$. Adding all the joint pmf values must sum to 1:

$$
\begin{aligned}
\{\Omega\} &= \bigcup_x \bigcup_y \{\mathrm{X} = x\} \cap \{\mathrm{Y} = y\} \\
\mathrm{P}(\{\Omega\}) &= 1 \\
\implies 1 &= \mathrm{P}((\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) \cup \ldots \cup (\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) + \ldots + \mathrm{P}(\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= (p - \frac{1}{16}) + (\frac{1}{4} - p) + (0) + (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p) \\
1 &= -\frac{1}{16} + \frac{4}{16} + \frac{7}{16} + \frac{1}{16} + \frac{1}{16} + \frac{4}{16} \\
1 &= 1
\end{aligned}
$$

Unfortunately, this tells us no information about p. From the definition of probability, $\mathrm{P}(\{c\})$ for $c \in \Omega$ must be greater or equal to 0, $\mathrm{P}(\{c \in \Omega\}) \geq 0$. This can be used to restrict the possible values of p:

$$
\begin{aligned}
\mathrm{P}(A \subseteq \Omega) &\geq 0 \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
p - \frac{1}{16} &\geq 0 \\
p &\geq \frac{1}{16} \\
\implies \mathrm{P}(\{\mathrm{X} = 0\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
\frac{1}{4} - p &\geq 0 \\
p &\leq \frac{1}{4} \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = 1\}) &\geq 0 \\
p + \frac{1}{16} &\geq 0 \\
p &\leq \frac{1}{16}
\end{aligned}
$$

$$
\implies p \in [\frac{1}{16}, \frac{1}{4}] \tag{1}
$$

Therefore, $\frac{1}{16} \leq p \leq \frac{1}{4}$, and can be any value within this range.

1.2 Part b)

Aim is to find $\mathrm{P}(\{\mathrm{X} = \mathrm{Y}\})$:

$$
\begin{aligned}
\mathrm{P}(\{\mathrm{X} = \mathrm{Y}\}) &= \sum_{a} \mathrm{P}(\{\mathrm{X} = a\} \cap \{\mathrm{Y} = a\}) \\
&= \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) + \mathrm{P}(\{\mathrm{X} = 0\} \cap \{\mathrm{Y} = 0\}) + \mathrm{P}(\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\}) \\
&= (p - \frac{1}{16}) + (\frac{3}{16}) + (\frac{1}{4} - p) \\
&= \frac{6}{16} = \frac{3}{}
\end{aligned}
$$

1.3 Part c)

The marginal pdf of X is $f_{\mathrm{X}}(x)$, which is equal to $\mathrm{P}(\{\mathrm{X} = x\})$ and can be manually evaluated:

$$
\begin{aligned}
\mathrm{P}(\{\mathrm{X} = -1\}) &= \sum_{y} \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = y\}) \\
&= (p - \frac{1}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) \\
&= 2p + \frac{1}{8} \\
\mathrm{P}(\{\mathrm{X} = 0\}) &= \sum_{y} \mathrm{P}(\{\mathrm{X} = 0\} \cap \{\mathrm{Y} = y\}) \\
&= (\frac{1}{4} - p) + (\frac{3}{16}) + (\frac{1}{16}) \\
&= -p + \frac{1}{2} \\
\mathrm{P}(\{\mathrm{X} = 1\}) &= \sum_{y} \mathrm{P}(\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = y\}) \\
&= (0) + (\frac{1}{8}) + (\frac{1}{4} - p) \\
&= -p + \frac{3}{8} \\
\implies f_{\mathrm{X}}(x) = \mathrm{P}(\{\mathrm{X} = x\}) &= \begin{cases} 2p + \frac{1}{8} & x = -1 \\ -p + \frac{1}{2} & x = 0 \\ -p + \frac{3}{8} & x = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}(\{\mathrm{Y}=-1\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=-1\}) \\
&= (p-\frac{1}{16}) + (\frac{1}{8}) + (p+\frac{1}{16}) \\
&= p+\frac{1}{8} \\
\mathrm{P}(\{\mathrm{Y}=0\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=0\}) \\
&= (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) \\
&= \frac{7}{16} \\
\mathrm{P}(\{\mathrm{Y}=1\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=1\}) \\
&= (p+\frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4}-p) \\
&= \frac{6}{18} = \frac{3}{8} \\
\implies f_{\mathrm{Y}}(x) = \mathrm{P}(\{\mathrm{Y}=y\}) &= \begin{cases} p+\frac{1}{8} & y=-1 \\ \frac{7}{16} & y=0 \\ \frac{3}{8} & y=1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

1.4 Part d)

X and Y are independant if

$$
\mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=y\}) = \mathrm{P}(\{\mathrm{X}=x\})\cdot\mathrm{P}(\{\mathrm{Y}=y\}) = f_{\mathrm{X}}(x)\cdot f_{\mathrm{Y}}(y) \tag{2}
$$

for all possible values x and y. Therefore, this must be true for $x=-1$ and $y=1$:

$$
\begin{aligned}
\mathrm{LHS} &= \mathrm{P}(\{\mathrm{X}=-1\}\cap\{\mathrm{Y}=1\}) \\
&= p-\frac{1}{16}
\end{aligned}
\qquad
\begin{aligned}
\mathrm{RHS} &= \mathrm{P}(\{\mathrm{X}=-1\})\mathrm{P}(\{\mathrm{Y}=1\}) \\
&= (2p+\frac{1}{8})(\frac{3}{8}) \\
&= \frac{3}{4}p+\frac{3}{64}
\end{aligned}
$$

As shown above, LHS and RHS are only equal for zero or one values of p. Letting LHS = RHS, we can find this exact value (or lack thereof):

$$
\begin{aligned}
p-\frac{1}{16} &= \frac{3}{4}p+\frac{3}{64} \\
\frac{1}{4}p &= \frac{3}{64}+\frac{1}{16} \\
p &= \frac{7}{64}\cdot 4 = \frac{7}{16}
\end{aligned}
$$

Therefore LHS = RHS only when $p = \frac{7}{16}$, however from (1) this is not within the potential domain of p. Therefore LHS $\neq$ RHS, showing one counterexample to (2), hence X and Y are not independent.

1.5 Part e)

$$
\begin{aligned}
\mathrm{E(X)} &= \sum_x x\mathrm{P}(\{\mathrm{X} = x\}) \\
&= -1(2p + \frac{1}{8}) + 0(-p + \frac{1}{2}) + 1(-p + \frac{3}{8}) \\
&= -2p - \frac{1}{8} - p + \frac{3}{8} \\
\therefore \mathrm{E(X)} &= -3p + \frac{1}{4} \\
\mathrm{E(Y)} &= \sum_y y\mathrm{P}(\{\mathrm{Y} = y\}) \\
&= -1(p + \frac{1}{8}) + 0(\frac{7}{16}) + 1(\frac{3}{8}) \\
\therefore \mathrm{E(Y)} &= -p - \frac{1}{8} + \frac{3}{8} = -p + \frac{1}{4} \\
\mathrm{Cov(X, Y)} &= \mathrm{E}[(\mathrm{X} - \mathrm{E(X)}(\mathrm{Y} - \mathrm{E(Y)}] \\
\mathrm{Cov(X, Y)} &= \sum_{c \in \Omega} (\mathrm{X}(c) - \mathrm{E(X)}(\mathrm{Y}(c) - \mathrm{E(Y)})\mathrm{P}(\{c\}) \\
&= \sum_{x,y} (x + (+3p - \frac{1}{4}))(y + (p - \frac{1}{4})\mathrm{P}(\{\mathrm{X} = x\} \cap \{\mathrm{Y} = y\})
\end{aligned}
$$

Expanding this sum is tedious and results in nine trinomials. The following sum expansion significantly reduces the algebra necessary:

$$
\begin{aligned}
\mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= \mathrm{E}(\mathrm{XY}) - \mathrm{E}(\mathrm{X})\mathrm{E}(\mathrm{Y}) \\
\mathrm{E}(\mathrm{XY}) &= \sum_{c\in\Omega} \mathrm{X}(c)\mathrm{Y}(c)\mathrm{P}(\{c\}) \\
&= \sum_{x,y} xy\mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=y\}) \\
&= (-1)(-1)(p-\frac{1}{16}) + (-1)(0)(\frac{1}{4}-p) + (-1)(1)(0) \\
&+ (0)(-1)(\frac{1}{8}) + (0)(0)(\frac{3}{16}) + (0)(1)(\frac{1}{8}) \\
&+ (1)(-1)(p+\frac{1}{16}) + (1)(0)(\frac{1}{16}) + (1)(1)(\frac{1}{4}-p) \\
&= (p-\frac{1}{16}) - (p+\frac{1}{16}) + (\frac{1}{4}-p) \\
\therefore \mathrm{E}(\mathrm{XY}) &= -p+\frac{1}{8} \\
\implies \mathrm{Cov}(\mathrm{X},\mathrm{Y}) &= (-p+\frac{1}{8}) + (-3p+\frac{1}{4})(-p+\frac{1}{4}) \\
&= -p+\frac{1}{8}+3p-\frac{3}{4}-\frac{1}{4}p+\frac{1}{16} \\
&= (-1+3-\frac{1}{4})p + (\frac{1}{8}-\frac{3}{4}+\frac{1}{16}) \\
&= \frac{7}{4}p - \frac{9}{16}
\end{aligned}
$$