An Approach for the Reduction of Boundary Effects in Time-Frequency Representations: Part II

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I. Proof of Lemma 1

We have:

$$\frac{1}{K}\mathbf{X}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}' + \sigma^2\mathbf{I}}_{\triangleq S^{(0)}} + \mathbf{E}^{(0)} \tag{9}$$

$$\frac{1}{K}\mathbf{Y}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}^T + \sigma^2\mathbf{D}}_{\triangleq S^{(1)}} + \mathbf{E}^{(1)} \ , \tag{10}$$

where $\mathbf{E}^{(a)} = \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$ with:

$$\mathbf{E}_1^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{z}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] + \mathbf{w}[N_0+m+a+k]\mathbf{z}[N_0+m'+k] \ ,$$

and

$$\mathbf{E}_2^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{w}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] - \delta_{(m+a)m'} \ ,$$

with $a \in \{0,1\}$.

Remark 1. *The matrices $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ are said to be error matrices because:*

$$\mathbb{E}\{\mathbf{E}^{(0)}\} = \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0}$$
$$\mathbb{E}\{\mathbf{E}^{(1)}\} = \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0} \ .$$

Thus:

$$\tilde{\mathbf{A}} = (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}$$
$$\mathbf{A}_0 = \mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}}$$

Then:

$$\begin{aligned}\mathbf{h}^{(\ell)} &= \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)} \\ &= \mathbf{e}_M^T\left(\tilde{\mathbf{A}}^\ell - \tilde{\mathbf{A}}_0^\ell\right) \\ &= \mathbf{e}_M^T\left(\left((\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\right)^\ell - \tilde{\mathbf{A}}_0^\ell\right) \ .\end{aligned} \tag{11}$$

Thus, the randomness of $\mathbf{h}^{(\ell)}$ is completely originating from the random vector $\mathbf{g} \in \mathbb{R}^{M(M+1)}$ defined as:

$$\mathbf{g} = \mathrm{vec}\left(\begin{pmatrix}\mathbf{E}^{(0)} \\ \mathbf{e}_M^T\mathbf{E}^{(1)}\end{pmatrix}\right) \ .$$

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Thus, one can write $\mathbf{h}^{\ell}$ as $\mathbf{h}^{\ell} = f^{(\ell)}(g)$ where $f^{(\ell)}$ is a deterministic function:

$$\begin{aligned} f^{(\ell)} : \mathbb{R}^{M(M+1)} &\to \mathbb{R}^{M} \\ \mathbf{g} &\mapsto \mathbf{h}^{(\ell)} \ . \end{aligned}$$

Besides, by application of the central limit theorem, the random vector $\mathbf{g}$ converges in law to a Gaussian random vector:

$$\sqrt{K}\ \mathbf{g} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_0) \ .$$

Then, using Delta method, we have:

$$\sqrt{K}\ \mathbf{h}^{(\ell)} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, {\mathbf{F}^{(\ell)}}^T \mathbf{\Gamma}_0 \mathbf{F}^{(\ell)}) \ ,$$

where:

$$\mathbf{F}^{(\ell)}[m, m'] = \left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[m']}\right|_{\mathbf{g}=\mathbf{0}} \ .$$

II. Proof of Theorem 1

a) Expression of the bias $\boldsymbol{\mu}$.: Clearly, $\boldsymbol{\mu}[n] = 0$ when $n \in I$. When $n = N - 1 + \ell$, we have:

$$\begin{aligned} \boldsymbol{\mu}[n] &= \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[n] \\ &= \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K + \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[N-1+\ell] \end{aligned}$$

Let us first evaluate the expression of $\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K$. We have:

$$\begin{aligned} \mathbf{S}^{(a)}[m, m'] &= \sigma^2\delta_{(m+a)m'} + \sum_{j,j'=1}^{J} \frac{\Omega_j\Omega_{j'}}{K} \sum_{k=0}^{K-1} \cos\left(2\pi\frac{f_j}{f_\mathrm{s}}(N_0 + m + a + k)\right)\cos\left(2\pi\frac{f_{j'}}{f_\mathrm{s}}(N_0 + m' + k)\right) \\ &= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J} \frac{\Omega_j^2}{2K} \sum_{k=0}^{K-1} \cos\left(2\pi\frac{f_j}{f_\mathrm{s}}(m + a - m')\right) + \cos\left(2\pi\frac{f_j}{f_\mathrm{s}}(2k + m + a + m' + 2N_0)\right) \\ &= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\left(\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathrm{s}}(m + a - m')\right) + \underbrace{\frac{\Omega_j^2}{2K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathrm{s}}(2k + m + a + m' + 2N_0)\right)}_{=0 \text{ because } \frac{f_j}{f_\mathrm{s}} = \frac{p_j'}{K}}\right) \\ &= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J} \frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathrm{s}}(m + a - m')\right) \ . \end{aligned}$$

Thus, $\mathbf{S}^{(0)}$ is a circulant matrix and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U}\mathbf{\Lambda}^{(0)}\mathbf{U}^* \ ,$$

where $\mathbf{U}[m, m'] = \frac{1}{\sqrt{M}}e^{-2\mathrm{i}\pi mm'/M}$ and $\mathbf{\Lambda}^{(0)} = \mathrm{diag}(\lambda_0^{(0)}, \ldots, \lambda_{M-1}^{(0)})$ with:

$$\begin{aligned} \lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\sum_{q=0}^{M-1}\cos\left(2\pi\frac{f_j}{f_\mathrm{s}}q\right)e^{-2\mathrm{i}\pi qm/M} \\ &= \sigma^2 + \frac{M}{4}\sum_{j=1}^{J}\Omega_j^2(\delta_{m,p_j} + \delta_{m,M-p_j}) \ . \end{aligned}$$

Therefore:

$${\mathbf{S}^{(0)}}^{-1} = \mathbf{U}{\mathbf{\Lambda}^{(0)}}^{-1}\mathbf{U}^*$$

which leads to:

$$\mathbf{S}^{(0)^{-1}}[m,m'] = \frac{1}{\sigma^2}\delta_{m,m'} - \sum_{j=1}^{J} \frac{\Omega_j^2}{2\sigma^2(\sigma^2+\Omega_j^2 M/4)} \cos\left(2\pi p_j \frac{m-m'}{M}\right) \ ,$$

and, consequently:

$$\begin{aligned}\tilde{\mathbf{A}}_0[m,m'] &= \sum_{q=0}^{M-1} \mathbf{S}^{(1)}[m,q]\mathbf{S}^{(0)^{-1}}[q,m'] \\ &= \delta_{m+1,m'} + \sum_{j=1}^{J} \frac{2\Omega_j^2}{\Omega_j^2 M + 4\sigma^2} \cos\left(2\pi p_j \frac{m'}{M}\right) \delta_{m+1,M}\end{aligned} \tag{12}$$

Thus:

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}_0^{(1)}[m] &= \sum_{j=1}^{J} \frac{2\Omega_j^2}{\Omega_j^2 M + 4\sigma^2} \cos\left(2\pi p_j \frac{m}{M}\right) \\ &= \frac{2}{M}\sum_{j=1}^{J} \cos\left(2\pi p_j \frac{m}{M}\right) + o(\sigma) \ .\end{aligned}$$

Besides, from equation (12), we have

$$\tilde{\mathbf{A}}_0 \mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N-M+1] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K \end{pmatrix}$$

By induction, we have:

$$\tilde{\mathbf{A}}_0^{\ell} \mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N-M+\ell] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K \\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K \end{pmatrix} \ .$$

Then:

$$\begin{aligned}\boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K &= \tilde{\boldsymbol{\alpha}}_0^{(1)} \tilde{\mathbf{A}}_0^{\ell-1} \mathbf{z}_K \\ &= \sum_{m=0}^{M-\ell} \boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] + \sum_{m=M-\ell+1}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m]\boldsymbol{\alpha}_0^{(m-M+\ell)} \mathbf{z}_K\end{aligned} \tag{13}$$

But:

$$\begin{aligned}\boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K &= \sum_{m=0}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+m] \\ &= \sum_{j,j'=1}^{J} \Omega_{j'} \frac{2}{M} \underbrace{\sum_{m=0}^{M-1} \cos\left(2\pi p_j \frac{m}{M}\right) \cos\left(2\pi p_{j'} \frac{N+m}{M}\right)}_{=\delta_{j,j'} \frac{M}{2} \cos\left(2\pi p_j \frac{N}{M}\right)} + o(\sigma) \\ &= \sum_{j=1}^{J} \Omega_j \cos\left(2\pi p_j \frac{N}{M}\right) + o(\sigma) \\ &= \mathbf{z}[N] + o(\sigma)\end{aligned}$$

and, by induction from (13):

$$\boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K = \mathbf{z}[N-1+\ell] + o(\sigma) \tag{14}$$

Then:

$$\boldsymbol{\mu}[N-1+\ell] = \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} + o(\sigma) \ .$$

Besides, from Lemma 1, we have the following results:

$$\mathbb{E}\{\mathbf{h}^{(\ell)}\} \underset{K\to\infty}{=} o\left(\frac{1}{\sqrt{K}}\right)$$
$$\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} \underset{K\to\infty}{=} o\left(\frac{1}{\sqrt{K}}\right)$$

Consequently:

$$\boldsymbol{\mu}[N-1+\ell] \underset{K\to\infty}{\sim} o(\sigma) \ .$$

b) Expression of the covariance $\boldsymbol{\gamma}$.: Let us segregate the cases.
First, when $(n,n') \in I^2$, we clearly have $\boldsymbol{\gamma}[n,n'] = \sigma^2\delta_{n,m}$.
Second, we focus on the case where $n = N-1+\ell \geq N$ and $n' = N-1+\lambda \geq N$. Thus:

$$\begin{aligned}
\boldsymbol{\gamma}[n,n'] &= \mathbf{z}_K^T\mathbb{E}\left\{\boldsymbol{\alpha}^{(\ell)^T}\boldsymbol{\alpha}^{(\lambda)}\right\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\boldsymbol{\alpha}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\lambda)}\mathbf{w}_K\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K \\
&+ \sigma^2\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\boldsymbol{\alpha}^{(\lambda)}\mathbf{w}_K\} - \mathbf{z}[n]\mathbf{z}[n'] - \mathbf{z}[n]\boldsymbol{\mu}[n'] - \mathbf{z}[n]\boldsymbol{\mu}[n'] - \boldsymbol{\mu}[n]\boldsymbol{\mu}[n'] \\
&= \mathbf{z}_K^T\mathbb{E}\left\{\mathbf{h}^{(\ell)^T}\mathbf{h}^{(\lambda)}\right\}\mathbf{z}_K + \boldsymbol{\alpha}_0^{(\ell)}\sigma\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\}\mathbf{z}(n'] \\
&+ \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\lambda)}\mathbf{w}_K\}\mathbf{z}[n] + \sigma\boldsymbol{\alpha}_0^{(\lambda)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K \\
&+ \sigma\mathbb{E}\{\mathbf{h}^{(\lambda)}\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + \sigma^2\boldsymbol{\alpha}_0^{(\lambda)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} \\
&+ \sigma^2\left\langle\boldsymbol{\alpha}^{(\ell)},\boldsymbol{\alpha}^{(\ell)}\right\rangle + \sigma^2\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + o(\sigma^2)
\end{aligned}$$

From lemma 1, we have: To be continued...

Third, we focus on the case where $n = N-1+\ell \geq N$ and $n' \in I$. The case $n \in I$ and $n' = N-1+\lambda$ is directly derived from the current one. To be continued...