List of content

Financial industry; Work at banks

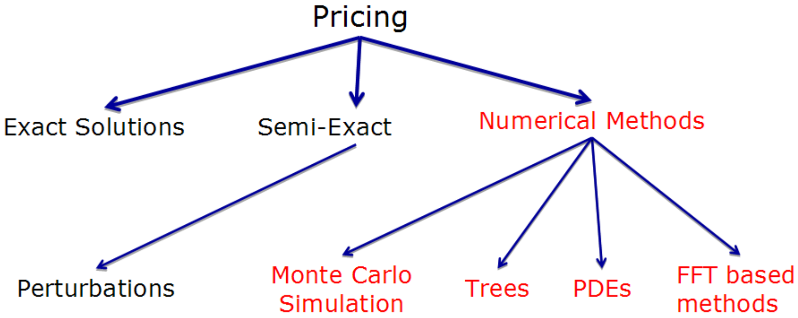
Front office $\Leftrightarrow$ Back office

Pricing and selling products $\Leftrightarrow$ Validation of prices, research into alternative models

- Pricing approach:
  1. Start with some financial product
  2. Model asset prices involved (SDEs)
  3. Model product price correspondingly (P(I)DE or integral)
  4. Calibrate the model to market data (numerics, optimization)
  5. Price the product of interest (numerics, MC)
  6. Set up a hedge to remove the risk to the product (optimization)

Pricing Techniques

There are several different approaches for obtaining information on the prices of options:



Semi-Exact Solutions for option pricing

- It is generally difficult to find an analytic solution for multi-dimensional correlated stochastic differential equations;
- Monte-Carlo methods are straightforward but:
  - Depends on the sampling seed;
  - Involves sampling error;
  - Requires powerful computing machines;

Alternative methods need to be used!

- Alternatives to Monte-Carlo methods for pricing derivatives are for example Fourier based algorithms, that are based on determining the characteristic function.
- Although for complicated models the distribution is unknown analytically, the corresponding characteristic function can be often derived analytically/semi-analytically;

Motivation Fourier Methods

- Derive pricing methods that
  - are computationally fast
  - are not restricted to Gaussian-based models
  - should work as long as we have a characteristic function,

$$\phi(u) = \int_{-\infty}^{\infty} \mathrm{e}^{iux} f(x) \mathrm{d}x;$$

    (available for Lévy processes and also for Heston's model).
  - In probability theory a characteristic function of a continuous random variable X, equals the Fourier transform of the density of X.

Fourier Transformation

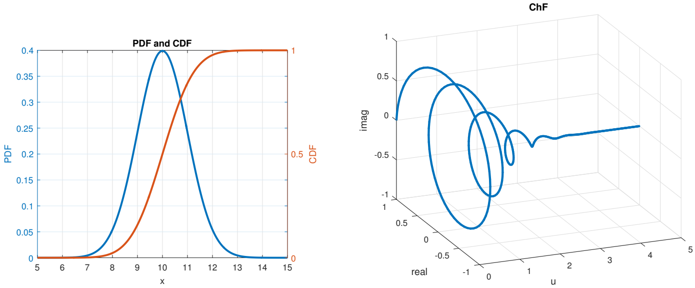


Figure: The CDF, PDF and characteristic function for an $\mathcal{N}(10, 1)$ random variable.

Fourier Transformation

- The continuous Fourier transform is one of the most important transforms in the signal analysis.
- It transforms one function into another, which is called the frequency domain representation of the original function (where the original function is often a function in the time-domain).
- In this specific case, both domains are continuous and unbounded.
- There are several common conventions for defining the Fourier transform of a complex-valued Lebesgue integrable functions.
- In communications and signal processing,

Fourier Transformation

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ which is in L^1, i.e.,

$$\int_{-\infty}^{+\infty} |f(x)| \, \mathrm{d}x < \infty,$$

and if $f(x)$ is continuous, then the Fourier transform of $f(x)$ is defined as:

$$\phi(u) = \mathbb{E}\left[\mathrm{e}^{iuX}\right] = \int_{-\infty}^{+\infty} \mathrm{e}^{iux} f(x) \mathrm{d}x = \int_{-\infty}^{+\infty} \mathrm{e}^{iux} \mathrm{d}F(x),$$

where $x \in \mathbb{R}$.

Characteristic Function and Useful Properties

- A useful fact regarding $\phi_X(u)$ is that it uniquely determines the distribution function of X. Moreover, the moments of random variable X can also be derived by $\phi_X(u)$, as

$$\mathbb{E}\left[X^k\right] = \frac{1}{i^k}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\phi_X(u)\Big|_{u=0},$$

  with i again the imaginary unit, for $k \in \{0, 1, ...\}$, assuming $\mathbb{E}[|X|^k] < \infty$.

- For $X = \log Y$, the corresponding characteristic function reads:

$$\begin{aligned}\phi_{\log Y}(u) &= \mathbb{E}\left[\mathrm{e}^{iu\log Y}\right] = \int_0^\infty \mathrm{e}^{iu\log y} f_Y(y)\mathrm{d}y \\ &= \int_0^\infty y^{iu} f_Y(y)\mathrm{d}y. \qquad (1)\end{aligned}$$

  Note that we use $\log Y \equiv \log_e Y \equiv \ln Y$. By setting $u = -ik$, we have:

$$\phi_{\log Y}(-ik) = \int_0^\infty y^k f_Y(y)\mathrm{d}y \stackrel{\text{def}}{=} \mathbb{E}[Y^k]. \qquad (2)$$

ChF for the Black-Scholes Model

With a substitution, $X(t) = \log S(t)$, we have:

$$U(X,t) = V(S,t).$$

So:

$$\begin{cases} \dfrac{\partial V}{\partial t} = \dfrac{\partial U}{\partial t}, \\ \dfrac{\partial V}{\partial S} = \dfrac{\partial U}{\partial X}\dfrac{\partial X}{\partial S} = \dfrac{1}{S}\dfrac{\partial U}{\partial X}, \\ \dfrac{\partial^2 V}{\partial S^2} = -\dfrac{1}{S^2}\dfrac{\partial U}{\partial X} + \dfrac{1}{S^2}\dfrac{\partial^2 U}{\partial X^2} \end{cases}$$

The pricing PDE now reads:

$$\frac{\partial U}{\partial t} + rS\frac{1}{S}\frac{\partial U}{\partial X} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\frac{\partial U}{\partial X} + \frac{1}{S^2}\frac{\partial^2 U}{\partial X^2}\right) - rU = 0,$$

which simply becomes:

$$\frac{\partial U}{\partial t} + r\frac{\partial U}{\partial X} + \frac{1}{2}\sigma^2\left(-\frac{\partial U}{\partial X} + \frac{\partial^2 U}{\partial X^2}\right) - rU = 0.$$

An example: Black-Scholes Case

By setting

$$\tau = T - t,$$

we have:

$$-\frac{\partial U}{\partial \tau} + (r - \frac{1}{2}\sigma^2)\frac{\partial U}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial X^2} - rU = 0.$$

By the results of Duffie-Pan-Singleton, we probe the following solution:

$$U(X,t) := \phi(u,\tau) = \mathrm{e}^{A(u,\tau)+B(u,\tau)X},$$

with boundary condition: $\phi(u,0) = \mathrm{e}^{iuX}$. By partial differentiation we have:

$$\begin{cases} \dfrac{\partial \phi}{\partial \tau} = \phi\left(\dfrac{\partial A}{\partial \tau} + X\dfrac{\partial B}{\partial \tau}\right), \\ \dfrac{\partial \phi}{\partial X} = \phi B, \\ \dfrac{\partial^2 \phi}{\partial X^2} = \phi B^2. \end{cases} \tag{3}$$

An example: Black-Scholes Case

Now, by substituting these quantities in the pricing PDE we have:

$$-\phi\left(\frac{\partial A}{\partial\tau}+X\frac{\partial B}{\partial\tau}\right)+\left(r-\frac{1}{2}\sigma^2\right)\phi B+\frac{1}{2}\sigma^2\phi B^2-r\phi=0,$$

or

$$-\left(\frac{\partial A}{\partial\tau}+X\frac{\partial B}{\partial\tau}\right)+\left(r-\frac{1}{2}\sigma^2\right)B+\frac{1}{2}\sigma^2 B^2-r=0.$$

From above we obtain the set of ODEs in the following way:

$$\begin{cases}\dfrac{\partial B}{\partial\tau}=0,\\[1ex] \dfrac{\partial A}{\partial\tau}=\left(r-\dfrac{1}{2}\sigma^2\right)B+\dfrac{1}{2}\sigma^2 B^2-r.\end{cases}\tag{4}$$

An example: Black-Scholes Case

By using the boundary conditions we find

$$\begin{cases} B(u,\tau) = iu, \\ A(u,\tau) = \left(r - \frac{1}{2}\sigma^2\right) iu\tau - \frac{1}{2}\sigma^2 u^2 \tau - r\tau. \end{cases} \tag{5}$$

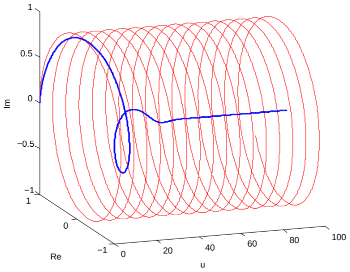
So the obtained solution (characteristic function) is given by

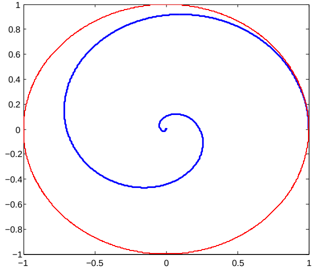
$$\phi(u,\tau) = \mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)iu\tau - \frac{1}{2}\sigma^2 u^2 \tau - r\tau + iuX}.$$

Example: Black-Scholes model

The characteristic function for the Black-Scholes asset price is given by:

$$\phi(u,\tau) = \exp\left(i(\log(S_0) + (r - \frac{1}{2}\sigma^2)\tau)u - \frac{1}{2}\sigma^2\tau u^2 \right),$$





Class of Affine Diffusion (AD) processes

More generally: Suppose we have the following system of SDEs:

$$\mathrm{d}\mathbf{X}(t) = \bar{\mu}(\mathbf{X}(t))\mathrm{d}t + \bar{\sigma}(\mathbf{X}(t))\mathrm{d}\widetilde{\mathbf{W}}(t),$$

with independent Brownian motions $\widetilde{\mathbf{W}}(\mathbf{t})$. For processes in the affine diffusion (AD) class it is assumed that drift, volatility, and interest rate components are of the affine form, i.e.

$$\begin{aligned}
\bar{\mu}(\mathbf{X}(t)) &= a_0 + a_1\mathbf{X}(t) \text{ for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n},\\
\bar{\sigma}(\mathbf{X}(t))\bar{\sigma}(\mathbf{X}(t))^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}(t), (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n},\\
r(\mathbf{X}(t)) &= r_0 + r_1^T\mathbf{X}(t), \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n.
\end{aligned}$$

Characteristic function for AD

Duffie, Pan and Singleton (2000) have shown that for affine diffusion processes the discounted characteristic function, defined as:

$$\phi(\mathbf{X}(\mathbf{t}),\mathbf{t},\mathbf{T},\mathbf{u}) \equiv \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(\mathbf{X_s})ds}\mathrm{e}^{i\mathbf{u}\mathbf{X_T}}\middle|\mathcal{F}(t)\right] \text{ for } \mathbf{u}\in\mathbb{C}^{\mathbf{n}},$$

with boundary condition:

$$\phi(\mathbf{X_T},\mathbf{T},\mathbf{T},\mathbf{u}) = \mathrm{e}^{i\mathbf{u}^T\mathbf{X_T}},$$

has a solution of the following form:

$$\phi(\mathbf{X}(\mathbf{t}),\mathbf{t},\mathbf{T},\mathbf{u}) = \mathrm{e}^{A(\mathbf{u},t,T)+\mathbf{B}(\mathbf{u},t,T)^T\mathbf{X}(\mathbf{t})},$$

How to find the coefficients $A(\mathbf{u},t,T)$ and $\mathbf{B}(\mathbf{u},\mathbf{t},\mathbf{T})^{\mathbf{T}}$?

Characteristic function for AD

The coefficients $A(\mathbf{u}, t, T)$ and $\mathbf{B}(\mathbf{u}, \mathbf{t}, \mathbf{T})^{\mathbf{T}}$ have to satisfy the following system of Riccati-type ODEs[1]:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} A(\mathbf{u}, \tau) &= -r_0 + \mathbf{B}^T a_0 + \frac{1}{2}\mathbf{B}^T c_0 \mathbf{B}, \\
\frac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{B}(\mathbf{u}, \tau) &= -r_1 + a_1^T \mathbf{B} + \frac{1}{2}\mathbf{B}^T c_1 \mathbf{B}.
\end{aligned}
$$

[1] Note that we do not consider jumps.

An example: Black-Scholes

For a given stock-process

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t),$$

with the money savings account $M(t)$:

$$\mathrm{d}M(t) = rM(t)\mathrm{d}t,$$

the pricing PDE is given by:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{6}$$

Black Scholes Model

For GBM we have the following SDE:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t),$$

The process is not affine because of

$$\bar{\sigma}(S(t))\bar{\sigma}(S(t)) = \sigma^2 S(t)^2,$$

To consider the process into the affine class we define:

$$x(t) = \log S(t),$$

which gives following SDE

$$\mathrm{d}\log S(t) = \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W^{\mathbb{Q}}(t)$$

Black Scholes Model

The model is in the AD class of processes, moreover we have:

$$\bar{\mu}(x(t)) = \underbrace{r - \frac{1}{2}\sigma^2}_{a_0} + \underbrace{0}_{a_1} x(t),$$

$$\bar{\sigma}(x(t))\bar{\sigma}(x(t)) = \underbrace{\sigma^2}_{c0} + \underbrace{0}_{c_1} x(t),$$

and

$$r(x(t)) = r + \underbrace{0}_{r_1} x(t)$$

Black Scholes Model

In order to find the characteristic function:

$$\phi(u,\tau) = \mathrm{e}^{A(u,\tau)+B(u,\tau)x_0}$$

we set up the system of ODEs

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}B(u,\tau) &= -r_1 + a_1 B(u,\tau) + \frac{1}{2}B(u,\tau)c_1 B(u,\tau) \\ \frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) &= -r_0 + a_0 B(u,\tau) + \frac{1}{2}B(u,\tau)c_0 B(u,\tau) \end{cases}$$

which reads:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}B(u,\tau) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}\tau}A(u,\tau) &= -r + \left(r - \frac{1}{2}\sigma^2\right) B(u,\tau) + \frac{1}{2}\sigma^2 B(u,\tau)B(u,\tau) \end{cases}$$

Black Scholes Model

By taking the boundary conditions:

$$B(u,0) = iu,$$

and

$$A(u,0) = 0,$$

we obtain:

$$\begin{cases} B(u,\tau) = iu, \\ A(u,\tau) = \left[-r + iu\left(r - \frac{1}{2}\sigma^2\right) - \frac{1}{2}u^2\sigma^2\right]\tau. \end{cases}$$

The characteristic function for BM is now given by:

$$\phi(u,\tau) = \mathrm{e}^{iu \log S_0 + iu\left(r - \frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}u^2\sigma^2\tau - r\tau}$$

2D uncorrelated GBM

- The 2D uncorrelated GBM process, with $\mathbf{S}(t) = [S_1(t), S_2(t)]$, is not affine, and therefore the logarithmic transformation, $\mathbf{X}(t) = [\log S_1(t), \log S_2(t)] =: [X_1(t), X_2(t)]$, should be performed, i.e.

$$\begin{cases} \mathrm{d}X_1(t) &= \left(r - \frac{1}{2}\sigma_1^2\right)\mathrm{d}t + \sigma_1 \mathrm{d}\widetilde{W}_1(t), \\ \mathrm{d}X_2(t) &= \left(r - \frac{1}{2}\sigma_2^2\right)\mathrm{d}t + \sigma_2 \mathrm{d}\widetilde{W}_2(t), \end{cases} \tag{7}$$

  with independent Brownian motions $\widetilde{W}_1(t)$, $\widetilde{W}_2(t)$.

- In matrix notation, this system reads,

$$\begin{aligned} \begin{bmatrix} \mathrm{d}X_1(t) \\ \mathrm{d}X_2(t) \end{bmatrix} &= \begin{bmatrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{bmatrix}\mathrm{d}t + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}\begin{bmatrix} \mathrm{d}\widetilde{W}_1(t) \\ \mathrm{d}\widetilde{W}_2(t) \end{bmatrix} \Leftrightarrow \\ \mathrm{d}\mathbf{X}(t) &= \bar{\boldsymbol{\mu}}(t, \mathbf{X}(t))\mathrm{d}t + \bar{\boldsymbol{\sigma}}(t, \mathbf{X}(t))\mathrm{d}\widetilde{\mathbf{W}}(t). \end{aligned} \tag{8}$$

- Following the definitions for the AD processes, we here find:

$$a_0 = \begin{bmatrix} r - \frac{1}{2}\sigma_1^2 \\ r - \frac{1}{2}\sigma_2^2 \end{bmatrix}, \quad r_0 = r, \quad \text{and } c_0 \quad = \quad \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}.$$

  Moreover, $a_1 = 0$, $c_1 = 0$ and $r_1 = 0$.

- For the affine system of SDEs, we can now easily derive the 2D characteristic function, with $\mathbf{u} = [u_1, u_2]$, and $\tau = T - t$,

$$\phi_{\mathbf{X}}(\mathbf{u}; t, T) = \mathrm{e}^{\bar{A}(\mathbf{u},\tau)+\bar{B}_1(\mathbf{u},\tau)X_1(t)+\bar{B}_2(\mathbf{u},\tau)X_2(t)}, \tag{9}$$

  with

$$\phi_{\mathbf{X}}(\mathbf{u}; T, T) = \mathrm{e}^{iu_1X_1(T)+iu_2X_2(T)}. \tag{10}$$

  In the present setting, we thus have $\bar{A}(\mathbf{u}, 0) = 0$, $\bar{B}_1(\mathbf{u}, 0) = iu_1$ and $\bar{B}_2(\mathbf{u}, 0) = iu_2$.

- Since a_1, c_1 and r_1 are zero vectors and matrices, the system of the ODEs is given by:

$$\frac{\mathrm{d}\bar{B}_1}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}\bar{B}_2}{\mathrm{d}\tau} = 0, \tag{11}$$

and

$$\begin{aligned}\frac{\mathrm{d}\bar{A}}{\mathrm{d}\tau} &= -r + iu_1\left(r - \frac{1}{2}\sigma_1^2\right) + iu_2\left(r - \frac{1}{2}\sigma_2^2\right)\\ &+ \frac{1}{2}(iu_1)^2\sigma_1^2 + \frac{1}{2}(iu_2)^2\sigma_2^2,\end{aligned}$$

so that $\bar{B}_1(\mathbf{u},\tau) = iu_1, \bar{B}_2(\mathbf{u},\tau) = iu_2$, and the solution for $\bar{A}(\mathbf{u},\tau)$ reads:

$$\begin{aligned}\bar{A}(\mathbf{u},\tau) &= -r\tau + iu_1\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)\tau + iu_2\left(\mu_2 - \frac{1}{2}\sigma_2^2\right)\tau\\ &- \frac{1}{2}u_1^2\sigma_1^2\tau - \frac{1}{2}u_2^2\sigma_2^2\tau.\end{aligned} \tag{12}$$

- The 2D characteristic function is therefore given by:

$$\phi_{\mathbf{X}}(\mathbf{u}; t, T) = \mathrm{e}^{iu_1X_1(t)+iu_2X_2(t)+\bar{A}(\mathbf{u},\tau)},$$

with the function $\bar{A}(\mathbf{u},\tau)$ from (12).

Affine Jump Diffusion Setting

- The stochastic model of interest can now be expressed by the following stochastic differential form:

$$\mathrm{d}\mathbf{X}(t) = \bar{\boldsymbol{\mu}}(t, \mathbf{X}(t))\mathrm{d}t + \bar{\boldsymbol{\sigma}}(t, \mathbf{X}(t))\mathrm{d}\widetilde{\mathbf{W}}(t) + \mathbf{J}(t)^{\mathrm{T}}\mathrm{d}\mathbf{X}_{\mathcal{P}}(t), \quad (13)$$

where $\widetilde{\mathbf{W}}(t)$ is an $\mathcal{F}(t)$-standard column vector of *independent* Brownian motions in $\mathbb{R}^n$, $\bar{\boldsymbol{\mu}}(t, \mathbf{X}(t)) : \mathbb{R} \to \mathbb{R}^n$, $\bar{\boldsymbol{\sigma}}\,(t, \mathbf{X}(t)) : \mathbb{R} \to \mathbb{R}^{n \times n}$, and $\mathbf{X}_{\mathcal{P}}(t) \in \mathbb{R}^n$ is a vector of orthogonal Poisson processes, characterized by an intensity vector $\bar{\boldsymbol{\xi}}(t, \mathbf{X}(t)) \in \mathbb{R}^n$.

Affine Jump Diffusion Setting

- For processes in the AJD class, the drift term $\bar{\boldsymbol{\mu}}(t,\mathbf{X}(t))$, covariance matrix $\bar{\boldsymbol{\sigma}}(t,\mathbf{X}(t))\bar{\boldsymbol{\sigma}}(t,\mathbf{X}(t))^{\mathrm{T}}$, and interest rate component $\bar{\boldsymbol{r}}(t,\mathbf{X}(t))$ (as explained previously) should be affine, but also the jump intensity should be of *the affine form*, i.e.

$$\bar{\xi}(t,\mathbf{X}(t)) = l_0 + l_1\mathbf{X}(t), \text{ with } (l_0, l_1) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{14}$$

- It can be shown that in this class, for a state vector $\mathbf{X}(t)$, the discounted characteristic function is also of the following form:

$$\phi_{\mathbf{X}}(\mathbf{u}; t, T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s + i\mathbf{u}^{\mathrm{T}}\mathbf{X}(T)}\Big|\mathcal{F}(t)\right] = \mathrm{e}^{\bar{A}(\mathbf{u},\tau)+\bar{\mathbf{B}}^{\mathrm{T}}(\mathbf{u},\tau)\mathbf{X}(t)},$$

with the expectation under risk-neutral measure $\mathbb{Q}$.

Affine Jump Diffusion Setting

- The coefficients $\bar{A}(\mathbf{u},\tau)$ and $\bar{\mathbf{B}}^{\mathrm{T}}(\mathbf{u},\tau)$ have to satisfy the following complex-valued *Riccati* ODEs, see the work by Duffie-Pan-Singleton:

$$\boxed{\begin{aligned}\frac{\mathrm{d}\bar{A}}{\mathrm{d}\tau} &= -r_0 + \bar{\mathbf{B}}^{\mathrm{T}}a_0 + \frac{1}{2}\bar{\mathbf{B}}^{\mathrm{T}}c_0\bar{\mathbf{B}} + l_0^{\mathrm{T}}\mathbb{E}\left[\mathrm{e}^{\mathbf{J}(\tau)\bar{\mathbf{B}}} - 1\right],\\ \frac{\mathrm{d}\bar{\mathbf{B}}}{\mathrm{d}\tau} &= -r_1 + a_1^{\mathrm{T}}\bar{\mathbf{B}} + \frac{1}{2}\bar{\mathbf{B}}^{\mathrm{T}}c_1\bar{\mathbf{B}} + l_1^{\mathrm{T}}\mathbb{E}\left[\mathrm{e}^{\mathbf{J}(\tau)\bar{\mathbf{B}}} - 1\right],\end{aligned}} \tag{15}$$

  where the expectation, $\mathbb{E}[\cdot]$ in (15), is taken with respect to the jump amplitude $\mathbf{J}(t)$.
- The dimension of the (complex-valued) ODEs for $\bar{\mathbf{B}}(\mathbf{u},\tau)$ corresponds to the dimension of the state vector $\mathbf{X}(t)$. The interpretation of these ODEs for the AJD models remains the same as in the case of the affine diffusion models. We clarify the expression in (15) by means of the following example.

AJD Example

- Earlier, we have seen that for a stock price, driven by the following SDE,

$$\frac{\mathrm{d}S(t)}{S(t)} = \left(r - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right]\right)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}(t) + \left(\mathrm{e}^J - 1\right)\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t), \quad (16)$$

  the corresponding option pricing PIDE is given by:

$$\frac{\partial V}{\partial t} + \left(r - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right]\right) S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r + \xi_p) V + \xi_p \mathbb{E}\left[V(t, S\mathrm{e}^J)\right] = 0.$$

- According to the affinity conditions in (13), this model does not belong to the class of affine jump diffusions. We therefore consider the same model under log-asset transformation, $X(t) = \log S(t)$, for which the dynamics read,

$$\mathrm{d}X(t) = \left(r - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right] - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}(t) + J\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t).$$

AJD Example

- For $V(\tau, X)$, and $\tau := T - t$, we find the following option pricing PIDE:

$$\begin{aligned}\frac{\partial V}{\partial \tau} &= \left(r - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right] - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial X^2}\\ &- (r+\xi_p)V + \xi_p\mathbb{E}\left[V(T-\tau, X+J)\right].\end{aligned} \tag{17}$$

- The discounted characteristic function is now of the following form,

$$\phi_X := \phi_X(u; t, T) = \mathrm{e}^{\bar{A}(u,\tau)+\bar{B}(u,\tau)X(t)},$$

  with initial condition $\phi_X(u; T, T) = \mathrm{e}^{iuX(0)}$.
- Substitution of all derivatives results in,

$$\frac{\partial \phi_X}{\partial \tau} = \phi_X\left(\frac{\mathrm{d}\bar{A}}{\mathrm{d}\tau} + \frac{\mathrm{d}\bar{B}}{\mathrm{d}\tau}\right), \;\; \frac{\partial \phi_X}{\partial X} = \phi_X\bar{B}, \;\; \frac{\partial^2 \phi_X}{\partial x^2} = \phi_X\bar{B}^2,$$

  and, because the expectation in (17) is taken only with respect to jump size $F_J(y)$,

$$\begin{aligned}\mathbb{E}\left[\phi_{X+J}\right] &= \mathbb{E}\left[\exp\left(\bar{A}(u,\tau) + \bar{B}(u,\tau)(X+J)\right)\right]\\ &= \phi_X \cdot \mathbb{E}\left[\exp\left(\bar{B}(u,\tau)J\right)\right].\end{aligned} \tag{18}$$

AJD Example

- In the PIDE in (17), this gives us:

$$
\begin{aligned}
-\left(\frac{\mathrm{d}\bar{A}}{\mathrm{d}\tau}+X\frac{\mathrm{d}\bar{B}}{\mathrm{d}\tau}\right)+\left(r-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]-\frac{1}{2}\sigma^2\right)\bar{B}\\
+\frac{1}{2}\sigma^2\bar{B}^2-(r+\xi_p)+\xi_p\mathbb{E}\left[\exp\left(\bar{B}\cdot J\right)\right]=0.
\end{aligned}
$$

and the following system of ODEs needs to be solved,

$$
\begin{cases}
\dfrac{\mathrm{d}\bar{B}}{\mathrm{d}\tau}=0,\\
\dfrac{\mathrm{d}\bar{A}}{\mathrm{d}\tau}=\left(r-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]-\frac{1}{2}\sigma^2\right)\bar{B}+\frac{1}{2}\sigma^2\bar{B}^2-(r+\xi_p)+\xi_p\mathbb{E}\left[\mathrm{e}^{\bar{B}\cdot J}\right].
\end{cases}
$$

- With the solution, $\bar{B}(u,\tau)=iu$, we find,

$$
\bar{A}(u,\tau)=\left(r-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]-\frac{1}{2}\sigma^2\right)iu\tau-\frac{1}{2}\sigma^2u^2\tau-(r+\xi_p)\tau+\xi_p\tau\mathbb{E}\left[\mathrm{e}^{iuJ}\right].
$$

- The system of ODEs forms an example of the ODE representation in (15), using $r_0=r$, $r_1=0$, $a_0=-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]-\frac{1}{2}\sigma^2$, $a_1=0$, $c_0=\sigma^2$, $c_1=0$, $l_0=\xi_p$, $l_1=0$.