Introduction to Algorithms
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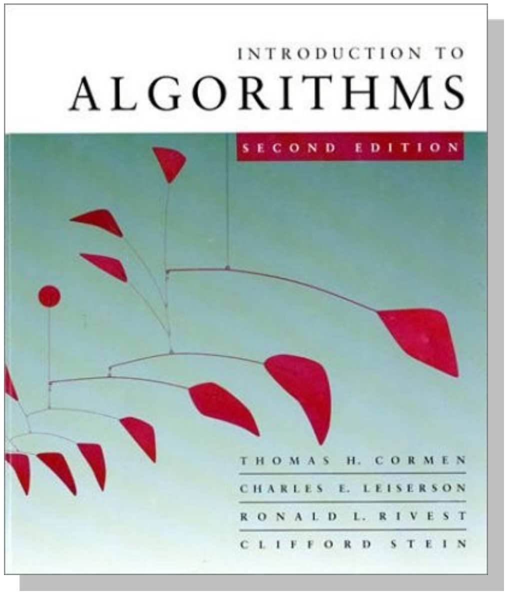

LECTURE 1
Analysis of Algorithms
- Bubble sort
- Asymptotic analysis
- Merge sort
- Recurrences

Prof. Charles E. Leiserson

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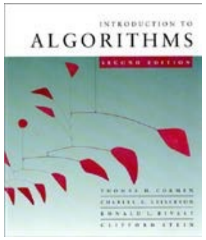

Analysis of algorithms

The theoretical study of computer-program performance and resource usage.

What's more important than performance?

- modularity
- correctness
- maintainability
- functionality
- robustness
- user-friendliness
- programmer time
- simplicity
- extensibility
- reliability

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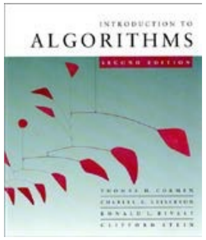

Why study algorithms and performance?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!

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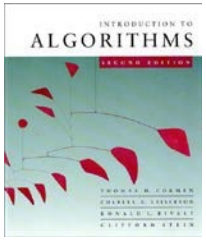

The problem of sorting

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

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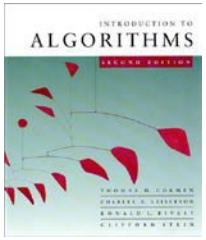
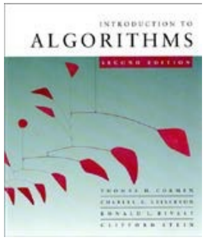


Bubble Sort

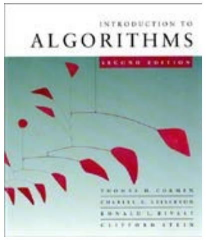


"Bubbling Up" the Largest Element

Traverse a collection of elements

- **Move from the front to the end**
- **"Bubble" the largest value to the end using pair-wise comparisons and swapping**

77	42	35	12	101	5



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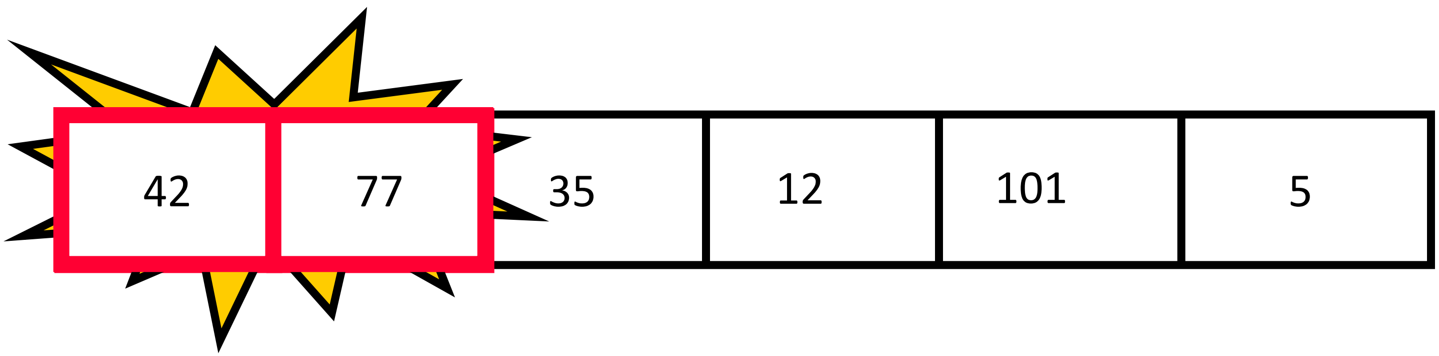

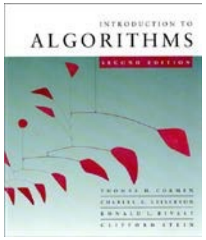

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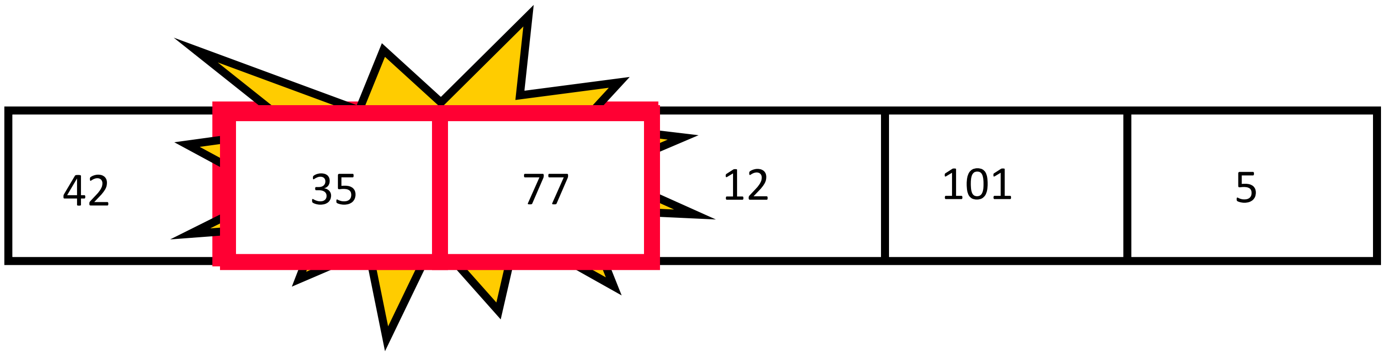

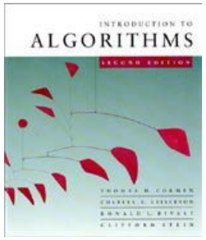

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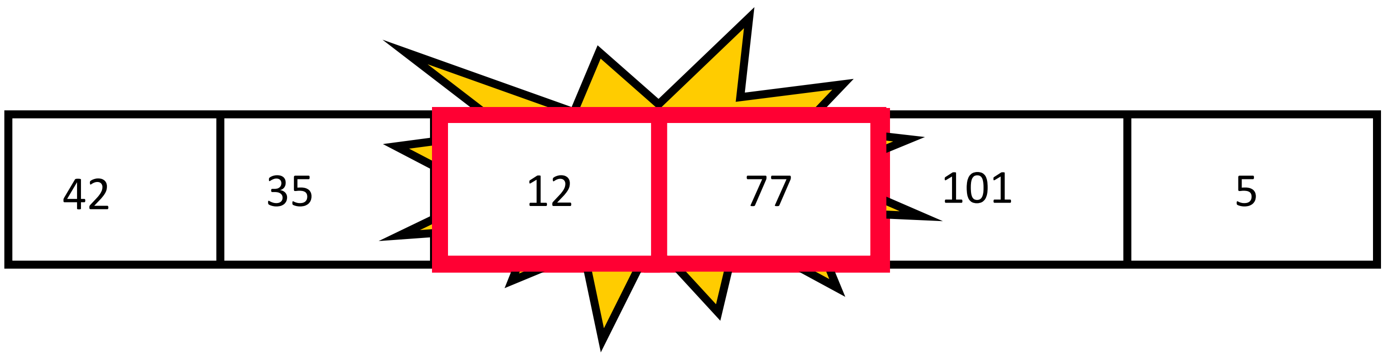

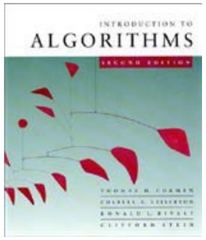

"Bubbling Up" the Largest Element

Traverse a collection of elements

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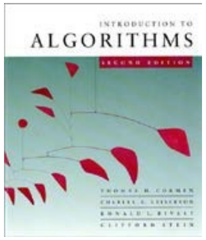
42	35	12	77	101	5

No need to swap



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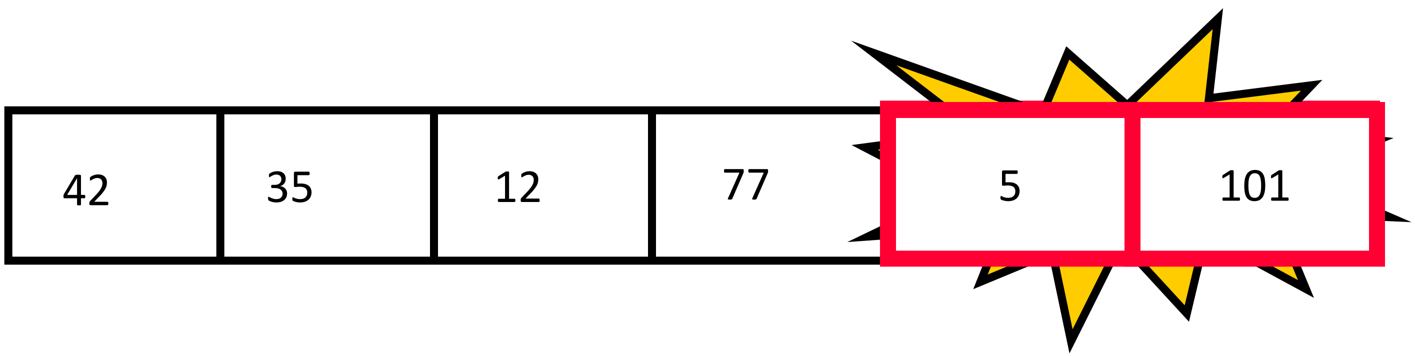
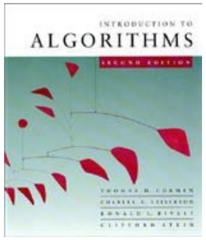




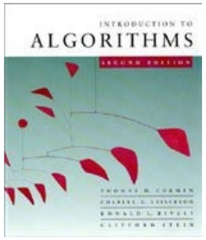
"Bubbling Up" the Largest Element

Traverse a collection of elements

- Move from the front to the end
- "Bubble" the largest value to the end using pair-wise comparisons and swapping

42	35	12	77	5	101

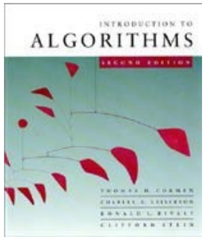
Largest value correctly placed



The “Bubble Up” Algorithm

```
index <- 1
last_compare_at <- n – 1

loop
  exitif(index > last_compare_at)
  if(A[index] > A[index + 1]) then
    Swap(A[index], A[index + 1])
  endif
  index <- index + 1
endloop
```

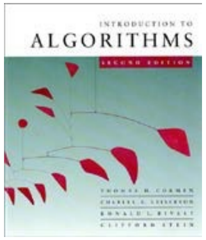


Items of Interest

- **Notice that only the largest value is correctly placed**
- **All other values are still out of order**
- **So we need to repeat this process**

42	35	12	77	5	101

Largest value correctly placed



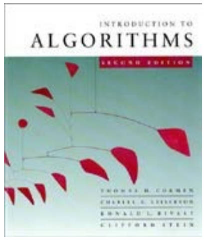
Repeat "Bubble Up" How Many Times?

- **If we have N elements…**

- **And if each time we bubble an element, we place it in its correct location…**

- **Then we repeat the "bubble up" process N – 1 times.**

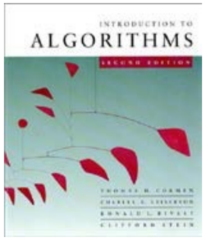
- **This guarantees we'll correctly place all N elements.**



“Bubbling” All the Elements

N - 1

42	35	12	77	5	101
35	12	42	5	77	101
12	35	5	42	77	101
12	5	35	42	77	101
5	12	35	42	77	101



Reducing the Number of Comparisons

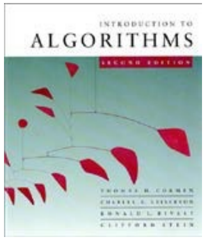
77	42	35	12	101	5

42	35	12	77	5	101

35	12	42	5	77	101

12	35	5	42	77	101

12	5	35	42	77	101

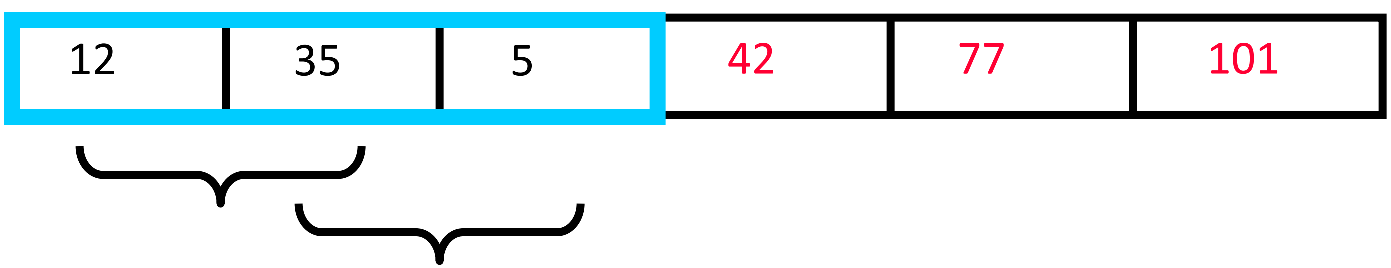


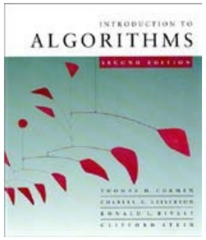
Reducing the Number of Comparisons

On the Nth "bubble up", we only need to do MAX-N comparisons.

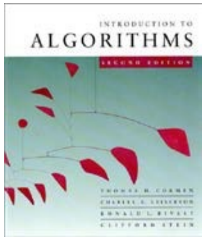
For example:

- This is the 4th "bubble up"
- MAX is 6
- Thus we have 2 comparisons to do





Putting It All Together

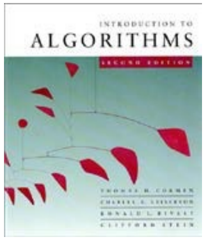



```
procedure Bubblesort(A)
  to_do, index isoftype Num
  to_do <- N - 1

  loop
    exitif(to_do = 0)
    index <- 1
    loop
      exitif(index > to_do)
      if(A[index] > A[index + 1]) then
        Swap(A[index], A[index + 1])
      endif
      index <- index + 1
    endloop
    to_do <- to_do - 1
  endloop
endprocedure // Bubblesort
```

Inner loop

Outer loop

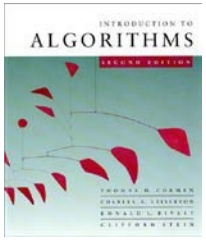


Already Sorted Collections?

- **What if the collection was already sorted?**
- **What if only a few elements were out of place and after a couple of "bubble ups," the collection was sorted?**

- **We want to be able to detect this and "stop early"!**

5	12	35	42	77	101

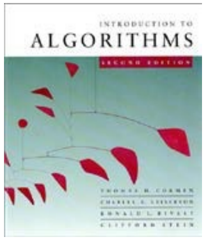


Using a Boolean "Flag"

- We can use a boolean variable to determine if any swapping occurred during the "bubble up."

- If no swapping occurred, then we know that the collection is already sorted!

- This boolean "flag" needs to be reset after each "bubble up."




```
did_swap: Boolean
did_swap <- true

loop
  exitif ((to_do = 0) OR NOT(did_swap))
  index <- 1
  did_swap <- false
  loop
    exitif(index > to_do)
    if(A[index] > A[index + 1]) then
      Swap(A[index], A[index + 1])
      did_swap <- true
    endif
    index <- index + 1
  endloop
  to_do <- to_do - 1
endloop
```

Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.

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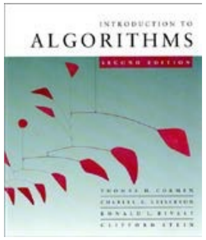

Kinds of analyses

Worst-case: (usually)

- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (bogus)

- Cheat with a slow algorithm that works fast on *some* input.

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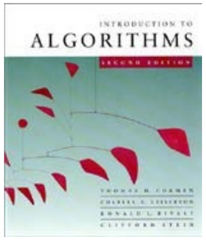

Machine-independent time

What is bubble sort's worst-case time?

- It depends on the speed of our computer:
  - relative speed (on the same machine),
  - absolute speed (on different machines).

BIG IDEA:

- Ignore machine-dependent constants.
- Look at ***growth*** of $T(n)$ as $n \to \infty$.

"Asymptotic Analysis"

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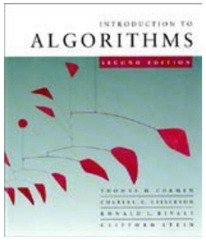

Θ-notation

Math:

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$

Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

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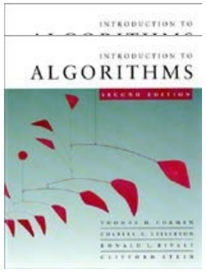


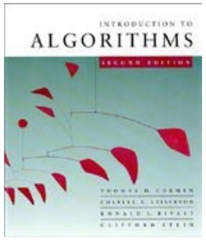
Other notations

$\mathrm{O}(g(n)) = \{ f(n)$: there exist positive constants c_2, and n_0 such that $0 \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$

$\Omega(g(n)) = \{ f(n)$: there exist positive constants c_1, and n_0 such that $0 \le c_2 g(n) \le f(n)$ for all $n \ge n_0 \}$

$\mathrm{o}(g(n)) = \{ f(n)$: for any $\varepsilon \ge 0$, there exist n_0 such that $0 \le g(n) \le \varepsilon f(n)$ for all $n \ge n_0 \}$

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$$



Examples

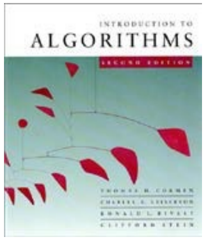
$$\sum_{i=1}^{n} i =$$

$$\sum_{i=1}^{n} i^2 =$$

$$\sum_{i=1}^{n} a^i \cdot c =$$

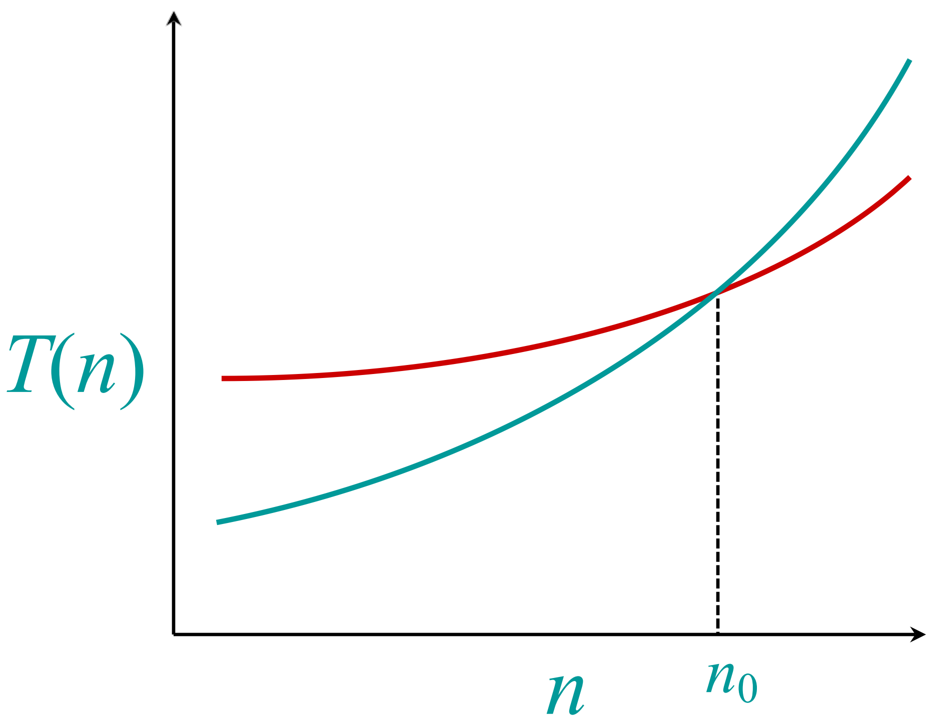
$$\sum_{i=1}^{n} \frac{1}{i} =$$

$$\sum_{i=1}^{n} \frac{1}{i^2} =$$



Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

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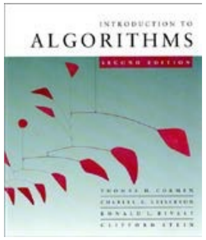

Insertion sort analysis

Worst case: Input reverse sorted.

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2) \qquad \text{[arithmetic series]}$$

Properties of Bubble Sort

- Bubble sort is a **stable** sorting algorithm.
- Bubble sort is an **in-place** sorting algorithm.
- Number of swaps in bubble sort = Number of inversion pairs present in the given array.
- Bubble sort is beneficial when array elements are less and the array is nearly sorted.

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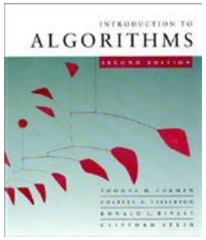

Merge sort

MERGE-SORT $A[1 \,.\,.\, n]$

1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key <u>subroutine</u>: MERGE

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Merging two sorted arrays

20	12
13	11
7	9
2	1

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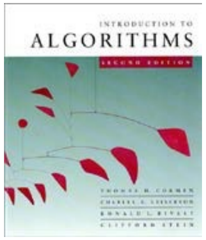
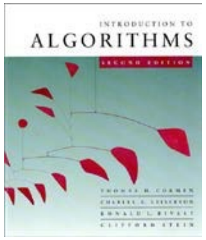


Merging two sorted arrays

20	12
13	11
7	9
2	1

1



Merging two sorted arrays

20	12		20	12
13	11		13	11
7	9		7	9
2	1		2	

1

Merging two sorted arrays

20	12
13	11
7	9
2	1

Output: 1

20	12
13	11
7	9
2	

Output: 2

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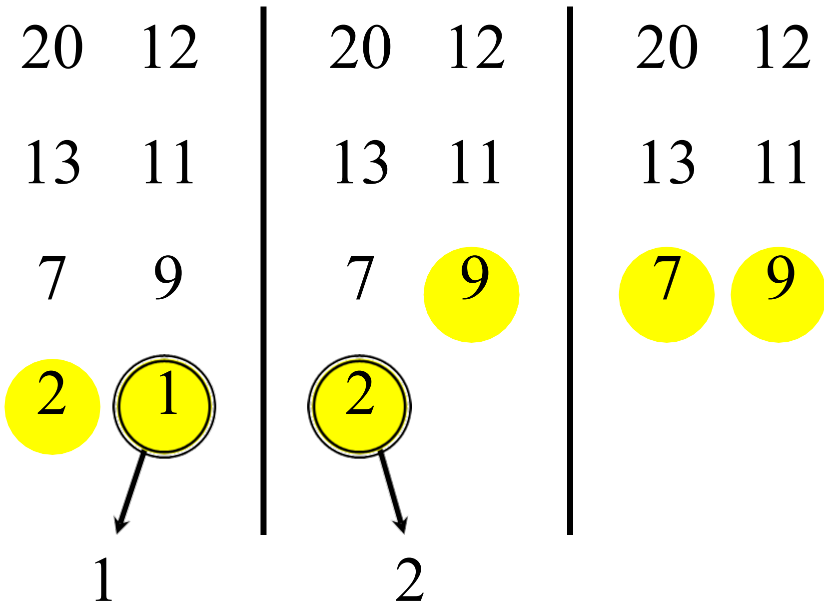
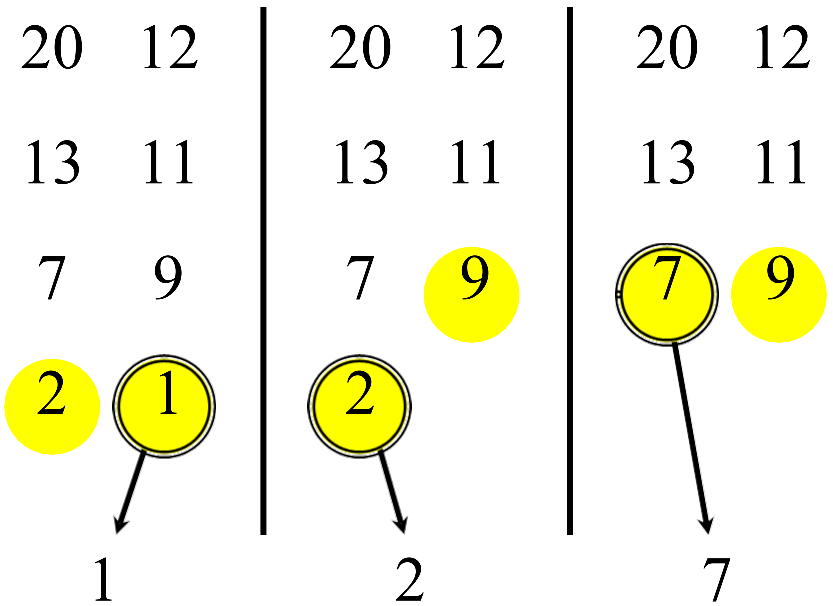
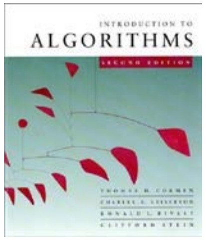
Merging two sorted arrays

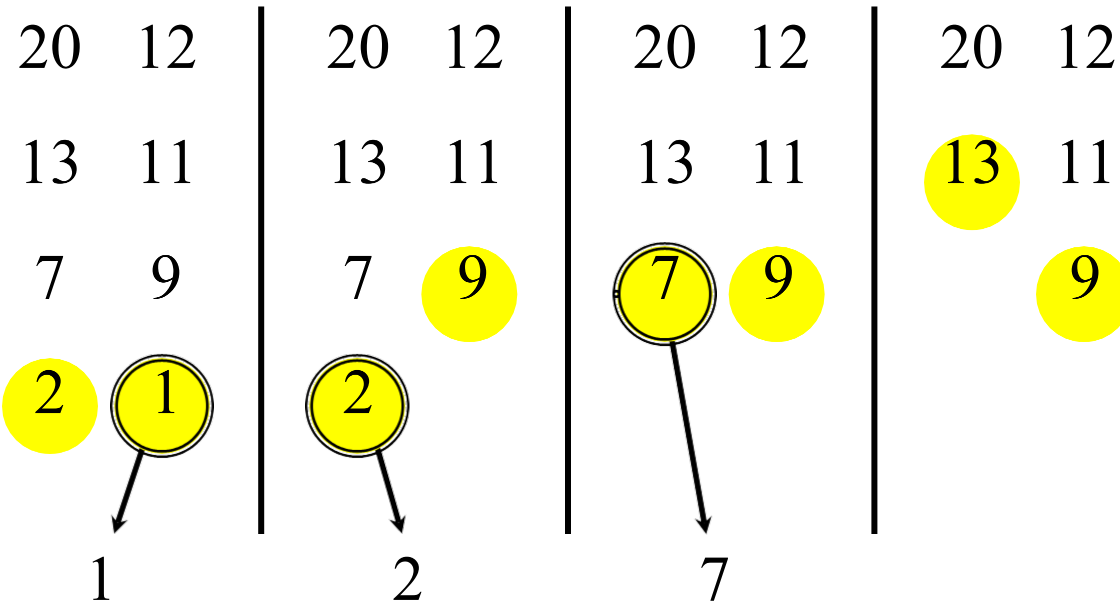


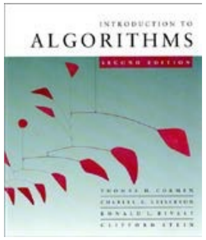
Merging two sorted arrays



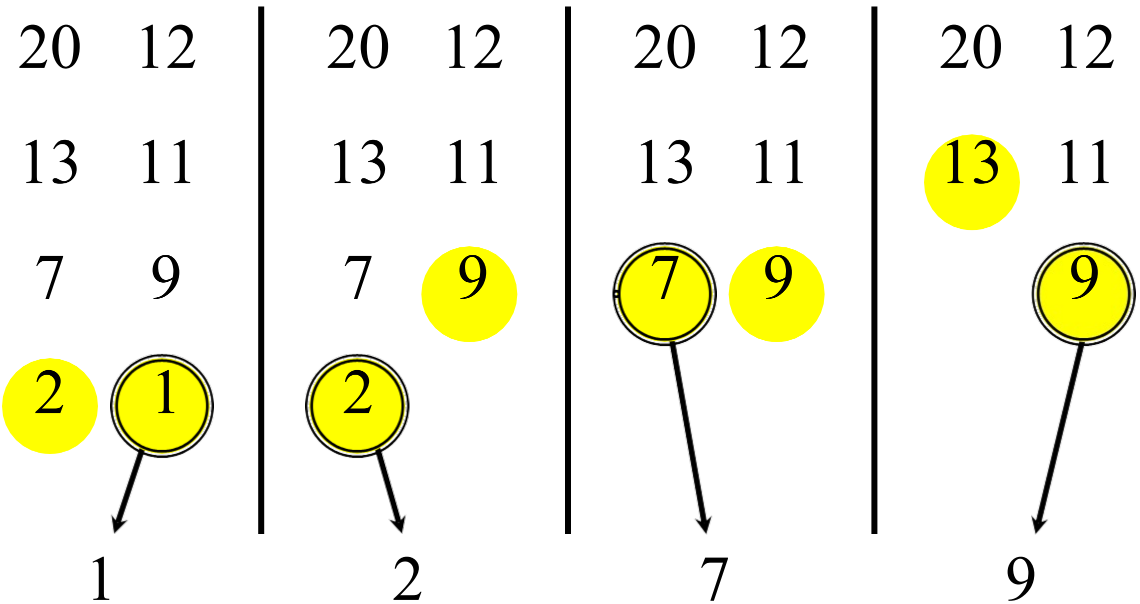


Merging two sorted arrays





Merging two sorted arrays



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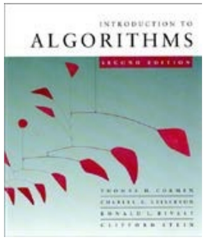
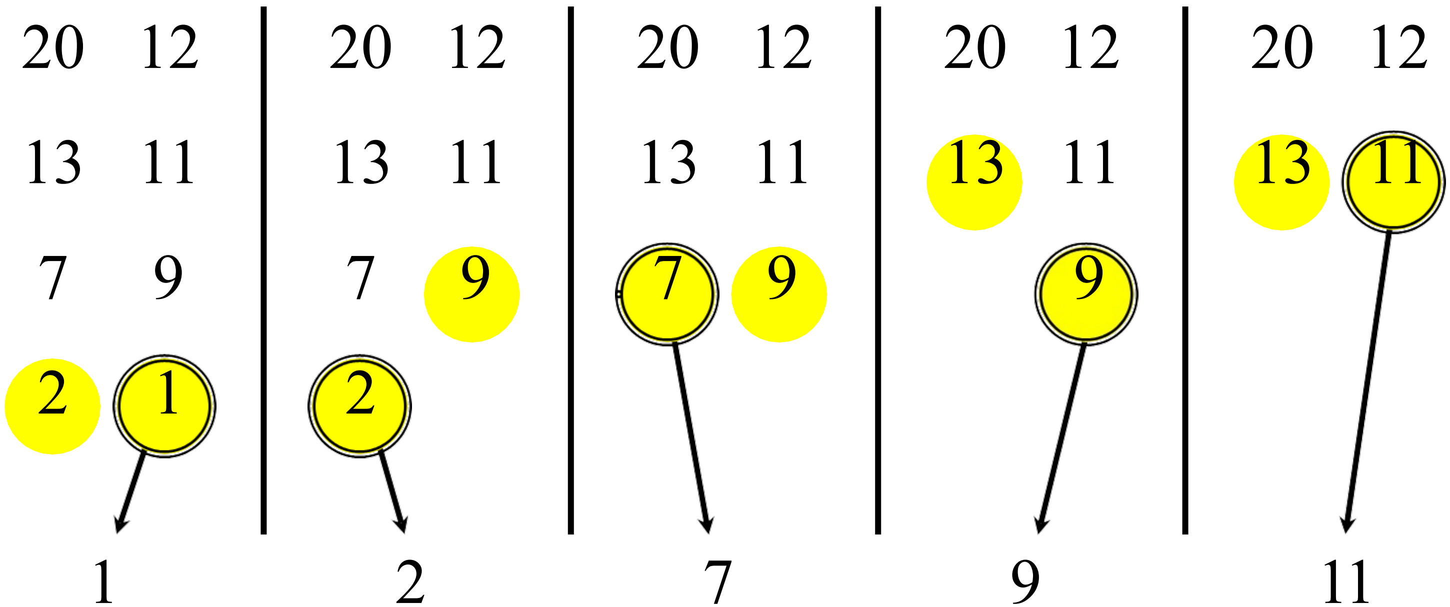


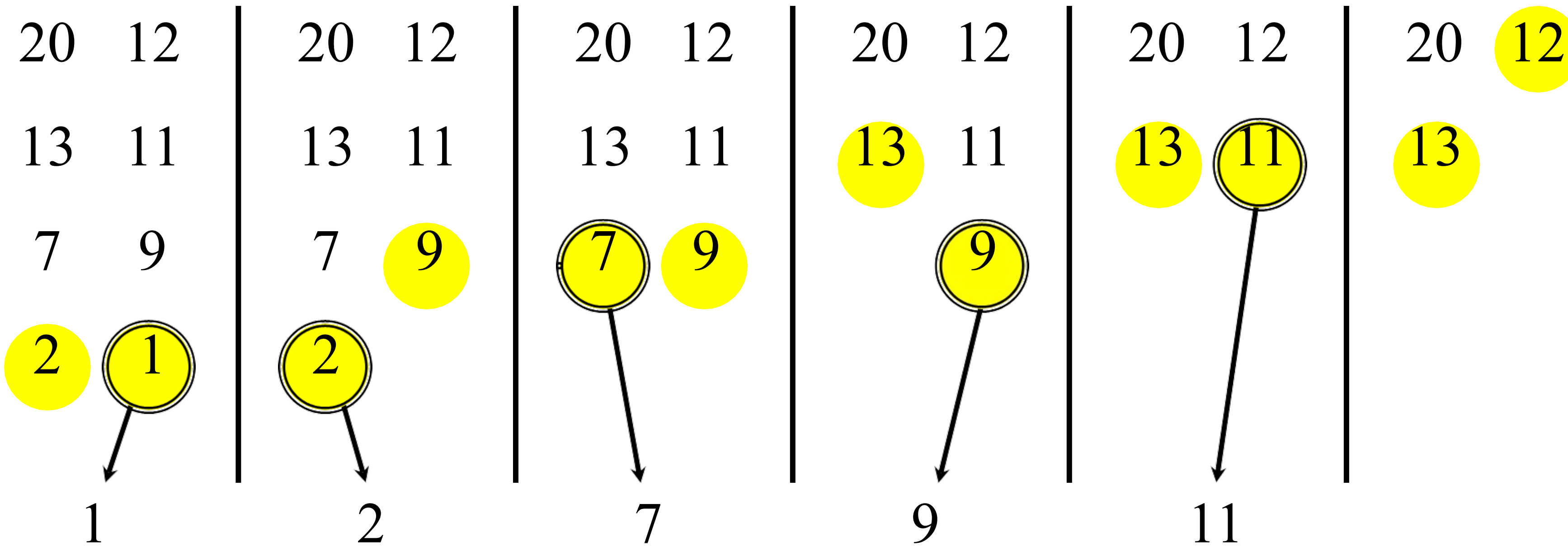
Merging two sorted arrays

20	12	20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11	13	11
7	9	7	9	7	9		9		
2	1	2							
1		2		7		9			

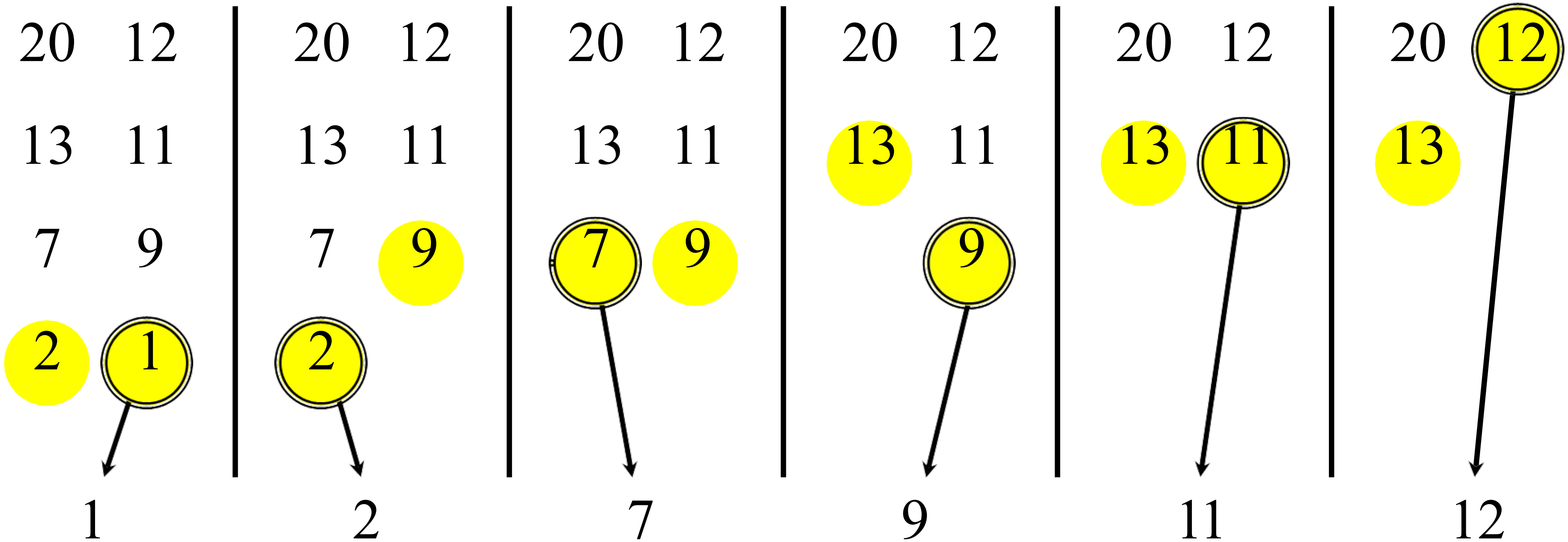
Merging two sorted arrays



Merging two sorted arrays



Merging two sorted arrays



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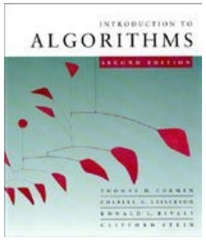
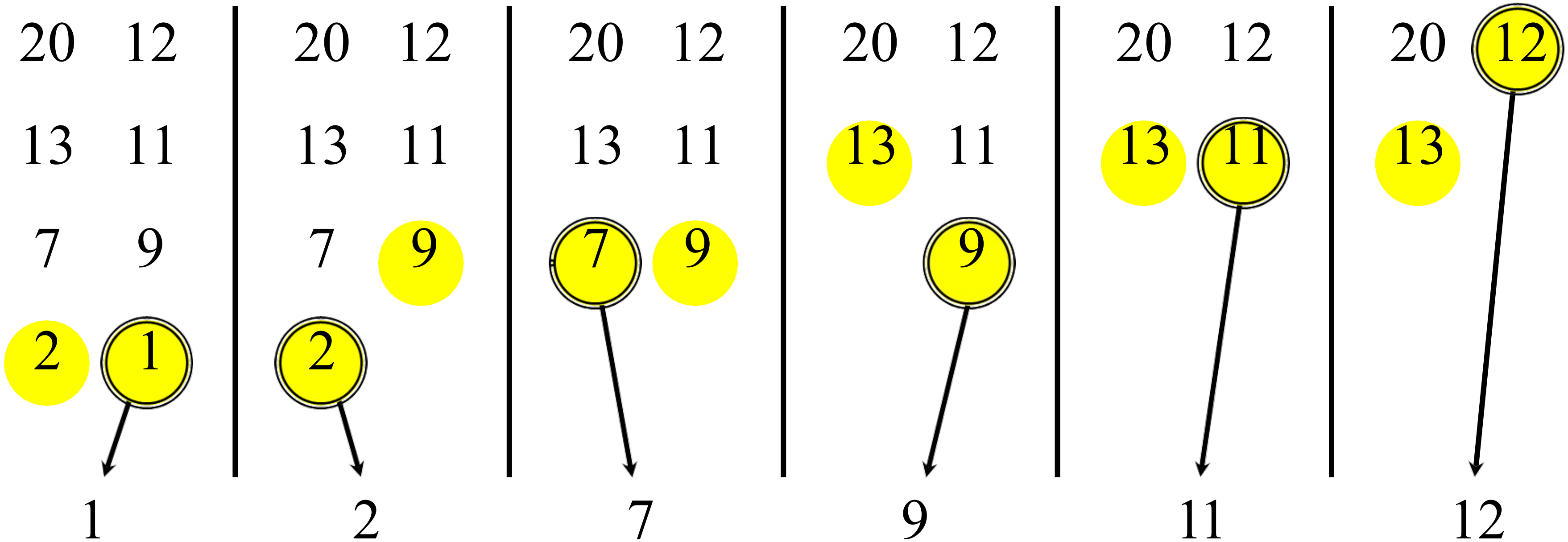


Merging two sorted arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).

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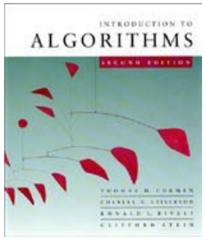

Analyzing merge sort

$T(n)$	MERGE-SORT $A[1\,.\,.\,n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1\,.\,.\lceil n/2\rceil]$ and $A[\,\lceil n/2\rceil{+}1\,.\,.\,n\,]$.
$\Theta(n)$	3. ***"Merge"*** the 2 sorted lists

Abuse

Sloppiness: Should be $T(\lceil n/2\rceil) + T(\lfloor n/2\rfloor)$, but it turns out not to matter asymptotically.

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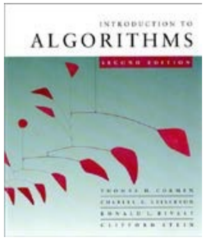

Recurrence for merge sort

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS and Lecture 2 provide several ways to find a good upper bound on $T(n)$.

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Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

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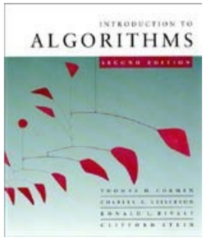

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$$T(n)$$

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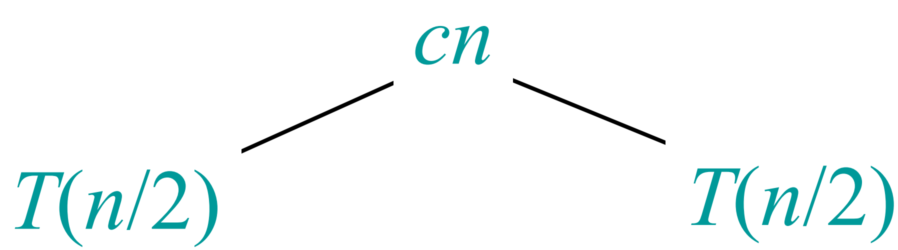

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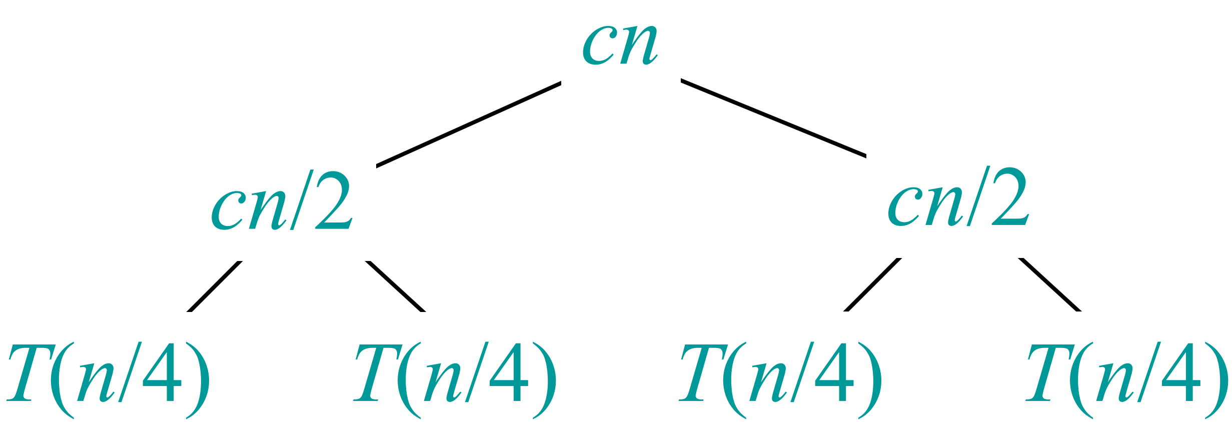

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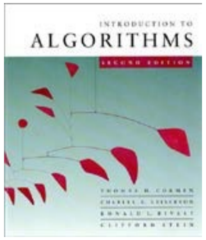

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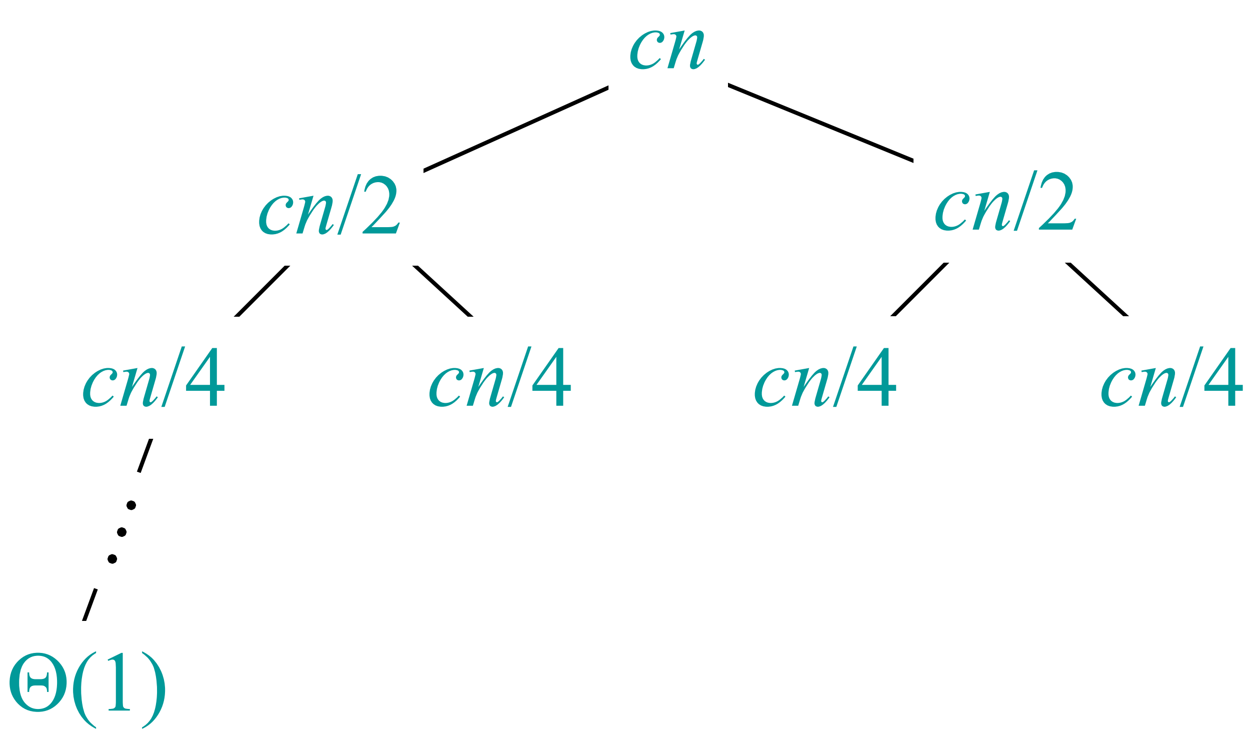

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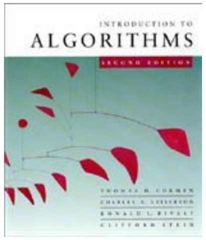

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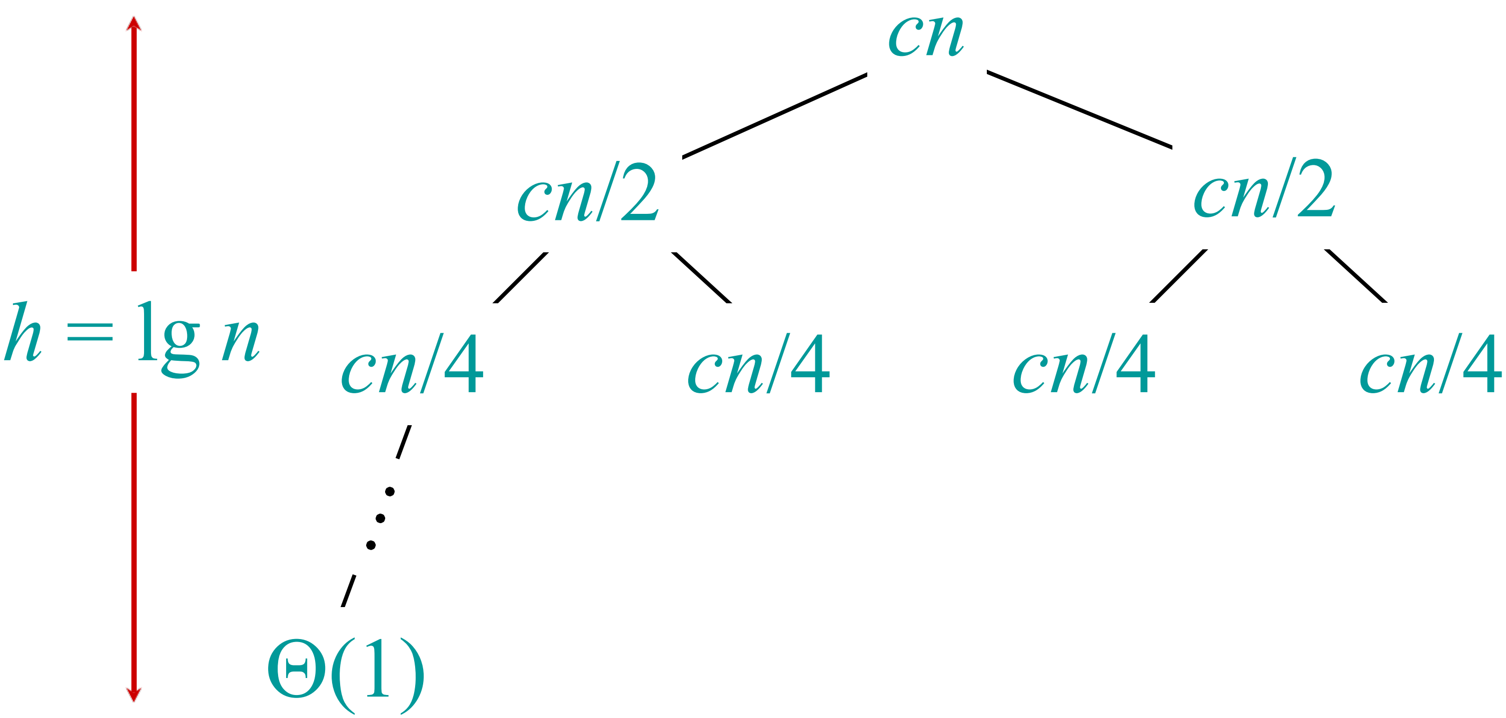

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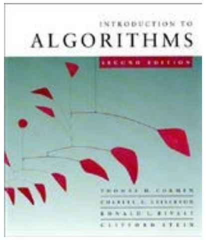

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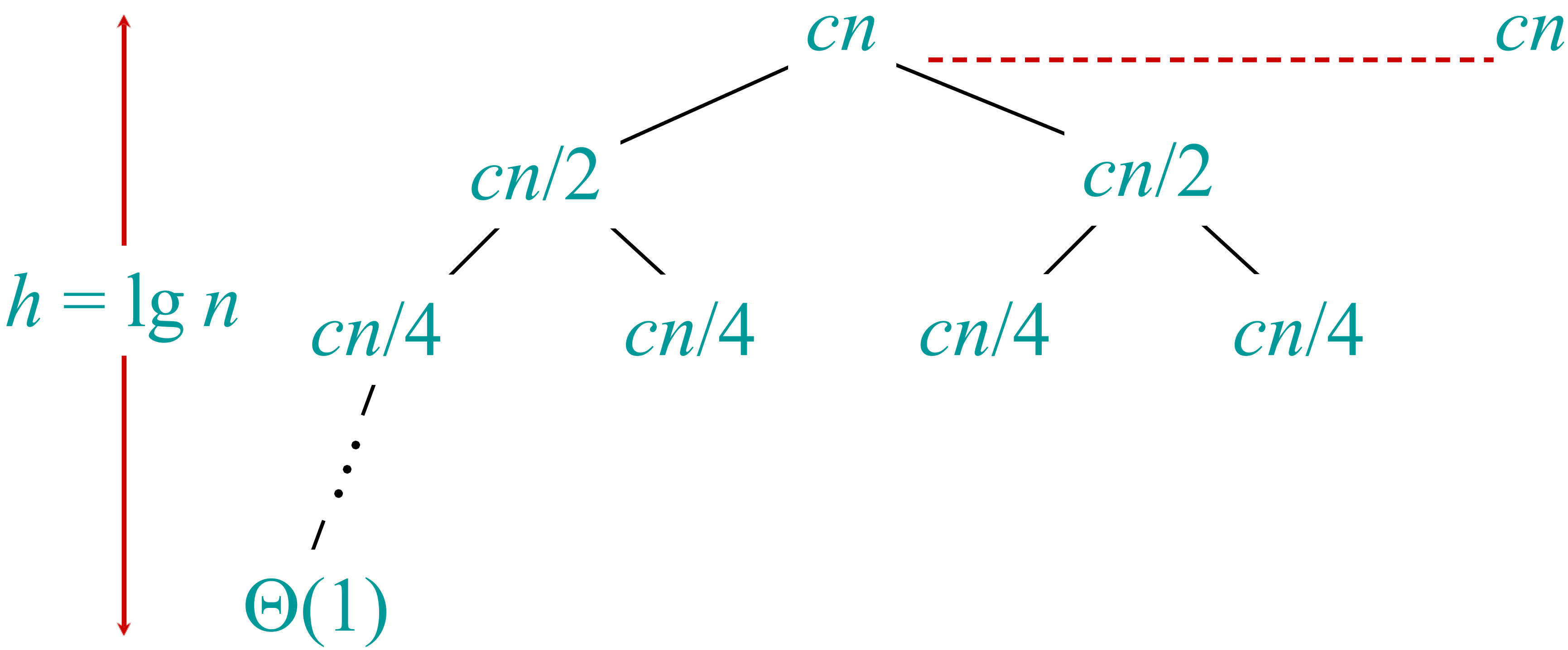

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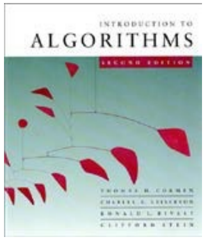

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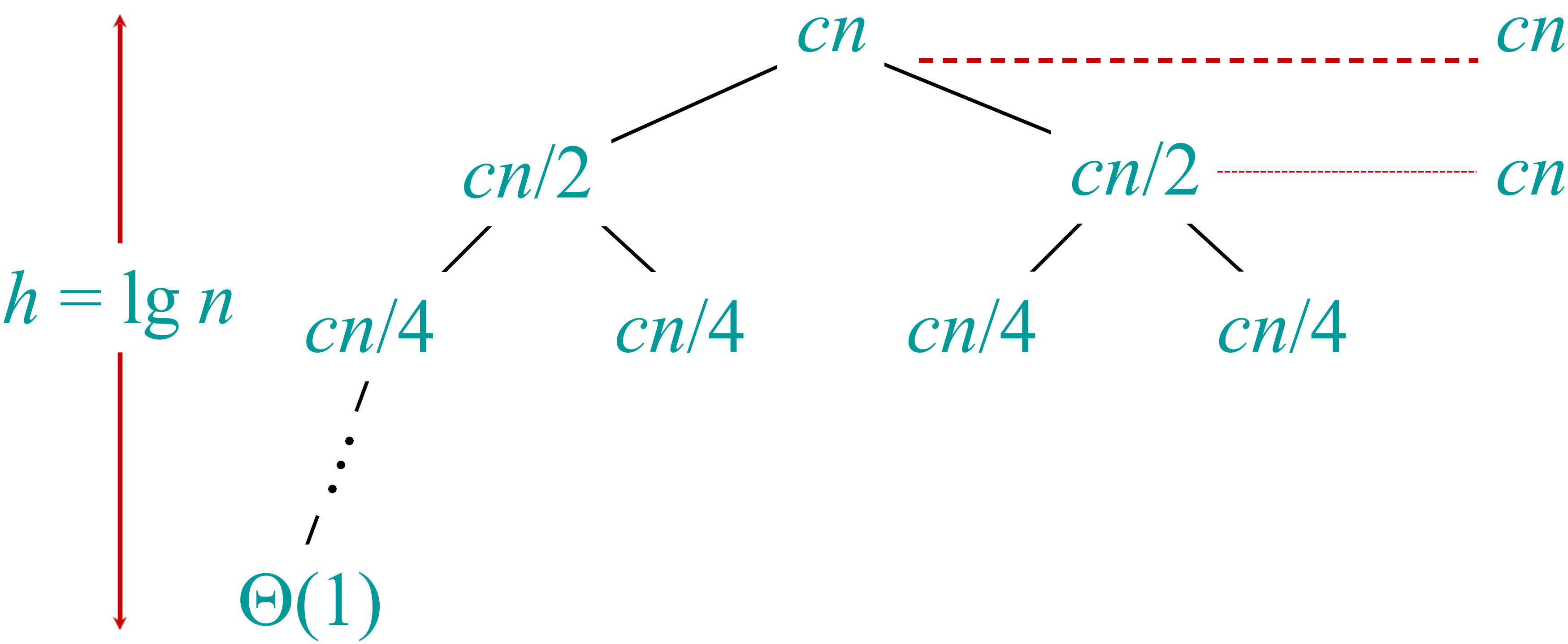

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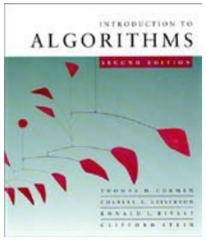

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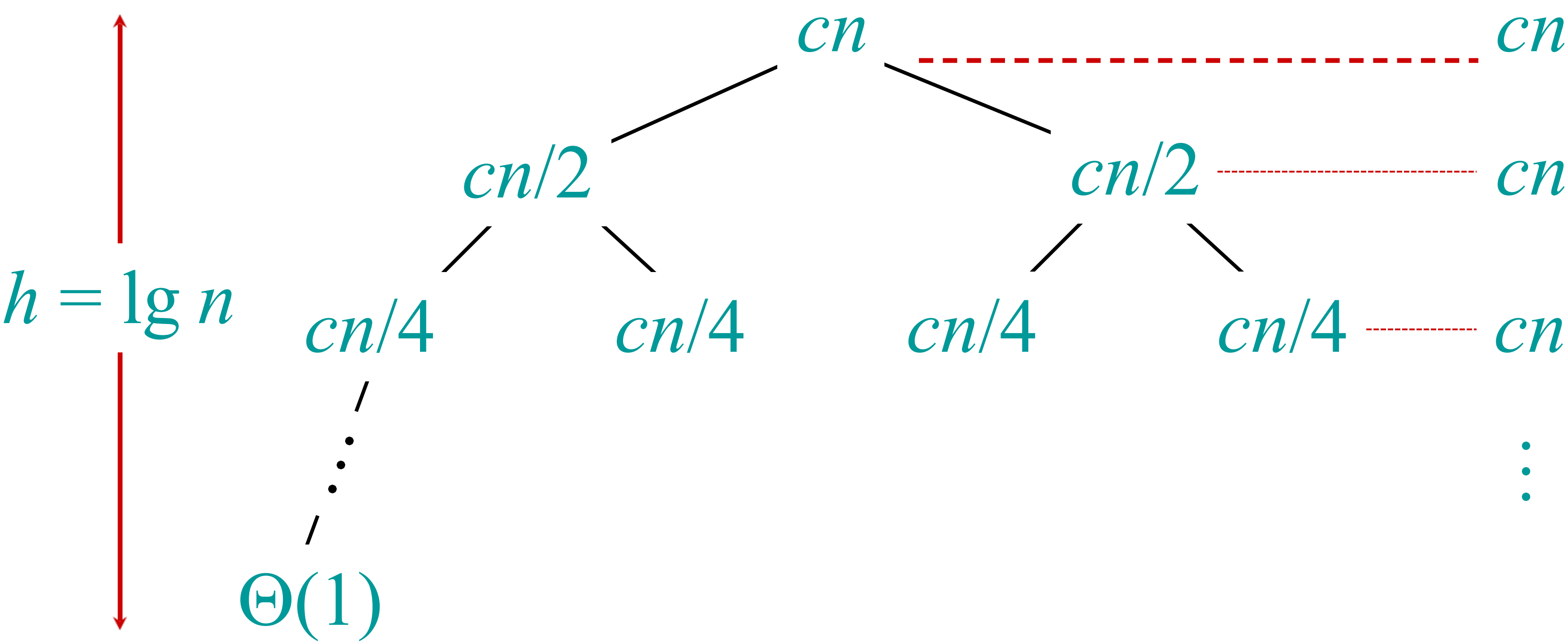

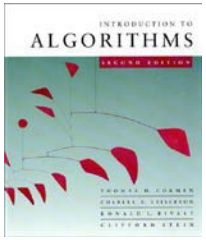

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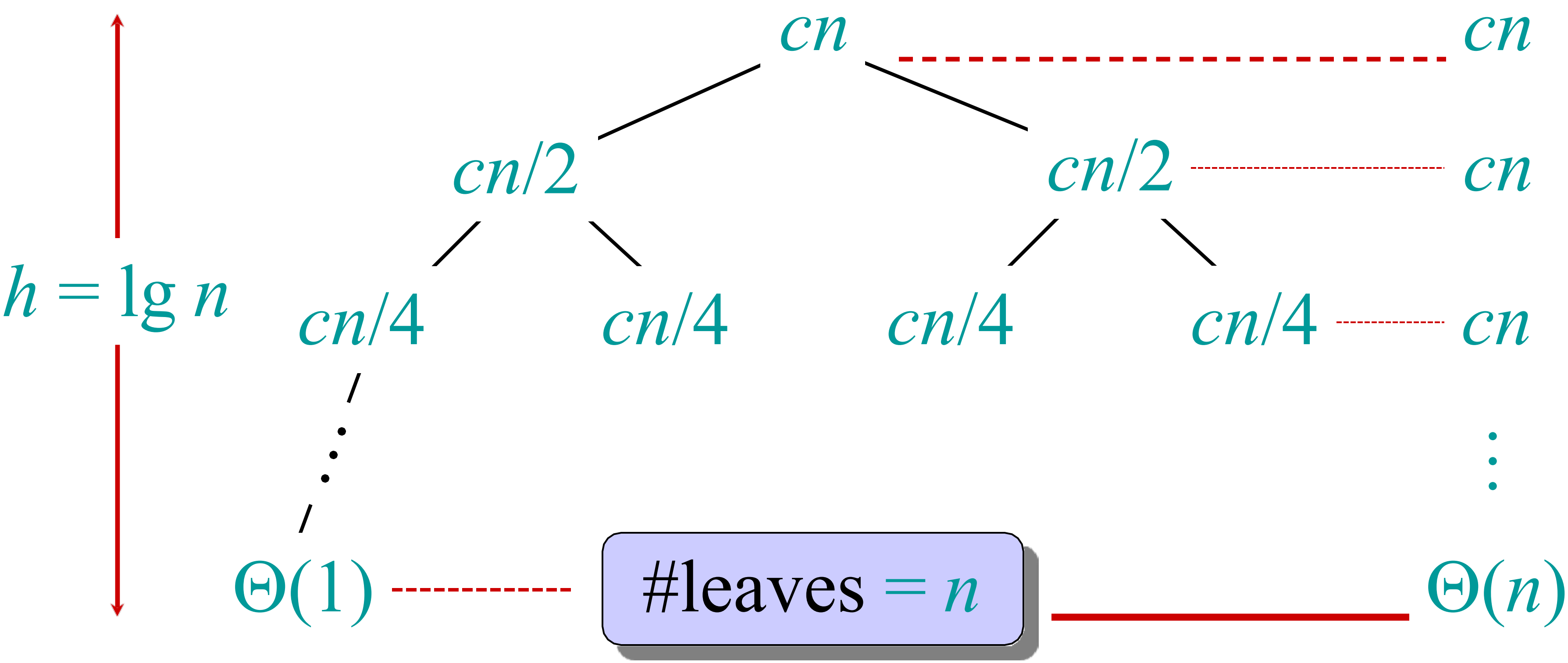

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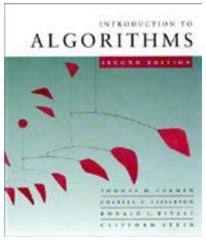

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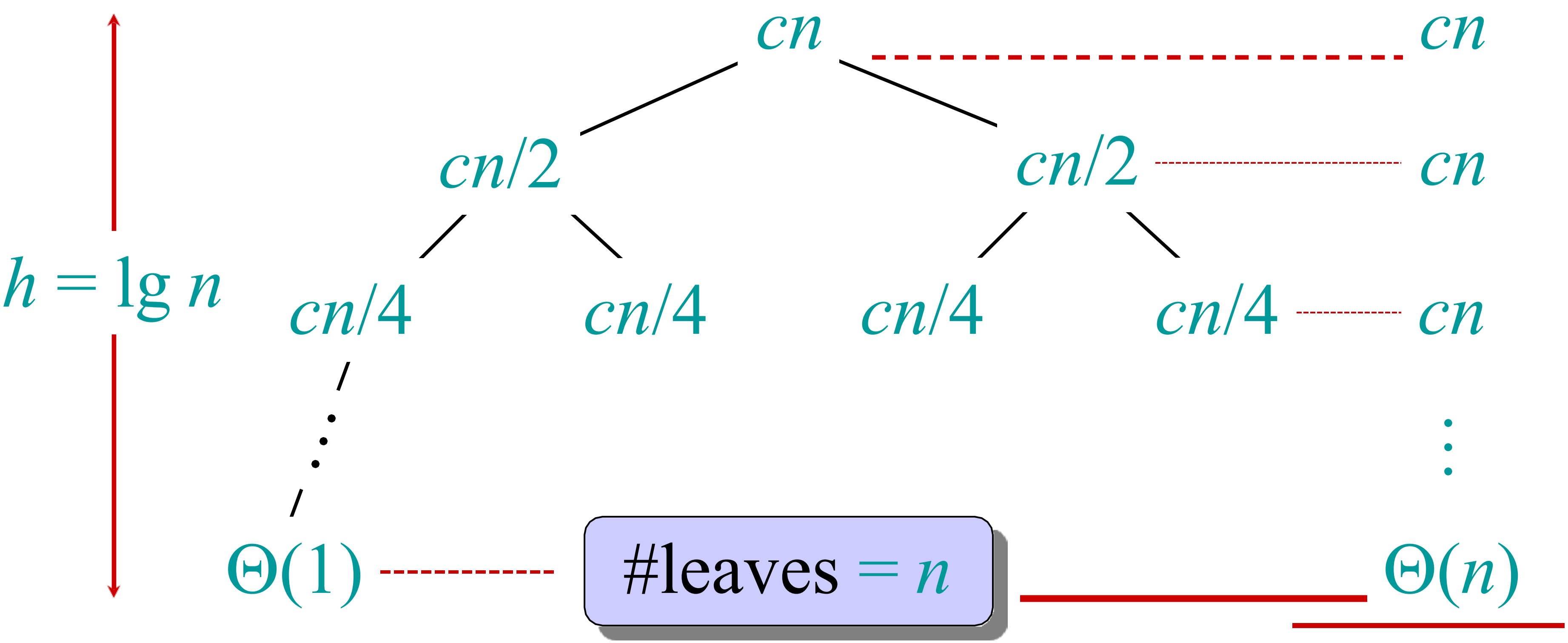

Total $= \Theta(n \lg n)$

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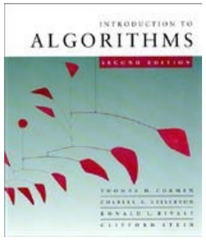

Conclusions

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.
- Go test it out for yourself!

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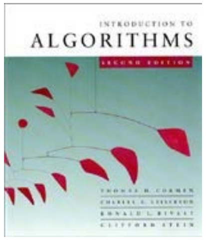

References

- Big-O Cheat Sheet: https://www.bigocheatsheet.com