1 Question 1

1.1 Part a)

Let Ω be the sample space. Therefore $\mathrm{P}(\{\Omega\}) = 1$. Adding all the joint pmf values must sum to 1:

$$
\begin{aligned}
\{\Omega\} &= \bigcup_x \bigcup_y \{\mathrm{X} = x\} \cap \{\mathrm{Y} = y\} \\
\mathrm{P}(\{\Omega\}) &= 1 \\
\implies 1 &= \mathrm{P}((\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) \cup \ldots \cup (\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) + \ldots + \mathrm{P}(\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= (p - \frac{1}{16}) + (\frac{1}{4} - p) + (0) + (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p) \\
1 &= -\frac{1}{16} + \frac{4}{16} + \frac{7}{16} + \frac{1}{16} + \frac{1}{16} + \frac{4}{16} \\
1 &= 1
\end{aligned}
$$

Unfortunately, this tells us no information about p. From the definition of probability, $\mathrm{P}(\{c\})$ for $c \in \Omega$ must be greater or equal to 0, $\mathrm{P}(\{c \in \Omega\}) \geq 0$. This can be used to restrict the possible values of p:

$$
\begin{aligned}
\mathrm{P}(A \subseteq \Omega) &\geq 0 \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
p - \frac{1}{16} &\geq 0 \\
p &\geq \frac{1}{16} \\
\implies \mathrm{P}(\{\mathrm{X} = 0\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
\frac{1}{4} - p &\geq 0 \\
p &\leq \frac{1}{4} \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = 1\}) &\geq 0 \\
p + \frac{1}{16} &\geq 0 \\
p &\leq \frac{1}{16}
\end{aligned}
$$

Therefore, $\frac{1}{16} \leq p \leq \frac{1}{4}$, and can be any value within this range.