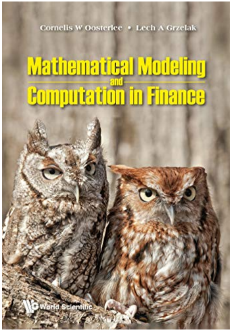
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- YouTube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

Delta Hedging under the Black-Scholes Model

- Based on the same strategy, consider the following portfolio:

$$\Pi(t,S) = V(t,S) - \Delta S. \tag{1}$$

- The objective of delta hedging is that the value of the portfolio does not change when the underlying asset moves, so the derivative of portfolio $\Pi(t,S)$ w.r.t S needs to be equal to 0, i.e.,

$$\frac{\partial \Pi(t,S)}{\partial S} = \frac{\partial V(t,S)}{\partial S} - \Delta = 0 \Rightarrow \Delta = \frac{\partial V}{\partial S}, \tag{2}$$

  with $V = V(t,S)$.
- Suppose we sold a call $V_c(t_0,S)$ at time t_0, with maturity T and strike K. By selling, we obtained a cash amount equal to $V_c(t_0,S)$ and perform a *dynamic* hedging strategy until time T. Initially, at the inception time, we have

$$\Pi(t_0,S) := V_c(t_0,S) - \Delta(t_0)S_0.$$

Delta Hedging under the Black-Scholes Model

- This value may be negative when $\Delta(t_0)S_0 > V_c(t_0, S)$.
- If funds are needed for buying $\Delta(t)$ shares, we make use of a *funding account*, $\text{PnL}(t) \equiv \text{P\&L}(t)$. $\text{PnL}(t)$ represents the total value of the option sold and the hedge, and it keeps track of the changes in the asset value $S(t)$.
- Typically the funding amount $\Delta(t_0)S_0$ is then obtained from a trading desk of a treasury department.
- Every day we may need to re-balance the position and hedge the portfolio. At some time $t_1 > t_0$, we then receive (or pay) interest over the time period $[t_0, t_1]$, which will amount to $\text{P\&L}(t_0)\mathrm{e}^{r(t_1-t_0)}$.

Delta Hedging under the Black-Scholes Model

- At t_1 we have $\Delta(t_0)S(t_1)$ which may be sold, and we will update the hedge portfolio. Particularly, we purchase $\Delta(t_1)$ stocks, costing $-\Delta(t_1)S(t_1)$. The overall P&L(t_1) account will become:

$$\text{P\&L}(t_1) = \underbrace{\text{P\&L}(t_0)\mathrm{e}^{r(t_1-t_0)}}_{\text{interest}} \underbrace{-\left(\Delta(t_1)-\Delta(t_0)\right)S(t_1)}_{\text{borrow}}. \quad (3)$$

- Assuming a time grid with $t_i = i\frac{T}{m}$, the following recursive formula for the m time steps is obtained,

$$\begin{aligned} \text{P\&L}(t_0) &= V_c(t_0,S)-\Delta(t_0)S(t_0), \quad (4)\\ \text{P\&L}(t_i) &= \text{P\&L}(t_{i-1})\mathrm{e}^{r(t_i-t_{i-1})}-\left(\Delta(t_i)-\Delta(t_{i-1})\right)S(t_i), \end{aligned}$$

for $i = 1,\ldots,m-1$.

Delta Hedging under the Black-Scholes Model

- At the option maturity time T, the option holder may exercise the option or the option will expire worthless. As the option writer, we will thus encounter a cost equal to the option's payoff at $t_m = T$, i.e. $\max(S(T) - K, 0)$.
- On the other hand, at maturity time we own $\Delta(t_m)$ stocks, that may be sold in the market. The value of the portfolio at maturity time $t_m = T$ is then given by:

$$\begin{aligned} \text{P\&L}(t_m) &= \text{P\&L}(t_{m-1})\mathrm{e}^{r(t_m - t_{m-1})} \\ &- \underbrace{\max(S(t_m) - K, 0)}_{\text{option payoff}} + \underbrace{\Delta(t_{m-1})S(t_m)}_{\text{sell stocks}}. \end{aligned} \quad (5)$$

Delta Hedging under the Black-Scholes Model

- In a perfect world, with continuous re-balancing, the P&L(T) would equal zero on average, i.e. $\mathbb{E}[\text{P\&L}(T)] = 0$.
- One may question the reasoning behind dynamic hedging if the profit made by the option writer on average equals zero. The profit in option trading, especially with OTC transactions, is to charge an additional fee (often called a "*spread*") at the start of the contract.
- At time t_0 the cost for the client is not $V_c(t_0, S)$ but $V_c(t_0, S) + spread$, where $spread > 0$ would be the profit for the writer of the option.

Example: Expected *P&L*

- We will give an example where $\mathbb{E}[\text{P\&L}(T)|\mathcal{F}(t_0)] = 0$. We consider a three-period case, with $t_0, t_1, t_2 := T$, and $\Delta(t_i)$ is a deterministic function, as in the Black-Scholes hedging case. At t_0 an option is sold, with expiry date t_2 and strike price K. In a three-period setting, we have three equations for $\text{P\&L}(t)$ related to the initial hedging, the re-balancing and the final hedging, i.e.,

$$
\begin{aligned}
\text{P\&L}(t_0) &= V_c(t_0, S) - \Delta(t_0)S(t_0),\\
\text{P\&L}(t_1) &= \text{P\&L}(t_0)\mathrm{e}^{r(t_1-t_0)} - \left(\Delta(t_1) - \Delta(t_0)\right)S(t_1),\\
\text{P\&L}(t_2) &= \text{P\&L}(t_1)\mathrm{e}^{r(t_2-t_1)} - \max(S(t_2) - K, 0) + \Delta(t_1)S(t_2).
\end{aligned}
$$

- After collecting all terms, we find,

$$
\begin{aligned}
\text{P\&L}(t_2) &= \Big[\left(V_c(t_0, S) - \Delta(t_0)S(t_0)\right)\mathrm{e}^{r(t_2-t_0)}\\
&- \left(\Delta(t_1) - \Delta(t_0)\right)S(t_1)\mathrm{e}^{r(t_2-t_1)}\Big]\\
&- \max(S(t_2) - K, 0) + \Delta(t_1)S(t_2).
\end{aligned}
$$

Example: Expected $P\&L$

- By the definition of a call option, we also have,

$$\mathbb{E}\left[\max(S(t_2)-K,0)\middle|\mathcal{F}(t_0)\right]=\mathrm{e}^{r(t_2-t_0)}V_c(t_0,S), \tag{6}$$

and because the discounted stock price, under the risk-neutral measure, is a martingale, $\mathbb{E}[S(t)|\mathcal{F}(s)]=S(s)\mathrm{e}^{r(t-s)}$, the expectation of the P&L is given by:

$$\begin{aligned}\mathbb{E}[\mathrm{P\&L}(t_2)|\mathcal{F}(t_0)] &= \big(V_c(t_0,S)-\mathbb{E}[\max(S(t_2)-K,0)|\mathcal{F}(t_0)]\\ &+ \Delta(t_1)\mathbb{E}[S(t_2)|\mathcal{F}(t_0)]-\Delta(t_0)S(t_0)\big)\cdot\mathrm{e}^{r(t_2-t_0)}\\ &- (\Delta(t_1)-\Delta(t_0))\,\mathbb{E}[S(t_1)|\mathcal{F}(t_0)]\mathrm{e}^{r(t_2-t_1)}.\end{aligned}$$

- Using the relation

$$\mathbb{E}[S(t_1)|\mathcal{F}(t_0)]\mathrm{e}^{r(t_2-t_1)}=\mathbb{E}[S(t_2)|\mathcal{F}(t_0)]=S(t_0)\mathrm{e}^{r(t_2-t_0)},$$

and by (6), the expression simplifies to,

$$\mathbb{E}[\mathrm{P\&L}(t_2)|\mathcal{F}(t_0)]=-\Delta(t_1)S(t_0)\mathrm{e}^{r(t_2-t_0)}+\Delta(t_1)\mathbb{E}[S(t_2)|\mathcal{F}(t_0)]=0.$$

Dynamic Hedging with the Black-Scholes: Experiment

- In this experiment we perform a dynamic hedge for a call option under the Black-Scholes model. For the asset price, the following model parameters are set, $S(t_0) = 1$, $r = 0.1$, $\sigma = 0.2$. The option's maturity is $T = 1$ and strike $K = 0.95$.
- On a time grid stock path $S(t_i)$ is simulated. Based on these paths, we perform the hedging strategy, according to Equations (4) and (5).
- In Figures three stock paths $S(t)$ are presented, for one of them the option would be in-the-money at time T (upper left), one ends out-of-money (upper right) and is at-the-money (lower left) at time T.

Dynamic Hedging with the Black-Scholes: Experiment

- In the three graphs $\Delta(t_i)$ (green line) behaves like the stock process $S(t)$, however when the stock $S(t)$ give a call price (pink line) deep in or out of the money, $\Delta(t_i)$ is either 0 or 1.

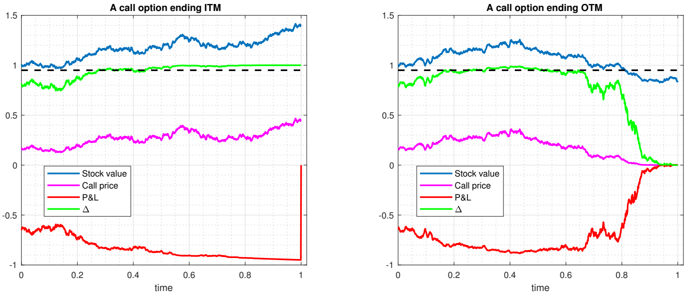


Figure: Delta hedging a call option. Blue: the stock path, pink: the value of a call option, red: P&L(t) portfolio, and green: Δ.

Dynamic Hedging with the Black-Scholes: Experiment

- The impact of the frequency of updating the hedge portfolio on the distribution of the P&L(T) is presented. Two simulations have been performed, one with 10 re-balancing actions during the option's life time and one with 2000 actions. It is clear that frequent re-balancing brings the variance of the portfolio P&L(T) down to almost 0.

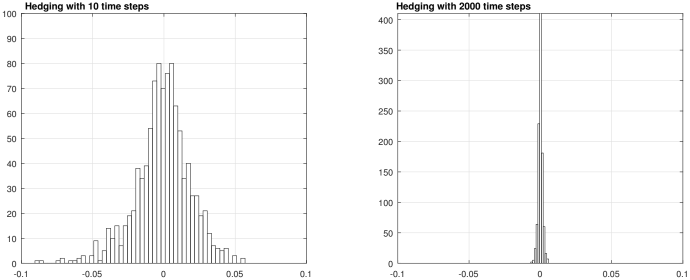


Figure: The impact of the re-balancing frequency on the variance of the P&L(T) portfolio. Left: 10 re-balancing times, Right: 2000 re-balancing times.

Dynamic Hedging in Presence of Jumps: Experiment

- Instead of the Black-Scholes model we consider a jump diffusion model:

$$\mathrm{d}S(t)/S(t) = \big(r - \xi_p\mathbb{E}[\mathrm{e}^J - 1]\big)\,\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{Q}}(t) + (\mathrm{e}^J - 1)\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t).$$

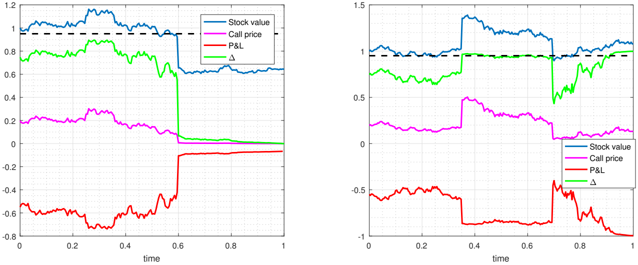


Figure: Delta hedging a call option for a stock with jumps, where we however work with the Black-Scholes delta. Blue: the stock path, pink: value of the call option, red: P&L(t) portfolio, and green: the Δ. Left: a path with one jump time, right: two occurring jumps.

Dynamic Hedging in Presence of Jumps: Experiment

- Let us not observe the impact of increased hedging frequency.

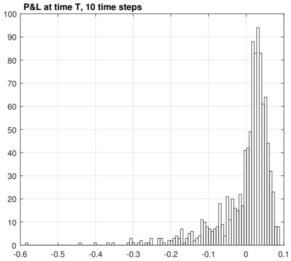


Figure: The impact of the hedging frequency on the variance of the P&L(T) portfolio, with the stock following a jump diffusion process. Left: 10 times hedging during the option lifetime, Right: 2000 times hedging.

- Let us check Python implementation.

Vega Hedging

- Vega is defined as a sensitivity of the value of a derivative with respect to volatility and for European Call and Put option it is defined as:

$$\text{vega} = \nu := \frac{\partial V}{\partial \sigma} = K\mathrm{e}^{-r(T-t_0)} f_{\mathcal{N}(0,1)}(d_2)\sqrt{T-t_0}.$$

- Vega is the same for Call and Put options.
- Since under the Black-Scholes model the volatility parameter σ is constant we do not need to vega hedge! In reality however σ would change.
- In practice to adjust the vega one needs to either buy or sell an option, or a derivative which is dependent on volatility: σ.
- Hedging of delta is typically more frequent than hedging of vega (hedging with options is considered to be expensive).
- Traders tend to perform hedging on a portfolio level. Then a benefit from netting can be seen and therefore the hedging costs are much lower.

Example for Delta and Vega Hedging

- Let us consider a portfolio of trades (stock + derivatives) with a delta of 50 and vega of 200.
- Our objective is to hedge Delta and Vega using underlying stock and a call option with a given strike K.
- Using the Black-Scholes model we computed that for the call option we have delta of 0.7 and vega of 10.
- Our objective is to find how many call options we need to buy to hedge our portfolio.
- We start with vega hedging. In order to mitigate vega effect of the portfolio we need to sell $200/10 = 20$ option contracts.
- Now we are vega-neutral but our delta position has changed and it is now equal to: $50 - 20x0.7 = 36$.
- Finally, to hedge the remaining delta effect we need to sell 36 stocks.

Gamma Hedging

- Gamma determines the change of Delta with respect to the price of the underlying asset.
- As for Vega, Gamma is the same for Calls and Puts and it is defined as:

$$\Gamma := \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2} = K\mathrm{e}^{-r(T-t_0)}\frac{f_{\mathcal{N}(0,1)}(d_2)}{S^2\sqrt{T-t_0}}.$$

- High gamma corresponds to high variation in delta, and hence more frequent re-balancing to maintain low delta.
- We cannot use stocks to hedge Gamma (Gamma for stocks is simply 0).
- As in Vega case we need to use options to hedge Gamma.

Monte Carlo Greeks I, Finite Differences

- With $V(\theta)$ (θ is a problem parameter) continuous and at least twice differentiable, we have for any $\Delta\theta > 0$,

$$V(\theta+\Delta\theta) = V(\theta) + \frac{\partial V}{\partial \theta}\Delta\theta + \frac{1}{2}\frac{\partial^2 V}{\partial \theta^2}\Delta\theta^2 + \dots. \tag{7}$$

- In the Monte Carlo framework, the sensitivity to θ is estimated as follows,

$$\frac{\partial V}{\partial \theta} \approx \frac{\bar{V}(\theta+\Delta\theta) - \bar{V}(\theta)}{\Delta\theta}, \tag{8}$$

  with $\bar{V}(\theta) = \frac{1}{N}\sum_{i=1}^{N} V_i(\theta)$.

- With the help of the Taylor expansion, we find, regarding central differencing,

$$\frac{\partial V}{\partial \theta} = \frac{V(\theta+\Delta\theta) - V(\theta-\Delta\theta)}{2\Delta\theta} + \mathcal{O}(\Delta\theta^2). \tag{9}$$

Black-Scholes delta and vega with finite differences

- For a call option the Black-Scholes delta and vega are simulated and compared to the analytic expressions.

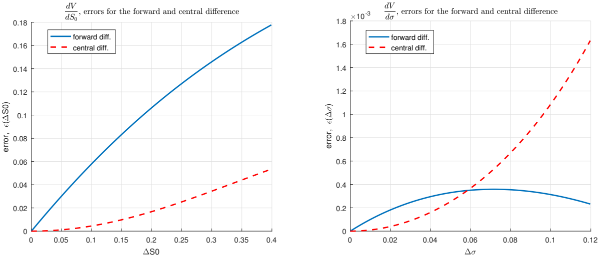


Figure: The governing parameters, $S(t_0) = 1$, $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = S(t_0)$.

Monte Carlo Greeks: Pathwise Sensitivities

- To efficiently estimate the sensitivities at time t_0, like with respect to $S(t_0)$ and also to other model parameters.
- The pathwise sensitivity method is applicable with continuous functions of the paramete of interest, and based on interchanging the differentiation and expectation operators,

$$\frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta}\mathbb{E}^{\mathbb{Q}}\left[\frac{V(T,S;\theta)}{M(T)}\Big|\mathcal{F}(t_0)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial \theta}\frac{V(T,S;\theta)}{M(T)}\Big|\mathcal{F}(t_0)\right]. \quad (10)$$

- Assuming a constant interest rate, we find,

$$\frac{\partial V}{\partial \theta} = \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial V(T,S;\theta)}{\partial S}\frac{\partial S}{\partial \theta}\Big|\mathcal{F}(t_0)\right]. \quad (11)$$

Example: Black-Scholes Delta and Vega

- As an example, we apply the pathwise sensitivity methodology from Equation (11) to a call option under the Black-Scholes model, i.e.,

  $V(T,S;\theta)=\max(S(T)-K,0),\text{with},S(T)=S(t_0)\mathrm{e}^{(r-\frac{1}{2}\sigma^2)(T-t_0)+\sigma(W(T)-W(t_0))}.$

- The derivative of the payoff with respect to $S(T)$ is given by,

$$\frac{\partial V}{\partial S(T)}=\mathbb{1}_{S(T)>K}, \tag{12}$$

and the necessary derivatives with respect to $S(t_0)$ and σ are as follows,

$$\begin{aligned}\frac{\partial S(T)}{\partial S(t_0)} &= \mathrm{e}^{(r-\frac{1}{2}\sigma^2)(T-t_0)+\sigma(W(T)-W(t_0))},\\ \frac{\partial S(T)}{\partial \sigma} &= S(T)\left(-\sigma(T-t_0)+W(T)-W(t_0)\right).\end{aligned}$$

Example: Black-Scholes Delta and Vega

- So, for the estimates of delta and vega we obtain,

$$
\begin{aligned}
\frac{\partial V}{\partial S(t_0)} &= \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{S(T)>K}\mathrm{e}^{(r-\frac{1}{2}\sigma^2)(T-t_0)+\sigma(W(T)-W(t_0))}\right] \\
&= \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)}{S(t_0)}\mathbb{1}_{S(T)>K}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial V}{\partial \sigma} &= \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{S(T)>K}S(T)\left(-\sigma(T-t_0)+W(T)-W(t_0)\right)\right] \\
&= \frac{\mathrm{e}^{-r(T-t_0)}}{\sigma}\mathbb{E}^{\mathbb{Q}}\left[S(T)\left(\log\left(\frac{S(T)}{S(t_0)}\right)-\left(r+\frac{1}{2}\sigma^2\right)(T-t_0)\right)\mathbb{1}_{S(T)>K}\right].
\end{aligned}
$$

Example: Black-Scholes Delta and Vega

- Numerical results

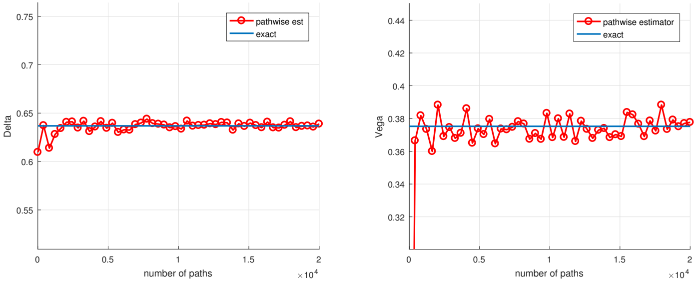


Figure: The Black-Scholes delta (left) and vega (right) estimated by the pathwise sensitivity method. The parameters are $S(t_0) = 1$, $r = 0.06$, $\sigma = 0.3$, $T = 1$, $K = S(t_0)$.

- Let us check the Python code.

Heston model's delta, pathwise sensitivity

- The pathwise sensitivity for Heston's delta parameter is given by,

$$\frac{\partial V}{\partial S(t_0)} = \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial V(T,S;S(t_0))}{\partial S(T)}\frac{\partial S(T)}{\partial S(t_0)}\Big|\mathcal{F}(t_0)\right]$$

- The derivative of the payoff function with respect to $S(T)$ reads,

$$\frac{\partial V(T,S;\theta)}{\partial S(T)} = \mathbb{1}_{S(T)>K}, \tag{13}$$

and the solution for the Heston process is then found to be,

$$S(T) = S(t_0)\exp\left[\int_{t_0}^{T}\left(r-\frac{1}{2}v(t)\right)\mathrm{d}t + \int_{t_0}^{T}\sqrt{v(t)}\mathrm{d}W_x(t)\right],$$

so that the sensitivity to $S(t_0)$ is given by,

$$\frac{\partial S(T)}{\partial S(t_0)} = \frac{S(T)}{S(t_0)}. \tag{14}$$

Heston model's delta, pathwise sensitivity

- As in the Black-Scholes model, the option Greek delta is given by the following expression,

$$\frac{\partial V}{\partial S(t_0)} \quad = \quad \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)}{S(t_0)}\mathbb{1}_{S(T)>K}\middle|\mathcal{F}(t_0)\right].$$

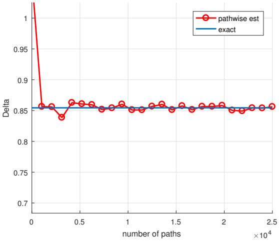


Figure: Convergence of the Heston model's delta, estimated by the pathwise sensitivity method, with parameters $\gamma = 0.5$, $\kappa = 0.5$, $\bar{v} = 0.04$, $\rho_{x,v} = -0.9$, $v_0 = 0.04$, $T = 1.0$, $S(t_0) = 100.0$, $r = 0.1$.

Monte Carlo Greeks III: Likelihood Ratio Method

- The basis of the method is formed by a differentiation of the probability density function.

$$V(\theta) \equiv V(t_0, S; \theta) \quad = \quad \mathrm{e}^{-r(T-t_0)} \int_{\mathbb{R}} V(T,z) f_{S(T)}(z;\theta) \mathrm{d}z, \tag{15}$$

- In order to compute the sensitivity with respect to parameter θ, the integration and differentiation operators are interchanged, i.e.,

$$\begin{aligned} \frac{\partial V}{\partial \theta} &= \mathrm{e}^{-r(T-t_0)} \frac{\partial}{\partial \theta} \int_{\mathbb{R}} V(T,z) f_{S(T)}(z;\theta) \mathrm{d}z \\ &= \mathrm{e}^{-r(T-t_0)} \int_{\mathbb{R}} V(T,z) \frac{\partial}{\partial \theta} f_{S(T)}(z;\theta) \mathrm{d}z. \end{aligned} \tag{16}$$

Likelihood Ratio Method

- The interchange of the integration and differentiation is typically fine, since probability density functions are usually smooth.

$$
\begin{aligned}
\frac{\partial V}{\partial \theta} &= \mathrm{e}^{-r(T-t_0)} \int_{\mathbb{R}} V(T,z) \frac{\partial f_{S(T)}(z;\theta)}{\partial \theta} \frac{f_{S(T)}(z;\theta)}{f_{S(T)}(z;\theta)} \mathrm{d}z \\
&= \mathrm{e}^{-r(T-t_0)} \int_{\mathbb{R}} V(T,z) \frac{\frac{\partial}{\partial \theta} f_{S(T)}(z;\theta)}{f_{S(T)}(z;\theta)} f_{S(T)}(z;\theta) \mathrm{d}z \\
&= \mathrm{e}^{-r(T-t_0)} \mathbb{E}^{\mathbb{Q}} \left[V(T,z) \frac{\frac{\partial}{\partial \theta} f_{S(T)}(z;\theta)}{f_{S(T)}(z;\theta)} \Big| \mathcal{F}(t_0) \right]. \qquad (17)
\end{aligned}
$$

- The ratio can be expressed as a logarithm, as follows,

$$
\frac{\partial V}{\partial \theta} = \mathrm{e}^{-r(T-t_0)} \mathbb{E}^{\mathbb{Q}} \left[V(T,z) \frac{\partial}{\partial \theta} \log f_{S(T)}(z;\theta) \Big| \mathcal{F}(t_0) \right]
$$

Likelihood Ratio Method

- Recall the lognormal PDF under the Black-Scholes model,

$$f_{S(T)}(x) = \frac{1}{\sigma x \sqrt{2\pi(T-t_0)}} \exp\left[-\frac{\left(\log \frac{x}{S(t_0)} - \left(r - \frac{1}{2}\sigma^2\right)(T-t_0)\right)^2}{2\sigma^2(T-t_0)} \right].$$

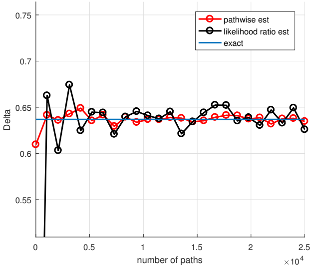
- The first derivatives with respect to $S(t_0)$ and σ are found to be,

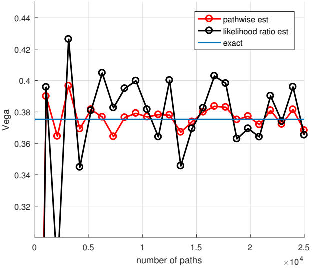
$$\begin{aligned}
\frac{\partial \log f_{S(T)}(x)}{\partial S(t_0)} &= \frac{\beta(x)}{S(t_0)\sigma^2(T-t_0)}, \beta(x) = \log \frac{x}{S(t_0)} - (r - \frac{1}{2}\sigma^2)(T-t_0) \\
\frac{\partial \log f_{S(T)}(x)}{\partial \sigma} &= -\frac{1}{\sigma} + \frac{1}{\sigma^3(T-t_0)}\beta^2(x) - \frac{1}{\sigma}\beta(x).
\end{aligned}$$

Likelihood Ratio Method

- The estimates for delta and vega are given by,

$$
\begin{aligned}
\frac{\partial V}{\partial S(t_0)} &= \frac{\mathrm{e}^{-r(T-t_0)}}{S(t_0)\sigma^2(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\max(S(T)-K,0)\beta(S(T))\right],\\
\frac{\partial V}{\partial \sigma} &= \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}[\max(S(T)-K,0)(-\frac{1}{\sigma}+\frac{1}{\sigma^3(T-t_0)}\beta^2(S(T))-\frac{1}{\sigma}\beta(S(T)
\end{aligned}
$$





- Superior convergence of the pathwise sensitivity method.