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Linear Algebra - Exercises 4

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I.BA_IMATH, Semesterweek 04

Please write down to solution of the exercises in a consise but comprehensible way. Numerical results should be accurate to 4 digits. Sketches should be correct qualitatively. At least 75% of the exercises have to be solve satisfactorily. Due time is one week after we have discussed the corresponding topic in class.

1 Translation is not linear

Show that the Translation by the vector $\mathbf{x}_0$

$$\mathbf{y} = T_{\mathbf{x}_0}(\mathbf{x}) \;=\; \mathbf{x} + \mathbf{x}_0$$

is not linear, i.e. show that the following does not hold:

$$\begin{aligned} T_{\mathbf{x}_0}(\mathbf{x}_1 + \mathbf{x}_2) &= T_{\mathbf{x}_0}(\mathbf{x}_1) + T_{\mathbf{x}_0}(\mathbf{x}_2) \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3, \\ T_{\mathbf{x}_0}(\alpha \mathbf{x}) &= \alpha T_{\mathbf{x}_0}(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^3 \text{ and } \forall \alpha \in \mathbb{R}. \end{aligned}$$

2 Composing translations

Write down the individual translation matrices

1. $\boldsymbol{T}_1$ translate by vector $\mathbf{x}_1 = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$
2. $\boldsymbol{T}_2$ translate by vector $\mathbf{x}_2 = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$
3. $\boldsymbol{T}_3$ translate by vector $\mathbf{x}_1 + \mathbf{x}_2$

How are these three matrices $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ and $\boldsymbol{T}_3$ related? Is there a general rule for the composition of translation matrices?

3 Inverse of the translation matrix

A translation in 3D by the vector $\mathbf{x}_0 = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}^T$ can be described in homogeneous coordinates by the 4×4 matrix

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Aufgabe 1

$$T_{x_0}(x_1+x_2) = \begin{bmatrix} 1 & 0 & (x_{11}+x_{21})+x_0 \\ 0 & 1 & (x_{12}+y_{22})+y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{x_0}(x_1) + T_{y_0}(x_2) = \begin{bmatrix} 1 & 0 & x_{11}+x_0 \\ 0 & 1 & x_{12}+x_0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & x_{21}+x_0 \\ 0 & 1 & x_{22}+x_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\neq$ sind nicht linear

$$T_{x_0}(\alpha x) = \begin{bmatrix} 1 & 0 & \alpha x + x_0 \\ 0 & 1 & \alpha x + y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\neq$ sind nicht linear

$$\alpha T_{x_0}(x) = \begin{bmatrix} 1 & 0 & x + x_0 \\ 0 & 1 & x + y_0 \\ 0 & 0 & 1 \end{bmatrix}$$

Aufgabe 2

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & y_2 \\ 0 & 0 & 1 & z_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_3 = \begin{bmatrix} 1 & 0 & 0 & x_1+x_2 \\ 0 & 1 & 0 & y_1+y_2 \\ 0 & 0 & 1 & z_1+z_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Composition of two translations T_{x_1} and T_{x_2} is given by
$T = T_{x_2} T_{x_1} = T_{x_1+x_2}$

somit gilt: $T_1 T_2 = T_2 T_1 = T_{1+2} = T_{2+1} = T_3$

Aufgabe 3

Es gilt für die Translationsmatrix T_{x_1} : $T = T_{x_1}^{-1} = T_{-x_1}$

somit ist:

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

zeigen, dass $TT^{-1} = T^{-1}T = I$ gilt:

$$T = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Find it's inverse $\boldsymbol{T}^{-1}$ and show that $\boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{I}$. Note: You can find the inverse by common sense!

4 Scaling 2D

in den Ursprung translatieren für Skalierung

The rectangle with the vertices $A(1,1)$, $B(3,1)$, $C(3,2)$, and $D(1,2)$ should be rescaled by a factor of 2 in the x- and a factor $1/2$ in the y-direction. The vertex A has to be held fixed. Calculate the scaling matrix and apply it the the rectangle. Draw the original and scaled rectangle.

5 Scaling 3D

Suppose You want to scale a 3D-object with the scaling center in $\begin{bmatrix} t_x & t_y & t_z & 1 \end{bmatrix}^T$ (homogeneous coord.) and scaling factors s_x, s_y and s_z in x-, y- and z-direction. Calculate the transformation matrix $\boldsymbol{S}_t$ and verify, that the center stays fixed under the transformation.

zeigen, dass Zentrum bei der Transformation nicht verschoben wird. D.h. es muss folgendes gelten:

$$T^{-1}ST \begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix}$$

sprich auf sich selbst abbilden

6 Rotation around (a_x, a_y) in 2D

T^{-1} R

You want to rotate a 2D-object around (a_x, a_y) by an angle θ. Try to find a general transformation matrix to accomplish this. Remember: You need a translation to the origin, the rotation and the translation back to (a_x, a_y) and the matrices are multiplied where the order of the matrices is reversed. $\frac{1}{T}$

7 Rotation around the origin in 3D

On the slides we have presented to 3×3 rotation matrix $\boldsymbol{Q}$ for a rotation around the axis $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$ by an angle θ. Check that $\boldsymbol{Q}\mathbf{a} = \mathbf{a}$, i.e. the direction of the rotation axis is invariant under this rotation.

8 Projection onto plane through the origin

What is the 3×3 projection matrix $\boldsymbol{I} - \mathbf{n}\mathbf{n}^T$ onto the plane $2x/3 + 2y/3 + z/3 = 0$? In homogeneous coordinates add 0, 0, 0, 1 as an extra row and column in $\boldsymbol{P}$. Project $(3,3,3)$ onto the plane.

9 Projection onto an arbitrary plane

With the same 4×4 matrix $\boldsymbol{P}$ from Exercise 8, multiply $\boldsymbol{T}_{+}\boldsymbol{P}\boldsymbol{T}_{-}$ to find the projection matrix onto the plane $2x/3 + 2y/3 + z/3 = 1$. The Translation $\boldsymbol{T}_{-}$ moves a point on that plane (choose a simple one) to $(0,0,0,1)$. The inverse matrix $\boldsymbol{T}_{+}$ moves it back. Project $(3,3,3)$ onto the plane.

Aufgabe 4

$A(1,1)$, $B(3,1)$, $C(3,2)$, $D(1,2)$

Als erstes müssen wir $A(1,1)$ in den Ursprung translatieren mit

$$T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Dann die Skalierungsmatrix $S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ anwenden

und dann zurück translatieren mit T^{-1}

$$\Rightarrow T^{-1}ST = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}} \underbrace{\begin{bmatrix} 2 & 0 & -2 \\ 0 & 0.5 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}}_{ST} = \underbrace{\begin{bmatrix} 2 & 0 & -1 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}}_{T^{-1}ST} \overbrace{\begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}}^{\text{Punkte}} = \begin{array}{cccc} A' & B' & C' & D' \end{array} \begin{bmatrix} 1 & 5 & 5 & 1 \\ 1 & 1 & 1.5 & 1.5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Aufgabe 5

$$T = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Punkte transformieren zu sich selbst

$$T^{-1}ST = \begin{bmatrix} s_x & 0 & 0 & t_x - s_x t_x \\ 0 & s_y & 0 & t_y - s_y t_y \\ 0 & 0 & s_z & t_z - s_z t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow T^{-1}ST \begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & t_x - s_x t_x \\ 0 & s_y & 0 & t_y - s_y t_y \\ 0 & 0 & s_z & t_z - s_z t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \\ 1 \end{bmatrix}$$

Aufgabe 6

Translation in den Ursprung: $T = \begin{bmatrix} 1 & 0 & -a_x \\ 0 & 1 & -a_y \\ 0 & 0 & 1 \end{bmatrix}$

Danach um den Winkel θ rotieren: $R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dann zurück translatieren mit $T^{-1} = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{bmatrix}$

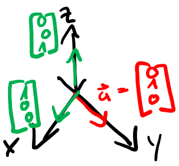
$$T^{-1}RT = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a_x \\ 0 & 1 & -a_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & -a_x\cos\theta + a_y\sin\theta \\ \sin\theta & \cos\theta & -a_x\sin\theta - a_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & a_x - a_x\cos\theta + a_y\sin\theta \\ \sin\theta & \cos\theta & a_y - a_x\sin\theta - a_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

Aufgabe 7

Wir nehmen die Drehung um die y-Achse, somit ist $Q_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

zeigen, dass $Qa = a$



$$Q_y a = \underbrace{\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{a} = a$$

Aufgabe 8

Es gilt $I - nn^T$ wobei n Einheitsnormalvektor

$\frac{2x}{3} + \frac{2y}{3} + \frac{z}{3} = 0$ somit ist $n = \frac{1}{\sqrt{9}} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$$P = I - nn^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4/9 & 4/9 & 2/9 \\ 4/9 & 4/9 & 2/9 \\ 2/9 & 2/9 & 1/9 \end{bmatrix} = \begin{bmatrix} 5/9 & -4/9 & -2/9 \\ -4/9 & 5/9 & -2/9 \\ -2/9 & -2/9 & 8/9 \end{bmatrix}$$

Projektions matrix $\rightarrow$ $= \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$

In homogenen Koordinaten

$$P_h = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -3 \\ -3 \\ 12 \\ 9 \end{bmatrix} = \begin{bmatrix} -3/9 \\ -3/9 \\ 12/9 \\ 9/9 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{4}{3} \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

wegen $\frac{1}{9}$!

Check: $\frac{2}{3} \cdot \left(-\frac{1}{3}\right) + \frac{2}{3}\left(-\frac{1}{3}\right) + \frac{1}{3}\left(\frac{4}{3}\right) \overset{!}{=} 0$

$-\frac{2}{9} - \frac{2}{9} + \frac{4}{9} = 0$ ✓

Aufgabe 9

$\frac{2x}{3} + \frac{2y}{3} + \frac{z}{3} = 1$

$\underbrace{T^{-1}}_{\text{move back}} \underbrace{P}_{\text{Projektion}} \underbrace{T}_{\text{move to } (0,0,0)}$

Point on plane $(1, 0, 0)$

$$T^{-1} P T = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 9 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 4 \\ -4 & 5 & -2 & 4 \\ -2 & -2 & 8 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

check: $\frac{1}{9} \begin{bmatrix} 5 & -4 & -2 & 4 \\ -4 & 5 & -2 & 4 \\ -2 & -2 & 8 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 \\ 0 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$= ✓$

Punkt bleibt wo er ist ✓

10 Reflection at a plane through the origin ✓

The 3×3 mirror matrix that reflects through the plane $\mathbf{n}^T \mathbf{v} = 0$ is $\boldsymbol{M} = \boldsymbol{I} - 2\mathbf{n}\mathbf{n}^T$. Find the reflection of the point $(3,3,3)$ in the plane of exercise 8.

11 Reflection at an arbitrary plane

Find the reflection of the point $(3,3,3)$ in the plane of exercise 9.

12 Affine Transformation in 2D

The unit square which is spanned by the three vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ which form the basis of the 2D cartesian coordinate system should be mapped, such that the origin moves to $(1,2)$ and

$$\boldsymbol{A} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$
$$\boldsymbol{A} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Compute the transformation 3×3 matrix $\boldsymbol{A}$ and check, that it performs right.

13 Affine Transformation in 3D

The unit cube which is spanned by the three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ which form the basis of the cartesian coordinate system should be mapped such that the origin moves the $(1,1,-1)$ and

$$\boldsymbol{A} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{A} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{A} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Compute the transformation 4×4 matrix $\boldsymbol{A}$ and check, that it performs right.

Viel Spass!

Aufgabe 10

$$P = I - 2nn^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\cdot\frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -8/9 & -8/9 & -4/9 \\ -8/9 & -8/9 & -4/9 \\ -4/9 & -4/9 & -2/9 \end{bmatrix} = \begin{bmatrix} 1/9 & -8/9 & -4/9 \\ -8/9 & 1/9 & -4/9 \\ -4/9 & -4/9 & 7/9 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix}$$

homogene Koordinaten

$$P_h = \frac{1}{9}\begin{bmatrix} 1 & -8 & -4 & 0 \\ -8 & 1 & -4 & 0 \\ -4 & -4 & 7 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -33 \\ -33 \\ -3 \\ 9 \end{bmatrix}$$

check: $d(3,3,3) = \frac{2}{3}3 + \frac{2}{3}3 + \frac{1}{3}3 = 5$

$d\ -\frac{1}{9}(33,33,3) = \frac{2}{3}\left(-\frac{33}{9}\right) + \frac{2}{3}\left(-\frac{33}{9}\right) + \frac{1}{3}\left(-\frac{3}{9}\right) = -5$ ✓

Aufgabe 11

Point on plane $(1,0,0)$

Standard Vorgehen:

$P = T^{-1} P T$ wobei $T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ dann gewohntes Vorgehen

Aufgabe 12

$A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = A'$ $\quad A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ $\quad A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$

Der Ursprung geht von $[0\ 0\ 1]^T$ nach $[1\ 2\ 1]^T$ also sind $a_{13} = 1$ und $a_{23} = 2$

$$A\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{13} \\ a_{21} + a_{23} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + 1 \\ a_{21} + 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} a_{11} = 1 \\ a_{21} = -1 \end{array}$$

$$A\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} + a_{13} \\ a_{22} + a_{23} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} + 1 \\ a_{22} + 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} a_{12} = -2 \\ a_{22} = 0 \end{array}$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Check: $\begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ ✓

Aufgabe 13

$$A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = A' \qquad A\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad A\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \qquad A\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Der Ursprung geht von $[0 \; 0 \; 0 \; 1]^T$ nach $[1 \; 1 \; -1 \; 1]^T$ also sind $a_{14} = 1$, $a_{24} = 1$, $a_{34} = -1$

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{14} \\ a_{21} + a_{24} \\ a_{31} + a_{34} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + 1 \\ a_{21} + 1 \\ a_{31} - 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} a_{11} = 1 \\ a_{21} = 0 \\ a_{31} = 1 \end{matrix}$$

$$A = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} + a_{14} \\ a_{22} + a_{24} \\ a_{32} + a_{34} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{12} + 1 \\ a_{22} + 1 \\ a_{32} - 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} a_{12} = -2 \\ a_{22} = 1 \\ a_{32} = 0 \end{matrix}$$

$$A = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{13} + a_{14} \\ a_{23} + a_{24} \\ a_{33} + a_{34} \\ 1 \end{bmatrix} = \begin{bmatrix} a_{13} + 1 \\ a_{23} + 1 \\ a_{33} - 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} a_{13} = -2 \\ a_{23} = -2 \\ a_{33} = 4 \end{matrix}$$

$$A = \begin{bmatrix} 1 & -2 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

P = Punkte

Check: $A P = P'$

$$\underbrace{\begin{bmatrix} 1 & -2 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1 & 2 & -1 & -1 \\ 1 & 1 & 2 & -1 \\ -1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{P'}$$

✓