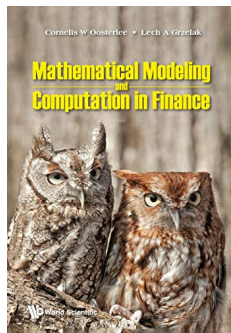


Materials for the course

The course is based on book “*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*”, by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go [here](#).



- ▶ YouTube Channel with courses can be found [here](#).
- ▶ Slides and the codes can be found [here](#).

List of content

- Key Elements when Pricing Derivatives
- Black-Scholes Implied Volatility
- Newton-Raphson Method and Implementation in Python
- Time-Dependent Volatility Parameter
- Implied Volatility Surface
- Deficiencies of the Black-Scholes Model

Block 1: Model Specification and Pricing PDE

We already know:

For a given market, described by the equations:

$$\begin{cases} dM(t) &= rM(t)dt \\ dS(t) &= \mu S(t)dt + \sigma S(t)dW^{\mathbb{P}}(t), \end{cases}$$

and a contingent claim of the form

$$\chi = V(T, S(T)),$$

the arbitrage free price is given, via Itô's Lemma, by $V(t, S)$, where function $V(t, S)$ satisfies the Black-Scholes equation:

$$\begin{cases} \frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 V}{\partial S^2} - rV &= 0 \\ V(T, S) &= \chi. \end{cases}$$

Block 2: Relation to Monte Carlo via Feynman-Kac Theorem

$V(t, S)$ is the unique solution of the final condition problem

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \\ V(T, S) = \text{given} \end{cases}$$

This solution can also be obtained as

$$V(t, S) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\{V(T, S(T)) | S(t)\}$$

with the sum of the first derivatives of the option square integrable.
Given the system of stochastic differential equations:

$$dS(t) = rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t).$$

Block 3: Pricing via Integrals

- ▶ Pricing under risk-neutral measure:

$$V(t, S) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \{ V(T, S(T)) | S(t) \}$$

- ▶ Quadrature:

$$V(t, S) = e^{-r(T-t)} \int_{\mathbb{R}} V(T, S(T)) f(S(T) | S(t)) dS(t)$$

- ▶ Transitional PDF, $f(S(T) | S(t))$, typically not available, but the characteristic function often is.

Analytic Solution of BS prices

Theorem (Black-Scholes formula)

The price of a European call option with strike price K and maturity T is given by the formula:

$$V_c(t, S(t)) = S(t) \cdot \mathcal{N}(d_1(t, S)) - e^{-r(T-t)} K \cdot \mathcal{N}(d_2(t, S(t))), \text{ with}$$

$$d_1(t, S(t)) = \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S(t)}{K} + \left(r + \frac{1}{2}\sigma^2 \right) (T-t) \right),$$

$$d_2(t, S(t)) = d_1(t, S(t)) - \sigma\sqrt{T-t},$$

where \mathcal{N} is the cumulative distribution function for standard normal distribution i.e., $\mathcal{N}(0, 1)$.

Summary of Black-Scholes model

In the Black-Scholes formula we have:

- ▶ time to maturity: T (known)
- ▶ strike : K (known)
- ▶ risk free rate: r (known)
- ▶ current underlying price: S_0 (known)

What about σ ? Risk is a driving factor for options so under normal circumstances the option's theoretical value is a monotonically increasing function of the volatility. This means there is a one-to-one relationship between **the option price** and **the volatility**.

How we can test whether this holds in the reality?

Option Quotations in the Market

- Let us refresh how options are quoted.

Calls									Puts									
	Bid	Ask	Last	Volume	Open Int	Open Nch	ImpVol Mid	Delta Mid	Strike		Bid	Ask	Last	Volume	Open Int	Open Nch	ImpVol Mid	Delta Mid
+ 31-Mar-2021																		
☞	143.4	144	111.05		41	0	20.4368	0.5176	3750	☞	143.1	143.7	225.67		43	0	20.4118	-0.4795
☞	140.2	141	115.5		70	0	20.3359	0.5119	3755	☞	144.9	145.6	213.73		1	0	20.3041	-0.4852
☞	137.3	137.5	144.81	9	6.951	-159	20.2454	0.5063	3800	☞	146.9	147.6	142.6		876	400	20.2282	-0.4908
☞	134	134.6	142.34		17	1	20.1083	0.5005	3805	☞	148.9	149.6	149.24	1	6	0	20.1199	-0.4966
☞	131.1	131.8	134.9	1	444	0	20.0313	0.4947	3810	☞	150.9	151.6	143.46	4	22	1	20.0219	-0.5023
+ 30-Jun-2021																		
☞	210.5	211.6			0	0	21.0256	0.5176	3750	☞	220.1	221.6	218.81	3	0	0	20.9539	-0.4763
☞	207.3	208.5			0	0	20.9436	0.5139	3755	☞	222	223.6	236.44		58	0	20.8820	-0.4801
☞	204.2	205.4	197.62	10	3.920	0	20.8647	0.5100	3800	☞	223.8	225.4	215.1	2	1,573	350	20.8085	-0.4839
☞	201.2	202.3			0	0	20.7889	0.5062	3805	☞	225.8	227.4	235.54		58	0	20.7234	-0.4878
☞	198.1	199.3			0	0	20.7114	0.5023	3810	☞	227.7	229.3	227.75		10	0	20.6415	-0.4916
+ 30-Sep-2021																		
☞	281.8	286.1	278.29		905	2	21.6095	0.5430	3750	☞	262.3	266.2	261.41	1	299	1	21.6301	-0.4478
☞	266	270.1	262.92		231	0	21.2738	0.5281	3775	☞	271.2	275.3	292.62		98	0	21.2921	-0.4627
☞	250.5	254.5	194.3		951	0	20.9421	0.5129	3800	☞	280.6	284.8	308.49		27	0	20.9630	-0.4780
☞	235.4	239.3	203.73		252	0	20.6141	0.4972	3825	☞	290.3	294.6	308.64		2	0	20.6299	-0.4936
☞	220.7	224.5	221.8		599	0	20.2888	0.4811	3850	☞	300.4	304.9	326.72		3	0	20.3034	-0.5097
+ 31-Dec-2021																		
☞	319.1	326.9	326.21	1	448	0	21.6803	0.5400	3750	☞	309.2	316.2	303.34		16	0	21.5757	-0.4476
☞	303.5	311.1	308.11		70	25	21.3895	0.5271	3775	☞	318.3	325.5	315.1		46	0	21.2776	-0.4606
☞	288.2	295.7	284.8		114	0	21.0950	0.5139	3800	☞	327.7	335.1	322		8	6	20.9803	-0.4738
☞	273.2	280.5	276.8	100	104	0	20.8013	0.5005	3825	☞	337.4	345	327.7		11	10	20.6831	-0.4874
☞	258.6	265.8	268.45	3	41	0	20.5173	0.4868	3850	☞	347.4	355.2	361.52		2	0	20.3853	-0.5012

Figure: Call and Put options for S&P index, spot is about 3800.

Market Implied volatility

Example:

Suppose we have given a standard call option V_C on 100 shares of company Z. The strike is \$75 and it expires in 55 days. The risk free rate is 5%. The current stock price is \$85, and from historical data we have obtained $\sigma_{hist} = 0.25$. So, the call price is given by BS model is:

$$BS(\sigma_{hist}, r, T, K, S_0) = BS(25\%, 5\%, \frac{55}{365}, 75, 85) = 10.8667$$

But in the market the price of such a call option is \$12.25.

What does it mean? An arbitrage?

Market Implied volatility

Example Continuation

Based on the standard BS pricing model, we find the volatility **implied** by the market price V_C to be **43.89%** i.e.,

$$\sigma_{market} = g(V_C^{mkt}) = 43.89\%.$$

In order to check the calculation we substitute the σ_{market} to the pricing model, i.e.,

$$\begin{aligned} V_C(t, S) &= BS(\sigma_{market}, r, T, K, S_0) \\ &= BS(43.89\%, 5\%, \frac{55}{365}, \$75, \$85) = \$12.25 \end{aligned}$$

How to find implied volatility, $\sigma_{impl} = \sigma_{market}$?

Market Implied volatility

Implied Volatility: "the wrong number in the wrong formula to get the right price". [Rebonato 1999]

Mathematically we have:

$$V_c(t, S) = BS(\sigma, r, T, K, S_0)$$

where BS is monotonically increasing in σ (higher volatility corresponds to higher prices). Now, assume the existence of some inverse function

$$g_\sigma(\cdot) = BS^{-1}(\cdot)$$

so that

$$\sigma_{impl} = g_\sigma(V_c^{mkt}, r, T, K, S_0)$$

By computing the implied volatility for traded options with different strikes and maturities, we can test the Black-Scholes model.

Solving the inverse pricing model function

How to find implied volatility?

The BS pricing function BS does not have a closed-form solution for its inverse $g_{\sigma}(\cdot)$. Instead, a root finding technique is used to solve the equation:

$$BS(\sigma_{impl}, r, T, K, S_0) - V_C^{mkt} = 0.$$

There are many ways to solve this equation, one of the most popular method are methods of "Newton" and "Brent"¹. Since the options prices can move very quickly, it is often important to use the most efficient method when calculating implied volatilities.

¹http://en.wikipedia.org/wiki/Brent's_method

Method of Newton

- Suppose an approximation x_n . Assume that g is differentiable, and write $\epsilon_n = x_{\text{ex}} - x_n$, a Taylor series expansion gives:

$$0 = g(x_{\text{ex}}) = g(x_n) + \epsilon_n g'(x_n) + \frac{\epsilon_n^2}{2} g''(x_n) + \dots$$

- Ignoring the terms of second order and higher,

$$x_{\text{ex}} \approx x_n - \frac{g(x_n)}{g'(x_n)}.$$

- Use this expression as our new approximation:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

- How fast does the error reduce with this approximation? With one additional term in the Taylor series expansion, we get:

$$\epsilon_{n+1} = x_{\text{ex}} - x_{n+1} = \epsilon_n + \frac{g(x_n) - g(x_{\text{ex}})}{g'(x_n)} \approx -\epsilon_n^2 \frac{g''(x_n)}{2g'(x_n)}$$

Method of Newton

One starts with an initial guess which is reasonably close to the true root. The function is then approximated by its tangent line, and one computes the x-intercept of this tangent line.

Suppose $g : [a, b] \rightarrow \mathbb{R}$ is a differentiable function. From basic calculus we have:

$$g'(x_n) = \frac{g(x_n) - 0}{x_n - x_{n+1}} = \frac{0 - g(x_n)}{x_{n+1} - x_n}, \quad n = 0, 1, \dots \quad (1)$$

which gives us the iteration:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \quad n = 0, 1, \dots \quad (2)$$

with some arbitrary initial value x_0 . In the case of BS we have:

$$\sigma_{n+1} = \sigma_n - \frac{BS(\sigma_n, \cdot) - V_C^{mkt}}{\frac{\partial BS(\sigma_n, \cdot)}{\partial \sigma_n}}, \quad n = 0, 1, \dots$$

Method of Newton

In Equation (1) we ignore the $\mathcal{O}(h^2)$ term, so we expect the error $x_n - x_{\text{ex}}$ squares as n increases to $n + 1$; that is: if

$$x_n - x_{\text{ex}} = \mathcal{O}(h)$$

then

$$x_{n+1} - x_{\text{ex}} = \mathcal{O}(h^2)$$

The analysis can be formalized to give the following result:

Theorem

Suppose g has a continuous second derivative, and suppose $x_{\text{ex}} \in \mathbb{R}$ satisfies $g(x_{\text{ex}}) = 0$ and $g'(x_{\text{ex}}) \neq 0$. Then there exists a $\delta > 0$ such that for $|x_0 - x_{\text{ex}}| < \delta$ the sequence given in (2) is defined for all $n > 0$, $\lim_{n \rightarrow +\infty} |x_n - x_{\text{ex}}| = 0$, and there exists a constant C such that:

$$|x_{n+1} - x_{\text{ex}}| \leq C|x_n - x_{\text{ex}}|^2.$$

Method of Newton

- ▶ Implementation in Python.

Implied volatility and Black-Scholes

Why is the implied volatility important?

Implied volatility: Model

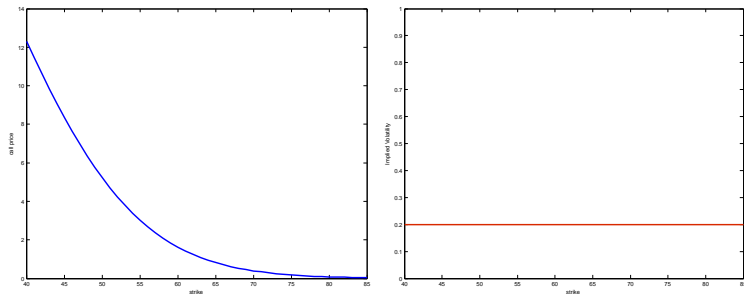


Figure: MODEL-LEFT: BS Call Prices, RIGHT: Implied Volatilities.

Implied volatility and Black-Scholes

Why is the implied volatility important?

Implied volatility: Market

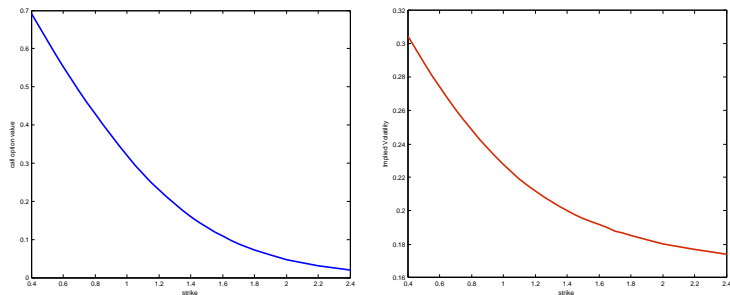


Figure: MARKET DATA– LEFT: Market Call Prices, RIGHT: Implied Volatilities.

Implied Volatility Shapes

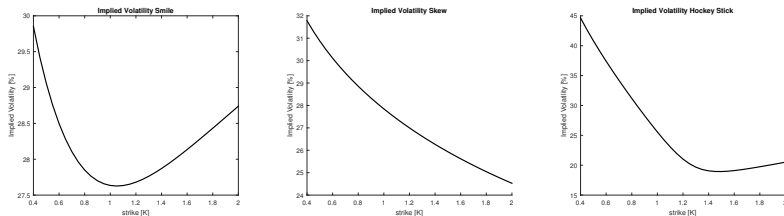


Figure: Typical implied volatility shapes: a smile, a skew and the so-called hockey stick. The hockey stick can be seen as a combination of the implied volatility smile and the skew.

Implied Volatility Term Structure

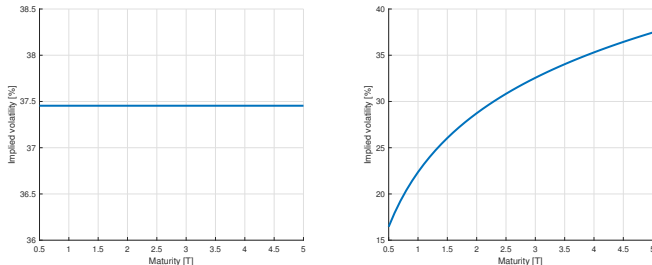


Figure: Comparison of the volatility term structure (for ATM volatilities) for Black-Scholes model with constant volatility σ_* (left-side picture) versus a model with time-dependent volatility $\sigma(t)$ (right-side picture).

Implied Volatility Surface

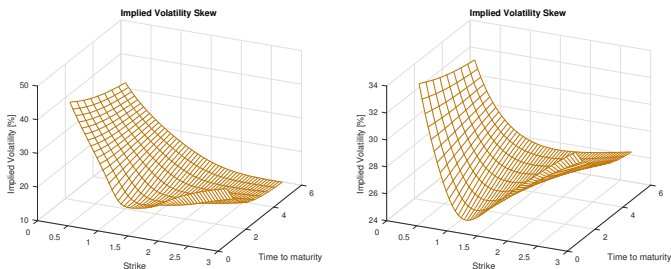


Figure: Implied volatility surfaces. A pronounced smile for the short maturity and a pronounced skew for longer maturities T (left side figure), and a pronounced smile over all maturities (right side).

Deficiencies of the Black-Scholes Model

- ▶ The Black-Scholes model, and its notion of hedging option contracts by stocks and money, forms the foundation of modern finance.

However:

- ▶ Delta hedging is supposed to be a continuous process, but in practice it is a discrete process (a hedged portfolio is typically updated once a week or so).
- ▶ Empirical studies of financial time series have revealed that the normality assumption for asset prices cannot capture *heavy tails* and *asymmetries*, present in log-asset returns in practice.
- ▶ The volatility is supposed to be a known deterministic function of time, which is inconsistent since numerical inversion of the Black-Scholes equation based on market prices from different strikes and fixed maturity, produces a so-called *volatility skew or smile*.

Deficiencies of the Black-Scholes Model

- ⇒ The idea of implied volatility does not fit to the Black-Scholes model
 - ▶ Look for market consistent asset price models.
- ⇒ Use a **local volatility**, a **model with jumps**, or **stochastic volatility** to better fit market data, and incorporate smile effects