MAT185 Linear Algebra
Assignment 4

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf**. Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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Student number: ____________________

Full Name: ____________________

Student number: ____________________

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ**.
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) ____________________

2) ____________________

Preamble:

Standard basis:

When considering the straight-line movement of an object in $^3\mathbb{R}$, this movement can easily be described using the standard basis vectors

$$\mathbf{e}_x = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{e}_y = \begin{bmatrix}0\\1\\0\end{bmatrix}, \text{and } \mathbf{e}_z = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

As shown in Figure 1, the location of an arbitrary point $\mathbf{r}$ in $^3\mathbb{R}$ can than be expressed as a linear combination of these standard basis vectors and the coordinates $x, y,$ and z:

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$$

A movement of the point $\mathbf{r}(t)$ along a path can be described by the time-dependent coordinates $x(t), y(t),$ and $z(t)$. The basis vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are constant and not time-dependent. In that case, the linear combination of the basis vectors is

$$\mathbf{r}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z.$$

When calculating the velocity $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \dot{\mathbf{r}}(t)$ or acceleration $\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{r}}(t) = \ddot{\mathbf{r}}(t)$ in this basis, only the time-dependency of the coordinates has to be considered.

Cylindrical basis:

When describing the movement of an object on a circular path, it is typically easier to express this object by a different coordinate system with the cylindrical basis vectors $\{\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z\}$ which also span $^3\mathbb{R}$ (see Figure 2). Again, an arbitrary point $\mathbf{r}$ in $^3\mathbb{R}$ can be described as a linear combination of these basis vectors with the coordinates r and z (the coordinate for $\mathbf{e}_\varphi$ is 0). In this cylindrical basis as shown in Figure 2, $\mathbf{e}_r$ always points in the direction of the projection of $\mathbf{r}$ on the plane spanned by $\{\mathbf{e}_r, \mathbf{e}_\varphi\}$ (here: $\text{span}\{\mathbf{e}_r, \mathbf{e}_\varphi\} = \text{span}\{\mathbf{e}_x, \mathbf{e}_y\}$). The angle between $\mathbf{e}_r$ and the x-axis of the standard basis is called φ. The vector $\mathbf{e}_\varphi$ is always orthogonal to $\mathbf{e}_r$ in this plane. Additionally, the basis vector $\mathbf{e}_z$ is always pointing upwards.

A movement of a point $\mathbf{r}(t)$ along a path can than be described as a linear-combination of the cylindrical basis vectors and time-dependent coordinates $r(t)$ and $z(t)$:

$$\mathbf{r}(t) = r(t)\mathbf{e}_r + z(t)\mathbf{e}_z$$

For example, if $\mathbf{r}(t)$ moves along a circular path, $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are always following this rotation. Since $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ are rotating with the point $\mathbf{r}(t)$, the described time-dependency of the basis vectors has to be considered when calculating the velocity $\dot{\mathbf{r}}(t)$ or acceleration $\ddot{\mathbf{r}}(t)$ in this basis. This observation will be the subject of this assignment.

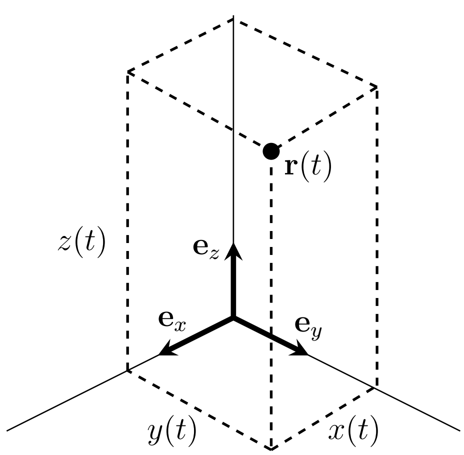

Figure 1: Standard basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ in $^3\mathbb{R}$ and the coordinates in terms of this basis $x(t)$, $y(t)$, and $z(t)$.

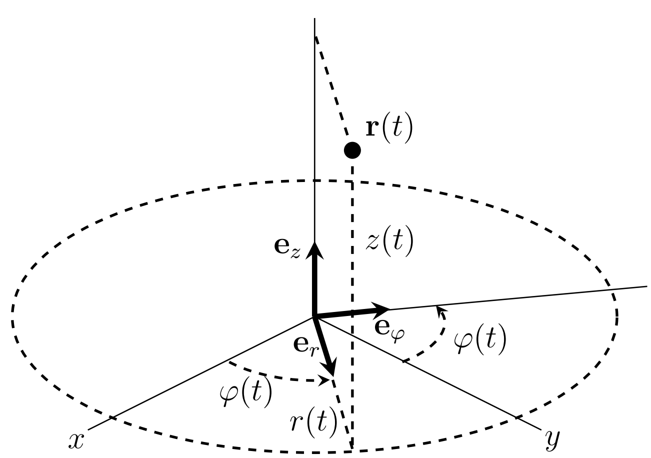

Figure 2: Cylindrical basis $\{\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z\}$ and the coordinates in terms of this basis $r(t)$ and $z(t)$, and the angle $\varphi(t)$.

1.

(a) Express the velocity $\dot{\mathbf{r}}(t)$ of $\mathbf{r} \in {}^3\mathbb{R}$ in Figure 1 in terms of the standard basis vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the coordinates x, y, and z. Additionally, describe $\dot{\mathbf{r}}$ in Figure 2 in terms of the cylindrical basis vectors $\{\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z\}$, their time derivatives, and the coordinates r, and z.

To express the velocity $\dot{\mathbf{r}}(t)$, we begin by taking the derivative of $\mathbf{r}$ with respect to time.
Note that $\frac{d\mathbf{e}_x}{dt} = \frac{d\mathbf{e}_y}{dt} = \frac{d\mathbf{e}_z}{dt} = 0$ since the basis vectors are time independent.

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \frac{d}{dt}\left(x(t)\mathbf{e}_x + y(t)\mathbf{e}_y + z(t)\mathbf{e}_z\right) \\ &= \frac{d}{dt}x(t)\mathbf{e}_x + 0 + \frac{d}{dt}y(t)\mathbf{e}_y + 0 + \frac{d}{dt}z(t)\mathbf{e}_z + 0 \\ &= \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z\end{aligned}$$

For the cyclindrical basis, only the $\mathbf{e}_z$ basis vector is time independent:

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \frac{d}{dt}\left(r(t)\mathbf{e}_r + z(t)\mathbf{e}_z\right) \\ &= \frac{d}{dt}r(t)\mathbf{e}_r + r(t)\frac{d}{dt}\mathbf{e}_r + \frac{d}{dt}z(t)\mathbf{e}_z + z(t)(0) \\ &= \dot{r}\mathbf{e}_r + r(t)\dot{\mathbf{e}}_r + \dot{z}\mathbf{e}_z \\ &= \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r + \dot{z}\mathbf{e}_z\end{aligned}$$

(b) Determine the transformation matrix $\mathbf{P}$ mapping the standard basis to the cylindrical basis:

$$\begin{bmatrix}\mathbf{e}_r \\ \mathbf{e}_\varphi \\ \mathbf{e}_z\end{bmatrix} = \mathbf{P}\begin{bmatrix}\mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z\end{bmatrix}$$

To determine the transformation matrix P, we must express the cylindrical basis vectors as a linear combination of the standard basis vectors. From Figure 1 & 2, we can see by using trigonometry that:

$$\begin{aligned}\mathbf{e}_r &= \cos(\varphi)\mathbf{e}_x + \cos(\frac{\pi}{2} - \varphi)\mathbf{e}_y = \cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y \\ \mathbf{e}_\varphi &= \cos(\frac{\pi}{2} + \varphi)\mathbf{e}_x + \cos(\varphi)\mathbf{e}_y = -\sin(\varphi)\mathbf{e}_x + \cos(\varphi)\mathbf{e}_y \\ \mathbf{e}_z &= \mathbf{e}_z\end{aligned}$$

Writing the cyclindrical basis vectors as a linear combination of the standard basis vectors, we get:

$$\begin{aligned}\mathbf{e}_r &= \cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y + 0 \cdot \mathbf{e}_z \\ \mathbf{e}_\varphi &= -\sin(\varphi)\mathbf{e}_x + \cos(\varphi)\mathbf{e}_y + 0 \cdot \mathbf{e}_z \\ \mathbf{e}_z &= 0 \cdot \mathbf{e}_x + 0 \cdot \mathbf{e}_y + \mathbf{e}_z\end{aligned}$$

Expressing these equations in matrix form, we get:

$$\begin{bmatrix}\mathbf{e}_r \\ \mathbf{e}_\varphi \\ \mathbf{e}_z\end{bmatrix} = \begin{bmatrix}\cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z\end{bmatrix}$$

Therefore, the transformation matrix $\mathbf{P}$ is $\begin{bmatrix}\cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1\end{bmatrix}$.

2. As mentioned in the *preamble*, the vectors in the cylindrical basis have to be time-dependent if the point $\mathbf{r}(t)$ is moving on a path over time t. Therefore, the first time derivative (velocity) of the cylindrical basis vectors are not zero.

(a) Consider the linear transformation defined by

$$\begin{bmatrix} \dot{\mathbf{e}}_r \\ \dot{\mathbf{e}}_\varphi \\ \dot{\mathbf{e}}_z \end{bmatrix} = \mathbf{D} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\varphi \\ \mathbf{e}_z \end{bmatrix}, \qquad \text{with the notation } \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{e}_r \\ \mathbf{e}_\varphi \\ \mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_r \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_\varphi \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_z \end{bmatrix} = \begin{bmatrix} \dot{\mathbf{e}}_r \\ \dot{\mathbf{e}}_\varphi \\ \dot{\mathbf{e}}_z \end{bmatrix}$$

Determine the transformation matrix $\mathbf{D}$ for this transformation.

Hint: Start with the expression of the transformation in Question 1(b). Remember also that $\varphi = \varphi(t)$ is time-dependent.

From 1(b) we can find the time derivatives of the basis vectors:

$$\begin{aligned}
\dot{\mathbf{e}}_r &= \frac{d}{dt}(\cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y + 0 \cdot \mathbf{e}_z) \\
&= \frac{d}{dt}\cos(\varphi)\mathbf{e}_x + \cos(\varphi)\frac{d}{dt}\mathbf{e}_x + \frac{d}{dt}\sin(\varphi)\mathbf{e}_y + \sin(\varphi)\frac{d}{dt}\mathbf{e}_y + 0 \\
&= -\sin(\varphi)\dot{\varphi}\mathbf{e}_x + \cos(\varphi)\dot{\varphi}\mathbf{e}_y \\
&= \dot{\varphi}(-\sin(\varphi)\mathbf{e}_x + \cos(\varphi)\mathbf{e}_y) \\
&= \dot{\varphi}(\mathbf{e}_\varphi) \\
\dot{\mathbf{e}}_\varphi &= \frac{d}{dt}(-\sin(\varphi)\mathbf{e}_x + \cos(\varphi)\mathbf{e}_y + 0 \cdot \mathbf{e}_z) \\
&= \frac{d}{dt} - \sin(\varphi)\mathbf{e}_x + (-\sin(\varphi))\frac{d}{dt}\mathbf{e}_x + \frac{d}{dt}\cos(\varphi)\mathbf{e}_y + \cos(\varphi)\frac{d}{dt}\mathbf{e}_y + 0 \\
&= -\cos(\varphi)\dot{\varphi}\mathbf{e}_x - \sin(\varphi)\dot{\varphi}\mathbf{e}_y \\
&= -\dot{\varphi}(\cos(\varphi)\mathbf{e}_x + \sin(\varphi)\mathbf{e}_y) \\
&= -\dot{\varphi}(\mathbf{e}_r) \\
\dot{\mathbf{e}}_z &= \frac{d}{dt}(0 \cdot \mathbf{e}_x + 0 \cdot \mathbf{e}_y + 1 \cdot \mathbf{e}_z) \\
&= 0
\end{aligned}$$

Therefore we can express time derivatives of the cyclindrical basis vectors as a linear combination of the cyclindrical basis vectors:

$$\begin{aligned}
\dot{\mathbf{e}}_r &= 0 \cdot \mathbf{e}_r + \dot{\varphi} \cdot \mathbf{e}_\varphi + 0 \cdot \mathbf{e}_z \\
\dot{\mathbf{e}}_\varphi &= -\dot{\varphi} \cdot \mathbf{e}_r + 0 \cdot \mathbf{e}_\varphi + 0 \cdot \mathbf{e}_z \\
\dot{\mathbf{e}}_z &= 0 \cdot \mathbf{e}_r + 0 \cdot \mathbf{e}_\varphi + 0 \cdot \mathbf{e}_z
\end{aligned}$$

Therefore, the transformation matrix $\mathbf{D}$ is:

$$\mathbf{D} = \begin{bmatrix} 0 & -\dot{\varphi} & 0 \\ \dot{\varphi} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. As mentioned in the *preamble*, the vectors in the cylindrical basis have to be time-dependent if the point $\mathbf{r}(t)$ is moving on a path over time t. Therefore, the first time derivative (velocity) of the cylindrical basis vectors are not zero.

(b) Determine the first time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \dot{\mathbf{r}}$ of $\mathbf{r} \in {}^3\mathbb{R}$ expressed in the cylindrical basis from Question 1(a). For that, use the transformation calculated in Question 2(a) and express $\dot{\mathbf{r}}$ as a linear combination of $\mathbf{e}_r$, $\mathbf{e}_\varphi$, and $\mathbf{e}_z$.

From Question 1(a), we have the expression of the velocity $\dot{\mathbf{r}}$:

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + t\dot{\mathbf{e}}_r + \dot{z}\mathbf{e}_z$$

But from Question 2(a), we have the time derivatives of the basis vector: $\dot{\mathbf{e}}_r = -\dot{\varphi} \cdot \mathbf{e}_\varphi$. Substituting this into the above equation allows us the express the velocity $\dot{\mathbf{r}}$ as a linear combination of purely the cyclindrical basis vectors:

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r - r\dot{\varphi}\mathbf{e}_\varphi + \dot{z}\mathbf{e}_z$$

(c) Determine the second time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathbf{r}} = \ddot{\mathbf{r}}$ of $\mathbf{r} \in {}^3\mathbb{R}$ expressed in the cylindrical basis from Question 1(a). Use the transformation calculated in Question 2(a) and express $\ddot{\mathbf{r}}$ as a linear combination of $\mathbf{e}_r$, $\mathbf{e}_\varphi$, and $\mathbf{e}_z$.

We begin by taking the derivative of our result from Question 2(b):

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{d}{dt}\left(\dot{r}\mathbf{e}_r - r\dot{\varphi}\mathbf{e}_\varphi + \dot{z}\mathbf{e}_z\right) \\ &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r - \dot{r}\dot{\varphi}\mathbf{e}_\varphi - r\ddot{\varphi}\mathbf{e}_\varphi - r\dot{\varphi}\dot{\mathbf{e}}_\varphi + \ddot{z}\mathbf{e}_z\end{aligned}$$

From Question 2(a), we have the time derivatives of the basis vectors: $\dot{\mathbf{e}}_r = -\dot{\varphi}\mathbf{e}_\varphi$ and $\dot{\mathbf{e}}_\varphi = \dot{\varphi}\mathbf{e}_r$. Substituting these into the above equation allows us the express the acceleration $\ddot{\mathbf{r}}$ as a linear combination of purely the cyclindrical basis vectors:

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\mathbf{e}_r + \dot{r}(-\dot{\varphi}\mathbf{e}_\varphi) - \dot{r}\dot{\varphi}\mathbf{e}_\varphi - r\ddot{\varphi}\mathbf{e}_\varphi - r\dot{\varphi}(\dot{\varphi}\mathbf{e}_r) + \ddot{z}\mathbf{e}_z \\ &= \ddot{r}\mathbf{e}_r - \dot{r}\dot{\varphi}\mathbf{e}_\varphi - \dot{r}\dot{\varphi}\mathbf{e}_\varphi - r\ddot{\varphi}\mathbf{e}_\varphi - r\dot{\varphi}^2\mathbf{e}_r + \ddot{z}\mathbf{e}_z \\ &= \ddot{r}\mathbf{e}_r - 2\dot{r}\dot{\varphi}\mathbf{e}_\varphi - r\ddot{\varphi}\mathbf{e}_\varphi - r\dot{\varphi}^2\mathbf{e}_r + \ddot{z}\mathbf{e}_z \\ &= (\ddot{r} - r\dot{\varphi}^2)\mathbf{e}_r - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\mathbf{e}_\varphi + \ddot{z}\mathbf{e}_z\end{aligned}$$

3. We are now using the cylindrical basis to study a moth circling a light source in the dark. We can describe the path of the moth as a logarithmic spiral towards the light source. The moth flys with a constant angle α and with constant path velocity v in the direction of the light source (see Figure 3).

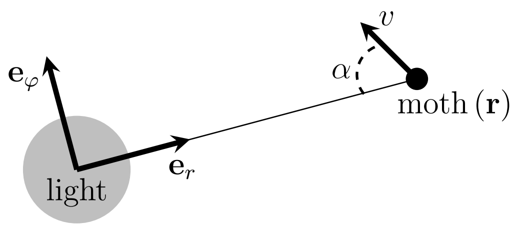

Figure 3: View from above of a moth circling a light source at hight $z = \text{const.} = 0$.

(a) Calculate the acceleration of the moth in $\mathbf{e}_r$ direction by using your result from Question 2(c). For that, break down the velocity components in Figure 3 for $\dot{\mathbf{r}}$ using the cylindrical basis.

The velocity of the moth $\dot{\mathbf{r}}$ can be expressed in the cylindrical basis using the velocity components as:

$$\dot{\mathbf{r}} = (-v\cos\alpha)\mathbf{e}_r + (v\sin\alpha)\mathbf{e}_\varphi$$

There is no velocity component in the $\mathbf{e}_z$ direction since the height of the month is constant. Comparing the expression derived for velocity if Question 2(b), we see:

$$\dot{r} = -v\cos\alpha \tag{1}$$

$$-r\dot{\varphi} = v\sin\alpha$$

$$\dot{\varphi} = -\frac{v\sin\alpha}{r} \tag{2}$$

Given $-v\cos\alpha$ is a constant, $\ddot{r}$ is 0. Using the expression for acceleration in the $\mathbf{e}_r$ direction from Question 2(c), we have:

$$\ddot{r} - r\dot{\varphi}^2 = 0 - r\left(-\frac{v\sin\alpha}{r}\right)^2$$

$$= -\frac{v^2\sin^2\alpha}{r}$$

(b) Interprete your result from Question 3(a). Discuss what happens if the moth gets close to the light source. How do you explain your observation that a moth drifts away at a critical distance from the light source and starts its approach again? No additional calculations are necessary.

Since $v^2\sin^2\alpha$ is a constant, the acceleration in the $\mathbf{e}_r$ direction is inversely proportional to the distance r from the light source. As the moth gets closer to the light source, r decreases and as a result, the moth accelerates more and more towards the light source (the negative direction of $\mathbf{e}_r$). As the moth gets farther from the light source, r increases and the acceleration decreases. This causes the moth to drift away from the light source because the acceleration in the $\mathbf{e}_r$ is approximately 0. But since the month still has an initial velocity in the $\mathbf{e}_r$ direction, it will start to spiral back towards the light source.