→ Eigen values / Vectors

$$Ax = b$$

$A \to$ function or operation

$x \to$ input

$b \to$ output

When output = scalar multiple of input

$$\boxed{Ax = \lambda x}$$

$\lambda \to$ Eigen value

Eigen vector = x_i for each λ_i

3

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \lambda = 1$$

$$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \lambda = -1$$

→ Solve $Ax = \lambda x$

$$\underline{\underline{(A - \lambda I)}}\, x = 0$$

matrix that is shifted version of A, must be singular (since $x \neq 0$)

$$\therefore \det(A - \lambda I) = 0$$

→ Ex. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 4)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 4, \ \lambda_2 = 2 \ //$$

E.Vector for $\lambda_1 = 4$

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similarly, for $\lambda_2 = 2$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Alternatively, for $\lambda_1 = 4$, $\lambda_2 = 2$

$$\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ vector 1

$$\begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

→ vector 2

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(a) Properties

(1) $n \times n$ matrix will have n E.values

(2) Trace = Sum of diagonals $= \sum_i \lambda_i$

(3) $\det(A) = \prod_i \lambda_i$ (product of λs)

→ For rotation matrix, no vector remains same, E.values are complex.

⑤ Diagonalization

say $S = \begin{bmatrix} | & | & | & & | \\ EV_1 & EV_2 & EV_3 & \cdots & EV_n \\ | & | & | & & | \end{bmatrix}$

What's A.S ?

$$A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 & \lambda x_2 & \cdots & \lambda x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & \cdot & 0 \\ 0 & \lambda_2 & & & \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & & & & \lambda_n \end{bmatrix}$$

$$= S \Lambda$$

Diagonal matrix

$\therefore \quad AS = S\Lambda$

$\therefore \quad S^{-1}AS = S^{-1}S\Lambda = \Lambda$

$\therefore \quad \boxed{A = S\Lambda S^{-1}}$

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda(S^{-1}S)\Lambda S^{-1}$$
$$= S\Lambda^2 S^{-1}$$

$$\boxed{A^k = S\Lambda^k S^{-1}}$$

Markov Matrices

$$A = \begin{bmatrix} 0.1 & 0.01 & 0.3 \\ 0.2 & 0.99 & 0.3 \\ 0.7 & 0 & 0.4 \end{bmatrix}$$

① All entries ≥ 0
② Columns add up to 1.

$$\boxed{\begin{array}{l} \text{One of the E.values} = 1 \\ \text{All other E.values} < 1 \end{array}}$$

Why $\lambda_1 = 1$?

$$A^T = \begin{bmatrix} \text{---} r_1 \text{---} \\ \text{---} r_2 \text{---} \\ \vdots \\ \text{---} r_n \text{---} \end{bmatrix} \quad \left.\begin{array}{l} \bullet \sum_i r_1 = 1 \\ \sum_i r_2 = 1 \\ \sum_i r_n = 1 \end{array}\right\} \text{each row adds to 1}$$

$$\therefore \quad A^T \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow \lambda_1 = 1$$

Since E.vectors of A and A^T the same.

③ Appⁿ of Markov Matrix

$$u_{k+1} = A u_k$$

$\uparrow$ Markov

③ Ex. churn of people between CS and ECE

$$\begin{bmatrix} u_{CS} \\ u_{ECE} \end{bmatrix}_{t=k+1} = A \begin{bmatrix} u_{CS} \\ u_{ECE} \end{bmatrix}_{t=k}$$

$$\begin{bmatrix} u_{CS} \\ u_{ECE} \end{bmatrix}_{k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{CS} \\ u_{ECE} \end{bmatrix}_{k}$$

i.e., $u_{CS}\Big|_{k+1} = 0.9\, u_{CS}\Big|_k + 0.2\, v_{ECE}\Big|_k$

Question $\Rightarrow$ What is $\begin{bmatrix} u_{CS} \\ u_{ECE} \end{bmatrix}$ at $k=100$?

say $u_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$

$$u_k = \frac{1000}{3} \lambda_1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} \lambda_2^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 1 \quad , \quad \lambda_2 = 0.7$$

$$u_k = \frac{1000}{3} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2000}{3} (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$u_\infty = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_{CS} \\ u_{ELE} \end{bmatrix} = \begin{bmatrix} 2000/3 \\ 1000/3 \end{bmatrix}$$