A NOVEL "PHASE GUESS" APPROACH WITH SEQUENTIALLY LINEARIZED DYNAMICS TO BUILD A LOW-FIDELITY SOLUTION SPACE OF FREE-FINAL TIME LOW THRUST OPTIMIZED TRAJECTORIES

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1. ABSTRACT

This paper attempts to overcome the challenges of solving free-final time low thrust trajectory optimization with a novel indirect approach sufficient to get a computationally expedient low fidelity solution space to reduce the search space in time, fuel spend and final state error for more complex, higher fidelity optimization algorithms. The fundamental idea is that for a minimum time problem optimized on Pontryagin's Minimum Principle (PMP), the Lagrangians (or co-states) of concern can be expressed as a simple harmonic oscillator function ("SHOF") in time. The paper suggests a structured way to sequentially linearize the dynamics, prioritize one of those Lagrangians, "guess" a suitable phase in its SHOF and derive phases for the other Lagranians SHOFs which fulfil the transversality conditions. The approach is tested on a hypothetical scenario of a missed thrust event (MTE) in Psyche mission approaching Mars for a flyby maneuver. The opportunities with and limitations of this approach are discussed and ideas for further improvements suggested.

2. INTRODUCTION

2.1. Low thrust trajectory optimization poses significant challenges due to the inherent nonlinearity of state equations. This complexity is exacerbated by the assumption, often made for cis-lunar trajectories, that the dynamics of Variable Specific Impulse (VSI) engines mirror those of Constant Specific Impulse (CSI) engines. While reasonable for certain scenarios, such assumptions prove inadequate for the intricate dynamics of deep space missions like Psyche [Ref: III, VI].

2.2. The introduction of free-final time problems, particularly in the pursuit of minimum-time trajectories, further complicates the optimization process. These scenarios involve split boundary conditions in second-order differential equations, necessitating the fulfillment of specific transversality conditions. This paper acknowledges the heightened complexity introduced by these conditions and emphasizes the need for innovative solutions to address these challenges.

2.3. In deep space missions, unforeseen missed thrust events (MTE) can have profound consequences. An MTE forces the spacecraft into a ballistic trajectory, injecting uncertainty into the time spent in this trajectory. The subsequent requirement to recompute a controlled low thrust trajectory transforms the problem into one with free-final time constraint. In past missions, mission designers have had to grapple with the task of ensuring mission resilience against not just one but potentially multiple MTEs [Ref: IV].

2.4. An important aspect of MTEs is that the effects of the event vary greatly depending on where in the trajectory in occurs. [Ref: IV]. Unexpected MTE can severely disrupt missions like Psyche which are deep space, low thrust, and have a small window of time to make a flyby maneuver around Mars to head to its final destination '16 Psyche' the metallic asteroid. Previous missions like Dawn resorted to using a "rolling coast" method to ensure that a minimum of 28 days of forced thrust shutdown time was possible at any point. [Ref: V].

2.5. The astrodynamics community traditionally relies on direct numerical approaches, such as Particle Swarm Optimization (PSO) or Sims-Flanagan Transcription (SMT), for solving split-boundary problems. Some papers propose alternative strategies involving forced shutdown planning and the plotting of virtual MTE recovery trajectories based on historical data [Ref: I, II, III, IV, V, VI, VII, VIII]. These approaches, particularly those rooted in minimum-fuel optimization, often grapple with computational expense. Further, their pursuit of local optima through iterative processes, even for minimum-fuel scenarios, poses challenges and prompts the exploration of alternative methodologies [Ref: I, II, III].

2.7. Indirect variational calculus-based approaches, however, are more promising alternatives for finding local optimal law. Even so, indirect approaches introduce their own set of challenges. The "curse of dimensionality" complicates trajectory optimization problems with split boundary conditions, accentuating the reliance on accurate initial guesses for Lagrangians and urging caution in the use of indirect methods. Recent efforts in solving minimum-time trajectory optimization problems, such as homotopy-based approaches, have shown promise but come at a high computational cost. The computation time ranges from 200 to 2000 seconds per month-long time of flight (TOF) optimization run, notwithstanding some shortcomings in convergence [Ref: VIII, IX].

As a side note, there were not sufficient computation time benchmarks found in our literature survey [Ref: V) conceded honestly that it was "difficult to describe" due to "varying sensitivities to MTEs"). Even so, the aim of this tool, as a low-fidelity solution space builder, is to keep the computation time to under 1 second per 1-month TOF (currently at 1.6875 per month on a i5 processor) without the need for multiple cores and add to its value as a component of the trajectory optimization software stack.

2.8. This paper introduces a novel indirect approach that aims to reconcile the challenges posed by complex dynamics within sequentially linearized state equations. Also, the "Phase Guess" method strategically addresses transversality conditions, albeit sub-optimally, by incorporating a phase guess into the optimization process. The objective is to generate a computationally expedient low fidelity solution space. The proposed "Phase Guess" approach positions itself as a valuable complement to computationally intensive transcription-based or homotopy-based approaches. By significantly reducing the grid of final-time and fuel-spend values required, it offers efficiency gains and extends its role as an integral component within the trajectory optimization software stack.

3. SEQUENTIALLY LINEARIZED DYNAMICS AND OPTIMAL CONTROL

3.1. State Equations (in 2 dimensions)

$$X = [\, x_1\ ,\ x_2\ ,\ x_3\ ,\ x_4\ ] = [\, r_x\ ,\ r_y\ ,\ v_x\ ,\ v_y\ ]$$

Where:

X is the state of the spacecraft in two vectors (position and velocity) expressed in two dimensions.

$$\dot{X} = AX + BU$$

Where:

$\dot{X}$ is the derivative in time of X

$$A = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ g & 0 & 0 & 0 \\ 0 & g & 0 & 0 \end{matrix}$$

Where:
$g = -\frac{\mu}{r\_mag}$; where μ is the gravitational parameter of the primary body and r_mag is the distance of the spacecraft from that primary body.

$$B = \begin{matrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

$U = \left[u_x ,\ u_y \right]$ is acceleration experienced by the spacecraft.

3.2. Terminal Conditions

Fixed-Final State X_f , Free-Final Time t_f

3.3. Cost Function

Minimum time optimization problem
$J = \int_{t_0}^{t_f} 1.dt$

3.4. Bounded Control and Bang-Bang Control Profile

$|u_x| \ ,\ |u_y| \leq u_{max}$
where u_{max} is the maximum acceleration thrust can provide based on factors like distance from the Sun (applicable for low thrust VSI SEP engines) and number of engines on.

Following the PMP to make the firstorder necessary conditions satisfied [Ref: VIII], the Euler-Lagrangian Equations (ELEs) and Transversality equations need to be solved to get functions in time for the Lagrangians (“Lags”) associated with u_x and u_y.

The Hamiltonian
$H = 1 + \lambda_1 x_3 + \lambda_2 x_4 + \lambda_3 (g x_1 + u_x) + \lambda_3 \left(g x_2 + u_y\right)$

Bang-Bang Control Profile:
$$u_x = \begin{cases} + u_{max} \ when \ \ \lambda_3 < 0 \\ - u_{max} \ when \ \ \lambda_3 > 0 \end{cases}$$
$$u_y = \begin{cases} + u_{max} \ when \ \ \lambda_4 < 0 \\ - u_{max} \ when \ \ \lambda_4 > 0 \end{cases}$$

Euler-Lagrange Equations (ELEs)
$\ddot{\lambda}_3 = g\, \lambda_3 \ ,\ \ \ddot{\lambda}_4 = g\, \lambda_4$

Transversality Equation
$H\left(*, t_f\right) + \frac{dh}{dt}\left(*, t_f\right) = 0$; where ‘ * ’ = states, co-states and controls in optimal profile
$\Rightarrow \dot{\lambda}_3 x_3 - \lambda_3 g x_1 + \dot{\lambda}_4 x_4 - \lambda_3 g x_2 - 1 = 0 \, , at\ t = t_f, assuming\ u_x = u_y = 0$ [Ref: VII]

3.5. The Challenge in Solving

Getting functions in time for λ_3 ("Lag3") and λ_4 ("Lag4"). Given the split boundary conditions, traditionally forced to use numerical methods which are computationally expensive (a problem accentuated for free-final-time condition).

3.6. Proposed Approach

Given Lag3 and Lag4 are related, if some logic can be used to make an initial guess for one, the other can be derived. The solution wouldn't perfectly fulfill necessary conditions but sufficient to significantly reduce the solution space of final time and fuel spend. We'll call this the "Phase Guess" approach.

4. PHASE GUESS APPROACH TO GET LAGRANGIAN 'SHOFs'

4.1. From the ELEs, it is evident that Lag3 and Lag4 can be expressed as SHOFs.

$$\lambda_i = C_i \cos\left(\sqrt{-g}\, t + \varphi_i\right) \quad \text{where } i = 3, 4$$

Given the Bang-Bang control profile, C_i can take any value greater than zero so long as the phase φ_i is correct. Therefore, the challenge reduces to determining suitable values for φ_i.

4.2. The "phase guess" approach provides for the following logic to prioritize which of the two undetermined SHOF phases must be fixed (or guessed)

i. At initial time t_0, for each dimension, the ideal (uncontrolled) thrust acceleration Q is calculated using the 2nd equation of motion assuming static state equation dynamics. As an example, in x-axis:

$$Q_x = \frac{v^2_{x\,at\,t_f} - v^2_{x\,at\,t_0}}{2\sqrt{r^2_{x\,at\,t_f} - r^2_{x\,at\,t_0}}} - g_x$$

ii. The dimension of which the magnitude of Q is the largest is "prioritized" i.e. the engine in that dimension must thrust for the longest time period before its corresponding Lag SHOF changes its sign. This time-period then must be exactly half of the SHOF time period, which makes the guess for φ_i either $\frac{\pi}{2}$ (if prioritized Q is positive) or $-\frac{\pi}{2}$ (if prioritized Q is negative).

iii. The phase for the other Lag is then derived from the Transversality Equation, re-written below in SHOF representation:

$$x_3\sqrt{-g}\sin\left(\sqrt{-g}\,t + \varphi_3\right) + g x_1\left(\cos\left(\sqrt{-g}\,t + \varphi_3\right)\right) + x_4\sqrt{-g}\sin\left(\sqrt{-g}\,t + \varphi_4\right) + g x_2\left(\cos\left(\sqrt{-g}\,t + \varphi_4\right)\right) + 1 = 0, \quad at\ t = t_f$$

5. ALGORITHM OF THE PHASE GUESS APPROACH WITH SEQUENTIALLY LINEARIZED DYNAMICS

The algorithm is summarized in the following 7 steps:

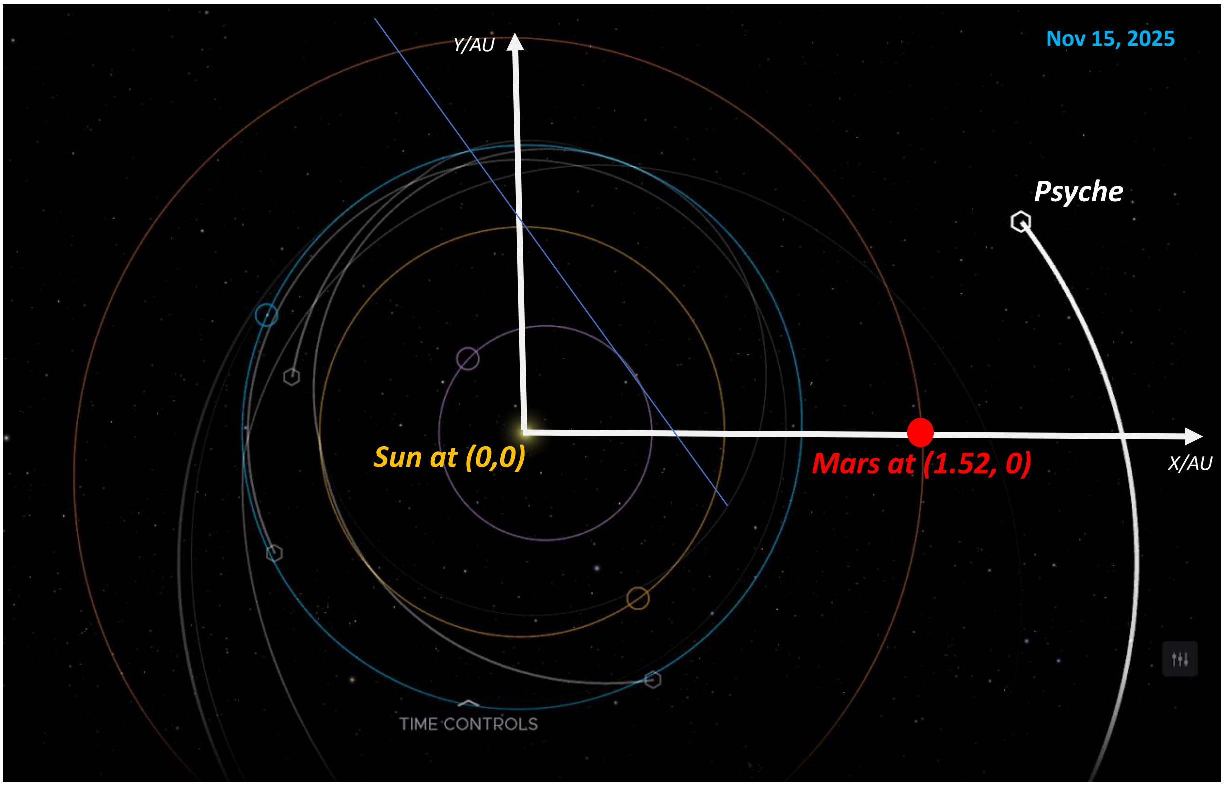
i. Get the initial state and final-fixed state of the spacecraft.
ii. Compute instantaneous values of state equations constants like g, u_{max}.
iii. Determine the dimension of which Q has the largest magnitude.
iv. For this same dimension's corresponding Lag, let SHOF phase be $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.
v. Compute SHOF phases for the other Lags from Transversality Equation.
vi. Propagate state until state equations constants need to be updated and go back to Step 1.
vii. End propagation until final-fixed state is reached or till a logical maximum time limit.

6. RESULTS FROM PSYCHE MTE SIMULATIONS

6.1. The inspiration behind this independent study was to support the astrodynamics community with optimization tools which make trajectories, deep space ones in particular, more resilient to unforeseen MTEs. As such, the phase guess approach was applied to the scenario where a few months before its scheduled flyby around Mars, Psyche spacecraft suffers an MTE to enter a Keplerian (no thrust) trajectory for an undetermined number of days.

6.2. Though Psyche mission's trajectory data is not available yet, the trajectory positions were derived from NASA's 'Eye on the Solar System' interactive.

Position of Psyche relative to Sun and Mars on Nov 15, 2025 adapted from NASA's interactive



6.3. The following spacecraft's specifications were derived from public websites of NASA and JPL.

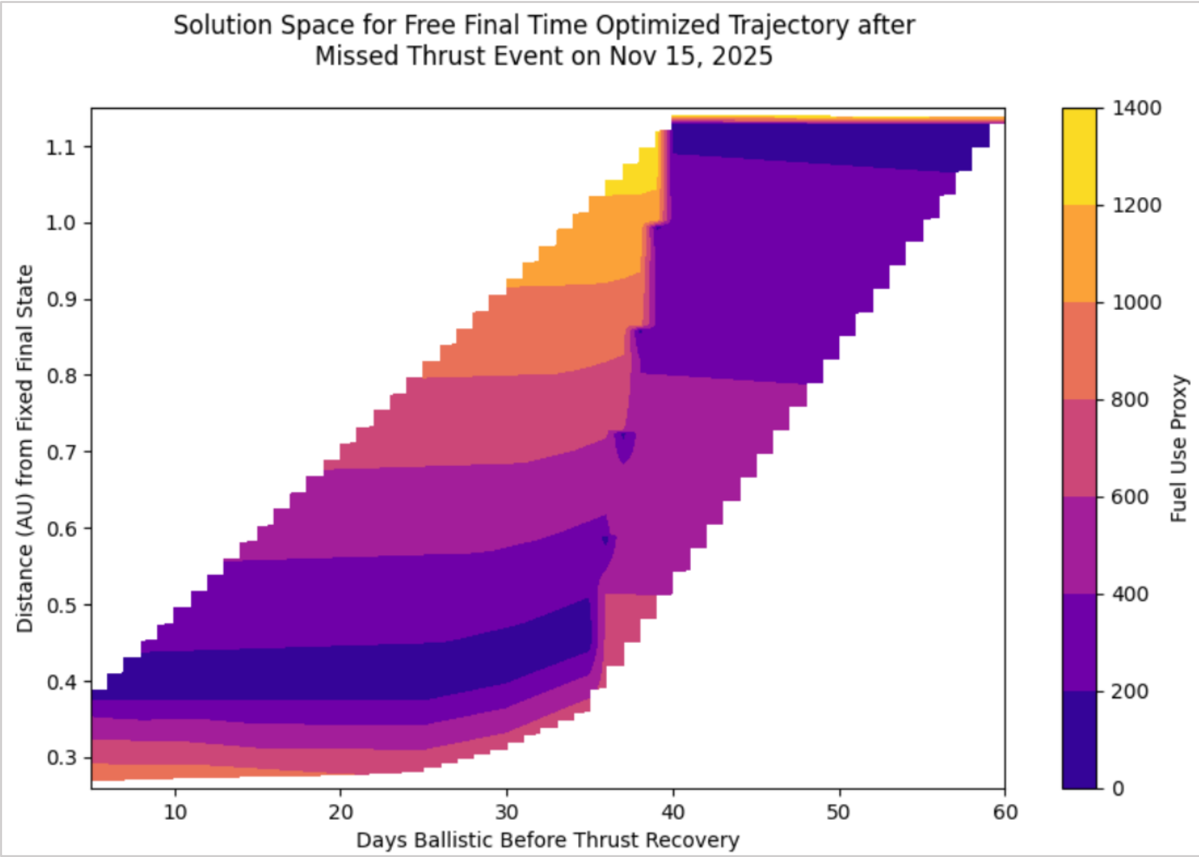
- Max thrust per engine (total 4 engines) = 240mN
- Solar panel area = 75 sqm
- Solar panel efficiency = 20% (LINK)
- Spacecraft weight (payload plus fuel) = 2,747 kg
- Fuel Consumption Rate = 0.35 kilograms to 1.3 kilograms of xenon a day
- Ejection velocity = 83 km/s

6.4. The desired outcome is to build, within seconds, a contour plot of days ballistic (following the MTE, before engine recovery), the distance from nominal flyby position and the expected fuel consumption (proxied as product of number of thrusts and electric power applied). The following four simulations present that outcome.

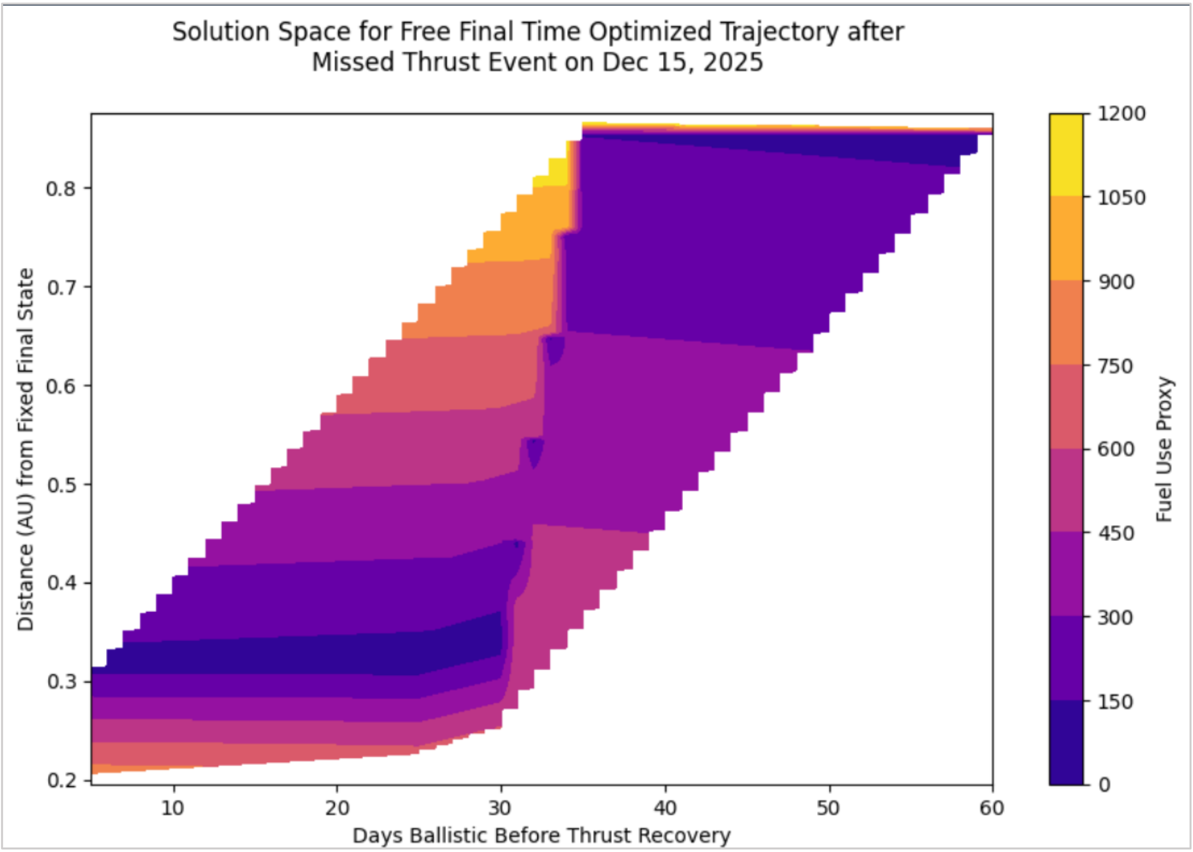
6.5. It should be noted that the reason why distances from fixed final state (Y-axis of contour plots) demonstrated below are relatively large is because of conservatively assumed values of the velocity vector at initial state and the maximum time of flight. In the lack of availability of real-world Psyche trajectory dataset, the initial velocity vector was assumed to be that of a circular orbit at initial position and time of flight was restricted to June 15, 2026. When these are modified to more realistic values, the distance from fixed final state reduces significantly (see Annex A).

Simulations over MTE dates ranging from Nov 15, 2025 to Feb 15, 2026

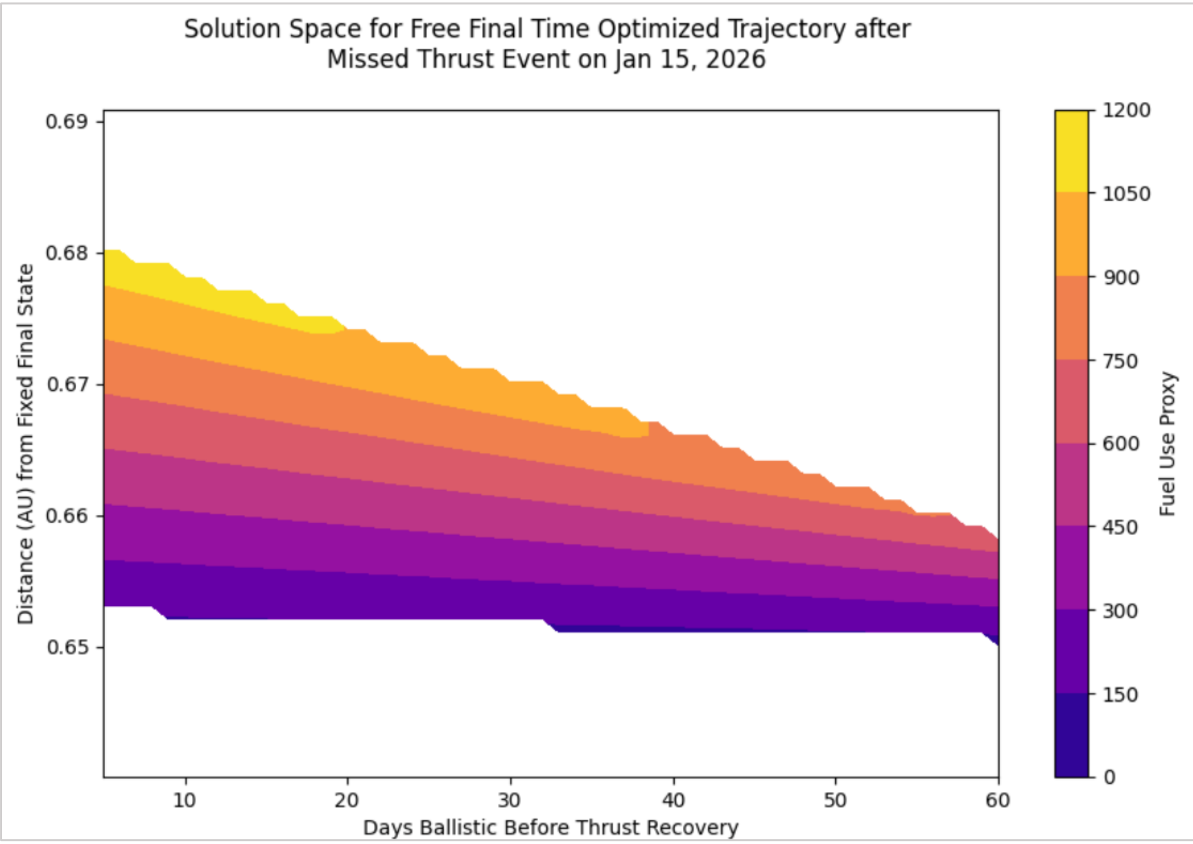
Simulation 1 – MTE on Nov 15, 2025 (computational time: 10.125 seconds)



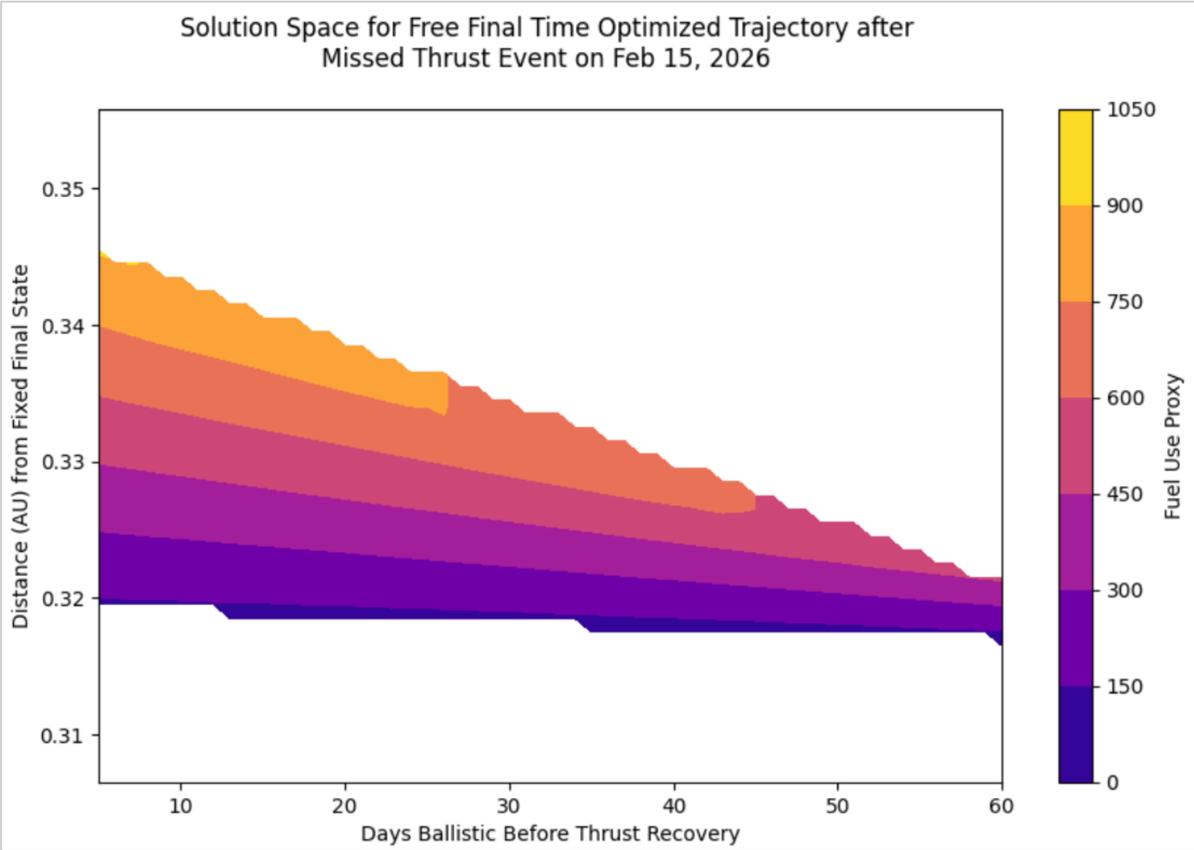
Simulation 2 – MTE on Dec 15, 2025 (computational time: 8.455 seconds)



Simulation 3 – MTE on Jan 15, 2026 (computational time: 10.369 seconds)



Simulation 4 – MTE on Feb 15, 2026 (computational time: 7.527 seconds)



7. LIMITATIONS AND FURTHER IMPROVEMENTS

7.1. The current iteration of this novel "Phase Guess" approach faces a limitation in offering a logically optimal control law for scenarios involving free-final states. To address this constraint, the implementation of a shooting method may be necessary. Utilizing a shooting method could facilitate the determination of values for all Lagrangian SHOF phases, ensuring a more comprehensive and optimized solution.

7.2. Despite these limitations, the "Phase Guess" approach exhibits promise in deep space mission applications. For instance, beside resilience to MTEs, this approach to optimize trajectories on minimum time performance can give missions' scientific teams the leverage to tweak parameters for incorporating intermediate planetary flybys. This flexibility contrasts with existing trajectory optimization approaches, which often rely on a priori selection of final states and times of flight. By allowing dynamic adjustments to mission parameters, the "Phase Guess" approach contributes to a more adaptive and versatile trajectory optimization framework for deep space exploration.

7.3. The effectiveness of the "Phase Guess" approach has been primarily demonstrated in two-dimensional scenarios. To enhance its robustness and applicability, further testing is essential, particularly in three-dimensional settings. Additionally, the approach's performance should be evaluated under more complex dynamics, particularly incorporating third-body perturbations in deep space mission scenarios. This expansion ensures a thorough understanding of the approach's capabilities and limitations in diverse space mission contexts.

8. CONCLUSION

While the "Phase Guess" approach presents a novel and efficient solution space for certain trajectory optimization challenges, acknowledging and addressing its limitations ensures a more comprehensive and informed trajectory planning strategy. Continued research and development in the outlined areas promise advancements that contribute to the evolution of trajectory optimization methodologies, particularly in the context of deep space missions and dynamic mission parameter adjustments.

9. REFERENCES

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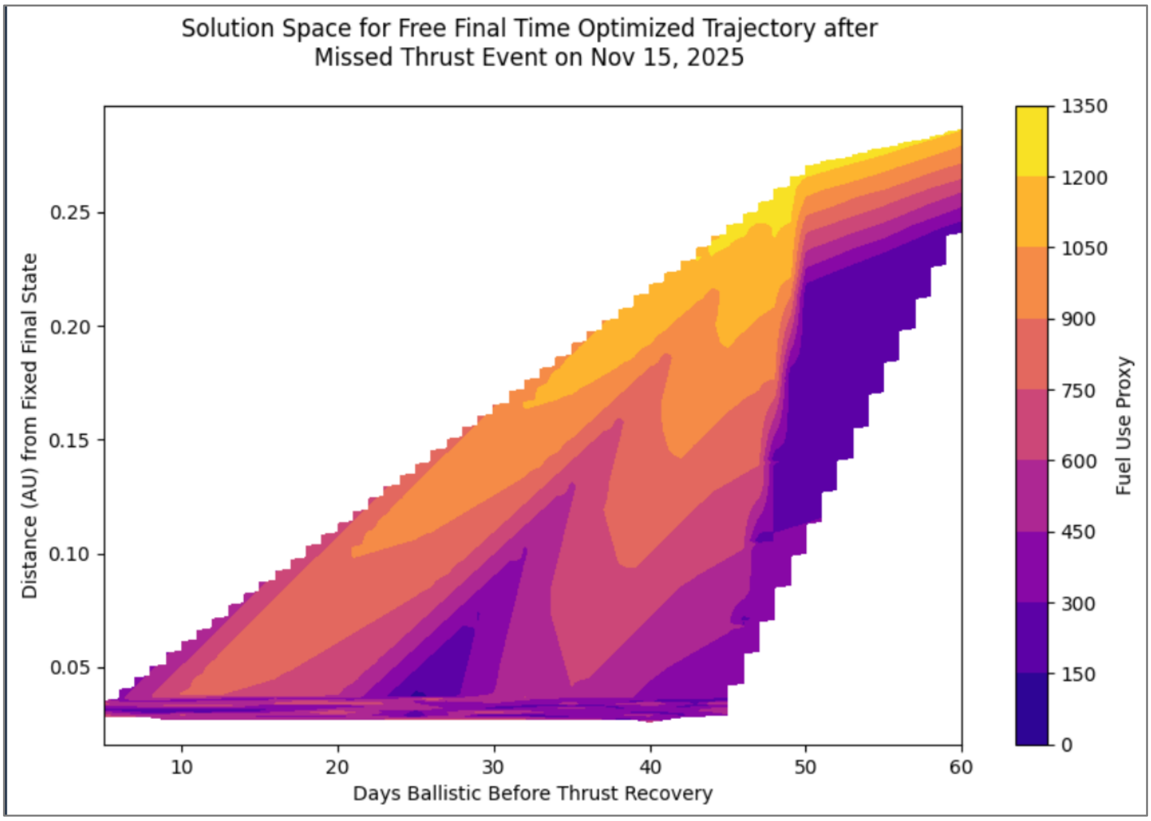
Annex A – Contour Plots with Less Conservative Initial Velocity Vector and Maximum Time of Flight

Velocity vector direction of initial state edited to cos 45, sin 45; max date Jun 15, 2026



Velocity vector direction of initial state edited to cos 45, sin 45; Aug 15 2026 max time