MAT185 Linear Algebra
Assignment 1

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) Tanakorn Wongaree

2) Adam Omarali

1. A *vector spyce* is a set V together with two operations called *vector addition* and *scalar multiplication* such that the following nine *axioms* hold:

(i) For all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} \in V$

(ii) For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$

(iii) There exists a vector $\mathbf{0} \in V$ with the property that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all vectors $\mathbf{x} \in V$

(iv) For each vector $\mathbf{x} \in V$, there exists a vector $-\mathbf{x} \in \mathbf{V}$ with the property that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$

(v) For all vectors $\mathbf{x} \in V$, and scalars $c \in \mathbb{R}$, $c\mathbf{x} \in V$

(vi) For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$

(vii) For all vectors $\mathbf{x}, \mathbf{y} \in V$, and scalars $c \in \mathbb{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$

(viii) For all vectors $\mathbf{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$

(ix) For all vectors $\mathbf{x} \in V$, $c\mathbf{x} = \mathbf{0}$ implies that either $c = 0$ or $\mathbf{x} = \mathbf{0}$

Prove that a *vector spyce* is a vector space.

To prove that *vectorspyce* is a vector space, we need to show that eight axioms for vector space hold. Most of it holds directly because of the axiom of *vectorspyce*.

Axiom [**AI**] holds as a result of **(i)**. Axiom [**AII**] holds because of **(ii)**. Axiom [**AIII**] is satisfied because of **(iii)**. Axiom [**AIV**] follows directly from **(iv)**. Axiom [**MI**] holds as a result of **(v)**. Axiom [**MII**] applies because of **(vi)**. Axiom [**MIII**] holds because of **(vii)** and **(viii)**.

The only axiom that cannot be applied directly is [**MIV**], For all vectors $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$.

Proof. Let $\mathbf{x} \in \mathbf{V}$.

$$\begin{aligned}
1(1\mathbf{x} + (-\mathbf{x})) &= 1(1\mathbf{x}) + \mathbf{1}(-\mathbf{x}) && \text{by } \textbf{(vii)} \\
&= (1 \cdot 1)\mathbf{x} + \mathbf{1}(-\mathbf{x}) && \text{by } \textbf{(vi)} \\
&= 1\mathbf{x} + \mathbf{1}(-\mathbf{x}) \\
&= 1(\mathbf{x} + (-\mathbf{x})) && \text{by } \textbf{(vii)} \\
&= 1 \cdot 0 && \text{by } \textbf{(iv)} \\
&= 0
\end{aligned}$$

Since $1(1\mathbf{x} + (-\mathbf{x})) = \mathbf{0}$ and $1 \neq 0$, from **(ix)** we know that

$$1\mathbf{x} + (-\mathbf{x}) = \mathbf{0} \tag{1}$$

Moreover,

$$\begin{aligned}
1\mathbf{x} &= 1\mathbf{x} + \mathbf{0} && \text{by } \textbf{(iii)} \\
&= 1\mathbf{x} + ((-\mathbf{x}) + (-(-\mathbf{x}))) && \text{by } \textbf{(iv)} \\
&= (1\mathbf{x} + (-\mathbf{x})) + (-(-\mathbf{x})) && \text{by } \textbf{(ii)} \\
&= \mathbf{0} + (-(-\mathbf{x})) && \text{by } (1) \\
&= (\mathbf{x} + (-\mathbf{x})) + (-(-\mathbf{x})) && \text{by } \textbf{(iv)} \\
&= \mathbf{x} + ((-\mathbf{x}) + (-(-\mathbf{x}))) && \text{by } \textbf{(ii)} \\
&= \mathbf{x} + \mathbf{0} && \text{by } \textbf{(iv)} \\
&= \mathbf{x} && \text{by } \textbf{(iii)}
\end{aligned}$$

□

Because the eight axioms of vector space hold, a *vector spyce* is a vector space.

2. Let $V = \{a_0 + a_1x + a_2x^2 \mid a_0 > 0\}$. Define vector addition in V by

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0b_0) + (a_1 + b_1 + 4)x + (a_2 + b_2)x^2$$

and scalar multiplication in V by

$$c(a_0 + a_1x + a_2x^2) = a_0^c + (ca_1 + 4(c-1))x + ca_2x^2$$

Verify that axiom MIII. (Medici, pp104) holds in V.

To verify axiom MIII, we must prove two statements:

(i) For all vectors $\mathbf{a}, \mathbf{b} \in V$, and scalars $c \in \mathbb{R}$, $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

Proof.

$$\begin{aligned} c(\mathbf{a} + \mathbf{b}) &= c((a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)) \\ &= c(a_0b_0 + (a_1 + b_1 + 4)x + (a_2 + b_2)x^2) \\ &= (a_0b_0)^c + c(a_1 + b_1 + 4) + 4(c-1))x + c(a_2 + b_2)x^2 \\ &= (a_0b_0)^c + (ca_1 + cb_1 + 4c + 4c - 4)x + c(a_2 + b_2)x^2 \end{aligned}$$

$$\begin{aligned} c\mathbf{a} + \mathbf{cb} &= [(a_0)^c + (ca_1 + 4c - 4)x + c(a_2)x^2] + [(b_0)^c + (cb_1 + 4c - 4)x + c(b_2)x^2] \\ &= c(a_0b_0 + (a_1 + b_1 + 4)x + (a_2 + b_2)x^2) \\ &= (a_0b_0)^c + c(a_1 + b_1 + 4) + 4(c-1))x + c(a_2 + b_2)x^2 \\ &= (a_0b_0)^c + (ca_1 + cb_1 + 4c + 4c - 4)x + c(a_2 + b_2)x^2 \end{aligned}$$

$$\therefore c\mathbf{a} + \mathbf{cb} = c(\mathbf{a} + \mathbf{b})$$

□

(ii) For all vectors $\mathbf{a} \in V$, and scalars $c, d \in \mathbb{R}$, $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

Proof.

$$\begin{aligned} (c + d)\mathbf{a} &= a_0^{c+d} + ((c + d)a_1 + 4(c + d - 1))x + (c + d)a_2x^2 \\ &= a_0^{c+d} + (ca_1 + da_1 + 4c + 4d - 4)x + (c + d)a_2x^2 \end{aligned}$$

$$\begin{aligned} c\mathbf{a} + \mathbf{da} &= [a_0^c + (ca_1 + 4c - 4)x + ca_2x^2] + [(a_0)^d + (da_1 + 4d - 4)x + da_2x^2] \\ &= a_0^c(a_0)^d + [(ca_1 + 4c - 4) + (da_1 + 4d - 4) + 4]x + (ca_2 + da_2)x^2 \\ &= a_0^{c+d} + (ca_1 + 4c + da_1 + 4d - 4)x + (c + d)a_2x^2 \\ &= a_0^{c+d} + (ca_1 + da_1 + 4c + 4d - 4)x + (c + d)a_2x^2 \end{aligned}$$

$$\therefore (c + d)\mathbf{a} = c\mathbf{a} + \mathbf{da}$$

□

3. We asked ChatGPT if $V = \{f \mid f' \text{ is constant}\}$ is a vector space. It's response is given below.

You

Is V={ f|f' is constant} a vector space?

ChatGPT

To determine if the set $V = \{f \mid f' \text{ is constant}\}$ is a vector space, we need to check the three vector space properties:

1. **Zero vector:** The constant function $f(x) = c$ has a derivative of $f'(x) = 0$, which is a constant. Therefore, the zero function is in V.
2. **Closure under addition:** Let f and g be functions in V with constant derivatives. Consider the function $h(x) = f(x) + g(x)$. The derivative of h is the sum of the derivatives of f and g, which are constants. So, h also has a constant derivative, and V is closed under addition.
3. **Closure under scalar multiplication:** Let f be a function in V with a constant derivative. If we multiply f by a scalar c, the derivative of cf is cf', which is still a constant. Thus, V is closed under scalar multiplication.

Since V satisfies all three vector space properties, it is indeed a vector space.

Identify *two* errors in ChatGPT's response and write *two* paragraphs, one for each error, each of which clearly identifies an error, and how you would correct them.

ChatGPT incorrectly claims there are three properties a vector space must satisfy as the foundation for its response. A vector space must satisfy these three properties (the presence of a unique zero vector, closure under vector addition and closure under scalar multiplication), but also 5 other properties or more accurately axioms that define a vector space. To resolve this issue, we would either check the remaining axioms for the set $V = \{f|f'isconstant\}$, which include additive associativity, presence of a unique additive negative, scalar associativity, scalar distributivity, and unitary. Alternatively, the 8 axioms can be verified for the less restrictive set $W = f$, and then a subspace test can be performed using the three properties ChatGPT has listed. If the three properties are satisfied, and since V is a subset of W, we can arrive at the answer $V \sqsubseteq W$ and that V is a vector space.

ChatGPT's proof for the presence of a unique zero vector is inaccurate and incomplete. Nowhere does it identify a singular zero vector ($\underline{\mathbf{0}}$) with the property $f + \underline{\mathbf{0}} = f$. Rather it identifies multiple, and therefore non unique, elements of V as constant functions. It confuses a unique zero derivative, which is not an element of V but rather a restriction on the set, with the actual zero vector. Take a linear function, $g(x) = ax$ where $g \in V, a \in R$. The zero vector provided, $\underline{\mathbf{0}} = f(x) = c$, does not satisfy the zero vector conditions since $g(x) + \underline{\mathbf{0}} \neq g(x)$. To resolve this, we can correctly identify the zero vector as $\underline{\mathbf{0}} = h(x) = 0 \in V$. $h(x)$ has a constant derivative of 0, and satisfies the property the $t + h = t$ where $t \in V$ for all t.

4. Let $c \in \mathbb{R}$, and consider the subset $W_c = \{ (x, y, z) \in \mathbb{R}^3 \mid x + y + z = c\,|z| \}$ of $\mathbb{R}^3$. Determine *all* values of c such that W is a subspace of $\mathbb{R}^3$. Your answer should clearly demonstrate that you've found *all* values of c such that W_c is a subspace. This includes demonstrating why W_c is not a subspace for certain values of c, if you think no such c exists.

Let $\mathbf{u} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \mathbf{W_c}$, $\mathbf{w} = (\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathbf{W_c}$, $\alpha \in \mathbb{R}$

We know that

$$x + y + z = c|z| \tag{2}$$

And

$$p + q + r = c|r| \tag{3}$$

SI: The zero vector for $\mathbb{R}^3$ is $(0, 0, 0) \in W_c$ $(0 + 0 + 0 = 0)$. This places no restrictions on c.

SII: $\mathbf{u} + \mathbf{w} = (\mathbf{x}, \mathbf{y}, \mathbf{z}) + (\mathbf{p}, \mathbf{q}, \mathbf{r}) = (\mathbf{x} + \mathbf{p}, \mathbf{y} + \mathbf{q}, \mathbf{z} + \mathbf{r}) \in \mathbf{W_c}$

for **[SII]** to hold,

$$\begin{aligned}
(x + p) + (y + q) + (z + r) &= c|z + r| \\
(x + y + z) + (p + q + r) &= c|z + r| \\
c|z| + c|r| &= c|z + r| && \text{by } \textbf{(2)} \text{ and } \textbf{(3)} \\
c(|z| + |r| - |z + r|) &= 0
\end{aligned}$$

This implies that either $c = 0$ or $|z| + |r| - |z + r| = 0$. However, the latter is not always true. For instance, if $z = 1$ and $r = -1$, then $2 = 0$, which is not true. Hence, c has to always equal 0. **[SII]** will not hold for other values of c.

SIII: $\alpha\mathbf{x} = (\alpha\mathbf{x}, \alpha\mathbf{y}, \alpha\mathbf{z}) \in \mathbf{W_c}$ Therefore, for **[SIII]** to hold

$$\alpha x + \alpha y + \alpha z = \alpha(x + y + z) = c|\alpha z|$$

By **(2)**,

$$\alpha(c|z|) = c|\alpha z|$$

Check $c = 0$,

$$0 = 0$$

Because the W_c passes all the subspace tests only when $c = 0$, c has to equal 0 for W_c to be a subspace of $\mathbb{R}^3$.