

# Lineare Algebra - Exercises 3

SVD 

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I.BA\_IMATH, Semesterweek 03

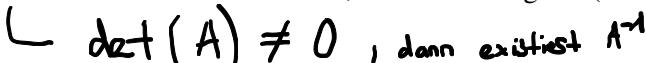
Please write down to solution of the exercises in a concise but comprehensible way. Numerical results should be accurate to 4 digits. Sketches should be correct qualitatively. At least 75% of the exercises have to be solved satisfactorily. Due time is one week after we have discussed the corresponding topic in class.

## 1 Determinant

Compute the determinant of the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}.$$

Which of the matrices have an inverse, i.e. are not singular (do not compute the inverse, though)?

  
**2 Again: Determinant**

First compute the determinant using Sarrus' formula. Then derive the reduced row echelon form of the matrices and again compute the determinant by multiplying the pivot elements.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix}$$

Is the matrix invertible (do not compute the inverse, though)?

## 3 Again: Determinant

Compute the determinant by Laplace expansion (along an appropriate column or row).

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Compare with the determinant you obtain, if using the reduced row echelon form of the matrix. Is the matrix invertible?

## Aufgabe 1

$$\det(A) = \det \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} = 1 \cdot 8 - 2 \cdot 3 = 8 - 6 = 2$$

$$\det(B) = \det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} = (1-\lambda) \cdot (8-\lambda) - 2 \cdot 3 = 8 - \lambda - 8\lambda + \lambda^2 - 6 = 2 - 9\lambda + \lambda^2$$

$$\det(C) = \det \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 3 & 6 \\ 0 & 1 \end{bmatrix} = \underline{\underline{-3}}$$

oder Produkt der Diagonalelemente, da obere Dreiecksmatrix  
 $\rightarrow 1 \cdot 3 \cdot (-1) = \underline{\underline{-3}}$

## Aufgabe 2

Sarrus

$$\det(A) = \det \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix} = 1 \cdot 1 \cdot 1 - 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 1 = 20 - 24 + 12 - 12 - 24 + 20 = \underline{\underline{-8}}$$

ref

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix} \xrightarrow[2_{13}(-3)]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 6 & 4 \end{bmatrix} \xrightarrow[2_{23}(-1.5)]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & -2 \end{bmatrix} = 1 \cdot 4 \cdot (-2) = \underline{\underline{-8}}$$

Sarrus

$$\det(B) = \det \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix} = 12 - 12 + 4 - 12 - 4 + 12 = \underline{\underline{0}}$$

ref

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix} \xrightarrow[2_{12}(-2)]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \end{bmatrix} \xrightarrow[2_{23}(-1)]{} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{\underline{0}}$$

sofort ersichtlich, dass  $\det 0$  sein muss, da ein Pivot 0 ist!

### Aufgabe 3

Laplace

$$\det(A) = \det \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

A1      A2

$$\begin{aligned} \text{A1} \quad \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} &= 1 \cdot \det \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} + 2 \cdot \det \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= -1 + 2 \cdot 1 \\ &= -1 + 2 = \underline{\underline{1}} \end{aligned}$$

$$\begin{aligned} \text{A2} \quad \det \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} &= 1 \cdot \det \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} - (-1) \det \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= 0 \cdot (-1) \cdot 1 = \underline{\underline{-1}} \end{aligned}$$

$$\det(A) = \underline{\underline{1}} - \underline{\underline{1}} = \underline{\underline{0}} \quad - \text{nicht invertierbar!}$$

ref

$$\begin{aligned} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} z_{12}(1) &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} z_{23}(1) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} z_{34}(1) \\ &= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 1 \cdot 1 \cdot 1 \cdot 0 = \underline{\underline{0}} \end{aligned}$$

Pivot ist null = det ist 0

## 4 Again: Determinant

If a  $4 \times 4$  matrix has  $\det(\mathbf{A}) = 1/2$ , find  $\det(2\mathbf{A})$  and  $\det(-\mathbf{A})$  and  $\det(\mathbf{A}^2)$  and  $\det(\mathbf{A}^{-1})$ .

You have to apply the rules on slide 6!

## 5 Eigenvalues and -vectors

Find the eigenvalues and eigenvectors of

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Check the trace and determinant.

## 6 Again: Eigenvalues and -vectors

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 3 & -0 \end{bmatrix}$$

Verify, that  $\mathbf{A} = S \Lambda S^{-1}$  where  $S$  contains the eigenvectors as columns. Make sure, they are in the same order as the eigenvalues in  $\Lambda$ .

Compute  $\exp(\mathbf{A})$  using this factorization.

Check Your results with Octave.

## 7 Again: Eigenvalues and -vectors

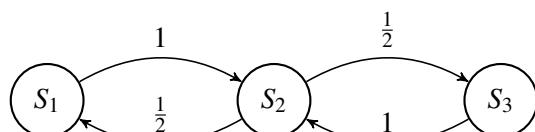
Find the eigenvalues of  $\mathbf{A}$  and  $\mathbf{B}$  (easy for triangular matrices) and  $\mathbf{A} + \mathbf{B}$ :

$$\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{A} + \mathbf{B}.$$

Are the eigenvalues of  $\mathbf{A} + \mathbf{B}$  equal (or not equal) to the eigenvalues of  $\mathbf{A}$  plus eigenvalues of  $\mathbf{B}$ ? Explain why!

## 8 Again: Eigenvalues and -vectors – Markov Chain

Write down the  $3 \times 3$  transition matrix  $\mathbf{P}$  for the Markov chain shown on the right. Make sure, the columns add to 1. The first column contains the probabilities for moving from state  $S_1$  to one of the states  $S_1, S_2$  and  $S_3$ .



## Aufgabe 4

$$\det(A) = \frac{1}{2}$$

Es gilt: if one row(column) is multiplied by a number  $k \in \mathbb{R}$  the determinant is multiplied by this number

$$\bullet \det(2A) = \det([2\alpha_1, 2\alpha_2, 2\alpha_3, 2\alpha_4]) = 2^4 \cdot \frac{1}{2} = \underline{\underline{8}}$$

$$\bullet \det(-A) = (-1)^4 \det(A) = \underline{\underline{\frac{1}{2}}}$$

Es gilt:  $\det(AB) = \det(A)\det(B)$ , also

$$\bullet \det(A^2) = \det(A)\det(A) = \frac{1}{2} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{4}}}$$

$$\bullet \det(A^{-1}) = \frac{1}{\det A} = \frac{1}{\frac{1}{2}} = \underline{\underline{2}}$$

## Aufgabe 5

$$A_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \det(A_1 - \lambda I_2) = 0$$

$$= \det \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix} = (3-\lambda) \cdot (-3-\lambda) - 16$$

$$= -9 + 3\lambda + 3\lambda - \lambda^2 - 16 = \underline{\underline{-25 + \lambda^2}} \Rightarrow \begin{array}{l} \lambda_1 = -5 \\ \lambda_2 = 5 \end{array}$$

$$U_5: \begin{bmatrix} 3-(-5) & 4 \\ 4 & -3-(-5) \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \xrightarrow{\text{Zeilen ausmultiplizieren}} = \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 2x + y = 0$$

$$= U_{-5} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

normiert

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$U_5: \begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \xrightarrow{\text{Zeilen ausmultiplizieren}} = \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow x + 2y = 0$$

$$= U_5 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

→ Octave

$$S = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \det(A_2) = 0, \text{ da in ref-Form 1 Pivot 0 ist}$$

$$\begin{aligned} \det(A_2 - \lambda I) = 0 &= \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} && 2-2\lambda-\lambda+\lambda^2 \\ &= (1-\lambda) \cdot \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} - (-1) \cdot \det \begin{bmatrix} -1 & 0 \\ -1 & 1-\lambda \end{bmatrix} && 2-3\lambda+\lambda^2-1 \\ &= -\lambda^3 + 4\lambda^2 - 4\lambda + 1 && (1-\lambda)(1-3\lambda+\lambda^2) \\ &= -\lambda^3 + 4\lambda^2 - 3\lambda && 1-3\lambda+\lambda^2-\lambda+2\lambda^2-3\lambda \\ &&& -\lambda^3 + 4\lambda^2 - 4\lambda + 1 \\ &&& \Rightarrow -\lambda(\lambda^2 - 4\lambda + 3) = -\lambda((\lambda-3)(\lambda-1)) = 0 \\ &&& \Rightarrow \lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 1 \end{aligned}$$

$$\text{check: } \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A_2)$$

$$0 \cdot 3 \cdot 1 = 0$$

$$0 = 0 \checkmark$$

$$U_0: \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{z_{12}(1)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{z_{23}(1)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{array}{l} x-y=0 \\ y-z=0 \\ \hline \end{array}} U_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad | \text{ normiert}$$

$$\begin{aligned} U_3: \begin{bmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 1-3 \end{bmatrix} &= \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{z_{12}(-\frac{1}{2})} \begin{bmatrix} -2 & -1 & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{z_1(-2)} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{z_2(-2)} \\ &= \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \xrightarrow{z_{23}(1)} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{z_{12}(-\frac{1}{2})} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{array}{l} 1x - 1z = 0 \\ 1y + 2z = 0 \\ \hline \end{array}} U_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} U_1: \begin{bmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{bmatrix} &= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{z_1(1)} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{z_{13}(1)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow{z_{23}(1)} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{z_2(1)} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{z_{21}(1)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\begin{array}{l} x+z=0 \\ y=0 \\ \hline \end{array}} U_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

→ use Octave to check :)

## Aufgabe 6

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \quad \det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 2-\lambda & 5 \\ 3 & -\lambda \end{bmatrix} = (2-\lambda)(-\lambda) - 15 = -2\lambda + \lambda^2 - 15 = \lambda^2 - 2\lambda - 15 = (\lambda-5)(\lambda+3) \Rightarrow \lambda_1 = 5, \lambda_2 = -3$$

$$U_5: \begin{bmatrix} 2-5 & 5 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 3 & -5 \end{bmatrix} \xrightarrow{\text{Zeil 2} \cdot (-1)} = \begin{bmatrix} -3 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow -3x + 5y = 0 \\ = U_5 = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{34}} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$U_{-3}: \begin{bmatrix} 2-(-3) & 5 \\ 3 & -(-3) \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 3 & 3 \end{bmatrix} \xrightarrow{\text{Zeil 2} \cdot (-0,6)} = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow 5x + 5y = 0 \\ = U_{-3} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = S \wedge S^{-1}$$

$$S = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}$$

$$\det(S) = -5 - 3 = -8$$

$$S^{-1} = -\frac{1}{8} \begin{bmatrix} -1 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & -\frac{5}{8} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & -\frac{5}{8} \end{bmatrix}$$

$$S \wedge = \begin{bmatrix} 25 & -3 \\ 15 & 3 \end{bmatrix}$$

$$S \wedge S^{-1} = \begin{bmatrix} \frac{25}{3} - \frac{5}{3} & \frac{25}{3} + \frac{5}{3} \\ \frac{15}{3} + \frac{3}{3} & \frac{15}{3} - \frac{15}{3} \end{bmatrix} = \begin{bmatrix} \frac{16}{3} & \frac{40}{3} \\ \frac{20}{3} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} = A$$

## Aufgabe 7

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) = 3 - 3\lambda - \lambda + \lambda^2 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1) = \lambda_1 = 3, \lambda_2 = 1$$

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \det(B - \lambda I) = 0$$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) \Rightarrow \lambda_1 = 1, \lambda_2 = 3$$

$$A+B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

$$\det(A+B-\lambda I) = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = (4-\lambda)^2 - 1 = \lambda^2 - 8\lambda + 15 = (\lambda-5)(\lambda-3) = 0 \Rightarrow \lambda_1 = 5, \lambda_2 = 3$$

also  $\lambda_A + \lambda_B \neq \lambda_{AB}$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

### Aufgabe 8



$$P = \begin{array}{c} \text{to state} \\ \text{state} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1/2 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1/2 & 0 \end{bmatrix}$$

wenn sich die Spalten einer stochastischen Matrix auf 1 addieren, dann ist der Eigenwert gerade = 1

$$\Rightarrow \lambda_1, \lambda_2, \lambda_3 = 1$$

$$U_{\lambda_1} \begin{bmatrix} -1 & 1/2 & 0 \\ 1 & -1 & 1 \\ 0 & 1/2 & -1 \end{bmatrix} z_{12}(1) = \begin{bmatrix} -1 & 1/2 & 0 \\ 0 & -1/2 & 1 \\ 0 & 1/2 & -1 \end{bmatrix} z_{23}(1) = \begin{bmatrix} -1 & 1/2 & 0 \\ 0 & -1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} -1x + 1/2y = 0 \\ -1/2y + 1z = 0 \end{array}$$

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = u_2 = \begin{bmatrix} 1/4 \\ 3/4 \\ 1/4 \end{bmatrix}$$

soll sich auf 1 aufaddieren

$$S = U \cdot S$$

$$P \cdot S = S$$

$$\begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ 3/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 3/4 \\ 3/8 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \\ 1/4 \end{bmatrix} \checkmark = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

Longterm probabilities

$$s_1 = 25\%, s_2 = 50\%, s_3 = 25\%$$

## 9 Singular Value Decomposition (SVD)

What is the long term probability for each state? You have find the eigenvector to the eigenvalue 1.

Note: it can be shown, that every stochastic matrix, i.e. one whos columns add to 1 has an eigenvalue 1.

## 9 Singular Value Decomposition (SVD)

Find the SVD of the singular matrix  $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ . The rank is  $r = 1$ .

## 10 Again: Singular Value Decomposition (SVD)

Find the eigenvalues  $\lambda_1 = \sigma_1^2$ ,  $\lambda_2 = \sigma_2^2$  and the corresponding unit eigenvectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  of  $A^T A$ . Then find  $\mathbf{u}_1 = A\mathbf{v}_1/\sigma_1$  and  $\mathbf{u}_2 = A\mathbf{v}_2/\sigma_2$ .

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Verify that  $\mathbf{u}_1$  is a unit eigenvector of  $AA^T$ . Complete the matrices  $U$ ,  $\Sigma$  and  $V$ . Write down orthonormal bases for the four fundamental subspaces of this  $A$ .

## 11 Again: Singular Value Decomposition (SVD)

Compute  $A^T A$  and  $AA^T$  and their eigenvalues and eigenvectors for this  $A$ :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Multiply the matrices  $U\Sigma V^T$  to recover  $A$ . Does  $\Sigma$  have the same shape as  $A$ ?

→ Octave

## 12 Again: Singular Value Decomposition (SVD)

Find the eigenvalues and unit eigenvectors of  $A^T A$  and  $AA^T$ . Keep each  $A\mathbf{v} = \sigma\mathbf{u}$ :

Fibonacci matrix       $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ .

Construct the SVD and verify that  $A$  equals  $U\Sigma V^T$ .

→ Octave

## 13 Concept of separable filters in image processing (optional)

A 2D filter  $F$  is said to be separable if it can be written as the convolution of two 1D filters  $G$  and  $H$ , i.e. as  $F = G \star H$ . If  $M_F$  is the filterkernel of  $F$ , then  $F$  is seperable, if and only if  $M_F$  has rank one. In this case we can write  $M_F$  as the product of a column vector  $M_G$  and a row vector  $M_H$ , i.e. every row (column) of  $M_F$  is a multiple of the row (column) vector  $M_H$  ( $M_G$ ).

→ Octave



## Aufgabe 9

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{rank } A = 1$$

$$A^T A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

$$\text{Eigenwert } A^T A = \det(A^T A - \lambda I) = 0$$

$$\begin{bmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{bmatrix} = (5-\lambda)(5-\lambda) - 25 \\ = 25 - 5\lambda - 5\lambda + \lambda^2 - 25 \\ = \lambda^2 - 10\lambda = \lambda(\lambda - 10) \\ \Rightarrow \lambda_1 = 10, \lambda_2 = 0$$

$\xrightarrow{\text{G1}}$

$$U_{10}: \begin{bmatrix} 5-10 & 5 \\ 5 & 5-10 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \xrightarrow{\text{Zeile 1} \leftrightarrow \text{Zeile 2}} = \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{Zeile 2} \cdot (-\frac{1}{5})} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 1x - 1y = 0 \\ u_{10} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\xrightarrow{\text{G2}}$

$$U_0: \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \xrightarrow{\text{Zeile 2} \cdot (-1)} = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{Zeile 1} \cdot (\frac{1}{5})} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 1x + 1y = 0 \Rightarrow u_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{es gilt } A^T A = V \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

1. Spalte von U:

$$u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

2. Spalte von U:

$$u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \quad \text{Wird so gewählt, dass die Spalten senkrecht aufeinander stehen!}$$

$$A = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \sqrt{10} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$U \quad \Sigma \quad V^T$

## Aufgabe 10

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad AA^T = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$$

Eigenwerte  $AA^T$ :  $\det(AA^T - \lambda I) = \begin{bmatrix} 5-\lambda & 15 \\ 15 & 45-\lambda \end{bmatrix} = (5-\lambda)(45-\lambda) - 225$

$$\begin{aligned} &= 225 - 5\lambda - 45\lambda + 225 - \lambda^2 \\ &= \lambda^2 - 50\lambda \\ &= \lambda(\lambda - 50) \Rightarrow \lambda_1 = 50, \lambda_2 = 0 \end{aligned}$$

$$U_{50}: \begin{bmatrix} -45 & 15 \\ 15 & -5 \end{bmatrix} z_{12}(\frac{1}{5}) = \begin{bmatrix} -45 & 15 \\ 0 & 0 \end{bmatrix}^{2(\frac{1}{5})} = \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} -3x + 1y &= 0 \\ x = s &\Rightarrow -3s + 1y = 0 \Rightarrow 1y = 3s \end{aligned}$$

$$v_1 = u_{50} = \begin{bmatrix} 3 \\ 3s \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$U_0: \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix} z_{12}(-\frac{1}{3}) = \begin{bmatrix} 5 & 15 \\ 0 & 0 \end{bmatrix} z_1(-\frac{1}{3}) = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} 1x + 3y &= 0 \\ y = s &\Rightarrow 1x + 3s \Rightarrow x = -3s \end{aligned}$$

$$v_2 = u_0 = \begin{bmatrix} -3s \\ s \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$V = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

$U$  berechnen:

$$U_1: = \frac{1}{\sqrt{\lambda_1}} A v_1 = \frac{1}{\sqrt{500}} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{500}} \begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} \frac{7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} \end{bmatrix}$$

$$u_1 = v_1 \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} \end{bmatrix}$$

$U_2: =$  Bestimmen einen Vektor der orthogonal zu  $u_1$  ist

$$\begin{bmatrix} -\frac{7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{7}{\sqrt{500}} & \frac{-7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} & \frac{21}{\sqrt{500}} \end{bmatrix} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$U$

$\Sigma$

$V^T$

4 Unterräume um A

$$A = U \Sigma V^T = [U_1 \ U_2] \Sigma [V_1 \ V_2]^T$$

$U_1$  ist eine Basis für  $C(A)$

$U_2$  ist eine Basis für  $N(A^T)$

$V_1$  ist eine Basis für  $C(A^T)$

$V_2$  ist eine Basis für  $N(A)$

## 14 Octave → separable Filters (IPCV)

As example let's look at the Gaussian filter, whose kernel is

$$\mathbf{M}_F = \begin{pmatrix} 0.011344 & 0.083820 & 0.011344 \\ 0.083820 & 0.619347 & 0.083820 \\ 0.011344 & 0.083820 & 0.011344 \end{pmatrix}$$

This is a rank one matrix and it can therefore be written as the product of a column vector and a row vector. In order to find these vectors we use the SVD.

Use the following Octave/Matlab commands to verify this:

```
octave:5> F = fspecial('gaussian')
F = 0.011344    0.083820    0.011344
      0.083820    0.619347    0.083820
      0.011344    0.083820    0.011344
octave:7> [U,S,V] = svd(gaussian)
U = -0.132923    0.958101   -0.253720
      -0.982173   -0.161664   -0.095924
      -0.132923    0.236446    0.962510
S = 6.4203e-01        0        0
      0    2.1032e-17        0
      0        0    2.6991e-34
V = -1.3292e-01   -6.9450e-01   -7.0711e-01
      -9.8217e-01   1.8798e-01   1.3878e-17
      -1.3292e-01   -6.9450e-01   7.0711e-01
octave:8> G = U(:,1) * sqrt(S(1,1))
G = -0.10651
      -0.78699
      -0.10651
octave:9> H = V(:,1)' * sqrt(S(1,1))
H = -0.10651   -0.78699   -0.10651
octave:10> G * H
ans = 0.011344    0.083820    0.011344
      0.083820    0.619347    0.083820
      0.011344    0.083820    0.011344
octave:11> F - G * H
ans = -2.4286e-17   -1.5266e-16   -2.4286e-17
      -1.3878e-17   -1.1102e-16   -1.3878e-17
      -3.4694e-18    0.0000e+00   -3.4694e-18
```

## 14 Octave → Octave

The following script allows to try a different number of singular values to reconstruct an image.

```
iclose all
clear all
clc

%reading and converting the image
inImage=imread('../figs/TheLetter_A.png');
inImage = 0.2989 * inImage(:,:,1) + 0.5870 * inImage(:,:,2) + 0.1140 * inImage(:,:,3);
inImageD=double(inImage);
[size_M,size_N] = size(inImage);
size_Orig = size_M*size_N;

% decomposing the image using singular value decomposition
[U,S,V]=svd(inImageD);
whos

% Using different number of singular values (diagonal of S) to compress and
% reconstruct the image
dispEr = [];
```

```

numSVals = [];
for N=1:5:41
    % store the singular values in a temporary var
    % jfb C = S;

    % discard the diagonal values not required for compression
    % jfb C(N+1:end,:) = 0;
    % jfb C(:,N+1:end) = 0;

    % Construct an Image using the selected singular values
    % D=U*C*V';
    D=U(:,1:N)*S(1:N,1:N)*V(:,1:N)';

    % display and compute error
    figure;
    buffer = sprintf('Image_output_using%d_singular_values_(compression_%f)',
        N, (size_M*N+N^2+size_N*N)/size_Orig)
    imshow(uint8(D));
    title(buffer);
    error=sum(sum((inImageD-D).^2));

    % store vals for display
    dispEr = [dispEr; error];
    numSVals = [numSVals; N];
end

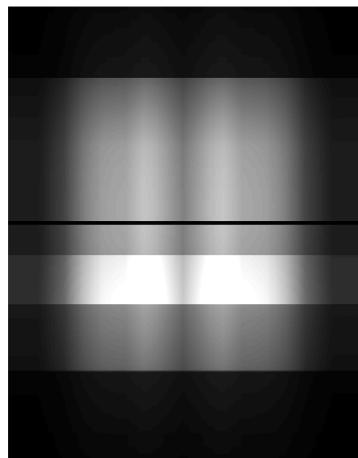
% display the error graph
figure;
title('Error_in_compression');
plot(numSVals, dispEr);
grid on
xlabel('Number_of_Singular_Values_used');
ylabel('Error_between_compress_and_original_image');

```

**Have fun!**

# Aufgabe 14

Image output using 1 singular values (compression 0.004756)



use SVD-demo.m Script