List of content

Models in Financial industry

- Pricing approach:
  1. Start with some financial product
  2. Model asset prices involved (SDEs)
  3. Calibrate the model to market data (numerics, optimization)
  4. Model product price correspondingly (P(I)DE or integral)
  5. Price the product of interest (numerics, MC)
  6. Set up a hedge to remove the risk to the product (optimization)

Options

Definition (Option)

Is a contract written by a seller, that gives the right (but not the obligation) to the holder to trade in the future the underlying asset at a previously agreed price.

Option styles:

- European option- an option that may be only exercised on expiration day;
- American option - an option that may be exercised on any trading day (also on the expiration);
- Bermudan option - an option that may be exercises only on specified dates;
- Exotic or path dependent options (value does not only depend on the current stock value, but, for example, also on the average value of the stock price development).

Binaries or Digitals

- Binary options have a discontinuous payoff at expiry.
- An example for a binary call is that the contract pays 1 at T, if the asset price is then greater than the exercise price K.
- The final condition is $V(S, T) = \mathcal{H}(S - K)$, where $\mathcal{H}(\cdot)$ is the Heaviside function.
- Variation: Asset or nothing, where Q is the asset itself.

Binary, compound, chooser

- Binary: Cash or Nothing: Pays out Q at expiry T if option is in the money $S > K$, otherwise expires worthless. Payoff:

$$V(S,T) = Q1_{S \geq K}$$

- Compound option: Call on a call: right to buy a 'call with maturity T and strike K' at time T_0 for the price K_0. Payoff:

$$V^{CC}(S, T_0, K_0, K, T) = \max\left[V^C(S, K, T) - K_0, 0\right]$$

- Chooser option: Gives the holder the right to choose whether the underlying option at time T_0 is a Call or a Put with the same strike K and maturity T. The payoff of a chooser option is

$$V^{CH}(S, K, T_0, T) = \max\left[V^C(S, K, T), V^P(S, K, T)\right]$$

Path-dependent options

- Barrier options (US, 1967) are options that either come alive or die when predetermined trigger points (barriers) are reached.
  Up-and-out call: Option is knocked out if S hits a certain barrier S_u.
  Payoff:

$$V(S,T) = \max\left(S-K,0\right) \text{ if } S \leq S_u; \text{ else ceases to exist}$$

- Lookback options: Path dependent options whose payoffs depend on the max or min of the asset during a certain period (lookback period $[T_0, T]$).
  Payoff European fixed strike lookback call: $(\max_{[T_0,T]}(S(t)) - K)^+$
  Payoff European fixed strike lookback put: $(K - \min_{[T_0,T]}(S(t)))^+$

Barrier options

- Barrier options come in two main varieties, the 'in' barrier option (or knock-in) and the 'out' barrier option (knock-out). The former only have a payoff if the barrier level is reached before expiry and the latter only have a payoff if the barrier is not reached.
- Barrier options are popular for various reasons.
- Usually, a purchaser has very precise views about the direction of the market. If she wants the payoff from a call option but does not want to pay for all the upside potential, believing that the upward movement of the underlying will be limited prior to expiry, then she may choose to buy an up-and-out call. It will be cheaper than a similar vanilla call, since the upside is severely limited.
- Conversely, an 'in' option will be bought by someone who believes that the barrier level will be realized. Again the option is cheaper then the equivalent vanilla option.

Different types of barriers

- The out option only pays off if a level is not reached. If the barrier is reached the option is said to have knocked out.
- The in option pays off as long as a level is reached before expiry. If the barrier is reached then the option is said to have knocked in.
- If the barrier is above the initial asset value, we have an up option
- If the barrier is below the initial value, we have a down option
- The payoffs are the usual ones
- Barrier can be time dependent

PDE for Barrier options

- These options satisfy the Black-Scholes equation, on a special domain, with special boundary conditions
- The details of the barrier feature come in through the specification of the boundary conditions
- If the asset reaches the barrier S_u in an 'out' barrier option then the contract becomes worthless:

$$V(S_u, t) = 0 \text{ for } t < T$$

- If we have a down-and-out option with a barrier at S_d we solve for $S_d < S < \infty$ with

$$V(S_d, t) = 0$$

'In' Barriers

- An 'in' option only has a payoff if the barrier is triggered. If the barrier is not triggered we have $V(S,T)=0$
- The value in the option is in the potential to hit the barrier. If the option is an up-and-in contract then on the upper barrier the contract must have the same value as a vanilla contract:

$$V(S_u,t)= \text{ value of standard option contract, function of } t$$

So,

$$V(S_u,t)=V_c(S_u,t) \text{ for } t<T$$

- Moverover, one can show that "in" + "out" = European.

Hedging barrier options

- Barrier options have discontinuous delta at the barrier

  For a knock-out, the option value is continuous, decreasing approximately continuously towards the barrier, then being zero beyond the barrier.
- A discontinuity in the delta means that the gamma is instantaneously infinite at the barrier. Delta hedging through the barrier is virtually impossible, and costly.
- There have been a number of suggestions made for ways to statically hedge barrier options. These methods try to mimic as closely as possible the value of a barrier option with vanilla calls and puts or with binary options.

Asian options

- Asian options: The option payoff depends on the average of the underlying:
  - $V(T) = \max\left(A(T) - K, 0\right)$: fixed strike call
  - $V(T) = \max\left(K - A(T), 0\right)$: fixed strike put
  - $V(T) = \max\left(S(T) - A(T), 0\right)$: floating strike call
  - $V(T) = \max\left(A(T) - S(T), 0\right)$: floating strike put
- Where $A(T)$ represents an average of stock $S(t)$ computed over certain period of time.

Types of averages

- Arithmetic average:

$$A_n = \frac{1}{n}\sum_{i=1}^{n} S(t_i).$$

- Geometric average: $A_n = \Pi_{i=1}^{n} S^{1/n}(t_i)$.
- Continuous average

$$A(t) = \frac{1}{t}\int_0^t S(u)\mathrm{d}u.$$

Asian Options; Option depending on continuous average

- The exercise price or the asset price is replaced by an average of the asset price:
- Final conditions for an arithmetic-average floating strike call:

$$V(S,T) = \max\left(S - \frac{1}{T}\int_0^T S(u)\mathrm{d}u, 0\right)$$

- for an arithmetic-average fixed strike call:

$$V(S,T) = \max\left(\frac{1}{T}\int_0^T S(u)\mathrm{d}u - K, 0\right)$$

- With a new variable: $I(t) := \int_0^t S(u)\mathrm{d}u$, one finds a similar Black-Scholes type equation for Asian options:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0$$

- There is no diffusion term in the I-direction.

Path dependency and the integral

- We start by assuming that the underlying asset follows the lognormal random walk:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t)$$

  Imagine a contract that pays at maturity T an amount that is a function of the path taken by the asset between zero and maturity
- Suppose that this path-dependent quantity can be represented by an integral of some function of the asset over the period zero to T:

$$I(T) = \int_0^T f(S, u)\mathrm{d}u$$

- Most path-dependent quantities in exotic derivative contracts can be written in this form with a suitable choice of $f(S, t)$.

- Prior to maturity we have information about the possible final value of S (at time T) in the present value of S at time t. For example, the higher S is today, the higher it will probably end up at maturity. Similarly we have information about the possible final value of $I(T)$ in the value of the integral to date:

$$I(t) = \int_0^t f(S, u)\mathrm{d}u$$

  As we get closer to maturity, we become more confident about the final value of $I(T)$.

- The value of the option is only a function of S and t, but also a function of I; I will be our new independent variable, called a state variable

- As we will use Itô's lemma, we need to know the stochastic differential equation satisfied by I. This is simply (incrementing t by $\mathrm{d}t$ we find that):

$$\mathrm{d}I(t) = f(S,t)\mathrm{d}t$$

- I is thus a smooth function, and the equation for $\mathrm{d}I$ does not contain stochastic terms

Examples

- An Asian option has a payoff that depends on the average of asset price over some period. If that period is from time zero to maturity and the average is arithmetic then:

$$I(t) = \int_0^t S(u)\mathrm{d}u$$

- The payoff may then be, (a floating strike put), for example,

$$\max\left(\frac{I(T)}{T} - S, 0\right)$$

- Another example: Imagine a contract that pays a function of the square of the underlying asset, but only counts those times for which the asset is below S_u. Then

$$I(t) = \int_0^t S^2(u)\mathcal{H}(S_u - S)\mathrm{d}u$$

where $\mathcal{H}$ is the Heaviside function.

Continuous sampling, The pricing equation

- We derive the pricing PDE for a contract that pays some function of new variable I
- The value of the contract is now a function of the three variables: $V(S, I, t)$.
- Set up a portfolio containing one of the path-dependent options and short a number Δ of the underlying asset:

$$\Pi = V(S, I, t) - \Delta \cdot S$$

- The change in the value of this portfolio is given by

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial I}\mathrm{d}I + \left(\frac{\partial V}{\partial S} - \Delta\right)\mathrm{d}S$$

- Choosing $\Delta = \partial V / \partial S$ to hedge the risk, we find:

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + f(S, t)\frac{\partial V}{\partial I}\right)\mathrm{d}t$$

- This change is risk free and thus earns the risk-free rate of interest r, leading to the PDE:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+f(S,t)\frac{\partial V}{\partial I}+rS\frac{\partial V}{\partial S}-rV=0$$

- This is to be solved subject to

$$V(S,I,T)=\text{ payoff}$$

- This completes the formulation of the valuation problem.

Examples of multi-asset options

- A basket option is an option whose payoff depends on the value of a portfolio (or basket) of assets. Basket options are growing in popularity as a means of hedging the risk of a portfolio and are highly interesting for banks nowadays.
- They are attractive because an option on a basket is cheaper than buying options on the individual assets. Furthermore, their payoff profile replicates the changes in a portfolio's value more closely than any combination of options on the underlying assets.
- Basket options: $V(\mathbf{S}(T), T) = \left(\sum_{i=1}^{d} w_i S_i - K\right)^+$
- Call option on the maximum of the underlying assets $V(\mathbf{S}(T), T) = (\max S_i - K)^+$
- Put option on the minimum of the underlying assets $V(\mathbf{S}(T), T) = (K - \min S_i)^+$
- Exchange option (two-asset): $V(\mathbf{S}(T), T) = (S_1 - S_2)^+$

Multi-d Black-Scholes equation

The PDE-method is based on the solution of the multi-dimensional Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{i=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{d}(r-\delta_i)\, S_i \frac{\partial V}{\partial S_i} - rV = 0 \quad (1)$$

With

- S_i, the value of underlying asset i
- σ_i volatility of asset i
- ρ_{ij} correlation between asset i and j
- r risk-free interest rate
- δ_i continuous dividend yield

Multi-asset options

- Multi-asset options are multi-dimensional. Using numerical techniques, the number of grid points grows exponentially $\Rightarrow$ Curse of dimensionality. Problems are not solvable on nowadays machines unless advanced techniques are used.
- Partitioning/Splitting and parallelization of the method reduces memory usage.
- Solution: Coordinate transform, grid adaptivity, applicability is limited.
- Resume to Monte Carlo techniques for these kinds of options!