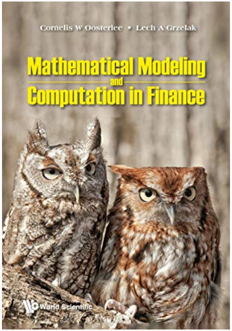
Materials for the course

The course is based on book "*Mathematical Modeling and Computation in Finance: With Exercises and Python and MATLAB Computer Codes*", by C.W. Oosterlee and L.A. Grzelak, World Scientific Publishing Europe Ltd, 2019. For more details go here.



- Youtube Channel with courses can be found here.
- Slides and the codes can be found here.

List of content

Caplets/Floorlets

- The price of a caplet, with a strike price K, is given by:

$$\begin{aligned} V^{\mathrm{CPL}}(t_0) &= N\tau_k\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_k)}\max\left(\ell_k(T_{k-1})-K,0\right)\Big|\mathcal{F}(t_0)\right] \\ &= N\tau_k P(t_0,T_k)\mathbb{E}^{T_k}\left[\max\left(\ell_k(T_{k-1})-K,0\right)\Big|\mathcal{F}(t_0)\right]. \end{aligned}$$

- Now, assuming that the libor rate follows a lognormal distribution, i.e.:

$$\boxed{\mathrm{d}\ell(t;T_{k-1},T_k)=\sigma_k\ell(t;T_{k-1},T_k)\mathrm{d}W^k(t).}$$

- If we consider Black's model dynamics to value this option, then the value of caplet k is given by:

$$\begin{aligned} \mathrm{Caplet}_k(t_0) &= N_k\tau_k P(t_0,T_k)\left[\ell(t_0;T_{k-1},T_k)N(d_1)-K_kN(d_2)\right],\ \text{with} \\ d_1 &= \frac{\log\left(\frac{\ell(t_0;T_{k-1},T_k)}{K}\right)+\frac{1}{2}\sigma_k^2(T_{k-1}-t_0)}{\sigma_k\sqrt{(T_{k-1}-t_0)}}, \\ d_2 &= d_1-\sigma_k\sqrt{T_{k-1}-t_0}. \end{aligned}$$

Caplets/Floorlets

- The price of a caplet, with a strike price K, is given by:

$$
\begin{aligned}
V^{\mathrm{CPL}}(t_0) &= N\tau_k\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_k)}\max\left(\ell_k(T_{k-1})-K,0\right)\Big|\mathcal{F}(t_0)\right]\\
&= N\tau_k P(t_0,T_k)\mathbb{E}^{T_k}\left[\max\left(\ell_k(T_{k-1})-K,0\right)\Big|\mathcal{F}(t_0)\right].
\end{aligned}
$$

- By the definition of the Libor rate, the (scaled) caplet valuation formula can be written as,

$$
\begin{aligned}
\frac{V^{\mathrm{CPL}}(t_0)}{P(t_0,T_k)} &= N\tau_k\mathbb{E}^{T_k}\left[\max\left(\frac{1}{\tau_k}\left(\frac{1}{P(T_{k-1},T_k)}-1\right)-K,0\right)\Big|\mathcal{F}(t_0)\right]\\
&= N\cdot\mathbb{E}^{T_k}\left[\max\left(\mathrm{e}^{-\bar{A}_r(\tau_k)-\bar{B}_r(\tau_k)r(T_{k-1})}-1-\tau_k K,0\right)\Big|\mathcal{F}(t_0)\right]\\
&= N\cdot\mathrm{e}^{-\bar{A}_r(\tau_k)}\mathbb{E}^{T_k}\left[\max\left(\mathrm{e}^{-\bar{B}_r(\tau_k)r(T_{k-1})}-\hat{K},0\right)\Big|\mathcal{F}(t_0)\right],
\end{aligned}
$$

with $\hat{K}=(1+\tau_k K)\mathrm{e}^{\bar{A}_r(\tau_k)}$.

- Alternative, based on the tower property, derivations can be found in the book on page 381.

Caplets/Floorlets

- From Lecture 5 we know that a European-style option is defined by the following equation:

$$V^{\text{ZCB}}(t_0, T_{k-1}) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_{k-1})}\max\left(\bar{\alpha}(P(T_{k-1}, T_k) - K), 0\right)\Big|\mathcal{F}(t_0)\right],$$

  with $\bar{\alpha} = 1$ for a call and $\bar{\alpha} = -1$ for a put option, strike price K and $\mathrm{d}M(t) = r(t)M(t)\mathrm{d}t$.
- The price of a caplet with a strike price K can be written as,

$$\begin{aligned} V^{\text{CPL}}(t_0) &= N\tau_i\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_i)}\max\left(\ell_i(T_{i-1}) - K, 0\right)\Big|\mathcal{F}(T_{i-1})\right]\right] \\ &= N\tau_i\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_{i-1})}\mathbb{E}^{\mathbb{Q}}\left[\frac{M(T_{i-1})}{M(T_i)}\max\left(\ell_i(T_{i-1}) - K, 0\right)\Big|\mathcal{F}(T_{i-1})\right]\right] \end{aligned}$$

Caplets/Floorlets

- After a change of measure, the inner expectation can be expressed as,

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{M(T_{i-1})}{M(T_i)}\max\left(\ell_i(T_{i-1})-K,0\right)\Big|\mathcal{F}(T_{i-1})\right]=P(T_{i-1},T_i)\max\left(\ell_i(T_{i-1})-K,\right.$$

- The caplet value can therefore be found as,

$$V^{\mathrm{CPL}}(t_0)=N\tau_i\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{M(T_{i-1})}P(T_{i-1},T_i)\max\left(\ell_i(T_{i-1})-K,0\right)\Big|\mathcal{F}(t_0)\right].$$

- For the (scaled) caplet value, this results in

$$\begin{aligned}\frac{V^{\mathrm{CPL}}(t_0)}{P(t_0,T_{i-1})} &= N\tau_i\cdot\mathbb{E}^{T_{i-1}}\left[P(T_{i-1},T_i)\max\left(\frac{1}{\tau_i}\left(\frac{1}{P(T_{i-1},T_i)}-1\right)-K,0\right)\right.\\ &= \hat{N}\cdot\mathbb{E}^{T_{i-1}}\left[\max\left(\frac{1}{\hat{K}}-P(T_{i-1},T_i)\right)\Big|\mathcal{F}(t_0)\right],\end{aligned}$$

with $\hat{N}=N\left(1+\tau_i K\right)$ and $\hat{K}=1+\tau_i K$.

Implied Volatility for Caplets/Floorlets

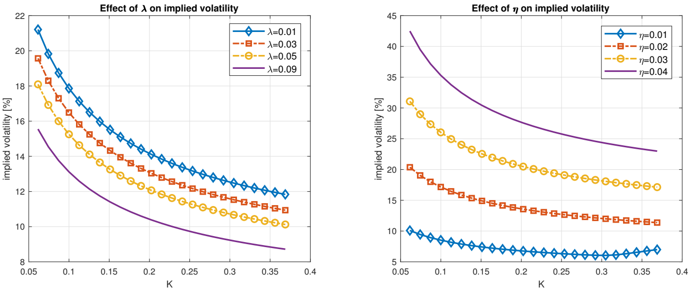


Figure: Effect of λ and η in the Hull-White model on the caplet implied volatilities.

Implied Volatility for Caplets/Floorlets

- The impact of the different Hull-White model parameters, λ, η, on the caplet implied volatility is presented.
- The mean reversion parameter λ appears to have a much smaller effect on the implied volatilities than the volatility parameter η.
- In practice, therefore, λ is often kept fixed whereas η is determined in a calibration process.
- Often the Hull-White model is extended and time-dependent volatility parameter $\eta(t)$ is used.
- The Hull-White model with time-dependent volatility parameter is the market practice for xVA. There the calibration is performed to (Bermudan) Swaptions.

Swaps

- Previously we have derived the value of a swap:

$$V_{m,n}^{\text{Swap}}(t_0) = \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\ell(t_0, T_{k-1}, T_k) - K \sum_{k=m+1}^{n} \tau_k P(t_0, T_k).$$

- The first summation can be further simplified, using the definition of the libor rate,

$$\begin{aligned}\sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\ell_k(t_0) &= \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\left[\frac{P(t_0, T_{k-1}) - P(t_0, T_k)}{\tau_k P(t_0, T_k)}\right]\\ &= \sum_{k=m+1}^{n} P(t_0, T_{k-1}) - P(t_0, T_k)\\ &= P(t_0, T_m) - P(t_0, T_n),\end{aligned}$$

  where in the last step the telescopic summation was recognized.
- The price of the swap is as follows:

$$V_{m,n}^{\text{Swap}}(t_0) = [P(t_0, T_m) - P(t_0, T_n)] - K \sum_{k=m+1}^{n} \tau_k P(t_0, T_k).$$

Swaps

- By setting the value of the swap to zero, entering into such a deal is *for free*. Moreover, the strike value for which the swap equals zero is called *swap rate* and is indicated by $S_{m,n}(t_0)$.
- By equating the swap value to zero we find,

$$S_{m,n}(t_0) = \frac{P(t_0, T_m) - P(t_0, T_n)}{\sum_{k=m+1}^{n} \tau_k P(t_0, T_k)} = \frac{P(t_0, T_m) - P(t_0, T_n)}{A_{m,n}(t_0)},$$

which, alternatively, by not simplifying the first summation in (1) we can write,

$$S_{m,n}(t_0) = \frac{1}{A_{m,n}(t_0)} \sum_{k=m+1}^{n} \tau_k P(t_0, T_k)\ell_k(t_0) = \sum_{k=m+1}^{n} \omega_k(t_0)\ell_k(t_0),$$

with $\omega_k(t_0) = \tau_k P(t_0, T_k)/A_{m,n}(t_0)$ and $\ell_k(t_0) := \ell(t_0, T_{k-1}, T_k)$.

- We can express the value of the swap as:

$$\boxed{V_{m,n}^{\text{Swap}}(t_0) = A_{m,n}(t_0)\big(S_{m,n}(t_0) - K\big).}$$

European Swaptions

- Swaptions are the options on interest rate swaps, i.e. a holder of a European swaption has the right, but not an obligation, to enter a swap at a future date at a given predetermined strike K.
- Similarly as for swaps the swaptions have payers and receivers who pay the fixed rate and receive the float leg (payer) or vice versa (receiver).
- In the standard setting the swaption maturity coincides with the first reset date of the underlying interest swap, i.e.: $T_0 = T_m$.
- Let time t_0 be today's date and T_m be some future date at which one has the option to enter into a swap deal, with the first reset date given by T_m.

European Swaptions

- The value of the deal at time T_m is then given by:

$$\begin{aligned} V^{\mathrm{S}}_{m,n}(T_m) &= \max\left(V^{\mathrm{Swap}}_{m,n}(T_m),0\right) \\ &= \max\left[\sum_{k=m+1}^{n}\tau_k P(T_m,T_k)\Big(\ell(T_m,T_{k-1},T_k)-K\Big),0\right], \end{aligned}$$

and today's discounted value is equal to:

$$V^{\mathrm{S}}_{m,n}(T_m)=\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_m)}\max\left(\sum_{k=m+1}^{n}\tau_k P(T_m,T_k)\Big(\ell(T_m,T_{k-1},T_k)-K\Big),0\right)\right].$$

- Using the annuity representation the value of the swaption is also equal to:

$$\begin{aligned} V^{\mathrm{S}}_{m,n}(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_m)}\max\left(A_{m,n}(T_m)\big(S_{m,n}(T_m)-K\big),0\right)\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{A_{m,n}(T_m)M(t_0)}{M(T_m)}\max\left(S_{m,n}(T_m)-K,0\right)\right], \end{aligned}$$

where $M(t_0)=1$ and where $t_0<T_m$ and $T_m<T_{m+1}$, with T_{m+1} the first payment date.

European Swaptions

- As discussed, annuity $A_{m,n}(T_m)$ is simply a combination of tradable zero-coupon bonds and can therefore be considered as a numéraire.
- This suggests that we can define a Radon-Nikodym derivative for changing measures, from the risk-neutral measure, $\mathbb{Q}$, associated with the money-savings account, $M(t)$, to the new annuity measure (also known as the swap measure), $\mathbb{Q}^{m,n}$, associated with the annuity $A_{m,n}(t)$, i.e.,

$$\boxed{\lambda_{\mathbb{Q}}^{m,n}(T_m) = \frac{\mathrm{d}\mathbb{Q}^{m,n}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}(T_m)} = \frac{A^{m,n}(T_m)}{A^{m,n}(t_0)}\frac{M(t_0)}{M(T_m)}.}$$

- The value of the swaption now is given by:

$$V_{m,n}^{S}(t_0) \quad = \quad \mathbb{E}^{m,n}\left[\frac{A_{m,n}(T_m)M(t_0)}{M(T_m)}\frac{A^{m,n}(t_0)}{A^{m,n}(T_m)}\frac{M(T_m)}{M(t_0)}\max\left(S_{m,n}(T_m)-K,0\right)\right].$$

- After simplifications we find:

$$\boxed{V_{m,n}^{S}(t_0) = A^{m,n}(t_0)\mathbb{E}^{m,n}\left[\max\left(S_{m,n}(T_m)-K,0\right)\right].}$$

European Swaptions under the Black-Scholes Model

- To avoid arbitrage the swap rate $S_{m,n}(T_m)$, i.e.

$$S_{m,n}(t) \quad = \quad \frac{P(t,T_m) - P(t,T_n)}{A_{m,n}(t)},$$

  has to be a martingale under the swap measure associated with annuity $A_{m,n}(t)$.
- This implies that the dynamics of the swap rate $S_{m,n}(t)$ under the swap measure, $\mathbb{Q}^{m,n}$, have to be driftless.
- A standard approach to price a swaption is via the Black-Scholes model, i.e, assuming GBM process for $S_{m,n}(t)$ under the annuity measure $A_{m,n}(t)$ with the dynamics given by:

$$\boxed{\mathrm{d}S_{m,n}(t) = \sigma_{m,n}S_{m,n}(t)\mathrm{d}W^{m,n}(t).}$$

- Note that the lognormal assumptions implies only positive swap rates!

European Swaptions under the Black-Scholes Model

- Then, the price of European Payer Swaption is given by:

$$\begin{aligned} V^{S}_{m,n}(t_0) &= A^{m,n}(t_0)\mathbb{E}^{m,n}\left[\max\left(S_{m,n}(T_m)-K,0\right)\right] \\ &= A^{m,n}(t_0)\left[S_{m,n}(t_0)F_{\mathcal{N}(0,1)}(d_1)-KF_{\mathcal{N}(0,1)}(d_2)\right]. \end{aligned}$$

- Then, the price of European Receiver Swaption is given by:

$$\begin{aligned} V^{S}_{m,n}(t_0) &= A^{m,n}(t_0)\mathbb{E}^{m,n}\left[\max\left(K-S_{m,n}(T_m),0\right)\right] \\ &= A^{m,n}(t_0)\left[KF_{\mathcal{N}(0,1)}(-d_2)-S_{m,n}(t_0)F_{\mathcal{N}(0,1)}(-d_1)\right]. \end{aligned}$$

- In the pricing two constants d_1 and d_2 are given by:

$$d_1=\frac{\log\left(\frac{S_{m,n}(t_0)}{K}\right)+\frac{1}{2}\sigma^2 T_m}{\sigma_{m,n}\sqrt{T_m}},\quad d_2=d_1-\sigma_{m,n}\sqrt{T_m}.$$

European Swaptions under the Hull-White model

- We need a slightly different representation for pricing with the Hull-White model.

$$\begin{aligned}\sum_{k=m+1}^{n}\tau_k P(T_m,T_k)\Big(\ell_k(T_m)-K\Big) &= 1-P(T_m,T_n)-K\sum_{k=m+1}^{n}\tau_k P(T_m,T_k)\\ &= 1-\sum_{k=m+1}^{n}c_k P(T_m,T_k),\end{aligned}$$

  with $c_k = K\tau_k$ for $k = m+1,\dots,n-1$ and $c_n = 1+K\tau_n$.
- This gives us the following expression for the swaption price:

$$\boxed{V^{\mathrm{S}}_{m,n}(t_0) = NP(t_0,T_m)\mathbb{E}^{T_m}\left[\max\left(1-\sum_{k=m+1}^{n}c_k P(T_m,T_k),0\right)\right].}$$

- Now we will use the definition of the ZCB under the Hull-White model.
- The pricing above is not easy as the distribution of the sum of $P(T_m,T_k)$ is unknown.

Jamshidian's Trick- Introduction

- Calulations of an expectation of the sum, is, due to linearity of expectation, a simple task:

$$\mathbb{E}\sum_{k=1}^{n}\psi_k(X)=\sum_{k=1}^{n}\mathbb{E}[\psi_k(X)].$$

- The same principle holds when dealing with min / max or any other function:

$$\mathbb{E}\sum_{k=1}^{n}\max(\psi_k(X)-K,0)=\sum_{k=1}^{n}\mathbb{E}[\max(\psi_k(X)-K,0)].$$

- Unfortunately, this principle cannot be used when we deal with a max / min of a sum, i.e.,

$$\mathbb{E}\max\left(\sum_{k=1}^{n}\psi_k(X)-K,0\right)\neq\sum_{k=1}^{n}\mathbb{E}\max\left(\psi(X)_k-K,0\right)$$

- Computation of the LHS is not trivial! F.Jamshidian has found an algorithm that allows us to compute the LHS, *almost* analytically.

Jamshidian's Trick

- In 1989 F.Jamshidian presented an approach for transforming the problem of calculating of maximum of a sum into a sum of certain maximums, i.e.: for

$$A = \max\Big(\sum_{k=1}^{n}\psi_k(X) - K, 0\Big),$$

where $\psi_k(X)$ is a monotone increasing or monotone decreasing sequence of functions $\psi_k(X) : \mathbb{R} \rightarrow \mathbb{R}^+$.

- Since each $\psi_k(X)$ is monotone in X the sum $\sum_{k=1}^{n}\psi_k(X)$ is as well this means that there exists X^* such that

$$\sum_{k=1}^{n}\psi_k(X^*) - K = 0,$$

where X^* needs typically be determined with some search algorithm like a Newton-Raphson discussed during previous lectures.

European Swaptions under the Hull-White model

- Since
$$\sum_{k=1}^{n}\psi_k(X^*)-K=0,\quad \text{thus}\quad K=\sum_{k=1}^{n}\psi_k(X^*).$$
- Now the maximum becomes:
$$A=\max\left(\sum_{k=1}^{n}\psi_k(X)-\sum_{k=1}^{n}\psi_k(X^*),0\right)=\max\left(\sum_{k=1}^{n}\left(\psi_k(X)-\psi_k(X^*)\right),0\right).$$
- The expression for A becomes:
$$A=\max\left(\sum_{k=1}^{n}\left(\psi_k(X)-\psi_k(X^*)\right),0\right)=\sum_{k=1}^{n}\left(\psi_k(X)-\psi_k(X^*)\right)1_{X>X^*},$$
which finally can be expressed as:
$$\boxed{A=\sum_{k=1}^{n}\max\left(\psi_k(X)-\psi_k(X^*),0\right).}$$
- This results is essenctal when pricing of Swaptions with the Hull-White Model.

Jamshidian's trick- Python Example

- Let us consider a sequence:

$$\psi_k(X) = \mathrm{e}^{-t_i|X|}, \quad t_i = \{0, 1, 2, 3, 4, \ldots, N\},$$

- Normal distributed random variable $X \sim \mathcal{N}(0, 1)$.
- Compute

$$A = \mathbb{E}\left[\max\left(\sum_{k=1}^{N} \psi_k(X) - K, 0\right)\right],$$

directly using Monte Carlo and by using Jamshidian's trick. Plot results against strike K.

python™

Applicability of the trick

- In Lecture 3 we have discussed the HW2F model. As we see the ZCBs/yield is not monotone therefore the Jamshidian's trick cannot be easily applied.

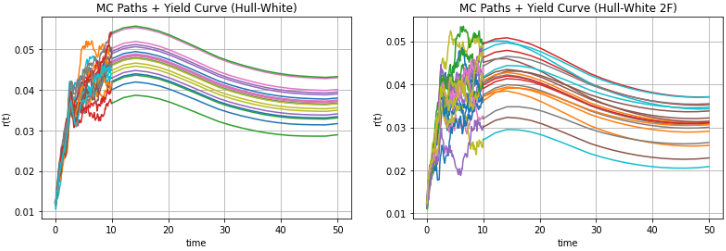


Figure: Dynamics of the yield curve for random market scenarios under the 1F Hull-White model.

European Swaptions under the Hull-White model

- Having this results in mind we can come back to the pricing of Swaptions under the Hull-White model.
- The Hull-White model is driven by the following dynamics:

$$\mathrm{d}r(t) = \lambda\left(\theta(t) - r(t)\right)\mathrm{d}t + \eta \mathrm{d}W^{\mathbb{Q}}(t),$$

  where function $\theta(t)$ is given explicitly in terms of the ZCBs, $P_{market}(t_0, T)$, and where $\lambda, \eta \in \mathbb{R}$.
- The zero-coupon bond $P(t, T)$ under the, affine, Hull-White process $r(t)$ can be expressed as

$$P(t, T) = \exp\left(A(t, T) + B(t, T)r(t)\right),$$

- Functions $A(t, T)$ and $B(t, T)$ can be simply found when calculating the following expectation

$$P(t, T) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\Big|\mathcal{F}(t_0)\right],$$

  with $r(t)$ to be driven by the Hull-White process.

European Swaptions under the Hull-White model

- The complete formulation of the ZCB under the Hull-White mode reads

$$\boxed{P(t,T) = \mathrm{e}^{A(t,T)+B(t,T)r(t)},}$$

- where functions $A(t,T)$ and $B(t,T)$ are, for $\tau = T - t$, given by:

$$\begin{aligned}
A(t,T) &= -\frac{\eta^2}{4\lambda^3}\left(3+\mathrm{e}^{-2\lambda\tau}-4\mathrm{e}^{-\lambda\tau}-2\lambda\tau\right)+\lambda\int_t^T\theta(z)B(z,T)\mathrm{d}z,\\
B(t,T) &= -\frac{1}{\lambda}\left(1-\mathrm{e}^{-\lambda(T-t)}\right).
\end{aligned}$$

- Ultimately, the formulation above can be further simplified and the integration over $\theta(t)$ can be avoided. Then the ZCB will explicitly depend on ZCBs from the market.
- We notice that for a given time t, $P(t,T_k)$ are monotone in $r(t)$ for any $T_k > k$.

European Swaptions under the Hull-White model

- Now, we apply Jamishidian's trick to pricing of swaptions under the Hull-White Model.

$$V^{S}_{m,n}(t_0)=\mathbb{E}^{\mathbb{Q}}\left[\frac{M(t_0)}{M(T_m)}\max\left[\sum_{k=m+1}^{n}\tau_k P(T_m,T_k)\Big(\ell(T_m,T_{k-1},T_k)-K\Big),0\right]\right].$$

- Since swaption expires at time T_M, we take the ZCB $P(t,T_m)$ as the new numéraire.
- After a change of measure, from the risk-free measure $\mathbb{Q}$ to the T_m-forward measure $\mathbb{Q}^{T_m}$, gives,

$$V^{S}_{m,n}(t_0)=P(t_0,T_m)\mathbb{E}^{T_m}\left[\max\left[\sum_{k=m+1}^{n}\tau_k P(T_m,T_k)\Big(\ell(T_m,T_{k-1},T_k)-K\Big),0\right]\right].$$

European Swaptions under the Hull-White model

- Recall that for $\ell_k(T_m) := \ell(T_m, T_{k-1}, T_k)$ the following relation has to hold:

$$\begin{aligned}\sum_{k=m+1}^{n} \tau_k P(T_m, T_k)\Big(\ell_k(T_m) - K\Big) &= 1 - P(T_m, T_n) - K \sum_{k=m+1}^{n} \tau_k P(T_m, T_k) \\ &= 1 - \sum_{k=m+1}^{n} c_k P(T_m, T_k),\end{aligned}$$

  with $c_i = K\tau_k$ for $i = m+1, \dots, n-1$ and $c_n = 1 + K\tau_n$.
- This gives us the following expression for the swaption price:

$$V^{S}_{m,n}(t_0) = NP(t_0, T_m)\mathbb{E}^{T_m}\left[\max\left(1 - \sum_{k=m+1}^{n} c_k P(T_m, T_k), 0\right)\right].$$

- Now we will use the definition of the ZCB under the Hull-White model.

Zero-Coupon Bond under the Hull-White model

- The pricing of Swaptions reads:

$$V^{\mathrm{S}}_{m,n}(t_0) = NP(t_0, T_m)\mathbb{E}^{T_m}\left[\max\left(1 - \sum_{k=m+1}^{n} c_k \mathrm{e}^{A(T_m,T_k)+B(T_m,T_k)r(T_m)}, 0\right)\right].$$

- Now we apply Jamshidian's trick we seek r^* such that

$$1 - \sum_{k=m+1}^{n} c_k \mathrm{e}^{A(T_m,T_k)+B(T_m,T_k)r^*} = 0.$$

- Now we can substitute this result for 1 in the pricing equation:

$$\begin{aligned} V^{\mathrm{S}}_{m,n}(t_0) &= NP(t_0, T_m)\mathbb{E}^{T_m}\left[\max\left(\sum_{k=m+1}^{n} c_k \mathrm{e}^{A+Br^*} - \sum_{k=m+1}^{n} c_k \mathrm{e}^{A+Br(T_m)}, 0\right)\right] \\ &= NP(t_0, T_m)\mathbb{E}^{T_m}\left[\max\left(\sum_{k=m+1}^{n} c_k \left(\mathrm{e}^{A+Br^*} - \mathrm{e}^{A+Br(T_m)}\right), 0\right)\right]. \end{aligned}$$

European Swaptions under the Hull-White model

- Now, using the Jamshidian's method we switch between max of sum to a sum of max:

$$
\begin{aligned}
V^{\rm S}_{m,n}(t_0) &= NP(t_0,T_m)\mathbb{E}^{T_m}\left[\max\left(\sum_{k=m+1}^{n} c_k\left(\mathrm{e}^{A+Br^*}-\mathrm{e}^{A+Br(T_m)}\right),0\right)\right]\\
&= NP(t_0,T_m)\sum_{k=m+1}^{n} c_k\mathbb{E}^{T_m}\left[\max\left(\mathrm{e}^{A+Br^*}-\mathrm{e}^{A+Br(T_m)},0\right)\right]
\end{aligned}
$$

- Commonly, it is rewritten for a strike $\hat{K}$ as

$$
\boxed{V^{\rm S}_{m,n}(t_0) = NP(t_0,T_m)\sum_{k=m+1}^{n} c_k\mathbb{E}^{T_m}\left[\max\left(\hat{K}-\mathrm{e}^{A+Br(T_m)},0\right)\right],}
$$

with

$$
\hat{K} = \mathrm{e}^{A+Br^*},
$$

and where a constant r^* chosen such that:

$$
\boxed{1-\sum_{k=m+1}^{n} c_k \exp\left(A(T_m,T_k)+B(T_m,T_k)r^*\right) = 0.}
$$

European Swaptions under the Hull-White model

- The task of pricing swaption is not complete yet as one still needs to determine the sum of expectations.
- We notice that each element of this sum is simply an European put option on the zero-coupon bond, i.e.:

$$V^{\mathrm{ZCB}}_{T_m,T_k}(t_0,K,-1) \quad = \quad P(t_0,T_m)\mathbb{E}^{T_m}\left[\max\left(K-\mathrm{e}^{A_r(\tau_k)+B_r(\tau_k)r(T_m)},0\right)\right],$$

  with $\tau_k = T_k - T_m$.
- Pricing of European-type options on the ZCB can be, under the Hull-White model, done analytically with the closed-form pricing solution.
- Closed form solution for pricing an option on a ZCB under the HW model is discussed in Lecture 4.

European Swaptions under the Hull-White model

- The final pricing equation for European swaptions then reads:

$$V_{m,n}^{\text{Swaption}}(t_0) = N \sum_{k=m+1}^{n} c_k V_{T_m,T_k}^{\text{ZCB}}(t_0, \hat{K}, -1),$$

with strike

$$\hat{K} := \exp\left(A_r(T_m, T_k) + B_r(T_m, T_k) r^*\right)$$

where a constant r^* is pre-calibrated by solving of the following equation:

$$1 - \sum_{k=m+1}^{n} c_k \exp\Big(A_r(T_m, T_k) + B_r(T_m, T_k) r^*\Big) = 0.$$

The Motivation Behind Negative Rates

- The financial crisis, which started 2007, exposed a lack of trust among counterparties.
- Since then incorporation of the probability of default in pricing models became standard.
- The lack of trust in the financial system reduced trading activities and kept the money in the pockets.
- To recover trust central banks decided to intervene and stimulate the monetary supply and demand.
- Lowering the interest rates was expected to encourage investors to borrow money at a low rate and invest into the economy, which would push the economy to grow.
- Since 2008 interest rates have gradually been lowered.

The Motivation Behind Negative Rates

- June 5th 2014, the rates set by ECB were negative for the first time: minus 10 basis points.
- Using negative rates is unconventional to "inspire" investors but it is not unprecedented (Switzerland, Sweden and Denmark also report negative rates).

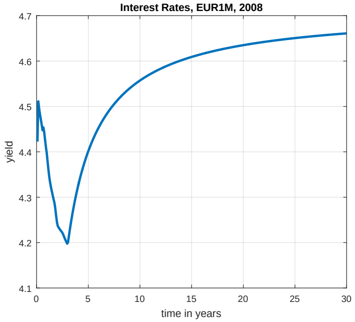


Figure: Both figures present the yield obtained from EUR 1M curve, left: 2008 and right: 2017.

Modeling of Volatility

- Each non-linear product relies on volatilities that are extracted from market quotes.
- Because of limited liquidity the volatilities are parameterized.
- Unfortunately, the parameterizations are often not arbitrage-free, especially not in the low/negative interest rate environment (this will be handled in a follow-up course).

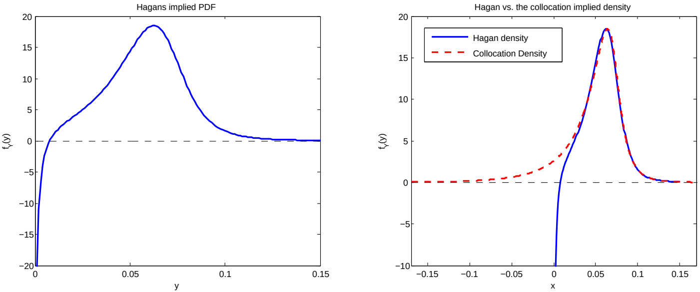


Figure: Probability density, with deterioration near zero; right: application of the collocation method

Modeling Challenges and Client Relations

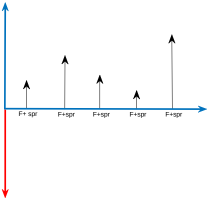
- Often, floating rate bonds are issued with coupons paid with a spread above the index.
- Shall the bond issuer request the payments from bond holders in the case the coupon payment is negative:

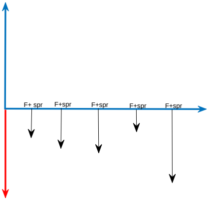
$$cpn_k = L(T_{k-1}, T_{k-1}, T_k) + spread < 0?$$

- To maintain a good client relation often coupons are calculated by

$$cpn_k = \max(L(T_{k-1}, T_{k-1}, T_k) + spread, 0).$$

- This however requires incorporation of **volatilities**





Pricing of Caplets under negative rates

- An obvious example for which the pricing under negative rates needs to be modified is the *valuation of caplets*.
- Based on the assumption of lognormal Libor rates, $\ell_k(t) := \ell(t; T_{k-1}, T_k)$, with the corresponding dynamics,

$$\mathrm{d}\ell_k(t) = \sigma_k \ell_k(t) \mathrm{d}W_k^k(t),$$

the pricing equation of a caplet is given by:

$$V_k^{\text{CPL}}(t_0) \;\; = \;\; N_k \tau_k P(t_0, T_k) \mathbb{E}^{T_k} \left[\max\left(\ell_k(T_{k-1}) - K, 0 \right) \Big| \mathcal{F}(t_0) \right].$$

- The solution is given by

$$V_k^{\text{CPL}}(t_0) \;\; = \;\; N_k \tau_k P(t_0, T_k) \left[\ell_k(t_0) N(d_1) - K_k N(d_2) \right],$$

with,

$$d_1 \;\; = \;\; \frac{\log\left(\frac{\ell_k(t_0)}{K}\right) + \frac{1}{2}\sigma_k^2 (T_k - t_0)}{\sigma_k \sqrt{T_k - t_0}}, \; d_2 = d_1 - \sigma_k \sqrt{T_k - t_0},$$

with $\ell_k(t_0) = \ell(t_0; T_{k-1}, T_k)$.

Pricing of Caplets under negative rates

- In a negative interest rate environment, the ZCBs $P(t_0, T_k)$ may get values that are higher than one unit of currency. This would imply that the Libor rate $\ell(t_0; T_{k-1}, T_k)$ would be negative. When the Libor rate is negative, the above pricing equation is not valid anymore, as the logarithm of a negative value is not well-defined.
- Instead of GBM dynamics, the arithmetic Brownian motion (ABM) dynamics, that also give rise to *negative realizations* could, for example, be chosen. Such a solution, although straight-forward, has a significant disadvantage, however, as the normal distribution has much flatter distribution tails as compared to a lognormal process.
- Instead of completely changing the underlying dynamics, the industrial standard for dealing with the negativity has become to "shift" the original process.

Pricing of Caplets under negative rates

- Caplets can be priced under a negative interest rate environment by an *adaptation* of the underlying dynamics of the Libor rate.
- This shifted process is defined, as follows,

$$\boxed{\hat{\ell}_k(t) = \ell_k(t) + \theta_k,}$$

where the process $\hat{\ell}_k(t)$ is governed by a lognormal process, with the following dynamics,

$$\boxed{\mathrm{d}\hat{\ell}(t; T_{k-1}, T_k) = \hat{\sigma}_k \hat{\ell}(t; T_{k-1}, T_k)\mathrm{d}W_k^k(t).}$$

- The concept of shifting a distribution is very similar to the idea of "displacement".

Pricing of Caplets under negative rates

- The Libor rate $\ell_k(t)$, as observed in the market, is simply given by $\ell_k(t) := \hat{\ell}_k(t) - \theta_k$.
  Shifting of the processes is equivalent to "moving" the probability density along the $x-$axis.
- The pricing of *caplets* is now given by,

$$
\begin{aligned}
V_k^{\text{CPL}}(t_0) &= N_k \tau_k P(t_0, T_k) \mathbb{E}^{T_k}\left[\max\left(\ell_k(T_{k-1}) - K, 0\right) \Big| \mathcal{F}(t_0)\right] \\
&= N_k \tau_k P(t_0, T_k) \mathbb{E}^{T_k}\left[\max\left(\hat{\ell}_k(T_{k-1}) - \theta_k - K, 0\right) \Big| \mathcal{F}(t_0)\right] \\
&= N_k \tau_k P(t_0, T_k) \mathbb{E}^{T_k}\left[\max\left(\hat{\ell}_k(T_{k-1}) - \hat{K}, 0\right) \Big| \mathcal{F}(t_0)\right],
\end{aligned}
$$

with $\hat{K} = K + \theta_k$.

Pricing of Caplets under negative rates

- Option pricing under shifted distributions is very convenient. The corresponding *solution* is in accordance with the unshifted variant,

$$V_k^{\text{CPL}}(t_0) \;\; = \;\; N_k\tau_k P(t_0,T_k)\left[\hat{\ell}_k(t_0)N(d_1) - \hat{K}_k N(d_2)\right],$$

with,

$$d_1 \;\; = \;\; \frac{\log\left(\frac{\hat{\ell}_k(t_0)}{\hat{K}}\right) + \frac{1}{2}\sigma_k^2(T_k - t_0)}{\sigma_k\sqrt{T_k - t_0}},\; d_2 = d_1 - \sigma_k\sqrt{T_k - t_0},$$

with $\hat{K} = K + \theta_k$ and $\hat{\ell}_k(t_0) = \ell_k(t_0) + \theta_k$.

Negative interest rates

- For decades the pricing models which generated negative rates were considered unrealistic and simplistic.
- The Hull-White model, which "suffered" from negative rates, has been widely used in the industry.

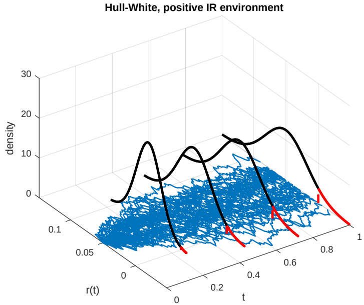
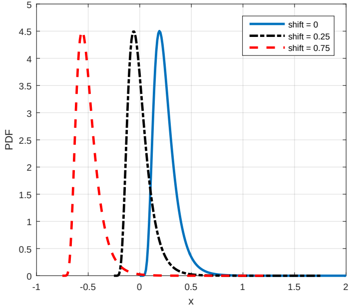


Figure: Both figures present the Monte Carlo Paths for the Hull-White model in a positive and negative interest rate environment.

Negative interest rates



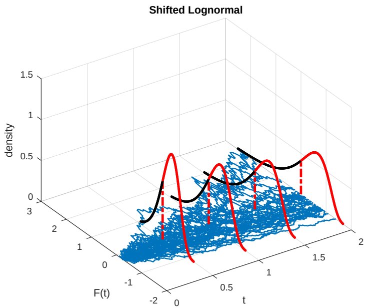


Figure: Shifted lognormal distribution used for pricing in negative interest rate environment.

Pricing under shifted distributions

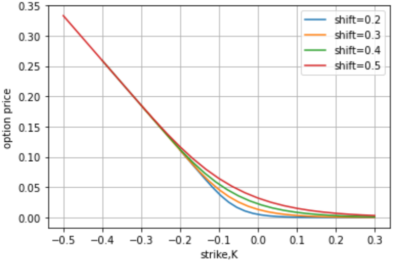


Figure: Shifted lognormal distribution used for pricing in negative interest rate environment.

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Summary

- Pricing of Caplets/Floorlets
- Pricing of Interest Rate Swaps
- Pricing of Swaptions under the Black-Scholes Model
- Jamshidian's Trick
- Swaptions under the Hull-White Model
- Negative Interest Rates
- Shifted Lognormal, Shifted Implied Volatility
- Summary of the Lecture + Homework

Homework Exercises

- **Exercise**
- Extend the code presented in slide 41 (pricing of caplets under shifted lognormal) and compute the corresponding implied volatilities.
- In the lecture we have presented how to use Jamshidian's trick. Apply the same strategy for pricing of Swaptions (the code for pricing of options on ZCBs is included in today's materials). Compare your results to Monte-Carlo.