

# An Approach for the Reduction of Boundary Effects in Time-Frequency Representations: Part II

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## I. PROOF OF LEMMA 1

### A. Notations

By definition the matrix  $\mathbf{X}$  and  $\mathbf{Y}$  and because of the noisy signal model (12), we have:

$$\frac{1}{K}\mathbf{X}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}\mathbf{Z}^T + \sigma^2\mathbf{I}}_{\triangleq \mathbf{S}^{(0)}} + \mathbf{E}^{(0)} \quad (27)$$

$$\frac{1}{K}\mathbf{Y}\mathbf{X}^T = \underbrace{\frac{1}{K}\mathbf{Z}'\mathbf{Z}^T + \sigma^2\mathbf{D}}_{\triangleq \mathbf{S}^{(1)}} + \mathbf{E}^{(1)}, \quad (28)$$

where  $\mathbf{E}^{(a)} = \sigma\mathbf{E}_1^{(a)} + \sigma^2\mathbf{E}_2^{(a)}$ , with:

$$\mathbf{E}_1^{(a)}[m, m'] = \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{z}[N_0 + m + a + k] \mathbf{w}[N_0 + m' + k] + \mathbf{w}[N_0 + m + a + k] \mathbf{z}[N_0 + m' + k],$$

and

$$\mathbf{E}_2^{(a)}[m, m'] = \frac{1}{K} \sum_{k=0}^{K-1} \mathbf{w}[N_0 + m + a + k] \mathbf{w}[N_0 + m' + k] - \delta_{(m+a)m'},$$

with  $a \in \{0, 1\}$ .

**Remark 1.** The matrices  $\mathbf{E}^{(0)}$  and  $\mathbf{E}^{(1)}$  are said to be error matrices because:

$$\begin{aligned} \mathbb{E}\{\mathbf{E}^{(0)}\} &= \mathbb{E}\{\mathbf{E}_1^{(0)}\} = \mathbb{E}\{\mathbf{E}_2^{(0)}\} = \mathbf{0} \\ \mathbb{E}\{\mathbf{E}^{(1)}\} &= \mathbb{E}\{\mathbf{E}_1^{(1)}\} = \mathbb{E}\{\mathbf{E}_2^{(1)}\} = \mathbf{0}. \end{aligned}$$

It follows from this:

$$\begin{aligned} \tilde{\mathbf{A}} &= (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1} \\ \mathbf{A}_0 &= \mathbf{S}^{(1)}\mathbf{S}^{(0)^{-1}}. \end{aligned}$$

Then:

$$\begin{aligned} \mathbf{h}^{(\ell)} &= \boldsymbol{\alpha}^{(\ell)} - \boldsymbol{\alpha}_0^{(\ell)} \\ &= \mathbf{e}_M^T (\tilde{\mathbf{A}}^\ell - \mathbf{A}_0^\ell) \\ &= \mathbf{e}_M^T \left( \left( (\mathbf{S}^{(1)} + \mathbf{E}^{(1)})(\mathbf{S}^{(0)} + \mathbf{E}^{(0)})^{-1} \right)^\ell - \mathbf{A}_0^\ell \right). \end{aligned} \quad (29)$$

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### B. Study of $\mathbf{h}^{(\ell)}$

The randomness of  $\mathbf{h}^{(\ell)}$  is completely originating from the error matrices. Besides, notice that the first  $M - 1$  rows in  $\mathbf{E}^{(1)}$  are equals to the last  $M - 1$  rows of  $\mathbf{E}^{(0)}$ . Thus, we gather all the sources of randomness into an vector  $\mathbf{g} \in \mathbb{R}^{M(M+1)}$ , containing  $M$  rows and defined as:

$$\mathbf{g} = \text{vec} \left( \begin{pmatrix} \mathbf{E}^{(0)} \\ \mathbf{e}_M^T \mathbf{E}^{(1)} \end{pmatrix} \right) .$$

Here, "vec" denotes the vectorization operator, concatenating the columns of a given matrix on top of one another. Then, by definition, we have:

$$\mathbf{g} = \sigma \mathbf{g}_1 + \sigma^2 \mathbf{g}_2 ,$$

where:

$$\begin{aligned} \mathbf{g}_1 &= \frac{1}{K} \sum_{k=0}^{K-1} \text{vec} \left( \tilde{\mathbf{z}}_k \mathbf{w}_k^T + \tilde{\mathbf{w}}_k \mathbf{z}_k^T \right) \\ \mathbf{g}_2 &= \frac{1}{K} \sum_{k=0}^{K-1} \text{vec} \left( \tilde{\mathbf{w}}_k \mathbf{w}_k^T - \tilde{\mathbf{I}} \right) . \end{aligned}$$

with  $\tilde{\mathbf{z}}_k^T = (\mathbf{z}_k^T \quad \mathbf{z}_{k+1}^T [M-1])$  and  $\tilde{\mathbf{w}}_k^T = (\mathbf{w}_k^T \quad \mathbf{w}_{k+1}^T [M-1])$ . Then,  $\mathbf{g}_1$  is a Gaussian random vector because it is a linear combination of Gaussian random vectors. Moreover, using the central limit theorem under weak dependence, we can show that  $\mathbf{g}_2$  also converges towards a Gaussian random vector as  $K \rightarrow \infty$ . Combining these two results gives the following result:

$$\sqrt{K} \mathbf{g} \xrightarrow[K \rightarrow \infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \Gamma_0) ,$$

where  $\Gamma_0 = \mathbb{E} \{ \mathbf{g} \mathbf{g}^T \}$  is a covariance matrix.

Furthermore, one can write  $\mathbf{h}^{(\ell)}$  as  $\mathbf{h}^{(\ell)} = f^{(\ell)}(\mathbf{g})$  where  $f^{(\ell)}$  is a deterministic function such that:

$$\begin{aligned} f^{(\ell)} : \mathbb{R}^{M(M+1)} &\rightarrow \mathbb{R}^M \\ \mathbf{g} &\mapsto \mathbf{h}^{(\ell)} . \end{aligned}$$

Then, as  $f^{(\ell)}$  is a differentiable function, using the Delta method gives:

$$\sqrt{K} \mathbf{h}^{(\ell)} \xrightarrow[K \rightarrow \infty]{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{F}^{(\ell)T} \Gamma_0 \mathbf{F}^{(\ell)}) ,$$

where  $\mathbf{F}^{(\ell)}$  is the Jacobian matrix such that:

$$\mathbf{F}^{(\ell)}[m, m'] = \left. \frac{\partial f_m^{(\ell)}}{\partial \mathbf{g}[m']} \right|_{\mathbf{g}=\mathbf{0}} .$$

## II. PROOF OF THEOREM 1

### A. Expression of the bias $\boldsymbol{\mu}$ .

Clearly,  $\boldsymbol{\mu}[n] = 0$  when  $n \in I$ . When  $n = N - 1 + \ell$ , we have:

$$\begin{aligned} \boldsymbol{\mu}[n] &= \mathbb{E} \{ \boldsymbol{\alpha}^{(\ell)} \} \mathbf{z}_K + \sigma \mathbb{E} \{ \boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K \} - \mathbf{z}[n] \\ &= \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K + \mathbb{E} \{ \mathbf{h}^{(\ell)} \} \mathbf{z}_K + \sigma \mathbb{E} \{ \mathbf{h}^{(\ell)} \mathbf{w}_K \} - \mathbf{z}[N - 1 + \ell] \end{aligned}$$

Let us first evaluate the expression of  $\alpha_0^{(\ell)} \mathbf{z}_K$ . We have:

$$\begin{aligned}
\mathbf{S}^{(a)}[m, m'] &= \sigma^2 \delta_{(m+a)m'} + \sum_{j,j'=1}^J \frac{\Omega_j \Omega_{j'}}{K} \sum_{k=0}^{K-1} \cos \left( 2\pi \frac{f_j}{f_s} (N_0 + m + a + k) + \varphi_j \right) \cos \left( 2\pi \frac{f_{j'}}{f_s} (N_0 + m' + k) + \varphi_{j'} \right) \\
&= \sigma^2 \delta_{(m+a)m'} + \sum_{j=1}^J \frac{\Omega_j^2}{2K} \sum_{k=0}^{K-1} \cos \left( 2\pi \frac{f_j}{f_s} (m + a - m') \right) + \cos \left( 2\pi \frac{f_j}{f_s} (2k + m + a + m' + 2N_0) \right) \\
&= \sigma^2 \delta_{(m+a)m'} + \sum_{j=1}^J \left( \frac{\Omega_j^2}{2} \cos \left( 2\pi \frac{f_j}{f_s} (m + a - m') \right) + \underbrace{\frac{\Omega_j^2}{2K} \sum_{k=0}^{K-1} \cos \left( 2\pi \frac{f_j}{f_s} (2k + m + a + m' + 2N_0) \right)}_{=0 \text{ because } \frac{f_j}{f_s} = \frac{p_j'}{K}} \right) \\
&= \sigma^2 \delta_{(m+a)m'} + \sum_{j=1}^J \frac{\Omega_j^2}{2} \cos \left( 2\pi \frac{f_j}{f_s} (m + a - m') \right) .
\end{aligned}$$

Thus,  $\mathbf{S}^{(0)}$  is a circulant matrix and is therefore diagonalizable in the Fourier basis:

$$\mathbf{S}^{(0)} = \mathbf{U} \mathbf{\Lambda}^{(0)} \mathbf{U}^* ,$$

where  $\mathbf{U}[m, m'] = \frac{1}{\sqrt{M}} e^{-2i\pi m m' / M}$  and  $\mathbf{\Lambda}^{(0)} = \text{diag}(\lambda_0^{(0)}, \dots, \lambda_{M-1}^{(0)})$  with:

$$\begin{aligned}
\lambda_m^{(0)} &= \sigma^2 + \sum_{j=1}^J \frac{\Omega_j^2}{2} \sum_{q=0}^{M-1} \cos \left( 2\pi \frac{f_j}{f_s} q \right) e^{-2i\pi q m / M} \\
&= \sigma^2 + \frac{M}{4} \sum_{j=1}^J \Omega_j^2 (\delta_{m,p_j} + \delta_{m,M-p_j}) .
\end{aligned}$$

Therefore:

$$\mathbf{S}^{(0)^{-1}} = \mathbf{U} \mathbf{\Lambda}^{(0)^{-1}} \mathbf{U}^*$$

which leads to:

$$\mathbf{S}^{(0)^{-1}}[m, m'] = \frac{1}{\sigma^2} \delta_{m,m'} - \sum_{j=1}^J \frac{\Omega_j^2}{2\sigma^2(\sigma^2 + \Omega_j^2 M/4)} \cos \left( 2\pi p_j \frac{m - m'}{M} \right) ,$$

and, consequently:

$$\begin{aligned}
\tilde{\mathbf{A}}_0[m, m'] &= \sum_{q=0}^{M-1} \mathbf{S}^{(1)}[m, q] \mathbf{S}^{(0)^{-1}}[q, m'] \\
&= \delta_{m+1,m'} + \sum_{j=1}^J \frac{2\Omega_j^2}{\Omega_j^2 M + 4\sigma^2} \cos \left( 2\pi p_j \frac{m'}{M} \right) \delta_{m+1,M}
\end{aligned} \tag{30}$$

Thus:

$$\begin{aligned}
\tilde{\alpha}_0^{(1)}[m] &= \sum_{j=1}^J \frac{2\Omega_j^2}{\Omega_j^2 M + 4\sigma^2} \cos \left( 2\pi p_j \frac{m}{M} \right) \\
&= \frac{2}{M} \sum_{j=1}^J \cos \left( 2\pi p_j \frac{m}{M} \right) + o(\sigma) .
\end{aligned}$$

Besides, from equation (30), we have

$$\tilde{\mathbf{A}}_0 \mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N - M + 1] \\ \vdots \\ \mathbf{z}[N - 1] \\ \alpha_0^{(1)} \mathbf{z}_K \end{pmatrix}$$

By induction, we have:

$$\tilde{\mathbf{A}}_0^\ell \mathbf{z}_K = \begin{pmatrix} \mathbf{z}[N - M + \ell] \\ \vdots \\ \mathbf{z}[N - 1] \\ \boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K \\ \vdots \\ \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K \end{pmatrix}.$$

Then:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K &= \tilde{\mathbf{A}}_0^{(1)} \tilde{\mathbf{A}}_0^{\ell-1} \mathbf{z}_K \\ &= \sum_{m=0}^{M-\ell} \boldsymbol{\alpha}_0^{(1)}[m] \mathbf{z}[N - M + \ell + m - 1] + \sum_{m=M-\ell+1}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m] \boldsymbol{\alpha}_0^{(m-M+\ell)} \mathbf{z}_K \end{aligned} \quad (31)$$

But:

$$\begin{aligned} \boldsymbol{\alpha}_0^{(1)} \mathbf{z}_K &= \sum_{m=0}^{M-1} \boldsymbol{\alpha}_0^{(1)}[m] \mathbf{z}[N - M + m] \\ &= \sum_{j,j'=1}^J \Omega_{j'} \frac{2}{M} \underbrace{\sum_{m=0}^{M-1} \cos\left(2\pi p_j \frac{m}{M}\right) \cos\left(2\pi p_{j'} \frac{N+m}{M} + \varphi_{j'}\right)}_{=\delta_{jj'} \frac{M}{2} \cos\left(2\pi p_j \frac{N}{M} + \varphi_j\right)} + o(\sigma) \\ &= \sum_{j=1}^J \Omega_j \cos\left(2\pi p_j \frac{N}{M} + \varphi_j\right) + o(\sigma) \\ &= \mathbf{z}[N] + o(\sigma) \end{aligned}$$

and, by induction from (31):

$$\boldsymbol{\alpha}_0^{(\ell)} \mathbf{z}_K = \mathbf{z}[N - 1 + \ell] + o(\sigma) \quad (32)$$

Then:

$$\boldsymbol{\mu}[N - 1 + \ell] = \mathbb{E}\{\mathbf{h}^{(\ell)}\} \mathbf{z}_K + \sigma \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K\} + o(\sigma).$$

Besides, from Lemma 1, we have the following results:

$$\begin{aligned} \mathbb{E}\{\mathbf{h}^{(\ell)}\} &\xrightarrow{K \rightarrow \infty} 0 \\ \mathbb{E}\{\mathbf{h}^{(\ell)} \mathbf{w}_K\} &\xrightarrow{K \rightarrow \infty} 0 \end{aligned}$$

Consequently:

$$\boldsymbol{\mu}[N - 1 + \ell] \underset{K \rightarrow \infty}{\sim} o(\sigma). \quad (33)$$

#### B. Expression of the covariance $\gamma$ .

We recall that  $n > N$ , we consequently denote  $n = N - 1 + \ell$ . To derive the covariance, let us segregate the cases.

a) If  $n' \in I^2$ : From Lemma 1, we have that  $h^{(\ell)}$  is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. Then, combining result (33) and equation (14), we obtain:

$$\gamma[n, n'] = \sigma \mathbb{E}\{\mathbf{w}[n'] \mathbf{h}^{(\ell)}\} \mathbf{z}_K + \sigma^2 \boldsymbol{\alpha}_0^{(\ell)} \mathbb{E}\{\mathbf{w}_K \mathbf{w}[n']\} + o(\sigma^2).$$

We remark that

$$\boldsymbol{\alpha}_0^{(\ell)} \mathbb{E}\{\mathbf{w}_K \mathbf{w}[n']\} = \boldsymbol{\alpha}_0^{(\ell)} [n' - (N - M)] \mathbb{1}_{(n' \geq N - M)}.$$

Moreover, using Delta method one can show that we have the asymptotic result:

$$\mathbb{E}\{\mathbf{w}[n'] \mathbf{h}^{(\ell)}\} \xrightarrow{K \rightarrow \infty} 0 \quad (34)$$

Finally:

$$\gamma[n, n'] \underset{K \rightarrow \infty}{\sim} \sigma^2 \boldsymbol{\alpha}_0^{(\ell)} [n' - (N - M)] \mathbb{1}_{(n' \geq N - M)} + o(\sigma^2).$$

b) If  $n' \geq N$ : From Lemma 1, we have that  $h^{(\ell)}$  is asymptotically Gaussian. Then, as a direct consequence of the Isserlis' theorem, odd-order moments are vanishing. Then, combining result (33) and equations (16) and (34), we obtain:

$$\begin{aligned}
\gamma[n, n'] &= \mathbf{z}_K^T \mathbb{E} \left\{ \boldsymbol{\alpha}^{(\ell)T} \boldsymbol{\alpha}^{(\lambda)} \right\} \mathbf{z}_K + \sigma \mathbb{E} \{ \boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K \boldsymbol{\alpha}^{(\lambda)} \} \mathbf{z}_K + \sigma \mathbb{E} \{ \boldsymbol{\alpha}^{(\lambda)} \mathbf{w}_K \boldsymbol{\alpha}^{(\ell)} \} \mathbf{z}_K \\
&\quad + \sigma^2 \mathbb{E} \{ \boldsymbol{\alpha}^{(\ell)} \mathbf{w}_K \boldsymbol{\alpha}^{(\lambda)} \mathbf{w}_K \} - \mathbf{z}[n] \mathbf{z}[n'] + o(\sigma^2) \\
&= \mathbf{z}_K^T \mathbb{E} \left\{ \mathbf{h}^{(\ell)T} \mathbf{h}^{(\lambda)} \right\} \mathbf{z}_K + \boldsymbol{\alpha}_0^{(\ell)} \sigma \mathbb{E} \{ \mathbf{w}_K \mathbf{h}^{(\lambda)} \} \mathbf{z}_K + \sigma \mathbb{E} \{ \mathbf{h}^{(\ell)} \mathbf{w}_K \} \mathbf{z}[n'] \\
&\quad + \sigma \mathbb{E} \{ \mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \} \mathbf{z}_K + \sigma \mathbb{E} \{ \mathbf{h}^{(\lambda)} \mathbf{w}_K \} \mathbf{z}[n] + \sigma \boldsymbol{\alpha}_0^{(\lambda)} \mathbb{E} \{ \mathbf{w}_K \mathbf{h}^{(\ell)} \} \mathbf{z}_K \\
&\quad + \sigma \mathbb{E} \{ \mathbf{h}^{(\lambda)} \mathbf{w}_K \mathbf{h}^{(\ell)} \} \mathbf{z}_K + \sigma^2 \boldsymbol{\alpha}_0^{(\ell)} \mathbb{E} \{ \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K \} + \sigma^2 \boldsymbol{\alpha}_0^{(\lambda)} \mathbb{E} \{ \mathbf{w}_K \mathbf{h}^{(\ell)} \mathbf{w}_K \} \\
&\quad + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \sigma^2 \mathbb{E} \{ \mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K \} + o(\sigma^2) \\
&\underset{K \rightarrow \infty}{\sim} \frac{1}{K} \mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell, \lambda)} \mathbf{z}_K + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \sigma^2 \mathbb{E} \{ \mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K \} + o(\sigma^2)
\end{aligned}$$

The Isserlis' theorem also gives the following result:

$$\begin{aligned}
\mathbb{E} \{ \mathbf{h}^{(\ell)} \mathbf{w}_K \mathbf{h}^{(\lambda)} \mathbf{w}_K \} &= \sum_{m, m'=0}^{M-1} \mathbb{E} \{ \mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m] \mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m'] \} \\
&= \sum_{m, m'=0}^{M-1} \mathbb{E} \{ \mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m] \} \mathbb{E} \{ \mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m'] \} + \mathbb{E} \{ \mathbf{h}^{(\ell)}[m] \mathbf{w}_K[m'] \} \mathbb{E} \{ \mathbf{h}^{(\lambda)}[m'] \mathbf{w}_K[m] \} \\
&\quad + \mathbb{E} \{ \mathbf{h}^{(\ell)}[m] \mathbf{h}^{(\lambda)}[m'] \} \underbrace{\mathbb{E} \{ \mathbf{w}_K[m] \mathbf{w}_K[m'] \}}_{=\delta_{m, m'}} \\
&\underset{K \rightarrow \infty}{\sim} \sum_{m=0}^{M-1} \mathbb{E} \{ \mathbf{h}^{(\ell)}[m] \mathbf{h}^{(\lambda)}[m] \} = \frac{1}{K} \text{Tr} \left( \boldsymbol{\Gamma}^{(\ell, \lambda)} \right) .
\end{aligned}$$

We conclude that:

$$\gamma[n, n'] \underset{K \rightarrow \infty}{\sim} \frac{1}{K} \mathbf{z}_K^T \boldsymbol{\Gamma}^{(\ell, \lambda)} \mathbf{z}_K + \sigma^2 \left\langle \boldsymbol{\alpha}^{(\ell)}, \boldsymbol{\alpha}^{(\lambda)} \right\rangle + \frac{\sigma^2}{K} \text{Tr} \left( \boldsymbol{\Gamma}^{(\ell, \lambda)} \right) + o(\sigma^2) .$$