Parallel & Distibuted Computing: Lecture 7

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October 17, 2017

Concepts and Terminology

1. Some General Parallel Terminology
2. Limits and Costs of Parallel Programming
3. Sequential implementation of domain integration of polynomials

Some General Parallel Terminology

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Often implemented by establishing a synchronization point within an application where a task may not proceed further until another task(s) reaches the same or logically equivalent point.

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- Coarse: relatively large amounts of computational work are done between communication events
- Fine: relatively small amounts of computational work are done between communication events

Observed Speedup & Parallel Overhead

Observed Speedup Observed speedup of a code which has been parallelized, defined as:

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- Task termination time

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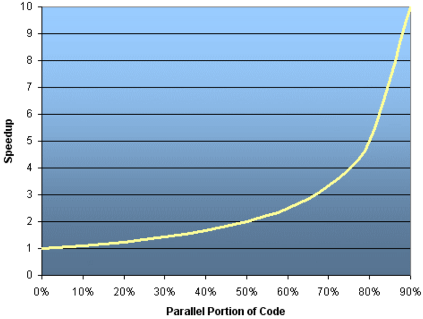
- Application algorithm
- Parallel overhead related
- Characteristics of your specific application

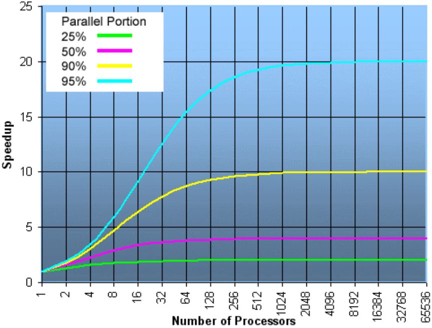
Limits and Costs of Parallel Programming

Amdahl's Law 1/4

Amdahl's Law states that potential program speedup is defined by the fraction of code (P) that can be parallelized:

$$\text{speedup} = \frac{1}{1-P}$$





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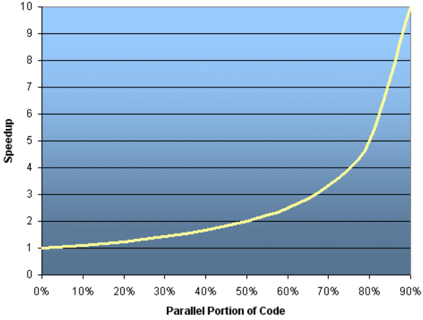

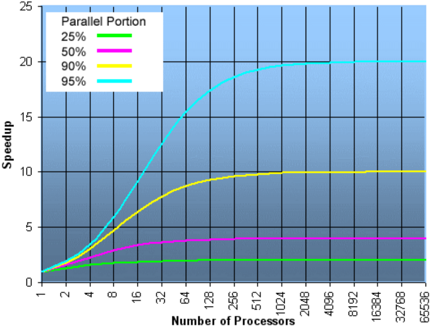

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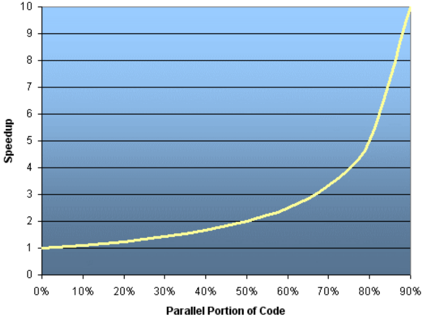

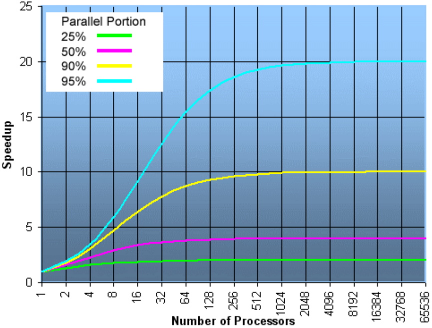

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- If all code is parallelized, $P = 1$ and the speedup $= \infty$ (in theory).
- If 50% of code can be parallelized, $max(\text{speedup}) = 2$, meaning the code may run twice as fast.

Amdahl's Law 2/4

Introducing the number N of processors performing the parallel fraction of work, the relationship can be modeled by:

$$\text{speedup} = \frac{1}{\frac{P}{N} + S}$$

where P = parallel fraction, N = number of processors and S = serial fraction.

Amdahl's Law 3/4

It soon becomes obvious that there are limits to the scalability of parallelism.

For example:

	speedup		
N	P = .50	P = .90	P = .99
10	1.82	5.26	9.17
100	1.98	9.17	50.25
1,000	1.99	9.91	90.99
10,000	1.99	9.91	99.02
100,000	1.99	9.99	99.90

Figure 1: Speedup table

Amdahl's Law 4/4

However, certain problems demonstrate increased performance by increasing the problem size. For example:

```
2D Grid Calculations     85 seconds   85%
Serial fraction          15 seconds   15%
```

We can increase the problem size by doubling the grid dimensions and halving the time step. This results in four times the number of grid points and twice the number of time steps. The timings then look like:

```
2D Grid Calculations     680 seconds   97.84%
Serial fraction           15 seconds    2.16%
```

Problems that increase the percentage of parallel time with their size are more scalable than problems with a fixed percentage of parallel time.

Complexity

- In general, parallel applications are much more complex than corresponding serial applications, perhaps an order of magnitude.

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- Adhering to "good" software development practices is essential when working with parallel applications - especially if somebody besides you will have to work with the software.

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- Hardware architectures are characteristically highly variable and can affect portability.

Resource Requirements

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- For short running parallel programs, there can actually be a decrease in performance compared to a similar serial implementation. The overhead costs associated with setting up the parallel environment, task creation, communications and task termination can comprise a significant portion of the total execution time for short runs.

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  - Amount of memory available on any given machine or set of machines
  - Processor clock speed
- Parallel support libraries and subsystems software can limit scalability independent of your application.

Sequential implementation of domain integration of polynomials

Domain integration of polynomials

Finite formulae for evaluation of integrals:

$$II_S \equiv \int\!\!\int_S f(\mathbf{p})\, dS, \qquad III_P \equiv \int\!\!\int\!\!\int_P f(\mathbf{p})\, dV, \tag{1}$$

The integrating function is a trivariate polynomial

$$f(\mathbf{p}) = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{m} \sum_{\gamma=0}^{p} a_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma},$$

where α, β, γ are non-negative integers. Since the extension to $f(\mathbf{p})$ is straightforward, we focus on integrals of monomials:

$$II_S^{\alpha\beta\gamma} \equiv \int\!\!\int_S x^{\alpha} y^{\beta} z^{\gamma}\, dS, \qquad III_P^{\alpha\beta\gamma} \equiv \int\!\!\int\!\!\int_P x^{\alpha} y^{\beta} z^{\gamma}\, dV. \tag{2}$$

From Cattani, Paoluzzi. "Boundary integration over linear polyhedra", CAD, 1990

Basic integration functions

structure product over a polyhedral surface S, open or closed, is a summation of structure products (3) over the 2-simplices of a triangulation K_2 of S:

$$II_S^{\alpha\beta\gamma} = \int\int_S x^\alpha y^\beta z^\gamma \, dS = \sum_{\tau \in K_2} II_\tau^{\alpha\beta\gamma}$$

```
function II(P, alpha, beta, gamma, signedInt=false)
    w = 0
    V, FV = P
    if typeof(P) == PyCall.PyObject
        if typeof(V) == Array{Any,2}
            V = V'
        end
        if typeof(FV) == Array{Any,2}
            FV = [FV[k,:] for k=1:size(FV,1)]
            FV = FV+1
        end
    end
    if typeof(FV) == Array{Int64,2}
        FV = [FV[:,k] for k=1:size(FV,2)]
    end
    for i=1:length(FV)
        tau = hcat([V[:,v] for v in FV[i]]...)
        if size(tau,2) == 3
            term = TT(tau, alpha, beta, gamma, signedInt)
            if signedInt
                w += term
            else
                w += abs(term)
            end
        elseif size(tau,2) > 3
            println("ERROR: FV[$(i)] is not a triangle")
        else
            println("ERROR: FV[$(i)] is degenerate")
        end
    end
    return w
```

Basic integration functions

$$III_P^{\alpha\beta\gamma} = \iiint_P x^\alpha y^\beta z^\gamma \, dx \, dy \, dz$$

$$= \frac{1}{\alpha+1} \sum_{\tau \in K_2} \left[\frac{(\mathbf{a} \times \mathbf{b})_x}{|\mathbf{a} \times \mathbf{b}|} \right]_\tau II_\tau^{\alpha+1,\beta,\gamma}$$

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    end
    for i=1:length(FV)
        tau = hcat([V[:,v] for v in FV[i]]...)
        vo,va,vb = tau[:,1],tau[:,2],tau[:,3]
        a = va - vo
        b = vb - vo
        c = cross(a,b)
        w += c[1]/vecnorm(c) * TT(tau, alpha+1, beta, gamma)
    end
    return w/(alpha + 1)
end
```

Basic integration functions

$$
\begin{aligned}
II_{\tau}^{\alpha\beta\gamma} \quad = \quad & |\mathbf{a}\times\mathbf{b}| \sum_{h=0}^{\alpha} \binom{\alpha}{h} x_o^{\alpha-h} \cdot \\
& \cdot \sum_{k=0}^{\beta} \binom{\beta}{k} y_o^{\beta-k} \cdot \\
& \cdot \sum_{m=0}^{\gamma} \binom{\gamma}{m} z_o^{\gamma-m} \cdot \\
& \cdot \sum_{i=0}^{h} \binom{h}{i} a_x^{h-i} b_x^{i} \cdot \\
& \cdot \sum_{j=0}^{k} \binom{k}{j} a_y^{k-j} b_y^{j} \cdot \\
& \cdot \sum_{l=0}^{m} \binom{m}{l} a_z^{m-l} b_z^{l} \cdot \\
& \cdot II^{\mu\nu}
\end{aligned}
$$

```
function TT(tau::Array{Float64,2}, alpha, beta, gamma, signedInt=false)
  vo,va,vb = tau[:,1],tau[:,2],tau[:,3]
  a = va - vo
  b = vb - vo
  s1 = 0.0
  for h=0:alpha
    for k=0:beta
      for m=0:gamma
        s2 = 0.0
        for i=0:h
          s3 = 0.0
          for j=0:k
            s4 = 0.0
            for l=0:m
              s4 += binomial(m,l) * a[3]^(m-l) * b[3]^l *
                M( h+k+m-i-j-l, i+j+l )
            end
            s3 += binomial(k,j) * a[2]^(k-j) * b[2]^j * s4
          end
          s2 += binomial(h,i) * a[1]^(h-i) * b[1]^i * s3;
        end
        s1 += binomial(alpha,h) * binomial(beta,k) * binomial(gamma,m) *
            vo[1]^(alpha-h) * vo[2]^(beta-k) * vo[3]^(gamma-m) * s2
      end
    end
  end
  c = cross(a,b)
  if signedInt == true
    return s1 * vecnorm(c) * sign(c[3])
  else return s1 * vecnorm(c) end
end
```

Basic integration functions

$$II^{\alpha\beta} = \frac{1}{\alpha+1}\sum_{h=0}^{\alpha+1}\binom{\alpha+1}{h}\frac{(-1)^h}{h+\beta+1},$$

```
function M(alpha, beta)
    a = 0
    for l=1:(alpha + 2)
        a += binomial(alpha+1,l) * (-1)^l/(l+beta+1)
    end
    return a/(alpha + 1)
end
```