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Applied Numerics - Exercise 3

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I.BA_IMATH, Semesterweek 10

The solution of the exercises should be presented in a clear and concise manner. Numerical results should be accurate to 4 digits. The exercises are accepted if You solve 75% of the exercises adequately. Please hand in the exercises no later than at the end of the last lecture in semesterweek 11.

1 Natural Cubic Spline

Compute a natural cubic spline through the points $(0,0)$, $(1,1)$ and $(2,1)$. First compute everything by hand and then check Your results using graphical means, i.e. plot the graph and make sure that $z_1 = z_3 = 0$ by taking the second derivative of the interpolating function.

2 Complex Numbers: Graphical Representation I

Draw the following complex numbers in the complex plane and connect the points one after the other: describe what You see!
(a) $z_1 = 1+2j$, $z_2 = -2+j$, $z_3 = -1-2j$, $z_4 = 2-j$
(b) $z_1 = 3+7j$, $z_2 = -3+5j$, $z_3 = -3+3j$, $z_4 = 3j$, $z_5 = -3-3j$, $z_6 = -3-6j$, $z_7 = 3-6j$.

Aufgabe 1

$(0,0)$, $(1,1)$, $(2,1)$ $\qquad h_i = 1$ for $i = 1,2$

$$z_1 + 4z_2 + z_3 = \underbrace{\frac{6}{h^2}}_{\text{ist so definiert, glaube ich jedenfalls}} \left(y_3 - 2y_2 + y_1\right) = 6\left(1 - 2\cdot 1 + 0\right) = -6$$

$z_1 = z_3 = 0$; da natural spline somit gilt $4z_2 = -6$ oder $z_2 = -\frac{3}{2}$

$$C_1 = \frac{y_2 - y_1}{h_1} - \frac{h_1}{6}(z_2 - z_1) = 1 - \frac{1}{6}\left(-\frac{3}{2}\right) = \frac{5}{4}$$

$$C_2 = \frac{y_3 - y_2}{h_2} - \frac{h_2}{6}(z_3 - z_2) = 0 - \frac{1}{6}\left(\frac{3}{2}\right) = -\frac{1}{4}$$

$$D_1 = y_1 - \frac{h_1^2}{6} z_1 = 0 - \frac{1}{6}\cdot 0 = 0$$

$$D_2 = y_2 - \frac{h_2^2}{6} z_2 = 1 - \frac{1}{6}\left(-\frac{3}{2}\right) = \frac{5}{4}$$

Wir finden folgende Splines:

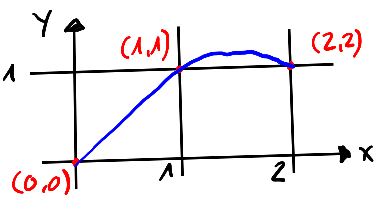
$$P_1(x) = \frac{z_1}{6h}(x_2 - x)^3 + \frac{z_2}{6h}(x - x_1)^3 + C_1(x - x_1) + D_1 \qquad x_1 \leq x \leq x_2$$

$$= -\frac{1}{4}x^3 + \frac{5}{4}x \qquad 0 \leq x \leq 1$$

$$P_2(x) = \frac{z_2}{6h}(x_3 - x)^3 + \frac{z_3}{6h}(x - x_2)^3 + C_2(x - x_2) + D_2 \qquad x_2 \leq x \leq x_3$$

$$= -\frac{1}{4}(2-x)^3 - \frac{1}{4}(x-1) + \frac{5}{4} \qquad 1 \leq x \leq 2$$

Sollte etwa so aussehen:

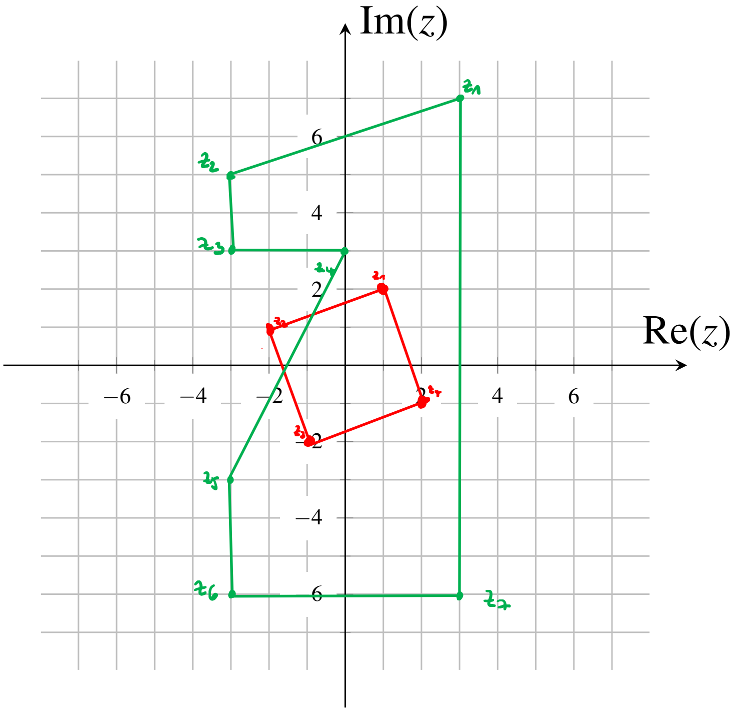


Wir können das Ganze überprüfen, in dem wir die Ableitungen betrachten

$P_1''(0) = 0$ ✓

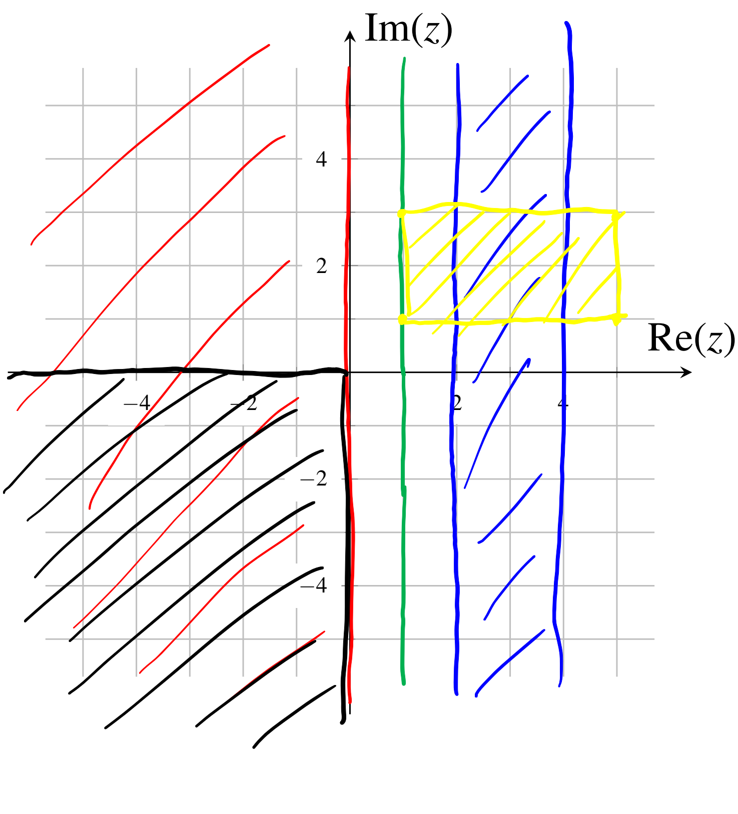
$P_2''(2) = 0$ ✓

Passt



3 Complex Numbers: Graphical Representation II

Draw the following sets of complex numbers in the complex plane:
(a) $\{z \in \mathbb{C} | \mathrm{Re}(z) = 1\}$, (b) $\{z \in \mathbb{C} | \mathrm{Re}(z) \leq 0\}$, (c) $\{z \in \mathbb{C} | 2 \leq \mathrm{Re}(z) \leq 4\}$,
(d) $\{z \in \mathbb{C} | 1 \leq \mathrm{Re}(z) \leq 5 \quad \textit{and} \quad 1 \leq \mathrm{Im}(z) \leq 3\}$, (e) $\{z \in \mathbb{C} | \mathrm{Re}(z)\mathrm{Im}(z) \leq 0\}$.



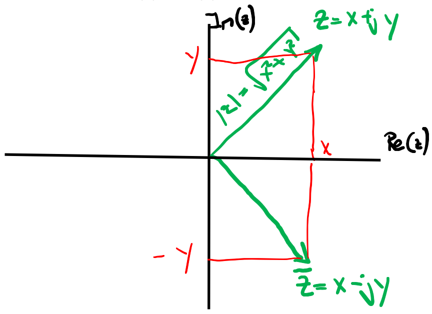
4 Complex Numbers: Complex Conjugate Numbers

The **complex conjugate** of the number $z = x + jy$ is the number $\bar{z} = x - jy$. Verify, that the complex conjugate is the number z mirrored on the real axis (x-axis) in the complex plane. Show that $z\bar{z} = \bar{z}z = x^2 + y^2 = |z|^2$

Show, that for each complex number $z = x + jy$ the following holds:

(a) $z + \bar{z} = 2\mathrm{Re}(z)$ (b) $z - \bar{z} = 2j\mathrm{Im}(z)$

(c) $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$



where $\mathrm{Re}(z) = x$ is the **real part of** z and $\mathrm{Im}(z) = y$ is the **imaginary part of** z.

5 Complex Numbers: Addition and Substraction

Compute (a) $(8+2j)+(7+3j)$, (b) $(11-15j)+(-3+8j)$ and (c) $98-(99-100j)$.

6 Complex Numbers: Multiplikation

Compute (a) $8 \cdot 5j$, (b) $(-7-12j)5j$, (c) $(8+2j)(7+3j)$ and $(13+17j)(13-17j)$.

7 Complex Numbers: Powers

Given $z_1 = 7-5j$, $z_2 = 2+j$, $z_3 = -5+2j$, $z_4 = -10-3j$, $z_5 = 8$ and $z_6 = 8j$ compute (a) $z_1 - z_3 - z_5$, (b) $z_1 z_3 z_4$ (c) $z_1^2 + z_2^2$, (d) $jz_4 - z_3 z_6$, (e) $\mathrm{Re}(z_1^2 z_3)$, and (f) $\mathrm{Im}(2z_2 - 3z_3)$.

8 Complex Numbers: Division

For $a, b, c, d \in \mathbb{R}$ represent the following numbers in the form $z = x + jy$

(a) $(7a+3bj)(4c-5dj)$, (b) $(a+bj)/(c-dj)$, (c) $j(a+bj)+(a-bj)/j$, (d) $\overline{(b-cj)}(b-cj)^{-1}$ and (e) $aj + \overline{j/a} + a/j + j/a$.

9 Complex Numbers: Polar Form

Every complex number $z = x + jy$ can be written in **polar form** $z = r(\cos\phi + j\sin\phi)$ where ϕ is the **argument** of the complex number z and $r = |z|$ is the **modulus** of a complex number z, also called the **complex norm**. Find the modulus $|z|$ and the argument ϕ (in radians) of the following numbers:

(a) $z_1 = 1+2j$, $z_2 = -2+j$, $z_3 = -1-2j$, $z_4 = 2-j$

(b) $z_1 = 3+7j$, $z_2 = -3+5j$, $z_3 = -3+3j$, $z_4 = 3j$, $z_5 = -3-3j$, $z_6 = -3-6j$, $z_7 = 3-6j$, $z_8 = 3-2j$.

Note: Do use the calculator only for checking the correctness of Your result.

$2-j$ $r = \sqrt{2^2+1} = \sqrt{5}$

$\sqrt{5} \cdot e^{j}$

Aufgabe 4

(a) $z + \bar{z} = 2\operatorname{Re}(z) = (x + jy) + (x - jy) = 2x$

(b) $z - \bar{z} = 2j\operatorname{Im}(z) = (x + jy) - (x - jy) = 2jy = 2j\operatorname{Im}(z)$

(c) $\underset{\underset{z^{-1}}{\parallel}}{\dfrac{1}{z}} = \dfrac{1}{z\bar{z}}\bar{z} = \dfrac{\bar{z}}{|z|^2}$

Aufgabe 5

(a) $(8+2j) + (7+3j) = 15 + 5j$

(b) $(11 - 15j) + (-3 + 8j) = 8 - 7j$

(c) $98 - (99 - 100j) = -1 + 100j$

Aufgabe 6

$j^2 = -1$

(a) $8 \cdot 5j = (8 + 0j)(0 + 5j) = 40j = (0 + 40j)$

(b) $(-7 - 12j)5j = -35j - 60j^2 = 60 - 35j$

(c) $(8+2j)(7+3j) = 56 + 24j + 14j + 6j^2 = 50 + 38j$

(d) $(13 + 17j)(13 - 17j) = 169 - 289j^2 = 169 + 289 = 458$

Aufgabe 7

$z_1 = 7 - 5j, z_2 = 2 + j, z_3 = -5 + 2j, z_4 = -10 - 3j, z_5 = 8, z_6 = 8j$

(a) $z_1 - z_3 - z_5 = (7 - 5j) - (-5 + 2j) - (8) = 4 - 7j$

(b) $z_1 z_3 z_4 = (7 - 5j) \cdot (-5 + 2j) \cdot (-10 - 3j) = (-25 + 39j)(-10 - 3j) = 367 - 315j$

(c) $z_1^2 + z_2^2 = (7 - 5j)^2 + (2 + j)^2 = (24 - 70j) + (3 + 4j) = 27 - 66j$

(d) $j \cdot z_4 - z_3 \cdot z_6 = (0 + j)(-10 - 3j) - ((-5 + 2j)(0 + 8j)) = (3 - 10j) - (-16 - 40j) = 3 - 10j + 16 + 40j = 19 + 30j$

(e) $\operatorname{Re}(\overbrace{z_1^2 \cdot z_3}^{x}) \Rightarrow x = (24 - 70j) \cdot (-5 + 2j) = -120 + 48j + 350j - 140j^2 = 20 + 398j \Rightarrow \operatorname{Re}(x) = \operatorname{Re}(20 + 398j) = 20$

(f) $\operatorname{Im}(\overbrace{2z_2 - 3z_3}^{x}) \Rightarrow x = (4 + 2j) - (15 + 6j) = -11 - 4j \Rightarrow \operatorname{Im}(x) = \operatorname{Im}(-11 - 4j) = -4$

Aufgabe 8

(a) $(7a + 3bj)(4c - 5dj) = (28ac + 15bd) + j(-35ad + 12bc)$

(b) $\dfrac{(a + bj)}{(c - dj)} = \dfrac{(a + bj)\ (c + dj)}{(c - dj)(c + dj)} = \dfrac{ac + adj + bcj + bdj^2}{c^2 + d^2} = \dfrac{(ac - bd) + j(ad + bc)}{c^2 + d^2} = \dfrac{ac - bd}{c^2 + d^2} + j\dfrac{(bc + ad)}{c^2 + d^2}$

(c) $j(a + bj) + \dfrac{(a - bj)}{(0 + j)} = -b + aj + \dfrac{(a - bj)(0 - j)}{(0 + j)(0 - j)} = -b + aj + (-b - aj) = -2b$

(d) $\overline{(b - cj)}(b - cj)^{-1} = \dfrac{(b + cj)}{(b - cj)} = \dfrac{(b + cj)(b + cj)}{(b - cj)(b + cj)} = \dfrac{(b^2 - c^2) + 2bcj}{b^2 + c^2} = \dfrac{b^2 - c^2}{b^2 + c^2} + j\dfrac{2bc}{b^2 + c^2}$

(e) $aj + \dfrac{j}{a} + \dfrac{a}{j + \frac{j}{a}}$ ToDo – Bürgler würde sagen HA!

Aufgabe 9 und 10

Allg. $z = x + jy$ kann geschrieben werden als

$z = r e^{j\phi}$ wobei $r = \sqrt{x^2 + y^2}$ und

$\phi = \operatorname{Arg}(z)$ i.e. $\tan \phi = \frac{y}{x}$

(a) $z_1 = 1 + 2j = \sqrt{5} \cdot e^{j \cdot 1.1071}$ $\quad r = \sqrt{1^2 + 2^2} = \sqrt{5}$

$e^{j\phi} = \sqrt{5}\left(\cos(1.1071) + j \cdot \sin(1.1071)\right)$ $\quad \tan \phi = \frac{2}{1} = 2 \Rightarrow \arctan(2) = 1.1071 = \phi$

bc. $x > 0$

$z_2 = -2 + j = \sqrt{5} \cdot e^{j \cdot 2.6779}$ $\quad r = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

$\tan \phi = \frac{1}{-2} \Rightarrow \arctan(-\frac{1}{2}) = -0.4636 + \pi = \phi = 2.6779$

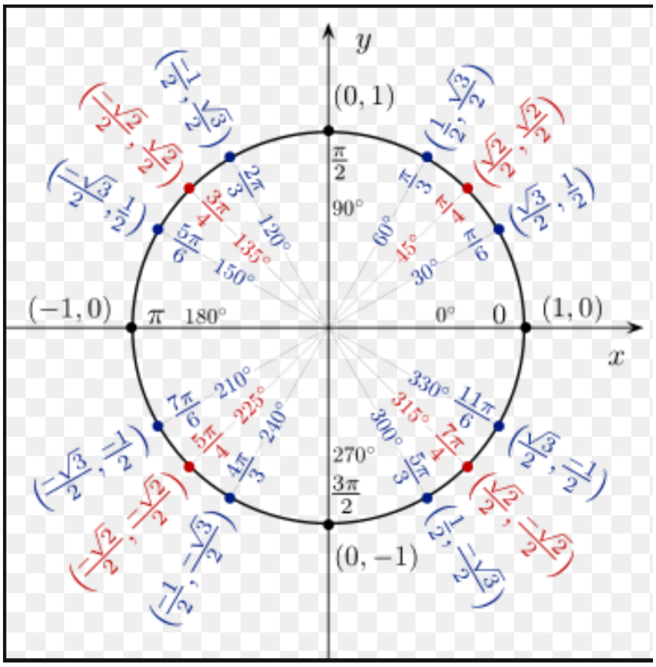
$e^{j\phi} = \sqrt{5}\left(\cos(2.6779) + j \cdot \sin(2.6779)\right)$

$z_3 = -1 - 2j = \sqrt{5} \cdot e^{j \cdot (-2.0344)}$ $\quad r = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$ $\quad \arctan(\frac{-2}{-1}) = 1.1071 - \pi = -2.0344$

$e^{j\phi} = \sqrt{5}\left(\cos(-2.0344) + j \cdot \sin(-2.0344)\right)$

$z_4 = 2 - j = \sqrt{5} \cdot e^{j \cdot (-0.4636)}$ $\quad r = \sqrt{2^2 + (-1)^2} = \sqrt{5}$ $\quad \arctan(\frac{-1}{2}) = -0.4636$

$e^{j\phi} = \sqrt{5}\left(\cos(-0.4636) + j \cdot \sin(-0.4636)\right)$



$$\operatorname{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

431 × 162

10 Complex Numbers: Exponential Form

Using the Euler-Formula $e^{j\phi} = \cos\phi + j\sin\phi$ we can further simplify the polar form:

$$z = r(\cos\phi + j\sin\phi) = re^{j\phi}$$

Note: the number $e^{j\phi}$ lives on the unit circle and the angle between the x-axis and z is equal to ϕ. Write the number z_1, z_2, …, z_8 from the previous example in Exponential form.

11 Complex Numbers: Multiplication and Division in Exponential Form

Multiplication or division of two complex number $z_1 = r_1 e^{j\phi_1}$ and $z_2 = r_2 e^{j\phi_2}$ in exponential form is simple:

$$z = z_1 z_2 = r_1 e^{j\phi_1} r_2 e^{j\phi_2} = r_1 r_2 e^{j(\phi_1+\phi_2)} = re^{j\phi}$$

where $r = r_1 r_2$ and $\phi = \phi_1 + \phi_2$. Hence the modulus of the product is the product of the moduli and the argument of the product is the sum of the arguments. Questions:

1. Using the geometrical representation of complex number in the complex plane describe what happens, if one multiplies $z = re^{j\phi}$ with the imaginary unit j?.
2. Furthermore, compute the product of $z_1 = 1 + 2j$ and $z_2 = -2 + j$ in two different ways, i.e. in (i) cartesian coordinates and (ii) exponential form.

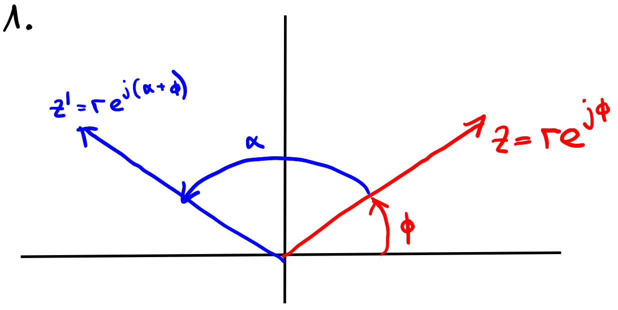
12 Trigonometric functions

Compute a complex and a real trigonometric polynomial which passes through the points $(0,8)$, $(\pi/2,6)$, $(\pi,4)$, and $(3\pi/2,2)$ and which is 2π-periodic.

Have Fun!

Aufgabe 11

1.



2. $z_1 = 1 + 2j \qquad z_2 = -2 + j$

(i) $z_1 z_2 = (1+2j)(-2+j) = -4 - 3j$

(ii) $\sqrt{5}\cdot e^{j\cdot 1.1071} \cdot \sqrt{5}\cdot e^{j\cdot 2.6779} = 5 \cdot e^{j(1.1071 + 2.6779)}$

Aufgabe 12

$$T_4(x) = c_0 + c_1 \cdot \underbrace{e^{jx}}_{\cos x \,+\, j\cdot \sin x} + c_2 \cdot e^{2jx} + c_3 \cdot e^{3jx}$$

$$T_4(0) = f(0) = f_0$$
$$T_4(\pi/2) = f(\pi/2) = f_1$$
$$T_4(\pi) = f(\pi) = f_2$$
$$T_4(3\pi/2) = f(3\pi/2) = f_3$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{j\pi/2} & e^{j\pi} & e^{j\frac{3\pi}{2}} \\ 1 & e^{j\pi} & e^{2j\pi} & e^{3j\pi} \\ 1 & e^{j\frac{3\pi}{2}} & e^{3j\pi} & e^{9j\frac{\pi}{2}} \end{bmatrix}}_{=:F} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{=:c} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}}_{=:f}$$

$$/\!/ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$\cos(3\pi/2) + j \sin(3\pi/2)$

Inverse für F berechnen

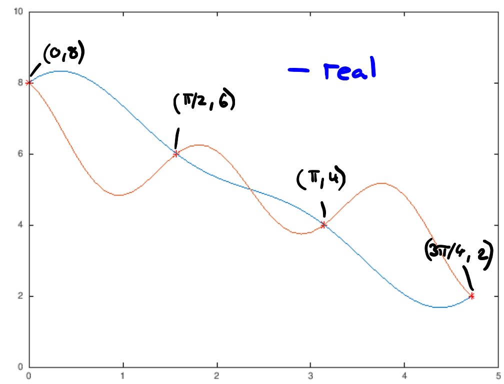
$$F^{-1} = \frac{1}{4}\bar{F} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \overline{e^{j\pi/2}} & \overline{e^{j\pi}} & \overline{e^{j\frac{3\pi}{2}}} \\ 1 & \overline{e^{j\pi}} & \overline{e^{2j\pi}} & \overline{e^{3j\pi}} \\ 1 & \overline{e^{j\frac{3\pi}{2}}} & \overline{e^{3j\pi}} & \overline{e^{9j\frac{\pi}{2}}} \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Nun folgendes einfügen und auflösen $f_0 = 8,\ f_1 = 6,\ f_2 = 4,\ f_3 = 2$

mit Octave, wobei dabei ein Vorzeichen falsch war!

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 8 \\ 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1-j \\ 1 \\ 1+j \end{bmatrix} \Rightarrow$$

$$T_4(x) = c_0 + c_1 \cdot e^{jx} + c_2 \cdot e^{2jx} + c_3 \cdot e^{3jx}$$
$$T_4(x) \overset{!}{=} 5 + (1-j)e^{jx} + e^{2jx} + (1+j)e^{3jx}$$



Check

$a_0 = 2c_0 = 10$

$a_1 = 2\mathrm{Re}(c_1) = 2$

$b_1 = -2\mathrm{Im}(c_1) = +2$

$a_2 = 2c_2 = 2$

$$\tilde{T}_4(x) = \frac{a_0}{2} + a_1 \cdot \cos x + b_1 \cdot \sin x + \frac{a_2}{2}\cos 2x$$
$$\overset{!}{=} 5 + 2\cos(x) + 2\sin(x) + \cos(2x)$$

↑ real trigonometric polynom

↑ von Folie, keine Ahnung warum genau das gilt. – gem. Bürgler sei dies eine Definition aus einem Buch.