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Block 1: Model Specification and Pricing PDE

We already know:
For a given market, described by the equations:

$$\begin{cases} \mathrm{d}M(t) &= rM(t)\mathrm{d}t \\ \mathrm{d}S(t) &= \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t), \end{cases}$$

and a contingent claim of the form

$$\chi = V(T, S(T)),$$

the arbitrage free price is given, via Itô's Lemma, by $V(t, S)$, where function $V(t, S)$ satisfies the Black-Scholes equation:

$$\begin{cases} \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV &= 0 \\ V(T, S) &= \chi. \end{cases}$$

Block 2: Relation to Monte Carlo via Feynman-Kac Theorem

$V(t,S)$ is the unique solution of the final condition problem

$$\begin{cases} \dfrac{\partial V}{\partial t} & + \quad \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0, \\ V(T,S) & = \quad \text{given} \end{cases}$$

This solution can also be obtained as

$$V(t,S) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\{V(T,S(T))|S(t)\}$$

with the sum of the first derivatives of the option square integrable. Given the system of stochastic differential equations:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t).$$

Block 3: Pricing via Integrals

- Pricing under risk-neutral measure:

$$V(t,S) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\{V(T,S(T))|S(t)\}$$

- Quadrature:

$$V(t,S) = \mathrm{e}^{-r(T-t)}\int_{\mathbb{R}} V(T,S(T))f(S(T)|S(t))\mathrm{d}S(t)$$

- Transitional PDF, $f(S(T)|S(t))$, typically not available, but the characteristic function often is.

Analytic Solution of BS prices

Theorem (Black-Scholes formula)

The price of a European call option with strike price K and maturity T is given by the formula:

$$
\begin{aligned}
V_c(t, S(t)) &= S(t) \cdot \mathcal{N}(d_1(t, S)) - \mathrm{e}^{-r(T-t)} K \cdot \mathcal{N}(d_2(t, S(t))), \text{ with} \\
d_1(t, S(t)) &= \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{K} + \left(r + \frac{1}{2}\sigma^2 \right)(T-t) \right), \\
d_2(t, S(t)) &= d_1(t, S(t)) - \sigma\sqrt{T-t},
\end{aligned}
$$

where $\mathcal{N}$ is the cumulative distribution function for standard normal distribution i.e., $\mathcal{N}(0,1)$.

Summary of Black-Scholes model

In the Black-Scholes formula we have:

- time to maturity: T (known)
- strike : K (known)
- risk free rate: r (known)
- current underlying price: S_0 (known)

What about σ ? Risk is a driving factor for options so under normal circumstances the option's theoretical value is a monotonically increasing function of the volatility. This means there is a one-to-one relationship between the option price and the volatility.

How we can test whether this holds in the reality?

Option Quotations in the Market

- Let us refresh how options are quoted.

Calls									Puts							
Bid	Ask	Last	Volume	Opn Int	Opint Nch	ImpVol Mid	Delta Mid	Strike	Bid	Ask	Last	Volume	Opn Int	Opint Nch	ImpVol Mid	Delta Mid
								31-Mar-2021								
143.4	144	111.05		41	0	20.4368	0.5176	3790	143.1	143.7	225.67		43	0	20.4118	-0.4795
140.2	141	115.5		70	0	20.3359	0.5119	3795	144.9	145.6	213.73		1	0	20.3041	-0.4852
137.3	137.9	144.81	9	6,951	-159	20.2454	0.5063	3800	146.9	147.6	142.6		876	400	20.2282	-0.4908
134	134.6	142.34		17	1	20.1083	0.5005	3805	148.9	149.6	149.24	1	6	0	20.1199	-0.4966
131.1	131.8	134.9	1	444	0	20.0313	0.4947	3810	150.9	151.6	143.46	4	22	1	20.0219	-0.5023
								30-Jun-2021								
210.5	211.6			0	0	21.0256	0.5176	3790	220.1	221.6	218.81	3	0	0	20.9539	-0.4763
207.3	208.5			0	0	20.9436	0.5139	3795	222	223.6	236.44		58	0	20.8820	-0.4801
204.2	205.4	197.62	10	3,920	0	20.8647	0.5100	3800	223.8	225.4	215.1	2	1,573	350	20.8085	-0.4839
201.2	202.3			0	0	20.7889	0.5062	3805	225.8	227.4	235.54		58	0	20.7234	-0.4878
198.1	199.3			0	0	20.7114	0.5023	3810	227.7	229.3	227.75		10	0	20.6415	-0.4916
								30-Sep-2021								
281.8	286.1	278.29		905	2	21.6065	0.5430	3750	262.3	266.2	261.41	1	299	1	21.6301	-0.4478
266	270.1	262.92		231	0	21.2738	0.5281	3775	271.2	275.3	292.62		98	0	21.2921	-0.4627
250.5	254.5	194.3		951	0	20.9421	0.5129	3800	280.6	284.8	308.49		27	0	20.9630	-0.4780
235.4	239.3	203.73		252	0	20.6141	0.4972	3825	290.3	294.6	308.64		2	0	20.6299	-0.4936
220.7	224.5	221.8		599	0	20.2888	0.4811	3850	300.4	304.9	326.72		3	0	20.3034	-0.5097
								31-Dec-2021								
319.1	326.9	326.21	1	448	0	21.6803	0.5400	3750	309.2	316.2	303.34		16	0	21.5757	-0.4476
303.5	311.1	308.11		70	25	21.3855	0.5271	3775	318.3	325.5	315.1		46	0	21.2776	-0.4606
288.2	295.7	284.8		114	0	21.0950	0.5139	3800	327.7	335.1	322		8	6	20.9803	-0.4738
273.2	280.5	276.8	100	104	0	20.8013	0.5005	3825	337.4	345	327.7		11	10	20.6831	-0.4874
258.6	265.8	268.45	3	41	0	20.5173	0.4868	3850	347.4	355.2	361.52		2	0	20.3853	-0.5012

Figure: Call and Put options for S&P index, spot is about **3800**.

Market Implied volatility

Example:

Suppose we have given a standard call option V_C on 100 shares of company Z. The strike is \$75 and it expires in 55 days. The risk free rate is 5%. The current stock price is \$85, and from historical data we have obtained $\sigma_{hist} = 0.25$. So, the call price is given by BS model is:

$$BS(\sigma_{hist}, r, T, K, S_0) = BS(25\%, 5\%, \frac{55}{365}, 75, 85) = 10.8667$$

But in the market the price of such a call option is \$12.25.

What does it mean? An arbitrage?

Market Implied volatility

Example Continuation

Based on the standard BS pricing model, we find the volatility implied by the market price V_C to be 43.89% i.e.,

$$\sigma_{market} = g(V_C^{mkt}) = 43.89\%.$$

In order to check the calculation we substitute the σ_{market} to the pricing model, i.e.,

$$\begin{aligned} V_c(t,S) &= BS(\sigma_{market}, r, T, K, S_0) \\ &= BS(43.89\%, 5\%, \frac{55}{365}, \$75, \$85) = \$12.25 \end{aligned}$$

How to find implied volatility, $\sigma_{impl} = \sigma_{market}$?

Market Implied volatility

Implied Volatility: *"the wrong number in the wrong formula to get the right price".* [Rebonato 1999]

Mathematically we have:

$$V_c(t, S) = BS(\sigma, r, T, K, S_0)$$

where BS is monotonically increasing in σ (higher volatility corresponds to higher prices). Now, assume the existence of some inverse function

$$g_\sigma(\cdot) = BS^{-1}(\cdot)$$

so that

$$\sigma_{impl} = g_\sigma(V_c^{mkt}, r, T, K, S_0)$$

By computing the implied volatility for traded options with different strikes and maturities, we can test the Black-Scholes model.

Solving the inverse pricing model function

How to find implied volatility?

The BS pricing function BS does not have a closed-form solution for its inverse $g_\sigma(\cdot)$. Instead, a root finding technique is used to solve the equation:

$$BS(\sigma_{impl}, r, T, K, S_0) - V_C^{mkt} = 0.$$

There are many ways to solve this equation, one of the most popular method are methods of "Newton" and "Brent"[1]. Since the options prices can move very quickly, it is often important to use the most efficient method when calculating implied volatilities.

[1] http://en.wikipedia.org/wiki/Brent's_method

Method of Newton

- Suppose an approximation x_n. Assume that g is differentiable, and write $\epsilon_n = x_{ex} - x_n$, a Taylor series expansion gives:

$$0 = g(x_{ex}) = g(x_n) + \epsilon_n g'(x_n) + \frac{\epsilon_n^2}{2} g''(x_n) + \ldots$$

- Ignoring the terms of second order and higher,

$$x_{ex} \approx x_n - \frac{g(x_n)}{g'(x_n)}.$$

- Use this expression as our new approximation:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

- How fast does the error reduce with this approximation? With one additional term in the Taylor series expansion, we get:

$$\epsilon_{n+1} = x_{ex} - x_{n+1} = \epsilon_n + \frac{g(x_n) - g(x_{ex})}{g'(x_n)} \approx -\epsilon_n^2 \frac{g''(x_n)}{2g'(x_n)}$$

Method of Newton

One starts with an initial guess which is reasonably close to the true root. The function is then approximated by its tangent line, and one computes the x-intercept of this tangent line.
Suppose $g : [a, b] \to \mathbb{R}$ is a differentiable function. From basic calculus we have:

$$g'(x_n) = \frac{g(x_n) - 0}{x_n - x_{n+1}} = \frac{0 - g(x_n)}{x_{n+1} - x_n},\ n = 0, 1, \ldots \tag{1}$$

which gives us the iteration:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)},\ n = 0, 1, \ldots \tag{2}$$

with some arbitrary initial value x_0. In the case of BS we have:

$$\sigma_{n+1} = \sigma_n - \frac{BS(\sigma_n, \cdot) - V_C^{mkt}}{\frac{\partial BS(\sigma_n, \cdot)}{\partial \sigma_n}},\ n = 0, 1, \ldots.$$

Method of Newton

In Equation (1) we ignore the $\mathcal{O}(h^2)$ term, so we expect the error $x_n - x_{ex}$ squares as n increases to $n+1$; that is:
if

$$x_n - x_{ex} = \mathcal{O}(h)$$

then

$$x_{n+1} - x_{ex} = \mathcal{O}(h^2)$$

The analysis can be formalized to give the following result:

Theorem

Suppose g has a continuous second derivative, and suppose $x_{ex} \in \mathbb{R}$ satisfies $g(x_{ex}) = 0$ and $g'(x_{ex}) \neq 0$. Then there exists a $\delta > 0$ such that for $|x_0 - x_{ex}| < \delta$ the sequence given in (2) is defined for all $n > 0$, $\lim_{n\to+\infty} |x_n - x_{ex}| = 0$, and there exists a constant C such that:

$$|x_{n+1} - x_{ex}| \leq C|x_n - x_{ex}|^2.$$

Method of Newton

- Implementation in Python.

Implied volatility and Black-Scholes

Why is the implied volatility important?

Implied volatility: Model

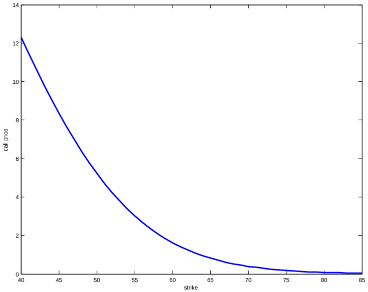

Figure: MODEL–LEFT: BS Call Prices, RIGHT: Implied Volatilities.

Implied volatility and Black-Scholes

Why is the implied volatility important?

Implied volatility: Market

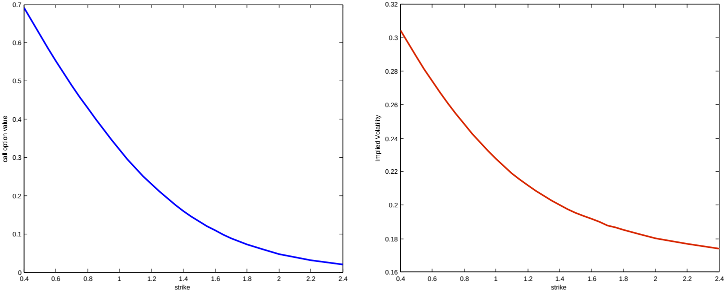

Figure: MARKET DATA– LEFT: Market Call Prices, RIGHT: Implied Volatilities.

Implied Volatility Shapes

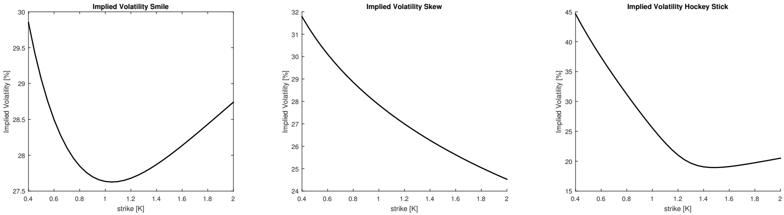

Figure: Typical implied volatility shapes: a smile, a skew and the so-called hockey stick. The hockey stick can be seen as a combination of the implied volatility smile and the skew.

Implied Volatility Term Structure

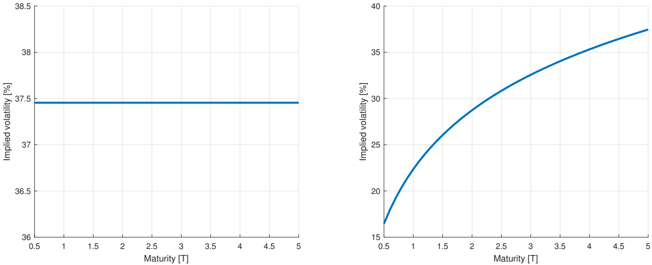

Figure: Comparison of the volatility term structure (for ATM volatilities) for Black-Scholes model with constant volatility σ_* (left-side picture) versus a model with time-dependent volatility $\sigma(t)$ (right-side picture).

Implied Volatility Surface

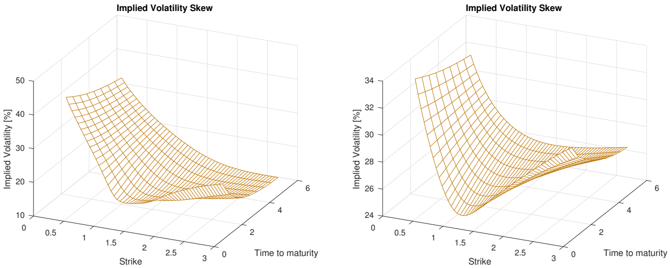

Figure: Implied volatility surfaces. A pronounced smile for the short maturity and a pronounced skew for longer maturities T (left side figure), and a pronounced smile over all maturities (right side).

Deficiencies of the Black-Scholes Model

- The Black-Scholes model, and its notion of hedging option contracts by stocks and money, forms the foundation of modern finance. However:
- Delta hedging is supposed to be a continuous process, but in practice it is a discrete process (a hedged portfolio is typically updated once a week or so).
- Empirical studies of financial time series have revealed that the normality assumption for asset prices cannot capture *heavy tails* and *asymmetries*, present in log-asset returns in practice.
- The volatility is supposed to be a known deterministic function of time, which is inconsistent since numerical inversion of the Black-Scholes equation based on market prices from different strikes and fixed maturity, produces a so-called *volatility skew or smile*.

Deficiencies of the Black-Scholes Model

$\Rightarrow$ The idea of implied volatility does not fit to the Black-Scholes model

- Look for market consistent asset price models.

$\Rightarrow$ Use a local volatility, a model with jumps, or stochastic volatility to better fit market data, and incorporate smile effects