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Applied Numerics - Exercise 2

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I.BA_IMATH, Semesterweek 09

The solution of the exercises should be presented in a clear and concise manner. Numerical results should be accurate to 4 digits. The exercises are accepted if You solve 75% of the exercises adequately. Please hand in the exercises no later than at the end of the last lecture in semesterweek 10.

1 Numerical Integration

Using the trapezoidal- and the Simpson-rule approximate the following integral using the given number of intervals. What is the minimal number of intervals to obtain an accuracy of 8 digits?

$$I = \int_{-1}^{2} e^{-x^2}\,\mathrm{d}x, \quad (n = 32, 64, 128, 256, 512, 1024)$$

2 Numerical Differentiation

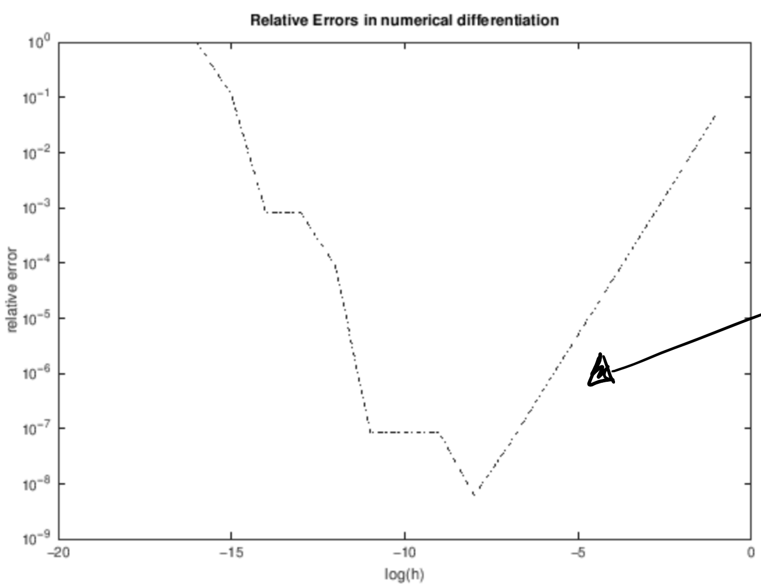
Numerical differentiation aims to compute the derivative of the function f at point x using

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Analysis tells us that the smaller $|h|$ the better the approximation. This exercise will be devoted to a detailed study of this approximation.
Let $f(x) = e^x$ and consider the approximation of the first derivative by means of the above difference quotient. Write a programm in Your preferred language that calculates $f'(x)$ numerically for $h = 10^{-n}$, $n = 1, 2, \ldots, 16$. Plot the relative error of the approximation $f'(0)$ for these values.

→ octave : df = (exp(x+h) - exp(h))/h

Aufgabe 2



instabil wegen numerischen Fehlern. Siehe letzte Woche: Subtraktion von fast gleichen Zahlen führt zu Fehlern!

Aufgabe 1

$$I = \int_{-1}^{2} e^{-x^2}\,dx$$

Trapezoidal rule

auf 8 Stellen $5 \cdot 10^{-9}$ genau:

$$f(x) = e^{-x^2}$$
$$f'(x) = -2x\,e^{-x^2}$$
$$f''(x) = 4x^2 e^{-x^2} - 2e^{-x^2} = e^{-x^2}(4x^2-2) \Rightarrow \max_{-1\le\xi\le2} |f''(\xi)| = 2$$

$$\varepsilon_T = \frac{(2-(-1))^3}{12n^2} \max_{-1\le\xi\le2} |f''(\xi)| = \frac{27}{12n^2} \cdot \max_{-1\le\xi\le2} |f''(\xi)|$$
$$= \frac{27}{12n^2} \cdot 2 \le 5 \cdot 10^{-9}$$
$$= \frac{4.5}{n^2} \le 5 \cdot 10^{-9} = n \ge \sqrt{\frac{4.5}{5\cdot 10^{-9}}}$$
$$n \ge 30'000$$

Simpson's rule

$$\varepsilon_T = \frac{(b-a)\,h^4}{180} \max_{x\in[a,b]} |f^{(4)}(\xi)|$$ ← Rechner verwendet :)

$$= \frac{3h^4}{180} \cdot 12 \le 5 \cdot 10^{-9}$$
$$= \frac{1h^4}{5} \le 5 \cdot 10^{-9} \quad \Big|\; h = \frac{(a-b)}{n}$$
$$= \frac{81}{5n^4} \le 5 \cdot 10^{-9}$$
$$= \frac{16.2}{n^4} \le 5 \cdot 10^{-9}$$
$$= n \ge \sqrt[4]{\frac{16.2}{5 \cdot 10^{-9}}} = 238.58\ldots$$

Exercise 1 - Trapezoidal and Simpson rule

In [25]:
```
format long
f = @(x) [exp(-x*x)];
a = [32,64,128,256,512,1024, 30000];
for i = a
    format long
    printf('TrapezoidalRule for n=%i \t intervall = %0.15e\n',i, TrapezoidalRule(f,-1,2,i))
endfor

using_quad = quad(f, -1,2)
```

```
TrapezoidalRule for n=32         intervall = 1.628312899375075e+00
TrapezoidalRule for n=64         intervall = 1.628757382371713e+00
TrapezoidalRule for n=128        intervall = 1.628868489202797e+00
TrapezoidalRule for n=256        intervall = 1.628896265039894e+00
TrapezoidalRule for n=512        intervall = 1.628903208944738e+00
TrapezoidalRule for n=1024       intervall = 1.628904944917545e+00
TrapezoidalRule for n=30000      intervall = 1.628905522900666e+00
using_quad =  1.628905523574849
```

In [24]:
```
format long
f = @(x) [exp(-x*x)];
a = [32,64,128,256,512,1024];
for i = a
    format long
    printf('Simpsons-Rule  for n=%i \t intervall = %0.15e\n',i, SimpsonsRule(f,-1,2,i))
endfor

using_quad = quad(f,-1,2)
```

```
Simpsons-Rule  for n=32         intervall = 1.628905839171525e+00
Simpsons-Rule  for n=64         intervall = 1.628905543370592e+00
Simpsons-Rule  for n=128        intervall = 1.628905524813158e+00
Simpsons-Rule  for n=256        intervall = 1.628905523652261e+00
Simpsons-Rule  for n=512        intervall = 1.628905523579687e+00
Simpsons-Rule  for n=1024       intervall = 1.628905523575152e+00
using_quad =  1.628905523574849
```

3 Numerical Interpolation in Newton Form

Given the date points $(-3,3)$, $(-1,2)$, $(0,0)$, and $(2,-2)$. Compute the interpolation polynomial in Newton Form by filling out the remaining parts in the following scheme

$$
\begin{array}{ccccc}
x_k & y_k & y[x_k,x_{k+1}] & y[x_k,x_{k+1},x_{k+2}] & y[x_k,x_{k+1},x_{k+2},x_{k+3}] \\
-3 & 3 & & & \\
 & & \ldots\ldots & & \\
-1 & 2 & & -1/2 & \\
 & & -2 & & \ldots\ldots \\
0 & 0 & & \ldots\ldots & \\
 & & -1 & & \\
2 & -2 & & &
\end{array}
$$

Complete the following polynomial and verify its correctness.

$$
\begin{aligned}
P_4(x) &= y_0 + y[x_0,x_1]\,(x-x_0) + y[x_0,x_1,x_2]\,(x-x_0)\,(x-x_1) + y[x_0,x_1,x_2,x_3]\,(x-x_0)\,(x-x_1)\,(x-x_2) \\
&= 3 - \ldots\,(x+3) - \frac{1}{2}\,(x+3)\,(x+1) + \ldots\,(x+3)\,(x+1)\,x.
\end{aligned}
$$

After You have made this calculation You notice, that a fifth data point $(1,1)$ was given. Do You have to repeat the whole calculation, or is there better method to fix the problem?

4 Lagrangian Form of the Interpolation Polynomial

Find an interpolation polynomial of as small degree as possible which passes through the points $(-6,-2)$, $(-5,-1)$, $(-3,2)$, $(-2,0)$, $(0,3)$, and $(2,-1)$.

5 Numerical Extrapolation

Use the results from exercise 1 to improve the accuracy of Simpson's rule using Richardson extrapolation. Notice that Simpson's rule is $\mathcal{O}(h^4)$, i.e. $s=4$ in the extrapolation formula, i.e. use

$$I_1(h,2h) = I(h) + \frac{I(h) - I(2h)}{2^s - 1}$$

with $s=4$. Would it suffice to use $n=512$ (or even $n=256$) intervals to obtain 14 correct digits?

Have Fun!

Aufgabe 3

$(-3,3), (-1,2), (0,0)$, und $(2,-2)$

x_k	y_k	$y[x_k, x_{k+1}]$	$y[x_k, x_{k+1}, x_{k+2}]$	$y[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$
-3	3			
		* $-\frac{1}{2}$		
-1	2		$-1/2$	
		-2		* $1/6$
0	0		* $1/3$	
		-1		
2	-2			

* $y[x_k, x_{k+1}] = \dfrac{y_{k+1} - y_k}{x_{k+1} - x_k} \Rightarrow \dfrac{2-3}{-1-(-3)} = \dfrac{-1}{2} = \underline{\underline{-\frac{1}{2}}}$

* $y[x_k, x_{k+1}, x_{k+2}] = \dfrac{y[x_{k+1}, x_{k+2}] - y[x_k - x_{k+1}]}{x_{k+2} - x_k} = \dfrac{-1-(-2)}{2-(-1)} = \dfrac{1}{3}$

* $y[x_k, x_{k+1}, x_{k+2}, x_{k+3}] = \dfrac{\frac{1}{3} - (-\frac{1}{2})}{2-(-3)} = \dfrac{\frac{5}{6}}{\frac{5}{1}} = \frac{5}{6} \cdot \frac{1}{5} = \frac{5}{30} = \frac{1}{6}$

$x^2 + 4x + 3$
$x^3 + 4x^2 + 3x$

$$P_4(x) = 3 - \frac{1}{2}(x+3) - \frac{1}{2}(x+3)(x+1) + \frac{1}{6}(x+3)(x+1)x$$

$$= 3 - \frac{1}{2}x - \frac{3}{2} - \frac{1}{2}x^2 - 2x - \frac{3}{2} + \frac{1}{6}x^3 + \frac{2}{3}x^2 + \frac{1}{2}x$$

$$= \underline{\underline{-2x + \frac{1}{6}x^2 + \frac{1}{6}x^3}}$$

→ neue Punkte hinzuzufügen ist einfach. Am Ende hinzufügen und dann gem. Vorgehen ausrechnen. Bereits berechnete Punkte bleiben.

Aber für grosse n numerisch instabil

Aufgabe 4

$(-6,-2), (-5,-1), (-3,2), (-2,0), (0,3)$, und $(2,-1)$

$$L_0 = \frac{x+5}{-6+5} \cdot \frac{x+3}{-6+3} \cdot \frac{x+2}{-6+2} \cdot \frac{x-0}{-6-0} \cdot \frac{x-2}{-6-2}$$

$$= -\frac{1}{576}(x+5)(x+3)(x+2)\,x\,(x-2)$$

$$L_1 = \frac{x+6}{-5+6} \cdot \frac{x+3}{-5+3} \cdot \frac{x+2}{-5+2} \cdot \frac{x-0}{-5-0} \cdot \frac{x-2}{-5-2}$$

$$= \frac{1}{210}(x+6)(x+3)(x+2)\,x\,(x-2)$$

$$L_2 = \frac{x+6}{-3+6} \cdot \frac{x+5}{-3+5} \cdot \frac{x+2}{-3+2} \cdot \frac{x-0}{-3-0} \cdot \frac{x-2}{-3-2}$$

$$= -\frac{1}{90}(x+6)(x+5)(x+2)\,x\,(x-2)$$

$$L_3 = \frac{x+6}{-2+6} \cdot \frac{x+5}{-2+5} \cdot \frac{x+3}{-2+3} \cdot \frac{x-0}{-2-0} \cdot \frac{x-2}{-2-2}$$

$$= \frac{1}{96}(x+6)(x+5)(x+3)\,x\,(x-2)$$

$$L_4 = \frac{x+6}{0+6} \cdot \frac{x+5}{0+5} \cdot \frac{x+3}{0+3} \cdot \frac{x+2}{0+2} \cdot \frac{x-2}{0-2}$$

$$= -\frac{1}{360}(x+6)(x+5)(x+3)(x+2)(x-2)$$

$$L_5 = \frac{x+6}{2+6} \cdot \frac{x+5}{2+5} \cdot \frac{x+3}{2+3} \cdot \frac{x+2}{2+2} \cdot \frac{x-0}{2-0}$$

$$= \frac{1}{2'240}(x+6)(x+5)(x+3)(x+2)\,x$$

$(-6,-2), (-5,-1), (-3,2), (-2,0), (0,3)$, und $(2,-1)$

$$P_5(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) + y_4 L_4(x) + y_5 L_5(x)$$

$$= -2 \cdot L_0(x) - 1 \cdot L_1(x) + 2\,L_2(x) + 0 \cdot L_3(x) + 3 \cdot L_4(x) - 1 \cdot L_5(x)$$

Aufgabe 5 Simpson's rule ist $\mathcal{O}(h^4)$ deshalb $s=4$

$$I_1(h,2h) = I(h) + \frac{I(h)-I(2h)}{2^s-1}$$

→ Berechnung in Octave

```
format long
f = @(x) [exp(-x*x)];
a = [2,4,8,16,32,64,128,256,512,1024];
S_old = 0;
for i = a
    format long
    S = SimpsonsRule(f,-1,2,i);
    printf('Simpsons-Rule  for n=%i \t intervall = %0.15e\n',i, S)

    % Richardson Extrapolation
    if (S_old > 0)
        S_extrapol = S + ((S - S_old)/(2^4-1));
        printf('with Extrapolation  for n=%i \t intervall = %0.15e\n\n',i, S_extrapol)
    endif
    S_old = S;

endfor

using_quad = quad(f,-1,2)
```

$$I(h) + \frac{I(h)-I(2h)}{2^4-1}$$

```
Simpsons-Rule  for n=2   intervall = 1.750699106172898e+00
Simpsons-Rule  for n=4   intervall = 1.634973611515320e+00
with Extrapolation  for n=4      intervall = 1.627258578538148e+00

Simpsons-Rule  for n=8   intervall = 1.628975770265992e+00
with Extrapolation  for n=8      intervall = 1.628575914182704e+00

Simpsons-Rule  for n=16          intervall = 1.628910490764503e+00
with Extrapolation  for n=16     intervall = 1.628906138797737e+00

Simpsons-Rule  for n=32          intervall = 1.628905839171525e+00
with Extrapolation  for n=32     intervall = 1.628905529065326e+00

Simpsons-Rule  for n=64          intervall = 1.628905543370592e+00
with Extrapolation  for n=64     intervall = 1.628905523650530e+00

Simpsons-Rule  for n=128         intervall = 1.628905524813158e+00
with Extrapolation  for n=128    intervall = 1.628905523575995e+00

Simpsons-Rule  for n=256         intervall = 1.628905523652261e+00
with Extrapolation  for n=256    intervall = 1.628905523574868e+00

Simpsons-Rule  for n=512         intervall = 1.628905523579687e+00
with Extrapolation  for n=512    intervall = 1.628905523574848e+00

Simpsons-Rule  for n=1024        intervall = 1.628905523575152e+00
with Extrapolation  for n=1024   intervall = 1.628905523574849e+00

using_quad =  1.628905523574849
```

ist bereits bei $n = 256$ auf 14 Stellen genau