Bonus Question

AI1110: Probability and Random Variables
Indian Institute of Technology, Hyderabad

CS22BTECH11001
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Given that a fair coin is marked 1 on one face and 6 on the other and a fair die is rolled. Find the probability that the sum turns up 3 and 12.

Solution. Let X be a random variable which represents the value on the coin.

Let Y be a random variable which represents the value of the die.

The Z-transform of X is defined as

$$M_X(z) = E\left[z^{-X}\right] = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{1}$$

$$\therefore M_X(z) = \frac{1}{2} \times \left(z^{-1}\right) + \frac{1}{2} \times \left(z^{-6}\right) \tag{2}$$

$$= \frac{z^{-1}}{2} + \frac{z^{-6}}{2} \tag{3}$$

Similarly,

$$M_Y(z) = \frac{1}{6} \times \left(z^{-1}\right) + \frac{1}{6} \times \left(z^{-2}\right) + \frac{1}{6} \times \left(z^{-3}\right) \tag{4}$$

$$+ \frac{1}{6} \times \left(z^{-4}\right) + \frac{1}{6} \times \left(z^{-5}\right) + \frac{1}{6} \times \left(z^{-6}\right)$$

$$= \frac{z^{-1}}{6} + \frac{z^{-2}}{6} + \frac{z^{-3}}{6} + \frac{z^{-4}}{6} + \frac{z^{-5}}{6} + \frac{z^{-6}}{6} \tag{5}$$

Now, As X and Y are independent

$$M_{X+Y}(z) = M_Y(z) M_X(z) \tag{6}$$

$$\therefore M_{X+Y}(z) = \left(\frac{z^{-1}}{6} + \frac{z^{-2}}{6} + \frac{z^{-3}}{6} + \frac{z^{-4}}{6} + \frac{z^{-5}}{6} + \frac{z^{-6}}{6}\right) \tag{7}$$

$$\times \left(\frac{z^{-1}}{2} + \frac{z^{-6}}{2}\right)$$

$$= \frac{z^{-2}}{12} + \frac{z^{-3}}{12} + \frac{z^{-4}}{12} + \frac{z^{-5}}{12} + \frac{z^{-6}}{12} + \frac{z^{-7}}{6} \tag{8}$$

$$+ \frac{z^{-8}}{12} + \frac{z^{-9}}{12} + \frac{z^{-10}}{12} + \frac{z^{-11}}{12} + \frac{z^{-12}}{12} \tag{9}$$

Therefore,

By comparing Coefficients,

$$p_{X+Y}(3) = \frac{1}{12} \tag{10}$$

$$p_{X+Y}(12) = \frac{1}{12} \tag{11}$$