

1 Question 1

1.1 Part a)

Let Ω be the sample space. Therefore $P(\{\Omega\}) = 1$. Adding all the joint pmf values must sum to 1:

$$\begin{aligned}\{\Omega\} &= \bigcup_x \bigcup_y \{X = x\} \cap \{Y = y\} \\ P(\{\Omega\}) &= 1 \\ \implies 1 &= P((\{X = -1\} \cap \{Y = -1\}) \cup \dots \cup (\{X = 1\} \cap \{Y = 1\})) \\ &= P(\{X = -1\} \cap \{Y = -1\}) + \dots + P(\{X = 1\} \cap \{Y = 1\}) \\ &= (p - \frac{1}{16}) + (\frac{1}{4} - p) + (0) + (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p) \\ 1 &= -\frac{1}{16} + \frac{4}{16} + \frac{7}{16} + \frac{1}{16} + \frac{1}{16} + \frac{4}{16} \\ 1 &= 1\end{aligned}$$

Unfortunately, this tells us no information about p . From the definition of probability, $P(\{c\})$ for $c \in \Omega$ must be greater or equal to 0, $P(\{c \in \Omega\}) \geq 0$. This can be used to restrict the possible values of p :

$$\begin{aligned}P(A \subseteq \Omega) &\geq 0 \\ \implies P(\{X = -1\} \cap \{Y = -1\}) &\geq 0 \\ p - \frac{1}{16} &\geq 0 \\ p &\geq \frac{1}{16} \\ \implies P(\{X = 0\} \cap \{Y = -1\}) &\geq 0 \\ \frac{1}{4} - p &\geq 0 \\ p &\leq \frac{1}{4} \\ \implies P(\{X = -1\} \cap \{Y = 1\}) &\geq 0 \\ p + \frac{1}{16} &\geq 0 \\ p &\leq \frac{1}{16} \\ \implies p &\in [\frac{1}{16}, \frac{1}{4}] \tag{1}\end{aligned}$$

Therefore, $\frac{1}{16} \leq p \leq \frac{1}{4}$, and can be any value within this range.

1.2 Part b)

Aim is to find $P(\{X = Y\})$:

$$\begin{aligned}
P(\{X = Y\}) &= \sum_a P(\{X = a\} \cap \{Y = a\}) \\
&= P(\{X = -1\} \cap \{Y = -1\}) + P(\{X = 0\} \cap \{Y = 0\}) + P(\{X = 1\} \cap \{Y = 1\}) \\
&= (p - \frac{1}{16}) + (\frac{3}{16}) + (\frac{1}{4} - p) \\
&= \frac{6}{16} = \frac{3}{8}
\end{aligned}$$

1.3 Part c)

The marginal pdf of X is $f_X(x)$, which is equal to $P(\{X = x\})$ and can be manually evaluated:

$$\begin{aligned}
P(\{X = -1\}) &= \sum_y P(\{X = -1\} \cap \{Y = y\}) \\
&= (p - \frac{1}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) \\
&= 2p + \frac{1}{8} \\
P(\{X = 0\}) &= \sum_y P(\{X = 0\} \cap \{Y = y\}) \\
&= (\frac{1}{4} - p) + (\frac{3}{16}) + (\frac{1}{16}) \\
&= -p + \frac{1}{2} \\
P(\{X = 1\}) &= \sum_y P(\{X = 1\} \cap \{Y = y\}) \\
&= (0) + (\frac{1}{8}) + (\frac{1}{4} - p) \\
&= -p + \frac{3}{8} \\
\Rightarrow f_X(x) = P(\{X = x\}) &= \begin{cases} 2p + \frac{1}{8} & x = -1 \\ -p + \frac{1}{2} & x = 0 \\ -p + \frac{3}{8} & x = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

$$\begin{aligned}
P(\{Y = -1\}) &= \sum_x P(\{X = x\} \cap \{Y = -1\}) \\
&= (p - \frac{1}{16}) + (\frac{1}{4} - p) + (0) \\
&= \frac{3}{16} \\
P(\{Y = 0\}) &= \sum_x P(\{X = x\} \cap \{Y = 0\}) \\
&= (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) \\
&= \frac{7}{16} \\
P(\{Y = 1\}) &= \sum_x P(\{X = x\} \cap \{Y = 1\}) \\
&= (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p) \\
&= \frac{6}{18} = \frac{3}{8} \\
\Rightarrow f_Y(x) = P(\{Y = y\}) &= \begin{cases} \frac{3}{16} & y = -1 \\ \frac{7}{16} & y = 0 \\ \frac{3}{8} & y = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

1.4 Part d)

X and Y are independant if

$$P(\{X = x\} \cap \{Y = y\}) = P(\{X = x\}) \cdot P(\{Y = y\}) = f_X(x) \cdot f_Y(y) \quad (2)$$

for all possible values x and y . Therefore, this must be true for $x = -1$ and $y = 1$:

$$\begin{aligned}
\text{LHS} &= P(\{X = -1\} \cap \{Y = 1\}) \\
&= p + \frac{1}{16} \\
\text{RHS} &= P(\{X = -1\})P(\{Y = 1\}) \\
&= (2p + \frac{1}{8})(\frac{3}{8}) \\
&= \frac{3}{4}p + \frac{3}{64}
\end{aligned}$$

As shown above, LHS and RHS are only equal for zero or one values of p . Letting LHS = RHS, we can find this exact value (or lack thereof):

$$\begin{aligned}
p + \frac{1}{16} &= \frac{3}{4}p + \frac{3}{64} \\
\frac{1}{4}p &= \frac{3}{64} - \frac{1}{16} \\
p &= -\frac{1}{64} \cdot 4 = -\frac{1}{16}
\end{aligned}$$

Therefore LHS = RHS only when $p = -\frac{1}{16}$, however from (1) this is not within the potential domain of p . Therefore LHS \neq RHS, showing one counterexample to (2), hence X and Y are not independent.

1.5 Part e)

$$\begin{aligned}
E(X) &= \sum_x xP(\{X = x\}) \\
&= -1(2p + \frac{1}{8}) + 0(-p + \frac{1}{2}) + 1(-p + \frac{3}{8}) \\
&= -2p - \frac{1}{8} - p + \frac{3}{8} \\
\therefore E(X) &= -3p + \frac{1}{4} \\
E(Y) &= \sum_y yP(\{Y = y\}) \\
&= -1(\frac{3}{16}) + 0(\frac{7}{16}) + 1(\frac{3}{8}) \\
\therefore E(Y) &= -\frac{3}{16} + \frac{3}{8} = \frac{3}{16} \\
\text{Cov}(X, Y) &= E[(X - E(X))(Y - E(Y))] \\
\text{Cov}(X, Y) &= \sum_{c \in \Omega} (X(c) - E(X))(Y(c) - E(Y))P(\{c\}) \\
&= \sum_{x,y} (x - (-3p + \frac{1}{4}))(y - (\frac{3}{16}))P(\{X = x\} \cap \{Y = y\})
\end{aligned}$$

Expanding this sum is tedious and results in nine trinomials. The following sum for the $\text{Cov}(X, Y)$ expansion significantly reduces the algebra necessary by computing $E(XY)$ instead:

$$\begin{aligned}
\text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
E(XY) &= \sum_{c \in \Omega} X(c)Y(c)P(\{c\}) \\
&= \sum_{x,y} xyP(\{X = x\} \cap \{Y = y\}) \\
&= (-1)(-1)(p - \frac{1}{16}) + (-1)(0)(\frac{1}{4} - p) + (-1)(1)(0) \\
&\quad + (0)(-1)(\frac{1}{8}) + (0)(0)(\frac{3}{16}) + (0)(1)(\frac{1}{8}) \\
&\quad + (1)(-1)(p + \frac{1}{16}) + (1)(0)(\frac{1}{16}) + (1)(1)(\frac{1}{4} - p) \\
&= (p - \frac{1}{16}) - (p + \frac{1}{16}) + (\frac{1}{4} - p) \\
&\therefore E(XY) = -p + \frac{1}{8} \\
\implies \text{Cov}(X, Y) &= (-p + \frac{1}{8}) - (-3p + \frac{1}{4})(\frac{3}{16}) \\
&= -p + \frac{1}{8} + \frac{9}{16}p - \frac{3}{64} \\
&\therefore \text{Cov}(X, Y) = -\frac{7}{16}p - \frac{5}{64}
\end{aligned}$$