Laplace Transform Lab: Solving ODEs using Laplace Transform in MATLAB

This lab will teach you to solve ODEs using a built in MATLAB Laplace transform function `laplace`.

There are five (5) exercises in this lab that are to be handed in. Write your solutions in a separate file, including appropriate descriptions in each step.

Include your name and student number in the submitted file.

Contents

Student Information

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```

Using symbolic variables to define functions

In this exercise we will use symbolic variables and functions.

```
syms t s x y

f = cos(t)
h = exp(2*x)
```

```
f =

cos(t)


h =

exp(2*x)
```

Laplace transform and its inverse

```
% The routine |laplace| computes the Laplace transform of a function

F=laplace(f)
```

```
F =

s/(s^2 + 1)
```

By default it uses the variable `s` for the Laplace transform But we can specify which variable we want:

```
H=laplace(h)
laplace(h,y)

% Observe that the results are identical: one in the variable |s| and the
% other in the variable |y|
```

```
H =

1/(s - 2)
```

```
ans =

1/(y - 2)
```

We can also specify which variable to use to compute the Laplace transform:

```
j = exp(x*t)
laplace(j)
laplace(j,x,s)

% By default, MATLAB assumes that the Laplace transform is to be computed
% using the variable |t|, unless we specify that we should use the variable
% |x|
```

```
j =

exp(t*x)


ans =

1/(s - x)


ans =

1/(s - t)
```

We can also use inline functions with `laplace`. When using inline functions, we always have to specify the variable of the function.

```
l = @(t) t^2+t+1
laplace(l(t))
```

```
l =

  function_handle with value:

    @(t)t^2+t+1


ans =

(s + 1)/s^2 + 2/s^3
```

MATLAB also has the routine `ilaplace` to compute the inverse Laplace transform

```
ilaplace(F)
ilaplace(H)
ilaplace(laplace(f))
```

```
ans =

cos(t)


ans =

exp(2*t)


ans =

cos(t)
```

If `laplace` cannot compute the Laplace transform, it returns an unevaluated call.

```
g = 1/sqrt(t^2+1)
G = laplace(g)
```

```
g =

1/(t^2 + 1)^(1/2)


G =

laplace(1/(t^2 + 1)^(1/2), t, s)
```

But MATLAB "knows" that it is supposed to be a Laplace transform of a function. So if we compute the inverse Laplace transform, we obtain the original function

```
ilaplace(G)
```

```
ans =

1/(t^2 + 1)^(1/2)
```

The Laplace transform of a function is related to the Laplace transform of its derivative:

```
syms g(t)
laplace(diff(g,t),t,s)
```

```
ans =

s*laplace(g(t), t, s) - g(0)
```

Exercise 1

Objective: Compute the Laplace transform and use it to show that MATLAB 'knows' some of its properties.

Details:

(a) Define the function `f(t)=exp(2t)*t^3`, and compute its Laplace transform `F(s)`. (b) Find a function `f(t)` such that its Laplace transform is `(s - 1)*(s - 2))/(s*(s + 2)*(s - 3)` (c) Show that MATLAB 'knows' that if `F(s)` is the Laplace transform of `f(t)`, then the Laplace transform of `exp(at)f(t)` is `F(s-a)`

(in your answer, explain part (c) using comments).

Observe that MATLAB splits the rational function automatically when solving the inverse Laplace transform.

```
% a
f = exp(2*t)*t^3;
F = laplace(f)
% b f(t) = -1/3 + 6/5 * exp(-2*t) + 2/15 * exp(3*t);
%c
f = sin(t);
a = 5;
F2 = laplace(f * exp(a*t))
% The laplace transform of sin(t) is F(s) = 1/(s^2 + 1). finding the laplace
% transform of sin(t)*exp(5*t) should give 1/((s-5)^2 + 1) = F(s-a), which is the
% given result
```

```
F =

6/(s - 2)^4


F2 =

1/((s - 5)^2 + 1)
```

Heaviside and Dirac functions

These two functions are builtin to MATLAB: `heaviside` is the Heaviside function `u_0(t)` at 0

To define `u_2(t)`, we need to write

```
f=heaviside(t-2)

% The Dirac delta function (at |0|) is also defined with the routine |dirac|

g = dirac(t-3)

% MATLAB "knows" how to compute the Laplace transform of these functions

laplace(f)
laplace(g)
```

```
f =

heaviside(t - 2)


g =

dirac(t - 3)


ans =

exp(-2*s)/s


ans =

exp(-3*s)
```

Exercise 2

Objective: Find a formula comparing the Laplace transform of a translation of `f(t)` by `t-a` with the Laplace transform of `f(t)`

Details:

- Give a value to `a`
- Let `G(s)` be the Laplace transform of `g(t)=u_a(t)f(t-a)` and `F(s)` is the Laplace transform of `f(t)`, then find a formula relating `G(s)` and `F(s)`

In your answer, explain the 'proof' using comments.

```
a = 5;
f1 = t;
f2 = t - a;
g = heaviside(t - a)*f2;
F = laplace(f1)
G = laplace(g)
F*exp(-s*a)
% The formula is given by G(s) = F(s) * exp(-s*a) as verified above

% The proof works by subsituting x = t - a
% This produces: G(s) = integral exp(-s(a + x)) * f(x) * dx
% Taking the constant term out: exp(-s*a) * integral f(x) * exp(-sx) * dx
% integral f(x) * exp(-sx) * dx = F(s)
% therefore G(s) = exp(-s*a) * F(s)
```

```
F =

1/s^2


G =

exp(-5*s)/s^2


ans =

exp(-5*s)/s^2
```

Solving IVPs using Laplace transforms

Consider the following IVP, `y''-3y = 5t` with the initial conditions `y(0)=1` and `y'(0)=2`. We can use MATLAB to solve this problem using Laplace transforms:

```
% First we define the unknown function and its variable and the Laplace
% tranform of the unknown

syms y(t) t Y s

% Then we define the ODE

ODE=diff(y(t),t,2)-3*y(t)-5*t == 0

% Now we compute the Laplace transform of the ODE.

L_ODE = laplace(ODE)

% Use the initial conditions

L_ODE=subs(L_ODE,y(0),1)
L_ODE=subs(L_ODE,subs(diff(y(t), t), t, 0),2)

% We then need to factor out the Laplace transform of |y(t)|

L_ODE = subs(L_ODE,laplace(y(t), t, s), Y)
Y=solve(L_ODE,Y)

% We now need to use the inverse Laplace transform to obtain the solution
% to the original IVP

y = ilaplace(Y)

% We can plot the solution

ezplot(y,[0,20])

% We can check that this is indeed the solution

diff(y,t,2)-3*y
```

```
ODE =

diff(y(t), t, t) - 3*y(t) - 5*t == 0


L_ODE =

s^2*laplace(y(t), t, s) - subs(diff(y(t), t), t, 0) - s*y(0) - 5/s^2 - 3*laplace(y(t), t, s) == 0


L_ODE =

s^2*laplace(y(t), t, s) - subs(diff(y(t), t), t, 0) - s - 5/s^2 - 3*laplace(y(t), t, s) == 0


L_ODE =

s^2*laplace(y(t), t, s) - s - 5/s^2 - 3*laplace(y(t), t, s) - 2 == 0


L_ODE =

Y*s^2 - s - 3*Y - 5/s^2 - 2 == 0


Y =

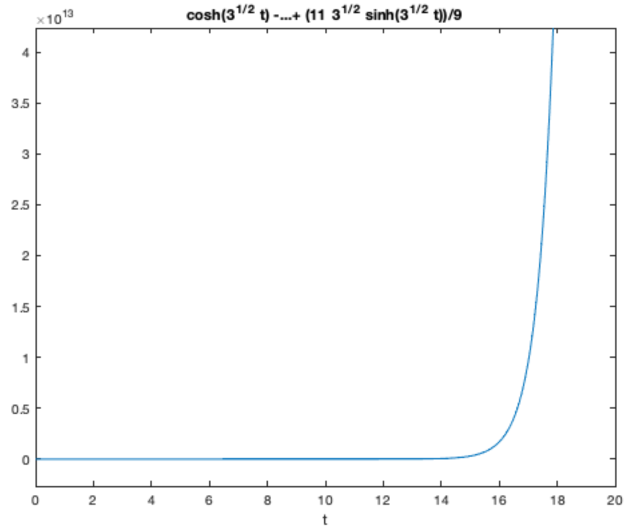
(s + 5/s^2 + 2)/(s^2 - 3)


y =

cosh(3^(1/2)*t) - (5*t)/3 + (11*3^(1/2)*sinh(3^(1/2)*t))/9


ans =

5*t
```



Exercise 3

Objective: Solve an IVP using the Laplace transform

Details: Explain your steps using comments

- Solve the IVP
- `y'''+2y''+y'+2*y=-cos(t)`
- `y(0)=0`, `y'(0)=0`, and `y''(0)=0`
- for `t` in `[0,10*pi]`
- Is there an initial condition for which `y` remains bounded as `t` goes to infinity? If so, find it.

```
syms y(t) t Y s
%define the ODE
ODE=diff(y(t),t,3)+2*diff(y(t),t,2)+diff(y(t),t,1)+2*y(t)+cos(t) == 0;
%apply the laplace transofrm to the ode
L_ODE = laplace(ODE)
L_ODE = subs(L_ODE, diff(y(t),t,2), 0); %sub the initial condition y''(0) = 0
L_ODE = subs(L_ODE, diff(y(t),t,1), 0); %sub the initial condition y'(0) = 0
L_ODE = subs(L_ODE, y(0), 0) %sub the initial condition y(0) = 0
L_ODE = subs(L_ODE, laplace(y(t), t, s), Y) %substitute Y as laplace(y(t), t, s) for clarity
Y = solve(L_ODE, Y) %solve for Y(s)
y = ilaplace(Y) %transform back to y(t)
ezplot(y, [0, 10*pi])
diff(y,t,3)+2*diff(y,t,2)+diff(y,t,1)+2*y
%there are no initial conditions that bound the solution since the terms
%that exponential grow (t*sin(t) / const_1) - (t*sint(t) / const_2) are not
%affected by the initial conditions.
```

```
L_ODE =

s*laplace(y(t), t, s) - y(0) - s*subs(diff(y(t), t), t, 0) - 2*s*y(0) - 2*subs(diff(y(t), t), t, 0) + s/(s^2 + 1) + 2*s^2*laplace(y(t), t, s) + s^3*la


L_ODE =

s*laplace(y(t), t, s) + s/(s^2 + 1) + 2*s^2*laplace(y(t), t, s) + s^3*laplace(y(t), t, s) + 2*laplace(y(t), t, s) == 0


L_ODE =

2*Y + Y*s + s/(s^2 + 1) + 2*Y*s^2 + Y*s^3 == 0


Y =

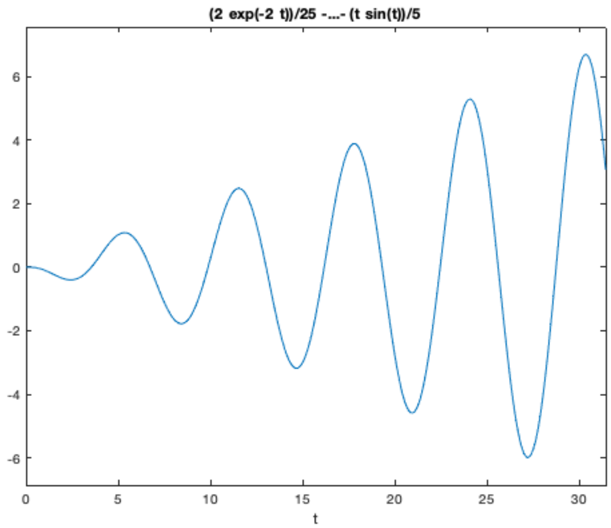
-s/((s^2 + 1)*(s^3 + 2*s^2 + s + 2))


y =

(2*exp(-2*t))/25 - (2*cos(t))/25 + (3*sin(t))/50 + (t*cos(t))/10 - (t*sin(t))/5
```

```
ans =

-cos(t)
```



Exercise 4

Objective: Solve an IVP using the Laplace transform

Details:

- Define
- `g(t) = 3 if 0 < t < 2`
- `g(t) = t+1 if 2 < t < 5`
- `g(t) = 5 if t > 5`
- Solve the IVP
- `y''+2y'+5y=g(t)`
- `y(0)=2 and y'(0)=1`
- Plot the solution for `t` in `[0,12]` and `y` in `[0,2.25]`.

In your answer, explain your steps using comments.

```
syms y(t) t Y s
g1 = 3 - 3 * heaviside(t - 2); %3 until t = 2
g2 = (t+1) * heaviside(t - 2) - (t+1) * heaviside(t-5); %t+1 starting at t=2 until t = 5
g3 = 5 * heaviside(t-5); %5 starting at t=5
g = g1 + g2 + g3; %combine the three function since they each only apply for the given intervals (zero elsewhere)
ODE = diff(y, t, 2) + 2 * diff(y, t, 1) + 5*y; %define the ode to solve
L_ODE = laplace(ODE); %apply the laplace transformation
L_ODE = subs(L_ODE, laplace(y(t), t, s), Y)
L_ODE = subs(L_ODE, diff(y, t, 1), 1); %initial condition for first derivate
L_ODE = subs(L_ODE, y(0), 2) %initial condition for function
Y = solve(L_ODE==laplace(g), Y) %solve for Y(s)
Y = simplify(Y) %simplify Y so the inverse laplace can be fully computed by matlab
y = ilaplace(Y) %transform back to y(t)
ezplot(y, [0, 12, 0, 2.25])
```

```
L_ODE =

5*Y - 2*y(0) + 2*Y*s - s*y(0) - subs(diff(y(t), t), t, 0) + Y*s^2


L_ODE =

5*Y - 2*s + 2*Y*s + Y*s^2 - 5


Y =

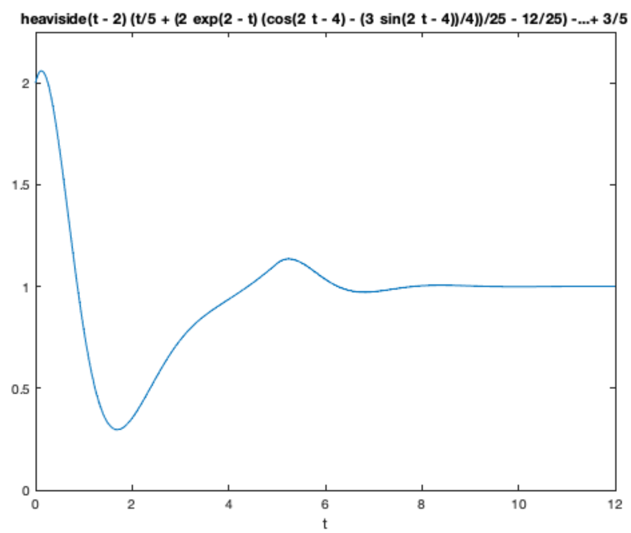
(exp(3*s) - s + 3*s*exp(5*s) + 5*s^2*exp(5*s) + 2*s^3*exp(5*s) - 1)/(5*s^2*exp(5*s) + 2*s^3*exp(5*s) + s^4*exp(5*s))
```

```
Y =

(exp(-5*s)*(exp(3*s) - s + 3*s*exp(5*s) + 5*s^2*exp(5*s) + 2*s^3*exp(5*s) - 1))/(s^2*(s^2 + 2*s + 5))


y =

heaviside(t - 2)*(t/5 + (2*exp(2 - t)*(cos(2*t - 4) - (3*sin(2*t - 4))/4))/25 - 12/25) - heaviside(t - 5)*(t/5 + (2*exp(5 - t)*(cos(2*t - 10) - (3*sin
```



Exercise 5

Objective: Use the Laplace transform to solve an integral equation

Verify that MATLAB knowns about the convolution theorem by explaining why the following transform is computed correctly.

```
syms t tau y(tau) s
I=int(exp(-2*(t-tau))*y(tau),tau,0,t)
laplace(I,t,s)
% L{y(t)} / (s+2) can be split into two functions:
% F(t) = L{y(t)} and G(t) = 1 / s+2
% taking the inverse laplace we know f(t) = y(t) and g(t) = exp(-2t)
% From the convolution theorm:
% (f * g) (t) = integral 0->t (f(tau) - g(t - tau)) * d(tau)
% Subing in the function f and g:
% = integral 0->t (y(tau) - exp(-2(t-tau))) * d(tau)
% = integral 0->t (y(tau) - exp(2(tau-t))) * d(tau)
% This is the original integral and therefore verifies matlab's knowledge
% of the colvolution theorm
```

```
I =

int(exp(2*tau - 2*t)*y(tau), tau, 0, t)


ans =

laplace(y(t), t, s)/(s + 2)
```

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