L2: Asymptotic Analysis

Data Structures and Algorithms

- **Algorithm:** Outline, the essence of a computational procedure, step-by-step instructions
- **Program:** an implementation of an algorithm in some programming language
- **Data structure:** Organization of data needed to solve the problem

Algorithmic problem

Specification of input → ? → Specification of output as a function of input

- Infinite number of input *instances* satisfying the specification.
  - For eg: A sorted, non-decreasing sequence of natural numbers of non-zero, finite length:
    - 1, 20, 908, 909, 100000, 1000000000.
    - 3.

Other boundary cases?

Algorithmic Solution

Input instance, adhering to the specification

Algorithm

Output related to the input as required

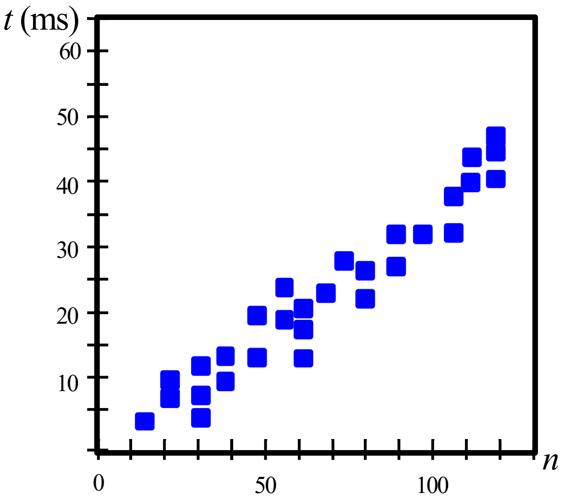
- Algorithm describes actions on the input instance
- many correct algorithms for the same algorithmic problem

What is a Good Algorithm?

- Efficient:
  - Running time
  - Space used
- Efficiency as a function of input size:
  - The number of bits in an input number
  - Number of data elements (numbers, points)

Measuring the Running Time

How should we measure the running time of an algorithm?



Experimental Study

- Write a program that implements the algorithm
- Run the program with data sets of varying size and composition.
- Use a system call to get an accurate measure of the actual running time.

Limitations of Experimental Studies

- It is necessary to implement and test the algorithm in order to determine its running time.
- Experiments done only on a limited set of inputs,
  - may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments needed

Beyond Experimental Studies

We will develop a general methodology for analyzing running time of algorithms. This approach

- Uses a high-level description of the algorithm instead of testing one of its implementations.
- Takes into account all possible inputs.
- Allows one to evaluate the efficiency of any algorithm in a way that is independent of the hardware and software environment.

Example

- **Algorithm** arrayMax(A, n):

  Input*:* An array A storing n integers.

  Output*:* The maximum element in A.

Pseudo-code (Functional / Recursive)

```
algorithm arrayMax(A[0..n-1])
{
   A[0]                                  if n=1
   max(arrayMax(A[0..n-2]), A[n-1])      otherwise
}
```

Pseudo-Code (imperative)

- A mixture of natural language and high-level programming concepts that describes the main ideas behind a generic implementation of a data structure or algorithm.
- Eg: **algorithm** arrayMax(A, n):

```
Input: An array A storing n integers.
Output: The maximum element in A.
currentMax ← A[0]
for i ← 1 to n-1 do
  if currentMax < A[i] then currentMax ← A[i]
return currentMax
```

Pseudo-Code

It is more structured than usual prose but less formal than a programming language

- Expressions:
  - use standard mathematical symbols to describe numeric and boolean expressions
  - use ← for assignment ("=" in Java)
  - use = for equality relationship ("==" in Java)

- Method Declarations:
  - **algorithm** name(param1, param2)

Pseudo Code

There are several styles commonly used.

- Programming Constructs:
  - decision structures: **if … then … [else …]**
  - while-loops: **while … do**
  - repeat-loops: **repeat … until …**
  - for-loop: **for … do**
  - array indexing: **A[i], A[i,j]**
- Methods:
  - calls: object method(args)
  - returns: **return** value

Analysis of Algorithms

- **Primitive Operation:** Low-level operation independent of programming language. Can be identified in pseudo-code. For eg:
  - Data movement (assign)
  - Control (branch, subroutine call, return)
  - arithmetic an logical operations (e.g. addition, comparison)
- By inspecting the pseudo-code, we can count the number of primitive operations executed by an algorithm.

Example: Sorting

INPUT

sequence of numbers

$a_1, a_2, a_3, \ldots., a_n$

2 5 4 10 7

OUTPUT

a permutation of the sequence of numbers

$b_1, b_2, b_3, \ldots., b_n$

2 4 5 7 10

Correctness (requirements for the output)

For any given input the algorithm halts with the output:

- $b_1 < b_2 < b_3 < \ldots. < b_n$
- $b_1, b_2, b_3, \ldots., b_n$ is a permutation of $a_1, a_2, a_3, \ldots., a_n$

Running time

Depends on

- number of elements (n)
- how (partially) sorted they are
- algorithm

Insertion Sort

A

3	4	6	8	9	7	2	5	1

1 j → n

← i

Strategy

- Start “empty handed”
- Insert a card in the right position of the already sorted hand
- Continue until all cards are inserted/sorted

INPUT: A[0..n-1] – an array of integers
OUTPUT: a permutation of A such that A[0]≤ A[1]≤ …≤A[n-1]

Pseudo-code (Functional / Recursive)

```
algorithm insertionSort(A[0..n-1])
{
   A[0]                                        if n=1
   insert(insertionSort(A[0..n-2]), A[n-1])    o.w.
}
algorithm insert(A[0..n-1], key)
{
   append(A[0..n-1], key)                 if key>=A[n-1]
   append(newarray(key), A[0])            if n=1&key<A[0]
   append(insert(A[0..n-2],key), A[n-1])  o.w.
}
```

Insertion Sort

A

3	4	6	8	9	7	2	5	1

1 j ⟶ n

⟵ i

Strategy

- Start “empty handed”
- Insert a card in the right position of the already sorted hand
- Continue until all cards are inserted/sorted

INPUT: A[0..n-1] – an array of integers
OUTPUT: a permutation of A such that A[0]≤ A[1]≤ …≤A[n-1]

```
for j←1 to n-1 do
    key ← A[j]
    //insert A[j] into the sorted sequence A[0..j-1]
    i←j-1
    while i>=0 and A[i]>key
        do A[i+1]←A[i]
           i--
    A[i+1]←key
```

Analysis of Insertion Sort

	cost	Times
for j←1 **to** *n-1* **do**	c_1	n
key←A[j]	c_2	n-1
//insert A[j] into the sorted sequence A[0..j-1]	0	n-1
i←j-1	c_3	n-1
while i>=0 **and** A[i]>key	c_4	$\sum_{j=1}^{n-1} t_j$
do A[i+1]←A[i]	c_5	$\sum_{j=1}^{n-1} (t_j - 1)$
i--	c_6	$\sum_{j=1}^{n-1} (t_j - 1)$
A[i+1] ← key	c_7	n-1

Total time = $n(c_1+c_2+c_3+c_7) + \sum^{n-1}_{j=1} t_j \ (c_4+c_5+c_6)$

$- (c_2+c_3+c_5+c_6+c_7)$

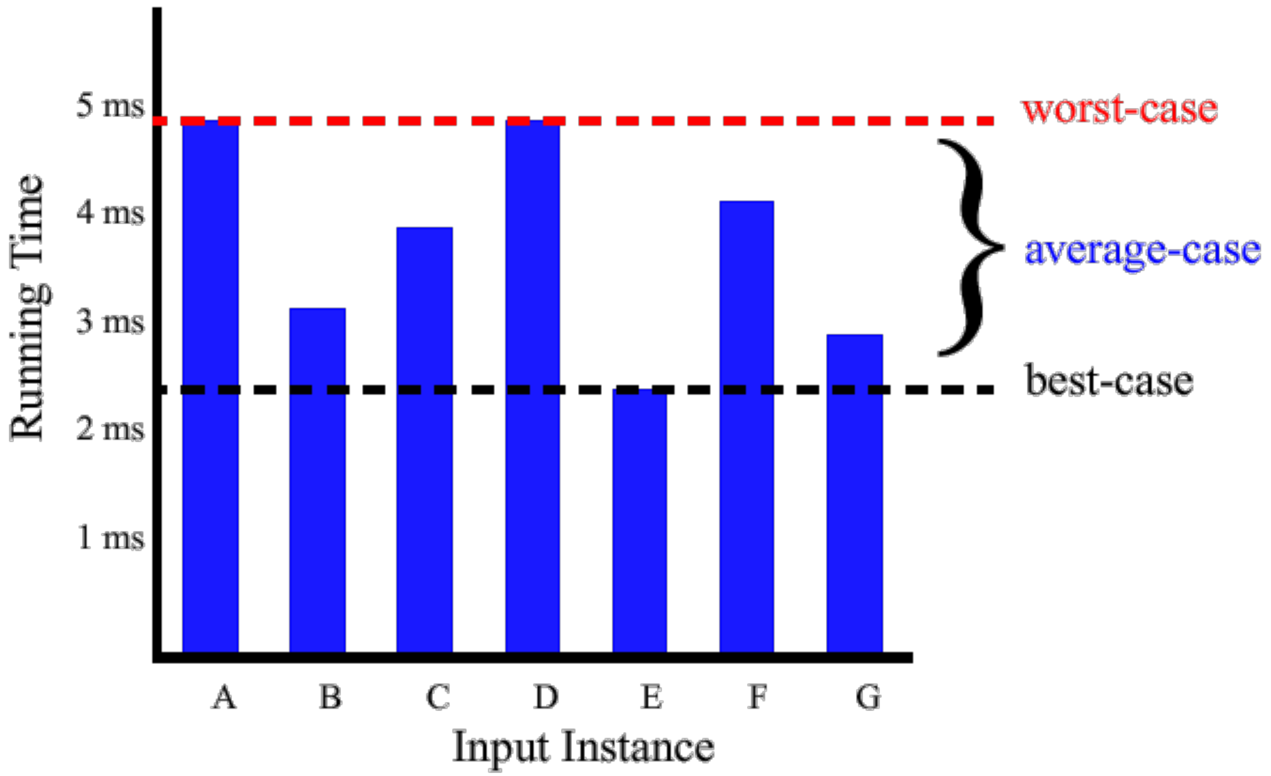
Best/Worst/Average Case

Total time = $n(c_1+c_2+c_3+c_7) + \sum^{n-1}_{j=1} t_j \ \ (c_4+c_5+c_6)$
$- (c_2+c_3+c_5+c_6+c_7)$

- **Best case**:
  - elements already sorted; $t_j=1$, running time = $f(n)$, i.e., *linear* time.
- **Worst case**:
  - elements are sorted in inverse order; $t_j=j+1$, running time = $f(n^2)$, i.e., *quadratic* time
- **Average case**:
  - $t_j=(j+1)/2$, running time = $f(n^2)$, i.e., *quadratic* time

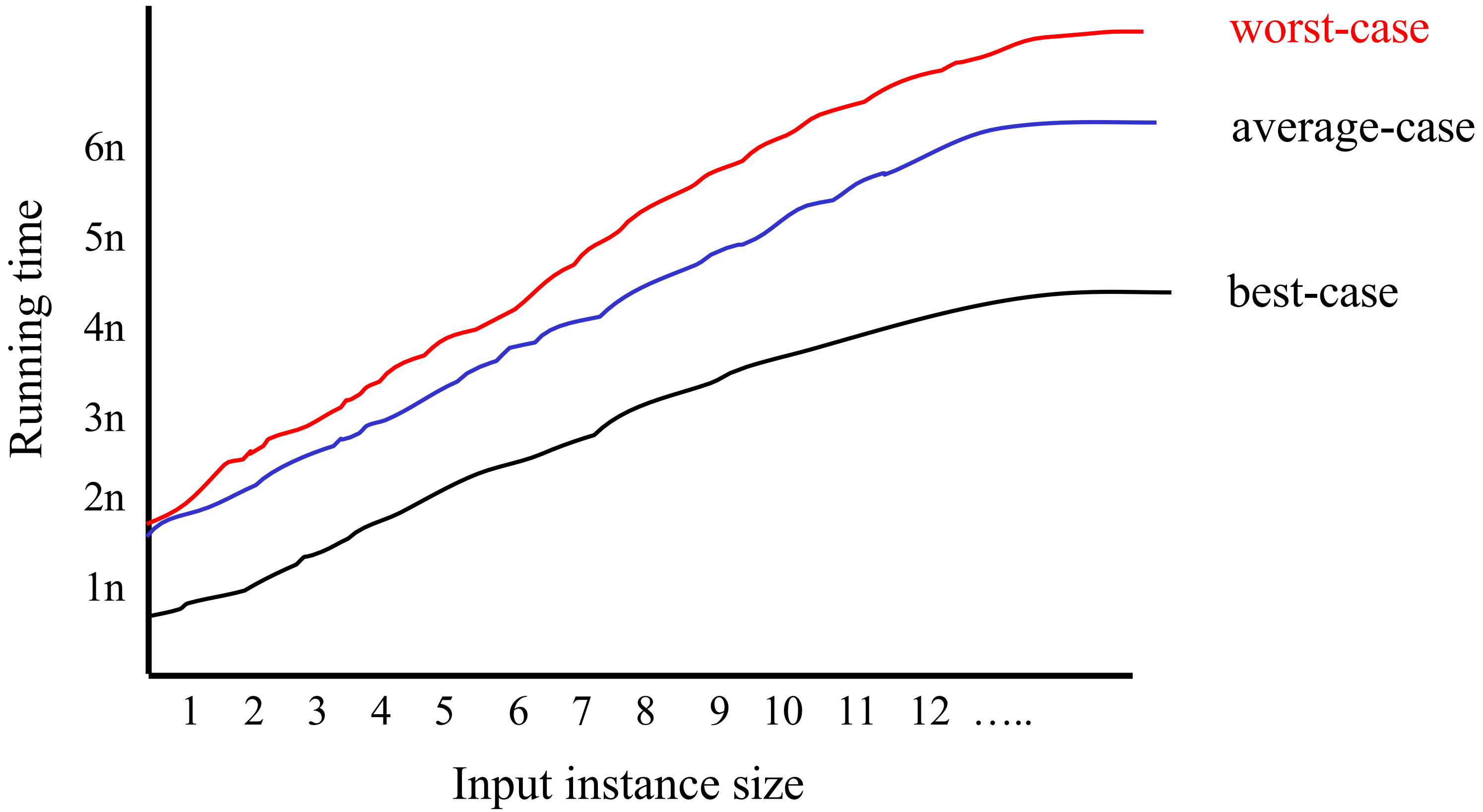
Best/Worst/Average Case (2)

- For a specific size of input n, investigate running times for different input instances:



Best/Worst/Average Case (3)

For inputs of all sizes:



Best/Worst/Average Case (4)

- **Worst case** is usually used: It is an upper-bound and in certain application domains (e.g., air traffic control, surgery) knowing the **worst-case** time complexity is of crucial importance

- For some algos **worst case** occurs fairly often

- **Average case** is often as bad as **worst case**

- Finding **average case** can be very difficult

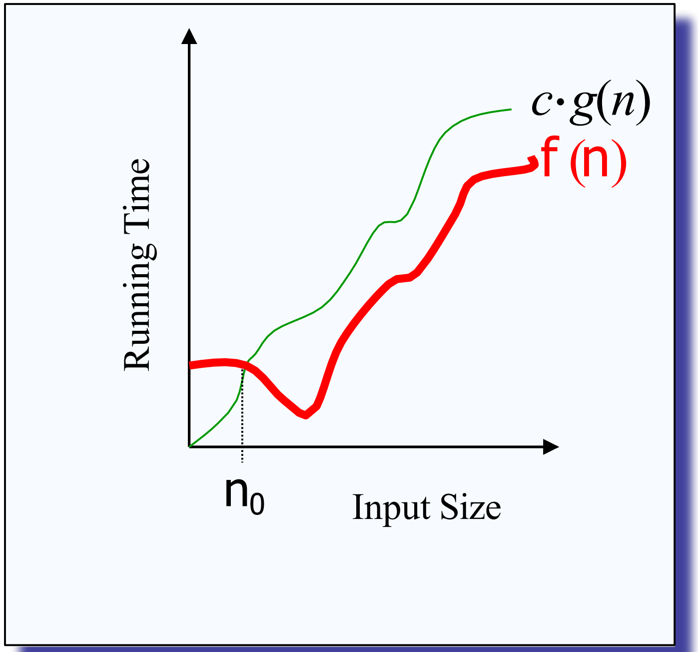
Asymptotic Analysis

- Goal: to simplify analysis of running time by getting rid of ”details”, which may be affected by specific implementation and hardware
  - like “rounding”: 1,000,001 ≈ 1,000,000
  - $3n^2 \approx n^2$

- Capturing the essence: how the running time of an algorithm increases with the size of the input *in the limit*.
  - Asymptotically more efficient algorithms are best for all but small inputs

Asymptotic Notation

- The "big-Oh" O-Notation
  - asymptotic upper bound
  - f(n) is O(g(n)), if there exists constants c and n_0, *s.t.* **$f(n) \le c\, g(n)$** for all $n \ge n_0$
  - f(n) and g(n) are functions over non-negative integers
- Used for *worst-case* analysis

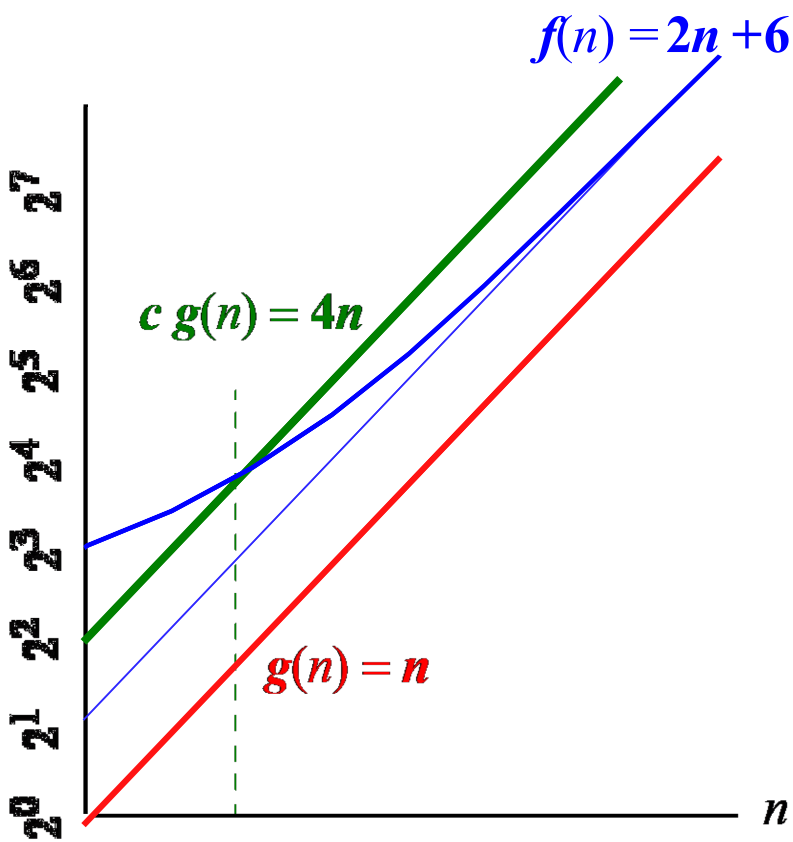


Example

For functions f(n) and g(n) there are positive constants c and n_0 such that: $f(n) \le c\, g(n)$ for $n \ge n_0$

conclusion:

2n+6 is O(n).



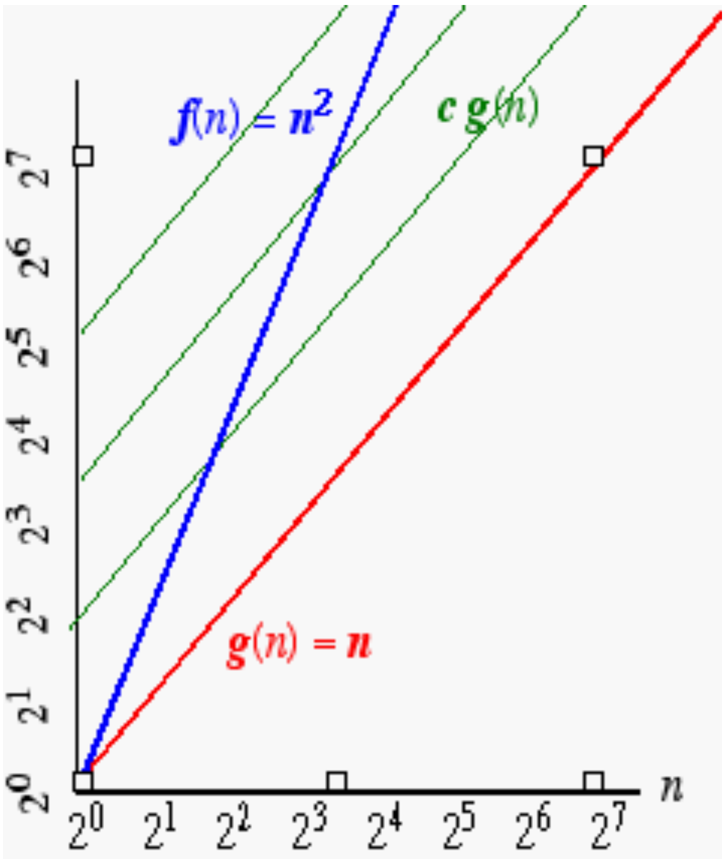
Another Example

On the other hand…

n^2 is not $O(n)$ because there is no c and n_0 such that:

$$n^2 \leq cn \text{ for } n \geq n_0$$

The graph to the right illustrates that no matter how large a c is chosen there is an n big enough that $n^2 > cn$).



Asymptotic Notation

- Simple Rule: Drop lower order terms and constant factors.
  - 50 $n \log n$ is $O(n \log n)$
  - $7n - 3$ is $O(n)$
  - $8n^2 \log n + 5n^2 + n$ is $O(n^2 \log n)$
- Note: Even though (50 $n \log n$) is $O(n^5)$, it is expected that such an approximation be of as small an order as possible

Asymptotic Analysis of Running Time

- Use O-notation to express number of primitive operations executed as function of input size.
- Comparing asymptotic running times
  - an algorithm that runs in $O(n)$ time is better than one that runs in $O(n^2)$ time
  - similarly, $O(\log n)$ is better than $O(n)$
  - hierarchy of functions: $\log n < n < n^2 < n^3 < 2^n$
- Caution! Beware of very large constant factors. An algorithm running in time 1,000,000 n is still $O(n)$ but might be less efficient than one running in time $2n^2$, which is $O(n^2)$

Example of Asymptotic Analysis

Algorithm prefixAverages1(X):

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that A[i] is the average of elements X[0], ... , X[i].

for i ← 0 **to** n-1 **do**

    a ← 0

    for j ← 0 **to** i **do**

        a ← a + X[j] ← 1 step

    A[i] ← a/(i+1)

return array A

i iterations with i=0,1,2...n-1

n iterations

Analysis: running time is $O(n^2)$

A Better Algorithm

Algorithm prefixAverages2(X):

Input: An *n*-element array X of numbers.

Output: An *n*-element array A of numbers such that A[*i*] is the average of elements X[0], ... , X[*i*].

s ← 0

for i ← 0 **to** n **do**

  s ← s + X[i]

  A[i] ← s/(i+1)

return array A

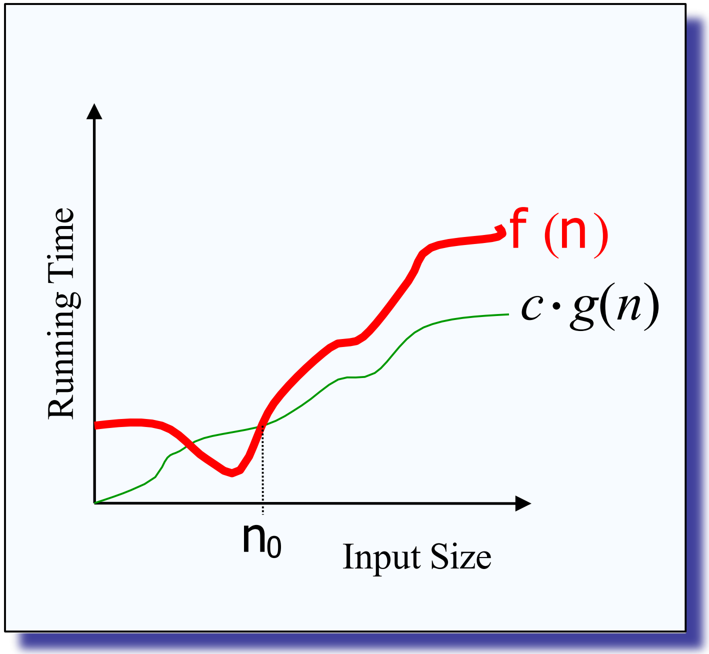
Analysis: Running time is O(n)

Asymptotic Notation *(terminology)*

- Special classes of algorithms:
  - Logarithmic: $O(\log n)$
  - Linear: $O(n)$
  - Quadratic: $O(n^2)$
  - Polynomial: $O(n^k)$, $k \geq 1$
  - Exponential: $O(a^n)$, $a > 1$
- "Relatives" of the Big-Oh
  - $\Omega(f(n))$: Big Omega -asymptotic *lower* bound
  - $\Theta(f(n))$: Big Theta -asymptotic *tight* bound

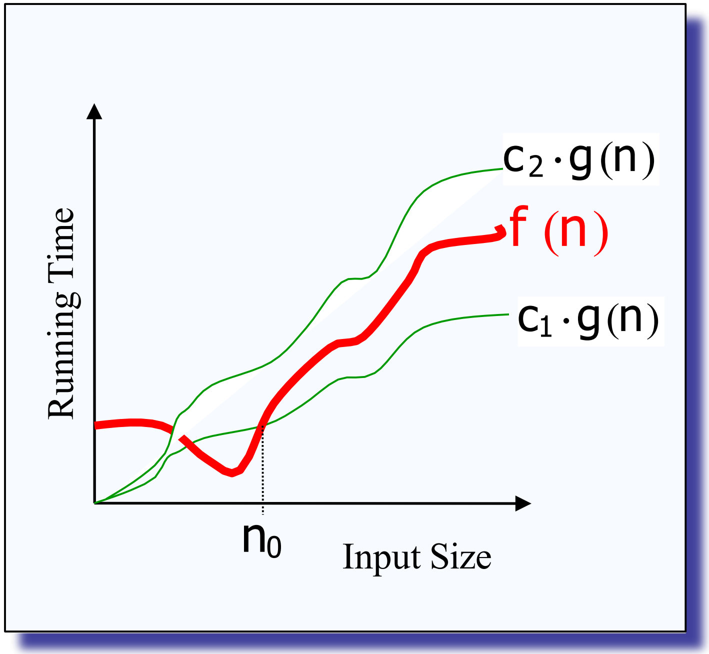
Asymptotic Notation

- The "big-Omega" Ω−Notation
  - asymptotic lower bound
  - *f(n)* is *Ω(g(n))* if there exists constants *c* and n_0, *s.t.* **c g(n) ≤ f(n)** for $n \geq n_0$
- Used to describe *best-case* running times or lower bounds for algorithmic problems
  - E.g., lower-bound for searching in an unsorted array is *Ω(n).*



Asymptotic Notation

- The "big-Theta" Θ− Notation
  - asymptotically tight bound
  - *f(n) is Θ(g(n))* if there exists constants c_1, c_2, and n_0, *s.t.* $\mathbf{c_1\, g(n) \le f(n) \le c_2\, g(n)}$ for $n \ge n_0$
- *f(n)* is Θ*(g(n))* if and only if *f(n)* is *O(g(n))* and *f(n)* is Ω*(g(n))*
- *O(f(n))* is often misused instead of Θ*(f(n))*



Asymptotic Notation

Two more asymptotic notations

- "Little-Oh" notation *f(n)* is *o(g(n))* non-tight analogue of Big-Oh
  - For every *c>0*, there should exist n_0, s.t. **f(n) ≤ c g(n)** for $n \geq n_0$
  - Used for **comparisons** of running times. If *f(n) is o(g(n))*, it is said that *g(n) dominates f(n).*
- "Little-omega" notation *f(n)* is *ω(g(n))* non-tight analogue of Big-Omega

Asymptotic Notation

- Analogy with real numbers
  - $f(n)$ is $O(g(n))$ $\cong$ $f \leq g$
  - $f(n)$ is $\Omega(g(n))$ $\cong$ $f \geq g$
  - $f(n)$ is $\Theta(g(n))$ $\cong$ $f = g$
  - $f(n)$ is $o(g(n))$ $\cong$ $f < g$
  - $f(n)$ is $\omega(g(n))$ $\cong$ $f > g$
- Abuse of notation: $f(n) = O(g(n))$ actually means $f(n) \in O(g(n))$

Comparison of Running Times

Running Time	Maximum problem size (n)		
	1 second	1 minute	1 hour
$400n$	2500	150000	9000000
$20n \log n$	4096	166666	7826087
$2n^2$	707	5477	42426
n^4	31	88	244
2^n	19	25	31