1 Question 1

1.1 Part a)

Let Ω be the sample space. Therefore $\mathrm{P}(\{\Omega\}) = 1$. Adding all the joint pmf values must sum to 1:

$$\begin{aligned}
\{\Omega\} &= \bigcup_x \bigcup_y \{\mathrm{X} = x\} \cap \{\mathrm{Y} = y\} \\
\mathrm{P}(\{\Omega\}) &= 1 \\
\implies 1 &= \mathrm{P}((\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) \cup \ldots \cup (\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) + \ldots + \mathrm{P}(\{\mathrm{X} = 1\} \cap \{\mathrm{Y} = 1\})) \\
&= (p - \frac{1}{16}) + (\frac{1}{4} - p) + (0) + (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4} - p) \\
1 &= -\frac{1}{16} + \frac{4}{16} + \frac{7}{16} + \frac{1}{16} + \frac{1}{16} + \frac{4}{16} \\
1 &= 1
\end{aligned}$$

Unfortunately, this tells us no information about p. From the definition of probability, $\mathrm{P}(\{c\})$ for $c \in \Omega$ must be greater or equal to 0, $\mathrm{P}(\{c \in \Omega\}) \geq 0$. This can be used to restrict the possible values of p:

$$\begin{aligned}
\mathrm{P}(A \subseteq \Omega) &\geq 0 \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
p - \frac{1}{16} &\geq 0 \\
p &\geq \frac{1}{16} \\
\implies \mathrm{P}(\{\mathrm{X} = 0\} \cap \{\mathrm{Y} = -1\}) &\geq 0 \\
\frac{1}{4} - p &\geq 0 \\
p &\leq \frac{1}{4} \\
\implies \mathrm{P}(\{\mathrm{X} = -1\} \cap \{\mathrm{Y} = 1\}) &\geq 0 \\
p + \frac{1}{16} &\geq 0 \\
p &\leq \frac{1}{16}
\end{aligned}$$

Therefore, $\frac{1}{16} \leq p \leq \frac{1}{4}$, and can be any value within this range.

1.2 Part b)

Aim is to find $\mathrm{P}(\{\mathrm{X} = \mathrm{Y}\})$:

$$
\begin{aligned}
\mathrm{P}(\{\mathrm{X}=\mathrm{Y}\}) &= \sum_{a} \mathrm{P}(\{\mathrm{X}=a\} \cap \{\mathrm{Y}=a\}) \\
&= \mathrm{P}(\{\mathrm{X}=-1\} \cap \{\mathrm{Y}=-1\}) + \mathrm{P}(\{\mathrm{X}=0\} \cap \{\mathrm{Y}=0\}) + \mathrm{P}(\{\mathrm{X}=1\} \cap \{\mathrm{Y}=1\}) \\
&= (p - \frac{1}{16}) + (\frac{3}{16}) + (\frac{1}{4} - p) \\
&= \frac{6}{16} = \frac{3}{\ }
\end{aligned}
$$

1.3 Part c)

The marginal pdf of X is $f_{\mathrm{X}}(x)$, which is equal to $\mathrm{P}(\{\mathrm{X}=x\})$ and can be manually evaluated:

$$
\begin{aligned}
\mathrm{P}(\{\mathrm{X}=-1\}) &= \sum_{y} \mathrm{P}(\{\mathrm{X}=-1\} \cap \{\mathrm{Y}=y\}) \\
&= (p - \frac{1}{16}) + (\frac{1}{8}) + (p + \frac{1}{16}) \\
&= 2p + \frac{1}{8} \\
\mathrm{P}(\{\mathrm{X}=0\}) &= \sum_{y} \mathrm{P}(\{\mathrm{X}=0\} \cap \{\mathrm{Y}=y\}) \\
&= (\frac{1}{4} - p) + (\frac{3}{16}) + (\frac{1}{16}) \\
&= -p + \frac{1}{2} \\
\mathrm{P}(\{\mathrm{X}=1\}) &= \sum_{y} \mathrm{P}(\{\mathrm{X}=1\} \cap \{\mathrm{Y}=y\}) \\
&= (0) + (\frac{1}{8}) + (\frac{1}{4} - p) \\
&= -p + \frac{3}{8} \\
\implies f_{\mathrm{X}}(x) = \mathrm{P}(\{\mathrm{X}=x\}) &= \begin{cases} 2p + \frac{1}{8} & x = -1 \\ -p + \frac{1}{2} & x = 0 \\ -p + \frac{3}{8} & x = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{P}(\{\mathrm{Y}=-1\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=-1\}) \\
&= (p-\frac{1}{16}) + (\frac{1}{8}) + (p+\frac{1}{16}) \\
&= p+\frac{1}{8} \\
\mathrm{P}(\{\mathrm{Y}=0\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=0\}) \\
&= (\frac{1}{8}) + (\frac{3}{16}) + (\frac{1}{8}) \\
&= \frac{7}{16} \\
\mathrm{P}(\{\mathrm{Y}=1\}) &= \sum_x \mathrm{P}(\{\mathrm{X}=x\}\cap\{\mathrm{Y}=1\}) \\
&= (p+\frac{1}{16}) + (\frac{1}{16}) + (\frac{1}{4}-p) \\
&= \frac{6}{18} = \frac{3}{8} \\
\implies f_{\mathrm{Y}}(x) = \mathrm{P}(\{\mathrm{Y}=y\}) &= \begin{cases} p+\frac{1}{8} & y=-1 \\ \frac{7}{16} & y=0 \\ \frac{3}{8} & y=1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$