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Lineare Algebra - Exercises 3

SVD

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I.BA_IMATH, Semesterweek 03

Please write down to solution of the exercises in a consise but comprehensible way. Numerical results should be accurate to 4 digits. Sketches should be correct qualitatively. At least 75% of the exercises have to be solve satisfactorily. Due time is one week after we have discussed the corresponding topic in class.

1 Determinant

Compute the determinant of the following matrices:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}.$$

Which of the matrices have an inverse, i.e. are not singular (do not compute the inverse, though)?

$\det(A) \neq 0$, dann existiert A^{-1}

2 Again: Determinant

First compute the determinant using Sarrus' formula. Then derive the reduced row echelon form of the matrices and again compute the determinant by multiplying the pivot elements.

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix}$$

Is the matrix invertible (do not compute the inverse, though)?

3 Again: Determinant

Compute the determinant by Laplace expansion (along an appropriate column or row).

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Compare with the determinant You obtain, if using the reduced row echelon form of the matrix. Is the matrix invertible?

Aufgabe 1

$$\det(A) = \det\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} = 1\cdot 8 - 2\cdot 3 = 8-6 = 2$$

$$\det(B) = \det\begin{bmatrix} 1-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} = (1-\lambda)\cdot(8-\lambda) - 2\cdot 3 = 8-\lambda-8\lambda+\lambda^2-6 = 2-9\lambda+\lambda^2$$

$$\det(C) = \det\begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix} = 1\cdot\det\begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} = \underline{\underline{-3}}$$

oder Produkt der Diagonalen, da obere Dreiecksmatrix

$$\rightarrow 1\cdot 3\cdot(-1) = \underline{\underline{-3}}$$

Aufgabe 2

Sarrus

$$\det(A) = \det\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix}\begin{matrix} 1 & -1 \\ 2 & 2 \\ 3 & 3 \end{matrix} = 20-24+12-12-24+20 = \underline{\underline{-8}}$$

ref

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix}\begin{matrix} \\ z_{12}(-2) \\ z_{13}(-3) \end{matrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 6 & 4 \end{bmatrix} z_{23}(-1.5) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & -2 \end{bmatrix} = 1\cdot 4\cdot(-2) = \underline{\underline{-8}}$$

=

Sarrus

$$\det(B) = \det\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix}\begin{matrix} 1 & -1 \\ 2 & 2 \\ 3 & 1 \end{matrix} = 12-12+4-12-4+12 = \underline{\underline{0}}$$

ref

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix}\begin{matrix} \\ z_{12}(-2) \\ z_{13}(-3) \end{matrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 0 \\ 0 & 4 & 0 \end{bmatrix} z_{23}(-1) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \underline{\underline{0}}$$

=

sofort ersichtlich, dass det 0 sein muss, da ein Pivot 0 ist!

Aufgabe 3

Laplace

$$\det(A) = \det\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = 1\cdot\underbrace{\det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{A1} + 1\cdot\underbrace{\det\begin{bmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{A2}$$

$$A1 \quad \det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = 1\cdot\det\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} + 2\cdot\det\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -1 + 2\cdot 1$$

$$= -1 + 2 = \underline{\underline{1}}$$

$$A2 \quad \det\begin{bmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} = 1\cdot\det\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \cdot (-1)\det\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 0\cdot(-1)\cdot 1 = \underline{\underline{-1}}$$

$\det(A) = 1 - 1 = \underline{\underline{0}}$ — nicht invertierbar!

ref

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} z_{12}(1) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} z_{23}(1) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} z_{34}(1)$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 1\cdot 1\cdot 1\cdot 0 = \underline{\underline{0}}$$

Pivot ist null → det ist 0

4 Again: Determinant

If a 4×4 matrix has $\det(\boldsymbol{A}) = 1/2$, find $\det(2\boldsymbol{A})$ and $\det(-\boldsymbol{A})$ and $\det(\boldsymbol{A}^2)$ and $\det(\boldsymbol{A}^{-1})$.

You have to apply the rules on slide 6!

5 Eigenvalues and -vectors

Find the eigenvalues and eigenvectors of

$$\boldsymbol{A}_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Check the trace and determinant.

6 Again: Eigenvalues and -vectors

Find the eigenvalues and eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ 3 & -0 \end{bmatrix}$$

Verify, that $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$ where $\boldsymbol{S}$ contains the eigenvectors as columns. Make sure, they are in the same order as the eigenvalues in $\boldsymbol{\Lambda}$.

Compute $\exp(\boldsymbol{A})$ using this factorization.

Check Your results with Octave.

7 Again: Eigenvalues and -vectors

Find the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$ (easy for triangular matrices) and $\boldsymbol{A}+\boldsymbol{B}$:

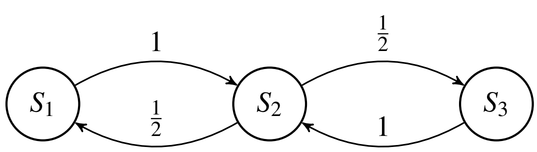
$$\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}+\boldsymbol{B}.$$

Are the eigenvalues of $\boldsymbol{A}+\boldsymbol{B}$ equal (or not equal) to the eigenvalues of $\boldsymbol{A}$ plus eigenvalues of $\boldsymbol{B}$? Explain why!

8 Again: Eigenvalues and -vectors

– Markov Chain

Write down the 3×3 transition matrix $\boldsymbol{P}$ for the Markov chain shown on the right. Make sure, the columns add to 1. The first column contains the probabilities for moving from state S_1 to one of the states S_1, S_2 and S_3.



Aufgabe 4

$\det(A) = \frac{1}{2}$

Es gilt: if one row (column) is multiplied by a number $k \in \mathbb{R}$ the determinant is multiplied by this number

- $\det(2A) = \det([2\vec{a}_1, 2\vec{a}_2, 2\vec{a}_3, 2\vec{a}_4]) = 2^4 \cdot \frac{1}{2} = \underline{\underline{8}}$
- $\det(-A) = (-1)^4 \det(A) = \underline{\underline{\frac{1}{2}}}$

Es gilt: $\det(AB) = \det(A)\det(B)$, also

- $\det(A^2) = \det(A)\det(A) = \frac{1}{2} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{4}}}$
- $\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{\frac{1}{2}} = \underline{\underline{2}}$

Aufgabe 5

$$A_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \det(A_1 - \lambda I_2) = 0$$

$-9 - 3\lambda + 3\lambda + \lambda^2$

$$= \det \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix} = (3-\lambda)\cdot(-3-\lambda) - 16 = -9 + \lambda^2 - 16 = \underline{\underline{-25 + \lambda^2}} \Rightarrow \begin{array}{l} \lambda_1 = -5 \\ \lambda_2 = 5 \end{array}$$

u_{-5}: $$\begin{bmatrix} 3-(-5) & 4 \\ 4 & -3-(-5) \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} Z_{12}(-\tfrac{1}{2}) = \begin{bmatrix} 8 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow 2x + y = 0$$

$$= u_{-5} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

normiert

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

u_5: $$\begin{bmatrix} 3-5 & 4 \\ 4 & -3-5 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} Z_{12}(2) = \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow 1x + 2y = 0$$

$$= u_5 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

→ Octave

$$S = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \det(A_2) = 0$$, da in ref-Form 1 Pivot 0 ist

$$2 - 2\lambda - \lambda + \lambda^2$$
$$2 - 3\lambda + \lambda^2 \quad -1$$
$$(1-\lambda)(1 - 3\lambda + \lambda^2)$$
$$1 - \cancel{3\lambda} + \cancel{\lambda^2} - \lambda + \cancel{3\lambda^2} - \cancel{\lambda^3}$$

$$\det(A_2 - \lambda I) = 0 = \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$= \underbrace{(1-\lambda) \cdot \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}}_{-\lambda^3 + 4\lambda^2 - 4\lambda + 1} \underbrace{- (-1) \cdot \det \begin{bmatrix} -1 & 0 \\ -1 & 1-\lambda \end{bmatrix}}_{-1+\lambda} \qquad -\lambda^3 + 4\lambda^2 - 4\lambda + 1$$

$$= -\lambda^3 + 4\lambda^2 - 4\lambda + 1 \quad -1 + \lambda$$

$$= \underline{\underline{-\lambda^3 + 4\lambda^2 - 3\lambda}}$$

$$\Rightarrow -\lambda(\lambda^2 - 4\lambda + 3) = -\lambda((\lambda-3)(\lambda-1)) = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 3, \lambda_1 = 1$$

check: $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A_2)$

$$0 \cdot 3 \cdot 1 \overset{!}{=} 0$$
$$0 \overset{!}{=} 0 \checkmark$$

$$u_0: \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} z_{12}(1) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} z_{23}(1) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x - y = 0 \\ y - z = 0 \end{matrix}$$

normiert

$$= u_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_3: \begin{bmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & -1 \\ 0 & -1 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} z_{12}(-\tfrac{1}{2}) = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -\tfrac{1}{2} & -1 \\ 0 & -1 & -2 \end{bmatrix} \overset{z_1(-2)}{=} \begin{bmatrix} 1 & \tfrac{1}{2} & 0 \\ 0 & -\tfrac{1}{2} & -1 \\ 0 & -1 & -2 \end{bmatrix} z_2(-2)$$

$$= \begin{bmatrix} 1 & \tfrac{1}{2} & 0 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} z_{23}(1) = \begin{bmatrix} 1 & \tfrac{1}{2} & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} z_{21}(-\tfrac{1}{2}) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} 1x - 1z = 0 \\ 1y + 2z = 0 \end{matrix}$$

$$= u_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$u_1: \begin{bmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & -1 \\ 0 & -1 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \overset{z_1(-1)}{=} \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} z_{23}(-1) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} z_2(-1)$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} z_{21}(1) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x + z = 0 \\ y = 0 \end{matrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

→ use Octave to check :)

Aufgabe 6

$A = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} \qquad \det(A - \lambda I) = 0$

$\det \begin{bmatrix} 2-\lambda & 5 \\ 3 & -\lambda \end{bmatrix} = (2-\lambda)(-\lambda) - 15 = -2\lambda + \lambda^2 - 15 \overset{!}{=} \lambda^2 - 2\lambda - 15 \overset{!}{=} (\lambda - 5)(\lambda + 3) \Rightarrow \lambda_1 = 5, \lambda_2 = -3$

$u_5: \begin{bmatrix} 2-5 & 5 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 3 & -5 \end{bmatrix} z_{12}(1) = \begin{bmatrix} -3 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow -3x + 5y = 0$

$\overset{!}{=} u_5 = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{34}} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$u_{-3}: \begin{bmatrix} 2-(-3) & 5 \\ 3 & -(-3) \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 3 & 3 \end{bmatrix} z_{12}(-0.6) = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow 5x + 5y = 0$

$\overset{!}{=} u_{-3}: \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A = S \Lambda S^{-1} \qquad S = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix} \qquad \det(S) = -5 - 3 = -8$

$S^{-1} = -\frac{1}{8} \begin{bmatrix} -1 & -1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & -\frac{5}{8} \end{bmatrix}$

$\begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} \frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & -\frac{5}{8} \end{bmatrix}$

$S\Lambda = \begin{bmatrix} 25 & -3 \\ 15 & 3 \end{bmatrix} \qquad S \Lambda S^{-1} = \begin{bmatrix} \frac{25}{8} - \frac{9}{8} & \frac{25}{8} + \frac{15}{8} \\ \frac{15}{8} + \frac{9}{8} & \frac{15}{8} - \frac{15}{8} \end{bmatrix} = \begin{bmatrix} \frac{16}{8} & \frac{40}{8} \\ \frac{24}{8} & 0 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 3 & 0 \end{bmatrix} = A$

Aufgabe 7

$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \qquad \det(A - \lambda I) = 0$

$\det \begin{bmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) = 3 - 3\lambda - \lambda + \lambda^2 = \lambda^2 - 4\lambda + 3 \overset{!}{=} (\lambda-3)(\lambda-1) = \lambda_1 = 3, \lambda_2 = 1$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \det(B - \lambda I) = 0$

$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) \Rightarrow \lambda_1 = 1, \lambda_2 = 3$

$A + B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

$$\det(A+B-\lambda I) = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} = (4-\lambda)^2 - 1$$

$$\overset{!}{=} \lambda^2 - 8\lambda + 15$$

$$\overset{!}{=} (\lambda-5)(\lambda-3) = 0$$

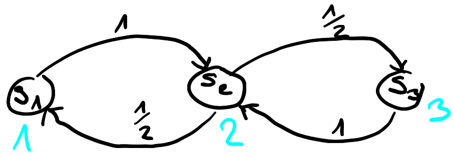
$$\overset{!}{\Rightarrow} \lambda_1 = 5 \,, \quad \lambda_2 = 3$$

also $\lambda_A + \lambda_B \neq \lambda_{AB}$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Aufgabe 8



from state

$$P = \text{to state} \; \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1/2 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1/2 & 0 \end{array}$$

wenn sich die Spalten einer stochastischen Matrix auf 1 addieren, dann ist der Eigenwert gerade $=1$

$\Rightarrow \lambda_1, \lambda_2, \lambda_3 = 1$

$$u_\lambda: \begin{bmatrix} -1 & 1/2 & 0 \\ 1 & -1 & 1 \\ 0 & 1/2 & -1 \end{bmatrix} z_{12}(1) = \begin{bmatrix} -1 & 1/2 & 0 \\ 0 & -1/2 & 1 \\ 0 & 1/2 & -1 \end{bmatrix} z_{23}(1) = \begin{bmatrix} -1 & 1/2 & 0 \\ 0 & -1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} -1x + \frac{1}{2}y = 0 \\ -\frac{1}{2}y + 1z = 0 \end{matrix}$$

$$u_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = u_\infty = \begin{bmatrix} 1/4 \\ 2/4 \\ 1/4 \end{bmatrix}$$

$S = u_\infty$

$P \cdot S = S$

$$\begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/4 \\ 2/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 2/8 \\ 2/4 \\ 2/8 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 2/4 \\ 1/4 \end{bmatrix} \checkmark$$

What is the long term probability for each state? You have find the eigenvector to the eigenvalue 1.

Note: it can be shown, that every stochastic matrix, i.e. one whos columns add to 1 has an eigenvalue 1.

9 Singular Value Decomposition (SVD)

Find the SVD of the singular matrix $\boldsymbol{A} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. The rank is $r = 1$.

10 Again: Singular Value Decomposition (SVD)

Find the eigenvalues $\lambda_1 = \sigma_1^2$, $\lambda_2 = \sigma_2^2$ and the corresponding unit eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ of $\boldsymbol{A}^T\boldsymbol{A}$. Then find $\mathbf{u}_1 = \boldsymbol{A}\mathbf{v}_1/\sigma_1$ and $\mathbf{u}_2 = \boldsymbol{A}\mathbf{v}_2/\sigma_2$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Verify that $\mathbf{u}_1$ is a unit eigenvector of $\boldsymbol{A}\boldsymbol{A}^T$. Complete the matrices $\boldsymbol{U}$, $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$. Write down orthonormal bases for the four fundamental subspaces of this $\boldsymbol{A}$.

11 Again: Singular Value Decomposition (SVD)

Compute $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$ and their eigenvalues and eigenvectors for this $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Multiply the matrices $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$ to recover $\boldsymbol{A}$. Does $\boldsymbol{\Sigma}$ have the same shape as $\boldsymbol{A}$?

→ Octave

12 Again: Singular Value Decomposition (SVD)

Find the eigenvalues and unit eigenvectors of $\boldsymbol{A}^T\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^T$. Keep each $\boldsymbol{A}\mathbf{v} = \sigma\mathbf{u}$:

$$\text{Fibonacci matrix} \qquad \boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Construct the SVD and verify that $\boldsymbol{A}$ equals $\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$.

→ Octave

13 Concept of separable filters in image processing (optional)

A 2D filter F is said to be separable if it can be written as the convolution of two 1D filters G and H, i.e. as $F = G \star H$. If M_F is the filterkernel of F, then F is seperable, if and only if M_F has rank one. In this case we can write M_F as the product of a column vector M_G and a row vector M_H, i.e. every row (column) of M_F is a multiple of the row (column) vector M_H (M_G).

→ Octave

Aufgabe 9

$$A = U \Sigma V^T$$

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \qquad \text{rank} = 1 \qquad A^TA = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

Eigenwerte $A^TA = \det(A^TA - \lambda I) = 0$

$$\begin{bmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{bmatrix} \overset{!}{=} (5-\lambda)(5-\lambda) - 25 \overset{!}{=} 25 - 5\lambda - 5\lambda + \lambda^2 - 25 \overset{!}{=} \lambda^2 - 10\lambda = \lambda(\lambda - 10)$$

$$\Rightarrow \lambda_1 = 10, \; \lambda_2 = 0$$

$\hat{=}\, \sigma_1^2$

$$u_{10}: \begin{bmatrix} 5-10 & 5 \\ 5 & 5-10 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} z_{12}(1) = \begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} z_1\left(-\tfrac{1}{5}\right) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow 1x - 1y = 0$$

$$u_{10} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\hat{=}\, \sigma_2^2$

$$u_0: \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} z_{12}(-1) = \begin{bmatrix} 5 & 5 \\ 0 & 0 \end{bmatrix} z_1\left(\tfrac{1}{5}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow 1x + 1y = 0 \Rightarrow u_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\rightarrow V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \text{es gilt } A^TA = V \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^T$$

1.Spalte von U:

$$u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

2. Spalte von U:

$$u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

wird so gewählt, dass die Spalten senkrecht aufeinander stehen!

$$A = \underbrace{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{V^T} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

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$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad AA^T = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$

Eigenwerte AA^T: $\det(AA^T - \lambda I) = \begin{bmatrix} 5-\lambda & 15 \\ 15 & 45-\lambda \end{bmatrix} \overset{!}{=} (5-\lambda)(45-\lambda) - 225$

$\overset{!}{=} 225 - 5\lambda - 45\lambda - 225 + \lambda^2$

$\overset{!}{=} \lambda^2 - 50\lambda$

$\overset{!}{=} \lambda(\lambda - 50) \Rightarrow \lambda_1 = 50, \lambda_2 = 0$

$u_{50}: \begin{bmatrix} -45 & 15 \\ 15 & -5 \end{bmatrix} Z_{12}\left(\frac{1}{3}\right) = \begin{bmatrix} -45 & 15 \\ 0 & 0 \end{bmatrix} Z_1\left(\tfrac{1}{15}\right) = \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow -3x + 1y = 0$

$x := s \Rightarrow -3s + 1y = 0 \Rightarrow 1y = 3s$

$v_1 = u_{50} \overset{!}{=} \begin{bmatrix} s \\ 3s \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$u_0: \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix} Z_{12}(-3) = \begin{bmatrix} 5 & 15 \\ 0 & 0 \end{bmatrix} Z_1\left(\frac{1}{5}\right) = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \Rightarrow 1x + 3y = 0$

$y := s \Rightarrow 1x + 3s \Rightarrow x = -3s$

$v_2 = u_0 \overset{!}{=} \begin{bmatrix} -3s \\ s \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$V = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$

U berechnen:

$u_1: = \frac{1}{\sqrt{\lambda_1}} A v_1 = \frac{1}{\sqrt{50}} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{500}} \begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} \frac{7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} \end{bmatrix}$

$u_1 = v_1 \Rightarrow \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} = \begin{bmatrix} \frac{7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} \end{bmatrix}$

$u_2: =$ Bestimmen einen Vektor der orthogonal zu u_1 ist

$\overset{!}{=} \begin{bmatrix} \frac{-7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} \end{bmatrix}$

$A = \begin{bmatrix} \frac{7}{\sqrt{500}} & \frac{-7}{\sqrt{500}} \\ \frac{21}{\sqrt{500}} & \frac{21}{\sqrt{500}} \end{bmatrix} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$

$U \qquad \Sigma \qquad V^T$

4 Unterräume von A

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \Sigma \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$$

U_1 ist eine Basis für $C(A)$

U_2 ist eine Basis für $N(A^T)$

V_1 ist eine Basis für $C(A^T)$

V_2 ist eine Basis für $N(A)$

→ separierbare Filter (IPCV)

As example let's look at the Gaussian filter, whose kernel is

$$\mathbf{M_F} = \begin{pmatrix} 0.011344 & 0.083820 & 0.011344 \\ 0.083820 & 0.619347 & 0.083820 \\ 0.011344 & 0.083820 & 0.011344 \end{pmatrix}$$

This is a rank one matrix and it can therefore be written as the product of a column vector and a row vector. In order to find these vectors we use the SVD.

Use the following Octave/Matlab commands to verify this:

```
octave:5> F = fspecial('gaussian')
F = 0.011344   0.083820   0.011344
    0.083820   0.619347   0.083820
    0.011344   0.083820   0.011344
octave:7> [U,S,V] = svd(gaussian)
U = -0.132923   0.958101  -0.253720
    -0.982173  -0.161664  -0.095924
    -0.132923   0.236446   0.962510
S = 6.4203e-01           0           0
             0   2.1032e-17           0
             0           0   2.6991e-34
V = -1.3292e-01  -6.9450e-01  -7.0711e-01
    -9.8217e-01   1.8798e-01   1.3878e-17
    -1.3292e-01  -6.9450e-01   7.0711e-01
octave:8> G = U(:,1) * sqrt(S(1,1))
G = -0.10651
    -0.78699
    -0.10651
octave:9> H = V(:,1)' * sqrt(S(1,1))
H = -0.10651  -0.78699  -0.10651
octave:10> G * H
ans = 0.011344   0.083820   0.011344
      0.083820   0.619347   0.083820
      0.011344   0.083820   0.011344
octave:11> F - G * H
ans = -2.4286e-17  -1.5266e-16  -2.4286e-17
      -1.3878e-17  -1.1102e-16  -1.3878e-17
      -3.4694e-18   0.0000e+00  -3.4694e-18
```

14 Octave → Octave

The following script allows to try a different number of singular values to reconstruct an image.

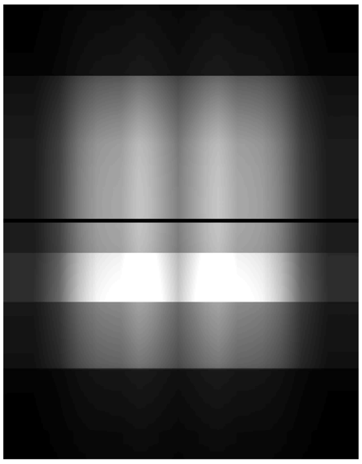
```
iclose all
clear all
clc

%reading and converting the image
inImage=imread('../figs/TheLetter_A.png');
inImage = 0.2989 * inImage(:,:,1) + 0.5870 * inImage(:,:,2) + 0.1140 *inImage(:,:,3);
inImageD=double(inImage);
[size_M,size_N] = size(inImage);
size_Orig = size_M*size_N;

% decomposing the image using singular value decomposition
[U,S,V]=svd(inImageD);
whos

% Using different number of singular values (diagonal of S) to compress and
% reconstruct the image
dispEr = [];
```

```
numSVals = [];
for N=1:5:41
    % store the singular values in a temporary var
    %jfb C = S;

    % discard the diagonal values not required for compression
    %jfb C(N+1:end,:)=0;
    %jfb C(:,N+1:end)=0;

    % Construct an Image using the selected singular values
    % D=U*C*V';
    D=U(:,1:N)*S(1:N,1:N)*V(:,1:N)';


    % display and compute error
    figure;
    buffer = sprintf('Image␣output␣using␣%d␣singular␣values␣(compression␣%f)',
                     N, (size_M*N+N^2+size_N*N)/size_Orig)
    imshow(uint8(D));
    title(buffer);
    error=sum(sum((inImageD-D).^2));

    % store vals for display
    dispEr = [dispEr; error];
    numSVals = [numSVals; N];
end

% dislay the error graph
figure;
title('Error␣in␣compression');
plot(numSVals, dispEr);
grid on
xlabel('Number␣of␣Singular␣Values␣used');
ylabel('Error␣between␣compress␣and␣original␣image');
```

Have fun!

Aufgabe 14

Image output using 1 singular values (compression 0.004756)



use SVD_demo.m Script