

Applied Numerics - Exercise 2

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I.BA_IMATH, Semesterweek 09

The solution of the exercises should be presented in a clear and concise manner. Numerical results should be accurate to 4 digits. The exercises are accepted if You solve 75% of the exercises adequately. Please hand in the exercises no later than at the end of the last lecture in semesterweek 10.

1 Numerical Integration

Using the trapezoidal- and the Simpson-rule approximate the following integral using the given number of intervals. What is the minimal number of intervals to obtain an accuracy of 8 digits?

$$I = \int_{-1}^2 e^{-x^2} dx, \quad (n = 32, 64, 128, 256, 512, 1024)$$

2 Numerical Differentiation

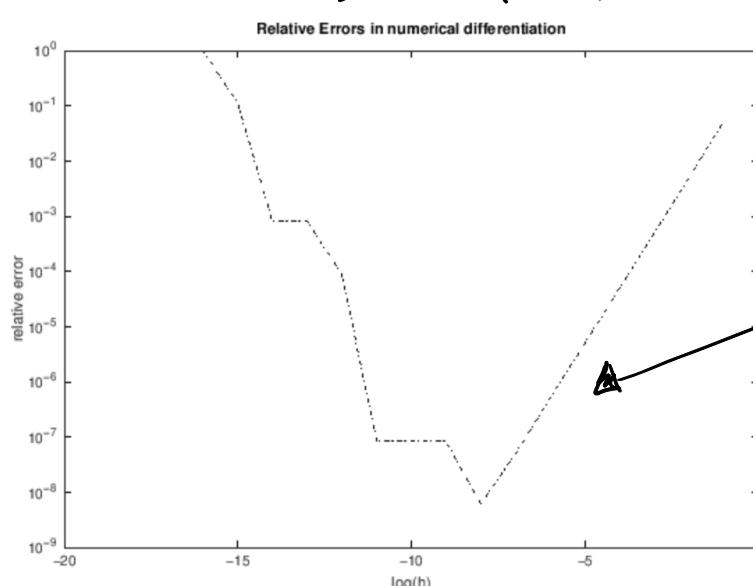
Numerical differentiation aims to compute the derivative of the function f at point x using

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Analysis tells us that the smaller $|h|$ the better the approximation. This exercise will be devoted to a detailed study of this approximation.

Let $f(x) = e^x$ and consider the approximation of the first derivative by means of the above difference quotient. Write a programm in Your preferred language that calculates $f'(x)$ numerically for $h = 10^{-n}$, $n = 1, 2, \dots, 16$. Plot the relative error of the approximation $f'(0)$ for these values.

→ octave : $df = (\exp(x+h) - \exp(h))/h$



instabil wegen
numerischen Fehlern.
Siehe letzte Woche:
Subtraktion von fast gleichen
Zahlen führt zu Fehlern!

$$8 \text{ digits: } \varepsilon_r = 5 \cdot 10^{-3}$$

$$f(x) = e^{-x^2}, f'(x) = -2x \cdot e^{-x^2}$$

$$f''(x) = \frac{-2e^{-x^2} + (-2x \cdot e^{-x^2}) \cdot (-2x)}{= -2e^{-x^2} + 4x^2 \cdot e^{-x^2} - 2e^{-x^2} \cdot (-4x)}$$

$$\max_{a \leq x \leq b} |f'(x)| = 2$$

$$\frac{(2 - (-1)^n)}{12 n^2} \max |f'(x)| \leq 5 \cdot 10^{-3}$$

$$\begin{aligned} \frac{27}{12 n^2} \cdot 2 &\leq 5 \cdot 10^{-3} \\ \frac{4.5}{n^2} &\leq 5 \cdot 10^{-3} \\ \frac{4.5}{5 \cdot 10^{-3}} &\leq n^2 \\ 30000 &\leq n \end{aligned}$$

$$\frac{(b-a) h^4}{180} \max_{x \in [a,b]} |f^{(4)}(x)| \geq 5 \cdot 10^{-3}$$

$$\begin{aligned} h &= 2^{-k} & k &= 10 & x &= 0 \\ h &= 2^{-10} & f(x) &= e^{-x} \end{aligned}$$

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{e^{-2^{-10}} - (-e^{-2^{-10}})}{2 \cdot (-)}$$

Aufgabe 1

$$I = \int_{-1}^2 e^{-x^2} dx$$

Trapezoidal rule

auf 8 Stellen $5 \cdot 10^{-3}$ genau:

$$\begin{aligned}\epsilon_T &= \frac{(2 - (-1))^3}{12n^2} \max_{-1 \leq \xi \leq 2} |f''(\xi)| = \frac{27}{12n^2} \max_{-1 \leq \xi \leq 2} |f''(\xi)| \\ &= \frac{27}{12n^2} \cdot 2 \leq 5 \cdot 10^{-3} \\ &= \frac{4.5}{n^2} \leq 5 \cdot 10^{-3} = n \geq \sqrt{\frac{4.5}{5 \cdot 10^{-3}}} \\ &\quad n \geq \underline{\underline{30000}}\end{aligned}$$

$$\begin{aligned}f(x) &= e^{-x^2} \\ f'(x) &= -2x e^{-x^2} \\ f''(x) &= \frac{4x^2 e^{-x^2} - 2e^{-x^2}}{e^{-x^2}} \stackrel{!}{=} 2 \Rightarrow \max_{-1 \leq \xi \leq 2} |f''(\xi)| = 2\end{aligned}$$

Simpson's rule

$$\begin{aligned}\epsilon_T &= \frac{(b-a)^4}{180} \max_{x \in [a,b]} |f^{(4)}(\xi)| \quad \leftarrow \text{Rechner verwendet :)} \\ &= \frac{3h^4}{180} \cdot 12 \leq 5 \cdot 10^{-3} \\ &= \frac{1h^4}{5} \leq 5 \cdot 10^{-3} \quad | h = \frac{(a-b)}{n} \\ &= \frac{81}{5n^4} \leq 5 \cdot 10^{-3} \\ &= \frac{16.2}{n^4} \leq 5 \cdot 10^{-3} \\ &= n \geq \sqrt[4]{\frac{16.2}{5 \cdot 10^{-3}}} = \underline{\underline{258.58}} \dots\end{aligned}$$

Exercise 1 - Trapezoidal and Simpson rule

```
In [25]: # format long
f = @(x) [exp(-x*x)];
a = [32, 64, 128, 256, 512, 1024, 30000];
for i = a
    format long
    printf('TrapezoidalRule for n=%i \t intervall = %0.15e\n', i, TrapezoidalRule(f, -1, 2, i))
endfor

using_quad = quad(f, -1, 2)

TrapezoidalRule for n=32
TrapezoidalRule for n=64
TrapezoidalRule for n=128
TrapezoidalRule for n=256
TrapezoidalRule for n=512
TrapezoidalRule for n=1024
TrapezoidalRule for n=30000
using_quad = 1.628905523574849
```

```
In [24]: # format long
f = @(x) [exp(-x*x)];
a = [32, 64, 128, 256, 512, 1024];
for i = a
    format long
    printf('Simpsons-Rule for n=%i \t intervall = %0.15e\n', i, SimpsonsRule(f, -1, 2, i))
endfor

using_quad = quad(f, -1, 2)

Simpsons-Rule for n=32
Simpsons-Rule for n=64
Simpsons-Rule for n=128
Simpsons-Rule for n=256
Simpsons-Rule for n=512
Simpsons-Rule for n=1024
using_quad = 1.628905523574849
```

3 Numerical Interpolation in Newton Form

Given the date points $(-3, 3)$, $(-1, 2)$, $(0, 0)$, and $(2, -2)$. Compute the interpolation polynomial in Newton Form by filling out the remaining parts in the following scheme

x_k	y_k	$y[x_k, x_{k+1}]$	$y[x_k, x_{k+1}, x_{k+2}]$	$y[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$
-3	3			
			
-1	2		-1/2	
		-2	
0	0		
		-1		
2	-2			

Complete the following polynomial and verify its correctness.

$$\begin{aligned} P_4(x) &= y_0 + y[x_0, x_1](x - x_0) + y[x_0, x_1, x_2](x - x_0)(x - x_1) + y[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \\ &= 3 - \dots (x + 3) - \frac{1}{2}(x + 3)(x + 1) + \dots (x + 3)(x + 1)x. \end{aligned}$$

After You have made this calculation You notice, that a fifth data point $(1, 1)$ was given. Do You have to repeat the whole calculation, or is there better method to fix the problem?

4 Lagrangian Form of the Interpolation Polynomial

Find an interpolation polynomial of as small degree as possible which passes through the points $(-6, -2)$, $(-5, -1)$, $(-3, 2)$, $(-2, 0)$, $(0, 3)$, and $(2, -1)$.

5 Numerical Extrapolation

Use the results from exercise 1 to improve the accuracy of Simpson's rule using Richardson extrapolation. Notice that Simpson's rule is $\mathcal{O}(h^4)$, i.e. $s = 4$ in the extrapolation formula, i.e. use

$$I_1(h, 2h) = I(h) + \frac{I(h) - I(2h)}{2^s - 1}$$

with $s = 4$. Would it suffice to use $n = 512$ (or even $n = 256$) intervals to obtain 14 correct digits?

Have Fun!

Aufgabe 3

$(-3, 3), (-1, 2), (0, 0)$, und $(2, -2)$

x_k	y_k	$y[x_k, x_{k+1}]$	$y[x_k, x_{k+1}, x_{k+2}]$	$y[x_k, x_{k+1}, x_{k+2}, x_{k+3}]$
-3	3	*	$-\frac{1}{2}$	
-1	2		-1/2	*
0	0		*	1/6
2	-2		1/3	

$$*\quad y[x_k, x_{k+1}] = \frac{y_{k+1} - y_k}{x_{k+1} - x_k} \Rightarrow \frac{2 - 3}{-1 - (-3)} = \frac{-1}{2} = \underline{\underline{-\frac{1}{2}}}$$

$$*\quad y[x_k, x_{k+1}, x_{k+2}] = \frac{y[x_{k+1}, x_{k+2}] - y[x_k, x_{k+1}]}{x_{k+2} - x_k} = \frac{-1 - (-2)}{2 - (-1)} = \frac{1}{3}$$

$$*\quad y[x_k, x_{k+1}, x_{k+2}, x_{k+3}] = \frac{\frac{1}{3} - (-\frac{1}{2})}{2 - (-3)} = \frac{\frac{5}{6}}{\frac{5}{1}} = \frac{5}{6} \cdot \frac{1}{5} = \frac{5}{30} = \frac{1}{6}$$

$$\begin{array}{l} x^2 + 4x + 3 \\ x^2 + 4x^2 + 3x \end{array}$$

$$\begin{aligned} P_4(x) &= 3 - \frac{1}{2}(x+3) - \frac{1}{2}(x+3)(x+1) + \frac{1}{6}(x+3)(x+1)x \\ &= 3 - \frac{1}{2}x - \frac{3}{2} - \frac{1}{2}x^2 - 2x - \frac{3}{2} + \frac{1}{6}x^2 + \frac{2}{3}x^2 + \frac{1}{2}x \\ &= \underline{\underline{-2x + \frac{1}{6}x^2 + \frac{1}{6}x^3}} \end{aligned}$$

→ neue Punkte hinzuzufügen ist einfach. Am Ende hinzufügen und dann gen. Vorgaben ausrechnen. Bereits berechnete Punkte bleiben.

Aber für grosse n numerisch instabil

Aufgabe 4

$(-6, -2), (-5, -1), (-3, 2), (-2, 0), (9, 3)$, und $(2, -1)$

$$L_0 = \frac{x+5}{-6+5} \cdot \frac{x+3}{-6+3} \cdot \frac{x+2}{-6+2} \cdot \frac{x-0}{-6-0} \cdot \frac{x-2}{-6-2}$$

$$= -\frac{1}{5!x} (x+5)(x+3)(x+2)x(x-2)$$

$$L_1 = \frac{x+6}{-5+6} \cdot \frac{x+3}{-5+3} \cdot \frac{x+2}{-5+2} \cdot \frac{x-0}{-5-0} \cdot \frac{x-2}{-5-2}$$

$$= \frac{1}{2!0} (x+6)(x+3)(x+2)x(x-2)$$

$$L_2 = \frac{x+6}{-3+6} \cdot \frac{x+5}{-3+5} \cdot \frac{x+2}{-3+2} \cdot \frac{x-0}{-3-0} \cdot \frac{x-2}{-3-2}$$

$$= -\frac{1}{3!0} (x+6)(x+5)(x+2)x(x-2)$$

$$L_3 = \frac{x+6}{-2+6} \cdot \frac{x+5}{-2+5} \cdot \frac{x+3}{-2+3} \cdot \frac{x-0}{-2-0} \cdot \frac{x-2}{-2-2}$$

$$= \frac{1}{3!6} (x+6)(x+5)(x+3)x(x-2)$$

$$L_4 = \frac{x+6}{0+6} \cdot \frac{x+5}{0+5} \cdot \frac{x+3}{0+3} \cdot \frac{x+2}{0+2} \cdot \frac{x-2}{0-2}$$

$$= -\frac{1}{360} (x+6)(x+5)(x+3)(x+2)(x-2)$$

$$L_5 = \frac{x+6}{2+6} \cdot \frac{x+5}{2+5} \cdot \frac{x+3}{2+3} \cdot \frac{x+2}{2+2} \cdot \frac{x-0}{2-0}$$

$$= \frac{1}{2!240} (x+6)(x+5)(x+3)(x+2)x$$

$(-6, -2), (-5, -1), (-3, 2), (-2, 0), (9, 3)$, und $(2, -1)$

$$P_5(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x) + y_3 L_3(x) + y_4 L_4(x) + y_5 L_5(x)$$

$$= -2 \cdot L_0(x) - 1 \cdot L_1(x) + 2 \cdot L_2(x) + 0 \cdot L_3(x) + 3 \cdot L_4(x) - 1 \cdot L_5(x)$$

Aufgabe 5 Simpson's rule ist $O(h^4)$ deshalb $s=4$

$$I_h(h, 2h) = I_h(h) + \frac{I(h) - I(2h)}{2^s - 1}$$

→ Berechnung in Octave

```
format long
f = @(x) [exp(-x*x)];
a = [2, 4, 8, 16, 32, 64, 128, 256, 512, 1024];
s_old = 0;
for i = a
    format long
    s = SimpsonsRule(f, -1, 2, i);
    printf('Simpsons-Rule for n=%i \t intervall = %0.15e\n', i, s)

    % Richardson Extrapolation
    if (s_old > 0)
        S_extrapol = s + ((s - s_old)/(2^4-1));
        printf('with Extrapolation for n=%i \t intervall = %0.15e\n\n', i, S_extrapol)
    endif
    s_old = s;
endfor

using_quad = quad(f, -1, 2)

Simpsons-Rule for n=2      intervall = 1.750699106172898e+00
Simpsons-Rule for n=4      intervall = 1.634973611515320e+00
with Extrapolation for n=4      intervall = 1.627258578538148e+00

Simpsons-Rule for n=8      intervall = 1.628975770265992e+00
with Extrapolation for n=8      intervall = 1.628575914182704e+00

Simpsons-Rule for n=16      intervall = 1.628910490764503e+00
with Extrapolation for n=16      intervall = 1.628906138797737e+00

Simpsons-Rule for n=32      intervall = 1.628905839171525e+00
with Extrapolation for n=32      intervall = 1.628905529065326e+00

Simpsons-Rule for n=64      intervall = 1.628905543370592e+00
with Extrapolation for n=64      intervall = 1.628905523650530e+00

Simpsons-Rule for n=128      intervall = 1.628905524813158e+00
with Extrapolation for n=128      intervall = 1.628905523575995e+00

Simpsons-Rule for n=256      intervall = 1.628905523652261e+00
with Extrapolation for n=256      intervall = 1.628905523574868e+00

Simpsons-Rule for n=512      intervall = 1.628905523579687e+00
with Extrapolation for n=512      intervall = 1.628905523574848e+00

Simpsons-Rule for n=1024      intervall = 1.628905523575152e+00
with Extrapolation for n=1024      intervall = 1.628905523574849e+00

using_quad = 1.628905523574849
```

ist bereits bei
n=256 auf
14 Stellen genau