An Efficient Forecasting Approach for the Real-Time Reduction of Boundary Effects in Time-Frequency Representations: Part II

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I. Proof of Lemma 1

A. Notations

By definition the matrix $\mathbf{X}$ and $\mathbf{Y}$ and because of the noisy signal model (13), we have:

$$\frac{1}{K}\mathbf{X}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}^T + \sigma^2\mathbf{I}}_{\triangleq S^{(0)}} + \mathbf{E}^{(0)} \tag{28}$$

$$\frac{1}{K}\mathbf{Y}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}'\mathbf{Z}^T + \sigma^2\mathbf{D}}_{\triangleq S^{(1)}} + \mathbf{E}^{(1)} \ , \tag{29}$$

where $\mathbf{E}^{(a)} = \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$, with:

$$\mathbf{E}_1^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{z}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] + \mathbf{w}[N_0+m+a+k]\mathbf{z}[N_0+m'+k] \ ,$$

and

$$\mathbf{E}_2^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{w}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] - \delta_{(m+a)m'} \ ,$$

with $a \in \{0,1\}$.

Remark 1. *The matrices $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ are said to be error matrices because:*

$$\mathbb{E}\{\mathbf{E}^{(0)}\} = \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0}$$
$$\mathbb{E}\{\mathbf{E}^{(1)}\} = \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0} \ .$$

It follows from this:

$$\tilde{\mathbf{A}} = (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}$$
$$\mathbf{A}_0 = \mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}} \ .$$

Then:

$$\begin{aligned}\mathbf{h}^{(\ell)} &= \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)} \\ &= \mathbf{e}_M^T\left(\tilde{\mathbf{A}}^\ell - \mathbf{A}_0^\ell\right) \\ &= \mathbf{e}_M^T\left(\left((\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\right)^\ell - \mathbf{A}_0^\ell\right) \ .\end{aligned} \tag{30}$$

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B. Study of $\mathbf{h}^{(\ell)}$

The randomness of $\mathbf{h}^{(\ell)}$ is completely originating from the error matrices. Besides, notice that the first $M-1$ rows in $\mathbf{E}^{(1)}$ are equals to the last $M-1$ rows of $\mathbf{E}^{(0)}$. Thus, we gather all the sources of randomness into an vector $\mathbf{g} \in \mathbb{R}^{M(M+1)}$, containing M rows and defined as:

$$\mathbf{g} = \mathrm{vec}\left(\begin{pmatrix} \mathbf{E}^{(0)} \\ \mathbf{e}_M^T \mathbf{E}^{(1)} \end{pmatrix}\right) .$$

Here, "vec" denotes the vectorization operator, concatenating the columns of a given matrix on top of one another. Then, by definition, we have:

$$\mathbf{g} = \sigma \mathbf{g}_1 + \sigma^2 \mathbf{g}_2 ,$$

where:

$$\mathbf{g}_1 = \frac{1}{K}\sum_{k=0}^{K-1} \mathrm{vec}\left(\tilde{\mathbf{z}}_k \mathbf{w}_k^T + \tilde{\mathbf{w}}_k \mathbf{z}_k^T\right)$$
$$\mathbf{g}_2 = \frac{1}{K}\sum_{k=0}^{K-1} \mathrm{vec}\left(\tilde{\mathbf{w}}_k \mathbf{w}_k^T - \tilde{\mathbf{I}}\right) .$$

with $\tilde{\mathbf{z}}_k^T = \begin{pmatrix}\mathbf{z}_k^T & \mathbf{z}_{k+1}[M-1]\end{pmatrix}$ and $\tilde{\mathbf{w}}_k^T = \begin{pmatrix}\mathbf{w}_k^T & \mathbf{w}_{k+1}[M-1]\end{pmatrix}$. Then, $\mathbf{g}_1$ is a Gaussian random vector because it is a linear combination of Gaussian random vectors. Moreover, using the central limit theorem under weak dependence, we can show that $\mathbf{g}_2$ also converges towards a Gaussian random vector as $K \to \infty$. Combining these two results gives the following result:

$$\sqrt{K}\ \mathbf{g} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_0) ,$$

where $\mathbf{\Gamma}_0 = \mathbb{E}\left\{\mathbf{g}\mathbf{g}^T\right\}$ is a covariance matrix.

Furthermore, one can write $\mathbf{h}^{(\ell)}$ as $\mathbf{h}^{(\ell)} = f^{(\ell)}(\mathbf{g})$ where $f^{(\ell)}$ is a deterministic function such that:

$$\begin{aligned} f^{(\ell)} : \mathbb{R}^{M(M+1)} &\to \mathbb{R}^M \\ \mathbf{g} &\mapsto \mathbf{h}^{(\ell)} . \end{aligned}$$

Then, as $f^{(\ell)}$ is a differentiable function, using the Delta method gives:

$$\sqrt{K}\ \mathbf{h}^{(\ell)} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, {\mathbf{F}^{(\ell)}}^T \mathbf{\Gamma}_0 \mathbf{F}^{(\ell)}) ,$$

where $\mathbf{F}^{(\ell)}$ is the Jacobian matrix such that:

$$\mathbf{F}^{(\ell)}[m, m'] = \left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[m']}\right|_{\mathbf{g}=\mathbf{0}} .$$

II. Proof of Theorem 1

A. Expression of the Bias $\boldsymbol{\mu}$.

Clearly, $\boldsymbol{\mu}[n] = 0$ when $n \in I$. When $n = N-1+\ell$, we have:

$$\begin{aligned} \boldsymbol{\mu}[n] &= \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[n] \\ &= \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K + \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[N-1+\ell] \end{aligned}$$

Let us first evaluate the expression of $\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K$. We have:

$$
\begin{aligned}
\mathbf{S}^{(a)}[m,m'] &= \sigma^2\delta_{(m+a)m'} + \sum_{j,j'=1}^{J}\frac{\Omega_j\Omega_{j'}}{K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(N_0+m+a+k)+\varphi_j\right)\cos\left(2\pi\frac{f_{j'}}{f_\mathsf{s}}(N_0+m'+k)+\varphi_{j'}\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right)+\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(2k+m+a+m'+2N_0)\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\left(\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right)+\frac{\Omega_j^2}{2K}\underbrace{\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(2k+m+a+m'+2N_0)\right)}_{=0\text{ because }\frac{f_j}{f_\mathsf{s}}=\frac{p_j'}{K}}\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right)\ .
\end{aligned}
$$

Thus, $\mathbf{S}^{(0)}$ is a circulant matrix and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U}\boldsymbol{\Lambda}^{(0)}\mathbf{U}^*\ ,$$

where $\mathbf{U}[m,m'] = \frac{1}{\sqrt{M}}e^{-2\mathrm{i}\pi mm'/M}$ and $\boldsymbol{\Lambda}^{(0)} = \mathrm{diag}(\lambda_0^{(0)},\dots,\lambda_{M-1}^{(0)})$ with:

$$
\begin{aligned}
\lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\sum_{q=0}^{M-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}q\right)e^{-2\mathrm{i}\pi qm/M}\\
&= \sigma^2 + \frac{M}{4}\sum_{j=1}^{J}\Omega_j^2(\delta_{m,p_j}+\delta_{m,M-p_j})\ .
\end{aligned}
$$

Therefore:

$$\mathbf{S}^{(0)^{-1}} = \mathbf{U}\boldsymbol{\Lambda}^{(0)^{-1}}\mathbf{U}^*$$

which leads to:

$$\mathbf{S}^{(0)^{-1}}[m,m'] = \frac{1}{\sigma^2}\delta_{m,m'} - \sum_{j=1}^{J}\frac{\Omega_j^2}{2\sigma^2(\sigma^2+\Omega_j^2M/4)}\cos\left(2\pi p_j\frac{m-m'}{M}\right)\ ,$$

and, consequently:

$$
\begin{aligned}
\tilde{\mathbf{A}}_0[m,m'] &= \sum_{q=0}^{M-1}\mathbf{S}^{(1)}[m,q]\mathbf{S}^{(0)^{-1}}[q,m']\\
&= \delta_{m+1,m'} + \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m'}{M}\right)\delta_{m+1,M} \qquad (31)
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\tilde{\boldsymbol{\alpha}}_0^{(1)}[m] &= \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m}{M}\right)\\
&= \frac{2}{M}\sum_{j=1}^{J}\cos\left(2\pi p_j\frac{m}{M}\right)+o(\sigma)\ .
\end{aligned}
$$

Besides, from equation (31), we have

$$\tilde{\mathbf{A}}_0\mathbf{z}_K = \begin{pmatrix}\mathbf{z}[N-M+1]\\ \vdots\\ \mathbf{z}[N-1]\\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K\end{pmatrix}$$

By induction, we have:

$$\tilde{\mathbf{A}}_0^{\ell}\mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N-M+\ell] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K \\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K \end{pmatrix} .$$

Then:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K &= \tilde{\boldsymbol{\alpha}}_0^{(1)}\tilde{\mathbf{A}}_0^{\ell-1}\mathbf{z}_K \\ &= \sum_{m=0}^{M-\ell}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] + \sum_{m=M-\ell+1}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\boldsymbol{\alpha}_0^{(m-M+\ell)}\mathbf{z}_K \end{aligned} \tag{32}$$

But:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K &= \sum_{m=0}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+m] \\ &= \sum_{j,j'=1}^{J}\Omega_{j'}\frac{2}{M}\underbrace{\sum_{m=0}^{M-1}\cos\left(2\pi p_j\frac{m}{M}\right)\cos\left(2\pi p_{j'}\frac{N+m}{M}+\varphi_{j'}\right)}_{=\delta_{j,j'}\frac{M}{2}\cos\left(2\pi p_j\frac{N}{M}+\varphi_j\right)} + o(\sigma) \\ &= \sum_{j=1}^{J}\Omega_j\cos\left(2\pi p_j\frac{N}{M}+\varphi_j\right) + o(\sigma) \\ &= \mathbf{z}[N] + o(\sigma) \end{aligned}$$

and, by induction from (32):

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K = \mathbf{z}[N-1+\ell] + o(\sigma) \tag{33}$$

Then:

$$\boldsymbol{\mu}[N-1+\ell] = \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} + o(\sigma) .$$

Besides, from Lemma 1, we have the following results:

$$\begin{aligned} \mathbb{E}\{\mathbf{h}^{(\ell)}\} &\underset{K\to\infty}{\longrightarrow} 0 \\ \mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} &\underset{K\to\infty}{\longrightarrow} 0 \end{aligned}$$

Consequently:

$$\boldsymbol{\mu}[N-1+\ell] \underset{K\to\infty}{\sim} o(\sigma) . \tag{34}$$

B. Expression of the Covariance $\boldsymbol{\gamma}$.

We recall that $n > N$, we consequently denote $n = N-1+\ell$. To derive the covariance, let us segregate the cases.

a) If $n' \in I^2$: From Lemma 1, we have that $h^{(\ell)}$ is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. Then, combining result (34) and equation (15), we obtain:

$$\boldsymbol{\gamma}[n,n'] = \sigma\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} + o(\sigma^2) .$$

We remark that

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} = \boldsymbol{\alpha}_0^{(\ell)}[n'-(N-M)]\mathbb{1}_{(n'\geq N-M)} .$$

Moreover, using Delta method one can show that we have the asymptotic result:

$$\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\} \underset{K\to\infty}{\longrightarrow} 0 \tag{35}$$

Finally:

$$\boldsymbol{\gamma}[n,n'] \underset{K\to\infty}{\sim} \sigma^2\boldsymbol{\alpha}_0^{(\ell)}[n'-(N-M)]\mathbb{1}_{(n'\geq N-M)} + o(\sigma^2) .$$

TABLE IV
ECG: Performance of the Boundary-Free TF Representations According to the Extension Method

Extension method	Averaged performance index D			
	STFT	SST	ConceFT	RS
SigExt	0.713	0.700	0.753	0.761
Symmetric	1.395	1.402	1.416	0.976
EDMD	0.856	0.792	0.559	0.688
GPR	0.723	0.734	0.734	0.761

b) If $n' \geq N$: From Lemma 1, we have that $h^{(\ell)}$ is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. Then, combining result (34) and equations (17) and (35), we obtain:

$$\begin{aligned}
\gamma[n,n'] &= \mathbf{z}_K^T \mathbb{E}\left\{\boldsymbol{\alpha}^{(\ell)^T}\boldsymbol{\alpha}^{(\lambda)}\right\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\boldsymbol{\alpha}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\lambda)}\mathbf{w}_K\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K \\
&\quad + \sigma^2\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\boldsymbol{\alpha}^{(\lambda)}\mathbf{w}_K\} - \mathbf{z}[n]\mathbf{z}[n'] + o(\sigma^2) \\
&= \mathbf{z}_K^T \mathbb{E}\left\{\mathbf{h}^{(\ell)^T}\mathbf{h}^{(\lambda)}\right\}\mathbf{z}_K + \boldsymbol{\alpha}_0^{(\ell)}\sigma\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\}\mathbf{z}[n'] \\
&\quad + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\lambda)}\mathbf{w}_K\}\mathbf{z}[n] + \sigma\boldsymbol{\alpha}_0^{(\lambda)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K \\
&\quad + \sigma\mathbb{E}\{\mathbf{h}^{(\lambda)}\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + \sigma^2\boldsymbol{\alpha}_0^{(\lambda)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} \\
&\quad + \sigma^2\left\langle\boldsymbol{\alpha}^{(\ell)},\boldsymbol{\alpha}^{(\lambda)}\right\rangle + \sigma^2\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + o(\sigma^2) \\
&\underset{K\to\infty}{\sim} \frac{1}{K}\mathbf{z}_K^T\boldsymbol{\Gamma}^{(\ell,\lambda)}\mathbf{z}_K + \sigma^2\left\langle\boldsymbol{\alpha}^{(\ell)},\boldsymbol{\alpha}^{(\lambda)}\right\rangle + \sigma^2\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + o(\sigma^2)
\end{aligned}$$

The Isserlis' theorem also gives the following result:

$$\begin{aligned}
\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} &= \sum_{m,m'=0}^{M-1}\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m]\mathbf{h}^{(\lambda)}[m']\mathbf{w}_K[m']\} \\
&= \sum_{m,m'=0}^{M-1}\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m]\}\mathbb{E}\{\mathbf{h}^{(\lambda)}[m']\mathbf{w}_K[m']\} + \mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m']\}\mathbb{E}\{\mathbf{h}^{(\lambda)}[m']\mathbf{w}_K[m]\} \\
&\qquad + \mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{h}^{(\lambda)}[m']\}\underbrace{\mathbb{E}\{\mathbf{w}_K[m]\mathbf{w}_K[m']\}}_{=\delta_{m,m'}} \\
&\underset{K\to\infty}{\sim} \sum_{m=0}^{M-1}\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{h}^{(\lambda)}[m]\} = \frac{1}{K}\mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)}\right) .
\end{aligned}$$

We conclude that:

$$\gamma[n,n'] \underset{K\to\infty}{\sim} \frac{1}{K}\mathbf{z}_K^T\boldsymbol{\Gamma}^{(\ell,\lambda)}\mathbf{z}_K + \sigma^2\left\langle\boldsymbol{\alpha}^{(\ell)},\boldsymbol{\alpha}^{(\lambda)}\right\rangle + \frac{\sigma^2}{K}\mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)}\right) + o(\sigma^2) .$$

III. Application to an Electrocardiogram

We provide here an additional implementation of BoundEffRed, applied to an electrocardiogram (ECG) dataset. The dataset is constructed from a 500-second-long ECG, sampled at $f_s = 200$ Hz, cut into 14 segments of 35 seconds each. Fig. 8 depicts one of these subsignals, together with the estimated 5 seconds extensions obtained by SigExt, EDMD, and GPR forecastings are superimposed to the ground-truth extension. The sharp and spiky ECG patterns make the AHM model too simplistic to describe this type of signal. Consequently, the forecast produced by SigExt is moderately satisfactory.

Table IV contains the performance index D of the boundary-free TF representations, averaged over the N subsignals, according to the extension method. As a result of the fair quality of the forecasts, the reduction of boundary effects is less significant than for PPG signal. Nevertheles, the results show that BoundEffRed has the same efficiency when the SigExt extension, the EDMD extension or the GPR extension is chosen. This justifies the choice of SigExt for real-time implementation.

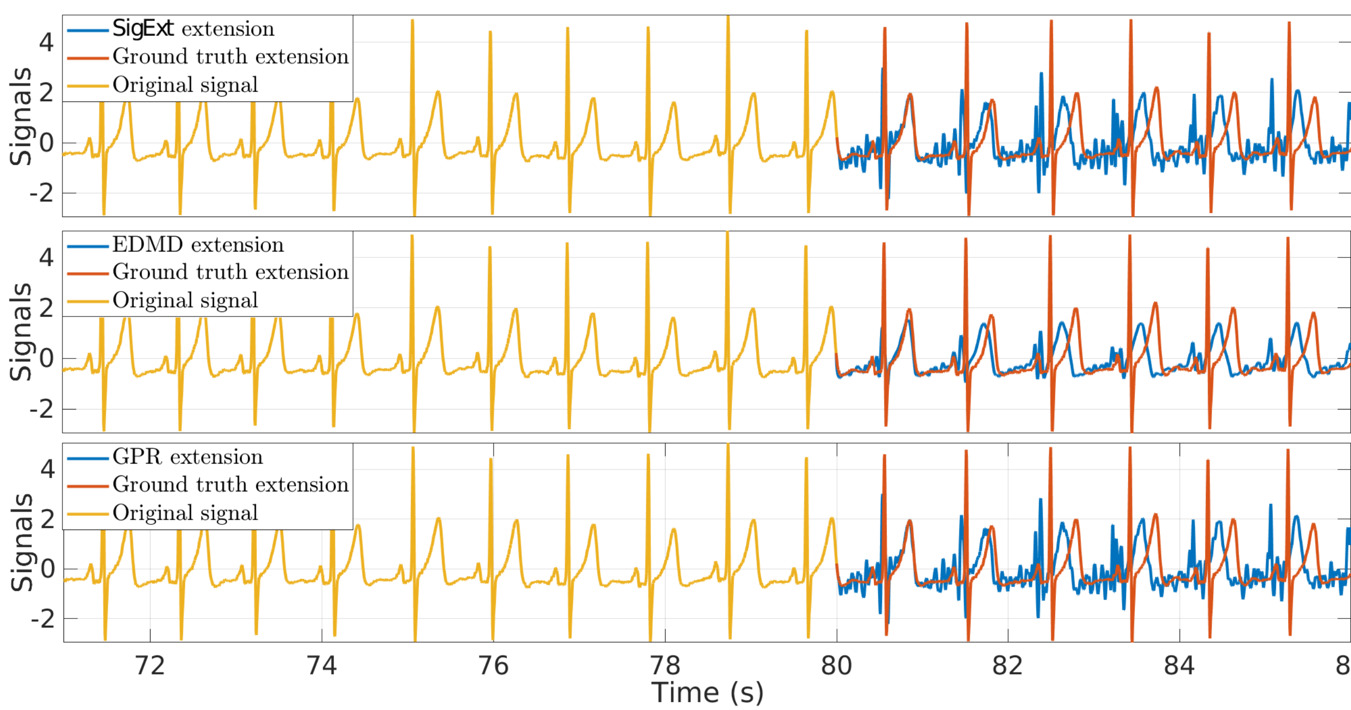

Fig. 8. Extended ECG (blue) obtained by the SigExt forecasting (top), the EDMD forecasting (middle), and the GPR forecasting (bottom), superimposed with the ground truth signal (red).