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Introduction to the Forward Start Options

- Forward-starting options, also called the performance options. The value of an option does not depend on the value a stock but on stocks performance.
- Forward starting options may be considered as European options with a future starting time.
- Whereas in European options the initial stock value, S_0, is known at initial time t_0, in the case of forward-starting options the initial stock value is unknown as it will be determined at some future time T_1.
- Forward-starting options do not depend on today's value of the underlying asset but on a value which is based on the performance of the asset over some future time period $[T_1, T_2]$.

Introduction to the Forward Start Options cont.

- Forward-starting options are often considered as building blocks for other financial derivatives, like cliquet (also called ratchet) options, consisting of a series of consecutive forward-start options. Cliquet options are encountered in the equity and interest rate markets.
- A useful feature of these Cliquet option contracts is that they allow for a periodic locking in of profits. If a certain asset performed well a period $[T_i, T_{i+1}]$, but performed poorly in $[T_{i+1}, T_{i+2}]$, an investor will receive the profit accumulated in the first period and does not loose money in the second period. As the profits can be limited as well, i.e. in the contract it can be specified that the profit cannot be larger than, say 20%, cliquet options may be a relatively cheap alternative for plain-vanilla European options.

Forward Start Options contract formulation

- We start with some general results regarding the pricing of forward-start options that do not depend on any particular model. In the subsections to follow different alternatives for the underlying asset dynamics will be discussed. For two maturities T_1 and T_2, so that $t_0 < T_1 < T_2$, we define a forward-starting option, as follows:

$$V(T_2; T_1, T_2) = \max\left(\frac{S(T_2) - S(T_1)}{S(T_1)} - K, 0\right),$$

  with a constant strike K.

- The contract value is determined by the percentage performance of underlying asset $S(t)$, measured at the future times T_1 and T_2. Obviously, for $t_0 = T_1$ the payoff collapses to a standard "scaled" European option with maturity at time T_2, and payoff equal to:

$$V(T_2; T_1, T_2) = \frac{1}{S_0}\max\left(S(T_2) - K^*, 0\right), \quad K^* = S_0(K+1).$$

Forward Start Options contract formulation cont.

- The payoff can also be expressed as:

$$V(T_2,S;T_1,T_2)=\max\left(\frac{S(T_2)}{S(T_1)}-(K+1),0\right). \tag{1}$$

As the contract will pay at time T_2, today's value is given by the following pricing equation:

$$V(t_0,S;T_1,T_2)=M(t_0)\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_2)}\max\left(\frac{S(T_2)}{S(T_1)}-k,0\right)\Big|\mathcal{F}(t_0)\right], \tag{2}$$

with $k=K+1$, and $M(t)$ the money-savings account.

Forward Start Options contract formulation cont.

- Under constant interest rates the pricing equation reads

$$\begin{aligned} V(t_0, S; T_1, T_2) &= \frac{M(t_0)}{M(T_2)}\mathbb{E}^{\mathbb{Q}}\left[\max\left(\frac{S(T_2)}{S(T_1)} - k, 0\right)\Big|\mathcal{F}(t_0)\right] \\ &= \frac{M(t_0)}{M(T_2)}\mathbb{E}^{\mathbb{Q}}\left[\max\left(\mathrm{e}^{x(T_1,T_2)} - k, 0\right)\Big|\mathcal{F}(t_0)\right] \end{aligned}$$

where

$$x(T_1, T_2) := \log X(T_1, T_2) = \log\frac{S(T_2)}{S(T_1)} = \log S(T_2) - \log S(T_1).$$

- Note that we haven't made any assumptions regarding dynamics of the stock process.

Characteristic Function for the Frwd-Start Options

- We can derive the characteristic function of $x(T_1, T_2)$:

$$\phi_x(u) \equiv \phi_x(u, t_0, T_2) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{iu(\log S(T_2)-\log S(T_1))}\Big|\mathcal{F}(t_0)\right].$$

By iterated expectations, we have

$$\phi_x(u) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{iu(\log S(T_2)-\log S(T_1))}\Big|\mathcal{F}(T_1)\right]\Big|\mathcal{F}(t_0)\right].$$

As $\log S(T_1)$ is measurable with respect to the filtration $\mathcal{F}(T_1)$, we can write

$$\phi_x(u) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-iu\log S(T_1)}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{iu\log S(T_2)}\Big|\mathcal{F}(T_1)\right]\Big|\mathcal{F}(t_0)\right].$$

- In terms of a *discounted characteristic function*, we need to insert an appropriate discounting term, i.e.

$$\phi_x(u) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-iu\log S(T_1)}\mathrm{e}^{r(T_2-T_1)}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T_2-T_1)}\mathrm{e}^{iu\log S(T_2)}\Big|\mathcal{F}(T_1)\right]\Big|\mathcal{F}(t_0)\right].$$

Characteristic Function for the Frwd-Start Options

- The inner expectation can be recognized as the discounted characteristic function of $X(T_2) = \log S(T_2)$, and therefore it follows that

$$\phi_x(u) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-iuX(T_1)}\mathrm{e}^{r(T_2-T_1)}\psi_X(u,T_1,T_2)\Big|\mathcal{F}(t_0)\right]. \quad (3)$$

  The function $\psi_X(u, T_1, T_2)$ will be detailed below, for two different asset dynamics, i.e. the Black-Scholes and the Heston dynamics.
- Until now, any particular dynamics for the underlying stock $S(t)$ have not been specified, except for the assumption of a constant interest rate. In the follow-up sections, we will consider pricing of forward start options under different asset dynamics, and first discuss forward start options under the Black-Scholes model.

Black-Scholes case

- Under the Black-Scholes model, the discounted characteristic function, $\psi_X(u, T_1, T_2)$, for the log-stock $X(t) = \log S(t)$, conditioned on the information available until time T_1 and has the following form here:

$$\psi_X(u, T_1, T_2) = \exp\left[\left(r - \frac{\sigma^2}{2}\right) iu\Delta T - \frac{1}{2}\sigma^2 u^2 \Delta T - r\Delta T + iuX(T_1)\right]. \quad (4)$$

for $\Delta T = T_2 - T_1$.

- By substitution of the forward characteristic function of the Black-Scholes model (4) into (3), the following expression for the *forward characteristic function* $\phi_x(u)$ of $x(T_1, T_2)$ is obtained,

$$\begin{aligned}\phi_x(u) &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\left(r-\frac{\sigma^2}{2}\right)iu(T_2-T_1)-\frac{1}{2}\sigma^2u^2(T_2-T_1)}\middle|\mathcal{F}(t_0)\right]\\ &= \exp\left(\left(r - \frac{\sigma^2}{2}\right) iu(T_2 - T_1) - \frac{1}{2}\sigma^2 u^2 (T_2 - T_1)\right). \quad (5)\end{aligned}$$

Forward Start Options under Black-Scholes Model

- This is, in fact, the characteristic function of *a normal density* with mean $(r-1/2\sigma^2)(T_2-T_1)$ and variance $\sigma^2(T_2-T_1)$. Expression for ChF does not depend on $S(t)$, but only on the interest rate and the volatility. It may be quite surprising that the stock has disappeared from the above expression, but this is a consequence of the fact that the ratio $S(T_2)/S(T_1)$, under the lognormal asset dynamics, gives as the solution:

$$\frac{S(T_2)}{S(T_1)} = \mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)(T_2-T_1)+\sigma(W(T_2)-W(T_1))}. \tag{6}$$

- The ratio $S(T_2)/S(T_1)$ can be simply expressed as:

$$\frac{S(T_2)}{S(T_1)} = \frac{S_0\mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)T_2+\sigma W(T_2)}}{S_0\mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)T_1+\sigma W(T_1)}} = \mathrm{e}^{\left(r-\frac{1}{2}\sigma^2\right)(T_2-T_1)+\sigma(W(T_2)-W(T_1))}. \tag{7}$$

- This result implies that we can find an analytic expression for the value of a forward-starting option under the Black-Scholes model.

Forward Start Options contract formulation cont.

Theorem (Forward-starting option under the BS model)

The price for a forward-starting option, defined in (1), under Black-Scholes dynamics for stock $S(t)$ is given by:

$$V(t_0, S; T_1, T_2) = \mathrm{e}^{-rT_2}\mathbb{E}^{\mathbb{Q}}\left[\max\left(\frac{S(T_2)}{S(T_1)} - k, 0\right)\middle|\mathcal{F}(t_0)\right]$$

with $k = K + 1$, has a closed-form solution, given by:

$$V(t_0, S; T_1, T_2) = \mathrm{e}^{-rT_1}F_{\mathcal{N}(0,1)}(d_1) - k\mathrm{e}^{-rT_2}F_{\mathcal{N}(0,1)}(d_2), \quad (8)$$

with

$$d_1 = \frac{\log\frac{1}{k} + (r + \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}, \quad d_2 = \frac{\log\frac{1}{k} + (r - \frac{1}{2}\sigma^2)(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}.$$

Proof.

Details of the proof can be found in the book. □

Forward Implied Volatility

- Based on the Black-Scholes-type equation for the valuation of forward start options, we may determine, for given market prices of forward start options, the corresponding *forward implied volatility*.
- In the same way as discussed in previous lectures, the Black-Scholes *forward* implied volatility $\sigma_{imp}^{\text{fwd}}$ is found by solving,

$$V^{\text{fwd}}(t_0, K^*, T_1, T_2, \sigma_{imp}^{\text{fwd}}) = V^{\text{fwd, mkt}}(K, T).$$

- Remember: term "Implied Volatility" is always used in the context of the Black-Scholes model, even if your pricing model is Heston, Bates, Local Volatility etc.

Frwd Start ChF for the Heston Model

- We have already seen that the pricing of forward start options, under the assumption of constant interest rates, boils down to finding an expression for the following characteristic function, see (3),

$$\phi_x(u) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-iuX(T_1)}\mathrm{e}^{r(T_2-T_1)}\psi_X(u, T_1, T_2)\Big|\mathcal{F}(t_0)\right]. \tag{9}$$

- To detail this characteristic function under the Heston SV model, an expression for $\psi_X(u, T_1, T_2)$ is required. This characteristic function, with vector $\mathbf{u}^T = (u, 0)^{\mathrm{T}}$, is of the following form:

$$\psi_X(u, T_1, T_2) = \mathrm{e}^{\bar{A}(u,\tau)+\bar{B}(u,\tau)X(T_1)+\bar{C}(u,\tau)v(T_1)}, \tag{10}$$

with $\tau = T_2 - T_1$ and the complex-valued functions $\bar{A}(u,\tau)$, $\bar{B}(u,\tau)$ and $\bar{C}(u,\tau)$, as presented in previous lecture.

Frwd Start ChF for the Heston Model

- Recall that for the Heston model, we have $\bar{B}(u,\tau) = iu$, which simplifies the expression for $\phi_x(u)$ in (9), as follows,

$$\phi_x(u) \quad = \quad \mathrm{e}^{\bar{A}(u,\tau)+r(T_2-T_1)}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\bar{C}(u,\tau)v(T_1)}\middle|\mathcal{F}(t_0)\right]. \qquad (11)$$

- Just like in the Black-Scholes case, the forward characteristic function under the Heston SV dynamics depends neither on the stock $S(t)$ nor on the log-stock $X(t) = \log S(t)$. It is completely determined in terms of the model's volatility.
- To complete the derivations for the Heston model, the expectation in (11) should be discussed. This expectation is *a representation of the moment-generating function* of $v(t)$. The theorem below provides us with the corresponding solution.

Moment Generating Function for the CIR Process

Theorem (Moment-generating function for the CIR process)

For a stochastic process $v(t)$, with the dynamics, given by,

$$\mathrm{d}v(t) = \kappa(\bar{v} - v(t))\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v(t), \quad v(t_0) = v_0,$$

for $t \geq t_0$, the moment-generating function (or Laplace transformation) has the following form:

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{uv(t)}\middle|\mathcal{F}(t_0)\right] = \left(\frac{1}{1-2u\bar{c}(t,t_0)}\right)^{\frac{1}{2}\delta}\exp\left(\frac{u\bar{c}(t,t_0)\bar{\kappa}(t,t_0)}{1-2u\bar{c}(t,t_0)}\right),$$

where the parameters, $\bar{c}(t,t_0)$, degrees of freedom δ, and noncentrality $\bar{\kappa}(t,t_0)$, are given by

$$\bar{c}(t,t_0) = \frac{\gamma^2}{4\kappa}(1-\mathrm{e}^{-\kappa(t-t_0)}), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2}, \quad \bar{\kappa}(t,t_0) = \frac{4\kappa v_0 \mathrm{e}^{-\kappa(t-t_0)}}{\gamma^2(1-\mathrm{e}^{-\kappa(t-t_0)})}, \tag{12}$$

Frwd Start ChF for the Heston Model

- The characteristic function $\phi_x(u)$, as it is given in (11) with $\mathbf{u}^\mathrm{T} = [u, 0]^\mathrm{T}$, is now completely determined, as

$$\begin{aligned}\phi_x(u) &= \mathrm{e}^{\bar{A}(u,\tau)+r(T_2-T_1)}\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{\bar{C}(u,\tau)v(T_1)}\Big|\mathcal{F}(t_0)\right] && (13)\\ &= \exp\left(\bar{A}(u,\tau)+r\tau+\frac{\bar{C}(u,\tau)\bar{c}(T_1,t_0)\bar{\kappa}(T_1,t_0)}{1-2\bar{C}(u,\tau)\bar{c}(T_1,t_0)}\right) && (14)\\ &\quad x\left(\frac{1}{1-2\bar{C}(u,\tau)\bar{c}(T_1,t_0)}\right)^{\frac{1}{2}\delta},\end{aligned}$$

with $\tau = T_2 - T_1$, parameters δ, $\bar{\kappa}(t, t_0)$, $\bar{c}(t, t_0)$ as in (12), $\bar{C}(u, \tau)$ is as in the derivation of this function for the Heston characteristic function.

Experiment: Forward Implied Volatility for Heston

- Forward implied volatilities under the Heston model] In this experiment we compute the forward implied volatilities for the Heston model. The following set of model parameters is chosen,

$$r = 0, \kappa = 0.6, \bar{v} = 0.1, \gamma = 0.2, \rho_{x,v} = -0.5, v(t_0) = 0.05.$$

In a first test, we set $T_1 = 1, 2, 3, 4$ and $T_2 = T_1 + 2$, while in the second test we take $T_1 = 1$ and $T_2 = 2, 3, 4, 5$.

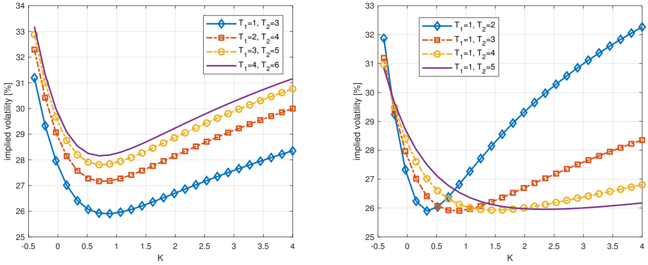

Figure: Forward implied volatilities for the Heston model.

The Bates Model

- Empirical studies have shown that the Heston model is not able to calibrate well to short-term implied volatilities.
- The Bates model also generalizes the Heston model by adding jumps to the Heston stock price process. The model is described by the following system of SDEs:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{S(t)} = \left(r - \xi_p \mathbb{E}[\mathrm{e}^J - 1]\right)\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_x(t) + \boxed{\left(\mathrm{e}^J - 1\right)\mathrm{d}X_{\mathcal{P}}(t),} \\ \mathrm{d}v(t) = \kappa\left(\bar{v} - v(t)\right)\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v(t), \end{cases}$$

with Poisson process $X_{\mathcal{P}}(t)$ with intensity ξ_p, and normally distributed jump sizes, J, with expectation μ_J and variance σ_J^2, i.e. $J \sim \mathcal{N}(\mu_j, \sigma_j^2)$. $X_{\mathcal{P}}(t)$ is assumed to be independent of the Brownian motions and of the jump sizes.

The Bates Model

- Under the log transformation, the Bates model reads

$$\begin{cases} \mathrm{d}X(t) = \left(r - \frac{1}{2}v(t) - \xi_p\mathbb{E}[\mathrm{e}^J - 1]\right)\mathrm{d}t + \sqrt{v(t)}\mathrm{d}W_x(t) + J\mathrm{d}X_{\mathcal{P}}(t), \\ \mathrm{d}v(t) = \kappa\left(\bar{v} - v(t)\right)\mathrm{d}t + \gamma\sqrt{v(t)}\mathrm{d}W_v(t). \end{cases}$$

- The PIDE for the Bates model, under the risk-neutral measure, can now be derived and is given, for $V = V(t, X)$, by

$$\frac{\partial V}{\partial t} + (r - \frac{1}{2}v - \xi_p\mathbb{E}[\mathrm{e}^J - 1])\frac{\partial V}{\partial X} + \kappa(\bar{v} - v(t))\frac{\partial V}{\partial v} + \frac{1}{2}\gamma^2 v\frac{\partial^2 V}{\partial v^2}$$
$$+\frac{1}{2}v\frac{\partial^2 V}{\partial X^2} + \rho_{x,v}\gamma v\frac{\partial^2 V}{\partial X \partial v} + \xi_p\mathbb{E}\left[V(t, X + J)\right] = (r + \xi_p)V.$$

The Bates Model

- The Bates model belongs to the class of affine jump diffusion processes. Therefore, we can derive the corresponding discounted characteristic function, for state-vector $\mathbf{X}(t) = [X(t), v(t)]^{\mathrm{T}}$. The model is based on the same covariance matrix as the Heston model.
- The jump component is given by

$$\bar{\xi}_p(t, \mathbf{X}(t)) = \xi_{p,0} + \xi_{p,1}\mathbf{X}(t) = \xi_p,$$

and a_0 and a_1, are given by

$$a_0 = \begin{bmatrix} r - \xi_p \mathbb{E}[\mathrm{e}^J - 1] \\ \kappa \bar{v} \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & -\kappa \end{bmatrix}. \tag{15}$$

The Bates Model

- With $\mathbf{J} = [J, 0]$ and $\bar{\mathbf{B}}(\mathbf{u}, \tau) = [\bar{B}(\mathbf{u}, \tau), \bar{C}(\mathbf{u}, \tau)]^{\mathrm{T}}$, the affinity relations in the class of AJD processes provide the following system of ODEs:

Lemma (Bates ODEs)

The functions $\bar{A}_{Bates}(\mathbf{u}, \tau)$, $\bar{B}(\mathbf{u}, \tau)$ and $\bar{C}(\mathbf{u}, \tau)$ satisfy the following system of ODEs:

$$
\begin{aligned}
\frac{\mathrm{d}\bar{B}}{\mathrm{d}\tau} &= 0,\\
\frac{\mathrm{d}\bar{C}}{\mathrm{d}\tau} &= \frac{1}{2}\bar{B}(\bar{B}-1) - \left(\kappa - \gamma\rho_{x,v}\bar{B}\right)\bar{C} + \frac{1}{2}\gamma^2\bar{C}^2,\\
\frac{\mathrm{d}\bar{A}_{Bates}}{\mathrm{d}\tau} &= \kappa\bar{v}\bar{C} + r(\bar{B}-1) - \xi_p\mathbb{E}[\mathrm{e}^J - 1]\bar{B} + \xi_p\mathbb{E}\left[\mathrm{e}^{J\bar{B}} - 1\right],
\end{aligned}
$$

with initial conditions $\bar{B}(\mathbf{u}, 0) = iu$, $\bar{C}(\mathbf{u}, 0) = 0$ and $A_{Bates}(\mathbf{u}, 0) = 0$, and the parameters κ,γ, $\bar{v}$, r and $\rho_{x,r}$ are as in the Heston model.

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The Bates Model

- The functions $\bar{B}(\mathbf{u},\tau)$ and $\bar{C}(\mathbf{u},\tau)$ are the same as for the standard Heston model. The Bates and the Heston model differ only in $\bar{A}_{\text{Bates}}(\mathbf{u},\tau)$, which in the Bates model contains jump components.
- As the jumps J in the Bates model are normally distributed, with mean μ_J and variance σ_J^2, the two expectations in the ODEs for $\bar{A}_{\text{Bates}}(\mathbf{u},\tau)$ are given by

$$\mathbb{E}\left[\mathrm{e}^{J}-1\right]=\mathrm{e}^{\mu_J+\frac{1}{2}\sigma_J^2}-1,\quad \mathbb{E}\left[\mathrm{e}^{iuJ}-1\right]=\mathrm{e}^{iu\mu_J-\frac{1}{2}\sigma_J^2u^2}-1,$$

  so that the ODE for $\bar{A}_{\text{Bates}}(\mathbf{u},\tau)$ reads:

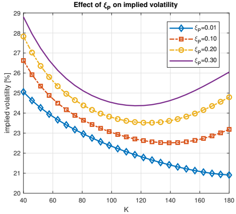
$$\frac{\mathrm{d}\bar{A}_{\text{Bates}}}{\mathrm{d}\tau}=\frac{\mathrm{d}\bar{A}}{\mathrm{d}\tau}-\xi_p iu\left(\mathrm{e}^{\mu_J+\frac{1}{2}\sigma_J^2}-1\right)+\xi_p\left(\mathrm{e}^{iu\mu_J-\frac{1}{2}\sigma_J^2u^2}-1\right),$$

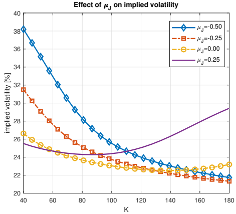
  with $\frac{\mathrm{d}\bar{A}}{\mathrm{d}\tau}$ as derived for the Heston model.
- The solution for $\bar{A}_{\text{Bates}}(\mathbf{u},\tau)$ can also easily be found, as

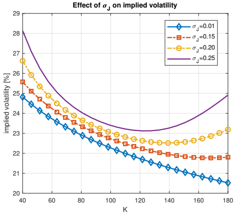
$$\bar{A}_{\text{Bates}}(\mathbf{u},\tau)=\bar{A}(\mathbf{u},\tau)-\xi_p iu\tau\left(\mathrm{e}^{\mu_J+\frac{1}{2}\sigma_J^2}-1\right)+\xi_p\tau\left(\mathrm{e}^{iu\mu_J-\frac{1}{2}\sigma_J^2u^2}-1\right).$$

The Bates Model

- In the experiments we use $T = 1$, $S_0 = 100$; the Heston model parameters have been set to $\kappa = 1.2$, $\bar{v} = 0.05$, $\gamma = 0.05$, $\rho_{x,v} = -0.75$, $v_0 = 0.05$ and $r = 0$. The three jump parameters, i.e. the intensity ξ_p, the jump mean μ_J and jump volatility σ_J, have been varied.







The Bates Model

- Figures shows quite similar patterns for the implied volatilities regarding parameter changes in ξ_p and σ_J. An increase of the parameter values increases the curvature of the implied volatilities. The plots also indicate that the implied volatilities from the Bates model can be higher than those obtained by the Heston model.
- The explanation lies in the fact that the additional, uncorrelated jump component is present in the dynamics of the Bates model, which may increase the *realized* stock variance. An increase of volatility of the stock increases the option price as the option is more likely to end in the money.
- In the case of parameter μ_J, which represents the *expected* location of the jumps, the implied volatilities reveal an irregular behavior. Parameter μ_J can take either positive or negative values, as it determines whether the stock paths will have upwards or downwards jumps.

Introduction to Variance Swaps

- Via financial products called variance swaps, we can actually trade the volatility, just like any other stock or commodity.
- Volatility can be measured, indirectly, by continuous observation of the stock performance. A variance swap payoff is defined as:

$$\begin{aligned} V(T,\sigma_v) &= \frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S(t_i)}{S(t_{i-1})}\right)^2 - K \qquad (16)\\ &=: \sigma_v^2(T) - K, \end{aligned}$$

  for asset $S(t)$ with a given time-grid $t_0 < t_1 < \cdots < t_n = T$, the strike level K, and σ_v^2 is the realized variance of the stock over the life of the swap. Integer 252 represents the number of the business days in a given year.
- Typically, the strike K is chosen in such a way that the value of the contract is initially equal to 0.

Introduction to Variance Swaps cont.

- Assuming the interest rate to be deterministic, the contract value at time t_0 reads:

$$V(t_0, \sigma_v) = \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\sigma_v^2(T) - K|\mathcal{F}(t_0)\right]. \tag{17}$$

- The strike value K at which the value of the contract initially equals to zero is given by,

$$\mathrm{e}^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}\left[\sigma_v^2(T) - K|\mathcal{F}(t_0)\right] = 0, \tag{18}$$

so, $K = \mathbb{E}^{\mathbb{Q}}\left[\sigma_v^2(T)|\mathcal{F}(t_0)\right]$.

- Let's relate the expression in (16) to the dynamics of the stock process $S(t)$. We consider at the term under the summation in(16), and find its limit as the time grid gets finer, i.e., $\delta_i = t_i - t_{i-1} \to 0$:

$$\log \frac{S(t_i)}{S(t_{i-1})} = \log S(t_i) - \log S(t_{i-1}) \xrightarrow{\delta_i \to 0} \mathrm{d}\log S(t),$$

so that the expression squared equals,

$$\left(\log \frac{S(t_i)}{S(t_{i-1})}\right)^2 \xrightarrow{\delta_i \to 0} (\mathrm{d}\log S(t))^2.$$

Introduction to Variance Swaps cont.

- Assuming the stock process to be governed by the following, possibly multidimensional, stochastic process:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t), \tag{19}$$

with a constant interest rate r and stochastic process $\sigma(t)$ for the volatility, the dynamics under the log-transformation yield,

$$\mathrm{d}\log S(t) = \left(r - \frac{1}{2}\sigma^2(t)\right)\mathrm{d}t + \sigma(t)\mathrm{d}W(t). \tag{20}$$

- By use of Itô's table, we find $(\mathrm{d}\log S(t))^2 = \sigma^2(t)\mathrm{d}t$, and thus

$$\sum_{i=1}^{n}\left(\log\frac{S(t_i)}{S(t_{i-1})}\right)^2 \to \int_{t_0}^{T}(\mathrm{d}\log S(t))^2 = \int_{t_0}^{T}\sigma^2(t)\mathrm{d}t. \tag{21}$$

- The term $252/n$ annualizes the realized variance (it returns the percentage of a year, being 252 business days) and which in the continuous case equals $1/(T-t_0)$.

Introduction to Variance Swaps cont.

- In the continuous case, the variance swap contract can be written as,

$$V(T) = \frac{1}{T-t_0}\int_{t_0}^{T}\sigma^2(t)\mathrm{d}t - K =: \sigma_v^2(T) - K. \quad (22)$$

- Having established the relation between discrete and the continuous equivalent, we can determine strike K at which the contract should be traded:

$$K = \mathbb{E}^{\mathbb{Q}}\left[\sigma_v^2(T)|\mathcal{F}(t_0)\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T-t_0}\int_{t_0}^{T}\sigma^2(t)\mathrm{d}t\Big|\mathcal{F}(t_0)\right]. \quad (23)$$

- Combining Equations (19) and (20), gives us:

$$\begin{aligned}\frac{\mathrm{d}S(t)}{S(t)} - \mathrm{d}\log S(t) &= \Big(r\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\Big) - \left[\left(r - \frac{1}{2}\sigma^2(t)\right)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\right] \\ &= \frac{1}{2}\sigma^2(t)\mathrm{d}t.\end{aligned}$$

Introduction to Variance Swaps cont.

- Equation (23) is given, in this case, by,

$$
\begin{aligned}
K &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T-t_0}\int_{t_0}^{T}\sigma^2(t)\mathrm{d}t\Big|\mathcal{F}(t_0)\right]\\
&= \frac{2}{T-t_0}\mathbb{E}^{\mathbb{Q}}\left[\int_{t_0}^{T}\frac{\mathrm{d}S(t)}{S(t)}-\mathrm{d}\log S(t)\Big|\mathcal{F}(t_0)\right]\\
&= \frac{2}{T-t_0}\mathbb{E}^{\mathbb{Q}}\left[\int_{t_0}^{T}\frac{\mathrm{d}S(t)}{S(t)}\Big|\mathcal{F}(t_0)\right]-\frac{2}{T-t_0}\mathbb{E}^{\mathbb{Q}}\left[\log\frac{S(T)}{S(t_0)}\Big|\mathcal{F}(t_0)\right].
\end{aligned}
$$

- After simplifications and interchanging integration and taking the expectation, we find:

$$
K = \frac{2}{T-t_0}\left\{r(T-t_0)-\mathbb{E}^{\mathbb{Q}}\left[\log\frac{S(T)}{S(t_0)}\Big|\mathcal{F}(t_0)\right]\right\}, \quad (24)
$$

where $S(T)/S(t_0)$ represents the rate of return of the underlying stock.