Supplementary material for "An Efficient Forecasting Approach for the Real-Time Reduction of Boundary Effects in Time-Frequency Representations"

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I. Proof of Lemma 1

Recall the model (13). Based on the definition of matrices $\mathbf{X}$ and $\mathbf{Y}$, we have:

$$\frac{1}{K}\mathbf{X}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}^T + \sigma^2\mathbf{I}}_{\triangleq S^{(0)}} + \mathbf{E}^{(0)} \tag{28}$$

$$\frac{1}{K}\mathbf{Y}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}'\mathbf{Z}^T + \sigma^2\mathbf{D}}_{\triangleq S^{(1)}} + \mathbf{E}^{(1)} \ , \tag{29}$$

where $\mathbf{E}^{(a)} := \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$,

$$\mathbf{E}_1^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{z}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] + \mathbf{w}[N_0+m+a+k]\mathbf{z}[N_0+m'+k] \ ,$$

and

$$\mathbf{E}_2^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{w}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] - \delta_{(m+a)m'} \ ,$$

with $a \in \{0,1\}$. We call $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ *error matrices* because:

$$\mathbb{E}\{\mathbf{E}^{(0)}\} = \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0}$$
$$\mathbb{E}\{\mathbf{E}^{(1)}\} = \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0} \ .$$

Thus,

$$\mathbf{A}_0 := \mathbf{S}^{(1)}{\mathbf{S}^{(0)}}^{-1} \ , \quad \tilde{\mathbf{A}} := (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1} \ .$$

As a result, for $\ell \in \mathbb{N}$,

$$\begin{aligned}\mathbf{h}^{(\ell)} &= \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)} \\ &= \mathbf{e}_M^T\left(\tilde{\mathbf{A}}^\ell - \mathbf{A}_0^\ell\right) \\ &= \mathbf{e}_M^T\left(\left((\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\right)^\ell - \mathbf{A}_0^\ell\right) \ .\end{aligned} \tag{30}$$

The randomness of $\mathbf{h}^{(\ell)}$ completely comes from the error matrices. Besides, notice that the first $M-1$ rows in $\mathbf{E}^{(1)}$ equal to the last $M-1$ rows of $\mathbf{E}^{(0)}$. We gather all sources of randomness into an vector $\mathbf{g} \in \mathbb{R}^{M(M+1)}$, containing M rows defined as

$$\mathbf{g} = \mathrm{vec}\left(\begin{bmatrix}\mathbf{E}^{(0)} \\ \mathbf{e}_M^T\mathbf{E}^{(1)}\end{bmatrix}\right) \ ,$$

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where "vec" denotes the vectorization operator, concatenating the columns of a given matrix on top of one another. Then, by definition, we have:

$$\mathbf{g} = \sigma \mathbf{g}_1 + \sigma^2 \mathbf{g}_2 \ ,$$

where:

$$\mathbf{g}_1 = \frac{1}{K}\sum_{k=0}^{K-1} \mathrm{vec}\left(\tilde{\mathbf{z}}_k \mathbf{w}_k^T + \tilde{\mathbf{w}}_k \mathbf{z}_k^T\right)$$
$$\mathbf{g}_2 = \frac{1}{K}\sum_{k=0}^{K-1} \mathrm{vec}\left(\tilde{\mathbf{w}}_k \mathbf{w}_k^T - \tilde{\mathbf{I}}\right) ,$$

where $\tilde{\mathbf{z}}_k^T = \begin{pmatrix}\mathbf{z}_k^T & \mathbf{z}_{k+1}[M-1]\end{pmatrix}$ and $\tilde{\mathbf{w}}_k^T = \begin{pmatrix}\mathbf{w}_k^T & \mathbf{w}_{k+1}[M-1]\end{pmatrix}$. Then, $\mathbf{g}_1$ is a Gaussian random vector because it is a linear combination of Gaussian random vectors. Moreover, using the central limit theorem under weak dependence, we can show that $\mathbf{g}_2$ also converges towards a Gaussian random vector as $K \to \infty$. Combining these two results leads to

$$\sqrt{K}\ \mathbf{g} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_0) \ ,$$

where $\mathbf{\Gamma}_0 = \mathbb{E}\left\{\mathbf{g}\mathbf{g}^T\right\}$ is a covariance matrix.

Furthermore, one can write $\mathbf{h}^{(\ell)}$ as $\mathbf{h}^{(\ell)} = f^{(\ell)}(\mathbf{g})$ where $f^{(\ell)}$ is a deterministic function such that:

$$\begin{aligned} f^{(\ell)} : \mathbb{R}^{M(M+1)} &\to \mathbb{R}^M \\ \mathbf{g} &\mapsto \mathbf{h}^{(\ell)} \ . \end{aligned}$$

Then, as $f^{(\ell)}$ is a differentiable function with non-zero differentiation at 0 [I assume that you have checked all conditions for the delta method. You may consider adding its formula to save the readers' time], using the Delta method gives:

$$\sqrt{K}\ \mathbf{h}^{(\ell)} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, {\mathbf{F}^{(\ell)}}^T \mathbf{\Gamma}_0 \mathbf{F}^{(\ell)}) \ ,$$

where $\mathbf{F}^{(\ell)}$ is the Jacobian matrix such that:

$$\mathbf{F}^{(\ell)}[m, m'] = \left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[m']}\right|_{\mathbf{g}=\mathbf{0}} \ .$$

II. Proof of Theorem 1

A. Expression of the bias $\boldsymbol{\mu}$.

Clearly, $\boldsymbol{\mu}[n] = 0$ when $n \in I$. When $n = N - 1 + \ell$, we have

$$\begin{aligned}\boldsymbol{\mu}[n] &= \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[n] \\ &= \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K + \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[N-1+\ell]\end{aligned}$$

Let us first evaluate the expression of $\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K$. We have

$$\begin{aligned}\mathbf{S}^{(a)}[m, m'] &= \sigma^2\delta_{(m+a)m'} + \sum_{j,j'=1}^{J} \frac{\Omega_j\Omega_{j'}}{K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(N_0+m+a+k)+\varphi_j\right)\cos\left(2\pi\frac{f_{j'}}{f_\mathsf{s}}(N_0+m'+k)+\varphi_{j'}\right) \\ &= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right) + \cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(2k+m+a+m'+2N_0)\right) \\ &= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\left(\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right) + \frac{\Omega_j^2}{2K}\underbrace{\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(2k+m+a+m'+2N_0)\right)}_{=0 \text{ because } \frac{f_j}{f_\mathsf{s}}=\frac{p_j'}{K}}\right) \\ &= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}(m+a-m')\right) \ .\end{aligned}$$

Thus, $\mathbf{S}^{(0)}$ is a circulant matrix and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U}\boldsymbol{\Lambda}^{(0)}\mathbf{U}^* \ ,$$

where $\mathbf{U}[m,m'] = \frac{1}{\sqrt{M}}e^{-2\mathrm{i}\pi mm'/M}$ and $\boldsymbol{\Lambda}^{(0)} = \operatorname{diag}(\lambda_0^{(0)},\ldots,\lambda_{M-1}^{(0)})$ with

$$\begin{aligned}\lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\sum_{q=0}^{M-1}\cos\left(2\pi\frac{f_j}{f_\mathsf{s}}q\right)e^{-2\mathrm{i}\pi qm/M} \\ &= \sigma^2 + \frac{M}{4}\sum_{j=1}^{J}\Omega_j^2(\delta_{m,p_j}+\delta_{m,M-p_j}) \ .\end{aligned}$$

Therefore,

$$\mathbf{S}^{(0)^{-1}} = \mathbf{U}\boldsymbol{\Lambda}^{(0)^{-1}}\mathbf{U}^* ,$$

which leads to

$$\mathbf{S}^{(0)^{-1}}[m,m'] = \frac{1}{\sigma^2}\delta_{m,m'} - \sum_{j=1}^{J}\frac{\Omega_j^2}{2\sigma^2(\sigma^2+\Omega_j^2M/4)}\cos\left(2\pi p_j\frac{m-m'}{M}\right) \ .$$

Consequently, we have

$$\begin{aligned}\tilde{\mathbf{A}}_0[m,m'] &= \sum_{q=0}^{M-1}\mathbf{S}^{(1)}[m,q]\mathbf{S}^{(0)^{-1}}[q,m'] \\ &= \delta_{m+1,m'} + \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m'}{M}\right)\delta_{m+1,M} \,.\end{aligned} \tag{31}$$

Thus,

$$\begin{aligned}\tilde{\boldsymbol{\alpha}}_0^{(1)}[m] &= \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m}{M}\right) \\ &= \frac{2}{M}\sum_{j=1}^{J}\cos\left(2\pi p_j\frac{m}{M}\right) + o(\sigma) \ .\end{aligned}$$

Besides, from equation (31), we have

$$\tilde{\mathbf{A}}_0\mathbf{z}_K = \begin{pmatrix}\mathbf{z}[N-M+1] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K\end{pmatrix} .$$

By induction, we have

$$\tilde{\mathbf{A}}_0^{\ell}\mathbf{z}_K = \begin{pmatrix}\mathbf{z}[N-M+\ell] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K \\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\end{pmatrix} .$$

Then,

$$\begin{aligned}\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K &= \tilde{\boldsymbol{\alpha}}_0^{(1)}\tilde{\mathbf{A}}_0^{\ell-1}\mathbf{z}_K \\ &= \sum_{m=0}^{M-\ell}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] + \sum_{m=M-\ell+1}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\boldsymbol{\alpha}_0^{(m-M+\ell)}\mathbf{z}_K \,.\end{aligned} \tag{32}$$

But,

$$\begin{aligned}
\boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K &= \sum_{m=0}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+m] \\
&= \sum_{j,j'=1}^{J} \Omega_{j'} \frac{2}{M} \underbrace{\sum_{m=0}^{M-1} \cos\left(2\pi p_j \frac{m}{M}\right) \cos\left(2\pi p_{j'} \frac{N+m}{M} + \varphi_{j'}\right)}_{=\delta_{j,j'} \frac{M}{2} \cos\left(2\pi p_j \frac{N}{M} + \varphi_j\right)} + o(\sigma) \\
&= \sum_{j=1}^{J} \Omega_j \cos\left(2\pi p_j \frac{N}{M} + \varphi_j\right) + o(\sigma) \\
&= \mathbf{z}[N] + o(\sigma)
\end{aligned}$$

By induction from (32), we have

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K = \mathbf{z}[N-1+\ell] + o(\sigma) \tag{33}$$

Then,

$$\boldsymbol{\mu}[N-1+\ell] = \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} + o(\sigma) \ .$$

Besides, from Lemma 1, we have the following results:

$$\begin{aligned}
\mathbb{E}\{\mathbf{h}^{(\ell)}\} &\underset{K\to\infty}{\longrightarrow} 0 \\
\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} &\underset{K\to\infty}{\longrightarrow} 0
\end{aligned}$$

[if it is possible to get the convergence rate, like $\mathbb{E}\{\mathbf{h}^{(\ell)}\} = O(1/K)$, that would be great!] Consequently,

$$\frac{1}{\sigma}\boldsymbol{\mu}[N-1+\ell] = o(1) \tag{34}$$

when $K \to \infty$.

B. *Expression of the covariance γ.*

Take $n > N$, and denote $n = N-1+\ell$. To derive the covariance, let us segregate the cases.

a) When $n' \in I$: From Lemma 1, we have that $h^{(\ell)}$ is asymptotically Gaussian when $K \to \infty$. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing [there is a gap here. Note that when K is finite, it is approximated by Gaussian but not Gaussian. The discrepancy should be made clear.]. Then, combining result (34) and equation (15), we obtain:

$$\gamma[n,n'] = \sigma\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} + o(\sigma^2)$$

when $K \to \infty$. We remark that

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} = \boldsymbol{\alpha}_0^{(\ell)}[n' - (N-M)]\mathbb{1}_{(n' \geq N-M)} \ .$$

Moreover, using Delta method one can show that we have the asymptotic result

$$\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\} \underset{K\to\infty}{\longrightarrow} 0\,. \tag{35}$$

As a result, we have

$$\gamma[n,n'] = \sigma^2\boldsymbol{\alpha}_0^{(\ell)}[n' - (N-M)]\mathbb{1}_{(n' \geq N-M)} + o(\sigma^2)$$

when $K \to \infty$. [Note that $o(\sigma^2)$ becomes 0 when $K \to \infty$. So use either "$\gamma[n,n'] = \sigma^2\boldsymbol{\alpha}_0^{(\ell)}[n' - (N-M)]\mathbb{1}_{(n' \geq N-M)} + o(\sigma^2)$ when $K \to \infty$" or "$\gamma[n,n'] \to \sigma^2\boldsymbol{\alpha}_0^{(\ell)}[n' - (N-M)]\mathbb{1}_{(n' \geq N-M)}$ when $K \to \infty$".]

b) When $n' \geq N$: From Lemma 1, we have that $h^{(\ell)}$ is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. [The same gap here.] Then, combining equations (17), (34) and (35), we obtain

$$\begin{aligned}
\gamma[n,n'] =& \mathbf{z}_K^T \mathbb{E}\left\{ {\boldsymbol{\alpha}^{(\ell)}}^T \boldsymbol{\alpha}^{(\lambda)} \right\} \mathbf{z}_K + \sigma \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K \boldsymbol{\alpha}^{(\lambda)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\boldsymbol{\alpha}^{(\lambda)} \mathbf{w}_K \boldsymbol{\alpha}^{(\ell)}\} \mathbf{z}_K \\
&+ \sigma^2 \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K \boldsymbol{\alpha}^{(\lambda)} \mathbf{w}_K\} - \mathbf{z}[n]\mathbf{z}[n'] + o(\sigma^2) \\
=& \mathbf{z}_K^T \mathbb{E}\left\{ {\mathbf{h}^{(\ell)}}^T \mathbf{h}^{(\lambda)} \right\} \mathbf{z}_K + \boldsymbol{\alpha}_0^{(\ell)} \sigma \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\lambda)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K\} \mathbf{z}[n'] \\
&+ \sigma \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\lambda)} \mathbf{w}_K\} \mathbf{z}[n] + \sigma \boldsymbol{\alpha}_0^{(\lambda)} \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\ell)}\} \mathbf{z}_K \\
&+ \sigma \mathbb{E}\{\mathbf{h}^{(\lambda)} \mathbf{w}_K \mathbf{h}^{(\ell)}\} \mathbf{z}_K + \sigma^2 \boldsymbol{\alpha}_0^{(\ell)} \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} + \sigma^2 \boldsymbol{\alpha}_0^{(\lambda)} \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\ell)} \mathbf{w}_K\} \\
&+ \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \sigma^2 \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} + o(\sigma^2) \\
=& \frac{1}{K} \mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell,\lambda)} \mathbf{z}_K + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \sigma^2 \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} + o(\sigma^2)
\end{aligned}$$

when $K \to \infty$ [in the last equation, make clear which term depends on which equation (17), (34) and (35).] [The previous or current asymptotic expressions with $\frac{1}{K}\mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell,\lambda)} \mathbf{z}_K$ are both not ideal. Either using your previous notation or following what I prefer here, the behavior of $\mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell,\lambda)} \mathbf{z}_K$ should be made clear. Is it bounded? If it is bounded, then use $= O(1/K)$ to replace $\to \frac{1}{K}\mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell,\lambda)} \mathbf{z}_K$. If not, how does it depend on K should be made clear.] [Moreover, if you prefer to use your previous notation, you may clarify what it means by $\sim$.]. The Isserlis' theorem also gives the following result [the same gap here.]

$$\begin{aligned}
\mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} =& \sum_{m,m'=0}^{M-1} \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m] \mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m']\} \\
=& \sum_{m,m'=0}^{M-1} \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m]\} \mathbb{E}\{\mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m']\} + \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m']\} \mathbb{E}\{\mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m]\} \\
&\quad + \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{h}^{(\lambda)}[m']\} \underbrace{\mathbb{E}\{\mathbf{w}_K[m] \mathbf{w}_K[m']\}}_{=\delta_{m,m'}} \\
\to& \sum_{m=0}^{M-1} \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{h}^{(\lambda)}[m]\} = \frac{1}{K} \mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)}\right)
\end{aligned}$$

when $K \to \infty$. We thus conclude that

$$\gamma[n,n'] \to \frac{1}{K} \mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell,\lambda)} \mathbf{z}_K + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \frac{\sigma^2}{K} \mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)}\right) + o(\sigma^2)$$

when $K \to \infty$.

III. Application to an Electrocardiogram

We provide here an additional implementation of BoundEffRed, applied to an electrocardiogram (ECG) dataset. The dataset is constructed from a 500-second-long ECG, sampled at $f_s = 200$ Hz, cut into 10 segments of 50 seconds each. Fig. 8 depicts the right boundary of one of these subsignals, together with the 6-second extensions estimated by SigExt (top panel), or EDMD (middle panel), or GPR (bottom panel). These extensions are superimposed to the ground-truth extension, plotted in red. The sharp and spiky ECG patterns make the AHM model too simplistic to describe this type of signal. Consequently, the forecast produced by SigExt is moderately satisfactory.

Table IV contains the median performance index D of the boundary-free TF representations, over the N subsignals evaluated, according to the extension method. As a result of the fair quality of the forecasts, the reduction of boundary effects is less significant than for PPG signal. Nevertheles, the results show that BoundEffRed has the same efficiency when the SigExt extension, the EDMD extension or the GPR extension is chosen. This justifies the choice of SigExt for real-time implementation.

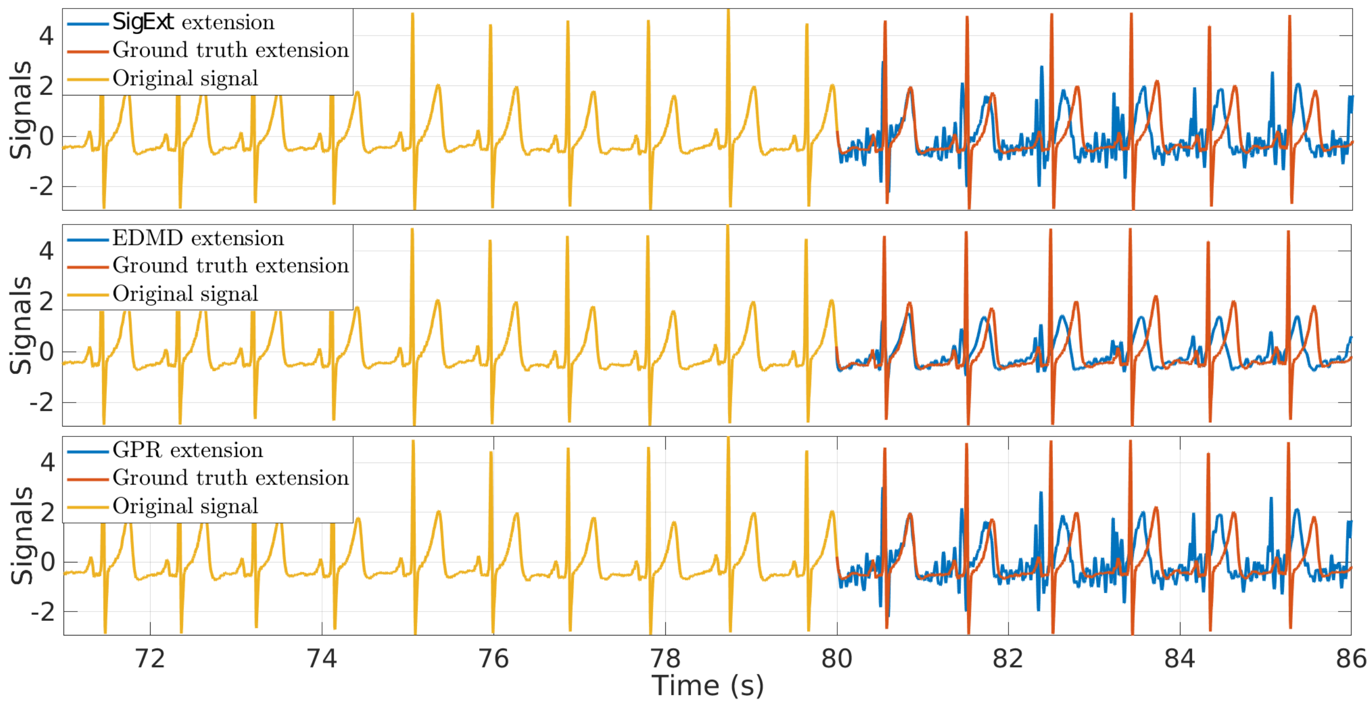

Fig. 8. Extended ECG (blue) obtained by the SigExt forecasting (top), the EDMD forecasting (middle), and the GPR forecasting (bottom), superimposed with the ground truth signal (red).

TABLE IV
ECG: PERFORMANCE OF THE BOUNDARY-FREE TF REPRESENTATIONS ACCORDING TO THE EXTENSION METHOD

Extension method	Median performance index D			
	STFT	SST	ConceFT	RS
SigExt	0.584	0.630	0.462	0.642
Symmetric	1.199	1.354	1.427	0.943
EDMD	0.538	0.558	0.496	0.714
GPR	0.639	0.588	0.485	0.616