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Multivariable Calculus - Exercise 3

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I.BA_IMATH, Semesterweek 07

The solution of the exercises should be presented in a clear and concise manner. Numerical results should be accurate to 4 digits. The exercises are accepted if You solve 75% of the exercises adequately. Please hand in the exercises no later than at the end of the last lecture in semesterweek 8.

1 Critical points I

Calculate the critical points of $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = -\sqrt{x^2 + y^2}$.

1. Draw the graph of the function and the contourlines to the levels -1, -2, -3, -4.
2. Describe the contourline to the level $-c$.
3. Describe the graph $\mathscr{G}(f)$.
4. Identify local (and global) maxima and minima (if there are any).

2 Critical points II

Calculate the critical points of $f : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto f(x, y) = x^2 - y^2$.

3 Critical points III

We can apply the concept of critical points to any real valued multivariable function $f : \mathbb{R}^n \to \mathbb{R}$. If f represents the room temperature, we can find candidates for minimal or maximal temperature using critical points.

Use the **distance function** $d(t, t') = \|\mathbf{x}(t) - \mathbf{y}(t')\|$ to find the shortest distance between the two straigth lines $g_1 : \mathbf{x}(t) = (3t, 2t, t)$ and $g_2 : \mathbf{y}(t') = (2t', 2t' + 3, 2t')$. Use the fact, that if $d(t, t')$ is extremal, than $(d(t, t'))^2$ is also extremal. Therefore You don't need to take a square root!

Aufgabe 1

$f: \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto f(x,y) = -\sqrt{x^2+y^2}$

1. Contour Lines

$x^2 + y^2 = c^2 \Rightarrow$

$c = -1$

$x^2 + y^2 = 1^2 = 1$

$y^2 = 1 - x^2$

$c = -2$

$x^2 + y^2 \overset{!}{=} 2^2$

$y^2 = 4 - x^2$

$c = -3$

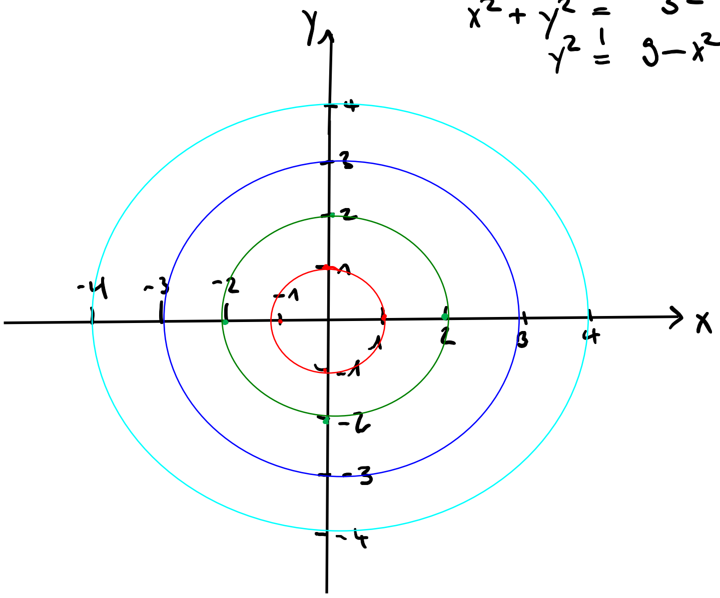
$x^2 + y^2 = 3^2 = 9$

$y^2 \overset{!}{=} 9 - x^2$

$c = -4$

$x^2 + y^2 = 4^2 = 16$

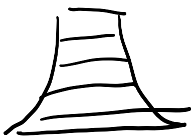
$y^2 \overset{!}{=} 16 - x^2$



2. $-c$ beschreibt die Gleichung $x^2 + y^2 = c^2$

3. hat Zylinderform

$-\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}} \; 2x$



4. $\nabla f(x,y) = \begin{bmatrix} \frac{-x}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} \end{bmatrix} = -\frac{1}{\sqrt{x^2+y^2}} \begin{bmatrix} x \\ y \end{bmatrix}$

↳ Stelle $P(0,0)$ wo Funktion nicht definiert ist, ist sie minimale.
Siehe Contour linien

Aufgabe 2

$f(x,y) = x^2 - y^2$

$\frac{\partial f}{\partial x} = 2x \Rightarrow 2x = 0 \qquad x = 0$

$\frac{\partial f}{\partial y} = -2y \Rightarrow -2y = 0 \qquad y = 0$

Kritischer Punkt ist

$\underline{\underline{(x,y) = (0,0)}}$

ist ein Sattelpunkt

Aufgabe 3

$x(t) = \begin{bmatrix} 3t \\ 2t \\ t \end{bmatrix} \qquad y(t') = \begin{bmatrix} 2t' \\ 2t'+3 \\ 2t' \end{bmatrix}$

$$(d(t,t'))^2 = \| x(t) - y(t') \|^2$$

$$= \begin{bmatrix} 3t & - & 2t' \\ 2t & - & 2t'+3 \\ t & - & 2t' \end{bmatrix}^2 = \begin{bmatrix} 3t & - & 2t' \\ 2t & - & 2t'+3 \\ t & - & 2t' \end{bmatrix} \begin{bmatrix} 3t & - & 2t' \\ 2t & - & 2t'+3 \\ t & - & 2t' \end{bmatrix}$$

$$= (3t - 2t')(3t - 2t') + (2t - 2t'+3)(2t - 2t'+3) + (t - 2t')(t - 2t')$$

$$= 9t^2 - 6tt' - 6tt' + 4t'^2 + 4t^2 - 4tt' + 6t - 4tt' + 4t'^2 - 6t' + 6t - 6t' + 9 + t^2 - 2tt' - 2tt' + 4t'^2$$

$$= 9 + 14t^2 + 12t'^2 - 24tt' + 12t - 12t'$$

$\frac{\partial d}{\partial t} = 28t - 24t' + 12$

$\frac{\partial d}{\partial t'} = 24t' - 24t - 12$

$$\nabla d = \begin{bmatrix} 28t - 24t' + 12 \\ -24t + 24t' - 12 \end{bmatrix} = \begin{bmatrix} 7t - 6t' + 3 \\ -6t + 6t' - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 \\ -6 & 6 \end{bmatrix} \begin{bmatrix} t \\ t' \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

4 Higher order partial derivatives I

1. Compute the four second order derivatives of the functions $f(x,y) = 3x^2y - 2xy + 5y^2$ and $g(x,y) = x^2 \cos y + y^2 \sin x$ and show, that the mixed partial derivatives are the same.
2. For $h(x,y) = y \sin x - x \sin y$ verify, that h_{xyy}, h_{yxy} and h_{yyx} are identical.

5 Classifying critical points

Compute and classify the critical points of $f(x,y) = 2x^2 - 3xy + 8y^2 + x - y$. Plot the contour diagram of f and show that the local extremum found is a global one.

6 Taylor Expansion

Find the linear, $L(x,y)$, and the quadratic, $Q(x,y)$, Taylor polynomials valid near $(1,0)$ of $f(x,y) = xe^{-y}$. Compare the values of the approximations at $(0.9, 0.2)$ with the exact value $f(0.9, 0.2)$. Compute the respective absolute and relative errors.

7 Optimization I

Find the point on the plane $3x + 2y + z = 1$ that is closest to the origin by minimizing the square of the distance. Use the squared distance to point (x,y,z) of the form $d^2(x,y,z) = x^2 + y^2 + z^2$.

8 Optimization II → Prüfung !

A closed rectangular box with faces parallel to the coordinate plane has one bottom corner at the origin and the opposite top corner in the first octant on the plane $3x + 2y + z = 1$. What is the maximum volume of such a box?

9 Linear Regression

Using linear regression find the equation of the line, that best approximates the points $(0,2)$, $(1,2)$, $(2,3)$ and $(3,3)$. Draw the line and the four points.

10 Constrained Optimization I

Use the Lagrangian function to find the maximum and minimum values of $f(x,y) = x^3 + y$ subject to the constraint $g(x,y) = 3x^2 + y^2 = 4$, if such values exist.

Aufgabe 4

1)

$f(x,y) = 3x^2y - 2xy + 5y^2$

$f_x(x,y) = 6xy - 2y$ $\qquad$ $f_y(x,y) = 3x^2 - 2x + 10y$

$f_{xx}(x,y) = 6y$ $\qquad$ $f_{yy}(x,y) = 10$

$f_{xy}(x,y) = 6x - 2 \quad = \quad f_{yx}(x,y) = 6x - 2 \quad \Rightarrow \quad f_{yx} = f_{xy}$

$g(x,y) = x^2 \cdot \cos y + y^2 \sin x$

$g_x(x,y) = 2x \cdot \cos y + y^2 \cdot \cos x$ $\qquad$ $g_y(x,y) = -x^2 \cdot \sin y + 2y \cdot \sin x$

$g_{xy}(x,y) = -2x \cdot \sin y + 2y \cdot \cos x \quad = \quad g_{yx}(x,y) = -2x \sin y + 2y \cdot \cos x \quad \Rightarrow \quad g_{xy} = g_{yx}$

2) $h(x,y) = y \cdot \sin x - x \sin y$

$h_x(x,y) = -y \cdot \cos x - \sin y$ $\qquad$ $h_y(x,y) = \sin x - x \cdot \cos y$ $\qquad$ $h_y(x,y) = \sin x - x \cdot \cos y$

$h_{xy}(x,y) = -\cos x - \cos y$ $\qquad$ $h_{yy}(x,y) = x \cdot \sin y$ $\qquad$ $h_{yx}(x,y) = \cos x - \cos y$

$h_{xyy}(x,y) = \sin y \quad = \quad h_{yyx}(x,y) = \sin y \quad = \quad h_{yxy}(x,y) = \sin y$

$h_{xyy} = h_{yyx} = h_{yxy}$

Aufgabe 5

$f(x,y) = 2x^2 - 3xy + 8y^2 + x - y$

$f_x(x,y) = 4x - 3y + 1$

$f_{xx}(x,y) = 4$

$f_y(x,y) = -3x + 16y - 1$

$f_{yy}(x,y) = 16$

$f_{yx}(x,y) = -3$

Es gilt: $D = f_{xx}(x_0,y_0)\, f_{yy}(x_0,y_0) - [f_{xy}(x_0,y_0)]^2$

$D = 4 \cdot 16 - (-3)^2$

$= 64 - 9 = \underline{\underline{55}}$

Da $D > 0$ und $f_{xx}(x_0,y_0) > 0$ ist f minimal in $\left(\frac{-52}{220}, \frac{1}{55}\right)$

Kritische Punkte finden:

$4x - 3y + 1 = 0$

$4x = 3y - 1$

$x = \frac{3y-1}{4} \quad \Big| \; y = \frac{1}{55}$

$x = \frac{3\left(\frac{1}{55}\right) - 1}{4}$

$x = \frac{-\frac{52}{55}}{\frac{4}{1}}$

$\underline{\underline{x = -\frac{52}{220}}}$

$-3x + 16y - 1 = 0 \quad \Big| \; x = \frac{3y-1}{4}$

$-3\left(\frac{3y-1}{4}\right) + 16y - 1 = 0$

$\frac{-9y+3}{4} + 16y - 1 = 0$

$\frac{55y}{4} - \frac{1}{4} = 0$

$\frac{55y}{4} = \frac{1}{4}$

$\underline{\underline{y = \frac{1}{55}}}$

Aufgabe 6

$f(x,y) = x \cdot e^{-y}$ Gesucht $L(x,y)$ und $Q(x,y)$ bei $(1,0)$

$f(1,0) = 1 \cdot e^{-0} = 1$

$f_x(x,y) = e^{-y} \Rightarrow f_x(1,0) = 1$

$f_y(x,y) = -xe^{-y} \Rightarrow f_y(1,0) = -1$

$f_{xx}(x,y) = 0$

$f_{yy}(x,y) = xe^{-y} \Rightarrow f_{yy}(1,0) = 1$

$f_{xy}(x,y) = -e^{-y} \Rightarrow f_{xy}(1,0) = -1$

$$L(\overset{a}{x},\overset{b}{y}) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= 1 + 1(x-1) - y$$
$$= 1 + x - 1 - y = \underline{\underline{x-y}}$$

$L(0.9, 0.2) = 0.9 - 0.2 = \underline{\underline{0.7}}$

$$Q(x,y) = L(x,y) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-a)^2$$
$$= x-y + 0 - 1(x-1)y + \frac{1}{2}y^2$$
$$= x - \cancel{y} - xy + \cancel{y} + \frac{1}{2}y^2$$
$$= x - xy + \frac{1}{2}y^2$$

$Q(0.9, 0.2) = 0.9 - 0.18 + 0.02 = \underline{\underline{0.74}}$

$f(0.9, 0.2) = 0.9 \cdot e^{-0.2} = \underline{\underline{0.7369}}$

Absoluter Fehler

$L(0.9, 0.2) \Rightarrow |0.7 - 0.7369| = 0.0369$

$Q(0.9, 0.2) \Rightarrow |0.74 - 0.7369| = 0.0031$

Relativer Fehler

$L \approx 5\%$

$Q \approx 0.42\%$

Aufgabe 7

$3x+2y+z=1$ nach z umstellen $\rightarrow z = 1-3x-2y$

$1-3x-2y-3x+9x^2+6xy-2y+6xy+4y^2$

$1-6x-4y+9x^2+4y^2+12xy$

$$d^2(x,y,z) = x^2+y^2+z^2 = x^2+y^2+(1-3x-2y)^2$$
$$= x^2+y^2+1-6x-4y+9x^2+4y^2+12xy$$
$$= 10x^2+5y^2+1-6x-4y+12xy$$
$$= 10x^2+5y^2+12xy-6x-4y+1$$

$\vec{\nabla} d^2 = 0$ also haben wir:

$$\frac{\partial d^2}{\partial x} = 20x+12y-6=0$$

nach x umstellen

$$x = \frac{-12y+6}{20} = \boxed{\frac{-6y+3}{10}}$$

$$\frac{\partial d^2}{\partial y} = 10y+12x-4=0$$

$$10y+12\left(\boxed{\frac{-6y+3}{10}}\right)-4=0$$

$$10y-\frac{72y}{10}+\frac{36}{10}-4=0$$

$$\frac{28}{10}y = \frac{4}{10}$$

$$28y = 4$$

$$y = \frac{4}{28} = \boxed{\frac{1}{7}}$$

$$x = \frac{-12\left(\boxed{\frac{1}{7}}\right)+6}{20} = \frac{-\frac{12}{7}+\frac{42}{7}}{20} = \frac{\frac{30}{7}}{\frac{20}{1}} = \frac{30}{140} = \frac{3}{14} = x$$

x und y in $z = 1-3x-2y$ einsetzen

$$z = 1-3x-2y = 1-\frac{9}{14}-\frac{4}{14} = \frac{1}{14} = z$$

Aufgabe 8

nach z umstellen

plane: $3x + 2y + z = 1$ $\longrightarrow$ $z = 1 - 3x - 2y$

$$V = x \cdot y \cdot z = xy(1 - 3x - 2y)$$

$$\overset{!}{=} xy - 3x^2y - 2xy^2 = V$$

Anstelle von D für die Distanz haben wir nun V für ein Volumen

Nun partielle Ableitungen gleich 0

$$\frac{\partial V}{\partial x} = y - 6xy - 2y^2 = 0$$

$$y(1 - 6x - 2y) \overset{!}{=} 0$$

$$\frac{\partial V}{\partial y} = x - 3x^2 - 4xy = 0$$

$$x(1 - 3x - 4y) \overset{!}{=} 0$$

$x, y = 0$ wären Lösungen, aber dann ist Volumen nicht maximal

$$\begin{array}{ll} \text{I} & 1 - 6x - 2y = 0 \\ \text{II} & 1 - 3x - 4y = 0 \end{array} \quad | \cdot (-2) \quad = \quad \begin{array}{ll} \text{I} & 1 - 6x - 2y = 0 \\ \text{II} & -1 + 9x + 0 = 0 \end{array} \Leftrightarrow x = \frac{1}{9}$$

x in I: $1 - \frac{6}{9} - 2y = 0$

$$\frac{3}{9} = 2y$$

$$y = \frac{3}{18} = \frac{1}{6}$$

x und y in $z = 1 - 3x - 2y$:

$$z = 1 - \frac{3}{9} - \frac{2}{6}$$

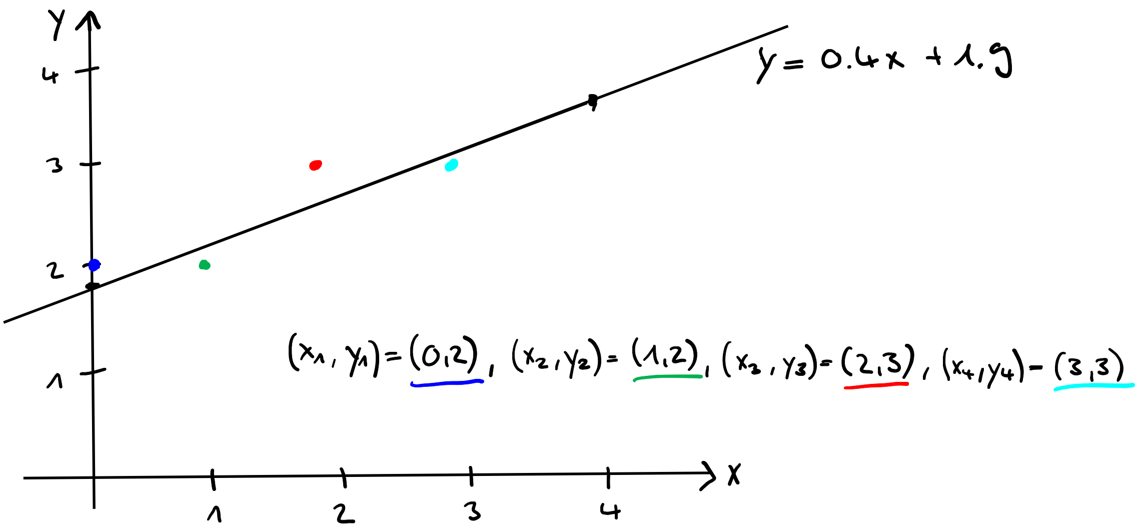
$$z = \frac{1}{3}$$

Somit ist das maximale Volumen:

$$V = \frac{1}{9} \cdot \frac{1}{6} \cdot \frac{1}{3} = \underline{\underline{\frac{1}{162}}}$$

Aufgabe 3

$(x_1, y_1) = \underline{(0,2)},\ (x_2, y_2) = \underline{(1,2)},\ (x_3, y_3) = \underline{(2,3)},\ (x_4, y_4) = \underline{(3,3)}$



$y = b + mx$

$$f(b,m) = \sum_{i=1}^{4} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{4} (y_i - \overbrace{(b + mx_i)}^{\hat{y}_i})^2$$

$$\overset{!}{=} (2 - (b + 0 \cdot m))^2 + (2 - (b + 1 \cdot m))^2 + (3 - (b + 2 \cdot m))^2 + (3 - (b + 3 \cdot m))^2$$

$$\overset{!}{=} (2 - b)^2 + (2 - b - m)^2 + (3 - b - 2m)^2 + (3 - b - 3m)^2$$

$$\frac{\partial f}{\partial b} = 2(2-b)(-1) + 2(2-b-m)(-1) + 2(3-b-2m)(-1) + 2(3-b-3m)(-1)$$

$$\overset{!}{=} -4 + 2b - 4 + 2b + 2m - 6 + 2b + 4m - 6 + 2b + 6m$$

$$\overset{!}{=} -20 + 8b + 12m = 0$$

$$\frac{\partial f}{\partial m} = 2(2-b-m)(-1) + 2(3-b-2m)(-2) + 2(3-b-3m)(-3)$$

$$\overset{!}{=} -4 + 2b + 2m - 12 + 4b + 8m - 18 + 6b + 18m$$

$$\overset{!}{=} -34 + 12b + 28m = 0$$

$$\begin{array}{ll} \mathrm{I} & -20 + 8b + 12m = 0 \quad | \cdot (-1.5) \\ \mathrm{II} & -34 + 12b + 28m = 0 \end{array} = \begin{array}{ll} \mathrm{I} & -20 + 8b + 12m = 0 \\ \mathrm{II} & -4 + 0 + 10m = 0 \end{array} \Leftrightarrow \underline{\underline{m = \frac{4}{10} = \frac{2}{5}}}$$

m in I: $-20 + 8b + \frac{24}{5} = 0$ $\qquad -\frac{100}{5} + \frac{24}{5}$

$$8b \overset{!}{=} \frac{76}{5}$$

$$\underline{\underline{b \overset{!}{=} \frac{76}{40} = 1.9}}$$

Gerade ist $\underline{\underline{y = 0.4x + 1.9}}$

Aufgabe 10

$f(x,y) = x^3 + y$

constraint $g(x,y) = 3x^2 + y^2 = 4 \Rightarrow 3x^2 + y^2 - 4 = 0$

es gilt: $\nabla f(x,y) + \lambda \nabla g(x,y) = 0$

$f_x(x,y) = 3x^2$

$f_y(x,y) = 1$

$g_x(x,y) = 6x$

$g_y(x,y) = 2y$

$L_x(x,y,\lambda) = f_x(x,y) + \lambda\, g_x(x,y) = 0 \Rightarrow 3x^2 + \lambda 6x = 0$

$L_y(x,y,\lambda) = f_y(x,y) + \lambda\, g_y(x,y) = 0 \Rightarrow 1 + \lambda 2y = 0$

$L_\lambda(x,y,\lambda) = g(x,y) = 0 \Rightarrow 3x^2 + y^2 - 4 = 0$

11 Constrained Optimization II

A company has a production function with three inputs x, y, and z given by

$$f(x,y,z) = 50x^{2/5}y^{1/5}z^{1/5}$$

The total budget is $24'000$ and the company can buy x, y, and z at 80, 12, 10 per unit, respectively. What combination of inputs will maximise production?

$$g(x,y,z) = 80x + 12y + 10z = 24'000$$

nach z umstellen:

$$10z = 24'000 - 80x - 12y$$

$$z = \frac{24'000}{10} - \frac{80x}{10} - \frac{12y}{10}$$

$$z = 2'400 - 8x - \frac{6}{5}y$$

Viel Spass!

Aufgabe 11

$$\nabla L(x,y,z,\lambda) = 0$$

$$L_x = 20x^{-3/5} \cdot y^{1/5} \cdot z^{1/5} - \lambda 80 = 0$$

$$L_y = 10x^{2/5} \cdot y^{-4/5} \cdot z^{1/5} - \lambda 12 = 0$$

$$L_z = 10x^{2/5} \cdot y^{1/5} \cdot z^{-4/5} - \lambda 10 = 0$$

$$L_\lambda = 24'000 - 80x - 12y - 10z = 0$$

Mit Octave lösen