An Approach for the Reduction of Boundary Effects in Time-Frequency Representations: Part II

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I. Proof of Lemma 1

A. Notations

We have:

$$\frac{1}{K}\mathbf{X}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}^T + \sigma^2\mathbf{I}}_{\triangleq S^{(0)}} + \mathbf{E}^{(0)} \tag{9}$$

$$\frac{1}{K}\mathbf{Y}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}'\mathbf{Z}^T + \sigma^2\mathbf{D}}_{\triangleq S^{(1)}} + \mathbf{E}^{(1)} \ , \tag{10}$$

where $\mathbf{E}^{(a)} = \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$ with:

$$\mathbf{E}_1^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{z}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] + \mathbf{w}[N_0+m+a+k]\mathbf{z}[N_0+m'+k] \ ,$$

and

$$\mathbf{E}_2^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{w}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] - \delta_{(m+a)m'} \ ,$$

with $a \in \{0,1\}$.

Remark 1. *The matrices $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ are said to be error matrices because:*

$$\mathbb{E}\{\mathbf{E}^{(0)}\} = \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0}$$
$$\mathbb{E}\{\mathbf{E}^{(1)}\} = \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0} \ .$$

It follows from this:

$$\tilde{\mathbf{A}} = (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}$$
$$\mathbf{A}_0 = \mathbf{S}^{(1)}{\mathbf{S}^{(0)}}^{-1} \ .$$

Then:

$$\begin{aligned}\mathbf{h}^{(\ell)} &= \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)} \\ &= \mathbf{e}_M^T\left(\tilde{\mathbf{A}}^\ell - \tilde{\mathbf{A}}_0^\ell\right) \\ &= \mathbf{e}_M^T\left(\left((\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1}\right)^\ell - \tilde{\mathbf{A}}_0^\ell\right) \ .\end{aligned} \tag{11}$$

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B. Study of $\mathbf{h}^{(\ell)}$

The randomness of $\mathbf{h}^{(\ell)}$ is completely originating from the error matrices. Besides, notice that the first $M-1$ rows in $\mathbf{E}^{(1)}$ are the same as the last $M-1$ rows of $\mathbf{E}^{(0)}$. Thus, we gather all the sources of randomness into an vector $\mathbf{g} \in \mathbb{R}^{M(M+1)}$ defined as:

$$\mathbf{g} = \text{vec}\left(\begin{pmatrix} \mathbf{E}^{(0)} \\ \mathbf{e}_M^T \mathbf{E}^{(1)} \end{pmatrix}\right) .$$

Here, "vec" denotes the vectorization operator, concatenating the columns of a given matrix on top of one another. Then, by definition, we have:

$$\mathbf{g} = \sigma \mathbf{g}_1 + \sigma^2 \mathbf{g}_2 ,$$

where:

$$\begin{aligned}
\mathbf{g}_1 &= \frac{1}{K} \sum_{k=0}^{K-1} \text{vec}\left(\tilde{\mathbf{z}}_k \mathbf{w}_k^T + \tilde{\mathbf{w}}_k \mathbf{z}_k^T\right) \\
\mathbf{g}_2 &= \frac{1}{K} \sum_{k=0}^{K-1} \text{vec}\left(\tilde{\mathbf{w}}_k \mathbf{w}_k^T - \tilde{\mathbf{I}}\right) .
\end{aligned}$$

with $\tilde{\mathbf{z}}_k^T = \begin{pmatrix} \mathbf{z}_k^T & \mathbf{z}_{k+1}[0] \end{pmatrix}$ and $\tilde{\mathbf{w}}_k^T = \begin{pmatrix} \mathbf{w}_k^T & \mathbf{w}_{k+1}[0] \end{pmatrix}$. Then, $\mathbf{g}_1$ is a Gaussian random vector as a linear combination of Gaussian random vectors. Moreover, using the central limit theorem under weak dependence, we can show that $\mathbf{g}_2$ also converges towards a Gaussian random vector as $K \to \infty$. Combining these two results gives the following result:

$$\sqrt{K}\ \mathbf{g} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{\Gamma}_0) ,$$

where $\mathbf{\Gamma}_0 = \mathbb{E}\left\{\mathbf{g}\mathbf{g}^T\right\}$ is a covariance matrix.

Furthermore, one can write $\mathbf{h}^{(\ell)}$ as $\mathbf{h}^{(\ell)} = f^{(\ell)}(\mathbf{g})$ where $f^{(\ell)}$ is a deterministic function such that:

$$\begin{aligned}
f^{(\ell)} : \mathbb{R}^{M(M+1)} &\to \mathbb{R}^M \\
\mathbf{g} &\mapsto \mathbf{h}^{(\ell)} .
\end{aligned}$$

Then, as $f^{(\ell)}$ is a differentiable function, using the Delta method gives:

$$\sqrt{K}\ \mathbf{h}^{(\ell)} \xrightarrow[K\to\infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{F}^{(\ell)T} \mathbf{\Gamma}_0 \mathbf{F}^{(\ell)}) ,$$

where $\mathbf{F}^{(\ell)}$ is the Jacobian matrix such that:

$$\mathbf{F}^{(\ell)}[m, m'] = \left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[m']}\right|_{\mathbf{g}=\mathbf{0}} .$$

II. Proof of Theorem 1

A. Expression of the bias $\boldsymbol{\mu}$.

Clearly, $\boldsymbol{\mu}[n] = 0$ when $n \in I$. When $n = N - 1 + \ell$, we have:

$$\begin{aligned}
\boldsymbol{\mu}[n] &= \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K + \sigma \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K\} - \mathbf{z}[n] \\
&= \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K + \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K\} - \mathbf{z}[N - 1 + \ell]
\end{aligned}$$

Let us first evaluate the expression of $\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K$. We have:

$$\begin{aligned}
\mathbf{S}^{(a)}[m,m'] &= \sigma^2\delta_{(m+a)m'} + \sum_{j,j'=1}^{J}\frac{\Omega_j\Omega_{j'}}{K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(N_0+m+a+k)\right)\cos\left(2\pi\frac{f_{j'}}{f_{\mathsf{s}}}(N_0+m'+k)\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2K}\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(m+a-m')\right)+\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(2k+m+a+m'+2N_0)\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\left(\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(m+a-m')\right)+\frac{\Omega_j^2}{2K}\underbrace{\sum_{k=0}^{K-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(2k+m+a+m'+2N_0)\right)}_{=0\text{ because }\frac{f_j}{f_{\mathsf{s}}}=\frac{p_j'}{K}}\right)\\
&= \sigma^2\delta_{(m+a)m'} + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}(m+a-m')\right)\ .
\end{aligned}$$

Thus, $\mathbf{S}^{(0)}$ is a circulant matrix and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U}\boldsymbol{\Lambda}^{(0)}\mathbf{U}^*\ ,$$

where $\mathbf{U}[m,m'] = \frac{1}{\sqrt{M}}e^{-2i\pi mm'/M}$ and $\boldsymbol{\Lambda}^{(0)} = \operatorname{diag}(\lambda_0^{(0)},\ldots,\lambda_{M-1}^{(0)})$ with:

$$\begin{aligned}
\lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^{J}\frac{\Omega_j^2}{2}\sum_{q=0}^{M-1}\cos\left(2\pi\frac{f_j}{f_{\mathsf{s}}}q\right)e^{-2i\pi qm/M}\\
&= \sigma^2 + \frac{M}{4}\sum_{j=1}^{J}\Omega_j^2(\delta_{m,p_j}+\delta_{m,M-p_j})\ .
\end{aligned}$$

Therefore:

$$\mathbf{S}^{(0)^{-1}} = \mathbf{U}\boldsymbol{\Lambda}^{(0)^{-1}}\mathbf{U}^*$$

which leads to:

$$\mathbf{S}^{(0)^{-1}}[m,m'] = \frac{1}{\sigma^2}\delta_{m,m'} - \sum_{j=1}^{J}\frac{\Omega_j^2}{2\sigma^2(\sigma^2+\Omega_j^2M/4)}\cos\left(2\pi p_j\frac{m-m'}{M}\right)\ ,$$

and, consequently:

$$\begin{aligned}
\tilde{\mathbf{A}}_0[m,m'] &= \sum_{q=0}^{M-1}\mathbf{S}^{(1)}[m,q]\mathbf{S}^{(0)^{-1}}[q,m']\\
&= \delta_{m+1,m'} + \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m'}{M}\right)\delta_{m+1,M}
\end{aligned}\tag{12}$$

Thus:

$$\begin{aligned}
\tilde{\boldsymbol{\alpha}}_0^{(1)}[m] &= \sum_{j=1}^{J}\frac{2\Omega_j^2}{\Omega_j^2M+4\sigma^2}\cos\left(2\pi p_j\frac{m}{M}\right)\\
&= \frac{2}{M}\sum_{j=1}^{J}\cos\left(2\pi p_j\frac{m}{M}\right)+o(\sigma)\ .
\end{aligned}$$

Besides, from equation (12), we have

$$\tilde{\mathbf{A}}_0\mathbf{z}_K = \begin{pmatrix}\mathbf{z}[N-M+1]\\ \vdots\\ \mathbf{z}[N-1]\\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K\end{pmatrix}$$

By induction, we have:

$$\tilde{\mathbf{A}}_0^{\ell}\mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N-M+\ell] \\ \vdots \\ \mathbf{z}[N-1] \\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K \\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K \end{pmatrix} .$$

Then:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K &= \tilde{\boldsymbol{\alpha}}_0^{(1)}\tilde{\mathbf{A}}_0^{\ell-1}\mathbf{z}_K \\ &= \sum_{m=0}^{M-\ell}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] + \sum_{m=M-\ell+1}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\boldsymbol{\alpha}_0^{(m-M+\ell)}\mathbf{z}_K \end{aligned} \tag{13}$$

But:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K &= \sum_{m=0}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+m] \\ &= \sum_{j,j'=1}^{J}\Omega_{j'}\frac{2}{M}\underbrace{\sum_{m=0}^{M-1}\cos\left(2\pi p_j\frac{m}{M}\right)\cos\left(2\pi p_{j'}\frac{N+m}{M}\right)}_{=\delta_{j,j'}\frac{M}{2}\cos\left(2\pi p_j\frac{N}{M}\right)} + o(\sigma) \\ &= \sum_{j=1}^{J}\Omega_j\cos\left(2\pi p_j\frac{N}{M}\right) + o(\sigma) \\ &= \mathbf{z}[N] + o(\sigma) \end{aligned}$$

and, by induction from (13):

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K = \mathbf{z}[N-1+\ell] + o(\sigma) \tag{14}$$

Then:

$$\boldsymbol{\mu}[N-1+\ell] = \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} + o(\sigma) .$$

Besides, from Lemma 1, we have the following results:

$$\mathbb{E}\{\mathbf{h}^{(\ell)}\} \underset{K\to\infty}{\longrightarrow} 0$$
$$\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} \underset{K\to\infty}{\longrightarrow} 0$$

Consequently:

$$\boldsymbol{\mu}[N-1+\ell] \underset{K\to\infty}{\sim} o(\sigma) .$$

B. Expression of the covariance γ.

We recall that $n > N$, we consequently denote $n = N-1+\ell$. To derive the covariance, let us segregate the cases.

a) If $n' \in I^2$: From Lemma 1, we have that $h^{(\ell)}$ is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. Then, combining result (II-A) and equation (**??**), we obtain:

$$\gamma[n,n'] = \sigma\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} + o(\sigma^2) .$$

We remark that

$$\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{w}[n']\} = \boldsymbol{\alpha}_0^{(\ell)}[n'-(N-M)]\mathbb{1}_{(n'\geq N-M)} .$$

Moreover, using Delta method one can show that we have the asymptotic result:

$$\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\} \underset{K\to\infty}{\longrightarrow} 0 \tag{15}$$

Finally:

$$\gamma[n,n'] \underset{K\to\infty}{\sim} \sigma^2\boldsymbol{\alpha}_0^{(\ell)}[n'-(N-M)]\mathbb{1}_{(n'\geq N-M)} + o(\sigma^2) .$$

b) If $n' \geq N$: From Lemma 1, we have that $h^{(\ell)}$ is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. Then, combining result (II-A) and equations (**??**) and (15), we obtain:

$$
\begin{aligned}
\gamma[n,n'] &= \mathbf{z}_K^T \mathbb{E}\left\{ {\boldsymbol{\alpha}^{(\ell)}}^T \boldsymbol{\alpha}^{(\lambda)} \right\} \mathbf{z}_K + \sigma \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K \boldsymbol{\alpha}^{(\lambda)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\boldsymbol{\alpha}^{(\lambda)} \mathbf{w}_K \boldsymbol{\alpha}^{(\ell)}\} \mathbf{z}_K \\
&\quad + \sigma^2 \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K \boldsymbol{\alpha}^{(\lambda)} \mathbf{w}_K\} - \mathbf{z}[n]\mathbf{z}[n'] + o(\sigma^2) \\
&= \mathbf{z}_K^T \mathbb{E}\left\{ {\mathbf{h}^{(\ell)}}^T \mathbf{h}^{(\lambda)} \right\} \mathbf{z}_K + \boldsymbol{\alpha}_0^{(\ell)} \sigma \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\lambda)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K\} \mathbf{z}[n'] \\
&\quad + \sigma \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\lambda)} \mathbf{w}_K\} \mathbf{z}[n] + \sigma \boldsymbol{\alpha}_0^{(\lambda)} \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\ell)}\} \mathbf{z}_K \\
&\quad + \sigma \mathbb{E}\{\mathbf{h}^{(\lambda)} \mathbf{w}_K \mathbf{h}^{(\ell)}\} \mathbf{z}_K + \sigma^2 \boldsymbol{\alpha}_0^{(\ell)} \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} + \sigma^2 \boldsymbol{\alpha}_0^{(\lambda)} \mathbb{E}\{\mathbf{w}_K \mathbf{h}^{(\ell)} \mathbf{w}_K\} \\
&\quad + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \sigma^2 \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} + o(\sigma^2) \\
&\underset{K\to\infty}{\sim} \frac{1}{K} \mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell,\lambda)} \mathbf{z}_K + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \sigma^2 \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} + o(\sigma^2)
\end{aligned}
$$

The Isserlis' theorem also gives the following result:

$$
\begin{aligned}
\mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K\} &= \sum_{m,m'=0}^{M-1} \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m] \mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m']\} \\
&= \sum_{m,m'=0}^{M-1} \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m]\} \mathbb{E}\{\mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m']\} + \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m']\} \mathbb{E}\{\mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m]\} \\
&\qquad + \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{h}^{(\lambda)}[m']\} \underbrace{\mathbb{E}\{\mathbf{w}_K[m] \mathbf{w}_K[m']\}}_{=\delta_{m,m'}} \\
&\underset{K\to\infty}{\sim} \sum_{m=0}^{M-1} \mathbb{E}\{\mathbf{h}^{(\ell)}[m] \mathbf{h}^{(\lambda)}[m]\} = \frac{1}{K} \mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)} \right) \ .
\end{aligned}
$$

We conclude that:

$$
\gamma[n,n'] \underset{K\to\infty}{\sim} \frac{1}{K} \mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell,\lambda)} \mathbf{z}_K + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \frac{\sigma^2}{K} \mathrm{Tr}\left(\boldsymbol{\Gamma}^{(\ell,\lambda)} \right) + o(\sigma^2) \ .
$$