Supplementary material for "An Efficient Forecasting Approach to Reduce Boundary Effects in Real-Time Time-Frequency Analysis"

Adrien Meynard, Hau-Tieng Wu

I. Proof of Lemma 1

Recall the model (13). Based on the definition of matrices $\mathbf{X}$ and $\mathbf{Y}$, we have:

$$\frac{1}{K}\mathbf{X}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}^T + \sigma^2\mathbf{I}}_{\triangleq S^{(0)}} + \mathbf{E}^{(0)} \tag{28}$$

$$\frac{1}{K}\mathbf{Y}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}'\mathbf{Z}^T + \sigma^2\mathbf{D}}_{\triangleq S^{(1)}} + \mathbf{E}^{(1)}\ , \tag{29}$$

where $\mathbf{E}^{(a)} := \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$,

$$\mathbf{E}_1^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{z}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] + \mathbf{w}[N_0+m+a+k]\mathbf{z}[N_0+m'+k]\ ,$$

and

$$\mathbf{E}_2^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{w}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] - \delta_{(m+a)m'}\ ,$$

with $a \in \{0,1\}$. We call $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ *error matrices* because:

$$\mathbb{E}\{\mathbf{E}^{(0)}\} = \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0}$$
$$\mathbb{E}\{\mathbf{E}^{(1)}\} = \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0}\ .$$

Thus,

$$\mathbf{A}_0 := \mathbf{S}^{(1)}{\mathbf{S}^{(0)}}^{-1}\ , \quad \tilde{\mathbf{A}} := (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\ .$$

As a result, for $\ell \in \mathbb{N}$,

$$\begin{aligned}\mathbf{h}^{(\ell)} &= \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)}\\ &= \mathbf{e}_M^T\left(\tilde{\mathbf{A}}^\ell - \mathbf{A}_0^\ell\right)\\ &= \mathbf{e}_M^T\left(\left((\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\right)^\ell - \mathbf{A}_0^\ell\right)\ .\end{aligned} \tag{30}$$

The randomness of $\mathbf{h}^{(\ell)}$ completely comes from the error matrices. Besides, notice that the first $M-1$ rows in $\mathbf{E}^{(1)}$ equal to the last $M-1$ rows of $\mathbf{E}^{(0)}$. We gather all sources of randomness into an vector $\mathbf{g} \in \mathbb{R}^{M(M+1)}$, containing M rows defined as

$$\mathbf{g} = \mathrm{vec}\left(\begin{bmatrix}\mathbf{E}^{(0)}\\ \mathbf{e}_M^T\mathbf{E}^{(1)}\end{bmatrix}\right)\ ,$$

A. Meynard and H.-T. Wu are with the Department of Mathematics, Duke University, Durham, NC, 27708 USA e-mail: adrien.meynard@duke.edu

where "vec" denotes the vectorization operator, concatenating the columns of a given matrix on top of one another. Then, we can write $\mathbf{h}^{(\ell)}$ as $\mathbf{h}^{(\ell)} = f^{(\ell)}(\mathbf{g})$ where $f^{(\ell)}$ is a deterministic function such that:

$$f^{(\ell)} : \mathbb{R}^{M(M+1)} \to \mathbb{R}^{M}$$
$$\mathbf{g} \mapsto \mathbf{h}^{(\ell)} \ .$$

The multivariate version of the delta method is applicable to $\mathbf{h}^{(\ell)}$ under the following conditions:

(i) $f^{(\ell)}$ is differentiable at the origin;

(ii) $\sqrt{K}\ \mathbf{g} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_0)$ where $\mathbf{\Gamma}_0$ is a covariance matrix.

When these conditions are satisfied, the delta method gives the asymptotic behavior of $\mathbf{h}^{(\ell)} = f^{(\ell)}(\mathbf{g})$, that is

$$\sqrt{K}\ \mathbf{h}^{(\ell)} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, {\mathbf{F}^{(\ell)}}^T \mathbf{\Gamma}_0 \mathbf{F}^{(\ell)}) \ , \tag{31}$$

where $\mathbf{F}^{(\ell)}$ is the Jacobian matrix of $f^{(\ell)}$ at the origin, that is

$$\mathbf{F}^{(\ell)}[m, m'] = \left. \frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[m']} \right|_{\mathbf{g}=\mathbf{0}} \ .$$

Condition (i) is satisfied. Indeed, the differentiation of $f^{(\ell)}$ can be checked as a composition of standard matrix derivation rules. Detail it

Concerning condition (ii), by definition of $\mathbf{g}$, we have:

$$\mathbf{g} = \sigma \mathbf{g}_1 + \sigma^2 \mathbf{g}_2 \ ,$$

where $\mathbf{g}_1$ and $\mathbf{g}_2$ are defined by

$$\mathbf{g}_1 = \frac{1}{K} \sum_{k=0}^{K-1} \operatorname{vec}\left(\tilde{\mathbf{z}}_k \mathbf{w}_k^T + \tilde{\mathbf{w}}_k \mathbf{z}_k^T \right) \quad \text{where} \quad \tilde{\mathbf{z}}_k^T = \begin{pmatrix} \mathbf{z}_k^T & \mathbf{z}_{k+1}[M-1] \end{pmatrix} \ ,$$
$$\mathbf{g}_2 = \frac{1}{K} \sum_{k=0}^{K-1} \operatorname{vec}\left(\tilde{\mathbf{w}}_k \mathbf{w}_k^T - \mathbf{I} \right) \quad \text{where} \quad \tilde{\mathbf{w}}_k^T = \begin{pmatrix} \mathbf{w}_k^T & \mathbf{w}_{k+1}[M-1] \end{pmatrix} \ .$$

First, $\mathbf{g}_1$ is intrinsically a Gaussian random vector since it is a linear combination of Gaussian random vectors. Second, using the central limit theorem under weak dependence, we can show that $\mathbf{g}_2$ also converges towards a Gaussian random vector as $K \to \infty$. Hence, we have

$$\sqrt{K}\ \mathbf{g} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_0) \ ,$$

where $\mathbf{\Gamma}_0$ is the limit covariance matrix so that $\mathbf{\Gamma}_0 = \lim_{K\to\infty} K\, \mathbb{E}\{\mathbf{g}\mathbf{g}^T\}$. Hence, conditions (i) and (i) are satisfied. The delta method can therefore be applied, and result (31) is valid.

II. Proof of Theorem 1

A. Expression of the Bias $\boldsymbol{\mu}$.

By definition of the measurement noise, $\boldsymbol{\mu}[n] = 0$ when $n \in I$. Outside the measurement interval I, denote by ℓ the index such that $n = N - 1 + \ell$. Then, given that $\mathbf{h}^{(\ell)} = \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)}$, we have

$$\begin{aligned} \boldsymbol{\mu}[n] &= \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K\} - \mathbf{z}[n] \\ &= \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K + \mathbb{E}\{\mathbf{h}^{(\ell)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K\} - \mathbf{z}[N-1+\ell] \ . \end{aligned} \tag{32}$$

Let us first evaluate the expression of $\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K$. We have

$$\begin{aligned}
\mathbf{S}^{(a)}[m,m'] &= \sigma^2\delta_{(m+a)m'} + \sum_{j,j'=1}^{J}\frac{\Omega_j\Omega_{j'}}{K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(N_0+m+a+k)+\varphi_j\right)\cos\left(2\pi\frac{f_{j'}}{f_{\mathsf{s}}}(N_0+m'+k)+\varphi_{j'}\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(m+a-m')\right)+\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(2k+m+a+m'+2N_0)\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\left(\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(m+a-m')\right)+\frac{\Omega_j^2}{2K}\underbrace{\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(2k+m+a+m'+2N_0)\right)}_{=0 \text{ because } \frac{f_j}{f_{\mathsf{s}}}=\frac{p_j'}{K}}\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(m+a-m')\right)\ . \qquad (33)
\end{aligned}$$

Thus, $\mathbf{S}^{(0)}$ is a circulant matrix and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U}\boldsymbol{\Lambda}^{(0)}\mathbf{U}^*\ ,$$

where $\mathbf{U}[m,m'] = \frac{1}{\sqrt{M}}e^{-2\mathrm{i}\pi mm'/M}$ and $\boldsymbol{\Lambda}^{(0)} = \mathrm{diag}(\lambda_0^{(0)},\ldots,\lambda_{M-1}^{(0)})$ with

$$\begin{aligned}
\lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\sum_{q=0}^{M-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}q\right)e^{-2\mathrm{i}\pi qm/M}\\
&= \sigma^2 + \frac{M}{4}\sum_{j=1}^{J}\Omega_j^2(\delta_{m,p_j}+\delta_{m,M-p_j})\ .
\end{aligned}$$

Therefore,

$$\mathbf{S}^{(0)^{-1}} = \mathbf{U}\boldsymbol{\Lambda}^{(0)^{-1}}\mathbf{U}^*\,,$$

which leads to

$$\mathbf{S}^{(0)^{-1}}[m,m'] = \frac{1}{\sigma^2}\delta_{m,m'} - \sum_{j=1}^{J}\frac{\Omega_j^2}{2\sigma^2(\sigma^2+\Omega_j^2M/4)}\cos\left(2\pi p_j\frac{m-m'}{M}\right)\ . \qquad (34)$$

Consequently, combining equations (33) and (34), we have

$$\begin{aligned}
\mathbf{A}_0[m,m'] &= \sum_{q=0}^{M-1}\mathbf{S}^{(1)}[m,q]\mathbf{S}^{(0)^{-1}}[q,m']\\
&= \delta_{m+1,m'} + \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m'}{M}\right)\delta_{m+1,M}\,. \qquad (35)
\end{aligned}$$

Thus $\boldsymbol{\alpha}_0^{(1)}$, the last row of $\mathbf{A}_0$, is written as

$$\begin{aligned}
\boldsymbol{\alpha}_0^{(1)}[m] &= \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m}{M}\right)\\
&= \frac{2}{M}\sum_{j=1}^{J}\cos\left(2\pi p_j\frac{m}{M}\right)+o(\sigma)\ .
\end{aligned}$$

Besides, from equation (35), we have

$$\mathbf{A}_0\mathbf{z}_K = \begin{pmatrix}\mathbf{z}[N-M+1]\\ \vdots\\ \mathbf{z}[N-1]\\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K\end{pmatrix}\ .$$

By induction, we have

$$\tilde{\mathbf{A}}_0^{\ell}\mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N-M+\ell] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K \\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K \end{pmatrix} .$$

Then,

$$\begin{aligned} \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K &= \boldsymbol{\alpha}_0^{(1)}\tilde{\mathbf{A}}_0^{\ell-1}\mathbf{z}_K \\ &= \sum_{m=0}^{M-\ell}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] + \sum_{m=M-\ell+1}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\boldsymbol{\alpha}_0^{(m-M+\ell)}\mathbf{z}_K \,. \end{aligned} \tag{36}$$

But,

$$\begin{aligned} \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K &= \sum_{m=0}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+m] \\ &= \sum_{j,j'=1}^{J}\Omega_{j'}\frac{2}{M}\underbrace{\sum_{m=0}^{M-1}\cos\left(2\pi p_j\frac{m}{M}\right)\cos\left(2\pi p_{j'}\frac{N+m}{M}+\varphi_{j'}\right)}_{=\delta_{j,j'}\frac{M}{2}\cos\left(2\pi p_j\frac{N}{M}+\varphi_j\right)} + o(\sigma) \\ &= \sum_{j=1}^{J}\Omega_j\cos\left(2\pi p_j\frac{N}{M}+\varphi_j\right)+o(\sigma) \\ &= \mathbf{z}[N]+o(\sigma) \end{aligned}$$

By induction from (36), we have

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K = \mathbf{z}[N-1+\ell]+o(\sigma) \ . \tag{37}$$

Then, inserting result (37) into equation (32) gives

$$\boldsymbol{\mu}[N-1+\ell] = \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\}+o(\sigma) \ .$$

Besides, from Lemma 1, we have the following result:

$$\mathbb{E}\{\mathbf{h}^{(\ell)}\} \underset{K\to\infty}{\longrightarrow} 0 \ .$$

[if it is possible to get the convergence rate, like $\mathbb{E}\{\mathbf{h}^{(\ell)}\} = O(1/K)$, that would be great!] Moreover, applying delta method to $(\mathbf{g}\ \mathbf{w}[n'])^T$, $\forall n' \in I$, one can show that we have the asymptotic result

$$\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\} \underset{K\to\infty}{\longrightarrow} 0 \,. \tag{38}$$

Consequently, $\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} \underset{K\to\infty}{\longrightarrow} 0$, and

$$\frac{1}{\sigma}\boldsymbol{\mu}[N-1+\ell] = o(1) \tag{39}$$

when $K \to \infty$.

B. Expression of the Covariance γ.

Take $n > N$, and denote $n = N-1+\ell$. To derive the covariance, let us segregate the cases.

a) When $n' \in I$: From Lemma 1, we have that $h^{(\ell)}$ is asymptotically Gaussian when $K \to \infty$. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing [there is a gap here. Note that when K is finite, it is approximated by Gaussian but not Gaussian. The discrepancy should be made clear.]. Then, inserting result (39) into equation (16), we obtain:

$$\gamma[n,n'] = \sigma\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} + o(\sigma^2) \tag{40}$$

when $K \to \infty$. In addition, we remark that

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} = \boldsymbol{\alpha}_0^{(\ell)}[n' - (N-M)]\mathbb{1}_{(n' \geq N-M)} \ .$$

Besides, according to result (38) the first term in (40) vanishes when $K \to \infty$. We thus have

$$\gamma[n,n'] = \sigma^2\boldsymbol{\alpha}_0^{(\ell)}[n' - (N-M)]\mathbb{1}_{(n' \geq N-M)} + o(\sigma^2)$$

when $K \to \infty$.

b) When $n' \geq N$: Inserting equation (39) into result (18), we obtain

$$\begin{aligned}
\gamma[n,n'] &= \mathbf{z}_K^T\mathbb{E}\left\{{\boldsymbol{\alpha}^{(\ell)}}^T\boldsymbol{\alpha}^{(\lambda)}\right\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\boldsymbol{\alpha}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\lambda)}\mathbf{w}_K\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K \\
&\quad + \sigma^2\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\boldsymbol{\alpha}^{(\lambda)}\mathbf{w}_K\} - \mathbf{z}[n]\mathbf{z}[n'] + o(\sigma^2) \\
&= \mathbf{z}_K^T\mathbb{E}\left\{{\mathbf{h}^{(\ell)}}^T\mathbf{h}^{(\lambda)}\right\}\mathbf{z}_K + \boldsymbol{\alpha}_0^{(\ell)}\sigma\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\}\mathbf{z}[n'] \\
&\quad + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\lambda)}\mathbf{w}_K\}\mathbf{z}[n] + \sigma\boldsymbol{\alpha}_0^{(\lambda)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K \\
&\quad + \sigma\mathbb{E}\{\mathbf{h}^{(\lambda)}\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + \sigma^2\boldsymbol{\alpha}_0^{(\lambda)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} \\
&\quad + \sigma^2\left\langle\boldsymbol{\alpha}^{(\ell)},\boldsymbol{\alpha}^{(\lambda)}\right\rangle + \sigma^2\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + o(\sigma^2) \ .
\end{aligned}$$

From Lemma 1, we have that $\mathbf{h}^{(\ell)}$ is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. [The same gap here.] Considering this property and equation (38) gives

$$\gamma[n,n'] = \frac{1}{K}\mathbf{z}_K^T\boldsymbol{\Gamma}^{(\ell,\lambda)}\mathbf{z}_K + \sigma^2\left\langle\boldsymbol{\alpha}^{(\ell)},\boldsymbol{\alpha}^{(\lambda)}\right\rangle + \sigma^2\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} + o(\sigma^2)$$

when $K \to \infty$. The Isserlis' theorem also gives the following result [the same gap here.]

$$\begin{aligned}
\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\lambda)}\mathbf{w}_K\} &= \sum_{m,m'=0}^{M-1}\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m]\mathbf{h}^{(\lambda)}[m']\mathbf{w}_K[m']\} \\
&= \sum_{m,m'=0}^{M-1}\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m]\}\mathbb{E}\{\mathbf{h}^{(\lambda)}[m']\mathbf{w}_K[m']\} + \mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m']\}\mathbb{E}\{\mathbf{h}^{(\lambda)}[m']\mathbf{w}_K[m]\} \\
&\qquad + \mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{h}^{(\lambda)}[m']\}\underbrace{\mathbb{E}\{\mathbf{w}_K[m]\mathbf{w}_K[m']\}}_{=\delta_{m,m'}} \\
&= \sum_{m=0}^{M-1}\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{h}^{(\lambda)}[m]\} = \frac{1}{K}\mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)}\right)
\end{aligned}$$

when $K \to \infty$. We thus conclude that

$$\gamma[n,n'] = \frac{1}{K}\mathbf{z}_K^T\boldsymbol{\Gamma}^{(\ell,\lambda)}\mathbf{z}_K + \sigma^2\left\langle\boldsymbol{\alpha}^{(\ell)},\boldsymbol{\alpha}^{(\lambda)}\right\rangle + \frac{\sigma^2}{K}\mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)}\right) + o(\sigma^2)$$

when $K \to \infty$.

III. Application to an Electrocardiogram

We provide here an additional implementation of BoundEffRed, applied to an electrocardiogram (ECG) dataset. The dataset is constructed from a 500-second-long ECG, sampled at $f_s = 200$ Hz, cut into 10 segments of 50 seconds each. Fig. 8 depicts the right boundary of one of these subsignals, together with the 6-second extensions estimated by SigExt (first panel), or EDMD (second panel), GPR (third panel), or TBATS (fourth panel). These extensions are superimposed to the ground-truth extension, plotted in red. The sharp and spiky ECG patterns make the AHM model too simplistic to describe this type of signal. Consequently, the forecast produced by SigExt is moderately satisfactory.

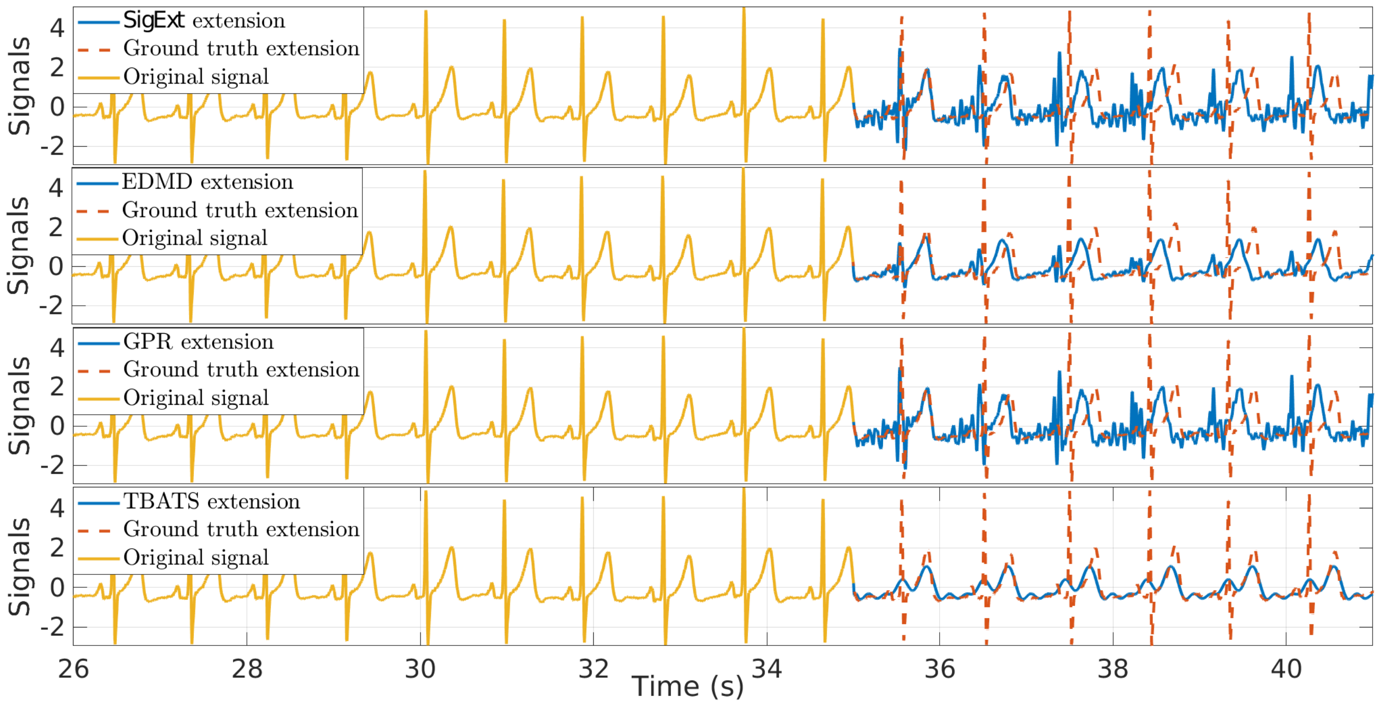

Fig. 8. Extended ECG (blue) obtained by the SigExt forecasting (first panel), the EDMD forecasting (second panel), the GPR forecasting (third panel), and the TBATS forecasting (fourth panel), superimposed with the ground truth signal (dashed red).

TABLE IV
ECG: Performance of the Boundary-Free TF Representations According to the Extension Method

Extension method	Median performance index D			
	STFT	SST	ConceFT	RS
SigExt	0.584	0.630	0.462	0.642
Symmetric	1.199	1.354	1.427	0.943
EDMD	0.538	0.558	0.496	0.714
GPR	0.639	0.588	0.485	0.616

Table IV contains the median performance index D of the boundary-free TF representations, over the N subsignals evaluated, according to the extension method. As a result of the fair quality of the forecasts, the reduction of boundary effects is less significant than for PPG signal. Nevertheles, the results show that BoundEffRed has the same efficiency when the SigExt extension, the EDMD extension or the GPR extension is chosen. Indeed, t-tests performed under the null hypothesis that the mean are equals, with a 5% significance level, show no statistical significant difference between SigExt and EDMD or GPR, regardless of the representation considered. This justifies the choice of SigExt for real-time implementation.