Tackling boundary effects on kernel-based representations (draft)

1 Introduction

In any digital acquisition system, the measured signals are sampled and of finite length. Then, when performing any kernel-based transform, such as synchrosqueezing transform, boundaries effect is going to affect the quality of the representation in the vicinity of the borders. Consequently, signal characteristics, like instantaneous frequencies, appear to be imprecisely determined near borders.

We propose here a method to limit the disruptions caused by the boundary effect. To reduce the influence of boundaries on the kernel-based representations, we proceed in two steps:

1. Extend the signal by forecasting it.

2. Run the kernel-based algorithm on the extended signal.

2 Algorithm

2.1 Forecasting

Notations. Let $x : \mathbb{R} \to \mathbb{R}$ denote a continuous-time signal. In this work, we consider a finite-length discretization of that one. Thus, the sampled signal $\mathbf{x}$, whose length is denoted by N, is such that

$$\mathbf{x}[n] = x\left(\frac{n}{f_{\mathsf{s}}}\right) , \quad \forall n \in \{0, \ldots, N-1\} ,$$

where f_{s} denotes the sampling frequency.

Let $M < N$, $K < N$. Then, for all $k \in \{0, \ldots, K-1\}$, we extract from $\mathbf{x} \in \mathbb{R}^N$ the sub-signal $\mathbf{x}_k \in \mathbb{R}^M$ given by:

$$\mathbf{x}_k = \begin{pmatrix} \mathbf{x}[N-K+(k-1)-(M-1)] \\ \vdots \\ \mathbf{x}[N-K+(k-1)] \end{pmatrix} .$$

These sub-signals are gathered into the matrix $\mathbf{X} \in \mathbb{R}^{M \times K}$ such that:

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}_0 & \cdots & \mathbf{x}_{K-1} \end{pmatrix} .$$

Notice that these sub-signals are overlapping each other. Indeed, $\mathbf{x}_{k+1}$ is a shifting of $\mathbf{x}_k$ from one sample.

We also consider the matrix $\mathbf{Y} \in \mathbb{R}^{M \times K}$ given by:

$$\mathbf{Y} = \begin{pmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_K \end{pmatrix} .$$

Dynamic model and forecasting. Establishing a dynamic model consists in determining the relation linking $\mathbf{Y}$ to $\mathbf{X}$, that is finding a function f so that

$$\mathbf{Y} = f(\mathbf{X}) .$$

We consider here a naive dynamic model, assuming that we have the following relation:

$$\mathbf{Y} = \mathbf{A}\mathbf{X} \,, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{M \times M}$. This is a linear dynamic model, which can be written equivalently in function of the sub-signals $\mathbf{x}_k$, as:

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k, \ , \forall k \in \{0, \ldots, K-1\} \,. \tag{2}$$

The forecasting method consists in estimating the unknown matrix $\mathbf{A}$. Indeed, let $\tilde{\mathbf{A}}$ denotes the estimate of $\mathbf{A}$, we then obtain the forecasting of the signal at time $\frac{N-1+\ell}{f_s}$ by:

$$\tilde{\mathbf{x}}[N-1+\ell] = \mathbf{e}_M^T \tilde{\mathbf{A}}^\ell \mathbf{x}_K \,, \tag{3}$$

where $\mathbf{e}_M$ is the vector of length M given by $\mathbf{e}_M = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^T$.

Model estimation. To estimate the matrix $\mathbf{A}$, we basically implement the least square estimator. Thus, we solve the following problem:

$$\tilde{\mathbf{A}} = \arg\min_{\boldsymbol{\alpha}} \mathcal{L}(\mathbf{A}) \,, \tag{4}$$

where the loss function $\mathcal{L}$ is given by:

$$\mathcal{L}(\mathbf{A}) = \|\mathbf{Y} - \mathbf{A}\mathbf{X}\|^2 = \sum_{k=0}^{K-1} \|\mathbf{x}_{k+1} - \mathbf{A}\mathbf{x}_k\|^2.$$

Therefore, solving the problem (4), i.e. $\nabla \mathcal{L}(\tilde{\mathbf{A}}) = \mathbf{0}$, gives the following estimate $\tilde{\mathbf{A}}$ of the dynamic model matrix $\mathbf{A}$:

$$\tilde{\mathbf{A}} = \mathbf{Y}\mathbf{X}^T (\mathbf{X}\mathbf{X}^T)^{-1} \,. \tag{5}$$

Remark 1. *This expression clearly shows that the matrix $\tilde{\mathbf{A}}$ takes the following form:*

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ \alpha_1 & \cdots & \cdots & \cdots & \alpha_M \end{pmatrix} .$$

Then, except the row vector $\boldsymbol{\alpha} = (\alpha_1 \cdots \alpha_M)$, the matrix $\mathbf{A}$ is fully determined by the dynamic model.

Signal extension. We can finally construct the extended signal $\tilde{\mathbf{x}} \in \mathbb{R}^{N+2L}$ concatenating the backward prediction $\tilde{\mathbf{x}}_{\mathrm{bw}}$, the observed signal $\mathbf{x}$, and the forward prediction $\tilde{\mathbf{x}}_{\mathrm{fw}}$. We summarize the extension step in Algorithm 1. Notice that we handle the backward estimation using the same strategy than described above, but applying it to the reverse signal $\mathbf{x}^{\mathrm{r}} = \begin{pmatrix} \mathbf{x}[N-1] & \cdots & \mathbf{x}[0] \end{pmatrix}^T$.

2.2 Kernel-based representation

Transform restriction. Let $\mathscr{F}_N : \mathbb{R}^N \to \mathbb{R}^{F \times N}$ generically denotes the kernel-based representation we are interested in. It can be, for instance, such as short-time Fourier transform (STFT), the continuous wavelet transform (CWT), the synchrosqueezing transform (SST), or the reassignment (RS). Here, F typically denotes the size of the representation along the frequency axis. Due to the boundary effects, the representation $\mathscr{F}_N(\mathbf{x})$ shows undesired patterns when approaching its edges.For example, the instantaneous frequencies highlighted by the SST can be blurred near that edges. To limit these phenomena, we apply the representation to the estimated extended signal $\tilde{\mathbf{x}}$. This strategy moves the boundary

Algorithm 1 Signal extension. $\tilde{\mathbf{x}} = \mathsf{SigExt}(\mathbf{x}, M, K, L)$

Inputs: $\mathbf{x}$, M, K, L

Forward forecasting.
- LS estimation of the forward matrix $\tilde{\mathbf{A}}_{\mathrm{fw}}$ via equation (5).
- Forward forecasting $\tilde{\mathbf{x}}_{\mathrm{bw}}$ obtained applying equation (3) with $\ell \in \{1, \dots, L\}$.

Backward forecasting.
- Reverse signal $\mathbf{x}$ to $\mathbf{x}^{\mathrm{r}}$.
- LS estimation of the backward matrix $\tilde{\mathbf{A}}_{\mathrm{bw}}$ via equation (5) applied to $\mathbf{x}^{\mathrm{r}}$.
- Reversed backward forecasting $\tilde{\mathbf{x}}^{\mathrm{r}}_{\mathrm{bw}}$ obtained applying equation (3) to $\mathbf{x}^{r}$ with $\ell \in \{1, \dots, L\}$.
- Reverse $\tilde{\mathbf{x}}^{\mathrm{r}}_{\mathrm{bw}}$ to obtain the estimate $\tilde{\mathbf{x}}_{\mathrm{bw}}$.

Output: Extended signal $\tilde{\mathbf{x}} = \begin{pmatrix}\tilde{\mathbf{x}}_{\mathrm{bw}} & \mathbf{x} & \tilde{\mathbf{x}}_{\mathrm{fw}}\end{pmatrix}^T$.

effects out of the time interval $I = [0, \frac{N-1}{f_s}]$. Finally, the boundary-effects insensitive representation $\mathscr{F}_N^{\mathrm{ext}} : \mathbb{R}^N \to \mathbb{R}^{F\times N}$ of $\mathbf{x}$ is given by:

$$\mathscr{F}_N^{\mathrm{ext}}(\mathbf{x})[f, n] = \mathscr{F}_{N+2L}(\tilde{\mathbf{x}})[f, L+n] \ , \quad \forall f \in \{0, \cdots, F-1\},\ n \in \{0, \cdots, N-1\} \ . \tag{6}$$

This amounts to restricting the representation $\mathscr{F}_{N+2L}(\tilde{\mathbf{x}})$ to the original measurement interval of $\mathbf{x}$. For the sake of simplicity, we denotes the restriction operator by $\mathcal{R}$, where $\mathcal{R} : \mathbb{R}^{F\times(N+2L)} \to \mathbb{R}^{F\times N}$. Consequently, we have:

$$\mathscr{F}_N^{\mathrm{ext}}(\mathbf{x}) = \mathcal{R}\left(\mathscr{F}_{N+2L}(\tilde{\mathbf{x}})\right) \ .$$

Conclusion. Finally, the global procedure we implement to reduce boundary effects on kernel-based representations is summarized by the pseudo-code of Algorithm 2.

Algorithm 2 Tackling boundary effects. $\mathbf{F}_{\mathbf{x}} = \mathsf{BoundEffRed}(\mathbf{x}, M, K, L, \mathscr{F})$

Inputs: $\mathbf{x}$, M, K, L, $\mathscr{F}$

Forecasting step.
- Signal extension: $\tilde{\mathbf{x}} = \mathrm{SigExt}(\mathbf{x})$.

Representation step.
- Representation evaluation: $\mathscr{F}_{N+2L}(\tilde{\mathbf{x}})$.
- Restriction of $\mathscr{F}_{N+2L}(\tilde{\mathbf{x}})$ to the central time interval (see (6)) to obtain $\mathbf{F}_{\mathbf{x}} = \mathscr{F}_N^{\mathrm{ext}}(\mathbf{x})$.

Output: Signal representation $\mathbf{F}_{\mathbf{x}}$

3 Theoretical performance

3.1 General signal model

We model the observed signal as a given waveform corrupted by an additive Gaussian white noise. Therefore, the measured discrete signal $\mathbf{x}$ is written as:

$$\mathbf{x} = \mathbf{z} + \sigma\mathbf{w} \ ,$$

where $\mathbf{z}$ is a deterministic signal, $\mathbf{w}$ is a (Gaussian) white noise, whose variance is denoted by σ^2.

3.2 Forecasting error

On the forecasting interval, we decompose the estimated signal $\tilde{\mathbf{x}}$ as follows:

$$\tilde{\mathbf{x}}[N-1+\ell] = \mathbf{z}[N-1+\ell] + \boldsymbol{\epsilon}[\ell] \,,$$

where $\boldsymbol{\epsilon} \in \mathbb{R}^L$ is a the forward forecasting error, which would ideally behave like the measurement noise $\sigma\mathbf{w}$. To evaluate the actual behavior of the forward forecasting error $\boldsymbol{\epsilon}$, we determine its two first moments.

1. The mean, which is also the estimation bias, is such that:

$$\begin{aligned}\boldsymbol{\mu}[\ell] &\triangleq \mathbb{E}\{\boldsymbol{\epsilon}[\ell]\} \\ &= \mathbb{E}\{\mathbf{e}_M^T \tilde{\mathbf{A}}^\ell\}\mathbf{z}_K + \mathbb{E}\{\mathbf{e}_M^T \tilde{\mathbf{A}}^\ell \mathbf{w}_K\} - \mathbf{z}[N-1+\ell] \\ &= \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[N-1+\ell] \end{aligned} \tag{7}$$

   where $\boldsymbol{\alpha}^{(\ell)} \in \mathbb{R}^{1\times M}$ denotes the last row of $\tilde{\mathbf{A}}^\ell$.

2. The covariance is given by:

$$\gamma[\ell,\lambda] \triangleq \mathbb{E}\{(\boldsymbol{\epsilon}[\ell]-\boldsymbol{\mu}[\ell])\,(\boldsymbol{\epsilon}[\lambda]-\boldsymbol{\mu}[\lambda])\}$$

   By means of the Cauchy-Schwarz inequality the covariance, we provide an upper bound on the covariance:

$$|\gamma[\ell,\lambda]| \leq \sqrt{\gamma[\ell,\ell]\gamma[\lambda,\lambda]} \,. \tag{8}$$

   We then focus on the determination of $\gamma[\ell,\ell]$. Thus:

$$\begin{aligned}\gamma[\ell,\ell] &= \mathbb{E}\{\boldsymbol{\epsilon}[\ell]^2\} - \boldsymbol{\mu}[\ell]^2 \\ &= \mathbf{z}[N-1+\ell]^2 - 2\mathbf{z}[N-1+\ell]\mathbb{E}\{\tilde{\mathbf{x}}[N-1+\ell]\} + \mathbb{E}\{\tilde{\mathbf{x}}[N-1+\ell]^2\} - \boldsymbol{\mu}[\ell]^2 \,.\end{aligned}$$

   But:

$$\mathbb{E}\{\tilde{\mathbf{x}}[N-1+\ell]^2\} = \mathbb{E}\{(\boldsymbol{\alpha}^{(\ell)}\mathbf{z}_K)^2\} + 2\sigma\mathbb{E}\{(\boldsymbol{\alpha}^{(\ell)}\mathbf{z}_K)(\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K)\} + \sigma^2\mathbb{E}\{(\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K)^2\}$$

   The mean square error can therefore be written as follows:

$$\begin{aligned}\gamma[\ell,\ell] =& -\mathbf{z}[N-1+\ell]^2 - 2\mathbf{z}[N-1+\ell]\boldsymbol{\mu}[\ell] + \mathbb{E}\{(\boldsymbol{\alpha}^{(\ell)}\mathbf{z}_K)^2\} \\ &+ 2\sigma\mathbb{E}\{(\boldsymbol{\alpha}^{(\ell)}\mathbf{z}_K)(\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K)\} + \sigma^2\mathbb{E}\{(\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K)^2\} - \boldsymbol{\mu}[\ell]^2 \,.\end{aligned} \tag{9}$$

Decomposition of $\boldsymbol{\alpha}^{(\ell)}$. For all $m, m' \in \{0,\ldots,M-1\}$, we have:

$$(\mathbf{X}\mathbf{X}^T)[m,m'] = \sum_{k=0}^{K-1} \mathbf{x}[\underbrace{N-K-M}_{\triangleq N_0}+m+k]\mathbf{x}[N-K-M+m'+k] = K(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}) \tag{10}$$

$$(\mathbf{Y}\mathbf{X}^T)[m,m'] = \sum_{k=0}^{K-1} \mathbf{x}[N_0+m+1+k]\mathbf{x}[N_0+m'+k] = K(\mathbf{S}^{(1)}+\mathbf{E}^{(1)}) \,, \tag{11}$$

where

$$\mathbf{S}^{(a)}[m,m'] = \sigma^2\delta_{(m+a)m'} + \frac{1}{K}\sum_{k=0}^{K-1}\mathbf{z}[N_0+m+a+k]\mathbf{z}[N_0+m'+k] \,,$$

and $\mathbf{E}^{(a)} = \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$ with:

$$\mathbf{E}_1^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1}\mathbf{z}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] + \mathbf{w}[N_0+m+a+k]\mathbf{z}[N_0+m'+k] \,,$$

and

$$\mathbf{E}_2^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{w}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] - \delta_{(m+a)m'}\ ,$$

with $a \in \{0,1\}$.

Remark 2. *The matrices $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ are said to be error matrices because:*

$$\mathbb{E}\{\mathbf{E}^{(0)}\} = \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0}$$
$$\mathbb{E}\{\mathbf{E}^{(1)}\} = \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0}\,.$$

Thus:

$$\tilde{\mathbf{A}} = (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\,.$$

Let $\tilde{\mathbf{A}}_0 = \mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}}$. Then:

$$\begin{aligned}\boldsymbol{\alpha}^{(\ell)} &= \mathbf{e}_M^T\tilde{\mathbf{A}}^\ell \\ &= \boldsymbol{\alpha}_0^{(\ell)} + \mathbf{h}^{(\ell)}\ ,\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\alpha}_0^{(\ell)} &= \mathbf{e}_M^T\tilde{\mathbf{A}}_0^\ell \\ \mathbf{h}^{(\ell)} &= \mathbf{e}_M^T\left(\left((\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\right)^\ell - \tilde{\mathbf{A}}_0^\ell\right)\end{aligned} \tag{12}$$

Because σ is small, we develop the error vector $\mathbf{h}^{(\ell)}$ using a second-order Taylor expansion with respect to σ. Then, a development of the expression (12) gives:

$$\begin{aligned}\mathbf{h}^{(\ell)} &= \mathbf{e}_M^T\left(\left(\left(\mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}} + \mathbf{E}^{(1)}\mathbf{S}^{(0)^{-1}}\right)\left(\mathbf{I} + \sigma(\mathbf{E}_1^{(0)} + \sigma\mathbf{E}_2^{(0)})\mathbf{S}^{(0)^{-1}}\right)^{-1}\right)^\ell - \tilde{\mathbf{A}}_0^\ell\right) \\ &= \mathbf{e}_M^T\left(\left(\left(\mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}} + \mathbf{E}^{(1)}\mathbf{S}^{(0)^{-1}}\right)\left(\mathbf{I} - \sigma(\mathbf{E}_1^{(0)} + \sigma\mathbf{E}_2^{(0)})\mathbf{S}^{(0)^{-1}} + \sigma^2\mathbf{E}_1^{(0)}\mathbf{S}^{(0)-1}\mathbf{E}_1^{(0)}\mathbf{S}^{(0)-1} + \mathbf{o}(\sigma^2)\right)\right)^\ell - \tilde{\mathbf{A}}_0^\ell\right) \\ &= \mathbf{e}_M^T\Big(\Big(\mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}} + \sigma\mathbf{E}_1^{(1)}\mathbf{S}^{(0)^{-1}} + \sigma^2\mathbf{E}_2^{(1)}\mathbf{S}^{(0)^{-1}} - \sigma\mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}}\mathbf{E}_1^{(0)}\mathbf{S}^{(0)^{-1}} - \sigma^2\mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}}\mathbf{E}_2^{(0)}\mathbf{S}^{(0)^{-1}} \\ &\qquad - \sigma^2\mathbf{E}_1^{(1)}\mathbf{S}^{(0)-1}\mathbf{E}_1^{(0)}\mathbf{S}^{(0)-1} + \sigma^2\mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}}\mathbf{E}_1\mathbf{S}^{(0)-1}\mathbf{E}_1^{(0)}\mathbf{S}^{(0)-1} + \mathbf{o}(\sigma^2)\Big)^\ell - \tilde{\mathbf{A}}_0^\ell\Big) \\ &= \mathbf{e}_M^T\left(\tilde{\mathbf{A}}_0^\ell\left(\mathbf{I} + \sigma\tilde{\mathbf{A}}_0^{-1}\mathbf{Q}_1 + \sigma^2\tilde{\mathbf{A}}_0^{-1}\mathbf{Q}_2 + \mathbf{o}(\sigma^2)\right)^\ell - \tilde{\mathbf{A}}_0^\ell\right)\ ,\end{aligned}$$

where:

$$\begin{aligned}\mathbf{Q}_1 &= \left(\mathbf{E}_1^{(1)} - \tilde{\mathbf{A}}_0\mathbf{E}_1^{(0)}\right)\mathbf{S}^{(0)^{-1}} \\ \mathbf{Q}_2 &= \left(\mathbf{E}_2^{(1)} - \tilde{\mathbf{A}}_0\mathbf{E}_2^{(0)} - \mathbf{E}_1^{(1)}\mathbf{S}^{(0)-1}\mathbf{E}_1^{(0)} + \tilde{\mathbf{A}}_0\mathbf{E}_1^{(0)}\mathbf{S}^{(0)-1}\mathbf{E}_1^{(0)}\right)\mathbf{S}^{(0)^{-1}}\,.\end{aligned}$$

Then:

$$\mathbf{h}^{(\ell)} = \boldsymbol{\alpha}_0^{(\ell-1)}\left(\ell\sigma\mathbf{Q}_1 + \ell\sigma^2\mathbf{Q}_2 + \sigma^2\frac{\ell(\ell-1)}{2}\mathbf{Q}_1\tilde{\mathbf{A}}_0^{-1}\mathbf{Q}_1 + \mathbf{o}(\sigma^2)\right)\ .$$

Furthermore, from equation (7) we have:

$$\boldsymbol{\mu}[\ell] = \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K + \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[N-1+\ell]$$

with:

$$\begin{aligned}
\mathbb{E}\left\{\mathbf{h}^{(\ell)}\right\} &= \ell\sigma^2 \boldsymbol{\alpha}_0^{(\ell-1)} \left(\mathbb{E}\{\mathbf{Q}_2\} + \frac{(\ell-1)}{2}\mathbb{E}\{\mathbf{Q}_1 \tilde{\mathbf{A}}_0^{-1}\mathbf{Q}_1\}\right) + \mathbf{o}(\sigma^2) \\
\sigma\mathbb{E}\left\{\mathbf{h}^{(\ell)}\mathbf{w}_K\right\} &= \ell\sigma^2 \boldsymbol{\alpha}_0^{(\ell-1)} \mathbb{E}\{\mathbf{Q}_1\mathbf{w}_K\} + \mathbf{o}(\sigma^2)
\end{aligned}$$

Then:

$$\boldsymbol{\mu}[\ell] = \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K - \mathbf{z}[N-1+\ell] + \ell\sigma^2 \boldsymbol{\alpha}_0^{(\ell-1)}\mathbf{u} + o(\sigma^2) \tag{13}$$

where

$$\mathbf{u} = \mathbb{E}\left\{\mathbf{Q}_1\mathbf{w}_K\right\} - \left(\mathbb{E}\{\mathbf{Q}_2\} + \frac{(\ell-1)}{2}\mathbb{E}\{\mathbf{Q}_1\tilde{\mathbf{A}}_0^{-1}\mathbf{Q}_1\}\right)\mathbf{z}_K\,. \tag{14}$$

Besides

$$\begin{aligned}
\gamma[\ell,\ell] =& -\mathbf{z}[N-1+\ell]^2 - 2\mathbf{z}[N-1+\ell]\boldsymbol{\mu}[\ell] + (\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K)^2 + 2\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \mathbb{E}\{(\mathbf{h}^{(\ell)}\mathbf{z}_K)^2\} \\
&+ 2\sigma\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} + 2\sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{z}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} + 2\sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{z}_K\boldsymbol{\alpha}_0^{(\ell)}\mathbf{w}_K\} + \sigma^2\mathbb{E}\{(\mathbf{h}^{(\ell)}\mathbf{w}_K)^2\} \\
&+ 2\sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} + \sigma^2\|\boldsymbol{\alpha}_0^{(\ell)}\|^2 - \boldsymbol{\mu}[\ell]^2 \\
=& \, 2\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\mathbf{z}[N-1+\ell] - (\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K)^2 + \mathbf{z}[N-1+\ell]^2 + 2\left(\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K - \mathbf{z}[N-1+\ell]\right)\boldsymbol{\mu}[\ell] \\
&+ \mathbb{E}\{(\mathbf{h}^{(\ell)}\mathbf{z}_K)^2\} + 2\sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{z}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} + 2\sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{z}_K\boldsymbol{\alpha}_0^{(\ell)}\mathbf{w}_K\} + \sigma^2\mathbb{E}\{(\mathbf{h}^{(\ell)}\mathbf{w}_K)^2\} \\
&+ 2\sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} + \sigma^2\|\boldsymbol{\alpha}_0^{(\ell)}\|^2 - \boldsymbol{\mu}[\ell]^2
\end{aligned}$$

Therefore:

$$\gamma[\ell,\ell] = \sigma^2\mathbf{v}[\ell] + o(\sigma^2) \tag{15}$$

where

$$\mathbf{v}[\ell] = \|\boldsymbol{\alpha}_0^{(\ell)}\|^2 + 2\ell\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\boldsymbol{\alpha}_0^{(\ell-1)}\mathbf{Q}_1\}\mathbf{z}_K + \ell^2\mathbb{E}\{(\boldsymbol{\alpha}_0^{(\ell-1)}\mathbf{Q}_1\mathbf{z}_K)^2\} \tag{16}$$

3.3 Sum of sine waves

In that section, the deterministic part of the observed signal is assumed to be a multicomponent harmonic signal, that is a sum of sine waves. Then:

$$\mathbf{z}[n] = \sum_{j=0}^{J}\Omega_j \cos\left(2\pi f_j \frac{n}{f_\mathsf{s}}\right)\,,$$

where J denotes the number of components, Ω_j the amplitude of the j-th component, and f_j its frequency.

Assumption: $f_j = \dfrac{p_j}{M} f_\mathsf{s} = \dfrac{p'_j}{K} f_\mathsf{s}$ for some p_j, $p'_j \in \mathbb{N}^*$.

Evaluation of $\tilde{\mathbf{A}}_0$ and $\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K$.

$$\begin{aligned}
\mathbf{S}^{(a)}[m,m'] &= \sigma^2\delta_{(m+a)m'} + \sum_{j,j'=1}^{J}\frac{\Omega_j\Omega_{j'}}{K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(N_0+m+a+k)\right)\cos\left(2\pi\frac{f_{j'}}{f_\mathsf{s}}(N_0+m'+k)\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right)+\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(2k+m+a+m'+2N_0)\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\left(\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right)+\frac{\Omega_j^2}{2K}\underbrace{\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(2k+m+a+m'+2N_0)\right)}_{=0\text{ because }\frac{f_j}{f_\mathsf{s}}=\frac{p_j'}{K}}\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right)\ .
\end{aligned}$$

Thus, $\mathbf{S}^{(0)}$ is a circulant matrix and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U}\boldsymbol{\Lambda}^{(0)}\mathbf{U}^*\ ,$$

where $\mathbf{U}[m,m'] = \frac{1}{\sqrt{M}}e^{-2\mathrm{i}\pi mm'/M}$ and $\boldsymbol{\Lambda}^{(0)} = \mathrm{diag}(\lambda_0^{(0)},\ldots,\lambda_{M-1}^{(0)})$ with:

$$\begin{aligned}
\lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\sum_{q=0}^{M-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}q\right)e^{-2\mathrm{i}\pi qm/M}\\
&= \sigma^2 + \frac{M}{4}\sum_{j=1}^{J}\Omega_j^2(\delta_{m,p_j}+\delta_{m,M-p_j})\ .
\end{aligned}$$

Therefore:

$$\mathbf{S}^{(0)^{-1}} = \mathbf{U}\boldsymbol{\Lambda}^{(0)^{-1}}\mathbf{U}^*$$

which leads to:

$$\mathbf{S}^{(0)^{-1}}[m,m'] = \frac{1}{\sigma^2}\delta_{m,m'} - \sum_{j=1}^{J}\frac{\Omega_j^2}{2\sigma^2(\sigma^2+\Omega_j^2M/4)}\cos\left(2\pi p_j\frac{m-m'}{M}\right)\ ,$$

and, consequently:

$$\begin{aligned}
\tilde{\mathbf{A}}_0[m,m'] &= \sum_{q=0}^{M-1}\mathbf{S}^{(1)}[m,q]\mathbf{S}^{(0)^{-1}}[q,m']\\
&= \delta_{m+1,m'} + \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m'}{M}\right)\delta_{m+1,M} \qquad (17)
\end{aligned}$$

Thus:

$$\begin{aligned}
\tilde{\boldsymbol{\alpha}}_0^{(1)}[m] &= \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m}{M}\right)\\
&= \frac{2}{M}\sum_{j=1}^{J}\cos\left(2\pi p_j\frac{m}{M}\right)+o(\sigma^2)\ .
\end{aligned}$$

Besides, from equation (17), we have

$$\tilde{\mathbf{A}}_0 \mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N-M+1] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K \end{pmatrix}$$

By induction, we have:

$$\tilde{\mathbf{A}}_0^{\ell} \mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N-M+\ell] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K \\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K \end{pmatrix}$$

Then:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K &= \tilde{\boldsymbol{\alpha}}_0^{(1)} \tilde{\mathbf{A}}_0^{\ell-1} \mathbf{z}_K \\ &= \sum_{m=0}^{M-\ell} \boldsymbol{\alpha}_0^{(1)}[m] \mathbf{z}[N-M+\ell+m-1] + \sum_{m=M-\ell+1}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m] \boldsymbol{\alpha}_0^{(m-M+\ell)} \mathbf{z}_K \end{aligned} \tag{18}$$

But:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K &= \sum_{m=0}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m] \mathbf{z}[N-M+m] \\ &= \sum_{j,j'=1}^{J} \Omega_{j'} \frac{2}{M} \underbrace{\sum_{m=0}^{M-1} \cos\left(2\pi p_j \frac{m}{M}\right) \cos\left(2\pi p_{j'} \frac{N+m}{M}\right)}_{=\delta_{j,j'} \frac{2}{M} \cos\left(2\pi p_j \frac{N}{M}\right)} + o(\sigma^2) \\ &= \sum_{j=1}^{J} \Omega_j \cos\left(2\pi p_j \frac{N}{M}\right) + o(\sigma^2) \\ &= \mathbf{z}[N] + o(\sigma^2) \end{aligned}$$

and, by induction from (18):

$$\boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K = \mathbf{z}[N-1+\ell] + o(\sigma^2) \tag{19}$$

Evaluations of u and v. Because $\mathbb{E}\{\mathbf{Q}_2\} = -\mathbb{E}\{\mathbf{Q}_1 \mathbf{E}_1^{(0)}\} {\mathbf{S}^{(0)}}^{-1}$, we only need derive $\mathbf{Q}_1$ in order to determine $\mathbf{u}$ and $\mathbf{v}$ (see equations (14) and (16)). First, from equation (17) we have:

$$\begin{aligned} \mathbf{Q}_1 &= \left(\mathbf{E}_1^{(1)} - \tilde{\mathbf{A}}_0 \mathbf{E}_1^{(0)}\right) {\mathbf{S}^{(0)}}^{-1} \\ &= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{q} {\mathbf{S}^{(0)}}^{-1} \end{pmatrix} \end{aligned}$$

where $\mathbf{q} = \mathbf{e}_M^T \mathbf{E}_1^{(1)} - \boldsymbol{\alpha}_0^{(1)} \mathbf{E}_1^{(0)}$. Then $\mathbf{q} = \sum_{j=1}^{J} \mathbf{q}_j$ with:

$$
\begin{aligned}
\mathbf{q}_j[m] &= \frac{\Omega_j}{K} \sum_{k=0}^{K-1} \cos\left(2\pi p_j \frac{N-K+k}{M}\right) \mathbf{w}[N_0+m+k] + \mathbf{w}[N-K+k] \cos\left(2\pi p_j \frac{N_0+m+k}{M}\right) \\
&- \sum_{q=0}^{M-1} \frac{2}{M} \cos\left(2\pi p_j \frac{q}{M}\right) \frac{\Omega_j}{K} \sum_{k=0}^{K-1} \cos\left(2\pi p_j \frac{N-K+q+k}{M}\right) \mathbf{w}[N_0+m+k] + \mathbf{w}[N_0+q+k] \cos\left(2\pi p_j \frac{N-K+m+k}{M}\right) \\
&= \frac{\Omega_j}{K} \sum_{k=0}^{K-1} \left(\cos\left(2\pi p_j \frac{N-K+k}{M}\right) - \frac{2}{M} \underbrace{\sum_{q=0}^{M-1} \cos\left(2\pi p_j \frac{q}{M}\right) \cos\left(2\pi p_j \frac{N-K+q+k}{M}\right)}_{=\frac{M}{2}\cos\left(2\pi p_j \frac{M-K+k}{M}\right)} \right) \mathbf{w}[N_0+m+k] \\
&+ \frac{\Omega_j}{K} \sum_{k=0}^{K-1} \underbrace{\left(\mathbf{w}[N-K+k] - \frac{2}{M} \sum_{q=0}^{M-1} \cos\left(2\pi p_j \frac{q}{M}\right) \mathbf{w}[N_0+q+k] \right)}_{\triangleq \Delta_j[k]} \cos\left(2\pi p_j \frac{N_0+m+k}{M}\right) \\
&= \frac{\Omega_j}{K} \sum_{k=0}^{K-1} \Delta_j[k] \cos\left(2\pi p_j \frac{N_0+m+k}{M}\right)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\mathbf{q}_j {\mathbf{S}^{(0)}}^{-1}[m] &= \sum_{q=0}^{M-1} \mathbf{q}_j[q] {\mathbf{S}^{(0)}}^{-1}[q,m] \\
&= \frac{1}{\sigma^2} \mathbf{q}_j[m] - \frac{2}{M\sigma^2} \sum_{j'=1}^{J} \sum_{q=0}^{M-1} \mathbf{q}_j[q] \cos\left(2\pi p_{j'} \frac{q-m}{M}\right) + o(\sigma^2) \\
&= \frac{1}{\sigma^2} \mathbf{q}_j[m] - \frac{\Omega}{\sigma^2 K} \sum_{k=0}^{K-1} \Delta_{\mathbf{w}}[k] \sum_{j=1}^{J} \underbrace{\left(\frac{2}{M} \sum_{q=0}^{M-1} \cos\left(2\pi p_{j'} \frac{N_0+q+k}{M}\right) \cos\left(2\pi p_j \frac{q-m}{M}\right) \right)}_{=\delta_{j,j'} \cos\left(2\pi p_j \frac{N_0+m+q}{M}\right)} + o(\sigma^2) \\
&= \frac{1}{\sigma^2} \mathbf{q}_j[m] - \frac{1}{\sigma^2} \mathbf{q}_j[m] + o(\sigma^2) \\
&= o(\sigma^2)\,.
\end{aligned}
$$

Thus $\mathbf{Q}_1 = \mathbf{o}(\sigma^2)$ and $\mathbb{E}\{\mathbf{Q}_2\} = \mathbf{o}(\sigma^2)$. Consequently:

$$
\mathbf{u} = \mathbf{o}(\sigma^2) \tag{20}
$$

$$
\mathbf{v} = \|\boldsymbol{\alpha}_0^{(\ell)}\|^2 + \mathbf{o}(\sigma^2) \tag{21}
$$

Back to the forecasting error. From equations (13) and (15) combined with results (19), (20) and (21), we get:

$$
\begin{aligned}
\boldsymbol{\mu}[\ell] &= o(\sigma^2) \\
\boldsymbol{\gamma}[\ell,\ell] &= \sigma^2 \|\boldsymbol{\alpha}_0^{(\ell)}\|^2 + o(\sigma^2)
\end{aligned}
$$

Besides, when $\ell = 1$ we have:

$$
\|\boldsymbol{\alpha}_0^{(1)}\|^2 = \frac{4}{M^2} \sum_{j=1}^{J} \sum_{m=0}^{M-1} \cos\left(2\pi p_j \frac{m}{M}\right)^2 = J \frac{2}{M}
$$

Nevertheless, when $\ell \neq 1$, we cannot determine a closed-form expression for $\boldsymbol{\alpha}_0^{(\ell)}$ but we have a recurrence equation:

$$
\begin{aligned}
\boldsymbol{\alpha}_0^{(\ell)} &= \boldsymbol{\alpha}^{(1)} \tilde{\mathbf{A}}_0^{\ell-1} \\
&= \boldsymbol{\alpha}^{(1)} \begin{pmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & & & \ddots & 1 & \ddots & \vdots \\
\vdots & & & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\
 & & & \boldsymbol{\alpha}_0^{(1)} & & & \\
 & & & \vdots & & & \\
 & & & \boldsymbol{\alpha}_0^{(\ell-1)} & & &
\end{pmatrix} \\
&= \boldsymbol{\alpha}_0^{(1)\sharp(\ell-1)} + \sum_{q=1}^{\ell-1} \boldsymbol{\alpha}_0^{(1)}[M-q] \boldsymbol{\alpha}_0^{(\ell-q)}
\end{aligned}
$$

where

$$
\boldsymbol{\alpha}_0^{(1)\sharp(n)} = \begin{pmatrix} 0 & \cdots & 0 & \boldsymbol{\alpha}_0^{(1)}[0] & \cdots & \boldsymbol{\alpha}_0^{(1)}[M-1-n] \end{pmatrix}
$$

Then:

$$
\begin{aligned}
\|\boldsymbol{\alpha}_0^{(\ell)}\| &\leq \|\boldsymbol{\alpha}_0^{(1)\sharp(\ell-1)}\| + \sum_{q=1}^{\ell-1} \underbrace{|\boldsymbol{\alpha}_0^{(1)}[M-q]|}_{\leq J\frac{2}{M}} \|\boldsymbol{\alpha}_0^{(\ell-q)}\| \\
&\leq \|\boldsymbol{\alpha}_0^{(1)}\| + \sum_{q=1}^{\ell-1} \frac{2J}{M} \|\boldsymbol{\alpha}_0^{(\ell-q)}\| = \sqrt{\frac{2J}{M}} + \frac{2J}{M} \sum_{q=1}^{\ell-1} \|\boldsymbol{\alpha}_0^{(q)}\|
\end{aligned}
$$

By induction, we deduce from the previous inequality that

$$
\|\boldsymbol{\alpha}_0^{(\ell)}\| \leq \sqrt{\frac{2J}{M}} \left(1 + \frac{2J}{M}\right)^{\ell-1}
$$

Finally:

$$
\mathbf{v}[\ell] \leq \frac{2J}{M} \left(1 + \frac{2J}{M}\right)^{2\ell-2}
$$

Thus:

$$
\boldsymbol{\gamma}[\ell,\ell] \leq \sigma^2 \frac{2J}{M} \left(1 + \frac{J}{M}\right)^{2\ell-2} + o(\sigma^2)
$$

Conclusion:

$$
\begin{aligned}
\boldsymbol{\mu}[\ell] &= o(\sigma^2) \\
|\boldsymbol{\gamma}[\ell,\lambda]| &\leq \sigma^2 \frac{2J}{M} \left(1 + \frac{J}{M}\right)^{\ell+\lambda-2} + o(\sigma^2)
\end{aligned}
$$

More generally, the estimated extended signal $\tilde{\mathbf{x}}$ takes the following form:

$$
\tilde{\mathbf{x}}[n] = \mathbf{z}[n] + \boldsymbol{\rho}[n], \quad \forall\, n \in \{0, \dots, N-1+L\}\,.
$$

where:

$$
\mathbb{E}\{\boldsymbol{\rho}[n]\} = \begin{cases} 0 & \text{if} \quad n \in \{0, \dots, N-1\} \\ o(\sigma^2) & \text{else,} \end{cases}
$$

and:

$$
\begin{cases}
\mathrm{Cov}\left(\boldsymbol{\rho}[n], \boldsymbol{\rho}[n']\right) = \sigma^2 & \text{if} \quad (n, n') \in \{0, \ldots, N-1\}^2 \\
\mathrm{Cov}\left(\boldsymbol{\rho}[n], \boldsymbol{\rho}[n']\right) \leq \sigma^2 \dfrac{2J}{M}\left(1 + \dfrac{J}{M}\right)^{(n-N)^+ + (n'-N)^+} + o(\sigma^2) & \text{if} \begin{cases} (n \geq N) \cap (n' \geq N - M) \\ (n' \geq N) \cap (n \geq N - M) \end{cases} \\
\mathrm{Cov}\left(\boldsymbol{\rho}[n], \boldsymbol{\rho}[n']\right) \leq \sigma^2 \dfrac{2J}{M}\left(1 + \dfrac{J}{M}\right)^{-(n)^- - (n')^-} + o(\sigma^2) & \text{if} \begin{cases} (n < 0) \cap (n' \leq M - 1) \\ (n' < 0) \cap (n \leq M - 1) \end{cases} \\
\mathrm{Cov}\left(\boldsymbol{\rho}[n], \boldsymbol{\rho}[n']\right) = 0 & \text{else} ,
\end{cases}
$$

where $(n)^+$ (resp. $(n)^-$) denotes the positive part (resp. the negative part) of n.

3.4 Adaptive Harmonic model

In this section, the the deterministic part of the observed signal we handle follows the *adaptive harmonic model*, which in its is continuous-time version signal takes the following form:

$$z(t) = \sum_{j=1}^{J} a_j(t) \cos(2\pi\phi_j(t)) , \tag{22}$$

where a_p and ϕ_p' are smooth function. In other terms, we have:

$$|a_j'(t)| < \varepsilon_a, \quad \forall t \in \mathbb{R} , \tag{23}$$

$$|\phi_j''(t)| < \varepsilon_\phi, \quad \forall t \in \mathbb{R} , \tag{24}$$

for some positive constants ε_a and ε_ϕ.