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Multivariable Calculus - Exercise 1

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I.BA_IMATH, Semesterweek 05

Please write down to solution of the exercises in a consise but comprehensible way. Numerical results should be accurate to 4 digits. Sketches should be correct qualitatively. At least 75% of the exercises have to be solve satisfactorily. Due time is one week after we have discussed the corresponding topics in class.

1 Contourlines I

Draw the contour lines of the function $f(x,y) = 2x + 3y + 1$ in the rectangular domain $[0,6] \times [-3,3]$. You should be able to solve this problem without the help of the computer (or pocket calculator). Describe the contour lines.

2 Contourlines II

Draw the contour lines of the function $f(x,y) = \sqrt{x^2 + y^2}$ in the square domain $[-3,3]^2$. Describe the contour lines.

3 Contourlines III

Sketch the contourlines of the function $f : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto z = f(x,y) = x^2 - y^2$. What kind of curves do You see? Highlight the countourline with niveau $z = 1$. Describe the contour lines.

4 Partial derivatives I

Find V_r if $V = \frac{1}{3}\pi r^2 h$.

5 Partial derivatives II

Find all partial derivatives of $f(x,y,z) = \frac{x^2 y^3}{z}$.

Aufgabe 1

$f(x,y) = 2x+3y+1 \qquad \overset{x}{[0,6]} \times \overset{y}{[-3,3]}$

$f(x,y) = c$

für $c=0$

$2x+3y+1=0$

$3y \overset{!}{=} -2x-1$

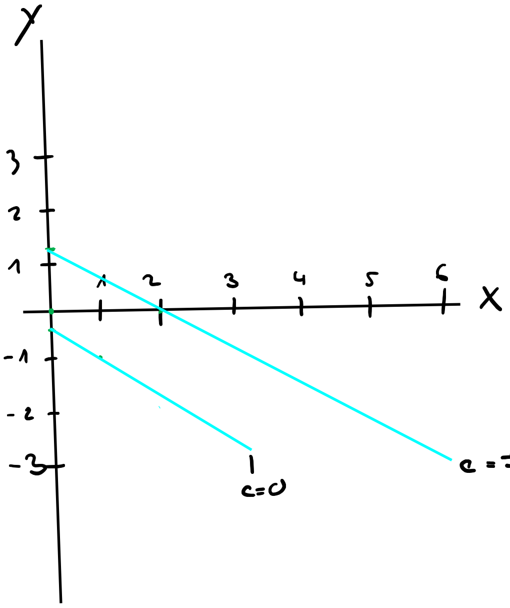
$y \overset{!}{=} \frac{-2x}{3} - \frac{1}{3}$

für $c=5$

$2x+3y+1=5$

$3y = -2x+4$

$y \overset{!}{=} \frac{-2x}{3} + \frac{4}{3}$



usw.

Aufgabe 2

$f(x,y) = \sqrt{x^2+y^2} \qquad [-3,3]^2$

$f(x,y) = c$

für $c=1$

$\sqrt{x^2+y^2} \overset{!}{=} c \quad | \cdot 1^2$

$x^2+y^2 = c^2$

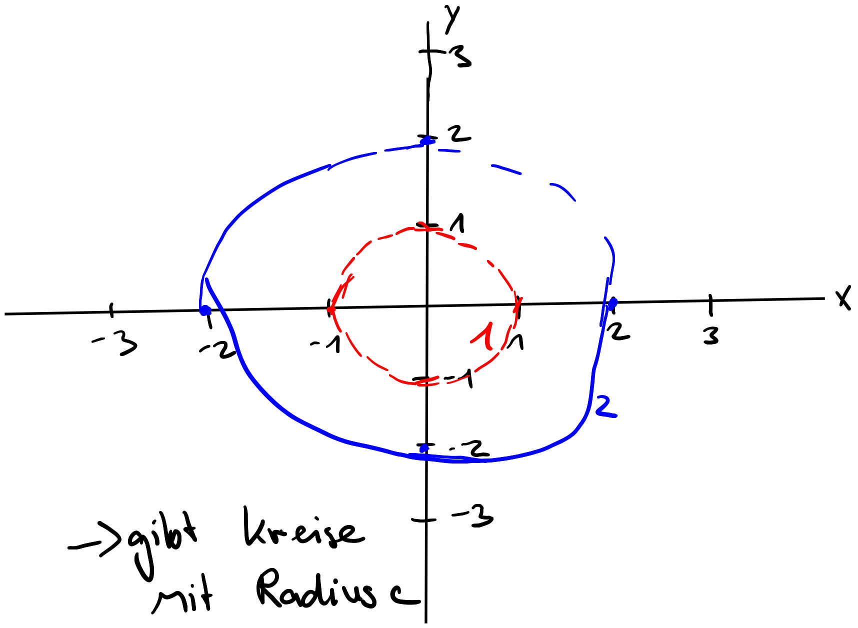
$x^2+y^2 \overset{!}{=} 1^2$

• $x^2 \overset{!}{=} 1^2 - y^2$

$c=2$

$x^2+y^2=4$

• $x^2 \overset{!}{=} 4-y^2$



→ gibt Kreise mit Radius c

Aufgabe 3

$f: \mathbb{R}^2 \to \mathbb{R}, \ (x,y) \longmapsto z = f(x,y) = x^2 - y^2$

$c=0$

$f(x,y)=0$

$x^2-y^2=0$

$(x-y)(x+y)=0$

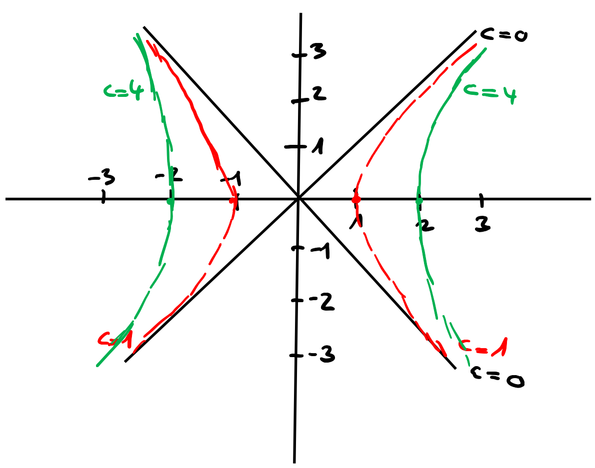
d.h. $x=y$

$c=1$

• $x^2-y^2=1$

$c=4$

• $x^2-y^2=4$



Aufgabe 4

$$V = \frac{1}{3}\pi r^2 h \quad \rightarrow \quad \frac{\partial V}{\partial r} = \frac{2}{3}\pi r h$$

Aufgabe 5

$$f(x,y,z) = \frac{x^2 y^3}{z} \quad \rightarrow \quad \frac{\partial f}{\partial x} = \frac{2xy^3}{z}$$

$$\frac{\partial f}{\partial y} = \frac{x^2 3y^2}{z}$$

$$\frac{\partial f}{\partial z} = x^2y^3z^{-1} = -1x^2y^3z^{-2} = -\frac{x^2y^3}{z^2}$$

Aufgabe 6

a)

$$z = 3x^2 + 4y^2 - axy$$

$$\frac{\partial z}{\partial x} = 6x - ay$$ → Ableitung bei P(1,2) muss positiv sein.

$$6 \cdot 1 - a \cdot 2 = 6 - 2a = 0$$

$$6 \overset{!}{=} 2a$$

$$3 \overset{!}{=} a$$

a muss also grösser als 3 sein

b) $$\frac{\partial z}{\partial y} = 8y - ax \quad \rightarrow \quad \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 8 \cdot 2 - a \cdot 1$$

$$\overset{!}{=} 16 - a > 0$$

wenn a < 13 ist, dann slopes the surface up

Aufgabe 7

(a) $$f(x,y) = \frac{3}{2}x^5 - \frac{4}{7}y^6 \quad \rightarrow \quad \frac{\partial f}{\partial x} = \frac{15}{2}x^4, \quad \frac{\partial f}{\partial y} = -\frac{24}{7}y^5 \qquad \nabla f(x,y) = \begin{bmatrix} \frac{15}{2}x^4 \\ -\frac{24}{7}y^5 \end{bmatrix}$$

(b) $$z = xe^y \quad \rightarrow \quad \frac{\partial z}{\partial x} = e^y, \quad \frac{\partial z}{\partial y} = xe^y \qquad \nabla z(x,y) = \begin{bmatrix} e^y \\ xe^y \end{bmatrix}$$

(c) $$z = \sin\left(\frac{x}{y}\right) \quad \rightarrow \quad \frac{\partial z}{\partial x} = \frac{1}{y}\cos\left(\frac{x}{y}\right) = \frac{\cos\left(\frac{x}{y}\right)}{y}$$

$$\frac{\partial z}{\partial y} = \text{innere Ableitung: } \left(\frac{x}{y}\right) = xy^{-1} = -xy^{-2} = \frac{-x}{y^2} \Rightarrow \frac{-x \cdot \cos\left(\frac{x}{y}\right)}{y^2}$$

äussere Ableitung: sin( ) = cos( )

$$\nabla z(x,y) = \begin{bmatrix} \frac{\cos\left(\frac{x}{y}\right)}{y} \\ \frac{-x \cdot \cos\left(\frac{x}{y}\right)}{y^2} \end{bmatrix}$$

(d) $$f(a,b) = \frac{2a+3b}{2a-3b} \quad \rightarrow \quad \frac{\partial f}{\partial a} = \frac{2(2a-3b) - 2(2a+3b)}{(2a-3b)^2}$$

$$\overset{!}{=} \frac{4a-6b-(4a+6b)}{(2a-3b)^2} = \frac{-12b}{(2a-3b)^2}$$

$$\frac{\partial f}{\partial b} \overset{!}{=} \frac{3(2a-3b) - (-3)(2a+3b)}{(2a-3b)^2}$$

$$= \frac{6a - 9b + 6a + 9b}{(2a-3b)^2} = \frac{12a}{(2a-3b)^2}$$

$$\nabla f(a,b) = \begin{bmatrix} \frac{-12b}{(2a-3b)^2} \\ \frac{12a}{(2a-3b)^2} \end{bmatrix}$$

6 Partial derivatives III

(a) The surface S is given, for some constant a, by

$$z = 3x^2 + 4y^2 - axy.$$

Find the values of a which ensure that S is sloping upward when we move in the positive x-direction from the point $(1,2)$.

(b) With the values of a from part (a), if you move in the positive y-direction from the point $(1,2)$, does the surface slope up or down? Explain.

7 Gradient I

Find the gradient of the following functions:

$$\text{(a)}\quad f(x,y) = \frac{3}{2}x^5 - \frac{4}{7}y^6, \quad \text{(b)}\quad z = xe^y, \quad \text{(c)}\quad z = \sin(x/y), \quad \text{(c)}\quad f(a,b) = \frac{2a+3b}{2a-3b}.$$

8 Gradient II

Find the gradient of the following functions at the point:

(a) $f(x,y) = x^2y + 7xy^3$, at $(1,2)$ (b) $f(r,h) = 2\pi rh + \pi r^2$, at $(2,3)$ (c) $f(m,n) = 5m^2 + 3n^4$, at $(5,2)$.

9 Directional derivative I

Calculate the directional derivative of the function $f(x,y) = x^2 + y^2$ at $\mathbf{x}_0 = \begin{bmatrix}1 & 0\end{bmatrix}^T$ in the direction of $\mathbf{e} = \begin{bmatrix}1 & 1\end{bmatrix}^T$. Draw the contourlines around that point and the gradient of f at that point. Check with the formula which uses the gradient of f to compute the directional derivative.

10 Directional derivative II

Calculate the directional derivative of the real valued function in 3D, $f(x,y,z) = z\sin x + \ln(x^2 - y^2)$ at $\mathbf{x}_0 = \begin{bmatrix}1 & 0 & 1\end{bmatrix}^T$ in the direction of $\mathbf{e} = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T$. Check with the formula which uses the gradient of f to compute the directional derivative.

11 Plotting with Octave → Octave

The following command plots $y = \sin x$ in the interval $[0, 2\pi]$.

```
x = linspace(0, 2*pi, 100);
y = sin(x);
plot(x, y);
```

Aufgabe 8

(a) $f(x,y) = x^2y + 7xy^3$, at $(1,2)$

$$\frac{\partial f}{\partial x} = 2xy + 7y^3 \quad , \quad \frac{\partial f}{\partial y} = x^2 + 21xy^2$$

$$\nabla f(x,y) = \begin{bmatrix} 2xy + 7y^3 \\ x^2 + 21xy^2 \end{bmatrix}$$

$$\nabla f(1,2) = \begin{bmatrix} 4 + 56 \\ 1 + 84 \end{bmatrix} = \begin{bmatrix} 60 \\ 85 \end{bmatrix}$$

(b) $f(r,h) = 2\pi rh + \pi r^2$, at $(2,3)$

$$\frac{\partial f}{\partial r} = 2\pi h + 2\pi r \quad , \quad \frac{\partial f}{\partial h} = 2\pi r$$

$$\nabla f(r,h) = \begin{bmatrix} 2\pi h + 2\pi r \\ 2\pi r \end{bmatrix} \quad \nabla f(2,3) = \begin{bmatrix} 6\pi + 4\pi \\ 4\pi \end{bmatrix} = \begin{bmatrix} 10\pi \\ 4\pi \end{bmatrix}$$

(c) $f(m,n) = 5m^2 + 3n^4$, at $(5,2)$

$$\frac{\partial f}{\partial m} = 10m \quad , \quad \frac{\partial f}{\partial n} = 12n^3 \qquad \nabla f(m,n) = \begin{bmatrix} 10m \\ 12n^3 \end{bmatrix} \qquad \nabla f(5,2) = \begin{bmatrix} 50 \\ 96 \end{bmatrix}$$

Aufgabe 9

$f(x,y) = x^2 + y^2$, $\quad x_0 = [1 \ 0]^T$, $\quad e = [1 \ 1]^T \;\text{—}\; \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

es gilt: $D_e f(x_0) = \nabla f(x_0) \cdot e = |\nabla f(x_0)| \cos\phi$

$$\frac{\partial f}{\partial x} = 2x \quad , \quad \frac{\partial f}{\partial y} = 2y \qquad \text{für } x_0 = [1 \ 0]^T \quad \left.\frac{\partial f}{\partial x}\right|_{(1,0)} = 2 \quad , \quad \left.\frac{\partial f}{\partial y}\right|_{(1,0)} = 0$$

$$D_e f(x_0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \sqrt{2}$$

Die Niveaulinien sind Kreise, da $x^2 + y^2 = z$. Kreise haben geraden den Radius $\sqrt{z}$

Aufgabe 10

$f(x,y,z) = z \sin x + \ln(x^2 - y^2)$ $\qquad x_0 = [1\ 0\ 1]^T \qquad e = [1\ 1\ 0]^T / \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$\frac{\partial f}{\partial x} = z \cos x + 2x \cdot \frac{1}{x^2 - y^2} = z \cos x + \frac{2x}{x^2 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{x^2 - y^2}$$

$$\frac{\partial f}{\partial z} = \sin x$$

für $x_0 = [1\ \ 0\ \ 1]^T$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,0,1)} = \cos(1) + 2$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,0,1)} = 0$$

$$\left.\frac{\partial f}{\partial z}\right|_{(1,0,1)} = \sin(1)$$

$$D_e f(x_0) = \begin{bmatrix} \cos(1) + 2 \\ 0 \\ \sin(1) \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{\cos(1) + 2}{\sqrt{2}}$$

Countourplots can be drawn as follows:

```
x = linspace(-2*pi,2*pi);
y = linspace(0,4*pi);
[X,Y] = meshgrid(x,y);
Z = X.^2-Y.^2;

figure
contour(X,Y,Z)
```

Plot Graph of a function from $\mathbb{R}^2$ to $\mathbb{R}$:

```
[X,Y] = meshgrid(-2:.2:2, -2:.2:2);
Z = X .* exp(-X.^2 - Y.^2);
mesh(X,Y,Z)
surf(X,Y,Z)
meshc(X,Y,Z)
axis off
```