Constructing a Pendulum to Determine its Period and Q-Factor

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October 4, 2023

1 Introduction

The purpose of the lab is to design a pendulum and measure various features with uncertainties to discover relationships.

One feature is the period of a pendulum which can be calculated given its length (L) in meters:

$$T = 2\sqrt{L} \tag{1}$$

Another feature is the position of a pendulum over time described by its angle with the vertical:

$$\theta(t) = \theta_0 \; e^{-t/\tau} \cos\left(2\pi \frac{t}{T} + \phi_0\right) \tag{2}$$

Where θ_0 is the initial angle in radians, t is the time in seconds, τ is the decay constant, T is the period in seconds, and ϕ_0 is a phase shift in radians to account for a starting delay.

We can also measure how quickly a pendulum decays via its Q-Factor, defined as:

$$Q = \pi \frac{\tau}{T} \tag{3}$$

2 Methods and Procedures

The pendulum was constructed from a light string with a length of 42.5 ± 0.1cm (measured from the pivot point to the mass). The mass it supports is a yellow foam ball with a radius of 5.95 ± 0.01cm.

A hook screwed into a structure was used to tie the string down with a simple knot. It was empirically found that tying the knot on the side of the hook rather than the bottom allowed for a tighter knot and less slippage. To allow for adjustable string lengths and maintain constant tension with a fixed pivot point, the remaining slack is taped down to the structure. The remaining end of the string is taped securely to the ball and the structure itself is secured to the table through tape and a mass.

The entire apparatus, displayed in Figure (1) is designed for repeatability by ensuring only the mass will move from the pivot point while the structure and knot remains in place from trial to trial.

Using a yellow ball is not crucial, but aids in assisting the tracking software by contrasting with dark surroundings.

2.1 Period

To measure the period (T), an iPad Air camera was used to take video at 60fps. This was held in place via a chair and weight such that it was perpendicular to the ground (a zero angle with the horizontal). It was placed at a distance where the whole apparatus could be viewed while the pendulum swung.

The period was measured by counting the number of frames to complete the first three cycles (at the start of the video) and then dividing by three. A cycle begins when the pendulum reaches its minimum distance from the ground (makes a zero angle with the vertical). The pendulum always passes through this point, which is also its equilibrium. Since other points may not be reached as the pendulum continues to oscillate (and decay is present as covered in *Results and Analysis*), tracking the period over time is less viable at other points. Averaging the period from multiple oscillations gives a more resistant value.

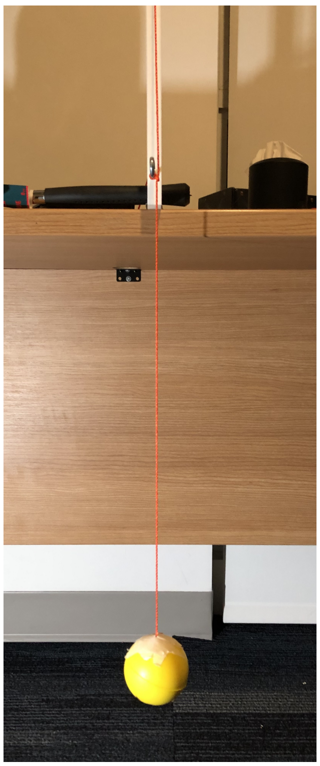

Figure 1: A foam ball is tied to a string, held in place on a hook via a knot. The remaining string is held in tension with tape (out of frame). A hammer and tape are used to secure the white structure to the desk.

It was decided to take 14 period measurements, 7 from positive angles and 7 from negative angles. The angle of release was measured live to give a rough estimate and ensure a range of values was measured. In the post analysis software *Tracker*[1], accurate values of the initial angle were measured with a protractor. Angles measured ranged from -1.36 to 1.44 $\pm$ 0.01 rad with each trial being changed by 0.17 $\pm$ 0.08 rad (10 $\pm$ 5 degrees).

After raising the ball to a roughly accurate angle, it was released multiple times until the motion was parallel to the wall with negligible z-axis movement (where the z-axis is coming out of the video). Releasing all fingers from the ball at the same time aids with reducing rotational motion.

2.2 Q-Factor and Angle

A somewhat small angle of 0.33 $\pm$ 0.01 rad was chosen to empirically measure the Q-Factor. For more precise and accurate measurements, the same iPad Air camera was used to film at 240fps in slo-mo mode. The process for releasing the ball remains the same as measuring the period. A longer video was recorded.

To measure the angle over time, the Autotracker feature in *Tracker* was used. The template was set to Evolve: 40%, Tether: 5%, Automark: 4 and a Step Size: 5.

3 Results and Analysis

Empirical data describing the effect of the initial angle of release on the pendulum's period identifies a non-constant value, contradicting Equation (1). Figure 2's shape and best-fit equation show that the period increases quadratically as the initial angle increases. For smaller angles closer to the best-fit vertex, the periods can be described by a line as a result of the small angle approximation being closer to the given angle. This is also represented in the small quadratic constant value of the equation's best fit ($a = 0.08 \pm 0.01$).

From the graph's general shape, we can also verify the symmetry of the pendulum, where releasing the ball from two angles with different signs but the same magnitudes yields similar periods. In addition, the best-fit equation's linear constant is experimentally zero, centring the parabola at approximately (0, 0).

Equation (1) is most accurate for small angles by comparing the graph and theoretical value for the period ($T = 2\sqrt{L} = 1.3s$)

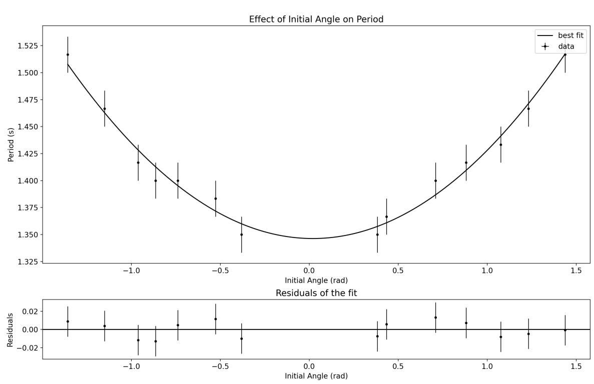

Figure 2: The plot shows how the period varies based on the initial angle of release. The best fit equation is approximately $T = 1.346 + 0.085\theta^2$. The linear term is smaller than the uncertainty (initial angle uncertainty ± 0.008) and therefore, experimentally zero.

Next, to measure the Q-Factor, either the angle over time can be measured (Figure 3), or the amplitudes for each oscillation (Figure 5). By fitting the angle over time to Equation (2), parameters to solve for Equation (3) can be calculated. Figure 3 fits best to sinusoidal exponential decay. The sinusoidal function describes the pendulum's repeated motion of swinging back and forth. The exponential decay recognizes that forces exist which reduce the amplitude less and less over time.

Figure (3) shows that amplitude decreases exponentially over time, which can be thought of as smaller and smaller angles of release. According to Figure (2), different angles of release result in different periods. And therefore, as the amplitude changes, the period also changes slightly, but less and less at smaller angles where the decay is minimal.

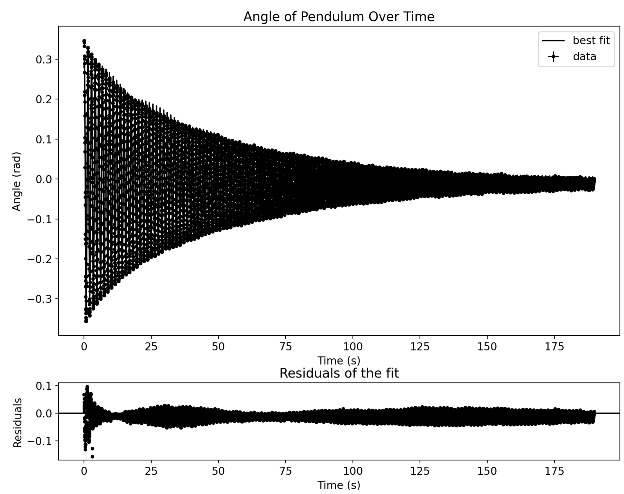

Figure 3: The plot shows how the position of the pendulum, described as an angle from the vertical, changes over time and multiple oscillations. The best fit line is approximated by Equation (2) which describes the angle as a function of time. The best fit parameters are $\theta_0 = 0.327 \pm 0.001, \tau = 46.3 \pm 0.2, T = 1.3414 \pm 0.00003$. Due to the number of data points and small uncertainties, error bars are not visible. A zoomed-in version of the graph is given in the appendix.

Using Equation (3) and the best fit parameters the Q-Factor is given to be $Q = \pi(\frac{46.3 \pm 0.2}{1.3414 \pm 0.00003}) = 108.435 \pm 0.004$

We can also use the maximum to minimum distance in each oscillation to measure the amplitude. When the amplitude is roughly $e^{-\frac{\pi}{2}} \approx 20\%$ of the initial amplitude, the number of oscillations gives $Q/2$.

Using Figure (3) to derive the amplitudes displayed in Figure (5), and comparing the amplitudes to 20% of the initial, yields a Q-Factor of 117.000 ± 0.006 (given from the amplitude uncertainty discussed in *Uncertainties*). This states that it takes approximately 117 oscillations for the pendulum to decrease its amplitude to $\tilde{4}\%$.

The two measured Q-Factors do not fall within each other's uncertainties but are reasonably close giving some confidence to the value.

Uncertainties

For Figure (3), measurement uncertainties exist for the initial angle and identifying when the pendulum has reached its lowest point. The measurement uncertainty due to the precision of device graduation for the initial angle is $\pm 0.1deg$, which is approximately $\pm 0.001rad$. There is also uncertainty around where the protractor is placed in *Tracker*. Looking at the average difference in angles for various measurements at the pivot point to the string gives an uncertainty of $\pm 0.5deg$ or $\pm 0.008rad$, which is larger than the previous uncertainty and therefore will be used. This is calculated by moving the protractor across various points on the vertices (the pivot point and the ball).

The period uncertainty is $\pm 1frame$ or $\pm 0.01s$ (at 60fps) since empirically, in each video, the instant at which the pendulum reaches the bottom lies between two frames. The time between two frames is $\frac{1}{60}s$. This uncertainty is somewhat large in Figure (2), and therefore, using a higher fps would help reduce the uncertainty.

Measuring the apparatus uncertainty (how much results change given the same trial conditions) was difficult due to live measurement errors and lack of precision. Three independent trials with angles between $1.06 \pm 0.01rad$ and $1.10 \pm 0.01rad$ yielded period values also with $\pm 1frame$, which falls in the period uncertainty discussed above.

For tracking the angle over time, the angle measured by *Tracker* has measurement uncertainties. Since Autotracker is being used, the point which is tracked on the ball changes from frame to frame. The main reason for choosing a higher fps is not to reduce time uncertainty, but to reduce the larger uncertainty of angle. At a lower fps, *Tracker* will choose points across the entire diameter, leading to a significant difference in angles. At 240fps, a random sample of $n = 10$ frames is taken and angles are adjusted to the center. In addition, the coordinate axis and identification of the ball's centre hold uncertainties in angle measurement. In adjusting each of these uncertainties to find the range of angle variation, the uncertainty given is $\pm 0.005rad$. This is the largest uncertainty.

The amplitude uncertainty stems from the angle that could be travelled in between frames when at its highest extremes. This uncertainty is already very small since the ball's angular displacements are smallest at the highest points. The uncertainty given by the angular velocity over a 5 frame interval at 240fps is $\pm 0.001rad$. However, measurements also differ from tracking uncertainty previously mentioned, creating a total uncertainty of $\pm 0.006rad$. The number of oscillations is a counted value and therefore has no uncertainty.

To reduce uncertainties related to the initial angle, clear markings can be placed on the pivot point and ball to ensure the angle is measured in the same place, reducing the amount of angle variation. A more graduated protractor would also help but likely have significant costs.

Additional uncertainties assumed to be negligible include movement in the z-direction and slightly rotated videos since it has already been minimized by the experimental setup and post analysis (by slightly rotating images to become horizontal).

4 Conclusion

This lab has explored how a pendulum's period is dependent on its initial angle of release through a quadratic relationship. The period calculations are fairly confident when using a small enough fps. It is also observed that the amplitude exponentially decays, affecting the release angle over time and therefore the period. To more accurately measure the period, taking multiple period calculations at small angles, where the decay is visibly smaller, is best. The Q-Factor is also calculated within a reasonable range using a formulaic and empircal method.

5 Appendix

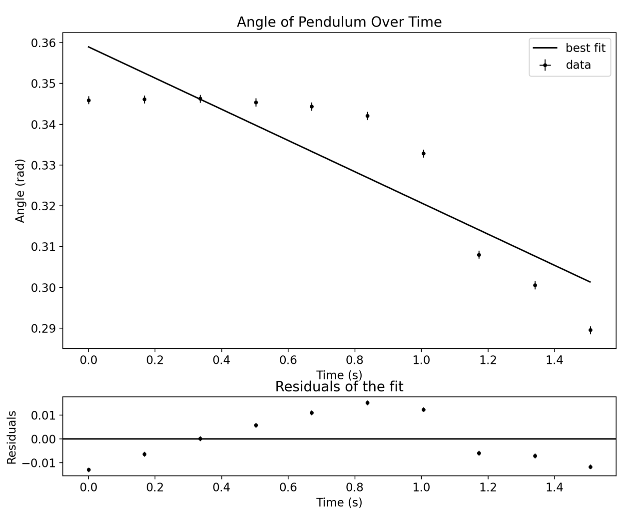

Figure 4: A zoomed in version of Figure 3 with visible error bars. The angle uncertainty is given by $\pm 0.006 rad$.

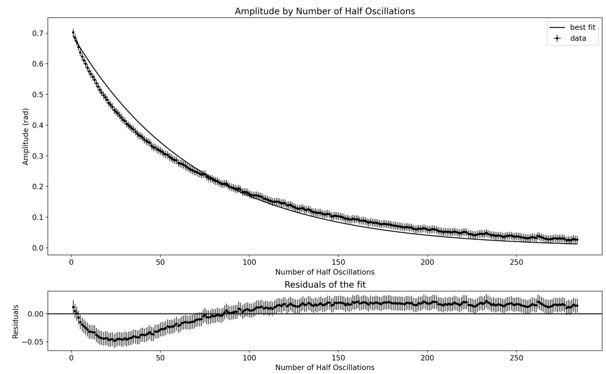

Figure 5: This graph confirms how the amplitude decays exponentially as the number of oscillations increases. The amplitude is recorded every time a maximum and minimum angle is reached, which is only half an oscillation. So finding the value that's 20% of the initial amplitude actually gives the Q-Factor.

References

[1] Brown, D. (2023). Tracker. https://physlets.org/tracker/