

# Multivariable Calculus - Exercise 3

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I.BA\_IMATH, Semesterweek 07

The solution of the exercises should be presented in a clear and concise manner. Numerical results should be accurate to 4 digits. The exercises are accepted if You solve 75% of the exercises adequately. Please hand in the exercises no later than at the end of the last lecture in semesterweek 8.

## 1 Critical points I

Calculate the critical points of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto f(x, y) = -\sqrt{x^2 + y^2}$ .

1. Draw the graph of the function and the contourlines to the levels  $-1, -2, -3, -4$ .
2. Describe the contourline to the level  $-c$ .
3. Describe the graph  $\mathcal{G}(f)$ .
4. Identify local (and global) maxima and minima (if there are any).

## 2 Critical points II

Calculate the critical points of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x, y) \mapsto f(x, y) = x^2 - y^2$ .

## 3 Critical points III

We can apply the concept of critical points to any real valued multivariable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . If  $f$  represents the room temperature, we can find candidates for minimal or maximal temperature using critical points.

Use the **distance function**  $d(t, t') = \|\mathbf{x}(t) - \mathbf{y}(t')\|$  to find the shortest distance between the two straight lines  $g_1 : \mathbf{x}(t) = (3t, 2t, t)$  and  $g_2 : \mathbf{y}(t') = (2t', 2t' + 3, 2t')$ . Use the fact, that if  $d(t, t')$  is extremal, than  $(d(t, t'))^2$  is also extremal. Therefore You don't need to take a square root!

# Aufgabe 1

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, (x,y) \mapsto f(x,y) = -\sqrt{x^2 + y^2}$$

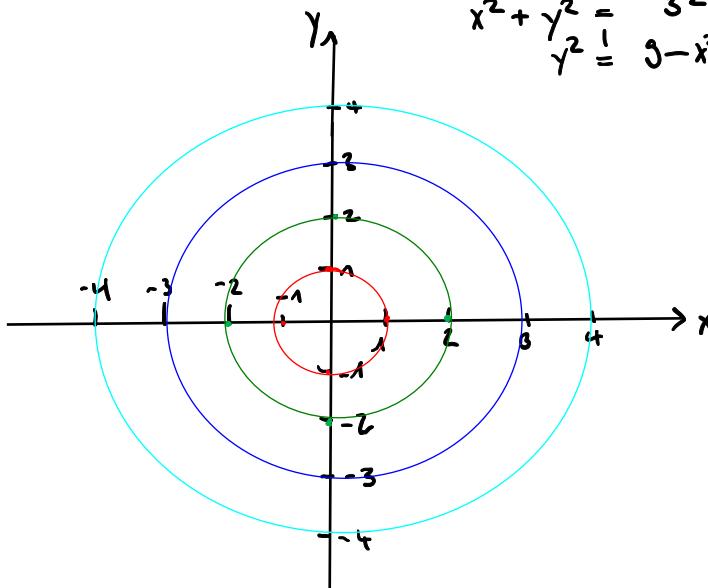
1. Contour Lines

$$x^2 + y^2 = c^2 \Rightarrow \frac{c = -1}{x^2 + y^2} = 1^2 = 1 \\ y^2 = 1 - x^2$$

$$\frac{c = -2}{x^2 + y^2} = \frac{2^2}{y^2} = \frac{4}{4 - x^2}$$

$$\frac{c = -3}{x^2 + y^2} = \frac{3^2}{y^2} = \frac{9}{9 - x^2}$$

$$\frac{c = -4}{x^2 + y^2} = \frac{4^2}{y^2} = \frac{16}{16 - x^2}$$



2.  $-c$  beschreibt die Gleichung  $x^2 + y^2 = c^2$

3. hat Zylinderform

$$-\frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} 2x$$



4.

$$\nabla f(x,y) = \begin{bmatrix} -x \\ \sqrt{x^2 + y^2} \\ -y \\ \sqrt{x^2 + y^2} \end{bmatrix} = -\frac{1}{\sqrt{x^2 + y^2}} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\hookrightarrow$  Stelle  $P(0,0)$  wo Funktion nicht definiert ist, ist sie minimale. Siehe Contour lines

## Aufgabe 2

$$f(x,y) = x^2 - y^2$$

$$\frac{\partial f}{\partial x} = 2x \Rightarrow 2x = 0 \quad x=0$$

$$\frac{\partial f}{\partial y} = -2y \Rightarrow -2y = 0 \quad y=0$$

Kritischer Punkt ist

$$(x,y) = (0,0)$$

ist ein Sattelpunkt

## Aufgabe 3

$$x(t) = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} \quad y(t) = \begin{bmatrix} 2+t \\ 2t+3 \\ 2t \end{bmatrix}$$

$$(d(t,t'))^2 = \| x(t) - y(t') \|^2$$

$$= \begin{bmatrix} 3t & - 2t' \\ 2t & - 2t'+3 \\ t & - 2t' \end{bmatrix}^2 = \begin{bmatrix} 3t & - 2t' \\ 2t & - 2t'+3 \\ t & - 2t' \end{bmatrix} \begin{bmatrix} 3t & - 2t' \\ 2t & - 2t'+3 \\ t & - 2t' \end{bmatrix}$$

$$= (3t - 2t')(3t - 2t') + (2t - 2t'+3)(2t - 2t'+3) + (t - 2t')(t - 2t')$$

$$= \underline{9t^2} - \underline{6t t'} - \underline{6t t'} + \underline{4t'^2} + \underline{4t'^2} - \underline{4t t'} + \underline{6t} - \underline{4-t'} + \underline{4t^2} - \underline{6t'} + \underline{6t} - \underline{6t'} + \underline{3} + \underline{t^2} - \underline{2t'} - \underline{2t'} + \underline{4t'^2}$$

$$= 9t^2 + 14t'^2 + 12t'^2 - 24t t' - 12t + 12t'$$

$$\frac{\partial d}{\partial t} = 28t - 24t' - 12$$

$$\frac{\partial d}{\partial t'} = 24t' - 24t + 12$$

$$\nabla d = \begin{bmatrix} 28t - 24t' - 12 \\ 24t' - 24t + 12 \end{bmatrix} = \begin{bmatrix} 7t - 6t' - 3 \\ 6t' - 6t + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -7 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} t' \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Die einzige Lösung ist  $(t, t') = (0, -1/2)$

## 4 Higher order partial derivatives I

1. Compute the four second order derivatives of the functions  $f(x,y) = 3x^2y - 2xy + 5y^2$  and  $g(x,y) = x^2 \cos y + y^2 \sin x$  and show, that the mixed partial derivatives are the same.
2. For  $h(x,y) = y \sin x - x \sin y$  verify, that  $h_{xyy}$ ,  $h_{yxy}$  and  $h_{yyx}$  are identical.

## 5 Classifying critical points

Compute and classify the critical points of  $f(x,y) = 2x^2 - 3xy + 8y^2 + x - y$ . Plot the contour diagram of  $f$  and show that the local extremum found is a global one.

## 6 Taylor Expansion

Find the linear,  $L(x,y)$ , and the quadratic,  $Q(x,y)$ , Taylor polynomials valid near  $(1,0)$  of  $f(x,y) = xe^{-y}$ . Compare the values of the approximations at  $(0.9,0.2)$  with the exact value  $f(0.9,0.2)$ . Compute the respective absolute and relative errors.

## 7 Optimization I

Find the point on the plane  $3x + 2y + z = 1$  that is closest to the origin by minimizing the square of the distance. Use the squared distance to point  $(x,y,z)$  of the form  $d^2(x,y,z) = x^2 + y^2 + z^2$ .

## 8 Optimization II → Prüfung !

A closed rectangular box with faces parallel to the coordinate plane has one bottom corner at the origin and the opposite top corner in the first octant on the plane  $3x + 2y + z = 1$ . What is the maximum volume of such a box?

## 9 Linear Regression

Using linear regression find the equation of the line, that best approximates the points  $(0,2)$ ,  $(1,2)$ ,  $(2,3)$  and  $(3,3)$ . Draw the line and the four points.

## 10 Constrained Optimization I

Use the Lagrangian function to find the maximum and minimum values of  $f(x,y) = x^3 + y$  subject to the constraint  $g(x,y) = 3x^2 + y^2 = 4$ , if such values exist.



## Aufgabe 4

1)

$$f(x,y) = 3x^2y - 2xy + 5y^2$$

$$\begin{aligned} f_x(x,y) &= 6xy - 2y & f_y(x,y) &= 3x^2 - 2x + 10y \\ f_{xx}(x,y) &= 6y & f_{yy}(x,y) &= 10 \\ f_{xy}(x,y) &= 6x - 2 \quad \underline{=} \quad f_{yx}(x,y) = 6x - 2 \quad \Rightarrow \quad f_{xx} = f_{xy} \end{aligned}$$

$$g(x,y) = x^2 \cdot \cos y + y^2 \sin x$$

$$\begin{aligned} g_x(x,y) &= 2x \cdot \cos y + y^2 \cdot \cos x & g_y(x,y) &= -x^2 \cdot \sin y + 2y \cdot \sin x \\ g_{xy}(x,y) &= -2x \cdot \sin y + 2y \cdot \cos x \quad \underline{=} \quad g_{yx}(x,y) = -2x \cdot \sin y + 2y \cdot \cos x \quad \Rightarrow \quad g_{xy} = g_{yx} \end{aligned}$$

$$2) \quad h(x,y) = y \cdot \sin x - x \cdot \sin y$$

$$\begin{aligned} h_x(x,y) &= -y \cdot \cos x - \sin y & h_y(x,y) &= \sin x - x \cdot \cos y & h_{yy}(x,y) &= \sin x - x \cdot \cos y \\ h_{xy}(x,y) &= -\cos x - \cos y & h_{yy}(x,y) &= x \cdot \sin y & h_{yx}(x,y) &= \cos x - \cos y \\ h_{yyy}(x,y) &= \sin y \quad \underline{=} \quad h_{yyx}(x,y) = \sin y & h_{yyx}(x,y) &= \sin y & h_{yyx}(x,y) &= \sin y \end{aligned}$$

$$h_{yyy} = h_{yyx} = h_{yyx}$$

## Aufgabe 5

$$f(x,y) = 2x^2 - 3xy + 8y^2 + x - y$$

$$f_x(x,y) = 4x - 3y + 1$$

$$f_{xx}(x,y) = 4$$

$$f_y(x,y) = -3x + 16y - 1$$

$$f_{yy}(x,y) = 16$$

$$f_{yx}(x,y) = -3$$

$$\text{Es gilt: } D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$$

$$\begin{aligned} D &= 4 \cdot 16 - (-3)^2 \\ &= 64 - 9 = \underline{\underline{55}} \end{aligned}$$

Da  $D > 0$  und  $f_{xx}(x_0, y_0) > 0$  ist  $f$  minimal  
in  $(-\frac{13}{55}, \frac{1}{55})$

Kritische Punkte finden:

$$\begin{aligned} 4x - 3y + 1 &= 0 & -3x + 16y - 1 &= 0 & \mid x = \frac{3y-1}{4} \\ 4x &\underline{=} 3y-1 & -3\left(\frac{3y-1}{4}\right) + 16y - 1 &= 0 \\ x &= \frac{3y-1}{4} & \frac{-9y+3}{4} + 16y - 1 &= 0 \\ x &= \frac{3(4y-1)}{4} & \frac{55y}{4} - \frac{1}{4} &= 0 \\ x &= \frac{-52}{4} & \frac{55y}{4} &= \frac{1}{4} \\ x &= \underline{\underline{-\frac{52}{4}}} & y &= \frac{1}{55} \\ x &= \underline{\underline{-\frac{52}{220}}} = \underline{\underline{-\frac{13}{55}}} & \end{aligned}$$

## Aufgabe 6

$$f(x,y) = x \cdot e^{-y} \quad \text{Gesucht } L(x,y) \text{ und } Q(x,y) \text{ bei } (1,0)$$

$$f(1,0) = 1 \cdot e^0 = 1$$

$$f_x(x,y) = e^{-y} \Rightarrow f_x(1,0) = 1 \quad f_{xx}(x,y) = 0$$

$$f_y(x,y) = -xe^{-y} \Rightarrow f_y(1,0) = -1 \quad f_{yy}(x,y) = xe^{-y} \Rightarrow f_{yy}(1,0) = 1$$

$$f_{xy}(x,y) = -e^{-y} \Rightarrow f_{xy}(1,0) = -1$$

$$\begin{aligned} L(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &= 1 + 1(x-1) - y \\ &\underline{=} 1 + x - 1 - y = \underline{\underline{x-y}} \end{aligned}$$

$$L(0.9, 0.2) = 0.9 - 0.2 = \underline{\underline{0.7}}$$

$$Q(x,y) = L(x,y) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

$$= x-y + 0 - 1(x-1)y + \frac{1}{2}y^2$$

$$= x - y - xy + y + \frac{1}{2}y^2$$

$$\underline{=} x - xy + \frac{1}{2}y^2$$

$$Q(0.9, 0.2) = 0.9 - 0.18 + 0.02 = \underline{\underline{0.74}}$$

$$f(0.9, 0.2) = 0.9 \cdot e^{-0.2} = \underline{\underline{0.7369}}$$

Absolute Fehler

$$|L(0.9, 0.2) - 0.7369| = 0.0369$$

$$|Q(0.9, 0.2) - 0.74| = 0.0031$$

Relatives Fehler

$$L \approx 5\%$$

$$Q \approx 0.42\%$$

## Aufgabe 7

$$3x + 2y + z = 1 \quad \begin{array}{l} \text{nach } z \text{ umstellen} \\ \rightarrow z = 1 - 3x - 2y \end{array} \quad \begin{array}{l} 1 - 3x - 2y - 3x + 9x^2 + 6xy - 2y + 6xy + 4y^2 \\ 1 - 6x - 4y + 9x^2 + 4y^2 + 12xy \end{array}$$

$$\begin{aligned} d^2(x, y, z) &= x^2 + y^2 + z^2 = x^2 + y^2 + (1 - 3x - 2y)^2 \\ &= x^2 + y^2 + 1 - 6x - 4y + 9x^2 + 4y^2 + 12xy \\ &= 10x^2 + 5y^2 + 1 - 6x - 4y + 12xy \\ &= 10x^2 + 5y^2 + 12xy - 6x - 4y + 1 \end{aligned}$$

$\nabla d^2 = 0$  also haben wir:

$$\frac{\partial d^2}{\partial x} = 20x + 12y - 6 = 0$$

nach x umstellen

$$x = \frac{-12y + 6}{20} = \frac{-6y + 3}{10}$$

$$x = -12 \left( \frac{y}{7} \right) + 6$$

$$= -\frac{12}{7} + \frac{42}{7}$$

$$= \frac{\frac{30}{7}}{\frac{20}{7}} = \frac{30}{140} = \underline{\underline{\frac{3}{14}}} = x$$

$$\frac{\partial d^2}{\partial y} = 10y + 12x - 4 = 0$$

$$10y + 12 \left( \frac{-6y + 3}{10} \right) - 4 = 0$$

$$10y - \frac{72y}{10} + \frac{36}{10} - 4 = 0$$

$$\frac{28y}{10} = \frac{4}{10}$$

$$28y = 4$$

$$y = \frac{4}{28} = \underline{\underline{\frac{1}{7}}}$$

x und y in  $z = 1 - 3x - 2y$  einsetzen

$$z = 1 - 3x - 2y$$

$$= 1 - \frac{9}{14} - \frac{4}{14}$$

$$= \underline{\underline{\frac{1}{14}}} = z$$

## Aufgabe 8

plane:  $3x + 2y + z = 1$       nach  $z$  umstellen  
 $\rightarrow z = 1 - 3x - 2y$

$$V = x \cdot y \cdot z = xy(1 - 3x - 2y)$$
$$\stackrel{!}{=} xy - 3x^2y - 2xy^2 = V$$

Anstelle von  $D$  für die Distanz  
haben wir nun  $V$  für ein  
Volumen

Nun partielle Ableitungen gleich 0

$$\frac{\partial V}{\partial x} = y - 6xy - 2y^2 = 0$$
$$\stackrel{!}{=} y(1 - 6x - 2y) = 0$$
$$\frac{\partial V}{\partial y} = x - 3x^2 - 4xy = 0$$
$$\stackrel{!}{=} x(1 - 3x - 4y) = 0$$

$x, y = 0$  wären Lösungen, aber dann ist Volumen nicht maximal

$$\begin{array}{lcl} I \quad 1 - 6x - 2y = 0 & | \cdot (-2) & = II \quad 1 - 6x - 2y = 0 \\ II \quad 1 - 3x - 4y = 0 & & | -1 + 9x + 0 = 0 \Leftrightarrow x = \frac{1}{3} \end{array}$$

$$x \text{ in } I: 1 - \frac{6}{3} - 2y = 0$$
$$\stackrel{!}{=} \frac{3}{9} = 2y$$
$$\stackrel{!}{=} y = \frac{3}{18} = \frac{1}{6}$$

$x$  und  $y$  in  $z = 1 - 3x - 2y$ :

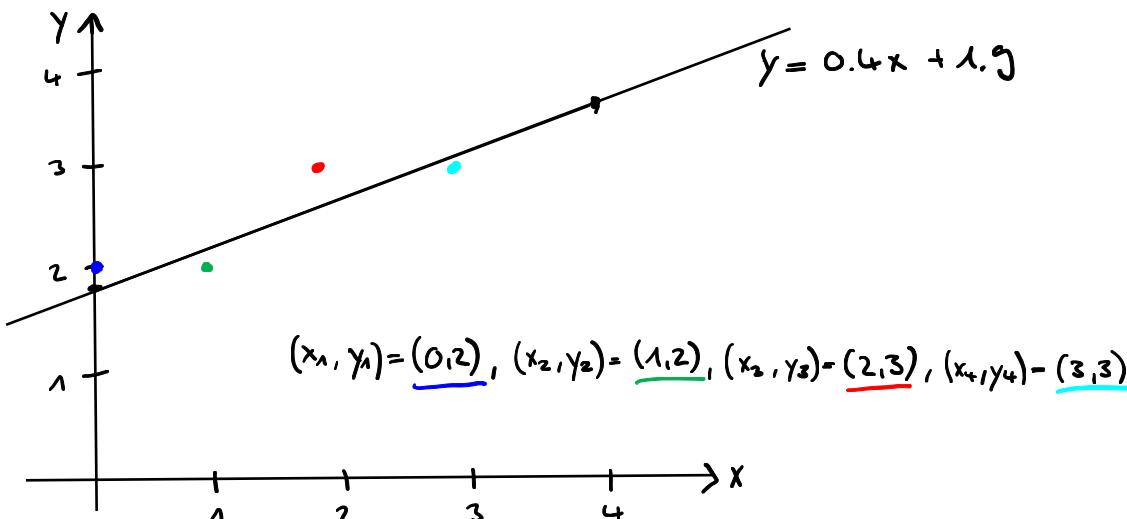
$$z = 1 - \frac{3}{9} - \frac{2}{6}$$
$$\stackrel{!}{=} z = \frac{1}{3}$$

Somit ist das maximale Volumen:

$$V_{\max} = \frac{1}{9} \cdot \frac{1}{6} \cdot \frac{1}{3} = \underline{\underline{\underline{\frac{1}{162}}}}$$

### Aufgabe 9

$$(x_1, y_1) = \underline{(0, 2)}, (x_2, y_2) = \underline{(1, 2)}, (x_3, y_3) = \underline{(2, 3)}, (x_4, y_4) = \underline{(3, 3)}$$



$$(x_1, y_1) = \underline{(0, 2)}, (x_2, y_2) = \underline{(1, 2)}, (x_3, y_3) = \underline{(2, 3)}, (x_4, y_4) = \underline{(3, 3)}$$

$$\begin{aligned} y &= b + mx \\ f(b, m) &= \sum_{i=1}^4 (y_i - \hat{y}_i)^2 = \sum_{i=1}^4 (y_i - (b + mx_i))^2 \\ &= (2 - (b + 0 \cdot m))^2 + (2 - (b + 1 \cdot m))^2 + (3 - (b + 2 \cdot m))^2 + (3 - (b + 3 \cdot m))^2 \\ &= (2 - b)^2 + (2 - b - m)^2 + (3 - b - 2m)^2 + (3 - b - 3m)^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial b} &= 2(2-b)(-1) + 2(2-b-m)(-1) + 2(3-b-2m)(-1) + 2(3-b-3m)(-1) \\ &= -4 + 2b - 4 + 2b + 2m - 6 + 2b + 4m - 6 + 2b + 6m \\ &= -20 + 8b + 12m = 0 \quad f_{bb}(b, m) = 8 \quad \text{und } f_{bm}(b, m) = 12 \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial m} &= 2(2-b-m)(-1) + 2(3-b-2m)(-2) + 2(3-b-3m)(-3) \\ &= -4 + 2b + 2m - 12 + 4b + 8m - 18 + 6b + 18m \\ &= -34 + 12b + 28m = 0 \quad f_{mm}(b, m) = 28 \end{aligned}$$

$$\begin{array}{lcl} I: -20 + 8b + 12m = 0 & | \cdot (-1.5) & = I: -20 + 8b + 12m = 0 \\ II: -34 + 12b + 28m = 0 & & = II: -4 + 0 + 10m = 0 \Leftrightarrow m = \underline{\underline{\frac{4}{10} = \frac{2}{5}}} \end{array}$$

$$\begin{aligned} m \text{ in } I: -20 + 8b + \frac{24}{5} &= 0 \\ 8b &= \frac{76}{5} \\ b &= \underline{\underline{\frac{76}{40} = 1.9}} \end{aligned}$$

$$\text{Gerade ist } \underline{\underline{y = 0.4x + 1.9}}$$

Überprüfen, ob minimal:

$$D = 8 \cdot 28 - (12)^2 = 80 \quad \text{Da } D > 0 \text{ und } f_{bb}(b, m) > 0 \text{ ist } y \text{ minimal}$$

## Aufgabe 10

$$f(x,y) = x^3 + y$$

$$\text{constraint } g(x,y) = 3x^2 + y^2 - 4 = 0$$

$$\text{es gilt: } \nabla f(x,y) - \lambda \nabla g(x,y) = 0$$

$$f_x(x,y) = 3x^2$$

$$f_y(x,y) = 1$$

$$g_x(x,y) = 6x$$

$$g_y(x,y) = 2y$$

$$L_x(x,y,\lambda) = f_x(x,y) - \lambda g_x(x,y) = 0 \Rightarrow 3x^2 - \lambda 6x = 0$$

$$L_y(x,y,\lambda) = f_y(x,y) - \lambda g_y(x,y) = 0 \Rightarrow 1 - \lambda 2y = 0$$

$$L_\lambda(x,y,\lambda) = -g(x,y) = 0 \Rightarrow -(3x^2 + y^2 - 4) = 0$$

Max.: 2 at  $(1,1)$  and  $(0,2)$

Min.: -2 at  $(-1,-1)$  and  $(0,-2)$

## 11 Constrained Optimization II

A company has a production function with three inputs  $x$ ,  $y$ , and  $z$  given by

$$f(x, y, z) = 50x^{2/5}y^{1/5}z^{1/5}$$

The total budget is 24'000 and the company can buy  $x$ ,  $y$ , and  $z$  at 80, 12, 10 per unit, respectively. What combination of inputs will maximise production?

$$\begin{aligned} g(x, y, z) &= 80x + 12y + 10z = 24'000 \\ \text{nach } z \text{ umstellen:} \\ 10z &= 24'000 - 80x - 12y \\ z &\stackrel{!}{=} \frac{24'000}{10} - \frac{80x}{10} - \frac{12y}{10} \\ z &= 2400 - 8x - \frac{6}{5}y \end{aligned}$$

### Aufgabe 11

$$\nabla L(x, y, z, \lambda) = 0$$

$$L_x = 20x^{-3/5} \cdot y^{1/5} \cdot z^{2/5} - \lambda 80 = 0$$

$$L_y = 10x^{2/5} \cdot y^{-4/5} \cdot z^{1/5} - \lambda 12 = 0$$

$$L_z = 10x^{2/5} \cdot y^{1/5} \cdot z^{-4/5} - \lambda 10 = 0$$

$$L_\lambda = 24'000 - 80x - 12y - 10z = 0$$

mit Octave lösen