

# Constructing a Pendulum to Determine its Period and Q Factor

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## 1 Introduction

By designing a two string pendulum attached to a yellow ball with a variable length, a given model is tested against empirical results to better inform the accuracy of the model.

The initial angle versus period, amplitude over time, length versus period and length versus Q Factor are analyzed with the aid of video tracking software Tracker[1].

## Equations

A power function describes the relationship between pendulum length ( $L$ ) and period  $T$ .

$$T = pL^q \quad (1)$$

The range of parameters  $p = 1.98 \pm 0.01 m^{-1}s$  and  $q = 0.474 \pm 0.007$  are empirically found to fall slightly outside the expected values of  $p = 2$  and  $q = 0.5$  given by [2].

A pendulum's initial angle release is found to have an even powered quadratic relationship with its period.

$$T = T_0(1 + B\theta + A\theta^2) \quad (2)$$

The odd power term  $B$  is found to be experimentally zero, while  $T_0 = 1.337 \pm 0.001s$ , and,  $A = 0.043 \pm 0.001 rad^{-2}$ . No asymmetry is found. Amplitude and period are not

found to be independent unless the small angle range of  $-0.5 \leq \theta \leq 0.5 rad$ .

The damped harmonic oscillator (Equation 3) accurately gives the angle over time ( $\pm 0.02 rad$ ) provided an initial angle ( $\theta_0$ ), the time ( $t$ ), the decay constant ( $\tau$ ), the period ( $T$ ) and a time offset ( $\phi$ ).

$$\theta(t) = \theta_0 e^{-t/\tau} \cos\left(2\pi \frac{t}{T} + \phi\right) \quad (3)$$

Equation 3 gives the decay constant  $\tau$ . This is used to calculate the Q Factor[2], a measurement of how quickly the pendulum decays.

$$Q = \pi \frac{\tau}{T} \quad (4)$$

The method of oscillations ( $Q = 84 \pm 2$ ) is used in combination with the equation ( $Q = 86.0 \pm 0.1$ ), providing the same Q value within uncertainties for a set angle of release and pendulum length.

It is found that the Q Factor has no relationship with the pendulum's length with low confidence due to high uncertainties ( $> 9$ ).

## 2 Setup and Methods

The pendulum was constructed from two light strings with variable lengths that support a yellow foam ball with a radius of  $5.95 \pm 0.01 cm$ .

By using two strings, oscillation in and out of the plane is reduced, which would otherwise affect the decay.

Two hooks screwed into the structure were used to tie the string down with a simple knot. Tying the knot on the side of the hook rather than the bottom was found to reduce slippage.

The same two strings were used to measure all relationships ranging from lengths  $17.6$  to  $80.5 \pm 0.5\text{cm}$ . Any extra slack was tied down at the top of the structure to maintain tension and reduce knot slippage.



Figure 1: A foam ball is tied to two strings and each string is held in place by a knot on a hook. The remaining string is held in tension with tape at the top of the structure. The entire structure is severely taped to the wall.

All videos were taken using an iPad Air camera at 60fps. This was held in place on a desk with a weight to hold it upright (zero angle with the horizontal). It was placed at a distance where the whole apparatus could be viewed while the pendulum swung.

## 2.1 Period

A period was measured by counting the number of frames to complete a cycle. A cycle begins when the pendulum reaches its lowest point. The pendulum always passes through this point, which is also its equilibrium. Since other points may not be reached as the pendulum continues to oscillate and decay, tracking the period over time is less viable at other points.

All presented period values are an average of the first three periods.

When comparing the initial angle to the period, 14 measurements were taken, 7 from positive angles and 7 from negative angles. The angle of release was measured live to give a rough estimate and ensure a range of values was measured. In the post analysis software Tracker, accurate values of the initial angle were measured with a protractor. Angles measured ranged from  $-1.36$  to  $1.42 \pm 0.05$  rad with each trial being changed by  $0.17 \pm 0.08$  rad.

The length of the pendulum was fixed for these trails at  $L = 43.5 \pm 0.3$  cm. All length measurements were taken using a 30 cm ruler. The ball was held at the top and bottom and released by hand.

## 2.2 Q Factor and Angle

All Q Factor Measurements were taken within the range of small angles (maximum angle of  $\theta = 26.5 \pm 0.05$  rad). All videos were recorded for at least 60s.

To measure the angle over time, the Auto-tracker feature in Tracker was used. The template was set to Evolve: 40%, Tether: 5%, Automark: 4 and a Step Size: 5. By using a yellow ball with white tape and a marked black dot, the reference image of the contrasting black dot was selected. This drastically improves the accuracy tracking with fewer low confidence frames (where the soft-

ware is unsure if it's tracking the right object). An axis was placed at the pivot point as a reference point to measure angles.

Using angle measurements, the amplitude and the number of oscillations are calculated from tracker data.

When comparing pendulum length to Q Factor and period, 8 lengths were considered ranging from  $17.6$  to  $80.5 \pm 0.5$  cm.

### 3 Results and Analysis

#### Initial Angle and Period

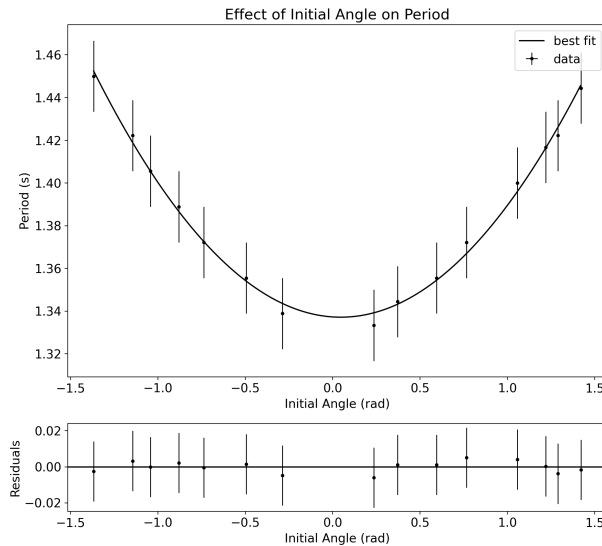


Figure 2: The plot shows how the period varies based on the initial angle of release for a pendulum of  $L = 43.5 \pm 0.5$  cm. The data was fit to the quadratic Equation 2, with parameters  $T_0 = 1.337 \pm 0.001s$ ,  $B = -0.003 \pm 0.007rad^{-1}$ ,  $A = 0.043 \pm 0.001rad^{-2}$ . A small angle range between  $-0.5 \leq \theta \leq 0.5rad$  is present.

From the fit, the odd term  $B$  is found to have a value smaller than its uncertainty, and therefore is experimentally zero. As a result, the pendulum is symmetric for positive and negative angles of release.

The quadratic fit is good since it captures all data points within their uncertainties, and has residuals  $< 0.02$ . The uncertainties of parameters are also low ( $\leq 0.001$ ), giving confidence to the fit. The small limiting measurement uncertainty of the period  $\pm 0.01s$  and angle  $\pm 0.008rad$  support the reasonableness of the results.

Considering all angles, the period varies outside each data's respective uncertainty. As the angle of initial release increases, and therefore the amplitude, the period also increases. Therefore, there is some dependence on amplitude and angle.

However, a small angle range is present:  $-0.5 \leq \theta \leq 0.5rad$ , which lies within uncertainties for the plotted values in the angle range. Since the period changes less drastically in this range, through the small angle approximation, releasing the ball within this range allows for period values independent of amplitude. Plotting the four points in this range,  $A = 0.008 \pm 0.001rad^{-2}$ , which will be considered small since it falls within the period measurement uncertainty of  $\pm 0.01s$ . This confirms the absence of the quadratic term in Equation 2, leaving a constant  $T = 1.328 \pm 0.002s$ .

#### Calculating Q Factor

The damped harmonic model given in Figure 3 is a good fit since each data point's uncertainties are captured by the fit and residuals are smaller than 0.02. The uncertainties of fitted parameters are also low ( $\leq 0.1$ ), giving confidence to the fit. The small limiting measurement uncertainty of the angle ( $\pm 0.008rad$ ) support the reasonableness of the results.

From the fitted parameters,  $Q$  can be calculated using Equation 4.  $Q = \pi \frac{36.3 \pm 0.1}{1.32489 \pm 0.00002} = 86.0 \pm 0.1$ .

The difference between maximum and minimum angles in each oscillation gives the am-

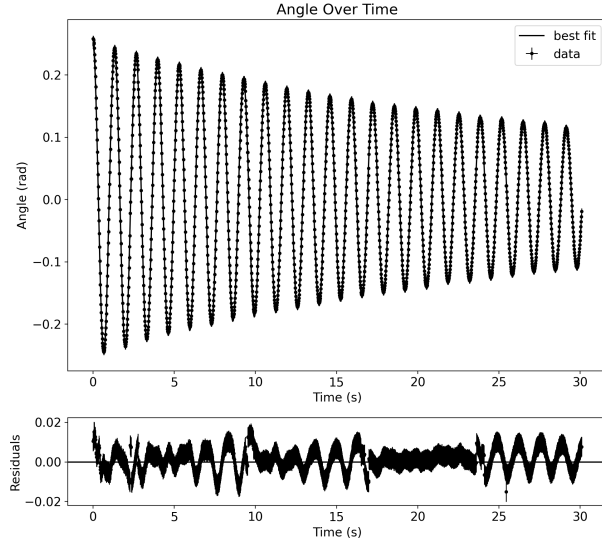


Figure 3: The plot shows how the position of the pendulum, described as an angle from the vertical, changes over time. The initial angle of release was chosen to be small,  $\theta_0 = 0.235 \pm 0.005 \text{ rad}$ . The best fit parameters for Equation 3 are  $\theta_0 = 0.2477 \pm 0.0004 \text{ rad}$ ,  $\tau = 36.3 \pm 0.1 \text{ s}$ ,  $T = 1.32489 \pm 0.00002 \text{ s}$ . A zoomed-in version of the graph with visible error bars is given in the appendix.

plitude. With amplitudes, we can use the number of oscillations to reach an amplitude  $e^{-\frac{\pi}{k}}\%$  of its initial amplitude. The number of oscillations gives  $\frac{Q}{k}$ . The mean Q value from counting oscillation in the k range [2, 8] is  $84 \pm 2$ .

The two methods of calculating Q yield the same value within uncertainties and therefore have evidence they agree with each other

## Length and Period

The power fit captures all data points within their uncertainties with low residuals ( $< 0.025$ ). The log-log plot performs even better at fitting the data. The low uncertainties of fitted parameters ( $\leq 0.1$ ) also give confidence to the fit. Therefore, the power relationship between length and period of a pendulum is accurate. The small limiting measurement

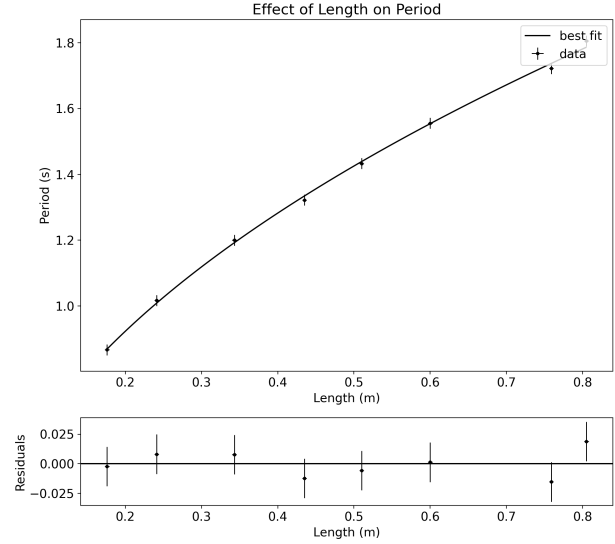


Figure 4: The plot shows how the period changes based on a pendulum's length. It is fitted to Equation 1 with parameters  $p = 1.98 \pm 0.01 \text{ m}^{-1} \text{ s}$  and  $q = 0.474 \pm 0.007$

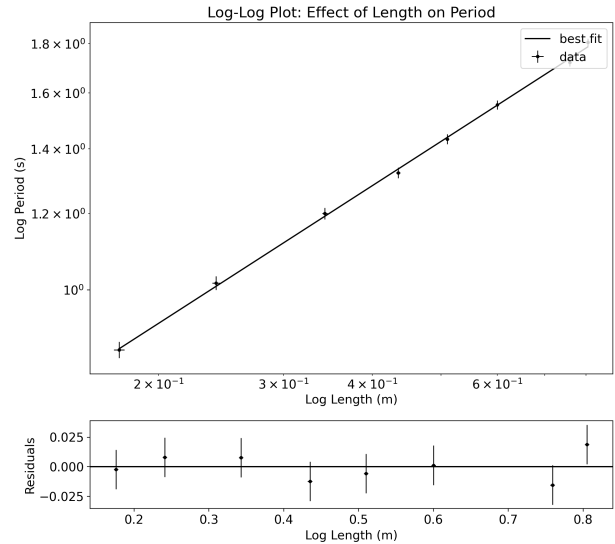


Figure 5: Figure 5: The plot displays how the period changes based on a pendulum's length as a log-log plot fit to a linear function. The function of  $y = ax + b$  has parameters  $a = 1.98 \pm 0.01$  and  $b = 0.474 \pm 0.007$ .

uncertainty of the length  $\pm 0.003 \text{ m}$  and period  $\pm 0.01 \text{ s}$  support the reasonableness of the results.

Given the parameters p and q, and their low

range of fitted uncertainties, the parameters differ from the expected values of  $p = 2$  and  $q = 0.5$  given by [2]. One reason for this is the fitted uncertainties are smaller than measurement uncertainties.

### 3.1 Q Factor and Length

Since the uncertainty for calculating Q Factors by Equation 4 ( $\pm 0.1$ ) is less than the uncertainty by counting oscillations ( $\pm 2$ ), Equation 4 is used.

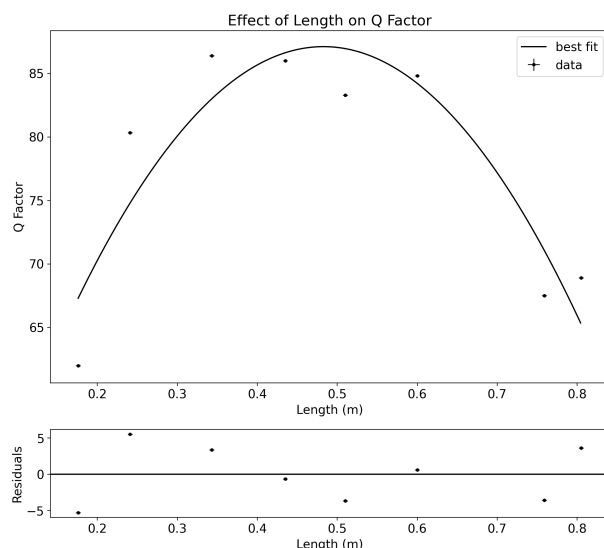


Figure 6: The pendulum's Q Factor and Length are plotted and fitted using a quadratic function  $ax^2 + bx + c$ . Where  $a = -210 \pm 42m^{-2}$ ,  $b = 200 \pm 43m^{-1}$ ,  $c = 30 \pm 9$ . The y uncertainty is given by  $\tau$ 's fitted uncertainty of  $\pm 0.1$ .

Given the data points, a quadratic fit most resembles the trend between the Q Factor and length. Despite the similar trend, the best fit does not capture the values within their y uncertainties ( $\pm 0.1$ ). In addition, the residuals of the plot are high ( $> 5$ ) and outside the uncertainties. The uncertainty of the coefficients is also high since they are above the measured uncertainties. This gives low confidence in the fit.

A reason this may have occurred is the slight

variation in angle ( $\pm 0.05rad$ ) for each length measurement. This variation may have an effect on  $\tau$  unlike how the small approximation was appropriate for the period  $T$ .

### 3.2 Uncertainties

For Figure 3, measurement uncertainties exist for the initial angle and identifying when the pendulum has reached its lowest point. The measurement uncertainty due to the precision of device graduation for the initial angle is  $\pm 0.1deg$ , which is approximately  $\pm 0.001rad$ .

There is also uncertainty around where the protractor is placed in Tracker. Looking at the range in angles for various measurements at the pivot point to the string gives an uncertainty of  $\pm 0.008rad$ , which is larger than the previous uncertainty and therefore will be used. This is calculated by moving the protractor across various points on the vertices (the pivot point and the ball).

The period uncertainty is  $\pm 1frame$  or  $\pm 0.01s$  (at 60fps) since empirically, in each video, the instant at which the pendulum reaches the bottom lies between two frames. The time between two frames is  $\frac{1}{60}s$ . A higher fps would help reduce this uncertainty.

Measuring the apparatus uncertainty (how much results change given the same trial conditions) was difficult due to live measurement errors and lack of precision. Three independent trials with angles between  $0.73 \pm 0.01rad$  and  $0.76 \pm 0.01rad$  yielded the same period values. Therefore, the experimental setup is not the largest uncertainty.

For tracking the angle over time, the angle measured by Tracker has measurement uncertainties. Since Autotracker is being used, the point which is tracked on the ball changes from frame to frame. At 60fps, a random sample of  $n = 10$  frames is taken and angles are adjusted to the center. In addition, the coordinate axis and identification of the ball's

centre itself hold uncertainties in angle measurement. In adjusting each of these uncertainties to find the range of angle variation, the uncertainty given is  $\pm 0.009 \text{ rad}$ . This is the largest uncertainty. This uncertainty applies to Figure 3.

Amplitude uncertainty stems from the angle that could be travelled in between frames when at its highest extremes. It is found that the ball is in between a maximum of 4 frames during its highest extremes. This uncertainty is already small since the ball's angular displacements are smallest at the highest points. The uncertainty given by the angular velocity  $\omega = 0.002$  over a 4 frame interval at 60fps is  $\pm 0.01 \text{ rad}$ . However, measurements also differ from tracking uncertainty previously mentioned but are smaller than the uncertainty from potential displacement. The number of oscillations is a counted value and therefore has no uncertainty.

The Q Factor by number of oscillations has a mean error of  $\pm 2$  using  $\sigma = 6$  and  $N = 7$ .

Length uncertainties arise from ruler precision and measurement errors. When measuring the same pendulum length three times, the range is found to be  $\pm 0.3 \text{ cm}$ , which is greater than the precision uncertainty  $\pm 0.01 \text{ cm}$ . This is mainly because a 30cm ruler was used, and therefore, the ruler had to be moved for some measurements. A measuring tape, rather than a rigid ruler that spans greater than string length could be used to reduce error.

To reduce uncertainties related to the initial angle, clear markings can be placed on the pivot point and ball to ensure the angle is measured in the same place, reducing the amount of angle variation. A more graduated protractor would also help but likely have significant costs.

Additional uncertainties involve slightly rotated videos since it has already been minimized by post analysis (by slightly rotating images to become horizontal). Movement in and out of the video plane has been reduced

by the two-string design.

## 4 Conclusion

A pendulum's period is dependent on the initial angle release based on the quadratic relationship  $T = 1.337 \pm 0.001 \times (1 + 0.043 \pm 0.001\theta^2)$ . The period is however independent of amplitude in the small angle range  $-0.5 \leq \theta \leq 0.5 \text{ rad}$  where the squared coefficient  $A \approx 0$ .

The angle over time is accurately modelled by the damped harmonic oscillator (Equation 3).  $\theta(t) = 0.2477 \pm 0.0004 \times e^{\frac{t}{36.3 \pm 0.1}} \times \cos(2\pi \frac{t}{1.32489 \pm 0.00002})$ . Angles are calculated  $\pm 0.02 \text{ rad}$  from their real values and the low coefficient uncertainties support the model.

This information was used to find a Q Factor. The analytical value using Equation 4 ( $Q = 86.0 \pm 0.1$ ) and mean value ( $Q = 84 \pm 2$ ) from counting the number of oscillations for various decaying amplitude values are found to agree.

The period of a pendulum is found to change based on its length where  $L = 1.98 \pm 0.01 L^{0.474 \pm 0.007}$ . This differs slightly from the expected values of  $p = 2$  and  $q = 0.5$ .

The above relationships have high confidence due to uncertainties  $\leq 0.1$ . The major experimental uncertainty that could be reduced is the angle measurements  $\pm 0.008$ . Precision of angle release may also benefit data related to Q Factor and Length, which is found to have no relationship with high uncertainties with fitted parameters.

## Appendix

## References

- [1] Brown, D. (2023). Tracker. <https://physlets.org/tracker/>

[2] Wilson, B. (2023). PHY180 Pendulum Project Guidelines. [https://q.utoronto.ca/courses/324650/files/27269654?module\\_item\\_id=4893671](https://q.utoronto.ca/courses/324650/files/27269654?module_item_id=4893671)

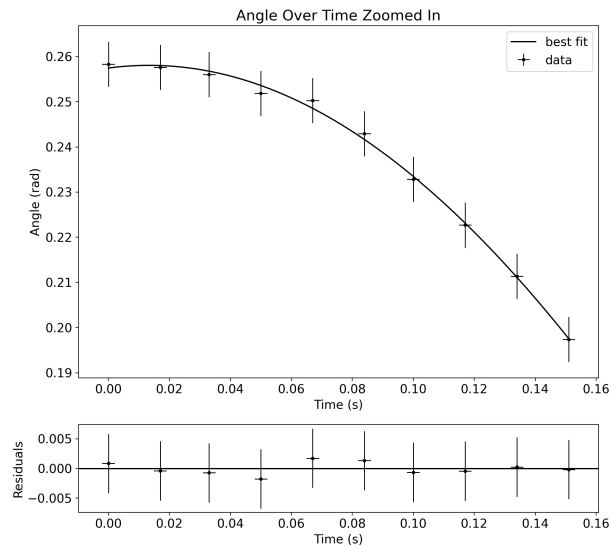


Figure 7: A zoomed-in version of Figure 3 with visible error bars. The angle uncertainty is  $\pm 0.008 \text{ rad}$

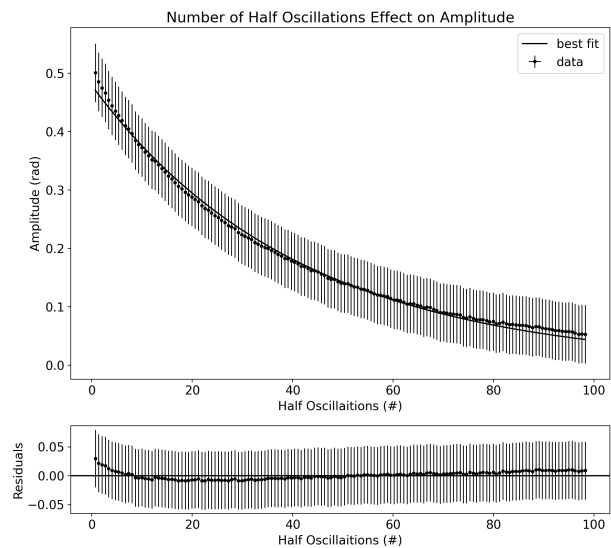


Figure 8: This graph confirms how the amplitude decays exponentially as the number of oscillations increases. The amplitude is recorded every time a maximum and minimum angle is reached, which is only half an oscillation.