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Applied Numerics - Exercise 3

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I.BA_IMATH, Semesterweek 10

The solution of the exercises should be presented in a clear and concise manner. Numerical results should be accurate to 4 digits. The exercises are accepted if You solve 75% of the exercises adequately. Please hand in the exercises no later than at the end of the last lecture in semesterweek 11.

1 Natural Cubic Spline

Compute a natural cubic spline through the points $(0,0)$, $(1,1)$ and $(2,1)$. First compute everything by hand and then check Your results using graphical means, i.e. plot the graph and make sure that $z_1 = z_3 = 0$ by taking the second derivative of the interpolating function.

2 Complex Numbers: Graphical Representation I

Draw the following complex numbers in the complex plane and connect the points one after the other: describe what You see!
(a) $z_1 = 1+2j$, $z_2 = -2+j$, $z_3 = -1-2j$, $z_4 = 2-j$
(b) $z_1 = 3+7j$, $z_2 = -3+5j$, $z_3 = -3+3j$, $z_4 = 3j$, $z_5 = -3-3j$, $z_6 = -3-6j$, $z_7 = 3-6j$.

Aufgabe 1 $(0,0),\ (1,1),\ (2,1)$ $z_1 = z_3 = 0$

Allg.

i	x_i	y_i	h_i
1	0	0	1
2	1	1	1
3	2	1	

$$\frac{h_{i-1}}{6} z_{i-1} + \left(\frac{h_{i-1}+h_i}{3}\right) z_i + \frac{h_i}{6} z_{i+1} = \frac{y_{i+1}-y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}$$

$$i=2:\ \frac{0}{6} z_1 + \left(\frac{1+1}{3}\right) z_2 + \frac{1}{6} z_3 \overset{!}{=} \frac{1-1}{1} - \frac{1-0}{1}$$

$$\frac{2}{3} z_2 + \frac{1}{6} z_3 \overset{!}{=} -1$$

$$-\frac{z_i}{2h_i}(x_{i+1}-x)^2 + \frac{z_{i+1}}{2h_i}(x-x_i)^2 - \frac{h_i}{6}(z_{i+1}-z_i)$$

$$\frac{z_i}{h_i}(x_{i+1}-x) + \frac{z_{i+1}}{h_i}(x-x_i)$$

$$P_1''(0) = \frac{z_1}{1}(1-0) + \frac{z_2}{1}(0-0) \overset{!}{=} z_1 = 0$$

$$P_2''(2) = \frac{z_2}{1}(2-2) + \frac{z_3}{1}(2-1) \overset{!}{=} z_3 = 0$$

$$\begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{6} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{5}{6} \cdot \frac{3}{2} \quad \frac{15}{12} \quad \frac{5}{4}$$

$a_i := y_i$

$b_i = \frac{y_{i+1}-y_i}{h_i} - \frac{h_i}{6}\cdot\left(s''_{i+1} + 2s''_i\right)$

$c_i = \frac{s''_i}{2}$

$d_i = \frac{s''_{i+1} - s''_i}{6h_i}$

$$\frac{2}{3} z_2 + \frac{1}{6} = -1$$

$$\frac{2}{3} z_2 \overset{!}{=} -\frac{5}{6}$$

$$z_2 \overset{!}{=} -\frac{5}{4}$$

N spline moment

$z_1 = 0$

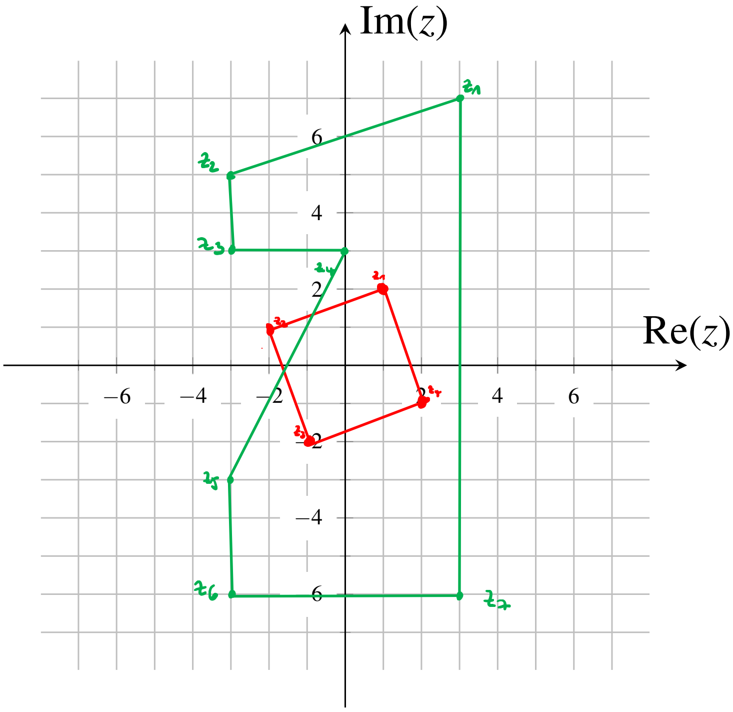
$z_3 = 0$

$P_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$

$b_i x^2$

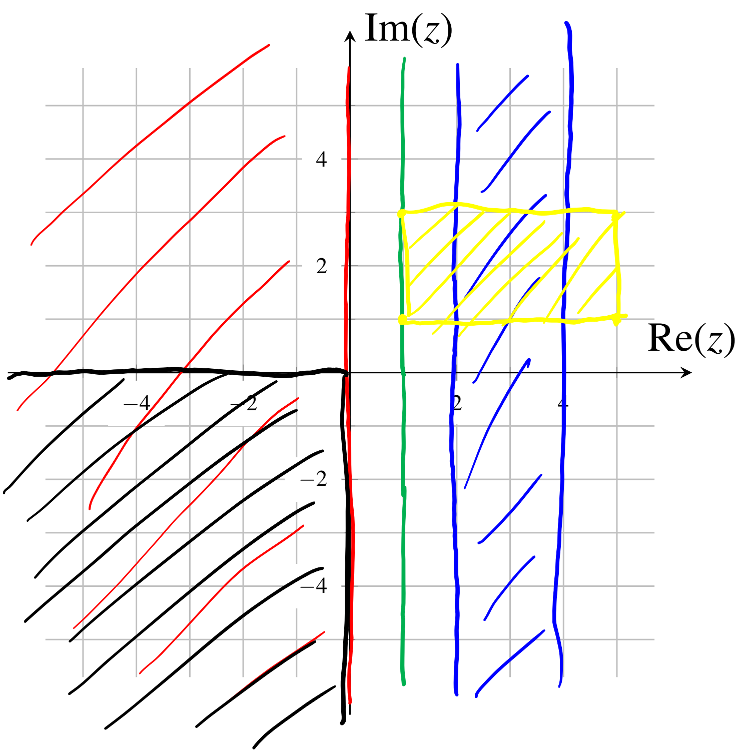
$$\frac{\frac{6}{7}}{\frac{6}{7}} \qquad \frac{36}{7}$$

→ keine Ahnung wie :C



3 Complex Numbers: Graphical Representation II

Draw the following sets of complex numbers in the complex plane:
(a) $\{z \in \mathbb{C} | \mathrm{Re}(z) = 1\}$, (b) $\{z \in \mathbb{C} | \mathrm{Re}(z) \leq 0\}$, (c) $\{z \in \mathbb{C} | 2 \leq \mathrm{Re}(z) \leq 4\}$,
(d) $\{z \in \mathbb{C} | 1 \leq \mathrm{Re}(z) \leq 5 \quad \textit{and} \quad 1 \leq \mathrm{Im}(z) \leq 3\}$, (e) $\{z \in \mathbb{C} | \mathrm{Re}(z)\mathrm{Im}(z) \leq 0\}$.



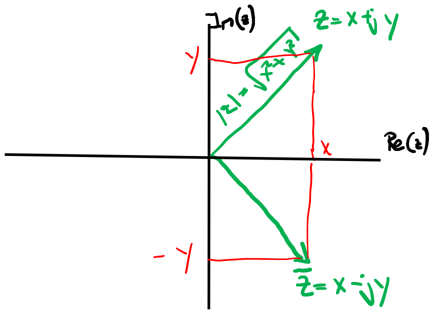
4 Complex Numbers: Complex Conjugate Numbers

The **complex conjugate** of the number $z = x + jy$ is the number $\bar{z} = x - jy$. Verify, that the complex conjugate is the number z mirrored on the real axis (x-axis) in the complex plane. Show that $z\bar{z} = \bar{z}z = x^2 + y^2 = |z|^2$

Show, that for each complex number $z = x + jy$ the following holds:

(a) $z + \bar{z} = 2\mathrm{Re}(z)$ (b) $z - \bar{z} = 2j\mathrm{Im}(z)$

(c) $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$



where $\mathrm{Re}(z) = x$ is the **real part of** z and $\mathrm{Im}(z) = y$ is the **imaginary part of** z.

5 Complex Numbers: Addition and Substraction

Compute (a) $(8+2j)+(7+3j)$, (b) $(11-15j)+(-3+8j)$ and (c) $98-(99-100j)$.

6 Complex Numbers: Multiplikation

Compute (a) $8 \cdot 5j$, (b) $(-7-12j)5j$, (c) $(8+2j)(7+3j)$ and $(13+17j)(13-17j)$.

7 Complex Numbers: Powers

Given $z_1 = 7-5j$, $z_2 = 2+j$, $z_3 = -5+2j$, $z_4 = -10-3j$, $z_5 = 8$ and $z_6 = 8j$ compute (a) $z_1 - z_3 - z_5$, (b) $z_1 z_3 z_4$ (c) $z_1^2 + z_2^2$, (d) $jz_4 - z_3 z_6$, (e) $\mathrm{Re}(z_1^2 z_3)$, and (f) $\mathrm{Im}(2z_2 - 3z_3)$.

8 Complex Numbers: Division

For $a,b,c,d \in \mathbb{R}$ represent the following numbers in the form $z = x + jy$
(a) $(7a+3bj)(4c-5dj)$, (b) $(a+bj)/(c-dj)$, (c) $j(a+bj)+(a-bj)/j$, (d) $\overline{(b-cj)}(b-cj)^{-1}$ and (e) $aj + \overline{j}/a + a/j + j/a$.

9 Complex Numbers: Polar Form

Every complex number $z = x + jy$ can be written in **polar form** $z = r(\cos\phi + j\sin\phi)$ where ϕ is the **argument** of the complex number z and $r = |z|$ is the **modulus** of a complex number z, also called the **complex norm**. Find the modulus $|z|$ and the argument ϕ (in radians) of the following numbers:

(a) $z_1 = 1+2j$, $z_2 = -2+j$, $z_3 = -1-2j$, $z_4 = 2-j$

(b) $z_1 = 3+7j$, $z_2 = -3+5j$, $z_3 = -3+3j$, $z_4 = 3j$, $z_5 = -3-3j$, $z_6 = -3-6j$, $z_7 = 3-6j$, $z_8 = 3-2j$.

Note: Do use the calculator only for checking the correctness of Your result.

Aufgabe 4

(a) $z+\bar{z}=2Re(z)=(x+jy)+(x-jy)=2x$

(b) $z-\bar{z}=2jIm(z)=(x+jy)-(x-jy)=2jy=2jIm(z)$

(c) $\underset{\underset{z^{-1}}{\parallel}}{\frac{1}{z}}=\frac{1}{z\bar{z}}\bar{z}=\frac{\bar{z}}{|z|^2}$

Aufgabe 5

(a) $(8+2j)+(7+3j)=15+5j$

(b) $(11-15j)+(-3+8j)=8-7j$

(c) $98-(99-100j)=-1+100j$

Aufgabe 6

$j^2=-1$

(a) $8\cdot 5j=(8+0j)(0+5j)=40j=(0+40j)$

(b) $(-7-12j)5j=-35j-60j^2=60-35j$

(c) $(8+2j)(7+3j)=56+24j+14j+6j^2=50+38j$

(d) $(13+17j)(13-17j)=169-289j^2=169+289=458$

Aufgabe 7

$z_1=7-5j, z_2=2+j, z_3=-5+2j, z_4=-10-3j, z_5=8, z_6=8j$

(a) $z_1-z_3-z_5=(7-5j)-(-5+2j)-(8)=4-7j$

(b) $z_1z_3z_4=(7-5j)\cdot(-5+2j)\cdot(-10-3j)=(-25+39j)(-10-3j)=367-315j$

(c) $z_1^2+z_2^2=(7-5j)^2+(2+j)^2=(24-70j)+(3+4j)=27-66j$

(d) $j\cdot z_4-z_3\cdot z_6=(0+j)(-10-3j)-((-5+2j)(0+8j))=(3-10j)-(-16-40j)=3-10j+16+40j=19+30j$

(e) $Re(\overbrace{z_1^2\cdot z_3}^{x}) \Rightarrow x=(24-70j)\cdot(-5+2j)=-120+48j+350j-140j^2=20+398j \Rightarrow Re(x)=Re(20+398j)=20$

(f) $Im(\overbrace{2z_2-3z_3}^{x}) \Rightarrow x=(4+2j)-(15+6j)=-11-4j \Rightarrow Im(x)=Im(-11-4j)=-4$

Aufgabe 8

(a) $(7a+3bj)(4c-5dj)=(28ac+15bd)+j(-35ad+12bc)$

(b) $\frac{(a+bj)}{(c-dj)}=\frac{(a+bj)\ (c+dj)}{(c-dj)(c+dj)}=\frac{ac+adj+bcj+bdj^2}{c^2+d^2}=\frac{(ac-bd)+j(ad+bc)}{c^2+d^2}$

(c) $j(a+bj)+\frac{(a-bj)}{(0+j)}=-b+aj+\frac{(a-bj)\ (0-j)}{(0+j)\ (0-j)}=-b+aj+(-b-aj)=-2b$

(d) $\overline{(b-cj)}\,(b-cj)^{-1}=\frac{(b+cj)}{(b-cj)}=\frac{(b+cj)\ (b+cj)}{(b-cj)\ (b+cj)}=\frac{(b^2-c^2)+j(2bc)}{b^2+c^2}$

(e) $aj+\frac{j}{a}+\frac{a}{j+\frac{j}{a}}$ ToDo

Aufgabe 9 und 10

Allg. $z = x + jy$ kann geschrieben werden als

$z = r e^{j\phi}$ wobei $r = \sqrt{x^2 + y^2}$ und

$\phi = \operatorname{Arg}(z)$ i.e. $\tan \phi = \frac{y}{x}$

(a) $z_1 = 1 + 2j = \sqrt{5} \cdot e^{j \cdot 1.1071}$ $\quad r = \sqrt{1^2 + 2^2} = \sqrt{5}$

$e^{j\phi} = \sqrt{5}\left(\cos(1.1071) + j \cdot \sin(1.1071)\right)$ $\quad \tan \phi = \frac{2}{1} = 2 \Rightarrow \arctan(2) = 1.1071 = \phi$

$z_2 = -2 + j = \sqrt{5} \cdot e^{j \cdot 2.6779}$ $\quad r = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

$\tan \phi = \frac{1}{-2} \Rightarrow \arctan(-\frac{1}{2}) = -0.4636 + \pi = \phi = 2.6779$ $\quad$ bc. $x > 0$

$e^{j\phi} = \sqrt{5}\left(\cos(2.6779) + j \cdot \sin(2.6779)\right)$

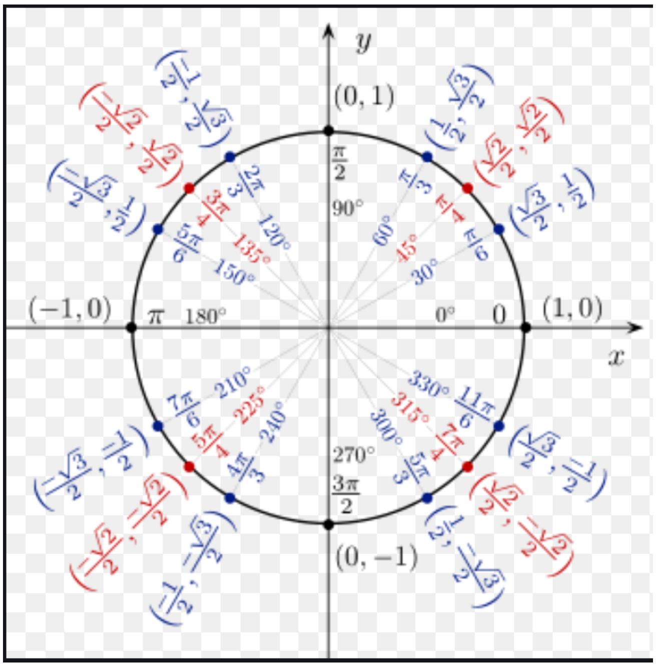
$z_3 = -1 - 2j = \sqrt{5} \cdot e^{j \cdot (-2.0344)}$ $\quad r = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$ $\quad \arctan\left(\frac{-2}{-1}\right) = 1.1071 - \pi = -2.0344$

$e^{j\phi} = \sqrt{5}\left(\cos(-2.0344) + j \cdot \sin(-2.0344)\right)$

$z_4 = 2 - j = \sqrt{5} \cdot e^{j \cdot (-0.4636)}$ $\quad r = \sqrt{2^2 + (-1)^2} = \sqrt{5}$ $\quad \arctan\left(\frac{-1}{2}\right) = -0.4636$

$e^{j\phi} = \sqrt{5}\left(\cos(-0.4636) + j \cdot \sin(-0.4636)\right)$

(b) ToDo



$$\operatorname{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

431 × 162

10 Complex Numbers: Exponential Form

Using the Euler-Formula $e^{j\phi} = \cos\phi + j\sin\phi$ we can further simplify the polar form:

$$z = r(\cos\phi + j\sin\phi) = re^{j\phi}$$

Note: the number $e^{j\phi}$ lives on the unit circle and the angle between the x-axis and z is equal to ϕ. Write the number z_1, z_2, …, z_8 from the previous example in Exponential form.

11 Complex Numbers: Multiplication and Division in Exponential Form

Multiplication or division of two complex number $z_1 = r_1 e^{j\phi_1}$ and $z_2 = r_2 e^{j\phi_2}$ in exponential form is simple:

$$z = z_1 z_2 = r_1 e^{j\phi_1} r_2 e^{j\phi_2} = r_1 r_2 e^{j(\phi_1+\phi_2)} = re^{j\phi}$$

where $r = r_1 r_2$ and $\phi = \phi_1 + \phi_2$. Hence the modulus of the product is the product of the moduli and the argument of the product is the sum of the arguments. Questions:

1. Using the geometrical representation of complex number in the complex plane describe what happens, if one multiplies $z = re^{j\phi}$ with the imaginary unit j?.
2. Furthermore, compute the product of $z_1 = 1 + 2j$ and $z_2 = -2 + j$ in two different ways, i.e. in (i) cartesian coordinates and (ii) exponential form.

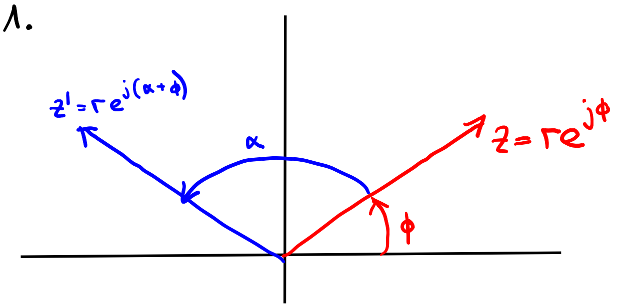
12 Trigonometric functions

Compute a complex and a real trigonometric polynomial which passes through the points $(0,8)$, $(\pi/2,6)$, $(\pi,4)$, and $(3\pi/2,2)$ and which is 2π-periodic.

Have Fun!

Aufgabe 11

1.



2. $z_1 = 1 + 2j \qquad z_2 = -2 + j$

(i) $z_1 z_2 = (1+2j)(-2+j) = -4 - 3j$

(ii) $\sqrt{5}\cdot e^{j\cdot 1.1071} \cdot \sqrt{5}\cdot e^{j\cdot 2.6779} = 5 \cdot e^{j(1.1071 + 2.6779)}$

Aufgabe 12

$\cos x + j \cdot \sin x$

$$T_4(x) = c_0 + c_1 \cdot e^{jx} + c_2 \cdot e^{2jx} + c_3 \cdot e^{3jx}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$\cos(3\pi/2) + j \sin(3\pi/2)$

$//$

$T_4(0) = f(0) = f_0$

$T_4(\pi/2) = f(\pi/2) = f_1$

$T_4(\pi) = f(\pi) = f_2$

$T_4(3\pi/2) = f(3\pi/2) = f_3$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\pi/2} & e^{j\pi} & e^{j\frac{3\pi}{2}} \\ 1 & e^{j\pi} & e^{2j\pi} & e^{3j\pi} \\ 1 & e^{j\frac{3\pi}{2}} & e^{3j\pi} & e^{9j\frac{\pi}{2}} \end{bmatrix}}_{=:F} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{=:c} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{=:f}$$

Inverse für F berechnen

$$F^{-1} = \frac{1}{4}\overline{F} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{e^{j\pi/2}} & \frac{1}{e^{j\pi}} & \frac{1}{e^{j\frac{3\pi}{2}}} \\ 1 & \frac{1}{e^{j\pi}} & \frac{1}{e^{2j\pi}} & \frac{1}{e^{3j\pi}} \\ 1 & \frac{1}{e^{j\frac{3\pi}{2}}} & \frac{1}{e^{3j\pi}} & \frac{1}{e^{9j\frac{\pi}{2}}} \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Nun folgendes einfügen und auflösen $f_0 = 8,\ f_1 = 6,\ f_2 = 4,\ f_3 = 2$

mit Octave, wobei dabei ein Vorzeichen falsch war!

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1-j \\ 1 \\ 1+j \end{bmatrix} \Rightarrow \begin{aligned} T_4(x) &= c_0 + c_1 \cdot e^{jx} + c_2 \cdot e^{2jx} + c_3 \cdot e^{3jx} \\ T_4(x) &\overset{!}{=} 5 + (1-j)e^{jx} + e^{2jx} + (1+j)e^{3jx} \end{aligned}$$

Check

$a_0 = 2c_0 = 10$

$a_1 = 2\mathrm{Re}(c_1) = 2$

$b_1 = -2\mathrm{Im}(c_1) = +2$

$a_2 = 2c_2 = 2$

Δ von Folie,

keine Ahnung warum genau das gilt.

$$\tilde{T}_4(x) = \frac{a_0}{2} + a_1 \cdot \cos x + b_1 \cdot \sin x + \frac{a_2}{2}\cos 2x$$

$$\overset{!}{=} 5 + 2\cos(x) + 2\sin(x) + \cos(2x)$$

↑ real trigonometric polynom