Constructing a Pendulum to Determine its Period and Q-Factor

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1 Introduction

By designing a two string pendulum attached to a yellow ball with a variable length, a given model is tested against empirical results to better inform the accuracy of the model.

The initial angle versus period, amplitude over time, length versus period and length versus Q-Factor are analyzed with the aid of video tracking software Tracker [1].

Equations

A pendulum's period is expected to be given by the following function [2]:

$$T = 2\sqrt{L} \tag{1}$$

By measuring the period across a varying length range and a fixed small angle range, the results are fitted to the following power function:

$$T = pL^q \tag{2}$$

The range of parameters p and q are empirically found to fall slightly outside the expected values but agree with the fit of a power function.

For assessing the relationship between the initial angle and period, an even power series is found to accurately describe how the period increases as the initial angle increases.

$$T = T_0(A + C\theta^2 + D\theta^4 ...) \tag{3}$$

However, it is found that for small angles, a constant period can be found within uncertainty values. Releasing within the range of -0.5 to 0.5 rad, where the amplitude and period change the least, allows for equation (2) to be used accurately.

The damped harmonic oscillator [2] provides angle measurements over time given an initial angle (θ_0), the time (t), the decay constant (τ), the period (T) and a time offset (ϕ).

$$\theta(t) = \theta_0 \, e^{-t/\tau} \cos\left(2\pi \frac{t}{T} + \phi\right) \tag{4}$$

Empirical measurements of angles over time agree with the exponential decay behaviour suggested by the model.

Equation 4 gives the decay constant τ. This is used to calculate the Q-Factor [2], a measurement of how quickly the pendulum decays.

$$Q = \pi \frac{\tau}{T} \tag{5}$$

It is found that the Q-Factor has no relationship with the pendulum's length.

2 Setup and Methods

The pendulum was constructed from two light strings with variable lengths that support a yellow foam ball with a radius of 5.95 $\pm$ 0.01cm. By using two strings, oscillation in and out of the plane is reduced, which would otherwise affect the decay.

Two hooks screwed into the structure were used to tie the string down with a simple knot. Tying the knot on the side of the hook rather than the bottom was found to reduce slippage.

The same two strings were used to measure all relationships ranging from lengths 17.6 to 80.5 ± 0.5cm. Any extra slack was tied down at the top of the structure to maintain tension and reduce knot slippage.

Figure 1: A foam ball is tied to two strings and each string is held in place by a knot on a hook. The remaining string is held in tension with tape at the top of the structure. The entire structure is severely taped to the wall.

All videos were taken using an iPad Air camera at 60fps. This was held in place on a desk with a weight to hold it upright (zero angle with the horizontal). It was placed at a distance where the whole apparatus could be viewed while the pendulum swung.

2.1 Period

A period was measured by counting the number of frames to complete a cycle. A cycle begins when the pendulum reaches its lowest point. The pendulum always passes through this point, which is also its equilibrium. Since other points may not be reached as the pendulum continues to oscillate and decay, tracking the period over time is less viable at other points.

All presented T values are an average of the first three periods.

When comparing the initial angle to the period, 14 measurements were taken, 7 from positive angles and 7 from negative angles. The angle of release was measured live to give a rough estimate and ensure a range of values was measured. In the post analysis software *Tracker*, accurate values of the initial angle were measured with a protractor. Angles measured ranged from -1.36 to 1.42 ± 0.05 rad with each trial being changed by 0.17 ± 0.08 rad.

The length of the pendulum was fixed for these trails at $L = 43.5 \pm 0.3$ cm. All length measurements were taken using a 30 cm ruler. The ball was held at the top and bottom and released by hand.

2.2 Q-Factor and Angle

All Q-Factor Measurements were taken within the range of small angles (maximum angle of $\theta = 26.5 \pm 0.05$ rad). All videos were recorded for at least 60s.

To measure the angle over time, the Autotracker feature in *Tracker* was used. The template was set to Evolve: 40%, Tether: 5%, Automark: 4 and a Step Size: 5. By using a yellow ball with white tape and a marked black dot, the reference image of the contrasting black dot was selected. This drastically improves the accuracy tracking with fewer low confidence frames (where the software is unsure if it's tracking the right object). An axis was placed at the pivot point as a reference point to measure angles.

Using angle measurements, the amplitude and the number of oscillations are calculated from tracker data.

When comparing pendulum length to Q-Factor and period, 8 lengths were considered ranging from 17.6 to 80.5 ± 0.5 cm.

3 Results and Analysis

Period and Initial Angle

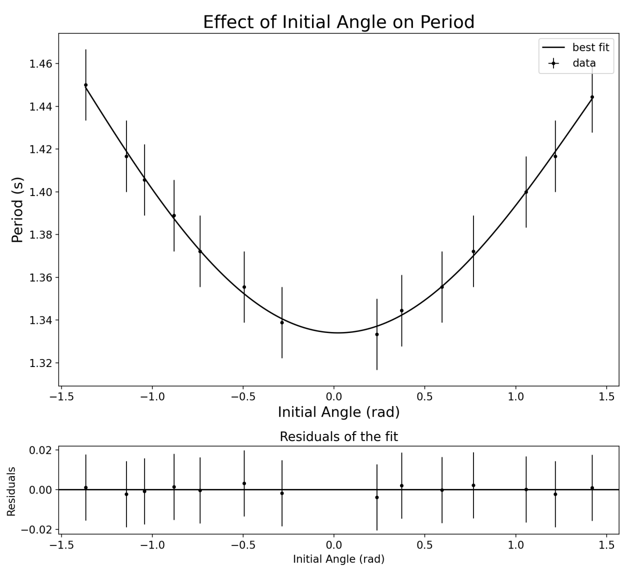

Figure 2: The plot shows how the period varies based on the initial angle of release for a pendulum of $L = 43.5 \pm 0.5$ cm. The data was fit to a quartic function $ax^4 + bx^3 + cx^2 + dx + e$ where $a = -0.005 \pm 0.001$, $c = 0.069 \pm 0.003$ and $e = 1.334 \pm 0.001$. The odd terms b and d are up to two times larger than their respective uncertainties, and therefore consistent with zero.

Comparing the quartic fit to the expected power series, it agrees with the absence of odd power terms. This also means the pendulum is symmetrical. For higher order even terms (a, b), their values are not experimentally 0, but get closer to it within uncertainties as the order increases. The residuals of the fit are within the period uncertainty of $\pm 0.03s$, meaning the function fitted is accurate.

A small angle range is present: $0.5 \leq \theta \geq 0.5rad$, where $T = 1.35s$, which lies within uncertainties for the plotted values in the range. Since the period changes less drastically in this range and can be approximated to a value, through the small angle approximation, releasing the ball within this range better agrees with the assumptions made by the model.

Calculating Q Factor

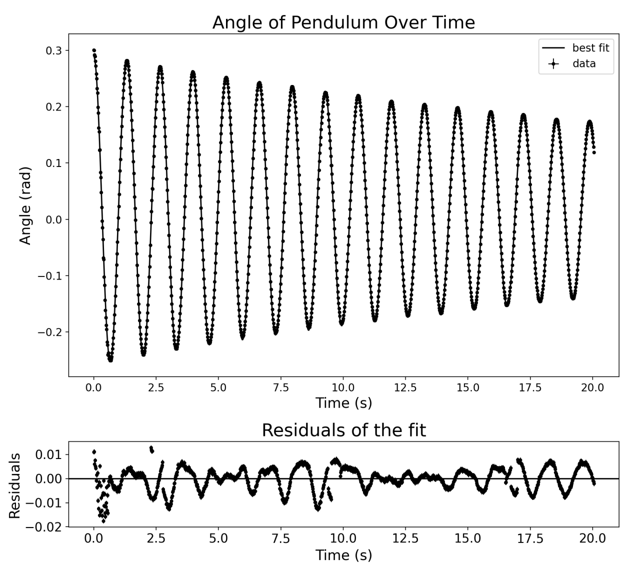

Figure 3: The plot shows how the position of the pendulum, described as an angle from the vertical, changes over time. The initial angle of release was chosen to be small, $\theta_o = 0.235 \pm 0.005$ rad. The best fit line is approximated by 4. The best fit parameters are $\theta_0 = 0.2737 \pm 0.0004rad, \tau = 34.7 \pm 0.2, T = 1.325 \pm 0.001s$. Due to the number of data points and small uncertainties, error bars are not visible. A zoomed-in version of the graph is given in the appendix.

Figure (3) shows that amplitude decreases exponentially over time, which can be thought of as smaller and smaller angles of release. According to Figure (2), different angles of release result in different periods. And therefore, as the amplitude changes, the period also changes slightly, but less and less at smaller angles where the decay is minimal.

Q-Factor is first measured using fitted parameters for the relationship between time and angle. The damped harmonic oscillator fit goes through all points, but its residuals do not fall within angle uncertainties ($\pm 0.009 rad$). It models the decaying behaviour of the pendulum due to air resistance and friction well, along with oscillation properties.

Using 5 we can numerically calculate the Q-Factor as $Q = \pi(\frac{34.7 \pm 0.2}{1.325 \pm 0.001}) = 82.2 \pm 0.2$.

The difference between maximum and minimum angles in each oscillation gives the amplitude. With amplitudes, we can use the number of oscillations to reach an amplitude $e^{-\frac{\pi}{4}} \approx 46\%$ of its initial amplitude. The number of oscillations gives $Q/4$.

Using Figure (3) to derive the amplitudes displayed in Figure (8), and comparing the amplitudes to 35% of the initial, yields a $Q = 86.00 \pm 0.01$. This states that it takes $\approx$91.50 oscillations for the pendulum to decrease its amplitude to $\approx$4%.

The two measured Q-Factors do not fall within each other's uncertainties and therefore show an inconsistency between the two methods.

3.1 Length and Period

Given the parameters p and q, and their range created by uncertainties, the parameters differ from the expected values of $p = 2$ and $q = 0.5$ given by equation 1. Given the current results and relatively low uncertainties, there is confidence in the results that empirical testing resembles a similar fit as the expected Equation 2, but with slightly different

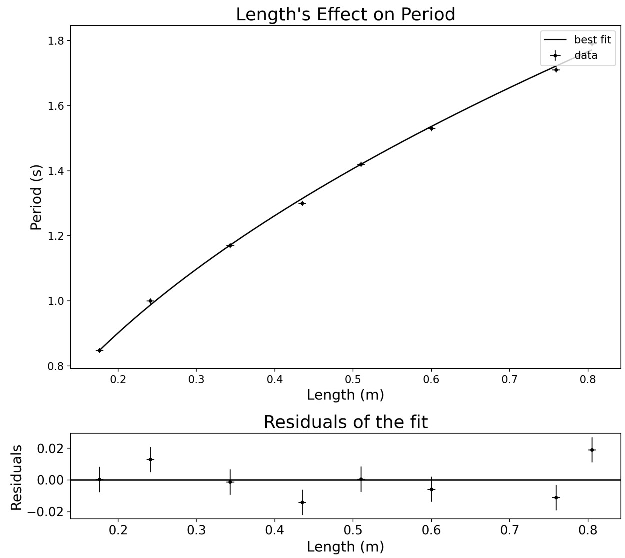

Figure 4: The plot shows how the period changes based on a pendulum's length. It is fit to Equation 2, a power function $T = pL^q$ with fitted parameters $p = 1.96 \pm 0.01$ and $q = 0.484 \pm 0.007$.

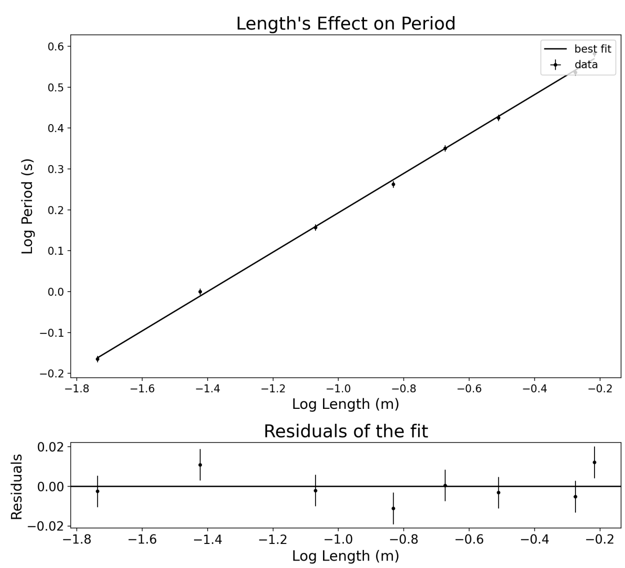

Figure 5: The plot also displays how the period changes based on a pendulum's length but as a log-log plot fit to a linear function. The function of $y = ax + b$ has parameters $a = 0.481 \pm 0.006$ and $b = 0.674 \pm 0.005$

fitted parameters.

3.2 Q Factor

Since the uncertainty for calculating Q-Factors by oscillations (± 0.01) is less than the uncertainty by fitted parameters (± 0.2), the method of oscillations is used. For this, the number of oscillations until the amplitude decays by $\approx 46\%$ is used.

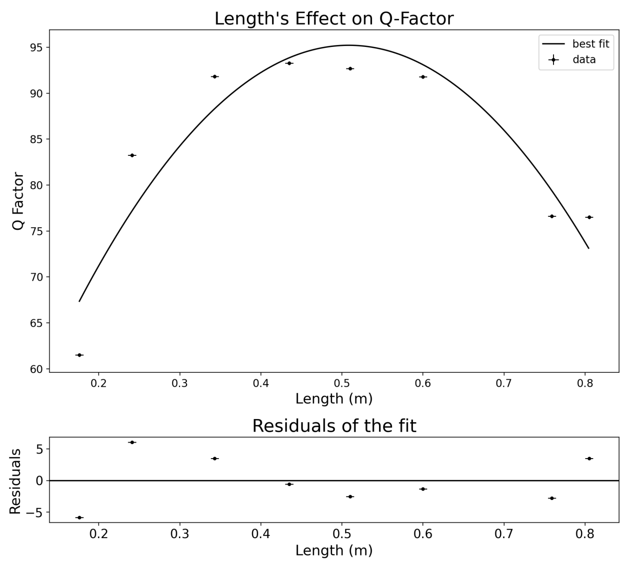

Figure 6: The pendulum's Q Factor and Length are plotted and fitted using a quadratic function $a * x^2 + b * x + c$. Where $a = -250 \pm 40, b = 25 \pm 40, c = 30 \pm 9$.

The large uncertainties and general lack of pattern in the data suggest that there is no relationship between Q-Factor and Length.

Uncertainties

For Figure 2, measurement uncertainties exist for the initial angle and identifying when the pendulum has reached its lowest point. The measurement uncertainty due to the precision of device graduation for the initial angle is $\pm 0.1 deg$, which is approximately $\pm 0.001 rad$. There is also uncertainty around where the protractor is placed in *Tracker*. Looking at the range in angles for various measurements at the pivot point to the string gives an uncertainty of $\pm 0.008 rad$, which is larger than the previous uncertainty and therefore will be used. This is calculated by moving the protractor across various points on the vertices (the pivot point and the ball).

The period uncertainty is $\pm 1 frame$ or $\pm 0.01 s$ (at 60fps) since empirically, in each video, the instant at which the pendulum reaches the bottom lies between two frames. The time between two frames is $\frac{1}{60} s$. A higher fps would help reduce this uncertainty.

Measuring the apparatus uncertainty (how much results change given the same trial conditions) was difficult due to live measurement errors and lack of precision. Three independent trials with angles between $0.73 \pm 0.01 rad$ and $0.76 \pm 0.01 rad$ yielded period values also with $\pm 0 frames$, which falls within the period uncertainty discussed above.

For tracking the angle over time, the angle measured by *Tracker* has measurement uncertainties. Since Autotracker is being used, the point which is tracked on the ball changes from frame to frame. At 60fps, a random sample of $n = 10$ frames is taken and angles are adjusted to the center. In addition, the coordinate axis and identification of the ball's centre itself hold uncertainties in angle measurement. In adjusting each of these uncertainties to find the range of angle variation, the uncertainty given is $\pm 0.009 rad$. This is the largest uncertainty. This uncertainty applies to the period vs angle graph and Q-Factor vs length graph.

Amplitude uncertainty stems from the angle that could be travelled in between frames when at its highest extremes. It is found that the ball is in between a maximum of 4 frames during its highest extremes. This uncertainty is already small since the ball's angular displacements are smallest at the highest points. The uncertainty given by the angular velocity ($\omega = 0.002$) over a 4 frame interval at 60fps is $\pm 0.01 rad$. However, measurements also differ from tracking uncertainty previously mentioned but are smaller than the uncertainty from potential displacement. The number of oscillations is a counted value and therefore has no uncertainty.

Finally, length uncertainties arise from ruler precision and measurement errors. When measuring the same pendulum length three times, the range is found to be ± 0.3 cm, which is greater than the precision uncertainty (± 0.01 cm). This is mainly because a 30cm ruler was used, and therefore, the ruler had to be moved for some measurements. A measuring tape, rather than a rigid ruler that spans greater than string length could be used to reduce error.

To reduce uncertainties related to the initial angle, clear markings can be placed on the pivot point and ball to ensure the angle is measured in the same place, reducing the amount of angle variation. A more graduated protractor would also help but likely have significant costs.

Additional uncertainties involve slightly rotated videos since it has already been minimized by post analysis (by slightly rotating images to become horizontal). Movement in and out of the video plane has been reduced by the two-string design.

4 Appendix

References

[1] Brown, D. (2023). Tracker. https://physlets.org/tracker/

[2] Wilson, B. (2023). PHY180 Pendulum Project Guidelines. https://q.utoronto.ca/courses/324650/files/27269654?module_item_id=4893671

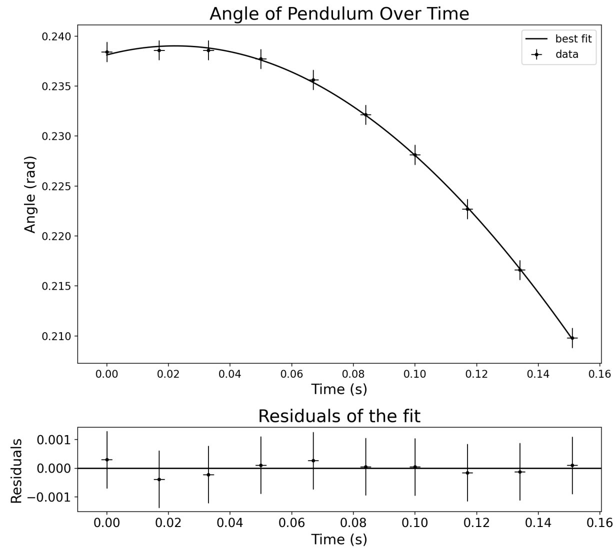

Figure 7: A zoomed-in version of Figure 3 with visible error bars. The angle uncertainty is given by $\pm 0.006 rad$.

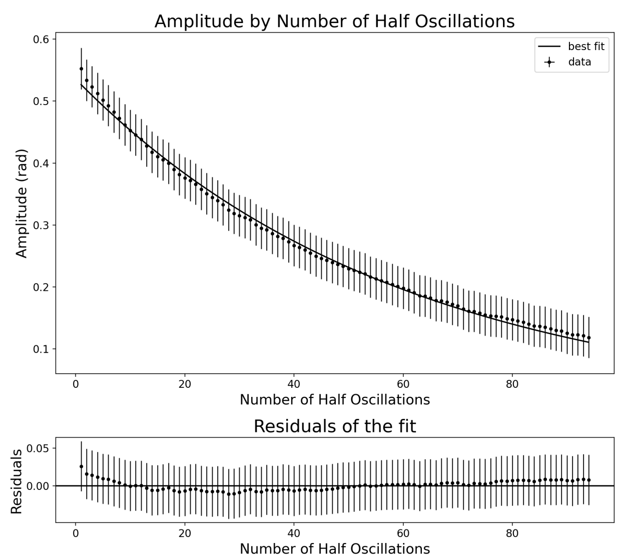

Figure 8: This graph confirms how the amplitude decays exponentially as the number of oscillations increases. The amplitude is recorded every time a maximum and minimum angle is reached, which is only half an oscillation. So finding the value that's 20% of the initial amplitude gives the Q-Factor.