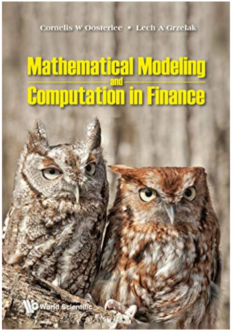
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- YouTube Channel with courses can be found here.
- Slides and the codes can be found here.

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Trading of Options and Hedging
Commodities
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Value of Call/Put Option and Hedging
Modeling of Asset Prices and Randomness
Stochastic Processes for Stock Prices
Itô's Lemma for Solving SDEs

Application for Trading

- Suppose you have stocks of a company, and you'd like to have cash in two years (to buy a house).
- You wish at least K euros for your stocks, but the stocks may drop in the coming years. How to assure K euros in two years?
- You can buy insurance against falling stock prices.
- This is the standard put option (i.e. the right to sell stock at a future time point).
- The uncertainty is in the stock prices.

Option Market

- There are many options depending on strike and maturity.

Calls									Puts							
Bid	Ask	Last	Volume	Opn Int	Opint Nch	ImpVol Mid	Delta Mid	Strike	Bid	Ask	Last	Volume	Opn Int	Opint Nch	ImpVol Mid	Delta Mid
								31-Mar-2021								
143.4	144	111.05		41	0	20.4368	0.5176	3790	143.1	143.7	225.67		43	0	20.4118	-0.4795
140.2	141	115.5		70	0	20.3359	0.5119	3795	144.9	145.6	213.73		1	0	20.3041	-0.4852
137.3	137.9	144.81	9	6,951	-159	20.2454	0.5063	3800	146.9	147.6	142.6		876	400	20.2282	-0.4908
134	134.6	142.34		17	1	20.1083	0.5005	3805	148.9	149.6	149.24	1	6	0	20.1199	-0.4966
131.1	131.8	134.9	1	444	0	20.0313	0.4947	3810	150.9	151.6	143.46	4	22	1	20.0219	-0.5023
								30-Jun-2021								
210.5	211.6			0	0	21.0256	0.5176	3790	220.1	221.6	218.81	3	0	0	20.9539	-0.4763
207.3	208.5			0	0	20.9436	0.5139	3795	222	223.6	236.44		58	0	20.8820	-0.4801
204.2	205.4	197.62	10	3,920	0	20.8647	0.5100	3800	223.8	225.4	215.1	2	1,573	350	20.8085	-0.4839
201.2	202.3			0	0	20.7889	0.5062	3805	225.8	227.4	235.54		58	0	20.7234	-0.4878
198.1	199.3			0	0	20.7114	0.5023	3810	227.7	229.3	227.75		10	0	20.6415	-0.4916
								30-Sep-2021								
281.8	286.1	278.29		905	2	21.6065	0.5430	3750	262.3	266.2	261.41	1	299	1	21.6301	-0.4478
266	270.1	262.92		231	0	21.2738	0.5281	3775	271.2	275.3	292.62		98	0	21.2921	-0.4627
250.5	254.5	194.3		951	0	20.9421	0.5129	3800	280.6	284.8	308.49		27	0	20.9630	-0.4780
235.4	239.3	203.73		252	0	20.6141	0.4972	3825	290.3	294.6	308.64		2	0	20.6299	-0.4936
220.7	224.5	221.8		599	0	20.2888	0.4811	3850	300.4	304.9	326.72		3	0	20.3034	-0.5097
								31-Dec-2021								
319.1	326.9	326.21	1	448	0	21.6803	0.5400	3750	309.2	316.2	303.34		16	0	21.5757	-0.4476
303.5	311.1	308.11		70	25	21.3855	0.5271	3775	318.3	325.5	315.1		46	0	21.2776	-0.4606
288.2	295.7	284.8		114	0	21.0950	0.5139	3800	327.7	335.1	322		8	6	20.9803	-0.4738
273.2	280.5	276.8	100	104	0	20.8013	0.5005	3825	337.4	345	327.7		11	10	20.6831	-0.4874
258.6	265.8	268.45	3	41	0	20.5173	0.4868	3850	347.4	355.2	361.52		2	0	20.3853	-0.5012

Figure: Call and Put options for S&P index, spot is about **3800**.

Hedging

- Example: Suppose a portfolio with S (shares) and V_P (puts). If the price of S falls, the value of the portfolio depends on the ratio of S and V_P.
- A ratio exists, which results in no movement in the value of the portfolio. This ratio is instantaneously risk-free.
- A reduction of risk, for example by combining a number of S and V_P in a portfolio is called hedging.
- What about the writers ? A writer of a call expects the value of a share to fall.
- However, writers of options use them as a hedge instrument: an insurance to reduce risk against unexpected movements in the market.

Commodities

- Commodities are raw products such as precious metals, oil, food products, etc.
- The prices of these products are unpredictable but often show seasonal effects. Scarcity of the product results in higher prices.
- Most trading is done on the futures market, making deals to buy or sell the commodity at some time in the future.
- Energy markets are commodity markets that deal specifically with the trade and supply of energy (electricity market, oil, gas).
- Energy markets have been liberalized in some countries; they are regulated by national and international authorities (including liberalized markets) to protect consumer rights and avoid oligopolies. Members of the European Union are required to liberalize their energy markets.

Application

- Consider an energy company, starting energy generation from a wind park at sea (with the tallest wind mills available).
- What are the risks involved, and how can we estimate them? (maintenance cost, failure due to too much wind, no wind, ...).
- How to deal with no wind, or too much wind?
- Uncertain wind, uncertain electricity prices
- There is a so-called energy market, where utilities can buy "insurance" (energy derivatives), so that they can buy and sell "power" when the generation is not stable.

Currencies

- The exchange rate is the rate at which one currency can be exchanged for another: This is the world of foreign exchange or FX. Some currencies are pegged together, others float freely.
- Consistency throughout the FX world: If it is possible to exchange dollars for euros and then euros for yen, this implies a relationship between the dollar/euro, euro/yen and dollar/yen rates.

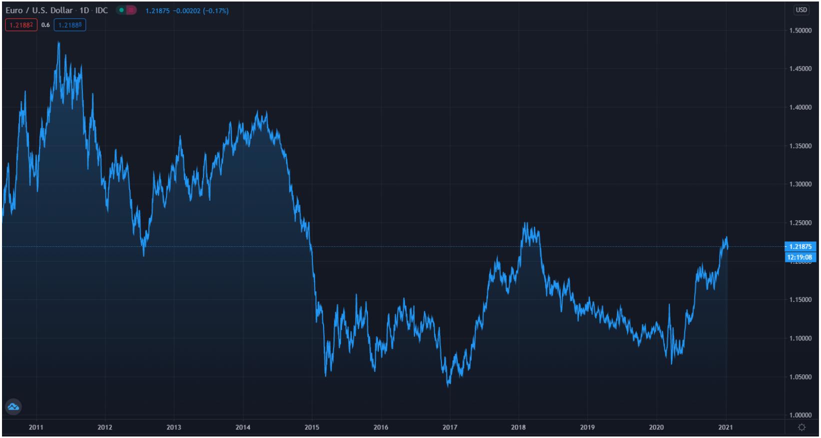
  If this relation moves out of line, it is possible to make arbitrage profits by exploiting the mispricing.
- Central banks can use interest rates as a tool for manipulating exchange rates.

Application

- A firm will have some business in America for several years.
- Investment may be in euros, and payment in the local currency.
- Firm's profit can be influenced negatively by the exchange rate.
- Banks sell insurance against changing FX rates. The option pays out in the best currency each year.
- Uncertain processes are the exchange rates, interest rate, but also the counterparty of the contract may go bankrupt!

FX market and volatility

- FX rate is stochastic and volatile;
- There are options on the FX rate, often used for either speculation or hedging.



Bitcoin

- Crypto currencies behave similarly as regular fiat currencies, but they are much more volatile.
- Bitcoin represents an FX rate of: BTC/USD.

Bitcoin and Option Market for Cryptos

- Crypto currencies also allow for pricing of options

Calls Underlying: SYN.BTC-31DEC21($36127.18) **31 Dec 2021** Expires In **353 days 12 hours 10 minutes** **Puts**

Last	Size	IV	Bid	Ask	IV	Size	Vol	Δ|Delta	Strike	Last	Size	IV	Bid	Ask	IV	Size	Vol	Δ|Delta
-	-	-	- -	- -	-	-	-	0.94	**12000**	0.0430	9.8	105.4%	0.0405 $1463.15	0.0525 $1896.68	115.7%	1.0	10.6	-0.06
0.6990	-	-	- -	- -	-	-	-	0.90	**16000**	0.0725	9.2	104.1%	0.0765 $2766.09	0.0970 $3507.33	116.2%	9.2	12.5	-0.10
0.6335	9.0	89.8%	0.5415 $19638.92	0.5955 $21597.37	115.0%	9.0	-	0.86	**20000**	0.1385	8.2	104.0%	0.1240 $4484.77	0.1440 $5208.12	113.2%	10.0	24.7	-0.14
0.6030	9.0	91.9%	0.5105 $18267.15	0.5685 $20342.56	116.0%	9.0	0.9	0.84	**22000**	0.1325	7.6	103.8%	0.1510 $5457.44	0.1740 $6288.71	113.5%	9.0	19.7	-0.16
0.5105	8.2	94.5%	0.4920 $17789.75	0.5430 $19633.81	114.2%	9.0	0.1	0.82	**24000**	0.1880	7.0	103.8%	0.1800 $6508.45	0.2030 $7340.08	112.7%	8.0	8.5	-0.18
0.5655	9.2	95.5%	0.4670 $16796.61	0.5195 $18684.88	114.4%	9.0	-	0.80	**26000**	0.2095	6.4	103.8%	0.2110 $7625.96	0.2340 $8457.23	112.1%	10.0	67.4	-0.20
0.4985	8.1	95.1%	0.4400 $15744.46	0.4970 $17784.08	114.4%	9.0	1.4	0.78	**28000**	0.2445	5.8	103.5%	0.2425 $8768.32	0.2675 $9672.27	112.1%	7.0	42.7	-0.22
0.4500	0.2	95.5%	0.4010 $14504.49	0.4415 $15969.40	108.0%	5.0	7.0	0.74	**32000**	0.3205	4.7	103.0%	0.3105 $11222.09	0.3335 $12053.36	110.2%	6.0	13.7	-0.26
0.3810	9.0	94.5%	0.3260 $11782.29	0.3750 $13553.25	108.0%	5.0	63.7	0.65	**40000**	0.4000	4.0	92.0%	0.4235 $15301.39	0.4930 $17812.48	111.1%	5.0	3.6	-0.35
0.3130	8.4	101.9%	0.2995 $10839.27	0.3405 $12323.11	112.7%	9.0	12.0	0.60	**48000**	0.6800	3.0	0.0%	0.3285 $11868.96	1.0170 $36745.01	223.5%	0.1	4.3	-0.40
0.2630	8.1	103.2%	0.2265 $8189.79	0.2610 $9437.25	112.0%	8.8	17.9	0.50	**64000**	1.2500	-	-	- -	1.2500 $44719.63	165.7%	1.0	1.0	-0.51
0.2255	8.5	110.8%	0.2030 $7340.08	0.2215 $8009.00	115.6%	8.5	33.6	0.44	**80000**	-	-	-	- -	- -	-	-	-	-0.57
0.1865	8.9	113.0%	0.1645 $5948.00	0.1835 $6635.00	118.1%	8.9	31.4	0.37	**100000**	-	0.1	0.0%	0.0005 $17.89	- -	-	-	-	-0.63
0.1450	9.2	114.9%	0.1375 $4967.49	0.1575 $5690.03	120.5%	20.2	33.5	0.32	**120000**	-	-	-	- -	- -	-	-	-	-0.68
0.1375	1.5	117.1%	0.1200 $4338.96	0.1300 $4700.54	120.1%	10.0	4.0	0.28	**140000**	-	-	-	- -	- -	-	-	-	-0.72
0.1245	9.6	117.8%	0.1025 $3707.17	0.1220 $4412.44	123.9%	21.6	41.1	0.25	**160000**	-	0.1	0.0%	0.0005 $17.89	- -	-	-	-	-0.75

- Let us take a look a bit at the history of options and types of options available in the market.

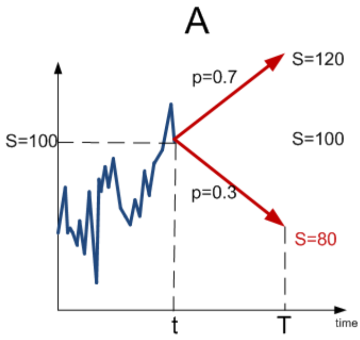
Financial derivatives, Options

- 26th April 1973, options were first officially traded on an exchange, The Chicago Board Options Exchange (CBOE) created standardized, listed options. Initially there were just calls on 16 stocks. Puts weren't introduced until 1977.
- Options are traded on CBOE, the American Stock Exchange, the Pacific Stock Exchange and the Philadelphia Stock Exchange. Worldwide, there are over 50 exchanges on which options are traded.
- Option styles:
  - European option- an option that may be only exercised on expiration day;
  - American option - an option that may be exercised on any trading day (also on the expiration);
  - Bermudan option - an option that may be exercises only on specified dates;
  - Exotic or path dependent options (value does not only depend on the current stock value, but, for example, also on the average value of the stock price development, OTC).

Back to standard options: Call

- European call option: At a prescribed time in the future, maturity or expiry date, the holder of the option may purchase a prescribed asset for a prescribed amount (the exercise or strike price).
  The other party of this contract (the writer, or seller) must sell the asset, if the holder decides to buy it.
- Portfolio: Set of securities (shares, options, . . .).
- Financial Engineering: Choosing suitable options in an optimal portfolio.
- Main interesting questions: How much would one pay for this right ? (What's the value of an option ?)
- How can the writer minimize the risk associated ?

The value of an option example:



For $K = 100$ which call option is more expensive $V_C^A < V_C^B$, $V_C^A > V_C^B$, $V_C^A = V_C^B$?

To find the answer we follow the reasoning of an option writer...

The value of an option Example cont.:

Writers, in order to hedge their short call option position buy some (Δ) stocks, so their position equals:

$$\Pi(t) = V_{C,0} - \Delta \cdot S_0,$$

Now, we consider two scenarios:

- Stock goes UP:

$$\begin{aligned} \Pi_{up} &= V_{C,0} - \Delta S_0 + \Delta S_{up} - S_{up} + K && (1)\\ &= V_{C,0} - \Delta S_0 + \Delta S_{up} - 20 && (2) \end{aligned}$$

- Stock goes DOWN:

$$\Pi_d = V_{C,0} - \Delta S_0 + \Delta S_d; \qquad (3)$$

Since writers don't want to take any risk:

$$\left\{ \begin{array}{rcl} V_{C,0} - \Delta S_0 + \Delta S_{up} - 20 &=& 0 \\ V_{C,0} - \Delta S_0 + \Delta S_d &=& 0; \end{array} \right. \qquad (4)$$

The value of an option Example cont.:

Simple algebra gives:

$$\Delta = \frac{20}{S_{up} - S_d}.$$

So now we have:

- Case A:

$$\Delta = \frac{1}{2},$$

  which gives $V_{C,0} = 10$

- Case B:

$$\Delta = \frac{4}{9},$$

  which gives $V_{C,0} = 11.1$.

How does a writer make money???

Options & Payoffs

The value of European call option at the expiry T is given by:

$$V_C(T, S(T)) = \max\left(S(T) - K, 0\right).$$

The value of European put option at the expiry T is given by:

$$V_P(T, S(T)) = \max\left(K - S(T), 0\right).$$

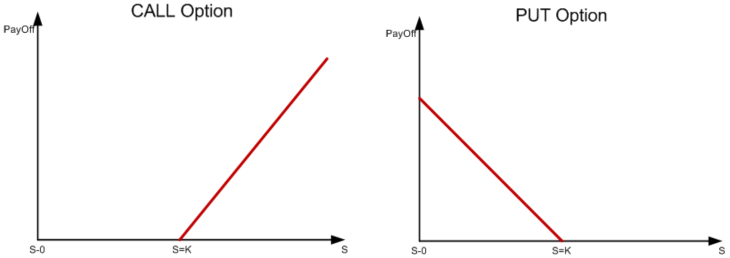


Figure: Payoff diagram for European Call (left), and European Put (right).

The value of an option

What determines the value of an option?

- what is the asset price today $S(t)$?
- how long there is until expiry $T - t$?
- how volatile is the asset $S(t)$?

General principles:

- The longer the time to expiry, the more time there is for the asset to rise or fall;
- The more the asset is volatile the higher the chance that it will rise or fall;

Put-Call Parity

A relation between value $V(t, S(t))$ of European call and put options with the same strike price K and expiry T. Suppose we have given two portfolios at time $t = 0$:

- Π_A : one call option plus Ke^{-rT} cash (invested in the bank);
- Π_B : one put option plus one unit of the asset.

At expiry portfolio Π_A is worth:

$$\max(S(T) - K, 0) + K = \max(S(T), K),$$

The portfolio Π_B is worth on expiry:

$$\max(K - S(T), 0) + S(T) = \max(K, S(T)).$$

So, we conclude:

$$V_C(t, S(t)) + Ke^{-r(T-t)} = V_P(t, S(t)) + S(t),$$

This relationship is the so-called Put-Call Parity.
Note that we did not make any assumptions about stock $S(t)$!

Arbitrage

- There is never an opportunity to make a risk-free profit that gives a greater return than that provided by a bank.

Example: Suppose we have given a portfolio:

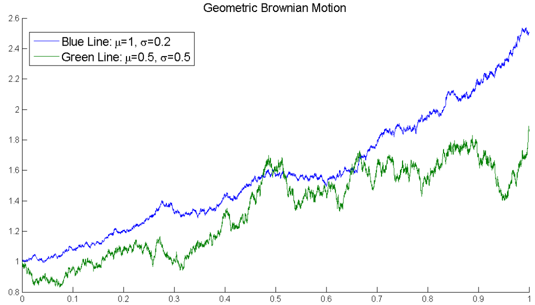
$$\Pi = V_P(t, S(t)) - V_C(t, S(t)) + S(t),$$

with $V_P(t, S(t))$ and $V_C(t, S(t))$ having expiry date T,and strike K.

Question: What is the "fair" price of this portfolio ?

Randomness

- Asset prices thus have a large element of randomness. This does *not* mean that we cannot model these prices, but it does mean that the modelling must be done in a probabilistic sense.
- A first model for asset prices, a stochastic differential equation (SDE), is geometric Brownian motion.
- We will also look in detail into so-called stochastic volatility SDE models, like the Heston dynamics.

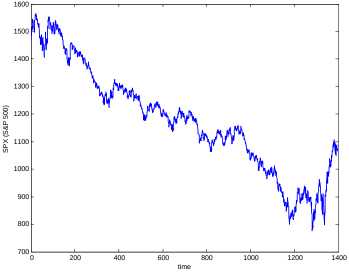


Modeling Asset Prices

When investing, the main concern is that the return on the investment is satisfactory. Suppose we have given asset S_t, then

$$\text{Return} = \frac{\text{Stock tomorrow} - \text{Stock today}}{\text{Stock today}} = \frac{S(t+\Delta) - S(t)}{S(t)}$$

Let's see it in practice! We take S&P index from 30-04-2002 to 09-11-2007 daily monitored.



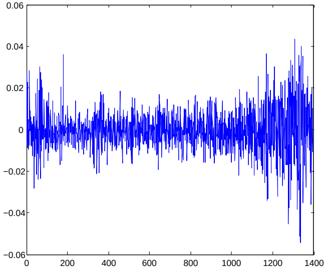
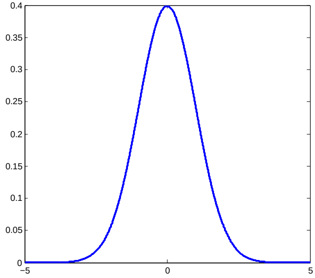
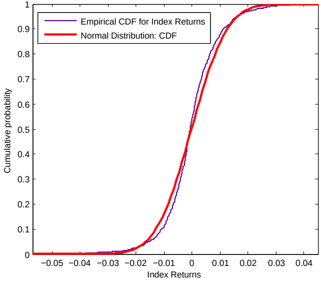


Figure: Left: S&P Index, Right: Index Return

Modeling Asset Prices

- From the data in this example we find that the mean is -0.000172 and the standard deviation is 0.0121;
- The distribution of the returns has been scaled and translated to give it a mean of zero, a standard deviation of one and an area under the curve of one.





Randomness of the stock prices

Daily returns for a certain asset look like noise!

- What can be then done?
- We can model the noise!

Definition (Wiener Process)

A stochastic process $W(t)$ for $t \in [0, \infty)$ is called a Wiener Process (or Brownian motion) if the following conditions are satisfied:

- *It starts at zero: $W_0 = 0$,*
- *It has stationary, independent increments,*
- *For every $t > 0$, $W(t)$ has a normal distribution with mean 0 and variance t,*
- *It has a.s. continuous paths with NO JUMPS.*

Wiener Process

A random variable $W(t)$ is known as a Wiener process, with:

- $\mathrm{d}W(t)$ is from normal distribution;
- the mean of $\mathrm{d}W(t) = 0$, i.e. $\mathbb{E}(\mathrm{d}W(t)) = 0$,
- the variance is $\mathrm{d}t$, i.e. $\mathrm{d}W(t) := \sqrt{\mathrm{d}t} \cdot Z$, where Z is normally distributed with mean equal to 0 and variance 1;

Stochastic Processes

Suppose we observe the stock price of a company at every fixed instance t after some initial time t_0 and known T.

- We can interpret the observed stock values as a realization $X(t)(\omega)$ of the random variable $X(t)$.
- We look for a model which takes into account almost continuous realizations of the stock prices.

Definition (Stochastic Process)

A stochastic process $X(t)$ is a collection of random variables

$$(X(t), t \in \mathcal{T}) = (X(t)(\omega), t \in \mathcal{T}, \omega \in \Omega)$$

A stochastic process $X(t)$ is a function of two variables:

- for a fixed instant t it's a random variable:

$$X(t) = X(t)(\omega) \text{ for } \omega \in \Omega,$$

- for a fixed random outcome $\omega \in \Omega$, it's a function of time

$$X(t) = X(t)(\omega), \text{ for } t \in \mathcal{T}.$$

Process trajectories

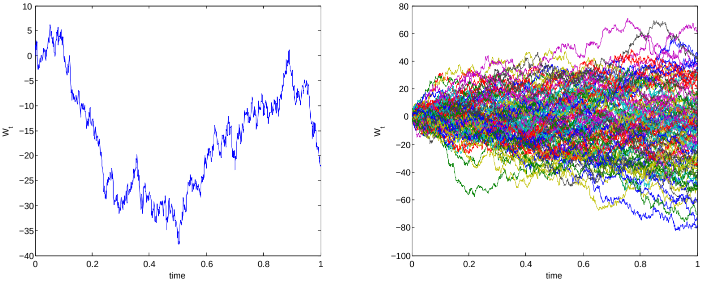


Figure: Sample paths of Brownian motion on $[0, 1]$. Left: 1 path, Right: 100 paths.

Modeling the stock prices/returns; Samuelson's model

The most popular stochastic process for generating stock prices is the Geometric Brownian Motion process (GBM):

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t),$$

which can be translated to:

$$S(t+\Delta t) = S(t) + \mu S(t)\Delta t + \sigma S(t)(W(t+\Delta t) - W(t)),$$

where:

- $\mu \mathrm{d}t$ is a deterministic return,
- $\sigma \mathrm{d}W(t)$ is the random change with $\mathrm{d}W(t)$ a sample from a normal distribution with mean 0 and variance dt.
- The SDE is the short form for:

$$S(t) = S_0 + \int_{t_0}^{t} \overline{\mu}(s, S(s))\mathrm{d}s + \int_{t_0}^{t} \overline{\sigma}(s, S(s))\mathrm{d}W(s),$$

  with $\overline{\mu}(t, S(t)) = \mu S(t)$ the drift and $\overline{\sigma}(t, S(t)) = \sigma S(t)$ the diffusion term.

$\Rightarrow$ Financial mathematics course!

Samuelson's model

Let us check the moments of $\mathrm{d}S(t)$:

- For the expectation we find:

$$\begin{aligned}\mathbb{E}_t\left(\mathrm{d}S(t)\right) &= \mu\mathbb{E}_t(S(t)\mathrm{d}t) + \sigma\mathbb{E}_t(S(t)\mathrm{d}W(t))\\ &= \mu S(t)\mathrm{d}t\end{aligned}$$

- The variance is equal to

$$\begin{aligned}\mathbb{V}\mathrm{ar}_t(\mathrm{d}S(t)) &= \mathbb{E}_t(\mathrm{d}S^2(t)) - (\mathbb{E}_t(\mathrm{d}S(t)))^2\\ &= \sigma^2 S(t)^2\mathrm{d}t\end{aligned}$$

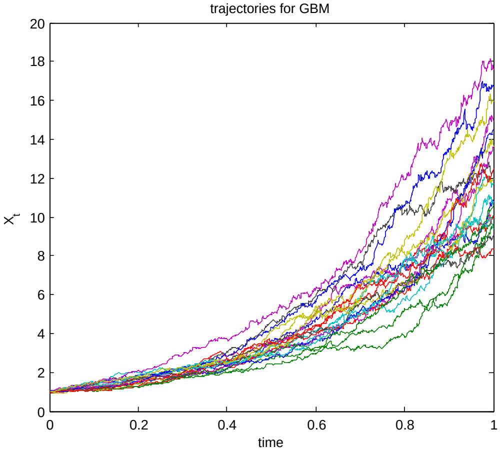
- The equation does not refer to past history of the asset price. Independence from the past is also known as the Markov property.

Properties

- This model fits real time data very well for equities and indices. (The agreement with currencies is less good, especially in the long term.)
- Discrepancies: real data appears to have a greater probability of large rises and falls than the model predicts.
- The drift is not apparent over short time scales for which the volatility dominates.
- Over long time scales, for instance decades, the drift becomes important.
- Alternatives: asset price model includes jumps or adopt stochastic volatility.

Modeling the returns: Continuation

$$S(t+\Delta t) = S(t) + \mu S(t)\Delta t + \sigma S(t)(W(t+\Delta t) - W(t)),$$



Types of processes

We compare the behaviour of the following SDEs:

- Arithmetic Brownian motion (ABM)

$$\mathrm{d}X(t) = \mu \mathrm{d}t + \sigma \mathrm{d}W(t),$$

- Geometric Brownian Motion (GBM)

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t),$$

- Ornstein-Uhlenbeck mean reverting processes (OU)

$$\mathrm{d}X(t) = \kappa(\theta - X(t))\mathrm{d}t + \sigma \mathrm{d}W(t),$$

where: μ, σ, κ and θ are known constants, and $W(t)$ is a Wiener process.

We also cover multidimensional extensions.
What is the difference between these processes?

Process trajectories

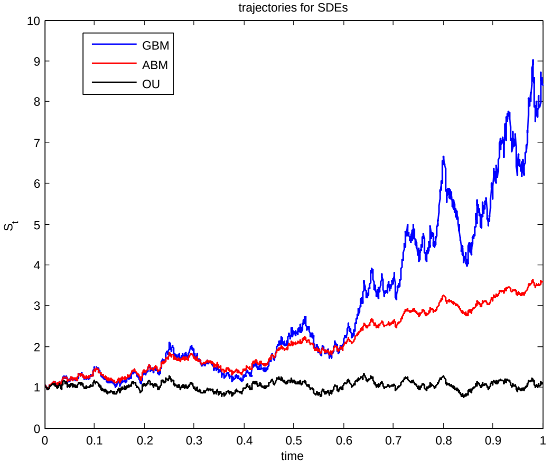


Figure: Process trajectories generated from the same random path. The parameters are: $\mu = 0.05$, $\sigma = 0.7$, $\kappa = 1.5$ and $\theta = 1$.

Ito's Lemma

Ito's Lemma is fundamental for stochastic processes. It helps in deriving solutions to stochastic differential equations (SDE).

Lemma (Ito's Lemma)

Suppose $X(t)$ follows an Ito process:

$$\mathrm{d}X(t) = \bar{\mu}(t, X(t))\mathrm{d}t + \bar{\sigma}(t, X(t))\mathrm{d}W(t), \text{ with } X_0 = x_0,$$

is the short form for:

$$X(t) = X_0 + \int_{t_0}^{t} \bar{\mu}(s, X(s))\mathrm{d}s + \int_{t_0}^{t} \bar{\sigma}(s, X(s))\mathrm{d}W_s,$$

with $\bar{\mu}(t, X(t))$ the drift term and $\bar{\sigma}(t, X(t))$ the diffusion term. Let now $g(t, X(t))$ be a function with continuous derivatives (up to second order). Then $Y(t) := g(t, X(t))$ follows an Ito process with the same Wiener process $W(t)$:

$$dY(t) = \left(\frac{\partial g}{\partial t} + \bar{\mu}\frac{\partial g}{\partial x} + \frac{1}{2}\bar{\sigma}^2\frac{\partial^2 g}{\partial x^2}\right)\mathrm{d}t + \bar{\sigma}\frac{\partial g}{\partial x}\mathrm{d}W(t).$$

Itô's Lemma

- By Itô's lemma we can handle $\mathrm{d}W(t)$ as $\mathrm{d}t \to 0$ (like Taylor's expansion for deterministic variables)
- Expansion for $g(t, X)$: $dg = \frac{\partial g}{\partial t}\mathrm{d}t + \frac{\partial g}{\partial X}\mathrm{d}X + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}\mathrm{d}X^2 + \ldots$.
- With $\mathrm{d}X(t) = \bar{\mu}(t, X(t))\mathrm{d}t + \bar{\sigma}(t, X(t))\mathrm{d}W(t)$ ($X(t)$ is a stochastic process), and the insight $\mathrm{d}W(t)^2 \to \mathrm{d}t \quad \text{as} \quad \mathrm{d}t \to 0$.

$$\mathrm{d}X(t)^2 = \bar{\sigma}^2(t, X(t))\mathrm{d}W(t)^2 + 2\bar{\mu}(t, X(t))\bar{\sigma}(t, X(t))\mathrm{d}t\mathrm{d}W(t) + \bar{\mu}^2(t, X(t))$$

- $\mathrm{d}W(t) = O(\sqrt{\mathrm{d}t})$, gives to leading order $\mathrm{d}X(t)^2 \to \bar{\sigma}^2(t, X(t))\mathrm{d}W(t)^2$

$$\begin{aligned} dg &= \frac{\partial g}{\partial t}\mathrm{d}t + \frac{\partial g}{\partial X}(\bar{\sigma}(t, X)\mathrm{d}W(t) + \bar{\mu}(t, X)\mathrm{d}t) + \frac{1}{2}\bar{\sigma}^2(t, X)\frac{\partial^2 g}{\partial X^2}\mathrm{d}t \\ &= (\frac{\partial g}{\partial t} + \bar{\mu}(t, X)\frac{\partial g}{\partial X} + \frac{1}{2}\bar{\sigma}^2(t, X)\frac{\partial^2 g}{\partial X^2})\mathrm{d}t + \bar{\sigma}(t, X)\frac{\partial g}{\partial X}\mathrm{d}W(t) \end{aligned}$$

- To be technically, mathematically correct, we should work with the SDE in integrated form and work with the rules from stochastic calculus: the end result is the same.

Ito's Lemma: Example

Example: We apply Itô's formula to the function

$$g(t, S) = \log S,$$

where $S(t)$ is given by:

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t),$$

By Ito's formula we have:

$$dg = \frac{\partial g}{\partial t}\mathrm{d}t + \frac{\partial g}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}(\mathrm{d}S)^2, \tag{5}$$

we have:

$$\frac{\partial g}{\partial t} = 0,\ \frac{\partial g}{\partial S} = \frac{1}{S} \text{ and } \frac{\partial^2 g}{\partial S^2} = -\frac{1}{S^2},$$

so:

$$dg = 0 \cdot \mathrm{d}t + \frac{1}{S}\mathrm{d}S - \frac{1}{2}\frac{1}{S^2}(\mathrm{d}S)^2, \tag{6}$$

Ito's Lemma: Example

With

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t),$$

$$\begin{aligned} dg(t,S) &= \frac{1}{S}\mathrm{d}S - \frac{1}{2}\frac{1}{S^2}(\mathrm{d}S)^2 \\ &= \frac{1}{S}\left(\mu S\mathrm{d}t + \sigma S\mathrm{d}W\right) - \frac{1}{2}\frac{1}{S^2}\left(\sigma^2 S^2\mathrm{d}t\right) \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W. \end{aligned}$$

We have found that

$$dg(t,S) = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W.$$

Distribution and density

- A normally distributed random variable X with expectation μ and variance σ^2 can be represented by the probability distribution:

$$F(x) = \mathrm{P}(X \leq x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp -(\frac{(z-\mu)^2}{2\sigma^2}) dz$$

- With $X(t) = g = \log S(t)$, we find $F_{X(t)}(x) =$

$$\frac{1}{\sigma\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{x} \exp -(\frac{(z-\log(S_0)-(\mu-\frac{1}{2}\sigma^2)(t-t_0))^2}{2\sigma^2(t-t_0)}) dz$$

- With probability density function:

$$f_{X(t)}(x) = \frac{1}{\sigma\sqrt{2\pi(t-t_0)}} \exp -(\frac{(x-\log(S_0)-(\mu-\frac{1}{2}\sigma^2)(t-t_0))^2}{2\sigma^2(t-t_0)})$$

Corresponding density

- Probability distribution for S: Define $S = \exp(X)$:

$$F_{S(t)}(x) = \quad \mathrm{P}(S \leq x) = \mathrm{P}(e^X \leq x) = \mathrm{P}(X \leq \log x) = F_{X(t)}(\log x)$$
$$= \frac{1}{\sigma\sqrt{2\pi(t-t_0)}} \quad \int_{-\infty}^{\log x} \exp -\left(\frac{(z - \log(S_0) - (\mu - \frac{1}{2}\sigma^2)(t-t_0))^2}{2\sigma^2(t-t_0)}\right) dz$$

- The probability density function for S , the log-normal density, reads

$$f_{S(t)}(x) = \frac{1}{\sigma x\sqrt{2\pi(t-t_0)}} \exp -\left(\frac{(\log x/S_0 - (\mu - \sigma^2/2)(t-t_0))^2}{2\sigma^2(t-t_0)}\right)$$

- Expectation, second moment and variance of $S(t)$:
  $\mathbb{E}[S(t)] = S_0 e^{\mu(t-t_0)}$;
  $\mathbb{E}[S(t)^2] = S_0^2 e^{(2\mu+\sigma^2)(t-t_0)}$;
  $\mathrm{var}[S(t)] = S_0^2 e^{2\mu}(e^{\sigma^2(t-t_0)-1})$.

Simulated Dynamics and Density

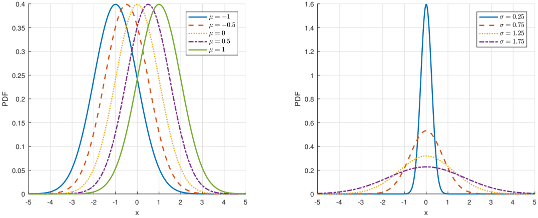


Figure: Probability density function for the normal distribution. Left: the impact of parameter μ, with $\sigma = 1$; Right: the impact of σ, using $\mu = 0$.

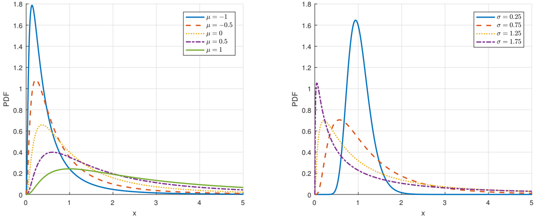


Figure: Probability density function for the lognormal distribution. Left: the impact of parameter μ, with $\sigma = 1$; Right: impact of parameter σ, with $\mu = 0$.

Simulated Dynamics and Density

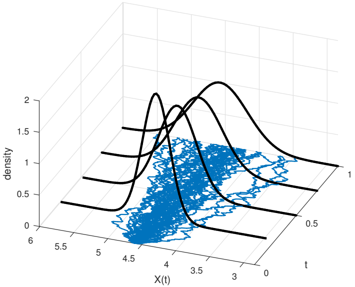


Figure: Paths and the corresponding densities. Left: $X(t) = \log S(t)$ and Right: $S(t)$ with the following configuration: $S_0 = 100$, $\mu = 0.05$, $\sigma = 0.4$; $T = 1$.