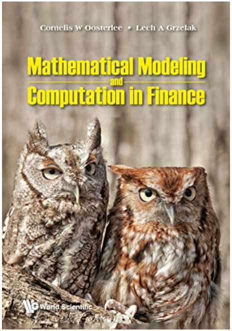
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- YouTube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

Simulated Dynamics and Density

$$\begin{aligned}
\mathrm{d}S(t) &= \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t),\\
X(t) &:= g(t, S(t)) = \log S(t),\\
S(t) &= \exp(X(t)).
\end{aligned}$$

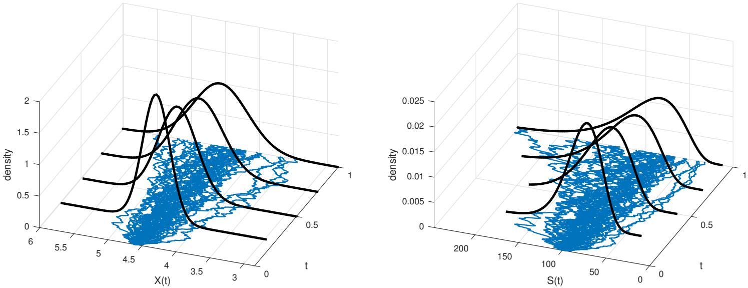


Figure: Paths and the corresponding densities. Left: $X(t) = \log S(t)$ and Right: $S(t)$ with the following configuration: $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.4$; $T = 1$.

Process Discretization and Implementation in Python

- The GBM process is given by the following SDE:

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t),$$

- Under the log-transformation we have

$$\mathrm{d}X(t) = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W(t).$$

- And the discretized version is given by:

$$X(t+\Delta t) = X(t) + \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma(W(t+\Delta t) - W(t)).$$

Black-Scholes model (1973)

We start with the assumption about the price stochastic process:

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{P}}(t),$$

and bank account:

$$\frac{\mathrm{d}M(t)}{M(t)} = r\mathrm{d}t,$$

where $\mathbb{P}$ represents real-world measure.

- We define $V(t, S(t))$ which represents the value of the option at time time t.
- Further we consider a trading strategy under which one holds one option and continuously trades in the stock in order to hold some $\Delta(t)$ shares.

We see that at time t, the value of the portfolio will be:

$$\Pi(t, S(t)) = V(t, S(t)) - \Delta(t)S(t).$$

Black-Scholes cont.

Since $S(t)$ is stochastic, our portfolio is as well, the dynamics of our portfolio $\Pi(t)$ we get from Ito's lemma:

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta(t)\mathrm{d}S(t). \tag{1}$$

We need to find $\mathrm{d}V$ so once more time by applying Ito's we have:

$$\begin{aligned}
\mathrm{d}V &= \frac{\partial V}{\partial S(t)}\mathrm{d}S(t) + \frac{\partial V}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 V}{\partial S(t)^2}\mathrm{d}S(t)^2 \\
\mathrm{d}V &= \left(\mu S(t)\frac{\partial V}{\partial S(t)} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V}{\partial S(t)^2}\right)\mathrm{d}t + \sigma S(t)\frac{\partial V}{\partial S(t)}\mathrm{d}W(t)
\end{aligned}$$

so now by using Equation (1) we have

Black-Scholes continuation

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta(t)\mathrm{d}S(t),$$

$$\begin{aligned}\mathrm{d}\Pi &= \left(\mu S(t)\frac{\partial V}{\partial S(t)} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S(t)^2}\right)\mathrm{d}t \\ &\quad + \sigma S(t)\frac{\partial V}{\partial S(t)}\mathrm{d}W(t) - \Delta(t)\left(\mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)\right).\end{aligned}$$

Since we want the risk to be hedged (remove the uncertainty), we set

$$\Delta(t) = \frac{\partial V}{\partial S(t)},$$

and further:

$$r\Pi\mathrm{d}t = \left(\mu S(t)\frac{\partial V}{\partial S(t)} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S(t)^2 \frac{\partial^2 V}{\partial S(t)^2}\right)\mathrm{d}t - \frac{\partial V}{\partial S(t)}\left(\mu S(t)\mathrm{d}t\right).$$

Black-Scholes continuation

So finally (with indep. variable $S = S(t)$):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r\Pi \quad = \quad 0.$$

Moreover we also know that:

$$\Pi = V - \frac{\partial V}{\partial S} S,$$

so:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - r\left(V - S\frac{\partial V}{\partial S}\right) \quad = \quad 0.$$

And finally:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Main Assumptions

- The asset price follows the lognormal random walk:

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t)$$

- Interest rate r and volatility σ are known functions of t.
- Transaction costs for hedging are not included in the model.
- No dividend is paid during the life of the option.
- There are no arbitrage possibilities.

$\Rightarrow$ Black-Scholes partial differential equation:
(for a European option)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Nobel prize in 1997 for Merton and Scholes (Black died in 1995).
- This is a parabolic partial differential equation

Important Quantities, Sensitivity Analysis

- The important quantities to be calculated are the so-called hedge parameters.
- Delta: $\Delta(S, t)$ the rate of change of the value of the option (or portfolio) with respect to S. The largest random component of a portfolio is eliminated. It indicates the number of shares, that should be kept with each option issued in order to cope with a loss in the case of exercise.
$$\Delta = \frac{\partial V}{\partial S}$$
- Gamma: indicates the change in Delta
$$\Gamma = \frac{\partial^2 V}{\partial S^2}$$
- If Gamma is low, it is only necessary to sometimes change the portfolio. If it is high, the portfolio under consideration results only for a very short period of time in a risk-less scenario.
- There are several other important hedging parameters.

What determines the value of an option ?

- The value of a call option is a function of parameters in the contract, such as the strike price K, and the time to maturity $T-t$, T is expiry time, t is the current time.
- The value will also depend on properties of the asset, such as its price, its volatility, as well as the risk-free rate of interest:

$$V(S,t;\sigma;K,T;r) \text{ or use } V(S,t)$$

- Semi-colons separate different types of variables and parameters
  - S and t are independent variables
  - σ is a parameter associated with the asset price
  - K and T are parameters associated with the particular contract;
  - r is a parameter associated with the currency
  - $\Rightarrow$ Note that parameter μ does not appear in the BS equation!

Option Quotations in the Market

- Let us refresh how options are quoted.

Calls									Puts							
Bid	Ask	Last	Volume	Opn Int	Opint Nch	ImpVol Mid	Delta Mid	Strike	Bid	Ask	Last	Volume	Opn Int	Opint Nch	ImpVol Mid	Delta Mid
								31-Mar-2021								
143.4	144	111.05		41	0	20.4368	0.5176	3790	143.1	143.7	225.67		43	0	20.4118	-0.4795
140.2	141	115.5		70	0	20.3359	0.5119	3795	144.9	145.6	213.73		1	0	20.3041	-0.4852
137.3	137.9	144.81	9	6,951	-159	20.2454	0.5063	3800	146.9	147.6	142.6		876	400	20.2282	-0.4908
134	134.6	142.34		17	1	20.1083	0.5005	3805	148.9	149.6	149.24	1	6	0	20.1199	-0.4966
131.1	131.8	134.9	1	444	0	20.0313	0.4947	3810	150.9	151.6	143.46	4	22	1	20.0219	-0.5023
								30-Jun-2021								
210.5	211.6			0	0	21.0256	0.5176	3790	220.1	221.6	218.81	3	0	0	20.9539	-0.4763
207.3	208.5			0	0	20.9436	0.5139	3795	222	223.6	236.44		58	0	20.8820	-0.4801
204.2	205.4	197.62	10	3,920	0	20.8647	0.5100	3800	223.8	225.4	215.1	2	1,573	350	20.8085	-0.4839
201.2	202.3			0	0	20.7889	0.5062	3805	225.8	227.4	235.54		58	0	20.7234	-0.4878
198.1	199.3			0	0	20.7114	0.5023	3810	227.7	229.3	227.75		10	0	20.6415	-0.4916
								30-Sep-2021								
281.8	286.1	278.29		905	2	21.6065	0.5430	3750	262.3	266.2	261.41	1	299	1	21.6301	-0.4478
266	270.1	262.92		231	0	21.2738	0.5281	3775	271.2	275.3	292.62		98	0	21.2921	-0.4627
250.5	254.5	194.3		951	0	20.9421	0.5129	3800	280.6	284.8	308.49		27	0	20.9630	-0.4780
235.4	239.3	203.73		252	0	20.6141	0.4972	3825	290.3	294.6	308.64		2	0	20.6299	-0.4936
220.7	224.5	221.8		599	0	20.2888	0.4811	3850	300.4	304.9	326.72		3	0	20.3034	-0.5097
								31-Dec-2021								
319.1	326.9	326.21	1	448	0	21.6803	0.5400	3750	309.2	316.2	303.34		16	0	21.5757	-0.4476
303.5	311.1	308.11		70	25	21.3855	0.5271	3775	318.3	325.5	315.1		46	0	21.2776	-0.4606
288.2	295.7	284.8		114	0	21.0950	0.5139	3800	327.7	335.1	322		8	6	20.9803	-0.4738
273.2	280.5	276.8	100	104	0	20.8013	0.5005	3825	337.4	345	327.7		11	10	20.6831	-0.4874
258.6	265.8	268.45	3	41	0	20.5173	0.4868	3850	347.4	355.2	361.52		2	0	20.3853	-0.5012

Figure: Call and Put options for S&P index, spot is about **3800**.

Martingale approach to option pricing

Definition (Martingale)

A stoch. process $X = \{X(t); t \geq 0\}$ *is a martingale wrt the* $\{\mathcal{F}(t)\}$ *if*

- X *is adapted to the filtration* $\mathcal{F}_{t\geq 0}$,
- *for all* t, $\mathbb{E}\left(|X(t)|\right) < \infty$,
- *for all* s *and* t *with* $s < t$ *we have* $\mathbb{E}\left(X(t)|\mathcal{F}(s)\right) = X(s)$.

Exercise: Show that $W(t)$ is a martingale:

- $W(s)$ is $\mathcal{F}(s)$-measurable
- $\mathbb{E}\left(|W(s)|\right) < \infty$
- for any $s \leq t$ we have

$$\mathbb{E}\left(W(t)|\mathcal{F}(s)\right) = \mathbb{E}\left(W(t) - W(s) + W(s)|\mathcal{F}(s)\right) = W(s)$$

Martingales: Example

Suppose we have a process $X(t) = \exp\left(\alpha W(t) - \alpha^2 \frac{t}{2}\right)$. The first two conditions are satisfied, and the final condition reads:

$$\begin{aligned}\mathbb{E}\left(X(t)|\mathcal{F}(s)\right) &= \exp\left(-\alpha^2\frac{t}{2}\right)\mathbb{E}\left(\mathrm{e}^{\alpha W(t)}|\mathcal{F}(s)\right)\\ &= \exp\left(-\alpha^2\frac{t}{2}+\alpha W(s)\right)\mathbb{E}\left(\mathrm{e}^{\alpha(W(t-s))}|\mathcal{F}(s)\right)\end{aligned}$$

Recall that if Y has a normal distribution with mean μ and variance σ^2, then $X = \exp(Y)$ has a lognormal distribution with expectation

$$\mathbb{E}(X) = \exp\left(\mu + \sigma^2/2\right).$$

So,

$$\begin{aligned}\mathbb{E}\left(X(t)|\mathcal{F}(s)\right) &= \exp\left(-\alpha^2\frac{t}{2}+\alpha W(s)\right)\exp\left(\alpha^2\frac{(t-s)}{2}\right)\\ &= \exp\left(\alpha W(s)-\alpha^2\frac{s}{2}\right) = X(s).\end{aligned}$$

Process $X(t)$ is a martingale with respect to filtration $\mathcal{F}(s)$

A stochastic integral process is a martingale

Theorem

Let $g \in \mathcal{L}^2$. For any $t \geq 0$

$$X(t) = \int_0^t g(s)\mathrm{d}W(s). \tag{2}$$

Then, the process $X = X(t); t \geq 0$ is a $\mathcal{F}(t)$-martingale.

Finding a risk-free drift

Suppose we have an asset process given by

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{P}}(t),$$

and a bank account:

$$\frac{\mathrm{d}M(t)}{M(t)} = r\mathrm{d}t.$$

We would like to find the discounted payoff under risk-neutral measure $\mathbb{Q}$, so the process $\frac{S(t)}{M(t)}$ has to be a martingale, i.e.

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{S(T)}{M(T)}|\mathcal{F}(t)\right) = \frac{S(t)}{M(t)}.$$

let us find a dynamics of $\frac{S(t)}{M(t)}$

Finding a risk-free drift continuation

Dynamics of $F = \frac{S(t)}{M(t)}$ we find from Itô's lemma (subscript S,M represent partial derivatives here):

$$\begin{aligned}
\mathrm{d}\frac{S}{M} &= F_S \mathrm{d}S + F_M \mathrm{d}M + \frac{1}{2}F_{SS}(\mathrm{d}S)^2 + \frac{1}{2}F_{MM}(\mathrm{d}M)^2 + F_{SM}\mathrm{d}M\mathrm{d}S \\
&= \frac{1}{M}\mathrm{d}S - \frac{S}{M^2}\mathrm{d}M + 0 + 0 + 0 \\
&= \frac{1}{M}\left(\mu S \mathrm{d}t + \sigma S \mathrm{d}W\right) - \frac{S}{M^2} rM\mathrm{d}t \\
&= \frac{S}{M}\left(\mu - r\right)\mathrm{d}t + \frac{1}{M}\sigma S \mathrm{d}W^{\mathbb{P}}
\end{aligned}$$

In order to make the dynamics of $\dfrac{S(t)}{M(t)}$ driftless (why ???) we need to have: $\mu = r$.

Martingale approach and option pricing

The notion of martingales gives an alternative method for deriving option pricing PDEs. We look for:

$$V(S(t),t) = M(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}V(S(T),T)\Big|\mathcal{F}(t)\right],$$

with, again, $M(t)$ the money-savings account at time t, $M(t)=1$ and

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}V(S(T),T)\Big|\mathcal{F}(t)\right] = \frac{V(S(t),t)}{M(t)}. \quad (3)$$

$\frac{V(S(t),t)}{M(t)}$ should be a martingale and its dynamics can be found by Itô's lemma,

$$d\left(\frac{V(S(t),t)}{M(t)}\right) = \frac{1}{M(t)}dV - r\frac{V(S(t),t)}{M(t)}dt. \quad (4)$$

For an infinitesimal change, dV of $V(S(t),t)$, we find

$$\mathrm{d}V = \left(\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S(t)} + \frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V}{\partial S(t)^2}\right)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)^{\mathbb{Q}}.$$

Martingale approach and option pricing

As $\frac{V(S(t),t)}{M(t)}$ should be a martingale, the martingale representation theorem indicates that the dynamics of $\frac{V(S(t),t)}{M(t)}$ *cannot contain any dt-terms*, which yields (with $S = S(t), M = M(t)$):

$$\frac{1}{M}\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) - r\frac{V}{M} = 0. \tag{5}$$

Multiplying both sides of (5) by M, gives us the Black-Scholes pricing PDE.

Simulation of Martingales in Python

- How to check whether a simulated process is a martingale or not?

Risk Neutral valuation

We already know:
For a given market, described by the equations:

$$\begin{cases} \mathrm{d}M(t) &= rM(t)\mathrm{d}t \\ \mathrm{d}S(t) &= \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{P}}(t), \end{cases}$$

and a contingent claim of the form

$$\chi = V(S(T), T),$$

the arbitrage free price is given, via Ito's Lemma, by $V(S,t)$, where function $V(S,t)$ satisfies the Black-Scholes equation:

$$\begin{cases} \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV &= 0 \\ V(S,T) &= \chi. \end{cases}$$

Feynman-Kac formula

Feynman-Kac established a link between partial differential equations (PDEs) and stochastic processes. It offers a method for solving certain PDEs by simulating random paths of a stochastic process. Suppose we are given the PDE:

$$\frac{\partial V}{\partial t} + \bar{\mu}(x,t)\frac{\partial V}{\partial x} + \frac{1}{2}\bar{\sigma}^2(x,t)\frac{\partial^2 V}{\partial x^2} = 0,$$

subject to the final condition $V(x,T) = \eta(x)$, then the Feynman-Kac formula reads:

$$V(x,t) = \mathbb{E}\left(\eta(X(T))|\mathcal{F}(t)\right)$$

where: $X(t)$ is an Ito process driven by the equation:

$$\mathrm{d}X(t) = \bar{\mu}(X(t),t)\mathrm{d}t + \bar{\sigma}(X(t),t)\mathrm{d}W(t),$$

with $W(t)$ is a Wiener process, with initial for $X(t) = x$.

proof of the Feynman-Kac formula.

We know the PDE for $V(x,t)$, so the Ito's dynamics for V are given by:

$$
\begin{aligned}
\mathrm{d}V &= \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 V}{\partial x^2}(\mathrm{d}x)^2 \\
\mathrm{d}V &= \left(\frac{\partial V}{\partial t} + \bar{\mu}(x,t)\frac{\partial V}{\partial x} + \frac{1}{2}\bar{\sigma}^2(x,t)\frac{\partial^2 V}{\partial x^2}\right)\mathrm{d}t + \bar{\sigma}(x,t)\frac{\partial V}{\partial x}\mathrm{d}W(t) \\
\mathrm{d}V &= \bar{\sigma}(x,t)\frac{\partial V}{\partial x}\mathrm{d}W(t)
\end{aligned}
$$

Integrating both sides one gets

$$
\int_t^T dV = V(X(T),T) - V(x,t) = \int_t^T \bar{\sigma}(x,t)\frac{\partial V}{\partial x}\mathrm{d}W(t)
$$

now by taking expectations, we have

$$
V(x,t) = \mathbb{E}\left(V(X(T),T)\right) = \mathbb{E}\left(\eta(X(T))\right)
$$

$\square$

Feynman-Kac in practice

Exercise: Solve the following PDE

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} &= 0 \\ V(x,T) &= x^2 \end{aligned}$$

where σ is a constant.

Answer: From Feynman-Kac we have:

$$V(x,t) = \mathbb{E}\left(\eta(X(T))|\mathcal{F}(t)\right) = \mathbb{E}\left(X(T)^2|\mathcal{F}(t)\right)$$

where:

$$\begin{aligned} \mathrm{d}X(s) &= 0 \cdot \mathrm{d}t + \sigma \mathrm{d}W(s) \\ X(t) &= x \end{aligned}$$

So we have: $X(T) = x + \sigma\left(W(T) - W(t)\right)$, and $X(T)$ has the distribution $N(x, \sigma\sqrt{T-t})$. Finally $V(t,x) = \sigma^2(T-t) + x^2$

Pricing: Feynman-Kac Theorem

$V(S,t)$ is the unique solution of the final condition problem

$$\begin{cases} \frac{\partial V}{\partial t} & + \quad \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \\ V(S,T) & = \quad \text{given} \end{cases}$$

This solution can also be obtained as

$$V(S,t) = \mathrm{e}^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[V(S(T),T)|S(t)]$$

with the sum of the first derivatives of the option square integrable.
Given the system of stochastic differential equations:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t).$$

A pricing approach

$$V(S_0, t_0) = \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}[V(S(T), T)|S_0]$$

Quadrature:

$$V(S_0, t_0) = \mathrm{e}^{-r(T-t_0)}\int_{\mathbb{R}} V(S(T), T)f(S(T)|S_0)\mathrm{d}S$$

- Trans. PDF, $f(S(T)|S_0)$, typically not available, but the characteristic function, $\hat{f}$, often is.

What about the drift? Which one to choose?

Risk Neutral probability and Option Pricing

- There are two major approaches to pricing an option. The first one is the PDE approach, the second one is the risk-neutral probability approach.
- The basic idea of the risk-neutral probability approach is the change the probability measure from the true (statistical) probability to risk-neutral probability.
- The difference of the two measures is the expected return of the stock:
  - In the true probability measure, the expected return is μ
  - In the risk-neutral probability measure, the expected stock return is risk-free rate r

Which one to choose μ or r ?

Definition of risk neutral measure

How the risk neutral probability is used in asset pricing theory.

- A risk neutral probability is the probability of an future event or state that both trading parties in the market agree upon.
  A simple example:
  - For a future event, two parties A and B enter into a contract, in which A pays B 1€ if it happens and 0€ if it doesn't.
  - For such an agreement, there is a price for B to pay A. If they agree that B pays 0.4€ to A, this means the two parties think that the probability of the event that happens is 40%. Otherwise, they won't reach that agreement and sign a contract.
  - This price reflects the common beliefs towards the probability that the event happens. 40% is the risk neutral probability of the event that happens.

Definition of risk-neutral measure

- It is not any historical statistic or prediction of any kind. It is not the true probability, either.
- One should ask what kind of information is offered from risk neutral probability and where can we find this measure in the real world?
- For the simple example mentioned above, once the price is established, the risk-neutral measure is also determined.
- Whenever you have a pricing problem in which the event is measurable under this measure, you have to use this measure to avoid arbitrage. If you don't, it's like you are simply giving out another price for the same event at the same time, which is an obvious arbitrage opportunity.

Risk Neutral valuation

Feynman-Kac:
The Black-Scholes equation is of a form which can be solved using a stochastic representation formula via the Feynman-Kac theorem.

Ito's Lemma:

$$g(t, S(t)) = \log S(t),$$

where

$$\mathrm{d}S(t) = rS(t)dt + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t),$$

$$\begin{aligned}
\mathrm{d}g(t,s) &= \frac{1}{S(t)}\mathrm{d}S(t) - \frac{1}{2}\frac{1}{S(t)^2}(\mathrm{d}S(t))^2 \\
&= \frac{1}{S(t)}\left(rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)\right) - \frac{1}{2}\frac{1}{S(t)^2}\left(\sigma^2 S^2(t)\mathrm{d}t\right) \\
&= \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{Q}}(t).
\end{aligned}$$

We found that

$$\mathrm{d}g(t, S(t)) = \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{Q}}(t).$$

Analytic Solution of BS prices

We have:

$$\begin{aligned}\int_{t_0}^{T} \mathrm{d}\log S(u) &= \int_{t_0}^{T}\left(r-\frac{1}{2}\sigma^2\right)\mathrm{d}t+\int_{t_0}^{T}\sigma \mathrm{d}W^{\mathbb{Q}}(t),\\ \log\frac{S(T)}{S_0} &= \left(r-\frac{1}{2}\sigma^2\right)(T-t_0)+\sigma\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t_0)\right),\end{aligned}$$

So, we find:

$$S(T)=S_0\exp\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t_0)+\sigma\left(W^{\mathbb{Q}}(T)-W^{\mathbb{Q}}(t_0)\right)\right).$$

Feynman-Kac theorem gives:

$$\begin{aligned}V(S_0,t_0) &= \mathrm{e}^{-r(T-t_0)}\int_{-\infty}^{+\infty}V(S(T),T)f_S(s)\mathrm{d}s\\ &= \mathrm{e}^{-r(T-t_0)}\int_{-\infty}^{+\infty}V(S_0\mathrm{e}^{Z},T)f_Z(z)\mathrm{d}z\end{aligned}$$

Analytic Solution of BS prices

where Z is a random variable with the distribution:

$$\mathcal{N}\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t_0), \sigma^2(T-t_0)\right).$$

If we now take

$$V(S(T),T)=\max(S(T)-K,0)=\max(S_0\mathrm{e}^z-K,0)$$

we have:

$$\mathbb{E}^{\mathbb{Q}}\left(\max(\mathrm{e}^z-K,0)|\mathcal{F}(t)\right)=0\cdot\mathbb{Q}\left(\mathrm{e}^z\leq K\right)+\int_{\log\frac{K}{S_0}}^{\infty}\left(S_0\mathrm{e}^z-K\right)f(z)\mathrm{d}z.$$

After simple calculations we end up with the Black-Scholes pricing theorem.

Analytic Solution of BS prices

Theorem (Black-Scholes formula)

The price of a European call option with strike price K and maturity T is given by the formula:

$$
\begin{aligned}
V(S_0, t_0) &= S_0\phi(d_1(t, S_0)) - \mathrm{e}^{-r(T-t_0)}K\phi(d_2(t, S_0)), \text{ with}\\
d_1(t_0, S_0) &= \frac{1}{\sigma\sqrt{T-t_0}}\left(\log\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t_0)\right),\\
d_2(t_0, S_0) &= d_1(t_0, S_0) - \sigma\sqrt{T-t_0},
\end{aligned}
$$

where ϕ is the cumulative distribution function for standard normal distribution i.e., $\mathcal{N}(0, 1)$.