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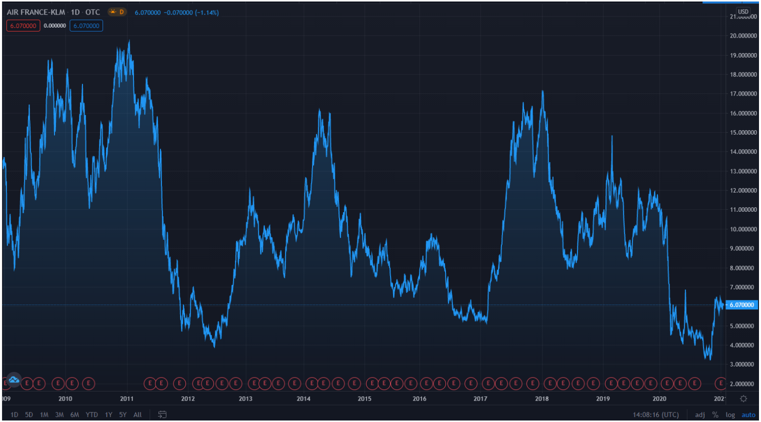
Deficiencies of the Black-Scholes Model

- $\Rightarrow$ The idea of implied volatility does not fit to the Black-Scholes model
- Look for market consistent asset price models.
- $\Rightarrow$ Use a local volatility, a model with jumps, or stochastic volatility to better fit market data, and incorporate smile effects

Jump processes

- Stock process often have jumps!



Jump processes

- *Jump-diffusion models and Lévy based models* are attractive because they explain the jump patterns exhibited by some stocks. The presence of jumps has been observed in the market, especially in times of financial turmoil, like in 1987, 2000 or 2008. Jump models are realistic when pricing options close to maturity time.
- Jump processes are superior to the Black-Scholes model in the sense that daily log-returns have heavy tails, and for longer time periods jump processes approach normality, which is consistent with empirical studies.
- Empirical densities are usually too peaked compared to the normal density; a phenomenon known as excess of kurtosis.
- With parameters, one can control kurtosis and asymmetry of the log-return density, and is able to fit the smile in the implied volatility.

Jump diffusion Process

- Extend the Black-Scholes model by independent jumps, driven by a Poisson process. $X(t) = \log S(t)$, under the real-world measure $\mathbb{P}$,

$$\boxed{\mathrm{d}X(t) = \mu \mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{P}}(t) + J\mathrm{d}X_{\mathcal{P}}(t),}$$

where μ is drift, σ volatility, $X_{\mathcal{P}}(t)$ a Poisson process and J gives the jump magnitude, governed by a distribution, F_J, of magnitudes. $W^{\mathbb{P}}(t)$ and $X_{\mathcal{P}}(t)$ are assumed to be *independent*.

Definition (Poisson random variable)

$X_{\mathcal{P}}$ counts the number of occurrences of an event during a given time period. Probability of observing $k \geq 0$ occurrences in a time period,

$$\mathbb{P}[X_{\mathcal{P}} = k] = \frac{\xi_p^k \mathrm{e}^{-\xi_p}}{k!}.$$

$\mathbb{E}[X_{\mathcal{P}}] = \xi_p$, average number of occurrences; $\mathbb{V}\mathrm{ar}[X_{\mathcal{P}}] = \xi_p$.

Poisson Process

Definition (Poisson process)

$X_{\mathcal{P}}(t), t \geq t_0 = 0$, with $\xi_p > 0$ is an integer-valued stochastic process,

- $X_{\mathcal{P}}(0) = 0$;
- $\forall\ t_0 = 0 < t_1 < \ldots < t_n$, the increments $X_{\mathcal{P}}(t_1) - X_{\mathcal{P}}(t_0), X_{\mathcal{P}}(t_2) - X_{\mathcal{P}}(t_1), \ldots, X_{\mathcal{P}}(t_n) - X_{\mathcal{P}}(t_{n-1})$ are independent random variables;
- for $s \geq 0, t > 0$ and $k \geq 0$, increments have the Poisson distribution:

$$\mathbb{P}[X_{\mathcal{P}}(s+t) - X_{\mathcal{P}}(s) = k] = \frac{(\xi_p t)^k \mathrm{e}^{-\xi_p t}}{k!}. \tag{1}$$

$X_{\mathcal{P}}(t)$ is a counting process, the number of *jumps* in a time period of length t specified via (1). Eq. (1) confirms stationary increments, since increments depend on the length of the interval and not on initial time. ξ_p is the *rate* of the Poisson process, i.e. it indicates the number of jumps in a time period.

Details

- The probability that exactly one event occurs in a small time interval, $\mathrm{d}t$, follows from (1) as

$$\mathbb{P}[X_{\mathcal{P}}(s+\mathrm{d}t)-X_{\mathcal{P}}(s)=1]=\frac{(\xi_p\mathrm{d}t)\mathrm{e}^{-\xi_p\mathrm{d}t}}{1!}=\xi_p\mathrm{d}t+o(\mathrm{d}t),$$

and the probability that no event occurs in $\mathrm{d}t$ is

$$\mathbb{P}[X_{\mathcal{P}}(s+\mathrm{d}t)-X_{\mathcal{P}}(s)=0]=\mathrm{e}^{-\xi_p\mathrm{d}t}=1-\xi_p\mathrm{d}t+o(\mathrm{d}t).$$

- In $\mathrm{d}t$, a jump will arrive with probability $\xi_p\mathrm{d}t$, resulting in:

$$\mathbb{E}[\mathrm{d}X_{\mathcal{P}}(t)]=1\cdot\xi_p\mathrm{d}t+0\cdot(1-\xi_p\mathrm{d}t)=\xi_p\mathrm{d}t,$$

where $\mathrm{d}X_{\mathcal{P}}(t)=X_{\mathcal{P}}(s+\mathrm{d}t)-X_{\mathcal{P}}(s)$. The expectation is given by

$$\mathbb{E}[X_{\mathcal{P}}(s+t)-X_{\mathcal{P}}(s)]=\xi_p t.$$

With $X_{\mathcal{P}}(0)=0$, the expected number of events in a time interval with length t, setting $s=0$, equals

$$\mathbb{E}[X_{\mathcal{P}}(t)]=\xi_p t. \tag{2}$$

Requirement

- If we define another process, $\bar{X}_{\mathcal{P}}(t) := X_{\mathcal{P}}(t) - \xi_p t$, then $\mathbb{E}[\mathrm{d}\bar{X}_{\mathcal{P}}(t)] = 0$, so that process $\bar{X}_{\mathcal{P}}(t)$, which is referred to as the *compensated* Poisson process, is a martingale.
- Given the following SDE:

$$\mathrm{d}X(t) = J(t)\mathrm{d}X_{\mathcal{P}}(t), \tag{3}$$

we may define the stochastic integral with respect to the Poisson process $X_{\mathcal{P}}(t)$, by

$$X(T) - X(t_0) = \int_{t_0}^{T} J(t)\mathrm{d}X_{\mathcal{P}}(t) := \sum_{k=1}^{X_{\mathcal{P}}(T)} J_k. \tag{4}$$

Variable J_k for $k \geq 1$ is an i.i.d. sequence of random variables with a jump-size probability distribution F_J, so that $\mathbb{E}[J_k] = \mu_J < \infty$.

Jump processes

- We present some discrete paths that have been generated by a Poisson process, with $\xi_p = 1$. The left-hand picture in Figure below displays the paths by $\mathrm{d}X_{\mathcal{P}}(t)$, whereas the right-hand picture shows the same paths for the compensated Poisson process, $-\xi_p \mathrm{d}t + \mathrm{d}X_{\mathcal{P}}(t)$.

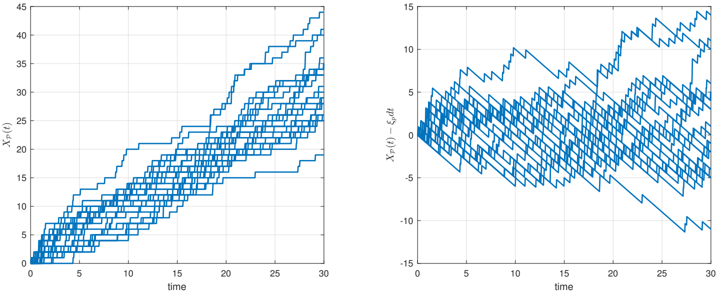


Figure: Monte Carlo paths for the Poisson (left) and the compensated Poisson process (right), $\xi_p = 1$.

Ito's Lemma and Jumps

- To derive the dynamics for $S(t) = \exp(X(t))$, a variant of Itô's lemma related to the Poisson process, needs to be employed.

Definition (Itô's lemma for Poisson process)

We consider a cádlág process, $X(t)$, defined as:

$$\mathrm{d}X(t) = \bar{\mu}(t, X(t))\mathrm{d}t + \bar{J}(t, X(t_-))\mathrm{d}X_{\mathcal{P}}(t), \quad \text{with} \quad X(t_0) \in \mathbb{R}, \tag{5}$$

where $\bar{\mu}, \bar{J} : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are deterministic, continuous functions
For differentiable function $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$, the Itô differential reads,

$$\begin{aligned} \mathrm{d}g(t, X(t)) &= \left[\frac{\partial g(t, X(t))}{\partial t} + \bar{\mu}(t, X(t))\frac{\partial g(t, X(t))}{\partial X}\right]\mathrm{d}t \qquad (6) \\ &+ \left[g\left(t, X(t_-) + \bar{J}(t, X(t_-))\right) - g(t, X(t_-))\right]\mathrm{d}X_{\mathcal{P}}(t), \end{aligned}$$

where the left limit is denoted by $X(t_-) := \lim_{s \to t} X(s), s < t$, so that, by the continuity of $\bar{J}(\cdot)$, its left limit equals $\bar{J}(t, X(t_-))$.

Ito's Lemma and Jumps

- An intuitive explanation for Itô's formula in the case of jumps is that when a jump takes place, i.e. $\mathrm{d}X_{\mathcal{P}}(t) = 1$, the process "jumps" from $X(t_{-})$ to $X(t)$, with the jump size determined by function $\bar{J}(t, X(t))$, resulting in the following relation:

$$g(t, X(t)) = g\left(t, X(t_{-}) + \bar{J}(t, X(t_{-})\right).$$

- After the jump at time t, the function $g(\cdot)$ is adjusted with the jump size, which was determined at time t_{-}.

Ito's Lemma and Jumps

- The dynamics of the combined process are given by:

$$\mathrm{d}X(t) = \bar{\mu}(t, X(t))\mathrm{d}t + \bar{J}(t, X(t_-))\mathrm{d}X_{\mathcal{P}}(t) + \bar{\sigma}(t, X(t))\mathrm{d}W(t), \quad (7)$$

- Assuming that the Poisson process $X_{\mathcal{P}}(t)$ is *independent* of the Brownian motion $W(t)$, the dynamics of $g(t, X(t))$ are given by:

$$\begin{aligned}
\mathrm{d}g(t, X(t)) &= \left[\frac{\partial g(t, X(t))}{\partial t} + \bar{\mu}(t, X(t))\frac{\partial g(t, X(t))}{\partial X}\right. \\
&+ \left.\frac{1}{2}\bar{\sigma}^2(t, X(t))\frac{\partial^2 g(t, X(t))}{\partial X^2}\right]\mathrm{d}t \\
&+ \Big[g\big(t, X(t_-) + \bar{J}(t, X(t_-))\big) - g\big(t, X(t_-)\big)\Big]\mathrm{d}X_{\mathcal{P}}(t) \\
&+ \bar{\sigma}(t, X(t))\frac{\partial g(t, X(t))}{\partial X}\mathrm{d}W(t), \qquad (8)
\end{aligned}$$

Itˆ's Table

- We made use of the Itô multiplication table, where the cross terms involving the Poisson process are also handled.
- An intuitive way to understand the Poisson process rule in the table is found in the notion that the term $\mathrm{d}X_{\mathcal{P}} = 1$ with probability $\xi_p \mathrm{d}t$, and $\mathrm{d}X_{\mathcal{P}} = 0$ with probability $(1 - \xi_p \mathrm{d}t)$, which implies that

$$
\begin{aligned}
(\mathrm{d}X_{\mathcal{P}})^2 &= \begin{cases} 1^2 & \text{with probability } \xi_p \mathrm{d}t, \\ 0^2 & \text{with probability } (1 - \xi_p \mathrm{d}t) \end{cases} \\
&= \mathrm{d}X_{\mathcal{P}}.
\end{aligned}
$$

	$\mathrm{d}t$	$\mathrm{d}W(t)$	$\mathrm{d}X_{\mathcal{P}}(t)$
$\mathrm{d}t$	0	0	0
$\mathrm{d}W(t)$	0	$\mathrm{d}t$	0
$\mathrm{d}X_{\mathcal{P}}(t)$	0	0	$\mathrm{d}X_{\mathcal{P}}(t)$

Table: Itô multiplication table for Poisson process.

Asset dynamics

- To apply Itô's lemma to the function $S(t) = \mathrm{e}^{X(t)}$, substitute $\bar{\mu}(t, X(t)) = \mu$, $\bar{\sigma}(t, X(t)) = \sigma$ and $\bar{J}(t, X(t_-)) = J$ in (8), giving,

$$\mathrm{d}\mathrm{e}^{X(t)} = (\mu \mathrm{e}^{X(t)} + \frac{1}{2}\sigma^2 \mathrm{e}^{X(t)})\mathrm{d}t + \sigma \mathrm{e}^{X(t)}\mathrm{d}W(t) + (\mathrm{e}^{X(t)+J} - \mathrm{e}^{X(t)})\mathrm{d}X_{\mathcal{P}}(t),$$

so that we obtain:

$$\frac{\mathrm{d}S(t)}{S(t)} = \left(\mu + \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W(t) + \left(\mathrm{e}^J - 1\right)\mathrm{d}X_{\mathcal{P}}(t).$$

$\Rightarrow$ Dynamics for stock $S(t)$ under the real-world measure $\mathbb{P}$.

Risk Neutral Measrue $\mathbb{Q}$

- Process $Y(t) := S(t)/M(t)$ should be a martingale, or, $dY(t) = \frac{dS(t)}{M(t)} - \frac{rS(t)dt}{M(t)}$, should have zero expectation,

$$\begin{aligned}\mathbb{E}\left[\mathrm{d}Y(t)\right] &= \mathbb{E}\left[\mu S(t) + \frac{1}{2}\sigma^2 S(t) - rS(t)\right]\mathrm{d}t + \mathbb{E}\left[\sigma S(t)\mathrm{d}W(t)\right] \\ &+ \mathbb{E}\left[\left(\mathrm{e}^J - 1\right)S(t)\mathrm{d}X_{\mathcal{P}}(t)\right].\end{aligned}$$

From the fact that all random components in the expression above are mutually independent, we get,

$$\begin{aligned}\mathbb{E}\left[\mathrm{d}Y(t)\right] &= \mathbb{E}\left[\mu S(t) + \frac{1}{2}\sigma^2 S(t) - rS(t)\right]\mathrm{d}t + \mathbb{E}\left[\left(\mathrm{e}^J - 1\right)S(t)\right]\xi_p\mathrm{d}t \\ &= \left(\mu - r + \frac{1}{2}\sigma^2 + \mathbb{E}\left[\xi_p(\mathrm{e}^J - 1)\right]\right)\mathbb{E}[S(t)]\mathrm{d}t.\end{aligned}$$

- By substituting $\mu = r - \frac{1}{2}\sigma^2 - \xi_p\mathbb{E}\left[\mathrm{e}^J - 1\right]$, we have $\mathbb{E}[\mathrm{d}Y(t)] = 0$.

Ito's Lemma and Jumps

- The term $\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]$ is the so-called *drift correction term*, which makes the process a martingale.
- The dynamics for stock $S(t)$ under the risk-neutral measure $\mathbb{Q}$ are therefore given by:

$$\boxed{\frac{\mathrm{d}S(t)}{S(t)}=\left(r-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]\right)\mathrm{d}t+\sigma\mathrm{d}W^{\mathbb{Q}}(t)+\left(\mathrm{e}^J-1\right)\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t).} \tag{9}$$

- The process in (9) is often presented in the literature as the *standard jump diffusion model*. The standard jump diffusion model is directly connected to the following $\mathrm{d}X(t)$ dynamics:

$$\mathrm{d}X(t)=\left(r-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]-\frac{1}{2}\sigma^2\right)\mathrm{d}t+\sigma\mathrm{d}W^{\mathbb{Q}}(t)+J\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t).$$

Jump processes

- In Figure below examples of paths for $X(t)$ in (16) and $S(t) = \mathrm{e}^{X(t)}$, as in (9), are presented. Here, the classical Merton model is used, $J \sim \mathcal{N}(\mu_J, \sigma_J^2)$, where the jumps are symmetric, as described in (13).

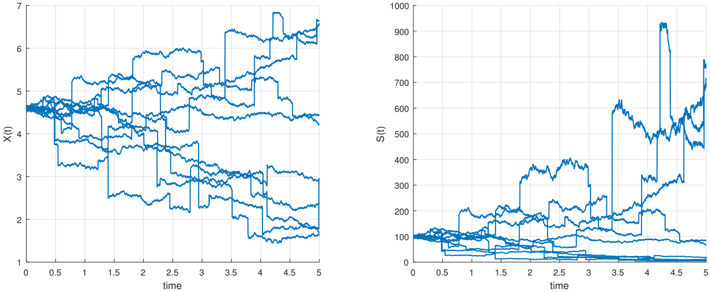


Figure: Left side: Paths of process $X(t)$ (16); Right side: $S(t)$ in (9) with $S(t_0) = 100$, $r = 0.05$, $\sigma = 0.2$, $\sigma_J = 0.5$, $\mu_J = 0$, $\xi_p = 1$ and $T = 5$.

Partial Integro-Differential Equations (PIDE)

- The option pricing PDE is again found with the martingale property. In the case of the jump diffusion process,

$$\mathrm{d}S(t) = \bar{\mu}(t, S(t))\mathrm{d}t + \bar{\sigma}(t, S(t))\mathrm{d}W^{\mathbb{Q}}(t) + \bar{J}(t, S(t))\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t), \quad (10)$$

where,

$$\begin{aligned} \bar{\mu}(t, S(t)) &:= \left(r - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right]\right) S(t), \quad \bar{\sigma}(t, S(t)) := \sigma S(t), \\ \bar{J}(t, S(t)) &:= (\mathrm{e}^J - 1)S(t). \end{aligned}$$

- The dynamics (10) are under measure $\mathbb{Q}$, so that we may apply the martingale approach to derive the option pricing equation.

$$\frac{V(t, S)}{M(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V(T, S)}{M(T)}\middle|\mathcal{F}(t)\right]. \quad (11)$$

PIDE

- With Itô's lemma, the dynamics of V/M are given by

$$\mathrm{d}\frac{V}{M} = \frac{1}{M}\mathrm{d}V - r\frac{V}{M}\mathrm{d}t.$$

- The dynamics of V are obtained by using Itô's lemma for the Poisson process, as presented in Equation (8). There, we set $g(t, S(t)) := V(t, S)$ and $\bar{J}(t, S(t)) := (\mathrm{e}^J - 1)S$, which implies:

$$\begin{aligned}\mathrm{d}V &= \left(\frac{\partial V}{\partial t} + \bar{\mu}(t, S)\frac{\partial V}{\partial S} + \frac{1}{2}\bar{\sigma}^2(t, S)\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \bar{\sigma}(t, S)\frac{\partial V}{\partial S}\mathrm{d}W^{\mathbb{Q}}(t)\\ &+ \left(V(t, S\mathrm{e}^J) - V(t, S)\right)\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t).\end{aligned}$$

PIDE

- After substitutions, the dynamics of V/M read:

$$\begin{aligned}\mathrm{d}\frac{V}{M} &= \frac{1}{M}\left(\frac{\partial V}{\partial t}+\bar{\mu}(t,S)\frac{\partial V}{\partial S}+\frac{1}{2}\bar{\sigma}^2(t,S)\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t+\frac{\bar{\sigma}(t,S)}{M}\frac{\partial V}{\partial S}\mathrm{d}W^{\mathbb{Q}}\\ &+ \frac{1}{M}\left(V(t,S\mathrm{e}^J)-V(t,S)\right)\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}-r\frac{V(t,S)}{M}\mathrm{d}t.\end{aligned}$$

- The jumps are independent of Poisson process $X_{\mathcal{P}}^{\mathbb{Q}}$ and Brownian motion $W^{\mathbb{Q}}$. Because V/M is a martingale, it follows that,

$$\left(\frac{\partial V}{\partial t}+\bar{\mu}(t,S)\frac{\partial V}{\partial S}+\frac{1}{2}\bar{\sigma}^2(t,S)\frac{\partial^2 V}{\partial S^2}-rV\right)\mathrm{d}t$$
$$+\mathbb{E}\left[\left(V(t,S\mathrm{e}^J)-V(t,S)\right)\right]\mathbb{E}\left[\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}\right]=0.$$

PIDE

- Based on Equation (2) and substitution of the expressions (11) in (12), the following pricing equation results:

$$\begin{aligned}\frac{\partial V}{\partial t}+\left(r-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]\right)S\frac{\partial V}{\partial S} \quad &+ \quad \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} \\ &- \quad (r+\xi_p)V+\xi_p\mathbb{E}\left[V(t,S\mathrm{e}^J)\right]=0,\end{aligned} \tag{12}$$

  which is a *partial integro-differential equation* (PIDE).
- We deal with partial derivatives, the expectation gives rise to an integral term.
- PIDEs are typically more difficult to be solved than PDEs, due to the presence of the additional integral term.
- An analytic expression has been found for the solution of (12) for Merton's model and Kou's model, the solution is given in the form of an infinite series.

PIDE

- For the jump diffusion process under measure $\mathbb{Q}$, we arrive at the following option valuation PIDE, in terms of the prices S,

$$\begin{cases} -\dfrac{\partial V}{\partial t} & = & \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r - \xi_p \mathbb{E}[\mathrm{e}^J - 1]) S \dfrac{\partial V}{\partial S} - (r + \xi_p) V \\ & + & \xi_p \displaystyle\int_0^\infty V(t, S\mathrm{e}^y) \mathrm{d}F_J(y), \quad \forall (t,S) \in [0,T) \times \mathbb{R}_+, \\ V(T,S) & = & \max(\bar{\alpha}(S(T) - K), 0), \quad \forall S \in \mathbb{R}_+, \end{cases}$$

with $\bar{\alpha} = \pm 1$ (call or put, respectively), and where $\mathrm{d}F_J(y) = f_J(y)\mathrm{d}y$.

PIDE

- In log-coordinates $X(t) = \log(S(t))$, the corresponding PIDE for $V(t,X)$ is given by

$$\begin{cases} -\dfrac{\partial V}{\partial t} & = & \dfrac{1}{2}\sigma^2\dfrac{\partial^2 V}{\partial X^2} + (r - \dfrac{1}{2}\sigma^2 - \xi_p\mathbb{E}[\mathrm{e}^J - 1])\dfrac{\partial V}{\partial X} - (r + \xi_p)V \\ & + & \xi_p \displaystyle\int_{\mathbb{R}} V(t, X+y)\mathrm{d}F_J(y), \quad \forall (t,X) \in [0,T) \times \mathbb{R}, \\ V(T,X) & = & \max\left(\bar{\alpha}(\exp\left(X(T)\right) - K), 0\right), \quad \forall X \in \mathbb{R}. \end{cases}$$

Jump distribution

- For the cumulative distribution function for the jump magnitude $F_J(x)$, two popular choices are:
  - $\Rightarrow$ *Classical Merton' s model*: The jump magnitude J is normally distributed, with mean μ_J and standard deviation σ_J. So, $\mathrm{d}F_J(x) = f_J(x)\mathrm{d}x$, where

$$f_J(x) = \frac{1}{\sigma_J\sqrt{2\pi}} \exp\left(-\frac{(x-\mu_J)^2}{2\sigma_J^2}\right). \tag{13}$$

  - $\Rightarrow$ *Non-symmetric double exponential* (Kou's model

$$f_J(x) = p_1\alpha_1 \mathrm{e}^{-\alpha_1 x}\mathbb{1}_{\{x\geq 0\}} + p_2\alpha_2\mathrm{e}^{\alpha_2 x}\mathbb{1}_{\{x<0\}}, \tag{14}$$

where p_1, p_2 are positive real numbers so that $p_1 + p_2 = 1$. To be able to integrate e^x over the real line it is required to have $\alpha_1 > 1$ and $\alpha_2 > 0$, and we obtain the expression

$$\mathbb{E}[\mathrm{e}^{J_k}] = p_1\frac{\alpha_1}{\alpha_1 - 1} + p_2\frac{\alpha_2}{\alpha_2 + 1}. \tag{15}$$

Feynman-Kac Theorem

- With r constant, $S(t)$ is governed by the following SDE

$$\frac{\mathrm{d}S(t)}{S(t)} = \left(r - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right]\right)\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}(t) + \left(\mathrm{e}^J - 1\right)\mathrm{d}X_{\mathcal{P}}^{\mathbb{Q}}(t),$$

  and option value $V(t,S)$ satisfies the following PIDE,

$$\begin{aligned}\frac{\partial V}{\partial t} + \left(r - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right]\right) S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \quad &- \quad (r+\xi_p)V \\ &+ \quad \xi_p \mathbb{E}\left[V(t, S\mathrm{e}^J)\right] = 0,\end{aligned}$$

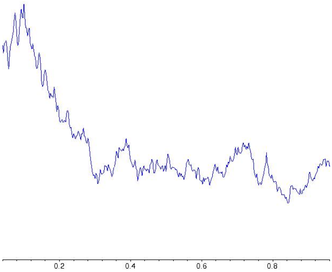
  with $V(T,S) = H(T,S)$, and the term $(\mathrm{e}^J - 1)$ representing the size of a proportional jump, the risk-neutral valuation formula determines the option value, i.e.

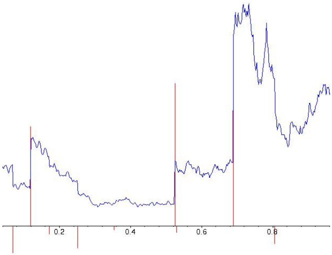
$$V(t,S) = M(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T)}H(T,S)|\mathcal{F}(t)\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}H(T,S)|\mathcal{F}(t)\right].$$

- The discounted expected payoff formula resembles the martingale property.

SDE Simulation

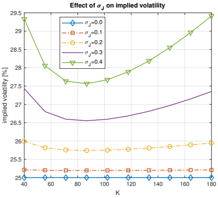
- Geom. Brownian motion vs. Jump Diffusion

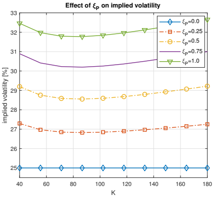


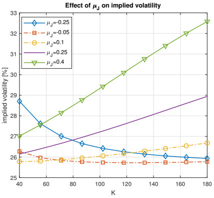


Implied volatility

- Illustrate the impact of the jump parameters in the Merton jump diffusion model on the implied volatilities. $J \sim \mathcal{N}(\mu_J, \sigma_J^2)$.
- The influence of ξ_p, σ_J and μ_J. Each parameter is varied individually, while the other parameters are fixed.
- σ_J has a significant impact on the curvature, ξ_p controls the overall level of the implied volatility, whereas μ_J influences the implied volatility slope (the skew).







Tower property

- We need the tower property of expectations for discrete RVs.
- Suppose $X_1, X_2, \ldots$ are independent random variables with mean μ and N_J is a nonnegative, integer-valued random variable, independent of X_i's. Then (Wald's equation),

$$\mathbb{E}\left[\sum_{i=1}^{N_J} X_i\right] = \mu \mathbb{E}\left[N_J\right]. \tag{16}$$

- Using the tower property for discrete RVs z_1 and z_2 gives,

$$\mathbb{E}\left[\mathbb{E}\left[z_1|z_2\right]\right] = \sum_z \mathbb{E}[z_1|z_2 = z]\,\mathbb{P}[z_2 = z],$$

so that,

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^{N_J} X_k\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{k=1}^{N_J} X_k \middle| N_J\right]\right] = \sum_{n=1}^{\infty} \mathbb{E}\left[\sum_{k=1}^{n} X_k \middle| N_J = n\right] \mathbb{P}[N_J = n] \\ &= \sum_{n=1}^{\infty} \mathbb{P}[N_J = n] \sum_{k=1}^{n} \mathbb{E}\left[X_k\right]. \qquad (17)\end{aligned}$$

Tower property

- Since the expectation for each X_k equals μ, we have:

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^{N_J} X_k\right] &= \sum_{n=1}^{\infty}\mathbb{P}[N_J = n]\sum_{k=1}^{n}\mu \\ &= \mu\sum_{n=1}^{\infty} n\mathbb{P}[N_J = n] \stackrel{\text{def}}{=} \mu\mathbb{E}[N_J],\end{aligned} \tag{18}$$

Characteristic function

- Merton's jump diffusion model under $\mathbb{Q}$ consists of a Brownian motion and a compound Poisson process, with $t_0 = 0$,

$$X(t) = X(t_0) + \bar{\mu}t + \sigma W(t) + \sum_{k=1}^{X_{\mathcal{P}}(t)} J_k. \tag{19}$$

  where, $\bar{\mu} = r - \frac{1}{2}\sigma^2 - \xi_p \mathbb{E}\left[\mathrm{e}^J - 1\right]$, $\sigma > 0$, BM $W(t)$, Poisson process $X_{\mathcal{P}}(t), t \geq 0$ with ξ_p and $\mathbb{E}[X_{\mathcal{P}}(t)|\mathcal{F}(0)] = \xi_p t$.
- In the Poisson process setting, the arrival of a jump is independent of the arrival of previous jumps, and the probability of two simultaneous jumps is equal to zero.
- Variable J_k, $k \geq 1$, is an i.i.d. sequence of random variables with a jump-size probability distribution F_J, so $\mathbb{E}[J_k] = \mu_J < \infty$.

Characteristic function

- With all sources of randomness *mutually independent*, the *characteristic function* of $X(t)$ looks,

$$\phi_X(u) := \mathbb{E}\left[\mathrm{e}^{iuX(t)}\right] = \mathrm{e}^{iuX(0)}\mathrm{e}^{iu\bar{\mu}t}\mathbb{E}\left[\mathrm{e}^{iu\sigma W(t)}\right]\cdot\mathbb{E}\left[\exp\left(iu\sum_{k=1}^{X_{\mathcal{P}}(t)}J_k\right)\right].$$

- As $W(t)\sim\mathcal{N}(0,t)$, it follows $\mathbb{E}\left[\mathrm{e}^{iu\sigma W(t)}\right]=\mathrm{e}^{-\frac{1}{2}\sigma^2u^2t}$. For the second expectation, consider the summation:

$$\mathbb{E}\left[\exp\left(iu\sum_{k=1}^{X_{\mathcal{P}}(t)}J_k\right)\right]=\sum_{n\geq0}\mathbb{E}\left[\exp\left(iu\sum_{k=1}^{X_{\mathcal{P}}(t)}J_k\right)\middle|X_{\mathcal{P}}(t)=n\right]\mathbb{P}[X_{\mathcal{P}}(t)=n]$$

which results from the tower property of expectations. We have,

$$\begin{aligned}\mathbb{E}\left[\exp\left(iu\sum_{k=1}^{X_{\mathcal{P}}(t)}J_k\right)\right] &= \sum_{n\geq0}\mathbb{E}\left[\exp\left(iu\sum_{k=1}^{n}J_k\right)\right]\frac{\mathrm{e}^{-\xi_pt}(\xi_pt)^n}{n!}\\ &= \sum_{n\geq0}\frac{\mathrm{e}^{-\xi_pt}(\xi_pt)^n}{n!}\left(\int_{\mathbb{R}}\mathrm{e}^{iux}f_J(x)\mathrm{d}x\right)^n \quad (20)\end{aligned}$$

Characteristic function

- The two nth powers at the right-hand side of (20) are seen as a Taylor expansion of an exponential. So,

$$
\begin{aligned}
\mathbb{E}\left[\exp\left(iu\sum_{k=1}^{X_{\mathcal{P}}(t)} J_k\right)\right] &= \mathrm{e}^{-\xi_p t}\sum_{n\geq 0}\frac{1}{n!}\left(\xi_p t\int_{\mathbb{R}}\mathrm{e}^{iux}f_J(x)\mathrm{d}x\right)^n \\
&= \exp\left(\xi_p t\int_{\mathbb{R}}\left(\mathrm{e}^{iux}f_J(x)\mathrm{d}x-1\right)\right) \\
&= \exp\left(\xi_p t\int_{\mathbb{R}}\left(\mathrm{e}^{iux}-1\right)f_J(x)\mathrm{d}x\right) \\
&= \exp\left(\xi_p t\mathbb{E}[\mathrm{e}^{iuJ}-1]\right),
\end{aligned}
$$

using $\int_{\mathbb{R}} f_J(x)\mathrm{d}x = 1$, and $J = J_k$ an i.i.d. sequence of RVs with CDF $F_J(x)$ and PDF $f_J(x)$.

Characteristic function

- The ChF can thus be written as:

$$
\begin{aligned}
\phi_X(u) &= \mathbb{E}\left[\mathrm{e}^{iuX(t)}\right] \qquad (21)\\
&= \exp\left(iu(X(0)+\bar{\mu}t)-\frac{1}{2}\sigma^2u^2t\right)\exp\left(\xi_p t\left(\mathbb{E}[\mathrm{e}^{iuJ}-1]\right)\right)\\
&= \exp\left(iu(X(0)+\bar{\mu}t)-\frac{1}{2}\sigma^2u^2t+\xi_p t\int_{\mathbb{R}}\left(\mathrm{e}^{iux}-1\right)f_J(x)\mathrm{d}x\right),
\end{aligned}
$$

with $\bar{\mu}=r-\frac{1}{2}\sigma^2-\xi_p\mathbb{E}\left[\mathrm{e}^J-1\right]$.

- For the Merton model, we have,

$$\mathbb{E}[\mathrm{e}^{iuJ}-1]=\mathrm{e}^{iu\mu_J-\frac{1}{2}u^2\sigma_J^2}-1,\ \mathbb{E}[\mathrm{e}^J-1]=\mathrm{e}^{\mu_J+\frac{1}{2}\sigma_J^2}-1.$$