

Multivariable Calculus - Exercise 1

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I.BA_IMATH, Semesterweek 05

Please write down to solution of the exercises in a concise but comprehensible way. Numerical results should be accurate to 4 digits. Sketches should be correct qualitatively. At least 75% of the exercises have to be solved satisfactorily. Due time is one week after we have discussed the corresponding topics in class.

1 Contourlines I

Draw the contour lines of the function $f(x,y) = 2x + 3y + 1$ in the rectangular domain $[0,6] \times [-3,3]$. You should be able to solve this problem without the help of the computer (or pocket calculator). Describe the contour lines.

2 Contourlines II

Prüfung

Draw the contour lines of the function $f(x,y) = \sqrt{x^2 + y^2}$ in the square domain $[-3,3]^2$. Describe the contour lines.

3 Contourlines III

Sketch the contourlines of the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}, (x,y) \mapsto z = f(x,y) = x^2 - y^2$. What kind of curves do you see? Highlight the countourline with niveau $z = 1$. Describe the contour lines.

4 Partial derivatives I

Find V_r if $V = \frac{1}{3}\pi r^2 h$.

5 Partial derivatives II

Find all partial derivatives of $f(x,y,z) = \frac{x^2 y^3}{z}$.

Aufgabe 1

$$f(x,y) = 2x + 3y + 1 \quad [0,6] \times [-3,3]$$

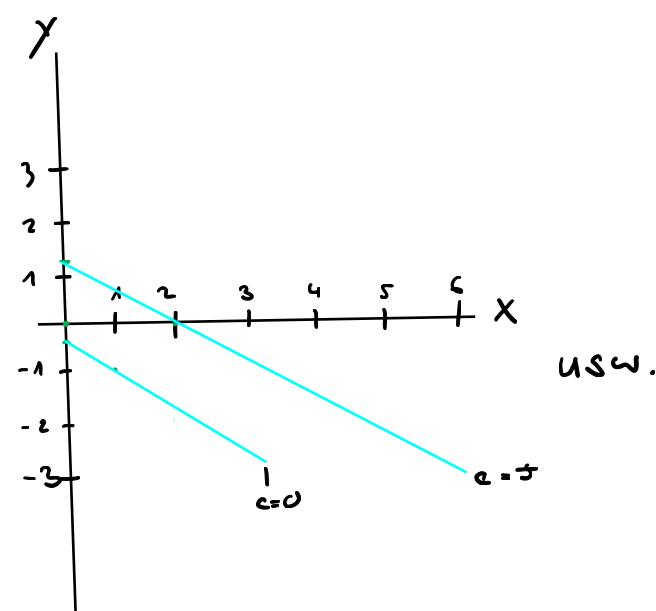
$$f(x,y) = c$$

für $c=0$

$$\begin{aligned} 2x + 3y + 1 &= 0 \\ 3y &\stackrel{!}{=} -2x - 1 \\ y &\stackrel{!}{=} \frac{-2x - 1}{3} \end{aligned}$$

für $c=5$

$$\begin{aligned} 2x + 3y + 1 &= 5 \\ 3y &\stackrel{!}{=} -2x + 4 \\ y &\stackrel{!}{=} \frac{-2x + 4}{3} \end{aligned}$$



usw.

Aufgabe 2

$$f(x,y) = \sqrt{x^2 + y^2} \quad [-3,3]^2$$

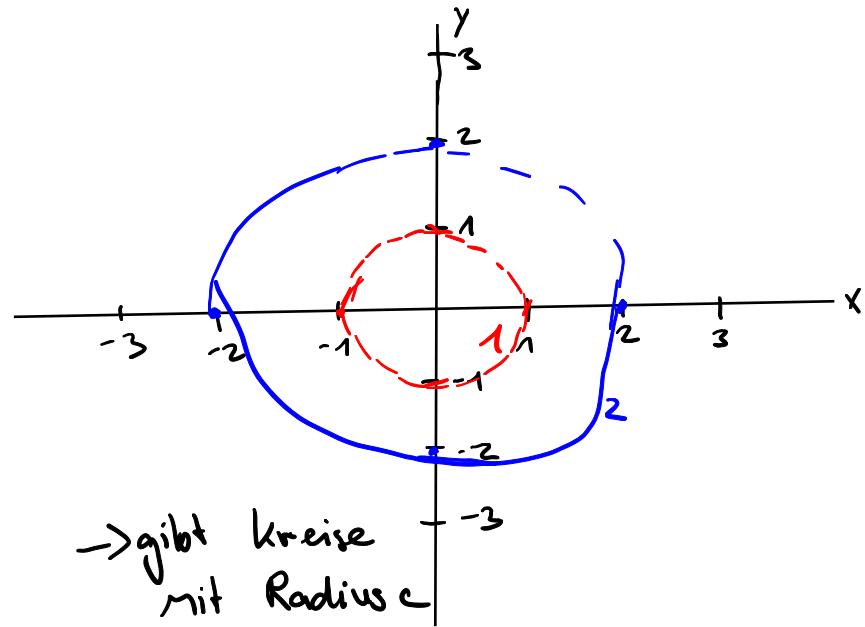
$$f(x,y) = c$$

für $c=1$

$$\begin{aligned} \sqrt{x^2 + y^2} &= c \quad | \cdot 1^2 \\ x^2 + y^2 &= c^2 \\ x^2 + y^2 &= 1^2 \\ x^2 &\stackrel{!}{=} 1^2 - y^2 \end{aligned}$$

$$c=2$$

$$\begin{aligned} x^2 + y^2 &= 4 \\ x^2 &\stackrel{!}{=} 4 - y^2 \end{aligned}$$



→ gibt Kreise mit Radius c

Aufgabe 3

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, (x,y) \mapsto z = f(x,y) = x^2 - y^2$$

$$c=0$$

$$f(x,y) = 0$$

$$x^2 - y^2 = 0$$

$$(x-y)(x+y) = 0$$

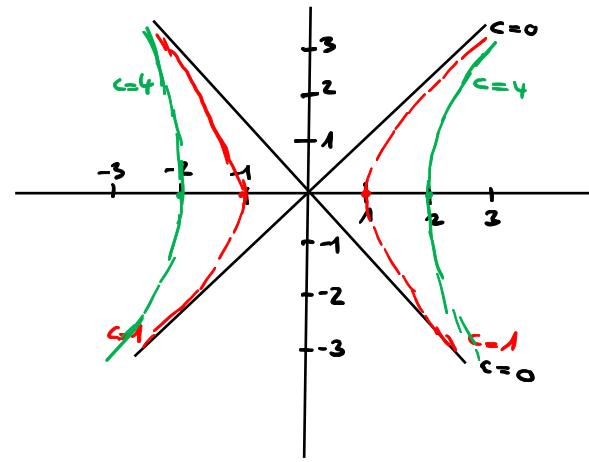
$$\text{d.h. } x=y$$

$$c=1$$

$$x^2 - y^2 = 1$$

$$c=4$$

$$x^2 - y^2 = 4$$



Aufgabe 4

$$V = \frac{1}{3} \pi r^2 h \rightarrow \frac{\partial V}{\partial r} = \frac{2}{3} \pi r h$$

Aufgabe 5

$$f(x, y, z) = \frac{x^2 y^3}{z} \rightarrow \begin{aligned} \frac{\partial f}{\partial x} &= \frac{2x y^3}{z} \\ \frac{\partial f}{\partial y} &= \frac{x^2 y^2}{z} \\ \frac{\partial f}{\partial z} &= x^2 y^3 z^{-1} = -1 x^2 y^3 z^{-2} = -\frac{x^2 y^3}{z^2} \end{aligned}$$

Aufgabe 6

a)

$$z = 3x^2 + 4y^2 - axy$$

$$\frac{\partial z}{\partial x} = 6x - ay \rightarrow \text{Ableitung bei } P(1,2) \text{ muss positiv sein.}$$

$$6 \cdot 1 - a \cdot 2 = 6 - 2a = 0$$

$$6 = 2a$$

$$3 = a$$

a muss also grösser als 3 sein

b) $\frac{\partial z}{\partial y} = 8y - ax$

$$\left. \frac{\partial z}{\partial y} \right|_{(1,2)} = \begin{cases} 8 \cdot 2 - a \cdot 1 \\ 16 - a \end{cases} = 16 - a > 0$$

wenn a < 16 ist, dann slopes the surface up

Aufgabe 7

$$(a) f(x, y) = \frac{3}{2}x^5 - \frac{4}{7}y^6 \rightarrow \frac{\partial f}{\partial x} = \frac{15}{2}x^4, \quad \frac{\partial f}{\partial y} = -\frac{24}{7}y^5 \quad \nabla f(x, y) = \begin{bmatrix} \frac{15}{2}x^4 \\ -\frac{24}{7}y^5 \end{bmatrix}$$

$$(b) z = xy \rightarrow \frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x \quad \nabla z(x, y) = \begin{bmatrix} y \\ xy \end{bmatrix}$$

$$(c) z = \sin\left(\frac{x}{y}\right) \rightarrow \frac{\partial z}{\partial x} = \frac{1}{y} \cos\left(\frac{x}{y}\right) = \frac{\cos\left(\frac{x}{y}\right)}{y}$$

$$\frac{\partial z}{\partial y} = \text{innere Ableitung: } \left(\frac{x}{y}\right)' = x y^{-2} = -x y^2 = \frac{-x}{y^2} \Rightarrow \frac{-x \cdot \cos\left(\frac{x}{y}\right)}{y^2}$$

$$\nabla z(x, y) = \begin{bmatrix} \frac{\cos\left(\frac{x}{y}\right)}{y} \\ -\frac{x \cdot \cos\left(\frac{x}{y}\right)}{y^2} \end{bmatrix}$$

$$(d) f(a, b) = \frac{2a+3b}{2a-3b} \rightarrow \begin{aligned} \frac{\partial f}{\partial a} &= \frac{2(2a-3b) - 2(2a+3b)}{(2a-3b)^2} \\ &= \frac{4a-6b - (4a+6b)}{(2a-3b)^2} = \frac{-12b}{(2a-3b)^2} \quad \nabla f(a, b) = \begin{bmatrix} \frac{-12b}{(2a-3b)^2} \\ \frac{12a}{(2a-3b)^2} \end{bmatrix} = \frac{12}{(2a-3b)^2} \begin{bmatrix} -b \\ a \end{bmatrix} \\ \frac{\partial f}{\partial b} &= \frac{3(2a-3b) - (-3)(2a+3b)}{(2a-3b)^2} \\ &= \frac{6a-9b + 6a+9b}{(2a-3b)^2} = \frac{12a}{(2a-3b)^2} \end{aligned}$$

6 Partial derivatives III

- (a) The surface S is given, for some constant a , by

$$z = 3x^2 + 4y^2 - axy.$$

Find the values of a which ensure that S is sloping upward when we move in the positive x -direction from the point $(1, 2)$.

- (b) With the values of a from part (a), if you move in the positive y -direction from the point $(1, 2)$, does the surface slope up or down? Explain.

7 Gradient I

Find the gradient of the following functions:

$$(a) \quad f(x, y) = \frac{3}{2}x^5 - \frac{4}{7}y^6, \quad (b) \quad z = xe^y, \quad (c) \quad z = \sin(x/y), \quad (d) \quad f(a, b) = \frac{2a+3b}{2a-3b}.$$

8 Gradient II

Find the gradient of the following functions at the point:

$$(a) \quad f(x, y) = x^2y + 7xy^3, \text{ at } (1, 2) \quad (b) \quad f(r, h) = 2\pi rh + \pi r^2, \text{ at } (2, 3) \quad (c) \quad f(m, n) = 5m^2 + 3n^4, \text{ at } (5, 2).$$

9 Directional derivative I Prafan

Calculate the directional derivative of the function $f(x, y) = x^2 + y^2$ at $\mathbf{x}_0 = [1 \ 0]^T$ in the direction of $\mathbf{e} = [1 \ 1]^T$. Draw the contourlines around that point and the gradient of f at that point. Check with the formula which uses the gradient of f to compute the directional derivative.

10 Directional derivative II

Calculate the directional derivative of the real valued function in 3D, $f(x, y, z) = z \sin x + \ln(x^2 - y^2)$ at $\mathbf{x}_0 = [1 \ 0 \ 1]^T$ in the direction of $\mathbf{e} = [1 \ 1 \ 0]^T$. Check with the formula which uses the gradient of f to compute the directional derivative.

11 Plotting with Octave → Octave

The following command plots $y = \sin x$ in the interval $[0, 2\pi]$.

```

1 x = linspace(0, 2*pi, 100);
2 y = sin(x);
3 plot(x, y);

```

$$f(x,y) = x^2 + y^2 \quad x_0 = [1 \ 0]^\top$$

$$e = [1 \ 1]^\top$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$D_e f(x_0) = \nabla f(x_0) \cdot e = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = 2x \Big|_{1,0} = 2$$

$$\frac{\partial f}{\partial y} = 2y \Big|_{1,0} = 0$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \sqrt{2}$$

Aufgabe 8

(a) $f(x,y) = x^2y + 7xy^3$, at $(1,2)$

$$\frac{\partial f}{\partial x} = 2xy + 7y^3, \quad \frac{\partial f}{\partial y} = x^2 + 21x y^2$$

$$\nabla f(x,y) = \begin{bmatrix} 2xy + 7y^3 \\ x^2 + 21xy^2 \end{bmatrix}$$

$$\nabla f(1,2) = \begin{bmatrix} 4 + 56 \\ 1 + 84 \end{bmatrix} = \begin{bmatrix} 60 \\ 85 \end{bmatrix}$$

(b) $f(r,h) = 2\pi rh + \pi r^2$, at $(2,3)$

$$\frac{\partial f}{\partial r} = 2\pi h + 2\pi r, \quad \frac{\partial f}{\partial h} = 2\pi r$$

$$\nabla f(r,h) = \begin{bmatrix} 2\pi h + 2\pi r \\ 2\pi r \end{bmatrix} \quad \nabla f(2,3) = \begin{bmatrix} 6\pi + 4\pi \\ 4\pi \end{bmatrix} = \begin{bmatrix} 10\pi \\ 4\pi \end{bmatrix}$$

(c) $f(m,n) = 5m^2 + 3n^4$, at $(5,2)$

$$\frac{\partial f}{\partial m} = 10m, \quad \frac{\partial f}{\partial n} = 12n^3$$

$$\nabla f(m,n) = \begin{bmatrix} 10m \\ 12n^3 \end{bmatrix}$$

$$\nabla f(5,2) = \begin{bmatrix} 50 \\ 96 \end{bmatrix}$$

Aufgabe 9

$$f(x,y) = x^2 + y^2, \quad x_0 = [1 \ 0]^T, \quad e = [1 \ 1]^T - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{es gilt: } D_e f(x_0) = \nabla f(x_0) \cdot e = |\nabla f(x_0)| e$$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y \quad \text{für } x_0 = [1 \ 0]^T \quad \left. \frac{\partial f}{\partial x} \right|_{(1,0)} = 2, \quad \left. \frac{\partial f}{\partial y} \right|_{(1,0)} = 0$$

$$D_e f(x_0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \sqrt{2}$$

Die Niveaulinien sind Kreise, da $x^2 + y^2 = z$. Kreise haben gerade den Radius \sqrt{z}

Aufgabe 10

$$f(x, y, z) = z \sin x + \ln(x^2 - y^2) \quad x_0 = [1 \ 0 \ 1]^T \quad e = [1 \ 1 \ 0]^T - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = z \cos x + 2x \cdot \frac{1}{x^2 - y^2} = z \cos x + \frac{2x}{x^2 - y^2}$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{x^2 - y^2}$$

$$\frac{\partial f}{\partial z} = \sin x$$

für $x_0 = [1 \ 0 \ 1]^T$

$$\left. \frac{\partial f}{\partial x} \right|_{(1,0,1)} = \cos(1) + 2$$

$$\left. \frac{\partial f}{\partial y} \right|_{(1,0,1)} = 0$$

$$\left. \frac{\partial f}{\partial z} \right|_{(1,0,1)} = \sin(1)$$

$$D_e f(x_0) = \begin{bmatrix} \cos(1) + 2 \\ 0 \\ \sin(1) \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{\cos(1) + 2}{\sqrt{2}}$$

11 Plotting with Octave

Countourplots can be drawn as follows:

```
1 x = linspace(-2*pi,2*pi);
2 y = linspace(0,4*pi);
3 [X,Y] = meshgrid(x,y);
4 Z = X.^2-Y.^2;
5
6 figure
7 contour(X,Y,Z)
```

Plot Graph of a function from \mathbb{R}^2 to \mathbb{R} :

```
1 [X,Y] = meshgrid(-2:.2:2, -2:.2:2);
2 Z = X .* exp(-X.^2 - Y.^2);
3 mesh(X,Y,Z)
4 surf(X,Y,Z)
5 meshc(X,Y,Z)
6 axis off
```