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→ Aufgabe 8 b und c als MEP - Vorbereitung

Multivariable Calculus - Exercise 2

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I.BA_IMATH, Semesterweek 06

Please write down to solution of the exercises in a consise but comprehensible way. Numerical results should be accurate to 4 digits. Sketches should be correct qualitatively. At least 75% of the exercises have to be solve satisfactorily. Due time is one week after we have discussed the corresponding topics in class.

1 Total differential I: The *Cow-culus* Exercise

A cow's udder is in the shape of a hemisphere.

1. If its diameter is measured to be 26 cm with a possible error of 0.5 cm, then use differentials to approximate the
   (a) propagated error
   (b) relative error
   (c) percent error

   in computing its volume.
2. Estimate the maximum allowable percent error in measuring the diameter if the error in computing the volume must not exceed 3%.

Note: In this exercise the diameter is the quantity being measured, while the volume is being computed. You should therefore express the volume as a function of the diameter.

2 Total differential II

(a) Find the (total) differential of $g(u,v) = u^2 + uv$.

(b) Use your answer to part (a) to estimate the change in g as you move from $(1,2)$ to $(1.2, 2.1)$.

3 Total differential III

In a room, the temperature is given by $T = f(x,t)$ degrees Celsius, where x is the distance from a heater (in meters) and t is the elapsed time (in minutes) since the heater has been turned on.

A person standing 3 m from the heater 5 min after it has been turned on observes the following:

Aufgabe 1

Volume hemisphere: $V = \frac{2}{3}\pi \cdot r^3 = \frac{2}{3}\pi \cdot \left(\frac{d}{2}\right)^3$

Diameter: $d = 26$ cm

Error: $e = 0.5$ cm

$\frac{\partial V}{\partial d} = \frac{2}{3}\pi \frac{d^3}{8} = \frac{2d^3}{24}\pi = \frac{6d^2}{24}\pi = \frac{d^2\pi}{4}$

$\frac{1}{12}d^3\pi$

1)

(a) Propagated error

$$\frac{dV}{e} = \frac{\partial V}{\partial d} \quad | \cdot e$$

$$dV = \frac{\partial V}{\partial d} \cdot e = \frac{d^2 \cdot \pi}{4} \cdot e = \frac{26^2 \cdot \pi \cdot 0.5}{4} = \underline{\underline{265.4646 \text{ cm}^3}}$$

(b) Relative error

$$\frac{dV}{V} = \frac{\frac{\partial V}{\partial d} \cdot e}{V} = \frac{\frac{\cancel{d^2} \cdot \cancel{\pi} \cdot e}{4}}{\frac{1}{12} d^{\cancel{3}} \cdot \cancel{\pi}} = \frac{\frac{1}{4}e}{\frac{1}{12}d} = 3\frac{e}{d} = 3 \cdot \frac{0.5}{26} = \underline{\underline{0.0577}}$$

(c) Percent error

$\underline{\underline{5.77\,\%}}$

2)

Es muss gelten:

$$\frac{\frac{\partial V}{\partial d} \cdot e}{V} \leq 0.03$$

gesucht

$$3 \cdot \frac{e}{26} \leq 0.03 \quad | \cdot 26$$

$$3e \leq 0.78 \quad | : 3$$

$$e \leq \underline{\underline{0.26 \text{ cm}}}$$

Der Fehler darf höchstens 0.26 cm betragen

Aufgabe 2

(a) $g(u,v) = u^2 + uv$

$$dg = \frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv = (2u+v)du + u \cdot dv$$

(b) (1,2) to (1.2, 2.1)

also sind $du = 0.2$ und $dv = 0.1$

$$(2 \cdot 1 + 2) \cdot 0.2 + 1 \cdot 0.1 = 0.8 + 0.1 = \underline{\underline{0.9}} = dg$$

überprüfen mit der Funktion:

$$g(1.2, 2.1) - g(1,2) = (1.44 + 2.52) - 3 = 3.96 - 3 = \underline{\underline{0.96}}$$

Aufgabe 3

$T = f(x,t)$

x: distance in m

t: elapsed time in minutes

Folgende Beobachtungen:

$$\left.\frac{df}{dt}\right|_{(3m, 5min)} = 1.2°C\ min^{-1}$$: Distanz konstant, deshalb x als Konstante betrachten

$$\left.\frac{df}{dx}\right|_{(3m, 5min)} = 2°C\ m^{-1}$$: Zeit konstant

→ Totales Differential

$dt = 6min - 5min = 1min$

$dx = 2.5m - 3m = -0.5m$

$$df = \frac{\partial f}{\partial t} \cdot dt + \frac{\partial f}{\partial x} \cdot dx = 1.2°C\,min^{-1} \cdot dt + 2°C\,m^{-1} \cdot dx$$

$$= 1.2°C\,min^{-1} \cdot 1min + 2°C\,m^{-1} \cdot (-0.5m)$$

$$= \underbrace{1.2°C\,min^{-1} - 2°C\,m^{-1} \cdot 0.5}_{\text{Keine Ahnung wie man das jetzt rechnen sollte :C}}$$

1. T is increasing $1.2\,°\mathrm{C}\,\mathrm{min}^{-1}$, and
2. by walking away from the heater, T decreases by $2\,°\mathrm{C}\,\mathrm{m}^{-1}$ as time is held constant.

Estimate (using the total differential) how much cooler or warmer it would be 2.5 m from the heater after 6 min.

4 Linearization I

Niveaulinie

From a differentiable function $f(x,y)$ we know $f(1,3) = 7$ and $\nabla f(1,3) = \left[2,\text{-}5\right]^T$.

(a) Find the equation of the tangent line to the level curve of f through the point $(1,3)$.

(b) Find the equation of the tangent plan to the surface $z = f(x,y)$ at the point $(1,3)$.

5 Linearization II

Suppose the gradient of f at $\mathbf{x}_0 = \left[x_0, y_0\right]^T$ is nonzero. Using linearization and the definition of the total derivative the change in f when we move from $\mathbf{x}_0$ to $\mathbf{x}$ is

$$\mathrm{d}f = \nabla f(\mathbf{x}_0) \bullet (\mathbf{x} - \mathbf{x}_0)$$

Therefore, the tangent line to the contourline satisfies

$$\mathrm{d}f = 0 \quad \Leftrightarrow \quad \nabla f(\mathbf{x}_0) \bullet (\mathbf{x} - \mathbf{x}_0) = 0$$

Therefore, the contourline through $\mathbf{x}_0$ is perpendicular to the gradient $\nabla f(\mathbf{x}_0)$ at this point.

Verify this for

(a) $f(x,y) = y - x^2$ and $\mathbf{x}_0 = \left[1,1\right]^T$.

(b) $g(x,y) = x^2 - y^2$ and $\mathbf{x}_0 = \left[2,\sqrt{3}\right]^T$.

Check by drawing the contour and the tangent line!

6 Newton-Raphson Method

Using the Newton-Raphson Method solve the following system of nonlinear equations

$$\begin{aligned} x - y + 1 &= 0 \\ x^2 - y - 1 &= 0 \end{aligned}$$

Use Octave/Matlab and verify that the programs on the slides work. Check the result by an analytical solution of this system of nonlinear equations. As the starting value use $(1,1)$. You must compute the Jacobian Matrix.

Aufgabe 4

$f(1,3) = 7$ und $\nabla f(1,3) = [2, -5]^T$

(a) $\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$

$$\Rightarrow \begin{bmatrix} 2 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-3 \end{bmatrix} = 2(x-1) - 5(y-3) = 0$$

$$2x - 2 - 5y + 15 \overset{!}{=} 0$$

$$2x - 5y + 13 \overset{!}{=} 0 \Rightarrow y = \frac{2x+13}{5}$$

(b) $L(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$

$$\Rightarrow L(x) = 7 + \begin{bmatrix} 2 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-3 \end{bmatrix}$$

$$= 7 + 2x - 5y + 13$$

$$\overset{!}{=} 2x - 5y + 20 = z$$

$\Rightarrow$ plane : $2x - 5y - z + 20$

überprüfen mit $P(1,3) = 7$

$$2 - 15 - z + 20 = 0$$

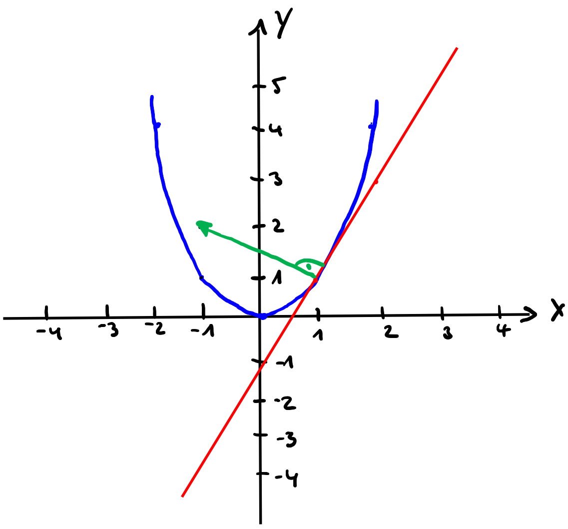
$$7 = z \quad ✓$$

Aufgabe 5

– zeigen, dass Niveaulinien und Gradient bei x_0 senkrecht zueinander sind

(a) $f(x,y) = y - x^2$ and $x_0 = [1,1]^T$

Contour line: $\underbrace{y - x^2}_{1-1=0} = 0 \Rightarrow y = x^2$



Tangente an Niveaulinie:

$$\vec{\nabla} f(x,y) = \begin{bmatrix} -2x \\ 1 \end{bmatrix} \Rightarrow \vec{\nabla} f(1,1) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \vec{\nabla} f(1,1) \cdot \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = -2x + 2 + y - 1$$

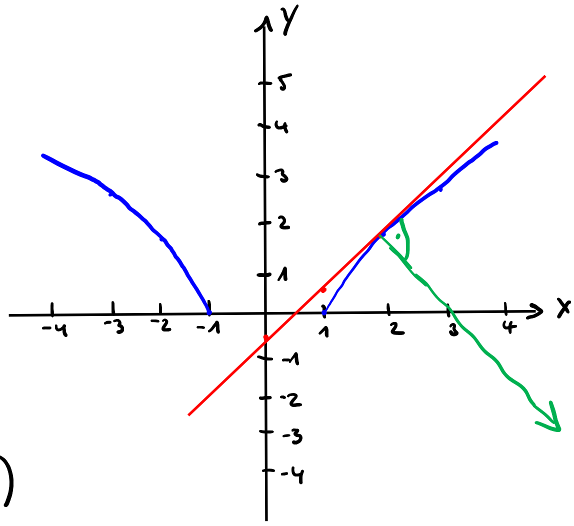
$$\overset{!}{=} -2x + y + 1 = 0$$

$$\rightarrow L(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0)$$

$$\overset{!}{=} 0 - 2x + y + 1$$

$$\overset{!}{=} -2x + y + 1 \Rightarrow y = 2x - 1$$

(b) $g(x,y) = x^2 - y^2$ and $x_0 = [2, \sqrt{3}]^T$ ← need to check again, bc $2^2 - \sqrt{3}^2 = ?$



Contour lines: $\underbrace{x^2 - y^2}_{4-3=1} = 1 \Rightarrow y^2 = x^2 - 1$

$y \overset{!}{=} \sqrt{x^2-1}$

Tangente an Niveaulinie:

$\vec{\nabla} g(x,y) = \begin{bmatrix} 2x \\ -2y \end{bmatrix} \Rightarrow \vec{\nabla} g(2, \sqrt{3}) = \begin{bmatrix} 4 \\ -2\sqrt{3} \end{bmatrix}$

$= \vec{\nabla} g(2, \sqrt{3}) \cdot \begin{bmatrix} x-2 \\ y-\sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -2\sqrt{3} \end{bmatrix} \begin{bmatrix} x-2 \\ y-\sqrt{3} \end{bmatrix} = 4(x-2) - 2\sqrt{3}(y - \sqrt{3})$

$= 4x - 8 \quad -2\sqrt{3}y + 6$

$\overset{!}{=} 4x - 2\sqrt{3}y - 2 = 0$

$\rightarrow L(x) = \nabla f(x_0) \cdot (x - x_0)$

$\overset{!}{=} 4x - 2\sqrt{3}y - 2$

$\overset{!}{=} -2\sqrt{3}y - 2 \Rightarrow y = \dfrac{4x-2}{2\sqrt{3}}$

Aufgabe 6

$$\begin{aligned} x - y + 1 &= 0 \\ x^2 - y - 1 &= 0 \end{aligned} \qquad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$k = 0$:

$$J_F(x_k) = \begin{bmatrix} \frac{\partial F_1}{\partial x}(x_k) & \frac{\partial F_1}{\partial y}(x_k) \\ \frac{\partial F_2}{\partial x}(x_k) & \frac{\partial F_2}{\partial y}(x_k) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2x_k & -1 \end{bmatrix}$$

$$J_F(x_0)\,\Delta x_0 = -F(x_0) \quad \Leftrightarrow \quad \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \Delta x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \Delta x_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$k = 1$:

$$x_1 = \underset{x_0}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \underset{\Delta x_0}{\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$J_F(x_1)\,\Delta x_1 = -F(x_1) \quad \Leftrightarrow \quad \begin{bmatrix} 1 & -1 \\ 6 & -1 \end{bmatrix} \Delta x_1 = \begin{bmatrix} 0 \\ -4 \end{bmatrix} \quad \Rightarrow \quad \Delta x_1 = \begin{bmatrix} -\frac{4}{5} \\ -\frac{4}{5} \end{bmatrix}$$

usw. $\rightarrow$ Octave :)

7 Chain-Rule I

Find $\mathrm{d}z/\mathrm{d}t$ using the chain rule if

(a) $z = xy^2, x = e^{-t}, y = \sin t$
(b) $z = \ln(x^2 + y^2), x = 1/t, y = \sqrt{t}$
(c) $z = (x + y)e^y, x = 2t, y = 1 - t^2$.

8 Chain-Rule II

Find $\partial z/\partial u = z_u$ and $\partial z/\partial v = z_v$ using the chain rule if

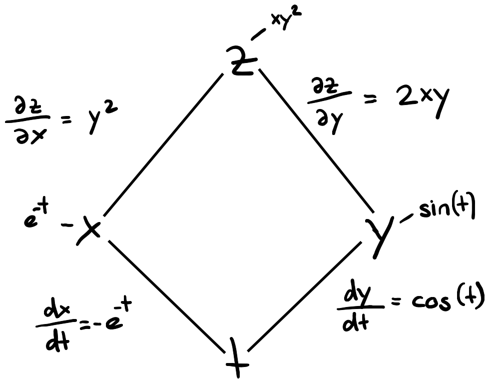
(a) $z = xe^y, x = \ln u, y = u$
(b) $z = xe^y, x = u^2 + v^2, y = u^2 - v^2$
(c) $z = \ln(xy), x = (u^2 + v^2)^2, y = (u^3 + v^3)^2$.

-> b und c in MEP Vorbereitung

Viel Spass!

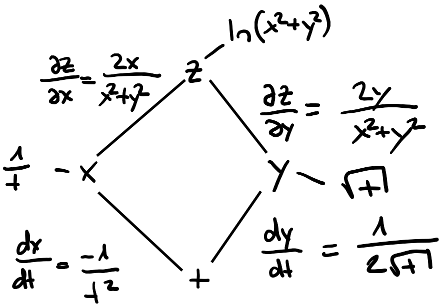
Aufgabe 7

(a) $z = xy^2$, $x = e^{-t}$, $y = \sin(t)$



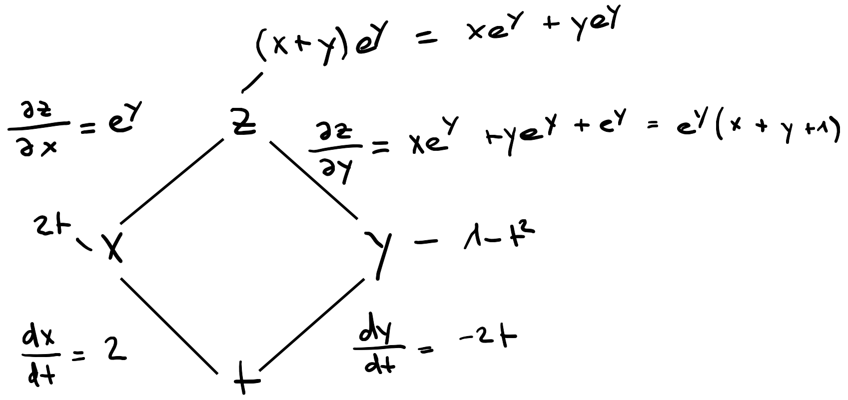
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= y^2 \cdot (-e^{-t}) + 2xy \cdot \cos(t)$$
$$= -y^2 e^{-t} + 2xy\cos(t)$$
$$= -(\sin(t))^2 \cdot e^{-t} + 2e^{-t} \cdot \sin(t)\cos(t)$$
$$= \underline{\underline{\sin(t)\, e^{-t}\left(-\sin(t) + 2\cos(t)\right)}}$$

(b) $z = \ln(x^2+y^2)$, $x = 1/t$, $y = \sqrt{t}$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= \frac{2x}{x^2+y^2} \cdot \frac{-1}{t^2} + \frac{2y}{x^2+y^2} \cdot \frac{1}{2\sqrt{t}}$$
$$= \frac{2}{(x^2+y^2)}\left(\frac{-x}{t^2} + \frac{y}{2\sqrt{t}}\right)$$
$$= \frac{2}{(\frac{1}{t})^2 + (\sqrt{t})^2}\left(\frac{\frac{1}{t}}{t^2} + \frac{\cancel{\sqrt{t}}}{2\cancel{\sqrt{t}}}\right)$$
$$= \frac{2}{\frac{1}{t^2} + t}\left(\frac{t^{-1}}{t^2} + \frac{1}{2}\right)$$
$$= \frac{2t^2}{1+t^3}\left(\frac{-1}{t^3} + \frac{1}{2}\right)$$
$$= \frac{2t^2}{1+t^3}\;\frac{-2+t^3}{2t^3}$$
$$= \frac{-4t^2 + 2t^5}{2t^3 + 2t^6} = \frac{-4t^2 + 2t^5}{2t^3(1+t^3)} = \underline{\underline{\frac{-2+t^3}{t(1+t^3)}}}$$

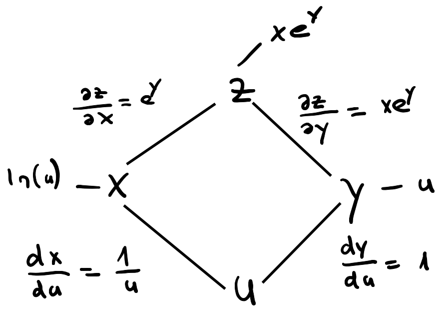
(c) $z = (x+y)e^y \,,\; x = 2t ,\; y = 1 - t^2$



$$
\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= e^y \cdot 2 + e^y(x+y+1)\cdot(-2t) \\
&= 2e^y - 2te^y(x+y+1) \\
&= 2e^y\left(1 - t(x+y+1)\right) \\
&= 2e^{(1-t^2)}\left(1 - t\left(2t + (1-t^2) + 1\right)\right) \\
&= 2e^{(1-t^2)}\left(1 - t(2t + 1 - t^2 + 1)\right) \\
&= 2e^{(1-t^2)}\left(1 - t(-t^2 + 2t + 2)\right) \\
&= 2e^{(1-t^2)}\left(1 + t^3 - 2t^2 - 2t\right) \\
&= \underline{\underline{2e^{(1-t^2)}\left(t^3 - 2t^2 - 2t + 1\right)}}
\end{aligned}
$$

Aufgabe 8

(a) $z = xe^y$, $x = \ln(u)$, $y = u$



$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}$$

$$= e^y \cdot \frac{1}{u} + xe^y \cdot 1$$

$$= \frac{e^y}{u} + xe^y$$

$$= e^y\left(\frac{1}{u} + x\right) = \underline{\underline{e^u\left(\frac{1}{u} + \ln(u)\right)}}$$