Supplementary Materials for "An Efficient Forecasting Approach to Reduce Boundary Effects in Real-Time Time-Frequency Analysis"

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I. Proof of Theorem 1

A. Preliminaries

1) Notations: Consider $\mathbf{z} \in \mathbb{R}^N$ the deterministic signal defined in (12), and denote by $\mathbf{z}_k \in \mathbb{R}^M$ the subsignal such that

$$\mathbf{z}_k[m] = \mathbf{z}[N-K-M+k+m]\,, \quad \forall m \in \{0,\dots,M-1\}\,,\ k \in \{0,\dots,K\}\ .$$

Define $\mathbf{Z} \in \mathbb{R}^{M\times K}$ and $\mathbf{Z}' \in \mathbb{R}^{M\times K}$, the matrices such that

$$\mathbf{Z} \triangleq \begin{pmatrix}\mathbf{z}_0 & \cdots & \mathbf{z}_{K-1}\end{pmatrix}\ ,$$
$$\mathbf{Z}' \triangleq \begin{pmatrix}\mathbf{z}_1 & \cdots & \mathbf{z}_{K}\end{pmatrix}\ .$$

Let $\mathbf{D} \in \mathbb{R}^{M\times M}$ be the matrix defined by $\mathbf{D}[m,m'] = \delta_{m+1,m'}$. Recall the model (14). Based on the definition of matrices $\mathbf{X}$ and $\mathbf{Y}$, we have:

$$\frac{1}{K}\mathbf{X}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}^T + \sigma^2\mathbf{I}}_{\triangleq \mathbf{S}^{(0)}} + \mathbf{E}^{(0)} \tag{27}$$

$$\frac{1}{K}\mathbf{Y}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}'\mathbf{Z}^T + \sigma^2\mathbf{D}}_{\triangleq \mathbf{S}^{(1)}} + \mathbf{E}^{(1)}\ , \tag{28}$$

where $\mathbf{E}^{(a)} \triangleq \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$,

$$\mathbf{E}_1^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{z}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] + \mathbf{w}[N_0+m+a+k]\mathbf{z}[N_0+m'+k]\ ,$$

and

$$\mathbf{E}_2^{(a)}[m,m'] = \frac{1}{K}\sum_{k=0}^{K-1} \mathbf{w}[N_0+m+a+k]\mathbf{w}[N_0+m'+k] - \delta_{(m+a)m'}\ ,$$

with $a \in \{0,1\}$. We call $\mathbf{E}^{(0)}$ and $\mathbf{E}^{(1)}$ *error matrices* because:

$$\mathbb{E}\{\mathbf{E}^{(0)}\} = \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0}$$
$$\mathbb{E}\{\mathbf{E}^{(1)}\} = \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0}\ .$$

Thus, the random matrix $\tilde{\mathbf{A}}$, defined in equation (9), is expressed in function of the above-defined matrices as:

$$\tilde{\mathbf{A}} = \left(\mathbf{S}^{(1)} + \mathbf{E}^{(1)}\right)\left(\mathbf{S}^{(0)} + \mathbf{E}^{(0)}\right)^{-1}\ .$$

Define $\mathbf{A}_0$ the deterministic matrix such that

$$\mathbf{A}_0 \triangleq \mathbf{S}^{(1)}{\mathbf{S}^{(0)}}^{-1}\ . \tag{29}$$

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We denote by $\boldsymbol{\alpha}_0^{(\ell)}$ the last row of $\mathbf{A}_0^{\ell}$. As a result, for $\ell \in \mathbb{N}^*$, the error vector $\mathbf{h}^{(\ell)}$ defined by $\mathbf{h}^{(\ell)} \triangleq \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)}$ satisfy the equation

$$\begin{aligned}\mathbf{h}^{(\ell)} &= \mathbf{e}_M^T \left(\tilde{\mathbf{A}}^{\ell} - \mathbf{A}_0^{\ell} \right) \\ &= \mathbf{e}_M^T \left(\left((\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1} \right)^{\ell} - \mathbf{A}_0^{\ell} \right) \ . \end{aligned} \tag{30}$$

The randomness of $\mathbf{h}^{(\ell)}$ completely comes from the error matrices. Besides, notice that the first $M-1$ rows in $\mathbf{E}^{(1)}$ equal to the last $M-1$ rows of $\mathbf{E}^{(0)}$. We gather all sources of randomness into a vector $\mathbf{g} \in \mathbb{R}^{M(M+1)}$, defined as

$$\mathbf{g} = \mathrm{vec}\left(\begin{bmatrix} \mathbf{E}^{(0)} \\ \mathbf{e}_M^T \mathbf{E}^{(1)} \end{bmatrix} \right) , \tag{31}$$

where "vec" denotes the vectorization operator, that concatenates the columns of a given matrix on top of one another. Then, we can write $\mathbf{h}^{(\ell)}$ as $\mathbf{h}^{(\ell)} = f^{(\ell)}(\mathbf{g})$ where $f^{(\ell)}$ is a deterministic function such that:

$$\begin{aligned} f^{(\ell)} : \mathbb{R}^{M(M+1)} &\to \mathbb{R}^M \\ \mathbf{g} &\mapsto \mathbf{h}^{(\ell)} \ . \end{aligned} \tag{32}$$

In the following paragraph, we provide some useful lemmas to prove Theorem 1.

2) Lemmas:

Lemma 1 (Expressions of $\mathbf{A}_0$ and $\mathbf{S}^{(0)^{-1}}$)**.** *Let $\mathbf{S}^{(0)}$ be the $M \times M$ matrix defined in* (27)*. Let $\mathbf{A}_0$ the $M \times M$ matrix defined in* (29)*. Assume the deterministic signal $\mathbf{z}$ takes the form* (12)*, and the observed noisy signal takes the form* (14)*. Then, the inverse of the matrix $\mathbf{S}^{(0)}$ is given by*

$$\mathbf{S}^{(0)^{-1}}[m, m'] = \frac{1}{\sigma^2}\delta_{m,m'} - \frac{2}{M\sigma^2}\sum_{j=1}^{J} \frac{1}{1 + \frac{4\sigma^2}{M\Omega_j^2}} \cos\left(2\pi p_j \frac{m - m'}{M} \right) , \tag{33}$$

and the matrix $\mathbf{A}_0$ is given by

$$\mathbf{A}_0[m, m'] = \delta_{m+1,m'} + \frac{2}{M}\sum_{j=1}^{J} \frac{1}{1 + \frac{4\sigma^2}{M\Omega_j^2}} \cos\left(2\pi p_j \frac{m'}{M} \right) \delta_{m+1,M} \, . \tag{34}$$

Let $\|\cdot\|_{\max}$ denote the maximum norm of a matrix, i.e. $\|\mathbf{M}\|_{\max} = \max_{n,n'} |\mathbf{M}[n, n']|$. Then,

$$\left\| \mathbf{S}^{(0)^{-1}} \right\|_{\max} \leq \frac{1}{\sigma^2}\left(1 + \frac{2J}{M} \right) , \tag{35}$$

$$\|\mathbf{A}_0\|_{\max} \leq \max\left(1, \frac{2J}{M} \right) \ . \tag{36}$$

Proof. It follows from the signal model (12) that the matrices $\mathbf{S}^{(0)}$ and $\mathbf{S}^{(1)}$ take the following form:

$$\begin{aligned} \mathbf{S}^{(a)}[m, m'] &= \sigma^2 \delta_{(m+a),m'} + \sum_{j,j'=1}^{J} \frac{\Omega_j \Omega_{j'}}{K} \sum_{k=0}^{K-1} \cos\left(2\pi \frac{f_j}{f_\mathrm{s}}(N_0 + m + a + k) + \varphi_j \right) \cos\left(2\pi \frac{f_{j'}}{f_\mathrm{s}}(N_0 + m' + k) + \varphi_{j'} \right) \\ &= \sigma^2 \delta_{(m+a),m'} + \sum_{j=1}^{J} \frac{\Omega_j^2}{2K} \sum_{k=0}^{K-1} \cos\left(2\pi \frac{f_j}{f_\mathrm{s}}(m + a - m') \right) + \cos\left(2\pi \frac{f_j}{f_\mathrm{s}}(2k + m + a + m' + 2N_0) \right) \\ &= \sigma^2 \delta_{(m+a),m'} + \sum_{j=1}^{J} \left(\frac{\Omega_j^2}{2} \cos\left(2\pi \frac{f_j}{f_\mathrm{s}}(m + a - m') \right) + \underbrace{\frac{\Omega_j^2}{2K} \sum_{k=0}^{K-1} \cos\left(2\pi \frac{f_j}{f_\mathrm{s}}(2k + m + a + m' + 2N_0) \right)}_{=0 \text{ because } \frac{f_j}{f_\mathrm{s}} = \frac{p_j'}{K}} \right) \\ &= \sigma^2 \delta_{(m+a),m'} + \sum_{j=1}^{J} \frac{\Omega_j^2}{2} \cos\left(2\pi \frac{f_j}{f_\mathrm{s}}(m + a - m') \right) \ . \end{aligned} \tag{37}$$

Thus, $\mathbf{S}^{(0)}$ is a circulant matrix, and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U}\boldsymbol{\Lambda}^{(0)}\mathbf{U}^* \ ,$$

where $\mathbf{U}[m,m'] = \frac{1}{\sqrt{M}} e^{-2\mathrm{i}\pi m m'/M}$ and $\mathbf{\Lambda}^{(0)} = \mathrm{diag}(\lambda_0^{(0)}, \ldots, \lambda_{M-1}^{(0)})$ with

$$\begin{aligned}\lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^{J} \frac{\Omega_j^2}{2} \sum_{q=0}^{M-1} \cos\left(2\pi \frac{f_j}{f_\mathsf{s}} q\right) e^{-2\mathrm{i}\pi q m/M} \\ &= \sigma^2 + \frac{M}{4} \sum_{j=1}^{J} \Omega_j^2 (\delta_{m,p_j} + \delta_{m,M-p_j}) \ .\end{aligned}$$

Therefore,

$$\mathbf{S}^{(0)^{-1}} = \mathbf{U} \mathbf{\Lambda}^{(0)^{-1}} \mathbf{U}^* \,,$$

which leads to

$$\mathbf{S}^{(0)^{-1}}[m,m'] = \frac{1}{\sigma^2} \delta_{m,m'} - \frac{2}{M\sigma^2} \sum_{j=1}^{J} \frac{1}{1 + \frac{4\sigma^2}{M\Omega_j^2}} \cos\left(2\pi p_j \frac{m-m'}{M}\right) \ . \tag{38}$$

Directly, we have:

$$\begin{aligned}\left|\mathbf{S}^{(0)^{-1}}[m,m']\right| &= \frac{1}{\sigma^2} \left| \delta_{m,m'} - \frac{2}{M} \sum_{j=1}^{J} \frac{1}{1 + \frac{4\sigma^2}{M\Omega_j^2}} \cos\left(2\pi p_j \frac{m-m'}{M}\right) \right| \\ &\le \frac{1}{\sigma^2} \left(1 + \frac{2}{M} \sum_{j=1}^{J} \frac{1}{1 + \frac{4\sigma^2}{M\Omega_j^2}} \right) \le \frac{1}{\sigma^2} \left(1 + \frac{2J}{M}\right) \ .\end{aligned}$$

Thus,

$$\left\| \mathbf{S}^{(0)^{-1}} \right\|_{\max} \le \frac{1}{\sigma^2} \left(1 + \frac{2J}{M}\right) \ .$$

Furthermore, combining equations (37) and (38), we have

$$\begin{aligned}\mathbf{A}_0[m,m'] &= \sum_{q=0}^{M-1} \mathbf{S}^{(1)}[m,q] \mathbf{S}^{(0)^{-1}}[q,m'] \\ &= \delta_{m+1,m'} + \frac{2}{M} \sum_{j=1}^{J} \frac{1}{1 + \frac{4\sigma^2}{M\Omega_j^2}} \cos\left(2\pi p_j \frac{m'}{M}\right) \delta_{m+1,M} \,.\end{aligned}$$

Directly, we have:

$$\left|\mathbf{A}_0[m,m']\right| \le \begin{cases} 1 & \text{if} \quad m < M-1 \,, \\ \frac{2}{M} \sum_{j=1}^{J} \frac{1}{1 + \frac{4\sigma^2}{M\Omega_j^2}} & \text{if} \quad m = M-1 \ . \end{cases}$$

Thus,

$$\|\mathbf{A}_0\|_{\max} \le \max\left(1, \frac{2J}{M}\right) \ . \ \blacksquare$$

Lemma 2 (Moments of $\mathbf{g}$)**.** *Let $\mathbf{g} \in \mathbb{R}^{M(M+1)}$ be the random vector defined in* (31)*. Assume the deterministic signal $\mathbf{z}$ takes the form* (12)*, and the observed noisy signal takes the form* (14)*. Then, as $K \to \infty$, the second-order moments of $\mathbf{g}$ are bounded as follows:*

$$\left|\mathbf{E}\{\mathbf{g}[r]\mathbf{g}[r']\}\right| \le \frac{1}{K} \left(C_\mathbf{z}^2 \sigma^2 + 2\sigma^4\right) + o\left(\frac{1}{K}\right) \,, \quad \forall (r,r') \in \{0, \ldots, M(M+1)-1\}^2 \,, \tag{39}$$

where $C_\mathbf{z} \triangleq 2\left(\sum_{j=1}^{J} \Omega_j\right)$. Besides, higher-order moments of $\mathbf{g}$ behave as $o\left(\frac{1}{K}\right)$.

Proof. In the following, for all $r \in \{0, \dots, M(M+1)-1\}$, we denote $\mathbf{g}[r] = \sigma \mathbf{g}_1[r] + \sigma^2 \mathbf{g}_2[r]$, where

$$\mathbf{g}_1[r] = \mathbf{E}_1^{(a_r)}[m_r, m'_r] = \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{z}_k[m_r + a_r]\mathbf{w}_k[m'_r] + \mathbf{w}_k[m_r + a_r]\mathbf{z}_k[m'_r] \,,$$
$$\mathbf{g}_2[r] = \mathbf{E}_2^{(a_r)}[m_r, m'_r] = \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{w}_k[m_r + a_r]\mathbf{w}_k[m'_r] - \delta_{m_r + a_r, m'_r} \,,$$

and m_r, m'_r, a_r are the corresponding coordinates of the matrices associated with r through the vectorization operation (31). Thus, order-two moments of this random vector is split as follows:

$$\mathbf{E}\{\mathbf{g}[r]\mathbf{g}[r']\} = \sigma^2 \mathbf{E}\{\mathbf{g}_1[r]\mathbf{g}_1[r']\} + \sigma^3 \mathbf{E}\{\mathbf{g}_1[r]\mathbf{g}_2[r']\} + \sigma^3 \mathbf{E}\{\mathbf{g}_2[r]\mathbf{g}_1[r']\} + \sigma^4 \mathbf{E}\{\mathbf{g}_1[r]\mathbf{g}_1[r']\} \ . \tag{40}$$

By definition of the signal $\mathbf{z}$ (see equation (12)), we have $|\mathbf{z}[n]| \le \sum_{j=1}^{J} \Omega_j$, for all $n \in \mathbb{N}$. Thus, by a direct bound, we have

$$\begin{aligned}
\left|\mathbf{E}\{\mathbf{g}_1[r]\mathbf{g}_1[r']\}\right| &= \Bigg|\frac{1}{K^2}\mathbb{E}\Bigg\{ \sum_{k,k'=0}^{K-1} \mathbf{z}_k[m_r + a_r]\mathbf{z}_{k'}[m_{r'} + a_{r'}]\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m'_{r'}] + \mathbf{z}_k[m'_r]\mathbf{z}_{k'}[m_{r'} + a_{r'}]\mathbf{w}_k[m_r + a_r]\mathbf{w}_{k'}[m'_{r'}] \\
&\qquad + \mathbf{z}_k[m_r + a_r]\mathbf{z}_{k'}[m'_{r'}]\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m_{r'} + a_{r'}] + \mathbf{z}_k[m'_r]\mathbf{z}_{k'}[m'_{r'}]\mathbf{w}_k[m_r + a_r]\mathbf{w}_{k'}[m_{r'} + a_{r'}]\Bigg\}\Bigg| \\
&\le \frac{1}{K^2}\left(\sum_{j=1}^{J}\Omega_j\right)^2 \sum_{k,k'=0}^{K-1} \left|\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m'_{r'}]\right\}\right| + \left|\mathbb{E}\left\{\mathbf{w}_k[m_r + a_r]\mathbf{w}_{k'}[m'_{r'}]\right\}\right| \\
&\qquad + \left|\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m_{r'} + a_{r'}]\right\}\right| + \left|\mathbb{E}\left\{\mathbf{w}_k[m_r + a_r]\mathbf{w}_{k'}[m_{r'} + a_{r'}]\right\}\right|
\end{aligned} \tag{41}$$

Besides, since $\mathbf{w}$ is a white noise,

$$\begin{aligned}
\frac{1}{K^2}\sum_{k,k'=0}^{K-1}\left|\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m'_{r'}]\right\}\right| &= \frac{1}{K^2}\sum_{k,k'=0}^{K-1}\delta_{k+m'_r,\,k'+m'_{r'}} \\
&= \frac{1}{K^2}\left(K - \left|m'_r - m'_{r'}\right|\right) \ .
\end{aligned}$$

Moreover, $0 \le \left|m'_r - m'_{r'}\right| \le M-1$. Thus,

$$\frac{1}{K} - \frac{M-1}{K^2} \le \frac{1}{K^2}\sum_{k,k'=0}^{K-1}\left|\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m'_{r'}]\right\}\right| \le \frac{1}{K} \ .$$

Therefore,

$$\frac{1}{K^2}\sum_{k,k'=0}^{K-1}\left|\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m'_{r'}]\right\}\right| = \frac{1}{K} + o\left(\frac{1}{K}\right) \ .$$

Similar calculations lead to the same results for the other three terms making up the sum (41). Therefore, we have:

$$\begin{aligned}
\left|\mathbf{E}\{\mathbf{g}_1[r]\mathbf{g}_1[r']\}\right| &\le \left(\sum_{j=1}^{J}\Omega_j\right)^2 \left(\frac{4}{K} + o\left(\frac{1}{K}\right)\right) \\
&\le \frac{C_{\mathbf{z}}^2}{K} + o\left(\frac{1}{K}\right) \ .
\end{aligned} \tag{42}$$

Besides, since odd-order moments of a zero-mean multivariate Gaussian random vector are zero, we have:

$$\begin{aligned}
\mathbf{E}\{\mathbf{g}_1[r]\mathbf{g}_2[r']\} &= \frac{1}{K^2}\sum_{k,k'=0}^{K-1} \mathbf{z}_k[m_r + a_r]\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m_{r'} + a_{r'}]\mathbf{w}_{k'}[m'_{r'}]\right\} + \mathbf{z}_k[m'_r]\mathbb{E}\left\{\mathbf{w}_k[m_r + a_r]\mathbf{w}_{k'}[m_{r'} + a_{r'}]\mathbf{w}_{k'}[m'_{r'}]\right\} \\
&\qquad - \delta_{m_{r'} + a_{r'}, m'_{r'}}\mathbf{E}\{\mathbf{g}_1[r]\} \\
&= 0 \ .
\end{aligned} \tag{43}$$

Similarly,

$$\mathbf{E}\{\mathbf{g}_2[r]\mathbf{g}_1[r']\} = 0 \ . \tag{44}$$

Besides, by a direct calculation, we have:

$$
\begin{aligned}
\mathbf{E}\{\mathbf{g}_2[r]\mathbf{g}_2[r']\} = \frac{1}{K^2}\sum_{k,k'=0}^{K-1} \mathbb{E}\left\{\mathbf{w}_k[m_r+a_r]\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m_{r'}+a_{r'}]\mathbf{w}_{k'}[m'_{r'}]\right\} - \delta_{m_r+a_r,m'_r}\frac{1}{K}\sum_{k'=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_{k'}[m_{r'}+a_{r'}]\mathbf{w}_{k'}[m'_{r'}]\right\} \\
- \delta_{m_{r'}+a_{r'},m'_{r'}}\frac{1}{K}\sum_{k=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m'_{r'}]\right\} + \delta_{m_r+a_r,m'_r}\delta_{m_{r'}+a_{r'},m'_{r'}} \\
= \frac{1}{K^2}\sum_{k,k'=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_k[m_r+a_r]\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m_{r'}+a_{r'}]\mathbf{w}_{k'}[m'_{r'}]\right\} - \delta_{m_r+a_r,m'_r}\delta_{m_{r'}+a_{r'},m'_{r'}}
\end{aligned}
\tag{45}
$$

Moreover, using the results of the Isserlis' theorem **[?]** to express fourth-order moments of a Gaussian random vector as a function of its second-order moments, we expand sum (45) as follows:

$$
\begin{aligned}
\mathbf{E}\{\mathbf{g}_2[r]\mathbf{g}_2[r']\} = \frac{1}{K^2}\sum_{k,k'=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_k[m_r+a_r]\mathbf{w}_k[m'_r]\right\}\mathbb{E}\left\{\mathbf{w}_{k'}[m_{r'}+a_{r'}]\mathbf{w}_{k'}[m'_{r'}]\right\} - \delta_{m_r+a_r,m'_r}\delta_{m_{r'}+a_{r'},m'_{r'}} \\
+ \frac{1}{K^2}\sum_{k,k'=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_k[m_r+a_r]\mathbf{w}_{k'}[m_{r'}+a_{r'}]\right\}\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m'_{r'}]\right\} \\
+ \frac{1}{K^2}\sum_{k,k'=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_k[m_r+a_r]\mathbf{w}_{k'}[m'_{r'}]\right\}\mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_{k'}[m_{r'}+a_{r'}]\right\} \\
= \frac{1}{K^2}\sum_{k,k'=0}^{K-1}\delta_{k+m_r+a_r,k'+m_{r'}+a_{r'}}\delta_{k+m'r,k'+m'_{r'}} + \delta_{k+m_r+a_r,k'+m'_{r'}}\delta_{k+m'_r,k'+m_{r'}+a_{r'}} \\
= \frac{1}{K^2}\left(\delta_{m'_r-m'_{r'},m_r+a_r-m_{r'}-a_{r'}}(K-|m'_r-m'_{r'}|) + \delta_{m_r+a_r-m'_{r'},m'_r-m_{r'}-a_{r'}}(K-|m_r+a_r-m'_{r'}|)\right) \\
= \frac{1}{K}\left(\delta_{m'_r-m'_{r'},m_r+a_r-m_{r'}-a_{r'}} + \delta_{m'_r+m'_{r'},m_r+a_r+m_{r'}+a_{r'}}\right) + o\left(\frac{1}{K}\right) .
\end{aligned}
$$

Therefore,

$$
\left|\mathbf{E}\{\mathbf{g}_2[r]\mathbf{g}_2[r']\}\right| \le \frac{2}{K} + o\left(\frac{1}{K}\right) . \tag{46}
$$

Thus, combining results (42), (43), (44) and (46) into expression (40) gives the following bound:

$$
\left|\mathbf{E}\{\mathbf{g}[r]\mathbf{g}[r']\}\right| \le \frac{1}{K}\left(C_{\mathbf{z}}^2\sigma^2 + 2\sigma^4\right) + o\left(\frac{1}{K}\right) .
$$

Concerning higher-order moments, let $T \ge 3$ denote the order of the moment defined by $\mathbb{E}\{\prod_{\theta=1}^T \mathbf{g}[r_\theta]\}$. Thus,

$$
\begin{aligned}
\mathbb{E}\left\{\prod_{\theta=1}^T \mathbf{g}[r_\theta]\right\} &= \mathbb{E}\left\{\prod_{\theta=1}^T\left(\sigma\mathbf{g}_1[r_\theta] + \sigma^2\mathbf{g}_2[r_\theta]\right)\right\} \\
&= \frac{1}{K^T}\mathbb{E}\left\{\prod_{\theta=1}^T\left(\sum_{k=0}^{K-1}\rho_{\theta,k}\left(\mathbf{w}[k+m'_{r_\theta}],\mathbf{w}[k+m_{r_\theta}+a_{r_\theta}]\right)\right)\right\} \\
&= \frac{1}{K^T}\sum_{k_1=0}^{K-1}\cdots\sum_{k_T=0}^{K-1}\mathbb{E}\left\{\prod_{\theta=1}^T\rho_{\theta,k_\theta}\left(\mathbf{w}[k_\theta+m'_{r_\theta}],\mathbf{w}[k_\theta+m_{r_\theta}+a_{r_\theta}]\right)\right\} ,
\end{aligned}
\tag{47}
$$

where

$$
\rho_{\theta,k}(u,v) = \sigma\mathbf{z}_k[m_{r_\theta}+a_{r_\theta}]u + \sigma\mathbf{z}_k[m'_{r_\theta}]v + \sigma^2 uv - \delta_{m_{r_\theta}+a_{r_\theta},m'_{r_\theta}} .
$$

Thus,

$$
\left|\mathbb{E}\left\{\prod_{\theta=1}^T\mathbf{g}[r_\theta]\right\}\right| \le \frac{\nu_K}{K^T}C_T ,
$$

where

$$
C_T = \max_{(k_1,\ldots,k_T)}\left|\mathbb{E}\left\{\prod_{\theta=1}^T\rho_{\theta,k_\theta}(\mathbf{w}[k_\theta+m'_{r_\theta}],\mathbf{w}[k_\theta+m_{r_\theta}+a_{r_\theta}])\right\}\right| ,
$$

and ν_K is the number of nonzero terms in the sum (47). Note that C_T is independent of K (but depends on $\mathbf{z}$ and σ). The behavior of the order T moment in function of K is therefore only determined by the ratio $\dfrac{\nu_K}{K^T}$.

Let us bound ν_K. Fix k_1. For each of the other indexes of summation k_θ ($\theta \in \{2,\ldots,T\}$), there are four values that make $\rho_{\theta,k_\theta}(\mathbf{w}[k_\theta+m'_{r_\theta}],\mathbf{w}[k_\theta+m_{r_\theta}+a_{r_\theta}])$ not independent of $\rho_{1,k_1}(\mathbf{w}[k_1+m'_{r_1}],\mathbf{w}[k_1+m_{r_1}+a_{r_1}])$. Indeed, since $\mathbf{w}$ is a white noise these quantities are independent except when k_θ is such that:

$$\begin{aligned}
k_\theta &= k_1 + m'_{r_1} - m'_{r_\theta}\\
k_\theta &= k_1 + m_{r_1} + a_{r_1} - m'_{r_\theta}\\
k_\theta &= k_1 + m'_{r_1} - m_{r_\theta} - a_{r_\theta}\\
k_\theta &= k_1 + m_{r_1} + a_{r_1} - m_{r_\theta} - a_{r_\theta}\ .
\end{aligned}$$

Consequently, for each value of k_1 there exist at least $(K-4)^{T-1}$ combinations of $(k_2,\ldots,k_T)$ where we have

$$\begin{aligned}
\mathbb{E}\left\{\prod_{\theta=1}^{T}\rho_{\theta,k_\theta}(\mathbf{w}[k_\theta+m'_{r_\theta}],\mathbf{w}[k_\theta+m_{r_\theta}+a_{r_\theta}])\right\} &= \mathbb{E}\left\{\rho_{1,k_1}(\mathbf{w}[k_1+m'_{r_1}],\mathbf{w}[k_1+m_{r_1}+a_{r_1}])\right\}\\
&\quad\times\mathbb{E}\left\{\prod_{\theta=2}^{T}\rho_{\theta,k_\theta}(\mathbf{w}[k_\theta+m'_{r_\theta}],\mathbf{w}[k_\theta+m_{r_\theta}+a_{r_\theta}])\right\} \qquad = 0\,,
\end{aligned}$$

because $\mathbb{E}\left\{\rho_{1,k_1}(\mathbf{w}[k_1+m'_{r_1}],\mathbf{w}[k_1+m_{r_1}+a_{r_1}])\right\} = 0$. Therefore, at least $K(K-4)^{T-1}$ of the sum (47) are zero. Because $T \geq 3$, we develop similar arguments on k_2 and k_3 to determine other cases where this correlation term vanishes. We subtract these cases to K^T, the total number of combinations of $(k_1,\ldots,k_T)$ to obtain the following maximum bound on the number of nonzero terms in the sum (47):

$$\nu_K \leq K^T - 3K(K-4)^{T-1} + 3K(K-4)(K-8)^{T-2} - K(K-4)(K-8)(K-12)^{T-3}\ .$$

Thus,

$$\begin{aligned}
\left|\mathbb{E}\left\{\prod_{\theta=1}^{T}\mathbf{g}[r_\theta]\right\}\right| &\leq C_T\frac{K^T - 3K(K-4)^{T-1} + 3K(K-4)(K-8)^{T-2} - K(K-4)(K-8)(K-12)^{T-3}}{K^T}\\
&\leq C_T\left(1-3\left(1-\frac{4}{K}\right)^{T-1}+3\left(1-\frac{4}{K}\right)\left(1-\frac{8}{K}\right)^{T-2}-\left(1-\frac{4}{K}\right)\left(1-\frac{8}{K}\right)\left(1-\frac{12}{K}\right)^{T-3}\right)\\
&\leq C_T\left(1-3+\frac{12(T-1)}{K}+3-\frac{12}{K}-\frac{24(T-2)}{K}-1+\frac{4}{K}+\frac{8}{K}+\frac{12(T-3)}{K}+o\left(\frac{1}{K}\right)\right)\\
&\leq C_T\ o\left(\frac{1}{K}\right)\ .
\end{aligned}$$

Therefore,

$$\left|\mathbb{E}\left\{\prod_{\theta=1}^{T}\mathbf{g}[r_\theta]\right\}\right| = o\left(\frac{1}{K}\right)\,,\quad \forall T\geq 3\ . \ \blacksquare$$

Lemma 3 (Bounds on the derivatives of $f^{(\ell)}$ at the origin)**.** *Let $f^{(\ell)}:\mathbb{R}^{M(M+1)}\to\mathbb{R}^M$ denote the multivariate function defined in* (32)*. Assume the deterministic signal* $\mathbf{z}$ *takes the form* (12)*, and the observed noisy signal takes the form* (14)*. Then, the first-order derivatives of $f^{(\ell)}$ at the origin are bounded as follows:*

$$\left|\left.\frac{\partial f_m^{(\ell)}}{\partial\mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}}\right| \leq \frac{d_{1,\mathbf{z},M,\ell}}{\sigma^2}\,,\quad \forall m\in\{0,\ldots,M-1\}\,,\ r\in\{0,\ldots,M(M+1)-1\}\,, \tag{48}$$

where

$$d_{1,\mathbf{z},M,\ell} \triangleq (2+(\ell-2)M)\,M^{\ell-1}\left(\max\left(1,\frac{2J}{M}\right)\right)^{\ell-1}\left(1+\max\left(1,\frac{2J}{M}\right)\right)\left(1+\frac{2J}{M}\right)\ .$$

Besides, the second-order derivatives of $f^{(\ell)}$ at the origin are bounded as follows:

$$\left|\left.\frac{\partial^2 f_m^{(\ell)}}{\partial\mathbf{g}[r]\partial\mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}}\right| \leq \frac{d_{2,\mathbf{z},M,\ell}}{\sigma^4}\,,\quad \forall m\in\{0,\ldots,M-1\}\,,\ (r,r')\in\{0,\ldots,M(M+1)-1\}^2\,, \tag{49}$$

where

$$d_{2,\mathbf{z},M,\ell} \triangleq \left(1+\frac{2J}{M}\right)^2\left(\max\left(1,\frac{2J}{M}\right)\right)^{\ell-2}\left(1+\max\left(1,\frac{2J}{M}\right)\right)\left(d_{2,M,\ell}+(d_{2,M,\ell}+2d'_{2,M,\ell})\max\left(1,\frac{2J}{M}\right)\right)\,,$$

and $d_{2,M,\ell}$ and $d'_{2,M,\ell}$ are only depending on M and ℓ.

Proof. Concerning the first-order derivative, from (30), we have:

$$\frac{\partial f^{(\ell)}}{\partial \mathbf{g}[r]} = \mathbf{e}_M^T \frac{\partial \tilde{\mathbf{A}}^\ell}{\partial \mathbf{g}[r]} = \sum_{\lambda=0}^{\ell-1} \mathbf{e}_M^T \tilde{\mathbf{A}}^\lambda \frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]} \tilde{\mathbf{A}}^{\ell-1-\lambda} .$$

Thus,

$$\left.\frac{\partial f^{(\ell)}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} = \sum_{\lambda=0}^{\ell-1} \mathbf{e}_M^T \mathbf{A}_0^\lambda \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \mathbf{A}_0^{\ell-1-\lambda} . \tag{50}$$

Furthermore,

$$\begin{aligned}
\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]} &= \frac{\partial \mathbf{E}^{(1)}}{\partial \mathbf{g}[r]} \left(\mathbf{S}^{(0)} + \mathbf{E}^{(0)}\right)^{-1} + \left(\mathbf{S}^{(1)} + \mathbf{E}^{(1)}\right) \frac{\partial \left(\mathbf{S}^{(0)} + \mathbf{E}^{(0)}\right)^{-1}}{\partial \mathbf{g}[r]} \\
&= \begin{cases} \mathbf{J}_{m_r,m'_r} \left(\mathbf{S}^{(0)} + \mathbf{E}^{(0)}\right)^{-1} & \text{if} \quad a_r = 1, \\ \left((1-\delta_{m_r,0})\mathbf{J}_{m_r-1,m'_r} + \left(\mathbf{S}^{(1)} + \mathbf{E}^{(1)}\right)\left(\mathbf{S}^{(0)} + \mathbf{E}^{(0)}\right)^{-1} \mathbf{J}_{m_r,m'_r}\right)\left(\mathbf{S}^{(0)} + \mathbf{E}^{(0)}\right)^{-1} & \text{else,} \end{cases}
\end{aligned}$$

where $\mathbf{J}_{m_r,m'_r} \in \mathbb{R}^{M\times M}$ is the matrix such that $\mathbf{J}_{m_r,m'_r}[m,m'] \triangleq \delta_{m,m_r}\delta_{m',m'_r}$. Thus,

$$\left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} = \begin{cases} \mathbf{J}_{m_r,m'_r} \mathbf{S}^{(0)^{-1}} & \text{if} \quad a_r = 1, \\ \left((1-\delta_{m_r,0})\mathbf{J}_{m_r-1,m'_r} + \mathbf{A}_0 \mathbf{J}_{m_r,m'_r}\right) \mathbf{S}^{(0)^{-1}} & \text{else.} \end{cases}$$

Then,

$$\left\| \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \right\|_{\max} \leq (1 + \|\mathbf{A}_0\|_{\max}) \left\| \mathbf{S}^{(0)^{-1}} \right\|_{\max} .$$

Given bounds (36) and (35), we have that:

$$\left\| \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \right\|_{\max} \leq \left(1 + \max\left(1, \frac{2J}{M}\right)\right)\left(1 + \frac{2J}{M}\right)\frac{1}{\sigma^2} . \tag{51}$$

Besides given expression (50), for all $r \in \{0, \ldots, M(M+1)-1\}$, we have:

$$\begin{aligned}
\left| \left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \right| &\leq 2M\|\mathbf{A}_0^{\ell-1}\|_{\max} \left\| \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \right\|_{\max} + M^2 \sum_{\lambda=1}^{\ell-2} \|\mathbf{A}_0^\lambda\|_{\max} \left\| \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \right\|_{\max} \|\mathbf{A}_0^{\ell-1-\lambda}\|_{\max} \\
&\leq (2 + (\ell-2)M)\, M^{\ell-1} \|\mathbf{A}_0\|_{\max}^{\ell-1} \left\| \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \right\|_{\max} .
\end{aligned}$$

Therefore, given bounds (36) and (51), we have:

$$\left| \left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \right| \leq \frac{d_{1,\mathbf{z},M,\ell}}{\sigma^2} .$$

Concerning the second-order derivative, we have:

$$\begin{aligned}
\frac{\partial^2 f^{(\ell)}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']} &= \sum_{\lambda=0}^{\ell-1} \mathbf{e}_M^T \frac{\partial \tilde{\mathbf{A}}^\lambda}{\partial \mathbf{g}[r']} \frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]} \tilde{\mathbf{A}}^{\ell-1-\lambda} + \mathbf{e}_M^T \tilde{\mathbf{A}}^\lambda \frac{\partial^2 \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']} \tilde{\mathbf{A}}^{\ell-1-\lambda} + \mathbf{e}_M^T \tilde{\mathbf{A}}^\lambda \frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]} \frac{\partial \tilde{\mathbf{A}}^{\ell-1-\lambda}}{\partial \mathbf{g}[r']} \\
&= \sum_{\lambda=1}^{\ell-1} \sum_{p=0}^{\lambda-1} \mathbf{e}_M^T \tilde{\mathbf{A}}^p \frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r']} \tilde{\mathbf{A}}^{\lambda-1-p} \frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]} \tilde{\mathbf{A}}^{\ell-1-\lambda} + \sum_{\lambda=0}^{\ell-1} \mathbf{e}_M^T \tilde{\mathbf{A}}^l \frac{\partial^2 \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']} \tilde{\mathbf{A}}^{\ell-1-\lambda} \\
&\quad + \sum_{\lambda=0}^{\ell-2} \sum_{p=0}^{\ell-\lambda-2} \mathbf{e}_M^T \tilde{\mathbf{A}}^\lambda \frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]} \tilde{\mathbf{A}}^p \frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r']} \tilde{\mathbf{A}}^{\ell-\lambda-2-p} .
\end{aligned}$$

Thus,

$$\left.\frac{\partial^2 f^{(\ell)}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}} = \sum_{\lambda=1}^{\ell-1}\sum_{p=0}^{\lambda-1} \mathbf{e}_M^T \mathbf{A}_0^p \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}} \mathbf{A}_0^{\lambda-1-p} \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \mathbf{A}_0^{\ell-1-\lambda}$$
$$+ \sum_{\lambda=0}^{\ell-1} \mathbf{e}_M^T \mathbf{A}_0^\lambda \left.\frac{\partial^2 \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}} \mathbf{A}_0^{\ell-1-\lambda} + \sum_{\lambda=0}^{\ell-2}\sum_{p=0}^{\ell-2-\lambda} \mathbf{e}_M^T \mathbf{A}_0^\lambda \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}} \mathbf{A}_0^p \left.\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}} \mathbf{A}_0^{\ell-\lambda-2-p} \ . \tag{52}$$

Besides, the second-order derivative of the matrix $\tilde{\mathbf{A}}$ is given by

$$\frac{\partial^2 \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']} = \begin{cases} 0 & \text{if } a_r = 1 \text{ and } a'_r = 1, \\ \mathbf{J}_{m_r,m'_r}\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1}\mathbf{J}_{m_{r'},m'_{r'}}\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1} & \text{if } a_r = 1 \text{ and } a'_r = 0, \\ \mathbf{J}_{m'_r,m'_{r'}}\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1}\mathbf{J}_{m_r,m'_r}\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1} & \text{if } a_r = 0 \text{ and } a'_r = 1, \\ \left((1-\delta_{m_r,0})\mathbf{J}_{m_r-1,m'_r} + \left(\mathbf{S}^{(1)}+\mathbf{E}^{(1)}\right)\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1}\mathbf{J}_{m_r,m'_r}\right) & \\ \quad\times\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1}\mathbf{J}_{m_{r'},m'_{r'}}\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1} & \\ +\left((1-\delta_{m_{r'},0})\mathbf{J}_{m_{r'}-1,m'_{r'}} + \left(\mathbf{S}^{(1)}+\mathbf{E}^{(1)}\right)\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1}\mathbf{J}_{m_{r'},m'_{r'}}\right) & \\ \quad\times\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1}\mathbf{J}_{m_r,m'_r}\left(\mathbf{S}^{(0)}+\mathbf{E}^{(0)}\right)^{-1} & \text{else.} \end{cases}$$

Thus, the second-order derivative of the matrix $\tilde{\mathbf{A}}$ at the origin is such that

$$\left.\frac{\partial^2 \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}} = \begin{cases} 0 & \text{if } a_r = 1 \text{ and } a_{r'} = 1, \\ \mathbf{J}_{m_r,m'_r}{\mathbf{S}^{(0)}}^{-1}\mathbf{J}_{m_{r'},m'_{r'}}{\mathbf{S}^{(0)}}^{-1} & \text{if } a_r = 1 \text{ and } a_{r'} = 0, \\ \mathbf{J}_{m_{r'},m'_{r'}}{\mathbf{S}^{(0)}}^{-1}\mathbf{J}_{m_r,m'_r}{\mathbf{S}^{(0)}}^{-1} & \text{if } a_r = 0 \text{ and } a_{r'} = 1, \\ \left((1-\delta_{m_r,0})\mathbf{J}_{m_r-1,m'_r} + \mathbf{A}_0\mathbf{J}_{m_r,m'_r}\right){\mathbf{S}^{(0)}}^{-1}\mathbf{J}_{m_{r'},m'_{r'}}{\mathbf{S}^{(0)}}^{-1} & \\ +\left((1-\delta_{m_{r'},0})\mathbf{J}_{m_{r'}-1,m'_{r'}} + \mathbf{A}_0\mathbf{J}_{m_{r'},m'_{r'}}\right){\mathbf{S}^{(0)}}^{-1}\mathbf{J}_{m_r,m'_r}{\mathbf{S}^{(0)}}^{-1} & \text{else.} \end{cases}$$

Then,

$$\left\|\left.\frac{\partial^2 \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']}\right|_{\mathbf{g}=0}\right\|_{\max} \le \begin{cases} \left\|{\mathbf{S}^{(0)}}^{-1}\right\|_{\max}^2 & \text{if } a_r = 1 \text{ or } a_{r'} = 1\,, \\ 2\left(1+\|\mathbf{A}_0\|_{\max}\right)\left\|{\mathbf{S}^{(0)}}^{-1}\right\|_{\max}^2 & \text{else} \end{cases}$$
$$\le 2\left(1+\max\left(1,\frac{2J}{M}\right)\right)\left(1+\frac{2J}{M}\right)^2\frac{1}{\sigma^4}\ . \tag{53}$$

Returning to equation (52), for all $r, r' \in \{0, \ldots, M(M+1)-1\}$ and $m \in \{0, \ldots, M-1\}$, we have:

$$\left|\left.\frac{\partial^2 f_m^{(\ell)}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}}\right| \le d_{2,M,\ell}\|\mathbf{A}_0\|_{\max}^{\ell-2}\left\|\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]}\right\|_{\max}\left\|\frac{\partial \tilde{\mathbf{A}}}{\partial \mathbf{g}[r']}\right\|_{\max} + d'_{2,M,\ell}\|\mathbf{A}_0\|_{\max}^{\ell-1}\left\|\frac{\partial^2 \tilde{\mathbf{A}}}{\partial \mathbf{g}[r]\mathbf{g}[r']}\right\|_{\max}\ ,$$

where $d_{2,M,\ell}$ and $d'_{2,M,\ell}$ are only depending on M and ℓ. Besides, given results (29), (51) and (53), we have:

$$\left|\left.\frac{\partial^2 f_m^{(\ell)}}{\partial \mathbf{g}[r]\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}}\right| \le \frac{d_{2,\mathbf{z},M,\ell}}{\sigma^4}\ . \ \blacksquare$$

B. Expression of the Bias $\boldsymbol{\mu}$.

By definition of the measurement noise, $\boldsymbol{\mu}[n] = 0$ when $n \in I$. Outside the measurement interval I, denote by ℓ the index such that $n = N-1+\ell$. Then, given that $\mathbf{h}^{(\ell)} = \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)}$, we deduce from expression (17) that

$$\begin{aligned} \boldsymbol{\mu}[n] &= \mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\boldsymbol{\alpha}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[n] \\ &= \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K + \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[N-1+\ell] \\ &\overset{\Delta}{=} \boldsymbol{\epsilon}_1[\ell] + \boldsymbol{\epsilon}_2[\ell] + \boldsymbol{\epsilon}_3[\ell]\ , \end{aligned} \tag{54}$$

where

$$\boldsymbol{\epsilon}_1[\ell] = \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K - \mathbf{z}[N-1+\ell]\,, \tag{55}$$

$$\boldsymbol{\epsilon}_2[\ell] = \mathbb{E}\{\mathbf{h}^{(\ell)}\}\mathbf{z}_K\,, \tag{56}$$

$$\boldsymbol{\epsilon}_3[\ell] = \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\}\ . \tag{57}$$

Let us first determine an upper bound on $|\boldsymbol{\epsilon}_1[n]|$. Since $\boldsymbol{\alpha}_0^{(1)}$ is the last row of $\mathbf{A}_0$, we deduce from the expression (34) of $\mathbf{A}_0$ that

$$\boldsymbol{\alpha}_0^{(1)}[m] = \frac{2}{M}\sum_{j=1}^{J}\frac{1}{1+\frac{4\sigma^2}{M\Omega_j^2}}\cos\left(2\pi p_j\frac{m}{M}\right)\ . \tag{58}$$

Besides, from equation (34), we also have

$$\mathbf{A}_0\mathbf{z}_K = \begin{pmatrix}\mathbf{z}[N-M+1]\\ \vdots \\ \mathbf{z}[N-1]\\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K\end{pmatrix}\ .$$

The upward-shift property is thus successively inducted when ℓ increases; that is,

$$\mathbf{A}_0^{\ell}\mathbf{z}_K = \begin{pmatrix}\mathbf{z}[N-M+\ell]\\ \vdots \\ \mathbf{z}[N-1]\\ \boldsymbol{\alpha}_0^{(1)}\mathbf{z}_K\\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\end{pmatrix}\ .$$

Then, $\boldsymbol{\alpha}_0^{(\ell)}$, the last row of $\mathbf{A}_0^{\ell}$ follows the following recurrence relation:

$$\begin{aligned}\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K &= \boldsymbol{\alpha}_0^{(1)}\tilde{\mathbf{A}}_0^{\ell-1}\mathbf{z}_K\\ &= \sum_{m=0}^{M-\ell}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] + \sum_{m=M-\ell+1}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\boldsymbol{\alpha}_0^{(m-M+\ell)}\mathbf{z}_K\,.\end{aligned}$$

Hence,

$$\begin{aligned}\boldsymbol{\epsilon}_1[\ell] &= \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K - \mathbf{z}[N-1+\ell]\\ &= \sum_{m=0}^{M}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] - \mathbf{z}[N-1+\ell] + \sum_{m=M-\ell+1}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\left(\boldsymbol{\alpha}_0^{(m-M+\ell)}\mathbf{z}_K - \mathbf{z}[N-M+\ell+m-1]\right)\\ &= \sum_{m=0}^{M}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] - \mathbf{z}[N-1+\ell] + \sum_{m=M-\ell+1}^{M-1}\boldsymbol{\alpha}_0^{(1)}[m]\boldsymbol{\epsilon}_1[m-M+\ell]\,.\end{aligned}$$

Besides, equation (58) gives

$$\begin{aligned}\sum_{m=0}^{M}\boldsymbol{\alpha}_0^{(1)}[m]\mathbf{z}[N-M+\ell+m-1] &= \sum_{j,j'=1}^{J}\Omega_{j'}\frac{2}{M}\frac{1}{1+\frac{4\sigma^2}{M\Omega_j^2}}\underbrace{\sum_{m=0}^{M-1}\cos\left(2\pi p_j\frac{m}{M}\right)\cos\left(2\pi p_{j'}\frac{N+m}{M}+\varphi_{j'}\right)}_{=\delta_{j,j'}\frac{M}{2}\cos\left(2\pi p_j\frac{N}{M}+\varphi_j\right)}\\ &= \sum_{j=1}^{J}\Omega_j\frac{1}{1+\frac{4\sigma^2}{M\Omega_j^2}}\cos\left(2\pi p_j\frac{N}{M}+\varphi_j\right)\ .\end{aligned}$$

Thus:

$$
\begin{aligned}
|\boldsymbol{\epsilon}_1[\ell]| &= \left| \sum_{j=1}^{J} \Omega_j \left(\frac{1}{1+\frac{4\sigma^2}{M\Omega_j^2}} - 1 \right) \cos\left(2\pi p_j \frac{N}{M} + \varphi_j\right) + \sum_{m=M-\ell+1}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m] \boldsymbol{\epsilon}_1[m-M+\ell] \right| \\
&\leq \sum_{j=1}^{J} \Omega_j \left| \frac{1}{1+\frac{4\sigma^2}{M\Omega_j^2}} - 1 \right| + \frac{2J}{M} \sum_{\lambda=1}^{\ell-1} |\boldsymbol{\epsilon}_1[\lambda]| \\
&\leq \frac{4\sigma^2}{M} \sum_{j=1}^{J} \frac{1}{\Omega_j} + \frac{2J}{M} \sum_{\lambda=1}^{\ell-1} |\boldsymbol{\epsilon}_1[\lambda]| \ .
\end{aligned} \tag{59}
$$

Then, by induction from the inequality (59), we have

$$
|\boldsymbol{\epsilon}_1[\ell]| \leq \frac{4\sigma^2}{M} \left(1 + \frac{2J}{M}\right)^{\ell-1} \sum_{j=1}^{J} \frac{1}{\Omega_j} \triangleq c^{(1)} \sigma^2 \ , \tag{60}
$$

where $c^{(1)} = \frac{4}{M}\left(1+\frac{2J}{M}\right)^{\ell-1} \sum_{j=1}^{J} \frac{1}{\Omega_j}$. Note that $c^{(1)}$ is not depending on σ or K.

Let us now determine an upper bound on $|\boldsymbol{\epsilon}_2[\ell]|$. Since $\mathbb{E}\{\mathbf{g}[r]\} = 0$ and moments of order 3 and higher behave as $o\left(\frac{1}{K}\right)$, a second-order Taylor expansion of $\mathbf{h}^{(\ell)}$ gives

$$
\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\} = \frac{1}{2} \sum_{r,r'=0}^{M(M+1)-1} \left. \frac{\partial^2 f_m^{(\ell)}}{\partial \mathbf{g}[r] \partial \mathbf{g}[r']} \right|_{\mathbf{g}=\mathbf{0}} \mathbb{E}\{\mathbf{g}[r]\mathbf{g}[r']\} + o\left(\frac{1}{K}\right) \ . \tag{61}
$$

Thus, given the bounds (39) on $|\mathbb{E}\{\mathbf{g}[r]\mathbf{g}[r']\}|$ and (49) on the second derivative of $f_m^{(\ell)}$, we have:

$$
\begin{aligned}
|\boldsymbol{\epsilon}_2[\ell]| &\leq C_{\mathbf{z}} \frac{M^3(M+1)^2}{4} \left\| \left. \frac{\partial^2 f^{(\ell)}}{\partial \mathbf{g}[r] \partial \mathbf{g}[r']} \right|_{\mathbf{g}=\mathbf{0}} \right\|_{\max} \frac{1}{K} \left(C_{\mathbf{z}}^2 \sigma^2 + 2\sigma^4 \right) + o\left(\frac{1}{K}\right) \\
&\leq \frac{1}{K} \left(c_1^{(2)} + \frac{c_2^{(2)}}{\sigma^2} \right) + o\left(\frac{1}{K}\right) \ ,
\end{aligned} \tag{62}
$$

where

$$
\begin{aligned}
c_1^{(2)} &\triangleq \frac{M^3(M+1)^2}{2} C_{\mathbf{z}} d_{2,\mathbf{z},M,\ell} \ , \\
c_2^{(2)} &\triangleq \frac{M^3(M+1)^2}{4} C_{\mathbf{z}}^3 d_{2,\mathbf{z},M,\ell} \ .
\end{aligned}
$$

Let us now determine an upper bound on $|\boldsymbol{\epsilon}_3[\ell]|$. A second-order Taylor expansion of $\mathbf{h}^{(\ell)}$ gives

$$
\mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K\} = \sum_{m=0}^{M-1} \sum_{r=0}^{M(M+1)-1} \left. \frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[r]} \right|_{\mathbf{g}=\mathbf{0}} \mathbb{E}\{\mathbf{g}[r]\mathbf{w}_K[m]\} + o\left(\frac{1}{K}\right) \tag{63}
$$

Indeed, the third-order moments $\mathbb{E}\{\mathbf{g}[r]\mathbf{g}[r']\mathbf{w}_K[m]\}$, behave as $o\left(\frac{1}{K}\right)$. Besides,

$$
\mathbb{E}\{\mathbf{g}[r]\mathbf{w}_K[m]\} = \sigma \mathbb{E}\{\mathbf{g}_1[r]\mathbf{w}_K[m]\} + \sigma^2 \mathbb{E}\{\mathbf{g}_2[r]\mathbf{w}_K[m]\} \ . \tag{64}
$$

Then,

$$
\begin{aligned}
|\mathbf{E}\{\mathbf{g}_1[r]\mathbf{w}_K[m]\}| &= \frac{1}{K} \left| \sum_{k=0}^{K-1} \mathbf{z}_k[m_r + a_r] \mathbb{E}\left\{\mathbf{w}_k[m'_r]\mathbf{w}_K[m]\right\} + \mathbf{z}_k[m'_r] \mathbb{E}\left\{\mathbf{w}_k[m_r + a_r]\mathbf{w}_K[m]\right\} \right| \\
&\leq \frac{2}{K} \left(\sum_{j=1}^{J} \Omega_j \right) = \frac{C_{\mathbf{z}}}{K} \ .
\end{aligned} \tag{65}
$$

$$\mathbf{E}\{\mathbf{g}_2[r]\mathbf{w}_K[m]\} = \frac{1}{K}\sum_{k,=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_k[m_r+a_r]\mathbf{w}_k[m'_r]\mathbf{w}_K[m]\right\} - \delta_{m_r+a_r,m'_r}\mathbb{E}\left\{\mathbf{w}_K[m]\right\}$$
$$= 0\ . \tag{66}$$

Thus, combining results (65) and (66) into expression (64) gives the following bound:

$$|\mathbf{E}\{\mathbf{g}[r]\mathbf{w}_K[m]\}| \leq C_{\mathbf{z}}\frac{\sigma}{K}\ . \tag{67}$$

Thus, given the bound (48) on the first derivative of $f_m^{(\ell)}$, and from (63) we have:

$$|\boldsymbol{\epsilon}_3[\ell]| \leq \sigma M^2(M+1)\frac{d_{1,\mathbf{z},M,\ell}}{\sigma^2}C_{\mathbf{z}}\frac{\sigma}{K} \triangleq \frac{c^{(3)}}{K}\ , \tag{68}$$

where $c^{(3)} \triangleq M^2(M+1)d_{1,\mathbf{z},M,\ell}C_{\mathbf{z}}$.

Thus, combining bounds (60) on $\boldsymbol{\epsilon}_1$, (62) on $\boldsymbol{\epsilon}_2$ and (68) on $\boldsymbol{\epsilon}_3$ gives the following bound of the bias:

$$|\boldsymbol{\mu}[n]| \leq c^{(1)}\sigma^2 + \frac{1}{K}\left(\frac{c_2^{(2)}}{\sigma^2} + c_1^{(2)} + c^{(3)}\right) + o\left(\frac{1}{K}\right)\ . \tag{69}$$

C. Expression of the Covariance $\boldsymbol{\gamma}$.

Outside the measurement interval I, denote by ℓ the index such that $n = N-1+\ell$. Then, given that $\mathbf{h}^{(\ell)} = \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)}$, we have

$$\begin{aligned}
\gamma[n,n] &= \left(\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\right)^2 + 2\left(\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\right)\mathbb{E}\left\{\mathbf{h}^{(\ell)}\right\}\mathbf{z}_K + \mathbb{E}\left\{\left(\mathbf{h}^{(\ell)}\mathbf{z}_K\right)^2\right\} + 2\sigma\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K + 2\sigma\boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\} \\
&\quad + 2\sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\left\|\boldsymbol{\alpha}_0^{(\ell)}\right\|^2 + \sigma^2\mathbb{E}\left\{\left(\mathbf{h}^{(\ell)}\mathbf{w}_K\right)^2\right\} + 2\sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} - \mathbf{z}[n]^2 - 2\mathbf{z}[n]\boldsymbol{\mu}[n] - \boldsymbol{\mu}[n]^2 \\
&= \sigma^2\left\|\boldsymbol{\alpha}_0^{(\ell)}\right\|^2 - \left(\mathbf{z}[n] - \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K + \mathbf{z}[n]\right)^2 + \mathbb{E}\left\{\left(\mathbf{h}^{(\ell)}\mathbf{z}_K\right)^2\right\} + 2\sigma\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K + 2\sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K \\
&\quad + \sigma^2\mathbb{E}\left\{\left(\mathbf{h}^{(\ell)}\mathbf{w}_K\right)^2\right\} + 2\sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\} \\
&\triangleq \eta_1[n] + \eta_2[n] + \eta_3[n] + \eta_4[n] + \eta_5[n] + \eta_6[n] + \eta_7[n]\ ,
\end{aligned}$$

where

$$\eta_1[n] = \sigma^2\left\|\boldsymbol{\alpha}_0^{(\ell)}\right\|^2, \qquad \eta_2[n] = -\left(\mathbf{z}[n] - \boldsymbol{\alpha}_0^{(\ell)}\mathbf{z}_K + \mathbf{z}[n]\right)^2, \qquad \eta_3[n] = \mathbb{E}\left\{\left(\mathbf{h}^{(\ell)}\mathbf{z}_K\right)^2\right\},$$
$$\eta_4[n] = 2\sigma\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K, \qquad \eta_5[n] = \sigma\mathbb{E}\{\mathbf{h}^{(\ell)}\mathbf{w}_K\mathbf{h}^{(\ell)}\}\mathbf{z}_K, \qquad \eta_6[n] = \sigma^2\mathbb{E}\left\{\left(\mathbf{h}^{(\ell)}\mathbf{w}_K\right)^2\right\},$$
$$\eta_7[n] = 2\sigma^2\boldsymbol{\alpha}_0^{(\ell)}\mathbb{E}\{\mathbf{w}_K\mathbf{h}^{(\ell)}\mathbf{w}_K\}\ .$$

Let us now determine an upper bound on each of these terms.

First, since $\|\boldsymbol{\alpha}_0^{(\ell)}\|^2 \leq M\|\mathbf{A}_0^{\ell}\|_{\max}^2$, we have:

$$\begin{aligned}
\eta_1[n] &\leq \sigma^2 M\left\|\mathbf{A}_0^{\ell}\right\|_{\max}^2 \\
&\leq \sigma^2 M^{\ell}\left\|\mathbf{A}_0\right\|_{\max}^{2\ell} \\
&\leq \sigma^2 M^{\ell}\max\left(1, \left(\frac{2J}{M}\right)^{2\ell}\right)\ .
\end{aligned} \tag{70}$$

Second, by definition of ϵ_2 and ϵ_3 (see expressions (56) and (57)), η_2 takes the following form:

$$\eta_2[n] = (\boldsymbol{\epsilon}_2[n] + \boldsymbol{\epsilon}_3[n])^2 = o\left(\frac{1}{K}\right)\ . \tag{71}$$

Third, second-order Taylor expansions of $\mathbf{h}^{(\ell)}$ give:

$$\begin{aligned}
\eta_3[n] &\leq \frac{C_{\mathbf{z}}^2}{4}\sum_{m,m'=0}^{M-1}\left|\mathbb{E}\left\{\mathbf{h}^{(\ell)}[m]\mathbf{h}^{(\ell)}[m']\right\}\right| \\
&\leq \frac{C_{\mathbf{z}}^2}{4}\sum_{r,r'=0}^{M(M+1)-1}\left|\left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}}\left.\frac{\partial f_{m'}^{(\ell)}}{\partial \mathbf{g}[r']}\right|_{\mathbf{g}=\mathbf{0}}\mathbb{E}\{\mathbf{g}[r]\mathbf{g}[r']\}\right| + o\left(\frac{1}{K}\right) \\
&\leq \frac{C_{\mathbf{z}}^2 M^2(M+1)^2}{4}\frac{d_{1,\mathbf{z},M,\ell}^2}{\sigma^4}\frac{1}{K}\left(C_{\mathbf{z}}^2\sigma^2+2\sigma^4\right) + o\left(\frac{1}{K}\right) \\
&\leq \frac{C_{\mathbf{z}}^2 M^2(M+1)^2 d_{1,\mathbf{z},M,\ell}^2}{4K}\left(\frac{C_{\mathbf{z}}^2}{\sigma^2}+2\right) + o\left(\frac{1}{K}\right)\ .
\end{aligned} \tag{72}$$

Fourth,

$$|\eta_4[n]| \leq \sigma C_{\mathbf{z}}\left\|\mathbf{A}_0^{\ell}\right\|_{\max}\sum_{m,m'=0}^{M-1}\left|\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m']\}\right|$$

But,

$$\begin{aligned}
\left|\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m']\}\right| &\leq \sum_{r=0}^{M(M+1)-1}\left|\left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}}\right|\left|\mathbb{E}\{\mathbf{g}[r]\mathbf{w}_K[m']\}\right| + o\left(\frac{1}{K}\right) \\
&\leq M(M+1)\frac{d_{1,\mathbf{z},M,\ell}}{\sigma^2}C_{\mathbf{z}}\frac{\sigma}{K} + o\left(\frac{1}{K}\right)
\end{aligned} \tag{73}$$

Thus,

$$|\eta_4[n]| \leq M^{\ell}(M+1)C_{\mathbf{z}}^2 d_{1,\mathbf{z},M,\ell}\left(\max\left(1,\frac{2J}{M}\right)\right)^{\ell}\frac{1}{K} + o\left(\frac{1}{K}\right)\ . \tag{74}$$

Fifth and sixth, second-order Taylor expansions of $\mathbf{h}^{(\ell)}$ give

$$\eta_5[n] = o\left(\frac{1}{K}\right)\ , \tag{75}$$

$$\eta_6[n] = o\left(\frac{1}{K}\right)\ . \tag{76}$$

Seventh,

$$\begin{aligned}
|\eta_7[n]| &\leq 2\sigma^2\left\|\mathbf{A}_0^{\ell}\right\|_{\max}\sum_{m,m'=0}^{M-1}\left|\mathbb{E}\{\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m]\mathbf{w}_K[m']\}\right| \\
&\leq 2\sigma^2\left\|\mathbf{A}_0^{\ell}\right\|_{\max}\sum_{m,m'=0}^{M-1}\sum_{r=0}^{M(M+1)-1}\left|\left.\frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[r]}\right|_{\mathbf{g}=\mathbf{0}}\right|\left|\mathbb{E}\{\mathbf{g}^{(\ell)}[r]\mathbf{w}_K[m]\mathbf{w}_K[m']\}\right| + o\left(\frac{1}{K}\right)\ .
\end{aligned}$$

But,

$$\mathbb{E}\{\mathbf{g}^{(\ell)}[r]\mathbf{w}_K[m]\mathbf{w}_K[m']\} = \sigma\mathbb{E}\{\mathbf{g}_1^{(\ell)}[r]\mathbf{w}_K[m]\mathbf{w}_K[m']\} + \sigma^2\mathbb{E}\{\mathbf{g}_2^{(\ell)}[r]\mathbf{w}_K[m]\mathbf{w}_K[m']\}\ .$$

As before, since $\mathbb{E}\{\mathbf{g}_1^{(\ell)}[r]\mathbf{w}_K[m]\mathbf{w}_K[m']\}$ is a third-order moment of a multivariate zero-mean Gaussian vector, it vanishes. And,

$$\begin{aligned}
\left|\mathbb{E}\{\mathbf{g}_2^{(\ell)}[r]\mathbf{w}_K[m]\mathbf{w}_K[m']\}\right| &= \frac{1}{K}\left|\sum_{k=0}^{K-1}\mathbb{E}\left\{\mathbf{w}_k[m_r+a_r]\mathbf{w}_k[m_r']\mathbf{w}_K[m]\mathbf{w}_K[m']\right\} - \delta_{m_r+a_r,m_r'}\mathbb{E}\left\{\mathbf{w}_K[m]\mathbf{w}_K[m']\right\}\right| \\
&= \frac{1}{K}\left|\sum_{k=0}^{K-1}\delta_{k+m_r+a_r,K+m}\delta_{k+m_r',K+m'} + \delta_{k+m_r+a_r,K+m'}\delta_{k+m_r',K+m}\right| \\
&\leq \frac{2}{K}\ .
\end{aligned}$$

Thus,

$$\begin{aligned}|\eta_7[n]| &\leq 2\sigma^2 \left\|\mathbf{A}_0^\ell\right\|_{\max} M^3(M+1)\frac{d_{1,\mathbf{z},M,\ell}}{\sigma^2}\frac{2\sigma^2}{K} + o\left(\frac{1}{K}\right) \\ &\leq 4M^{\ell+2}(M+1)d_{1,\mathbf{z},M,\ell}\left(\max\left(1,\frac{2J}{M}\right)\right)^\ell \frac{\sigma^2}{K} + o\left(\frac{1}{K}\right) \ . \end{aligned} \tag{77}$$

To conclude, we combine the expressions (70), (71), (72), (74), (75), (76), and (77) to determine an upper bound on the variance $\gamma[n,n]$. It follows:

$$\gamma[n,n] \leq c_0^{(n)}\sigma^2 + \frac{1}{K}\left(\frac{c_1^{(n)}}{\sigma^2} + c_2^{(n)} + c_3^{(n)}\sigma^2\right) + o\left(\frac{1}{K}\right) \ ,$$

where

$$\begin{aligned}c_0^{(n)} &= M^\ell \max\left(1, \left(\frac{2J}{M}\right)^{2\ell}\right) \\ c_1^{(n)} &= \frac{1}{4}C_{\mathbf{z}}^4 M^2(M+1)^2 d_{1,\mathbf{z},M,\ell}^2 \\ c_2^{(n)} &= \frac{1}{2}C_{\mathbf{z}}^2 M^2(M+1)^2 d_{1,\mathbf{z},M,\ell}^2 + M^\ell(M+1)C_{\mathbf{z}}^2 d_{1,\mathbf{z},M,\ell}\left(\max\left(1,\frac{2J}{M}\right)\right)^\ell \\ c_3^{(n)} &= 4M^{\ell+2}(M+1)d_{1,\mathbf{z},M,\ell}\left(\max\left(1,\frac{2J}{M}\right)\right)^\ell \ .\end{aligned}$$

a) If $n' \geq N$: When $n \geq N$ and $n' \geq N$, applying the Cauchy-Schwarz inequality, we obtain the following bound on the covariance $\gamma[n,n]$:

$$\begin{aligned}\left|\gamma[n,n']\right| &\leq \sqrt{\gamma[n,n]\gamma[n',n']} \\ &\leq \sqrt{c_0^{(n)}c_0^{(n')}}\sigma^2\sqrt{1 + \frac{1}{K\sigma^2}\left(\frac{1}{c_0^{(n)}}\left(\frac{c_1^{(n)}}{\sigma^2} + c_2^{(n)} + c_3^{(n)}\sigma^2\right) + \frac{1}{c_0^{(n')}}\left(\frac{c_1^{(n')}}{\sigma^2} + c_2^{(n')} + c_3^{(n')}\sigma^2\right)\right) + o\left(\frac{1}{K}\right)} \ .\end{aligned}$$

A first-order Taylor expansion of this bound as $K \to \infty$ gives

$$\begin{aligned}&\leq \sqrt{c_0^{(n)}c_0^{(n')}}\sigma^2 + \frac{1}{2K}\left(\sqrt{\frac{c_0^{(n')}}{c_0^{(n)}}}\left(\frac{c_1^{(n)}}{\sigma^2} + c_2^{(n)} + c_3^{(n)}\sigma^2\right) + \sqrt{\frac{c_0^{(n)}}{c_0^{(n')}}}\left(\frac{c_1^{(n')}}{\sigma^2} + c_2^{(n')} + c_3^{(n')}\sigma^2\right)\right) + o\left(\frac{1}{K}\right) \\ &\leq c_0^{(n,n')}\sigma^2 + \frac{1}{K}\left(\frac{c_1^{(n,n')}}{\sigma^2} + c_2^{(n,n')} + c_3^{(n,n')}\sigma^2\right) + o\left(\frac{1}{K}\right) \ ,\end{aligned}$$

where

$$\begin{aligned}c_0^{(n,n')} &= \sqrt{c_0^{(n)}c_0^{(n')}} \, , \\ c_p^{(n,n')} &= \frac{1}{2}\left(c_p^{(n)}\sqrt{\frac{c_0^{(n')}}{c_0^{(n)}}} + c_p^{(n')}\sqrt{\frac{c_0^{(n)}}{c_0^{(n')}}}\right) \ , \quad \forall p \in \{1,2,3\} \ .\end{aligned}$$

b) If $n' \in I$: When $n \geq N$ and $n' \in I$, equation (18) shows us that:

$$\begin{aligned}\gamma[n,n'] &= \sigma\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\}\mathbf{z}_K + \sigma^2\mathbb{E}\{\mathbf{w}[n']\boldsymbol{\alpha}_0^{(\ell)}\mathbf{w}_K\} + \sigma^2\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\mathbf{w}_K\} \\ &\triangleq \beta_1[n,n'] + \beta_2[n,n'] + \beta_3[n,n'] \ ,\end{aligned}$$

where

$$\begin{aligned}\beta_1[n,n'] &= \sigma\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\}\mathbf{z}_K \, , \\ \beta_2[n,n'] &= \sigma^2\mathbb{E}\{\mathbf{w}[n']\boldsymbol{\alpha}_0^{(\ell)}\mathbf{w}_K\} \, , \\ \beta_3[n,n'] &= \sigma^2\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}\mathbf{w}_K\} \ .\end{aligned}$$

Besides, thanks to the bound (73) we have:

$$\begin{aligned}\left|\beta_1[n,n']\right| &\leq \sigma\frac{C_{\mathbf{z}}}{2}\sum_{m=0}^{M-1}\left|\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}[m]\}\right| \\ &\leq \sigma\frac{C_{\mathbf{z}}}{2}M^2(M+1)\frac{d_{1,\mathbf{z},M,\ell}}{\sigma^2}C_{\mathbf{z}}\frac{\sigma}{K}+o\left(\frac{1}{K}\right) \\ &\leq M^2(M+1)\frac{C_{\mathbf{z}}^2 d_{1,\mathbf{z},M,\ell}}{2}\frac{1}{K}+o\left(\frac{1}{K}\right)\ .\end{aligned} \tag{78}$$

Besides,

$$\begin{aligned}\left|\beta_2[n,n']\right| &\leq \sigma^2\left\|\mathbf{A}_0^{\ell}\right\|_{\max}\sum_{m=0}^{M-1}\left|\mathbb{E}\{\mathbf{w}[n']\mathbf{w}_K[m]\}\right| \\ &\leq \sigma^2\left\|\mathbf{A}_0^{\ell}\right\|_{\max} \\ &\leq \sigma^2 M^{\ell-1}\left(\max\left(1,\frac{2J}{M}\right)\right)^{\ell}\ .\end{aligned} \tag{79}$$

Besides, identically to the bound (77) on η_7, we obtain

$$\begin{aligned}\left|\beta_3[n,n']\right| &\leq \sigma^2\sum_{m=0}^{M-1}\left|\mathbb{E}\{\mathbf{w}[n']\mathbf{h}^{(\ell)}[m]\mathbf{w}_K[m]\}\right| \\ &\leq \sigma^2 M^3(M+1)\frac{d_{1,\mathbf{z},M,\ell}}{\sigma^2}\frac{4\sigma^2}{K}+o\left(\frac{1}{K}\right) \\ &\leq M^3(M+1)d_{1,\mathbf{z},M,\ell}\frac{4\sigma^2}{K}+o\left(\frac{1}{K}\right)\ .\end{aligned} \tag{80}$$

Finally, we combine expressions (78), (79), and (80) to determine an upper bound on the variance $\gamma[n,n']$. It follows:

$$\left|\gamma[n,n']\right| \leq b_0^{(n,n')}\sigma^2+\frac{1}{K}\left(b_1^{(n,n')}+b_2^{(n)}\sigma^2\right)+o\left(\frac{1}{K}\right)\ ,$$

where

$$\begin{aligned}b_0^{(n,n')} &= M^{\ell-1}\left(\max\left(1,\frac{2J}{M}\right)\right)^{\ell} \\ b_1^{(n,n')} &= M^2(M+1)\frac{C_{\mathbf{z}}^2 d_{1,\mathbf{z},M,\ell}}{2} \\ b_2^{(n,n')} &= 4M^3(M+1)d_{1,\mathbf{z},M,\ell}\ .\end{aligned}$$

II. Application to an Electrocardiogram

We provide here an additional implementation of BoundEffRed, applied to an electrocardiogram (ECG) dataset. The dataset is constructed from a 500-second-long ECG, sampled at $f_s = 200$ Hz, cut into 10 segments of 50 seconds each. Fig. 8 depicts the right boundary of one of these subsignals, together with the 6-second extensions estimated by SigExt (first panel), or EDMD (second panel), GPR (third panel), or TBATS (fourth panel). These extensions are superimposed to the ground-truth extension, plotted in red. The sharp and spiky ECG patterns make the AHM model too simplistic to describe this type of signal. Consequently, the forecast produced by SigExt is moderately satisfactory.

Table IV contains the median performance index D of the boundary-free TF representations, over the N subsignals evaluated, according to the extension method. As a result of the fair quality of the forecasts, the reduction of boundary effects is less significant than for PPG signal. Nevertheles, the results show that BoundEffRed has the same efficiency when the SigExt extension, the EDMD extension or the GPR extension is chosen. Indeed, t-tests performed under the null hypothesis that the mean are equals, with a 5% significance level, show no statistical significant difference between SigExt and EDMD or GPR, regardless of the representation considered. This justifies the choice of SigExt for real-time implementation.

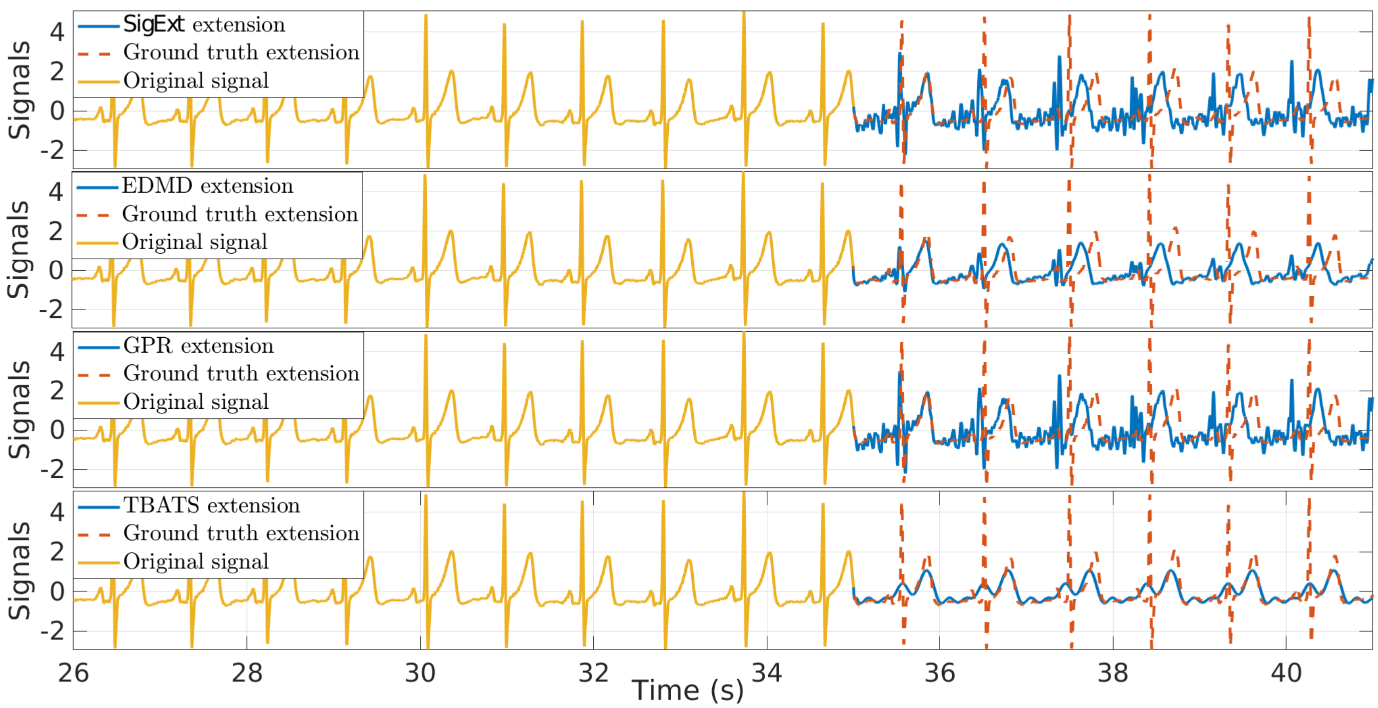

Fig. 8. Extended ECG (blue) obtained by the SigExt forecasting (first panel), the EDMD forecasting (second panel), the GPR forecasting (third panel), and the TBATS forecasting (fourth panel), superimposed with the ground truth signal (dashed red).

TABLE IV
ECG: PERFORMANCE OF THE BOUNDARY-FREE TF REPRESENTATIONS ACCORDING TO THE EXTENSION METHOD

Extension method	Median performance index D			
	STFT	SST	ConceFT	RS
SigExt	0.584	0.630	0.462	0.642
Symmetric	1.199	1.354	1.427	0.943
EDMD	0.538	0.558	0.496	0.714
GPR	0.639	0.588	0.485	0.616