MAT185 Linear Algebra
Assignment 2

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Preamble: An application of linear algebra to calculus.

Recall the technique of partial fractions decomposition to evaluate the integral of rational functions. For example, suppose we would like to evaluate the integral

$$\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx$$

We look for scalars a, b, and c such that

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

After some algebra, we find that $a = 2$, $b = 4$, and $c = 5$, and therefore,

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{2x+4}{x^2+3} + \frac{5}{x-2}$$

Then,

$$\begin{aligned}\int \frac{7x^2+7}{(x^2+3)(x-2)}\,dx &= \int \frac{2x+4}{x^2+3}\,dx + \int \frac{5}{x-2}\,dx \\ &= \ln(x^2+3) + \frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + 5\ln(x-2) + C\end{aligned}$$

where C is a constant.

In Question 1, we will use the theory of basis and dimension in linear algebra to explain why the partial fractions decomposition

$$\frac{7x^2+7}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

exists, thereby allowing us to solve the integral.

1. Let

$$V = \left\{ \frac{dx^2+ex+f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(a) Prove that $\dim V = 3$. Then, explain why a partial fractions decomposition of the form

$$\frac{dx^2+ex+f}{(x^2+3)(x-2)} = \frac{ax+b}{x^2+3} + \frac{c}{x-2}$$

is consistent with the dimension of V.

Use the page 3 to answer this question.

1(a)

By definition of dimension ($dimV = 3$), there are three vectors in any of V's bases. To prove this we will show there exists three linearly independent vectors that span V.
Assume these three vectors are $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \in \mathbf{V}$:

$$\mathbf{v_1} = \frac{x^2}{(x^2+3)(x-2)}, \mathbf{v_2} = \frac{x}{(x^2+3)(x-2)}, \mathbf{v_3} = \frac{1}{(x^2+3)(x-2)},$$

i. To show linear independence we will use its definition:

$$\begin{aligned}
&\lambda_1 \mathbf{v_1} + \lambda_\mathbf{2} \mathbf{v_2} + \lambda_\mathbf{3} \mathbf{v_3} = \mathbf{0} \\
&\lambda_1 \frac{x^2}{(x^2+3)(x-2)} + \lambda_2 \frac{x}{(x^2+3)(x-2)} + \lambda_3 \frac{1}{(x^2+3)(x-2)} = \mathbf{0}
\end{aligned}$$

Multiplying both sides by the denominator:

$$\lambda_1 x^2 + \lambda_2 x + \lambda_3 = \mathbf{0}$$

???The only value satisfying this equation is $\lambda_1 = \lambda_2 = \lambda_3 = 0$ Therefore, $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent.
ii. Proving span

$$\begin{aligned}
span\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \{\mathbf{v} | \mathbf{v} = \sum_{i=1}^{n} \lambda_i v_i, \lambda_i \in \mathbb{R}\} \\
&\lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \lambda_3 \mathbf{v_3} = \mathbf{0} \\
&= \lambda_1 \frac{x^2}{(x^2+3)(x-2)} + \lambda_2 \frac{x}{(x^2+3)(x-2)} + \lambda_3 \frac{1}{(x^2+3)(x-2)} = \mathbf{0} \\
&= \frac{\lambda_1 x^2 + \lambda_2 x + \lambda_3}{(x^2+3)(x-2)}
\end{aligned}$$

By choosing $\lambda_1 = d, \lambda_2 = e, \lambda_3 = f$, we see any vector $\frac{dx^2+ex+f}{(x^2+3)(x-2)}$ can be formed. Therefore $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ spans V.
iii. Since $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ span V and are linearly independent, no vectors need to be removed or added to the set. By Proof of Construction, a basis has been formed by $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ where $dim\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = 3$ since there are three linearly independent vectors.
Showing $\frac{ax+b}{x^2+3} + \frac{c}{x-2} | a, b, c \in \mathbb{R}$ also has dimension of 3:

$$\begin{aligned}
\frac{ax+b}{x^2+3} + \frac{c}{x-2} &= \frac{(ax+b)(x-2)}{(x^2+3)(x-2)} + \frac{c(x^2+3)}{(x^2+3)(x-2)} \\
&= \frac{(c+a)x^2}{(x^2+3)(x-2)} + \frac{(b-2a)x}{(x^2+3)(x-2)} + \frac{(3c-2b)}{(x^2+3)(x-2)}
\end{aligned}$$

This is the form of a linear combination of three vectors. Where $c + a = d$, $b - 2a = e$, $3c - 2b = f$, the linear combination spans V. To show linear independece, we can multiply by the common denominator and represent the coeffecient equations as a matrix.

$$\mathbf{Ax} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Matrix $\mathbf{A}$ has 3 linearly independent columns, and therefore a dimension of 3. As a result, the partial fraction decomposition forms three linearly independet vectors, which agree with the $dimV = 3$.

1. Let

$$V = \left\{ \frac{dx^2 + ex + f}{(x^2+3)(x-2)} \mid d, e, f \in \mathbb{R} \right\}$$

We define vector addition and scalar multiplication in V by the usual function addition and scalar multiplication. Then V is vector space.

(b) Using that dim $V = 3$ from part (a), explain why we do not expect a partial fractions decomposition of the form

$$\frac{dx^2 + ex + f}{(x^2+3)(x-2)} = \frac{a}{x^2+3} + \frac{b}{x-2}$$

to exist.

By proving $dim \frac{a}{x^2+3} + \frac{b}{x-2} < 3$, we know a solution of this form can not span V since it can not have a smaller dimension than the basis.

$$\begin{aligned}
\frac{a}{x^2+3} + \frac{b}{x-2} &= \frac{a(x-2)}{(x^2+3)(x-2)} + \frac{b(x^2+3)}{(x-2)(x^2+3)} \\
&= \frac{ax - 2a}{(x^2+3)(x-2)} + \frac{bx^2 + 3b}{(x-2)(x^2+3)} \\
&= \frac{ax - 2a + bx^2 + 3b}{(x^2+3)(x-2)} \\
&= \frac{bx^2}{(x^2+3)(x-2)} + \frac{ax}{(x^2+3)(x-2)} + \frac{-2a+3b}{(x^2+3)(x-2)}
\end{aligned}$$

Comparing to the form of V and multiplying out the denominator:

$$\frac{dx^2}{(x^2+3)(x-2)} + \frac{ex}{(x^2+3)(x-2)} + \frac{f}{(x^2+3)(x-2)} = \frac{bx^2}{(x^2+3)(x-2)} + \frac{ax}{(x^2+3)(x-2)} + \frac{-2a+3b}{(x^2+3)(x-2)}$$
$$dx^2 + ex + f = bx^2 + ax + (-2a + 3b)$$

To span V, we must show $b = d$, $a = e$, $-2a + 3b = f$. Representing these equations in Matrix form:

$$\mathbf{Ax} = \mathbf{b}$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Matrix $\mathbf{A}$ has a linearly dependent column of 0s, and therefore $dimA < 3$. This means the linear combination of the partial fraction decomposition also has a dimension less than 3.

$$dim(\{\frac{a}{x^2+3}, \frac{b}{x-2}\}) < 3$$

Therefore the partial fraction decomposition can not span V and the equality does not hold. This is numerical seen as any case where $f \neq 3b - 2a$, and there will be no solution to the $\mathbf{Ax} = \mathbf{b}$ system.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, we will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(a) If $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ all belong to W_1 explain why $W_1 \cap W_2$ contains a plane.

Given u_1, u_2, u_3 form a basis for W_1, the must be linearly independent and span W_2 by definition. Given $u_1, u_2, u_3 \in W_1$, $W_2 \in W_1$ since any vector $\mathbf{x} \in W_2$ and $\mathbf{x} \in span\{u_1, u_2, u_3\} \forall \mathbf{x}$ by closure under vector addition and scalar multiplication. If $W_2 \in W_1$, it follows $W_1 \cap W_2 = W_2$ and therefore $u_1, u_2, u_3 \in W_1 \cap W_2$.

The dimension of plane is 2, as it is a surface spanned by two linearly independent vectors. On other words, a plane's bases are formed by two linearly independent vectors.

Take two linearly independent vectors $u_1, u_2 \in W_1 \cup W_2$, which are also linearly independent being a basis for W_2. $span\{u_1, u_2\}$ forms a plane because it is all linear combinations of two linearly independent vectors and $span\{u_1, u_2\} \in W_1 \cup W_2$ as previously stated.

(b) Now suppose that not all of $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ belong to W_1. Say $\mathbf{u}_1 \notin W_1$. Prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}_1$ is a basis for $\mathbb{R}^4$.
Since w_1, w_2, w_3 form a basis for W_1, every vector $w \in W$ can be expressed by a linear combination:

$$w = \lambda_1 w1 + \lambda_2 w_2 + \lambda_3 w3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$$

Assume u_1 can be formed by a linear combination of w_1, w_2, w_3.

$$u = \lambda_1 w1 + \lambda_2 w_2 + \lambda_3 w3$$

Then, $u_1 \in W_1$ since all linear combinations of the basis vectors must be in W. Given $u_1 \notin W_1$, by contradiction, u_1 can not be formed by a linear combination of w_1, w_2, w_3 or $u_1 \notin span\{w_1, w_2, w_3\}$.
Consider the following linear combination where $\alpha_1, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and we assume $\alpha \neq 0$:

$$\begin{aligned}
\alpha_1 u_1 + \lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 &= 0 \\
\lambda_1 w_1 + \lambda_2 w_2 + \lambda_3 w_3 &= -\alpha_1 u_1 \\
-\frac{\lambda_1}{\alpha_1} w_1 - \frac{\lambda_2}{\alpha_1} w_2 - \frac{\lambda_3}{\alpha_1} w_3 &= u_1
\end{aligned}$$

Since $-\frac{\lambda_x}{\alpha_1} \in \mathbb{R}$, this is equation is a linear combination of w_1, w_2, w_3 that forms u_1. By the previous contradiction, this is not possible, and therefore, $\alpha = 0$. Additionaly, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ since they are linearly independent by the definition of a basis.
Therefore, u_1, w_1, w_2, w_3 are four linearly independent vectors. Since $dim\mathbb{R}^4 = 4$, there are four linearly independent vectors in the basis of $\mathbb{R}^4$. By the Algebra Triangle Theorm, the set $\{u_1, w_1, w_2, w_3\}$ must be a basis for W_1 since the four vectors are linearly independent.

2. Suppose that W_1 and W_2 are both three dimensional subspaces of $\mathbb{R}^4$. In this question, you will show that $W_1 \cap W_2$ contains a plane.

Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ be a basis for W_1, and let $\mathbf{u_1}, \mathbf{u}_2, \mathbf{u}_3$ be a basis for W_2.

(c) Using the assumption and conclusion from part (b), find two vectors in $W_1 \cap W_2$ and then prove that these two vectors span a plane.

From (b), there can only be 4 linearly independent vectors in $\mathbb{R}^4$. Since $W_1 \sqsubset \mathbb{R}^4$ and $W_2 \sqsubset \mathbb{R}^4$, only 4 vectors of $w_1, w_2, w_3, u_1, u_2, u_3$ can be linearly independent.

We know w_1, w_2, w_3, u_1 are linearly independent from (b). Therefore, u_2 and u_3 must be linearly dependent to two different vectors from w_1, w_2, w_3, u_1 since u_2 and u_3 are linearly independent themselves. We also know u_2 and u_3 can not be dependent to u_1 since by the definition of a basis, u_1, u_2, u_3 are linearly independent. Futhermore, $u_1 \notin W_1 \cap W_2$ given $u_1 \notin W_1$.

Let's pick w_2 to be linearly dependent to u_2 and w_3 to be linearly dependent to u_3. Since W_1, W_2 is closed under scalar multiplication, where $\alpha \in \mathbb{R}^4$, $\alpha w_2 \in W_1$ and $u_2 \in W_2$. As w_2 and u_2 are linearly dependent, u_2 can be expressed as $u_2 = \alpha w_2$. Therefore $u_2 \in W_1$. By the same arguement with u_3 and w_3, $u_3 \in W_1$.
By the reverse, $w_2 = \frac{1}{\alpha} u_2 \in W_1$. Therefore, $w_2 \in W_1$ and $w_3 \in W_2$.

Since $w_2, w_3, u_2, u_3 \in W_1$ and $w_2, w_3, u_2, u_3 \in W_2$, $w_2, w_3, u_2, u_3 \in W_1 \cup W_2$.

A $\mathbb{R}^4$ plane through the origin is formed from two linearly independent vectors $\in \mathbb{R}^4$ as dimplane $= 2$. To satisfy the condition these two vectors must be $\in W_1 \cap W_2$, either $\{w_2, w_3\}$ or $\{u_2, u_3\}$ can be picked.

Finally $span\{w_2, w_3\} \in W_1 \cap W_2$, and being two linearly independent vectors, must be a basis for a plane by the Algebra Triangle Theorm. Therefore, they must also span a plane by the definition of basis.