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Mesures spectrales d'opérateurs aléatoires

Spectral measures of random operators

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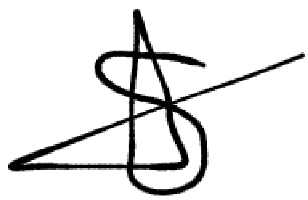

Liste de publications et participation aux conférences

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1. Arras, Adam, and Charles Bordenave. "Existence of absolutely continuous spectrum for Galton–Watson random trees." Communications in Mathematical Physics 403.1 (2023) : 495-527.

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5. GDR MEGA Probability Summer School, Aussois
6. Probability seminar, *IRMA*, Strasbourg
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It bugs me when people try to analyse jazz as an intellectual theorem. It's not. It's feeling.

Bill Evans

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Résumé et mots clés

Cette thèse se concentre sur l'analyse spectrale dans le cadre aléatoire et discret. Nous étudions l'opérateur des plus proches voisins agissant sur un graphe infini et considérons deux types d'aléas. Les graphes obtenus par un processus de branchement, appelés arbres de Galton-Watson. Les perturbations identiquement distribuées sur la diagonale encodant la présence d'un champ aléatoire, connue sous le nom du modèle d'impureté d'Anderson. Notre premier résultat est un nouveau critère de stabilité du spectre absolument continu sur les arbres, uniforme en le degré moyen du graphe. Notre deuxième résultat est une correspondance spectrale entre le modèle d'Anderson et les opérateurs de convolution déterministes sur les groupes. À la fin de cette thèse, nous discutons d'une approche de l'absence de spectres singuliers, appelée méthode du commutateur.

Mots clés : théorie spectrale des graphes, opérateurs aléatoires, processus de Galton-Watson, modèle d'Anderson.

Abstract and keywords

This thesis focuses on spectral analysis in a discrete random framework. We consider nearest neighbor operators acting on infinite graphs with two different types of randomness. On the one hand, the unimodular Galton-Watson model which is an random rooted tree obtained by a branching process. On the other hand, identically distributed diagonal perturbation that encodes the presence of a random field, known as the Anderson impurity model. Our first result is a new stability criterion for the absolutely continuous spectrum on the trees, uniform in the average degree of the graph. Our second result is a spectral correspondence between the Anderson model and the deterministic convolution operators on the groups. At the end of this thesis, we discuss an approach to the absence of singular spectra, called the commutator method.

Keywords: spectral graph theory, random operators, Galton–Watson process, Anderson model.

Contents

1 Introduction

Graphs have been a convenient mathematical structure to describe many physical phenomena. Dynamical processes such as the classical random walk, the heat diffusion and the quantum evolution often reduce to the study of spectral properties of the corresponding operators. Mixing properties for the classical random walk and heat diffusion are related to the global features of the Markov generator such as the existence of a spectral gap. In contrast, due to the underlying oscillatory phenomena, the mixing of the quantum evolution depends on a more subtle property of the spectrum: the regularity of the spectral measures. In the present work, we study spectral measures in two different frameworks:

— Galton-Watson random trees,

— Cayley graphs of groups.

The Galton-Watson tree is a recursive random process which naturally appears as the local limit of many finite graph models, when the number of edges scales as the number of vertices. For instance, the sparse Erdös-Renyi graph converges to the Galton-Watson model with Poisson offspring distribution. Cayley graphs are graphs together with a simply transitive group action, that preserves the graph structure. This is a convenient setting to study spectral measures through the lens of symmetries.

In the present introduction we will provide a detailed overview of the context and motivations, present our main contributions and finally discuss perspectives and open problems.

This thesis is composed of the following chapters:

Chapter I is a quick and formal introduction to spectral graph theory. Its main purpose is to set up the notations and motivate the work presented in the following chapters.

Chapter II provides a general criterion for the existence of an absolutely continuous component in the spectrum for the Galton-Watson process. It is a joint work with Charles Bordenave and has been published in Communications in Mathematical Physics.

Chapter III gives a general spectral correspondence between the random Schrödinger operator and the deterministic convolution operator on groups.

Chapter IV discusses the spectral measure of convolution operators. This is an ongoing work with Christophe Pittet.

1.1 Spectral theory of graphs

1.1.1 From graphs to operators

This thesis mostly deals with simple and undirected graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Since we do not allows for multiples edges, we consider edges as unordered pairs of vertices and write $x \sim y$ instead of $e = \{x, y\} \in \mathcal{E}$. An equivalent way to encode $\mathcal{G}$ is via the two dimensional array $A = (A_{xy})_{xy\in\mathcal{V}}$ with zero-one entries $A_{xy} = 1_{x\sim y}$. We will always assume that $\mathcal{G}$ is connected and locally finite, in particular $\mathcal{V}$ is at most countable. The powers of A are then well defined, and matrix multiplication implies that the entry $(A^k)_{xy}$ equals the number of walks from x to y of length k.

When one considers a function $\varphi\colon \mathcal{V} \to \mathbb{C}$, it is often convenient to impose the square summability condition to benefit from the theory of Hilbert spaces. This space is denoted $l^2(\mathcal{G})$ with canonical orthonormal basis $\delta_x = (1_{\{x=y\}})_{y\in\mathcal{V}}$. The array A naturally extends to a densely defined symmetric operator, called the adjacency operator. For finitely supported functions $\varphi \in l^2_c(\mathcal{G})$:

$$(A\varphi)(x) = \sum_{y\sim x} \varphi(y).$$

Spectral graph theory proposes to study various properties of $\mathcal{G}$ from this geometric viewpoint. More generally, physical problems involving graph structures often reduces to the spectral analysis of a local and self-adjoint operator. Throughout this thesis, we will adopt the notation of Schrödinger operators of the form *Laplacian plus potential* $H = L + V$:

$$(H\varphi)(x) = \sum_{y\sim x} L_{xy}\varphi(y) + v(x)\varphi(x). \tag{1.1}$$

Here $V = \mathrm{Diag}(v(x))_x$ is the multiplication operator by some potential $v\colon \mathcal{V} \to \mathbb{R}$. There are several choices for L. Important examples are the following:
— The Laplacian $D - A$,
— The combinatorial Laplacian $I - D^{-1/2}AD^{-1/2}$,
— The Markov operator : $D^{-1}A$,
where $\mathrm{Diag}(\deg(x))_x$ is the diagonal operator of vertex degree $\deg(x) = \#\{y \in \mathcal{V} | y \sim x\}$. Note that the Markov operator is in general symmetric for a different scalar product. When $\mathcal{G}$ is of constant degree, all these notions are equivalent since all these operators commutes. There are also some generalisation where L_{xy} is non-zero for non adjacent vertices, but decays with respect to the graphs distance.

1.1.2 Spectral measures of operators

For continuous spaces such as $\mathbb{R}^d$, most of the interesting operators are unbounded. This is also the case in our discrete setting $\mathcal{G}$, as soon as the sequence of degree or the potential is unbounded. The Hellinger–Toeplitz theorem asserts that on a Hilbert space $\mathcal{H}$, an everywhere defined symmetric operator is bounded [RS+80]. Therefore, any self-adjoint unbounded operator H has to come with an associated proper subset $D(H) \subset \mathcal{H}$, called its domain. We do not discuss the technicalities of self-adjointness here, and refer to [BLS11] for a treatment of this question in our context. We simply note that we will only consider operators of the form (1.1) that are essentially-self adjoint on compactly supported vectors, meaning that they admit a unique self-adjoint extension to some domain $D(H)$.

There are many different approaches to the spectral decomposition of self-adjoint operators, each of them corresponding to a formulation of the spectral theorem. We will employ the resolvent formalism to define the spectral measures μ_H^φ. The spectrum $\Sigma(H)$ is the set of complex parameters for which $H - z\colon D(H) \to \mathcal{H}$ fails to be invertible, and we recall that this is a closed subset of the real axis for self-adjoint operators. The resolvent set is the complement $\mathbb{C} \setminus \Sigma(H)$, on which we define the operator $G(z) = (H - z)^{-1}$. To each $\varphi \in l^2(\mathcal{V})$ corresponds the diagonal element of the resolvent $g_H^\varphi(z) = \langle \varphi, G(z)\varphi \rangle$. The first resolvent identity $G(z_2) - G(z_1) = (z_2 - z_1)G(z_2)G(z_1)$ shows that $G(z)$ is analytic and that $\mathbb{C}^+$ is stable by $g_H^\varphi(z)$. In particular, its imaginary part is a positive harmonic function. The Herglotz-Nevanlinna representation theorem [Sim19, Theorem 3.4] asserts that any such function admits an integral representation on the boundary of their domain, this is a special type of Poisson formulae. We obtain the following:

Proposition 1 (spectral theorem, measure formulation)**.** *Given a self-adjoint operator $(H, D(H))$ and a unit vector φ in $\mathcal{H}$, there is a unique Borel probability measure μ_H^φ such that:*

$$g_H^\varphi(z) = \langle \varphi, (H - z)^{-1}\varphi \rangle = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_H^\varphi(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}^+.$$

Polarisation identities allow to associate a Borel (complex) measure $\mu_H^{\varphi,\psi}()$ to each off-diagonal element of the resolvent $g_H^{\varphi,\psi}(z) = \langle \varphi, (H - z)^{-1}\psi \rangle$. On the space $l^2(\mathcal{G})$, we will mostly be concerned with canonical vectors, in which case we lighten the notation by writing g_H^x instead of $g_H^{\delta_x}$, similarly for $g_H^{x,y}$, μ_H^x and $\mu_H^{x,y}$. These measures are the cornerstone of spectral theory. Under mild assumption on a measurable function $f\colon \mathbb{R} \to \mathbb{C}$, an operator $f(H)$ is defined through

$$D(f(H)) = \left\{ \varphi \in \mathcal{H} \middle| f \in L^2(\mathbb{R}, \mu_H^\varphi) \right\}, \tag{1.2}$$

$$\langle \varphi, f(H)\psi \rangle = \int_{\mathbb{R}} f(\lambda)\mathrm{d}\mu_H^{\varphi,\psi}(\lambda). \tag{1.3}$$

An important class of such functions are characteristic function $1_A(\cdot)$ of a Borel set

A. In this case, the operator $1_A(H)$ is an orthogonal projections usually denoted $E_H(A)$. Since $\mu_H^\varphi(A) = \|E_H(A)\varphi\|^2$, the map $E\colon A \mapsto E_H(A)$ has properties similar to a probability measure. It sends the empty set and the set $\mathbb{R}$ respectively to the null and the identity operator, and is sigma additive for the strong operator topology. The symbol $E(A) = \int_A \mathrm{d}E$ has therefore a precise meaning and is called a projection valued measure. We obtain another formulation of the spectral theorem where the operator integral must be interpreted in a similar way to (1.3), we refer to [RS+80, Theorem VIII.6] for a precise definition.

Proposition 2 (spectral theorem, projection-valued measure formulation)**.** *There is a one to one correspondence between self-adjoint operators $(H, D(H))$ and projection valued measures $E(\cdot)$ on $\mathcal{H}$, given by*

$$H = \int_{\mathbb{R}} \lambda \mathrm{d}E(\lambda), \quad D(H) = \left\{ \varphi \in \mathcal{H} \middle| \int_{\mathbb{R}} \lambda^2 \langle \varphi, \mathrm{d}E(\lambda)\varphi \rangle < \infty \right\}.$$

Spectral measures in finite dimension. Since an Hermitian matrix $H \in \mathbb{C}^{n\times n}$ always admits othonormal basis of eigenvectors, spectral measures μ_H^φ have a simple interpretation in finite dimension. As the matrix H^k can be computed in an eigenbasis (φ_i) by the powers λ_i^k of the corresponding eigenvalues, one has $g_H^x(z) = \langle \delta_x, (H-z)^{-1}\delta_x \rangle = \sum_{i=1}^n (\lambda_i - z)^{-1} |\langle \varphi_i, \delta_x \rangle|^2$. It follows that the spectral measures μ_H^x are convex combination of Dirac masses supported on the eigenvalues, with weights depending on the localization of the eigenfunctions:

$$\mu_H^x = \sum_{i=1}^n |\varphi_i(x)|^2 \delta_{\lambda_i}. \tag{1.4}$$

An important observation is that, by averaging over any basis (δ_x), one recovers the empirical spectral distribution (ESD) of the matrix H:

$$\bar{\mu}_H := \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i} = \frac{1}{n} \sum_{x \in \mathcal{V}} \mu_H^x. \tag{1.5}$$

1.1.3 Lebesgue decomposition of spectral measures

In the infinite case, the behavior of the measures μ_H^x becomes richer. Particularly interesting is the unique Lebesgue decomposition into mutually singular measures:

$$\mu_H^x = \mu_{\mathrm{pp}}^x + \underbrace{\mu_{\mathrm{sc}}^x + \mu_{\mathrm{ac}}^x}_{\mu_{\mathrm{c}}^x}, \tag{1.6}$$

The purely atomic part μ_{pp} is supported on the eigenvectors of H, generalizing (1.4) to the infinite case. The continuous part μ_{c} is itself composed of an absolutely continuous part μ_{ac} and a singular continuous μ_{sc}. This decomposition can be lifted to the Hilbert as follows. One can define the projection valued measure E_H^s for $s \in \{\mathrm{pp}, \mathrm{sc}, \mathrm{ac}, \mathrm{c}\}$ through the equality $\mu_s^\varphi(A) = \|E_H^s(A)\varphi\|^2$, to obtain the direct sum

$$l^2(V) = \mathcal{H}_{\mathrm{pp}}(H) \oplus \mathcal{H}_{\mathrm{sc}}(H) \oplus \mathcal{H}_{\mathrm{ac}}(H),$$

where $\mathcal{H}_s(H) = \{\varphi \in l^2(\mathcal{V}) | \mu_H^\varphi = \mu_s^\varphi\}$. Similarly, the spectrum decomposes on non-necessarily disjoint subsets

$$\Sigma(H) = \Sigma_{\mathrm{pp}}(H) \cup \Sigma_{\mathrm{sc}}(H) \cup \Sigma_{\mathrm{ac}}(H). \tag{1.7}$$

An important motivation in studying these different types of spectra is that they lead to complementary behavior in quantum mechanics, as we explain in the next subsection.

The quantum mechanical viewpoint. Consider a single particle M on the vertex set of a given graph $\mathcal{G}$, moving according to the laws of quantum physics. Any unit vector $\varphi \in l^2(\mathcal{G})$ defines a possible state of the system, up to the global phase $e^{i\alpha}\varphi$. The probability $P_\varphi(M \in \Omega)$ to find the particle M in a given region $\Omega \subset \mathcal{V}$ is the sum of the one-site probabilities

$$P_\varphi(M \in \{x\}) = |\langle \delta_x, \varphi \rangle|^2 = |\varphi(x)|^2.$$

The time evolution of the system is governed by the Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = H\varphi_t, \tag{1.8}$$

where H is a self-adjoint operator of the form (1.1). Its solution is given in terms of the unitary group generated by H. From the Stone–Von Neumann theorem or simply by functional calculus

$$\varphi_t = e^{itH}\varphi. \tag{1.9}$$

Since $e^{itH} = \int_{\mathbb{R}} e^{it\lambda}\mathrm{d}E_H(\lambda)$, a surprising reformulation of the spectral theorem is that any quantum system is unitary equivalent to a sum of harmonics oscillators [1]. This is precisely the oscillatory nature of the dynamics that implies that the long time behavior of the system is sensible to the decomposition (1.6). The return probability of a particle initially at site x is

$$P_{\delta_x}^t(M \in \{x\}) = |\langle \delta_x, e^{itH}\delta_x \rangle|^2 = \left| \int_{\mathbb{R}} e^{it\lambda}\mathrm{d}\mu_H^x(\lambda) \right|^2 = |\widehat{\mu_H^x}(t)|^2,$$

where appears the Fourier transform of the spectral measure. It follows from Riemann-Lebesgue lemma that as soon as the measure μ_H^x is purely absolutely

1. More precisely, this is a direct integral.

continuous, the return probability goes to zero as the time goes to infinity. More generally, the limiting return probability always converges for the weaker notion of Cesàro limit. From Wiener's lemma, one has

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} P_\varphi^t(M \in \{x\}) \mathrm{d}t = \sum_{x\in\mathbb{R}} |\mu_{\mathrm{pp}}(\{x\})|^2.$$

Note that the l.h.s has the interpretation of the average time spent at x. In the above discussion, the site x does not plays any special role. We can start the dynamics from the initial state $\varphi \in l^2(\mathcal{V})$ and ask if the particle is at a given site $y \in \mathcal{V}$. An important observation is that the off-diagonal complex measure $\mu_H^{\delta_y,\varphi}$ is always absolutely continuous with respect to the spectral measure μ_H^φ of the initial state [2]. From this, one obtains the following characterisation, often called the R.A.G.E theorem due to its four contributors: Ruelle [Rue69], Amrein-Georgescu [AG73] and Enss [Ens78]).

Proposition 3 (Rage theorem). *Let* (Ω_r) *be any increasing sequence of finite subsets such that* $\cup_{\mathbb{N}}\Omega_r = \mathcal{V}$. *Then*

— $\varphi \in \mathcal{H}_c$ *iff the particle will spend a vanishing fraction of time in any compact region. More precisely*

$$\lim_{r\to\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} P_\varphi^t(M \in \Omega_r) \mathrm{d}t = 0,$$

— $\varphi \in \mathcal{H}_{\mathrm{pp}}$ *iff the particle will stay in any large enough compact region, up to an arbitrarily small probability for any* t. *More precisely*

$$\lim_{r\to\infty} \sup_{t\in\mathbb{R}} P_\varphi^t(M \notin \Omega_r) = 0.$$

Furthermore

— *If* $\varphi \in \mathcal{H}_{\mathrm{ac}}$, *then the particle will leave any compact region, i.e.*

$$\lim_{r\to\infty} \lim_{t\to\infty} P_\varphi^t(M \in \Omega_r) = 0.$$

When the spectral measure is continuous, there are also quantitative versions of the decays of the return probability, depending on the Hölder regularity of the measure [AW15]. For this reason, states with purely atomic spectral measure are often called bound states, since the particle has a tendency to remain localized. In contrast, states with absolutely continuous measure are referred as scattering states. Singular continuous states have somehow an intermediate behavior. Note that for many systems, such as the Galton-Watson random tree discussed later on in this thesis, a generic state is composed of a mixture of the different types of

2. This follows from $|\mu_H^{\psi,\varphi}(A)| = |\langle\psi, E_H(A)\varphi\rangle| = |\langle E_H(A)\psi, E_H(A)\varphi\rangle| \leq \sqrt{\mu_H^\psi(A)\mu_H^\varphi(A)}$

spectra.

1.2 Motivations

1.2.1 The Anderson model

In his seminal work [And58], the physicist P.W.Anderson has studied the transport properties of disordered crystals. This quantum analogue of random walks in a random environment is modeled as follows. One starts with a regular lattice $\mathcal{G} = \mathbb{Z}^d$ which encloses the global symmetry of the crystal. The presence of impurity at small scale is represented by an independent and identically distributed random field $(v(x,\omega))_{x\in\mathbb{Z}^d}$ over the vertices. Setting $H(\omega) = A + \lambda V$, one obtains a random Schrödinger operator of the form (1.1), where $\lambda \geq 0$ is a disorder parameter. At a fixed realisation $\omega \in \Omega$, one then considers the unitary evolution (1.9). Since the random operator is invariant by the translation group $\mathbb{Z}^d$, it follows from ergodicity theory that global features of $H(\omega)$ are almost sure [AW15, Theorem 3.10], including the decomposition (1.7). In the one dimensional case $\mathbb{Z}$, it turns out that even arbitrarily small randomness completely breaks down the transport properties of the medium. This translates for H to purely discrete spectrum and exponential decay of eigenfunctions. This phenomenon, that also occurs in any dimensions as soon as the disorder parameter is large enough, is referred as Anderson's localization [Sto11; Hun08]. It is by now reasonably understood from a mathematical viewpoint, since the work of Fröhlich and Spencer (via multi-scale analysis [FS83]) and Aizenman and Molchanov (via fractionnal moment method [AM93], see also [Gra94]). In contrary, the conjectured persistence at low disorder of transport in dimension $d \geq 3$, and therefore the stability of an absolutely continuous spectrum, remains a challenging open question despite considerable efforts of the community. There are known results if the potential decays at infinity [Bou02; Bou03], but the presence of absolutely continuous spectrum for a stationary potential has only been proven for trees. This has been achieved first by Klein [Kle98], on the infinite d-regular tree $\mathbb{T}_d, d \geq 3$, and generalized by different approaches since then [FHS07; AW15; KLW12; WA13]. Our contribution to this subtle question [AB23, Theorem 3], is a simplified and quantitative proof of Klein's theorem [Kle98], which is uniform in the degree of the regular tree lattice.

1.2.2 Graphs from the local viewpoint

In this paragraph, we discuss the notion of local weak limit graphs, introduced by Benjamini and Schramm [BS11] and further developed in [AS04; AL07]. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ together with a distinguished vertex o, we call unlabeled rooted graph the equivalent class $[\mathcal{G}, o]$ obtained by root preserving graph morphisms. The set of all such graphs $g = [\mathcal{G}, o]$, where $\mathcal{G}$ is further assumed to be connected and locally finite, is denoted G^*_{loc}. The starting point is to consider two such elements close if an observer standing at the root cannot distinguish between the two graphs by solely considering vertices up to some large but finite distance from his position. Concretely, this can be done with the distance

$$\mathrm{d}_{G^*_{loc}}(g, g') = \sum_{r=1}^{\infty} 2^{-r} 1_{\{g_r = g'_r\}},$$

where g_r is obtained from g by restriction to vertices located at a distance of at most r from the root. This turns G^*_{loc} into a complete separable metric space, i.e. a Polish space. Consider now the set $\mathcal{P}(G^*_{loc})$ of probability distribution on G^*_{loc} endowed with the notion of weak convergence. A sequence of distribution (μ_n) is said to converge towards μ_∞ iff for any continuous bounded real function $f \in C_b(G^*_{loc})$, one has

$$\lim_n \int_{G^*_{loc}} f(g) \mathrm{d}\mu_n(g) = \int_{G^*_{loc}} f(g) \mathrm{d}\mu_\infty(g).$$

It follows from the general theory that $\mathcal{P}(G^*_{loc})$ is also a Polish space, the topology being metrizable for the Lévy–Prokhorov distance that we do not discuss here, see e.g. [Bil13]. Introducing probability distribution over G^*_{loc} has the following benefit. There is now a canonical way to associate to each finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ an element of $\mathcal{P}(G^*_{loc})$, by simply choosing the root uniformly at random:

$$U(\mathcal{G}) = \frac{1}{|\mathcal{V}|} \sum_{x \in \mathcal{V}} \delta_{g_x}. \tag{1.10}$$

where $g_x = [\mathcal{G}_x, x]$ denotes the (equivalent class) of the connected component of x in $\mathcal{G}$, rooted at x. We emphasize that this is not an injective process. For example the disjoint union of two copies of the same graph has the same measure. A sequence $(\mathcal{G}_n)$ of finite graphs has a local weak (or Benjamini-Schramm) limit if the sequence $U(\mathcal{G}_n)$ converges in $\mathcal{P}(G^*_{loc})$. Consider a sequence $\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$ of d-regular graph of increasing size. Under the assumption that the short cycle density vanishes,

$$\forall k \geq 3\colon \lim_n \frac{\#\{\text{cycle of lengh } k \text{ in } \mathcal{G}_n\ \}}{|\mathcal{V}_n|} = 0, \tag{1.11}$$

the sequence $\mathcal{G}_n$ converges to the Dirac mass at the infinite d-regular rooted tree $\delta_{[\mathbb{T}_d, o]}$. Note that the short cycle assumption is generic as (1.11) is true with probability one for sequence of uniformly chosen d-regular graph [MWW04]. Many random graph models in the constant expected degree regime have a local weak limit. Many important graph features, such as the empirical spectral distribution, are continuous under this notion of convergence. The sparsity implies that, in most cases, the limiting object is a random rooted tree and is therefore often simpler to study due to its recursive structure. Among the possible limiting graphs, an important role is played by the (unimodular) Galton Watson tree, which constitutes the main object of interest of Chapter II. It appears as the local weak limit in the Erdős–Rényi model, high-dimensional lattice percolation, Stochastic block model and the configuration model under mild assumption on the given degree sequence.

1.2.3 Quantum percolation

Consider the random graph $\mathcal{G} \sim \mathrm{Perc}(\mathcal{G}_0, p)$ obtained by performing a bond percolation of intensity $0 \leq p \leq 1$ on a infinite transitive graph $\mathcal{G}_0$ such as $\mathbb{Z}^d$ or $\mathbb{T}_d$, and $A = A_{\mathcal{G}}(\omega)$ the associated adjacency random operator. The percolation threshold p_c is the largest intensity such that the graph almost surely consists of finite connected components. De Gennes, Lafore and Millot [DLM59b; DLM59a] introduced the following quantity under the name of quantum percolation threshold

$$p_q = \sup\{p \geq 0 \,|\, \mu_A^x \text{ has trivial } a.c. \text{ com with probability one}\}. \tag{1.12}$$

A simpler question is to study the averaged spectral measure $\bar{\mu}^x = \mathbb{E}\mu_A^x$. For a stationary random graph, this measure generalizes the identity (1.5) and is known as the density of states (DOS) measure in physics literature [KM07]. The associated mean percolation threshold

$$\bar{p}_q = \sup\{p \geq 0 \,|\, \bar{\mu}^x = \mathbb{E}\mu_A^x \text{ has a trivial continuous component}\}, \tag{1.13}$$

has been studied in [BSV17; CS21]. These two thresholds generalize the classical percolation problem since we have

$$p_c \leq \bar{p}_q \leq p_q.$$

Unlike the classical case, in quantum percolation there is no known monotonicity relating the intensity parameter p and the weight of the atoms of μ_A^x and $\bar{\mu}^x$. Physically, quantum percolation can be viewed as a limiting case of the Anderson model with a binary potential $v(x) \in \{0, \lambda\}$, in the limit $\lambda \to \infty$. However, the methods to deal with these two problems differ. Since the percolation Hamiltonian A has only zero-one entries, one can use algebraic tools such as the Lück's approximation theorem [ATV13; Bor16] (see also [Sal15; Sal19; CS21]). In [AB23, Theorem 1], our contribution to quantum percolation, we use the fact that eigenvalues of finite trees are supported on algebraic integers [AB23, Lemma 7] to prove that sub-critical Galton-Watson tree has non-trivial absolutely continuous part as soon as the relative variance of the offspring distribution is small.

2 Shrodinger operators on Galton-Watson tree

This chapter is a joint work with Charles Bordenave.

This chapter concerns spectral measures of Galton-Watson random trees. To illustrate our results, we plot the empirical spectral distribution (ESD) and the participation ratio (PR) of the corresponding eigenvectors for Erdös–Rényi random graphs of various average degrees. The participation ratio is defined as the quotient between l^4 and l^2 norms of the eigenvectors, and it measures their localization with respect to the canonical basis.

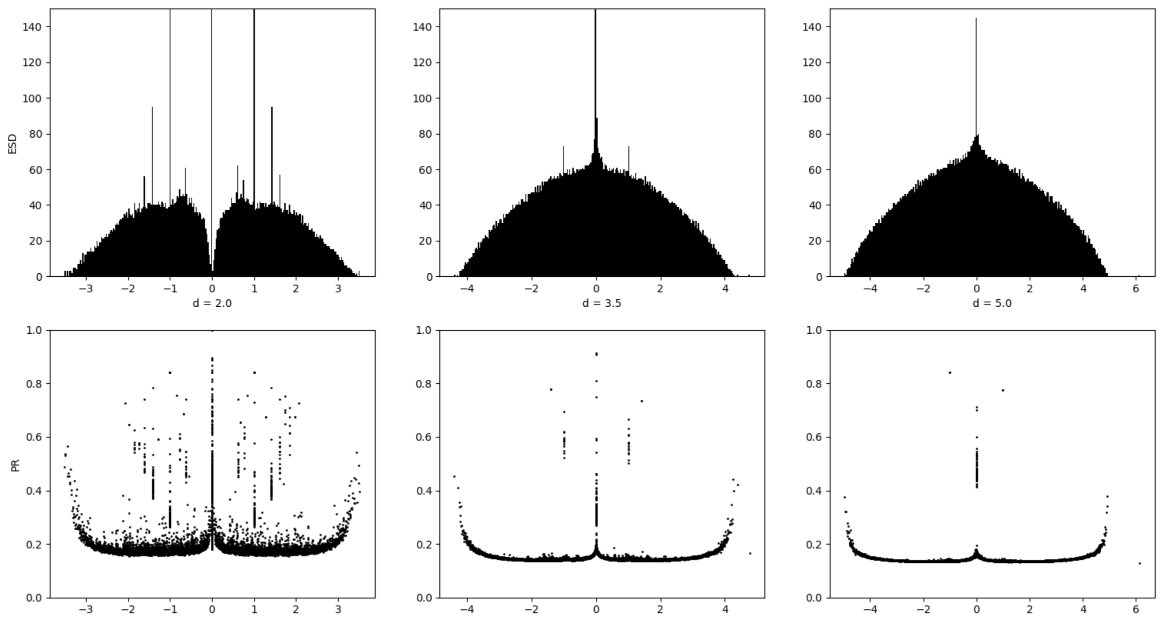

Figure 2.1 – ESD and PR for $\mathcal{G}(n, d/n)$ with $n = 10^4$ and $d \in \{2, 3.5, 5\}$.

For small expected degree, we observe spikes in the ESD, with a corresponding localized eigenvector. This phenomenon decreases as the expected degree increases.

We recall that the Erdös–Rényi model converges, in the regime of constant average degree, to the Galton-Watson tree with Poisson distribution. This suggests that for the Galton-Watson process, conditionally to its non-extinction, the spectral measure at any vertex is a mixture of a continuous and an atomic part. For infinite tree with leaves, it is not difficult to see that finitely-supported eigenfunctions can occurs, see for example [Sal15; Sal20]. We will show that if the relative variance of the offspring distribution is small, there is indeed an part absolutely continuous. The first section corresponds to the paper [AB23] and has been published in the academic journal Communications in Mathematical Physics. We have added some comments at the end of this chapter.

2.1 Absolutely continuous spectrum for Galton-Watson random tree

* * * * * * * * *

Abstract. We establish a quantitative criterion for an operator defined on a Galton-Watson random tree for having an absolutely continuous spectrum. For the adjacency operator, this criterion requires that the offspring distribution has a relative variance below a threshold. As a by-product, we prove that the adjacency operator of a supercritical Poisson Galton-Watson tree has a non-trivial absolutely continuous part if the average degree is large enough. We also prove that its Karp and Sipser core has purely absolutely continuous spectrum on an interval if the average degree is large enough. We finally illustrate our criterion on the Anderson model on a d-regular infinite tree with $d \geq 3$ and give a quantitative version of Klein's Theorem on the existence of absolutely continuous spectrum at disorder smaller than $C\sqrt{d}$ for some absolute constant C.

2.1.1 Introduction

In this paper, we study operators on an infinite random tree. We establish a quantitative criterion for the existence of an absolutely continuous spectrum which relies on the variance of the offspring distribution.

Galton-Watson trees. The simplest ensemble of random trees are the Galton-Watson trees. Let P be a probability distribution on the integers $\mathbb{N} = \{0, 1, \ldots\}$. We define $\mathrm{GW}(P)$ as the law of the tree $\mathcal{T}$ with root o obtained from a Galton-Watson branching process where the offspring distribution has law P (the root has a number of offspring N_o with distribution P and, given N_o, each neighbour of the root, has an independent number of offspring with distribution P, and so on). For example, if P is a Dirac mass at integer $d \geq 2$, then $\mathcal{T}$ is an infinite d-ary tree.

An important current motivation for studying random rooted trees is that they appear as the natural limiting objects for the Benjamini-Schramm topology of many finite graphs sequences, see for example [van17; Bor18]. The limiting random rooted graphs that can be limiting points for this topology must satisfy a stationarity property called unimodularity. The law $\mathrm{GW}(P)$ is generically not unimodular. To obtain a unimodular random rooted tree, it suffices to bias the law of the root vertex. More precisely, if $P_\star$ is a probability distribution on $\mathbb{N}$ with finite positive first moment $d_\star = \sum_k kP_\star(k)$, we define $\mathrm{UGW}(P_\star)$ as the law of the tree $\mathcal{T}$ with root o obtained from a Galton-Watson branching process where the root has offspring distribution $P_\star$ and all other vertices have offspring distribution $P = \widehat{P}_\star$ defined for all integer $k \geq 0$ by

$$P(k) = \frac{(k+1)P_\star(k+1)}{d_\star}. \tag{2.1}$$

Such a random rooted tree is also called a *unimodular Galton-Watson tree with degree distribution* $P_\star$ by distinction with the *Galton-Watson tree with offspring distribution* P defined above. For example, if $P_\star$ is a Dirac mass at $d_\star$ then $\mathcal{T}$ is an infinite $d_\star$-regular tree. If $P_\star = \mathrm{Poi}(d)$ is a Poisson distribution with mean d, then $\widehat{P}_\star = P_\star$ and the laws $\mathrm{UGW}(P_\star)$ and $\mathrm{GW}(P_\star)$ coincide. This random rooted tree is the Benjamini-Schramm limit of the Erdős-Rényi random graph with n vertices and connection probability d/n in the regime d fixed and n goes to infinity. More generally, uniform random graphs whose given degree sequence converge under mild assumptions to $\mathrm{UGW}(P_\star)$ where $P_\star$ is the limiting degree distribution, see for example [van17].

The *percolation problem* on such random trees is their extinction/survival probabilities, defined as the probability that the tree $\mathcal{T}$ is finite/infinite. We recall that for $\mathrm{GW}(P)$, with P different to a Dirac mass at 1, the survival probability is positive if and only if $d = \sum_k kP(k) > 1$. Obviously, for $\mathrm{UGW}(P_\star)$, the survival probability is positive if and only the survival probability of $\mathrm{GW}(P)$ is positive with $P = \widehat{P}_\star$ as in (2.1).

Adjacency operator. Let $\mathcal{T}$ be an infinite rooted tree which is locally finite, that is for any vertex $u \in \mathcal{T}$, the set of its neighbours denoted by $\mathcal{N}_u$ is finite. For $\psi \in \ell^2(\mathcal{T})$, with finite support, we set

$$(A\psi)(u) = \sum_{v \in \mathcal{N}_u} \psi(v), \quad u \in \mathcal{T}.$$

This defines an operator A on a dense domain of $\ell^2(\mathcal{T})$. Under mild conditions on $\mathcal{T}$, the operator A is essentially self-adjoint, see for example [BLS11, Proposition 3]. For example, if $\mathcal{T}$ has law $\mathrm{GW}(P)$ or $\mathrm{UGW}(P)$ with P having a finite first moment, then A is essentially self-adjoint with probability one, see [BLS11, Proposition 7] and [Bor18, Proposition 2.2] respectively. We may then consider the spectral decomposition of A and define, for any $v \in \mathcal{T}$, μ_v the *spectral measure at vector* δ_v which is a probability measure on $\mathbb{R}$ characterized for example through its Cauchy-Stieltjes transform: for all $z \in \mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$,

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu_v(\lambda)}{\lambda - z} = \langle \delta_v, (A - z)^{-1} \delta_v \rangle.$$

If $\mathcal{T}$ is random and o is the root of the tree, the *average spectral measure or density of states* is the deterministic probability measure $\bar{\mu} = \mathbb{E}\mu_o$. The probability measure $\bar{\mu}$ has been extensively studied notably when $\mathcal{T}$ is a Galton-Watson random tree: the atoms are studied in [BLS11; Sal15; Sal20], the existence of a continuous part in [BSV17] and the existence of an absolutely continuous part at $E = 0$ in [CS21], see [Bor18] for an introduction on the topic. We mention that for $\mathrm{GW}(P)$ or $\mathrm{UGW}(P_\star)$, $\bar{\mu}$ has a dense atomic part as soon as the tree has leaves (that is $P(0) > 0$).

Quantum percolation. Much less is known however on the random probability measure μ_o. Its decomposition into absolutely continuous and singular parts

$$\mu_o = \mu_o^{\mathrm{ac}} + \mu_o^{\mathrm{sg}}$$

is however of fundamental importance to understand the nature of the eigenwaves of A, notably in view of studying the eigenvectors of finite graph sequences which converge to $\mathcal{T}$, see notably [DP12; BL13; Bor15; BHY19; AS19a]. The *quantum percolation problem* is to determine whether μ_o^{ac} is trivial or not, it finds its origin in the works of De Gennes, Lafore and Millot [DLM59b; DLM59a]. The quantum percolation is a refinement of the classical percolation in the sense that a necessary condition for the non-triviality of μ_o^{ac} is that the tree $\mathcal{T}$ is infinite. We refer to [MS10; Bor18] for further references on quantum percolation.

In the present paper, we establish a general concentration criterion on the offspring distribution P of a Galton-Watson tree which guarantees the existence of an absolutely continuous part. Before stating our main results in the next section, we illustrate some of its consequences on $\mathrm{UGW}(\mathrm{Poi}(d))$ and $\mathrm{UGW}(\mathrm{Bin}(n, d/n))$ where for integer $n \geq 1$ and $0 \leq d \leq n$, $\mathrm{Bin}(n, d/n)$ is the Binomial distribution with parameters n and d/n. The random tree is $\mathrm{UGW}(\mathrm{Bin}(n, d/n))$ is particularly interesting because this is the random tree obtain after deleting independently each edge of the n-regular infinite tree with probability d/n. In the same vein, $\mathrm{UGW}(\mathrm{Poi}(d))$ is the Benjamini-Schramm limit of the bond-percolation with parameter d/n on the hypercube $\{0, 1\}^n$. As pointed above, in both cases $\bar{\mu}$ has a dense atomic part, so even if μ_o^{ac} is non-trivial, μ_o will contain an atomic part with positive probability.

Theorem 4. *Assume that $\mathcal{T}$ has law $\mathrm{UGW}(\mathrm{Poi}(d))$ or $\mathrm{UGW}(\mathrm{Bin}(n, d/n))$ with $1 \leq d \leq n$. For any $0 < \varepsilon < 1$, there exists $d_0 = d_0(\varepsilon) > 1$ such that if $d \geq d_0$ then, conditioned on non-extinction, with probability one, μ_o^{ac} is non-trivial and*

$\mathbb{E}[\mu_o^{\mathrm{ac}}(\mathbb{R})] \geq \varepsilon$.

In the proof of Theorem 4, we will exhibit a deterministic Borel set B of Lebesgue measure proportional to $\sqrt{d}$ such that, conditioned on non-extinction, with probability one, μ_o^{ac} has positive density almost everywhere on B. This theorem is consistent with the prediction in Harris [Har84] and Evangelou and Economou [EE92]. It should be compared to [Bor15] which proved a similar statement for $\mathrm{UGW}(\mathrm{Bin}(n,p))$ with n fixed and p close to 1, that is a random tree which is close to the n-regular infinite tree. To our knowledge, the present result establishes for the first time the presence of a non-trivial absolutely continuous part in a random tree which is not close to a deterministic tree. We note also that Theorem 4 implies as a corollary that the average spectral measure $\bar{\mu} = \mathbb{E}\mu_o$ has an absolutely continuous part of arbitrary large mass if d is large enough, this result was not previously known.

As pointed above, if $\mathcal{T}$ has law $\mathrm{UGW}(\mathrm{Poi}(d))$ or $\mathrm{UGW}(\mathrm{Bin}(n, d/n))$ then, due to leaves in $\mathcal{T}$, μ_o^{ac} has atoms. It was however suggested in Bauer and Golinelli [BG00; BG01] that the *Karp and Sipser core* of $\mathcal{T}$ should carry the absolutely continuous part of μ_o. This core is the random subforest of $\mathcal{T}$ obtained by iteratively repeating the following procedure introduced in [KS81]: pick a leaf of $\mathcal{T}$ and remove it together with its unique neighbour, see [BLS11] for a formal definition. The core, if any, is the infinite connected component that remains. For our purposes, it suffices to recall that for $d > d_{\mathrm{KS}}$, the core is non-empty with probability one and empty otherwise (for $\mathrm{Poi}(d)$, we have $d_{\mathrm{KS}} = e$). Moreover, for $d > d_{\mathrm{KS}}$, conditioned on the root being in the core, the connected component of the root in the core has law $\mathrm{UGW}(Q)$ where for $\mathrm{Poi}(d)$, Q is $\mathrm{Poi}(m(d))$ conditioned on being at least 2 while for $\mathrm{Bin}(n, d/n)$, Q is $\mathrm{Bin}(n, m_n(d)/n)$ conditioned on being at least 2. We have for $d > d_{\mathrm{KS}}$, $m(d), m_n(d) > 0$ and $m(d) \sim m_n(d) \sim d$ as d grows, see [AFP98; ZM06] or Lemma 16 in v1 arxiv version of [BLS11].

The following theorem supports the predictions of Bauer and Golinelli.

Theorem 5. *Assume that $\mathcal{T}$ has law $\mathrm{UGW}(Q_d)$ where Q_d is either $\mathrm{Poi}(d)$ or $\mathrm{Bin}(n, d/n)$ conditioned on being at least 2 with $0 < d \leq n$ and $n \geq 2$. The conclusion of Theorem 4 holds for Q_d. Moreover, for any $0 < E < 2$, there exists $d_1 = d_1(E) > 0$ such that if $d \geq d_1$ then, with probability one, on the interval $(-E\sqrt{d}, E\sqrt{d})$, μ_o is absolutely continuous with almost-everywhere positive density.*

This theorem should be compared with Keller [Kel12] which proves among other results a similar statement for $\mathrm{Bin}(n,p)$ conditioned on being at least 2 with n fixed and p close to 1, see also Remark (c) there. Theorem 5 proves that the core of $\mathrm{UGW}(\mathrm{Poi}(d))$ has purely absolutely continuous spectrum on a spectral interval of size proportional to $\sqrt{d}$ provided that d is large enough. We expect that this result is close to optimal: almost surely, the spectrum of $\mathrm{UGW}(\mathrm{Poi}(d))$ is $\mathbb{R}$ and the upper part of the spectrum should be purely atomic due to the presence of vertices of arbitrarily large degrees.

Let us also mention that for $\mathrm{Poi}(d)$ we expect that $d_1(0+) = 0$ in accordance to [BG00; BG01; CS21]. For the optimal value of $d_0(0+)$ in Theorem 4 for $\mathrm{Poi}(d)$, $d_0(0+) = e \approx 2.72$ seems plausible in view [BG00; BG01; CS21] but older references [Har84; EE92] predict that $d_0(0+) \approx 1.42$.

Anderson model. The focus of this paper is on random trees, we nevertheless conclude this introduction with an application of our main result to the Anderson model on the infinite d-regular tree, say $\mathcal{T}_d$. Let A be the adjacency operator of $\mathcal{T}_d$. Let $(V_u)_{u\in\mathcal{T}_d}$ be independent and identically distributed random variables. We consider the essentially self-adjoint operator on $\ell^2(\mathcal{T}_d)$,

$$H = A + \lambda V,$$

where $\lambda \geq 0$ and V is the multiplication operator $(V\psi)(u) = V_u\psi(u)$ for all $u \in \mathcal{T}_d$. This famous model was introduced by Anderson [And58] to describe the motion of an electron in a crystal with impurities. The study of such operators on the infinite regular tree was initiated in the influential works [AAT73; AT01]. We refer to the monograph [AW15] for further background and references.

As above, we denote by μ_o the spectral measure at the vector δ_o, where o is the root of $\mathcal{T}_d$. We prove the following statement:

Theorem 6. *Assume that V_o has finite fourth moment. For any $0 < \varepsilon < 1$ there exists $\lambda_0 = \lambda_0(\varepsilon) > 0$ such that for all $0 \leq \lambda \leq \lambda_0\sqrt{d}$ and $d \geq 3$, $\mathbb{E}[\mu_o^{\mathrm{ac}}(\mathbb{R})] \geq \varepsilon$. Moreover, for any $0 < E < 2$, there exists $\lambda_1 = \lambda_1(E) > 0$ such that for all $0 \leq \lambda \leq \lambda_1\sqrt{d}$ and $d \geq 3$, with probability one, on the interval $(-E\sqrt{d-1}, E\sqrt{d-1})$, μ_o is absolutely continuous with almost-everywhere positive density.*

For fixed $d \geq 3$, this theorem is contained in Klein [Kle98]. Other proofs of this important result have been proposed [ASW06; FHS07]. Beyond the proof, the novelty of Theorem 6 is the uniformity of the thresholds λ_0, λ_1 in $d \geq 3$. When V_o has a continuous density, the asymptotic for large d was settled in Bapst [Bap14] using the complementary criteria for pure point / absolutely continuous spectra established in [AM93; AW13].

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2.1.2 Main results

Model definition. Let $P_\star$ be a probability distribution on $\mathbb{N}$ with finite and positive second moment. We denote by P the size-biased version of $P_\star$ defined by (2.1). We denote by $N_\star$ and N random variables with laws $P_\star$ and P respectively.

We set
$$d_\star = \mathbb{E}N_\star \quad \text{and} \quad d = \mathbb{E}N = \frac{\mathbb{E}N_\star(N_\star - 1)}{\mathbb{E}N_\star}.$$
We use Neveu notation for indexing the vertices of a rooted tree. Let $\mathcal{V}$ be the set of finite sequence of positive integers $u = (i_1, \ldots, i_k)$, the empty sequence being denoted by o. We denote by $\mathcal{V}_k$ the sequences of length k. To each element of $\mathcal{V}$, we associate an independent random variable N_u where N_o has law $P_\star$ and N_u, $u \neq o$, has law P. We define the offsprings of $u \in \mathcal{V}_k$ as being the vertices $(u, 1), \ldots, (u, N_u) \in \mathcal{V}_{k+1}$ (by convention $(o, i) = i$). The connecting component of the root o in this genealogy defines a rooted tree $\mathcal{T}$. Its law is $\mathrm{UGW}(P_\star)$. More generally, for $u \in \mathcal{V}$, we denote by $\mathcal{T}_u$ the trees spanned by the descendants of u (so that $\mathcal{T}_o = \mathcal{T}$). If $u \neq o$, $\mathcal{T}_u$ has law $\mathrm{GW}(P)$ (to be precise up to the natural isomorphism which maps u to o). By construction, the degree of a vertex $u \in \mathcal{T}$ is $\deg(u) = \mathbf{1}_{u \neq o} + N_u$.

Let $(X_u)_{u \in \mathcal{V}}$ be independent and uniform random variables in some measured space. Then, we consider real random variables $V_u = f(X_u, N_u, \mathbf{1}_{u \neq o})$ for some measurable function f. Now, we introduce the operator defined on compactly supported vectors of $\ell^2(\mathcal{T})$,
$$H = \frac{A}{\sqrt{d}} - V, \tag{2.2}$$
where A is the adjacency operator and V the multiplication operator $V\psi(u) = V_u \psi(u)$. As pointed above, from [BLS11, Proposition 7], with probability one, the operators A and H are essentially self-adjoint on a dense domain of $\ell^2(\mathcal{T})$.

For example, if for all $u \in \mathcal{V}$, $V_u = 0$ then H is the adjacency matrix normalized by $1/\sqrt{d}$. If $V_u = (\deg(u) - d)/\sqrt{d}$ then $-H$ is the Laplacian operator on $\mathcal{T}$ shifted by d and normalized. If $V_u = f(X_u)$ for some measurable function f then we obtain the Anderson Hamiltonian operator on $\mathcal{T}$. These three examples are relevant for our study.

Resolvent recursive equation. We denote by μ_o the spectral measure at the root vector δ_o. For $z \in \mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$, the resolvent operator $(H - z)^{-1}$ is well-defined. The resolvent is the favorite tool to access the spectral measures on tree-like structures. We set
$$g_o(z) = \langle \delta_o, (H - z)^{-1} \delta_o \rangle = \int_{\mathbb{R}} \frac{d\mu_o(\lambda)}{\lambda - z}. \tag{2.3}$$
For $u \in \mathcal{T}$, we denote by H_u the restriction of H to $\ell^2(\mathcal{T}_u)$ (in particular, $H_o = H$). If $g_u(z) = \langle \delta_u, (H_u - z)^{-1} \delta_u \rangle$, then we have the following recursion:
$$g_u(z) = -\left(z + \frac{1}{d} \sum_{i=1}^{N_u} g_{ui}(z) + V_u \right)^{-1}, \tag{2.4}$$

see for example [AW15, Proposition 16.1]. By construction, all g_u with $u \neq o$, have the same law, let $g(z)$ be a random variable with law $g_u(z)$, $u \neq o$. For any $z \in \mathbb{H}$, we deduce the equations in distribution:

$$g_o(z) \stackrel{d}{=} -\left(z + \frac{1}{d}\sum_{i=1}^{N_\star} g_i(z) + v_\star\right)^{-1}, \tag{2.5}$$

and

$$g(z) \stackrel{d}{=} -\left(z + \frac{1}{d}\sum_{i=1}^{N} g_i(z) + v\right)^{-1}, \tag{2.6}$$

where $(N_\star, v_\star)$ has law (N_o, V_o), (N, v) has law (N_u, V_u), $u \neq o$, and $(g_i)_{i\geq 1}$ are independent iid copies of g.

The *semicircle distribution* is defined as

$$\mu_{sc} = \frac{1}{2\pi}\mathbf{1}_{\{|x|\leq 2\}}\sqrt{4-x^2}\mathrm{d}x.$$

Its Cauchy-Stieltjes transform, $\Gamma(z)$, satisfies, for all $z \in \mathbb{H}$,

$$\Gamma(z) = -\left(z + \Gamma(z)\right)^{-1}. \tag{2.7}$$

Similarly, we define the *modified Kesten-McKay distribution* with ratio $\rho = d_*/d$ as

$$\mu_\star = \frac{2\rho}{\pi}\mathbf{1}_{\{|x|\leq 2\}}\frac{\sqrt{4-x^2}}{2\rho^2(4-x^2)+x^2(2-\rho)^2}\mathrm{d}x,$$

(if $d_* = d+1$, we retrieve the usual Kesten-McKay distribution). Its Cauchy-Stieltjes transform is related to Γ through the identity for all $z \in \mathbb{H}$,

$$\Gamma_\star(z) = -\left(z + \rho\Gamma(z)\right)^{-1}. \tag{2.8}$$

In the statements below, we give a probabilistic upper bound on $|g(z) - \Gamma(z)|$ and $|g_o(z) - \Gamma_\star(z)|$ uniformly in z in the strip

$$\mathbb{H}_E = \{z \in \mathbb{H} : \Re z \in (-E, E),\, \Im z \leq 1\}, \tag{2.9}$$

for $0 < E < 2$ provided that v is small and the ratios N/d and $N_\star/d_\star$ are close enough to 1 in probability.

Trees with no leaves. For a given $z \in \mathbb{H}$, let $g(z)$ be a random variable in $\mathbb{H}$ satisfying the equation in distribution (2.6). We allow v to take value in $\mathbb{H}$ for a reason which will be explained below. Then, given a real number $p \geq 2$ and a parameter $\lambda \in (0,1)$, we denote by $\mathcal{E}_\lambda$ the event

$$\mathcal{E}_\lambda = \left\{ |v| < \lambda,\, N \geq 2,\, \frac{N}{d} \in (1-\lambda, 1+\lambda) \right\}, \tag{2.10}$$

and $\alpha_p(\lambda)$ the control parameter

$$\alpha_p(\lambda) = \mathbb{E}\left[\mathbf{1}_{\mathcal{E}_\lambda^c} (1+|v|)^{2p} \left(\frac{N}{d} + \frac{d}{N} \right)^p \right]. \tag{2.11}$$

Note that $\alpha_p(\lambda)$ is infinite if $\mathbb{E}N^p = \infty$ or $\mathbb{E}|v|^{2p} = \infty$. Also, Hölder inequality implies for example

$$\alpha_2(\lambda) \leq (\mathbb{P}(\mathcal{E}_\lambda^c))^{1/2} \left(\mathbb{E}\left[(1+|v|)^{12} \right] \right)^{1/3} \left(\mathbb{E}\left[\left(\frac{N}{d} + \frac{d}{N} \right)^{12} \right] \right)^{1/6}. \tag{2.12}$$

The following result asserts that if $\lambda + \alpha_p(\lambda)$ is small enough and $N \geq 1$ with probability one then $g(z)$ and $\Gamma(z)$ are close in L^p.

Theorem 7. *Let $0 < E < 2$, $p \geq 2$ and $(N, v) \in \mathbb{N} \times \mathbb{H}$ be pair of random variables with $\mathbb{E}N = d$. Assume $\mathbb{P}(N \geq 1) = 1$ and either v real or $\mathbb{E}N^{2p} < \infty$. There exist $\delta, C > 0$ depending only on (E, p) such that for any $g(z) \in \mathbb{H}$ verifying (2.6), if $\lambda + \alpha_p(\lambda) \leq \delta$ then for all $z \in \mathbb{H}_E$,*

$$\mathbb{E}\left[|g(z) - \Gamma(z)|^p \right] \leq C \sqrt{\lambda + \alpha_p(\lambda)},$$

$$\mathbb{E}\left[(\Im g(z))^{-p} \right] \leq C.$$

The strategy of proof is inspired by the geometric approach to Klein's Theorem initiated by Froese, Hasler and Spitzer in [FHS07] and refined in Keller, Lenz and Warzel [KLW12]. The main technical novelty is that Theorem 7 holds uniformly on the distribution of (N, v). To achieve this, we use some convexity and symmetries hidden in (2.6), as a result, our proof is at the same time simpler than [FHS07; KLW12] and more efficient.

We will obtain the following corollary of Theorem 7 which applies notably to operators on random trees with law $\mathrm{UGW}(P_\star)$ when the minimal degree is 2. Let $\mathcal{F}_\lambda$ and $\beta_p(\lambda)$ be

$$\mathcal{F}_\lambda = \left\{ \frac{|v_\star|}{\rho} < \lambda,\, \frac{N_\star}{d_\star} \in (1-\lambda, 1+\lambda) \right\},$$

$$\beta_p(\lambda) = \mathbb{E}\left[\mathbf{1}_{\mathcal{F}_\lambda^c} \left(1 + \left| \frac{v_\star}{\rho} \right| \right)^{2p} \left(\frac{N_\star}{d_\star} + \frac{d_\star}{N_\star} \right)^p \right],$$

where $\rho = d_\star / d$ and recall the definition of $\Gamma_\star$ in (2.8).

Corollary 8. *Let $0 < E < 2$, $p \geq 2$ and $(N_\star, v_\star)$, (N, v) be random variables in $\mathbb{N} \times \mathbb{H}$ with $\mathbb{E}N_\star = d_\star$, and $\mathbb{E}N = d$. Assume $\mathbb{P}(N_\star \geq 1) = \mathbb{P}(N \geq 1) = 1$ and either v real or $\mathbb{E}N^{2p} < \infty$. Let $\delta > 0$ be as in Theorem 7. There exists $C_\star > 0$ depending only on (E, p) such that for any $g_o(z)$ and $g(z)$ satisfying (2.5)-(2.6), if $\lambda + \alpha_p(\lambda) \leq \delta$ then for all $z \in \mathbb{H}_E$,*

$$\mathbb{E}\left[|g_o(z) - \Gamma_\star(z)|^p\right] \leq C_\star \rho^{-p} \max\left\{\varepsilon(\lambda), \sqrt{\varepsilon(\lambda)}\right\},$$

$$\mathbb{E}\left[(\Im g_o(z))^{-p}\right] \leq C_\star \left(1 + \beta_p(\lambda)\right)^p \left(\rho + \frac{1}{\rho}\right)^p,$$

where $\varepsilon(\lambda) = \big(1 + \lambda + \alpha_p(\lambda)\big)\big(1 + \lambda + \beta_p(\lambda)\big) - 1$.

We postpone to Section 2.1.5 some well-known consequences of this corollary on the spectral measure μ_o of H.

Trees with leaves. If $\mathbb{P}(N = 0) > 0$, the random tree $\mathcal{T}$ with law UGW$(P_\star)$ has leaves. For the adjacency operator, even when the tree is infinite, this creates atoms in the spectrum located at all eigenvalues of the finite pending trees which have positive probability, see [Bor18]. Nevertheless a strategy introduced in [Bor15] allows to use results such as Theorem 7 to prove existence of an absolutely continuous part.

For simplicity, we restrict ourselves to the adjacency operator A. As already pointed if $d = \mathbb{E}N > 1$, then $\mathcal{T}$ is infinite with positive probability. The probability of extinction π_e is the smallest solution in $[0, 1]$ of the equation $x = \varphi(x)$, where φ is the moment generating function of P:

$$\varphi(x) = \mathbb{E}\left[x^N\right] = \sum_{k=0}^{\infty} P_k x^k.$$

Note that if $\varphi_\star$ is the moment generating of $P_\star$ then $\varphi(x) = \varphi'_\star(x)/\varphi'_\star(1)$.

Next, we define the subtree $\mathcal{S} \subset \mathcal{T}$ of vertices $v \in \mathcal{T}$ such that $\mathcal{T}_v$ is an infinite tree. If $\mathcal{T}$ is infinite (that is, $o \in \mathcal{S}$), this set $\mathcal{S}$ is the infinite skeleton of $\mathcal{T}$ where finite pending trees are attached. If $v \in \mathcal{V}$, let N_v^s be the number of offsprings which are in $\mathcal{S}$. By construction, if $v \in \mathcal{S}$ then $N_v^s \geq 1$. Conditioned on $\mathcal{T}$ or $\mathcal{T}_v$, $v \neq o$, is infinite, the moment generating functions of N_o^s and N_v^s are given in terms of the extinction probability π_e by the formulas

$$\varphi_\star^s(x) = \frac{\varphi_\star((1 - \pi_e)x + \pi_e) - \varphi_\star(\pi_e)}{1 - \varphi_\star(\pi_e)} \quad \text{and} \quad \varphi^s(x) = \frac{\varphi((1 - \pi_e)x + \pi_e) - \pi_e}{1 - \pi_e}, \tag{2.13}$$

see Subsection 2.1.4 for more details. We denote by $N_\star^s$ and N^s random variables with moment generating functions $\varphi_\star^s$ and φ^s. We set $d_s = \mathbb{E}N^s = \varphi'(1 - \pi_e)$.

We then consider the rescaled adjacency operator

$$H = \frac{A}{\sqrt{d_s}},$$

and let $g_o(z) = \langle \delta_o, (H-z)^{-1}\delta_o \rangle$ be as in (2.3). By decomposing the recursion (2.5)-(2.6) over offspring in $\mathcal{S}$ and in $\mathcal{T}\backslash\mathcal{S}$, we will obtain the following Lemma which makes the connection between $g_o(z)$ and a recursion on $\mathcal{S}$ of the type studied above.

Lemma 9. *Assume $d > 1$. For each $z \in \mathbb{H}$, there exist random variables $v_\star(z)$ and $v(z)$ in $\mathbb{H}$, such that, conditioned on $\mathcal{T}$ is infinite,*

$$g_o(z) \stackrel{d}{=} -\left(z + \frac{1}{d_s} \sum_{i=1}^{N_\star^s} g_i^s(z) + v_\star(z) \right)^{-1},$$

and

$$g^s(z) \stackrel{d}{=} -\left(z + \frac{1}{d_s} \sum_{i=1}^{N^s} g_i^s(z) + v(z) \right)^{-1},$$

where $(g_i^s)_{i\geq 1}$ are independent iid copies of g^s, independent of $(N_\star^s, v_\star(z))$, $(N^s, v(z))$.

Since $N_s \geq 1$ with probability one, it is possible to use Theorem 7 provided that we have a moment estimate on $|v(z)|$ and $|v_\star(z)|$, see (2.12). This is the content of the following technical lemma. It is a refined quantified version of [Bor15, Proposition 22]. If $B \subset \mathbb{R}$ is a Borel set, we define

$$\mathbb{H}_B = \{z \in \mathbb{H} : \Re(z) \in B,\, \Im z \leq 1\}. \tag{2.14}$$

With our previous notation for $E > 0$, $\mathbb{H}_E = \mathbb{H}_{(-E,E)}$. We also denote by $\ell(\cdot)$ the Lebesgue measure on $\mathbb{R}$. Recall that N is the random variable with law P and $d = \mathbb{E}N$.

Lemma 10. *Let $\varepsilon > 0$. There exist universal constants $c_0, C > 0$ and a deterministic Borel set $B = B(\varepsilon) \subset \mathbb{R}$ with $\ell(B^c) \leq \varepsilon$ such that the following holds. Let $\pi_1 = \mathbb{P}(N \leq 1)$. If $4\pi_1 \mathbb{E}N^2 \leq d$ and $m \geq 1$ is an integer such that $\pi_1 \leq c_0^m$ then for any $z \in \mathbb{H}_B$,*

$$\mathbb{E}\left[|v(z)|^m\right] \leq \left(\frac{C}{\varepsilon d}\right)^m \frac{\mathbb{E}[N^m]}{d^m} d\pi_1,$$

and

$$\mathbb{E}\left[|v_\star(z)|^m\right] \leq \left(\frac{C}{\varepsilon d}\right)^m \frac{\mathbb{E}[N_\star^m]}{d^m} d\pi_1.$$

Lemma 10 implies that $v(z)$ and $v_\star(z)$ are small if N/d is concentrated around 1. As a byproduct of Lemma 9 and Lemma 10, we can use Theorem 7 and Corollary

8 to obtain an estimate on $|g^s(z) - \Gamma(z)|$ and $|g_o(z) - \Gamma_\star(z)|$ conditioned on non-extinction, for $z \in \mathbb{H}_E \cap \mathbb{H}_B$. We will see an application in the proof of Theorem 4. We note that from the proof, the Borel set B is fully explicit but not simple: B^c is dense in $\mathbb{R}$. We note also that the set B depends on ε and d_s (through a simple scaling of its complementary).

Discussion. The proofs in this paper are quantitative. For example, given (E, p), we could produce in principle numerical constants for the parameters δ and C in Theorem 7. Since these constants should not be optimal, in order to avoid unnecessarily complications of the proofs, we have not fully optimized our arguments and the statements above.

There is an important application of our main result for the eigenvectors of finite graphs converging in Benjamini-Schramm sense to $\mathrm{UGW}(P_\star)$. In [AS19a], the authors establish a form of quantum ergodicity for eigenvectors of finite graphs under some assumptions on the graph sequence. A key assumption called (Green) in [AS19a] is implied by Theorem 7 for p large enough (more precisely, the conclusion $\mathbb{E}(\Im g(z))^{-p} \le C$ for all $z \in \mathbb{H}_E$ gives a weak form of (Green) which is sufficient to apply the main result in [AS19a], see [AS19b, Subsection 3.2]). For example, the assumptions of [AS19a, Theorem 1.1] are met with probability one for a sequence of random graphs sampled uniformly among all graphs with a given degree sequence with degrees between 3 and a uniform bound D and whose empirical degree distribution converges to $P_\star$ for which we can apply the conclusion of Corollary 8 with p large enough. This answers Problem 4.4 and Problem 4.5 in [AS19b] for the weak form of (Green).

Organization of the rest of the paper. In Section 2.1.3, we prove Theorem 7 and Corollary 8. In Section 2.1.4, we prove Lemma 9 and Lemma 10. In the final Section 2.1.5, we gather some standard properties relating the spectral measures of an operator and its resolvent. As a byproduct, we prove Theorem 4, Theorem 5 and Theorem 6 from the introduction.

2.1.3 Proof of Theorem 7 and Corollary 8

Hyperbolic semi-metric

Froese-Hasler-Spitzer's strategy. Following [FHS07], we see $\mathbb{H}$ as the Poincaré half-plane model endowed with the usual hyperbolic distance $\mathrm{d}_{\mathbb{H}}$. The key observation is that the application $h \mapsto \hat{h} = -(z+h)^{-1}$, seen in $(\mathbb{H}, \mathrm{d}_{\mathbb{H}})$ acts as a contraction and thus determines $\Gamma(z)$ as its unique fixed point. As in [FHS07], we introduce the functional $\gamma(\cdot,\cdot)$ defined by

$$\gamma(g,h) = \frac{|g-h|^2}{\Im g \Im h}, \qquad g, h \in \mathbb{H},$$

which is related to the usual hyperbolic distance by $\mathrm{d}_{\mathbb{H}}(g,h) = \operatorname{arcosh}(1+\gamma\,(g,h)\,/2)$. The next lemma allows us to derive usual bounds from the ones obtained in the hyperbolic plane.

Lemma 11 (From hyperbolic to Euclidean bounds)**.** *For any $g,h \in \mathbb{H}$, we have*

$$|g-h|^2 \le |h|^2 \left(4\gamma^2(g,h) + 2\gamma(g,h)\right) \quad \text{and} \quad \frac{1}{\Im g} \le \frac{4\gamma(g,h)+2}{\Im h}.$$

Proof. We start with the second inequality. We may assume that $\Im g \le \Im h/2$, the bound is trivial otherwise. In particular we have $|g-h| \ge \Im h/2$, thus

$$\frac{1}{\Im g} \le \frac{4|g-h|^2}{\Im g \Im h^2} = \frac{4\gamma(g,h)}{\Im h}.$$

We obtain the claim bound. For the first inequality, considering the case $|g| \le 2|h|$ and $|g| \ge 2|h|$, we find similarly for any $g,h \in \mathbb{H}$

$$|g| \le 4\gamma\,(g,h)\,\Im h + 2|h|,$$

see [FHS07; KLW12]. From $|g-h|^2 = \gamma(g,h)\Im g \Im h$ and $\Im g \le |g|$, we get

$$|g-h|^2 \le 4\gamma^2(g,h)(\Im h)^2 + 2\gamma(g,h)|h|\Im h,$$

this implies the first bound. □

For ease of notation, we write

$$\gamma\,(h) = \gamma\,(h,\,\Gamma(z))\,, \quad h \in \mathbb{H},$$

where the dependency in z is implicit.

Our strategy is the following. Let $g = g(z)$ satisfying (2.6). For some transformation $\phi\colon \mathbb{H} \to \mathbb{H}$ verifying the equation in distribution $g \overset{d}{=} \phi(g)$, we want to obtain uniformly in z a contraction of the following type

$$\forall h \in \mathbb{H} : \gamma^p(\phi(h)) \le \left(1 - \varepsilon \mathbf{1}_{\{h \notin K\}}\right) \gamma^p(h)\,, \tag{2.15}$$

where K is a compact set. Denoting $R = \max_{h\in K} \gamma^p(h)$ and assuming that $\mathbb{E}\gamma^p(g)$ is finite, this implies that $\mathbb{E}\gamma^p(g) \le R/\varepsilon$. We will derive the desired bound on $\mathbb{E}|g-\Gamma|^p$ and $\mathbb{E}(\Im g)^{-p}$ from Lemma 11.

Properties of γ. The function $h \mapsto \gamma\,(h)$ has the following properties:

Lemma 12 (Convexity γ)**.**
(i) For any $h \in \mathbb{H}$, $\gamma\,(-(z+h)^{-1}) \le \gamma\,(h)$.
(ii) The function γ is a convex function.

(iii) For any integer $n \geq 1$ and $(h_1, \dots h_n) \in \mathbb{H}^n$:

$$\gamma\left(\frac{1}{n}\sum_{i=1}^{n} h_i\right) = \frac{1}{n}\sum_{i,j=1}^{n} \cos\alpha_{ij}\sqrt{q_i\gamma(h_i)}\sqrt{q_j\gamma(h_j)},$$

where $q_i = \frac{\Im h_i}{\sum_j \Im h_j}$ and $\alpha_{ij} = \arg(h_i - \Gamma)\overline{(h_j - \Gamma)}$ with the convention that $\arg 0 = \pi/2$.

Proof. Statement (i) comes from the fact that $h \mapsto \hat{h} = -(z+h)^{-1}$ is a contraction for the hyperbolic distance $\mathrm{d}_{\mathbb{H}}$ together with the fact that $\hat{\Gamma} = \Gamma$. Statement (ii) follows from (iii), Cauchy–Schwarz inequality and $\sum_i q_i = 1$,

$$\gamma\left(\frac{1}{n}\sum_{i=1}^{n} h_i\right) \leq \frac{1}{n}\sum_{i,j}\sqrt{q_i\gamma_i}\sqrt{q_j\gamma_j} \leq \frac{1}{n}\left(\sum_i \gamma_i\right)\left(\sum_i q_i\right) = \frac{1}{n}\sum_i \gamma(h_i).$$

The last statement (iii) follows by expanding the squared term in the definition of γ:

$$\gamma\left(\frac{1}{n}\sum_i h_i\right) = \frac{|\frac{1}{n}\sum_i h_i - \Gamma|^2}{\frac{1}{n}\sum_i \Im h_i \Im\Gamma} = \frac{1}{n\Im\Gamma}\frac{1}{\sum_j \Im h_j}\sum_{i,j}|h_i - \Gamma||h_j - \Gamma|\cos\alpha_{ij}.$$

We obtain the requested identity (iii). □

The next lemma gathers properties of γ. In the sequel, for $\lambda_0 > 0$, we denote by $\mathrm{Lip}_0(\lambda_0)$ the set of functions $f : [0, \lambda_0) \to \mathbb{R}^+$ such that for some $C > 0$, $f(t) \leq Ct$ for all $t \in [0, \lambda_0/2]$. Recall also the definition of $\mathbb{H}_E$ in (2.9).

Lemma 13 (Regularity of γ)**.** *Let $0 < E < 2$, $p \geq 2$ and $z \in \mathbb{H}_E$.*

(i) There exists a function $c \in \mathrm{Lip}_0(1)$ depending only on (E, p) such that for any $h \in \mathbb{H}, \lambda \in [0, 1)$, and $(u, v) \in \mathbb{R} \times \mathbb{H}$ with $|u|, |v| \leq \lambda$,

$$\gamma^q\big((1+u)h + v\big) \leq (1 + c(\lambda))\gamma^q(h) + c(\lambda), \qquad q \in [1, p]$$

(ii) There exists $C = C(E, p)$ such that for any $h \in \mathbb{H}, \mu > 0$ and $v \in \mathbb{H}$,

$$\gamma^p\big(\mu h + v\big) \leq C(1 + |v|^{2p})\left(\mu + \frac{1}{\mu}\right)^p (\gamma^p(h) + 1).$$

Proof. For statement (i) with $q = 1$ we adapt [KLW12, Lemma 1]. There are two cases to consider. If $|h-\Gamma| > \Im\Gamma/2$, then

$$\begin{aligned}\frac{\gamma\left((1+u)h+v\right)}{\gamma\left(h\right)} &= \left(\frac{|(1+u)h+v-\Gamma|}{|h-\Gamma|}\right)^2 \frac{\Im h}{\Im((1+u)h+v)} \\ &\leq \left(\frac{(1+\lambda)|h-\Gamma|+\lambda(1+|\Gamma|)}{|h-\Gamma|}\right)^2 \frac{1}{1-\lambda} \\ &\leq \left(1+\lambda+\frac{\lambda(1+|\Gamma|)}{\Im\Gamma/2}\right)^2 \frac{1}{1-\lambda}.\end{aligned}$$

Otherwise we have $|h-\Gamma| \leq \Im\Gamma/2$ and then $\Im h \geq \Im\Gamma/2$. We obtain similarly, using $2ab \leq a^2+b^2$ for real a, b at the third line,

$$\begin{aligned}\gamma\left((1+u)h+v\right) &\leq \frac{\left((1+\lambda)|h-\Gamma|+\lambda(1+|\Gamma|)\right)^2}{((1-\lambda)\Im h+\Im v)\Im\Gamma} \\ &\leq \frac{(1+\lambda)^2|h-\Gamma|^2+\lambda^2(1+|\Gamma|)^2+2\lambda(1+\lambda)|h-\Gamma|(1+|\Gamma|)}{(1-\lambda)\Im h\Im\Gamma} \\ &\leq \frac{(1+\lambda)^2+\lambda(1+\lambda)^2}{1-\lambda}\frac{|h-\Gamma|^2}{\Im h\Im\Gamma} + \frac{\lambda^2(1+|\Gamma|)^2+\lambda(1+|\Gamma|)^2}{(1-\lambda)(\Im\Gamma)^2/2} \\ &= \frac{(1+\lambda)^3}{1-\lambda}\gamma\left(h\right) + \frac{\lambda(1+\lambda)(1+|\Gamma|)^2}{(1-\lambda)(\Im\Gamma)^2/2}.\end{aligned}$$

We have $\Gamma(z) = (-z+\sqrt{z^2-4})/2$ where $\sqrt{\cdot}$ is the principal branch of the square root function (which maps $e^{i\theta}$ to $e^{i\theta/2}$ for $\theta \in [0,\pi]$). We deduce easily that $|\Gamma(z)| \leq 1$ and $\Im\Gamma(z)$ is lower bounded uniformly over $\mathbb{H}_E$. Thus, for some function $c(\lambda)$ in $\mathrm{Lip}_0(1)$ we have

$$\gamma\big((1+u)h+v\big) \leq (1+c(\lambda))\gamma(h)+c(\lambda). \tag{2.16}$$

Let $q > 1$, we claim that the general case follows up to a new function $\tilde{c}$. For any $x, c > 0$, from a Taylor expansion, we have for some $0 \leq \delta \leq c$,

$$((1+c)x+c)^q = (1+c)^q x^q + c(q-1)((1+c)x+\delta)^{q-1}.$$

Using that $a^{q-1} \leq a^q+1$, $(a+b)^q \leq 2^{q-1}(a^q+b^q)$ for $q \geq 1$, together with $0 \leq \delta \leq c$ we get

$$((1+c)x+\delta)^{q-1} \leq 2^{q-1}(1+c)^q x^q + 2^{q-1}c^q + 1.$$

This leads to

$$((1+c)x+c)^q \leq (1+c)^q(1+2^{q-1}(q-1)c)x^q + (2^{q-1}c^q+1)(q-1)c.$$

We get from (2.16)

$$\gamma^q\big((1+u)h+v\big) \le ((1+c(\lambda))\gamma(h)+c(\lambda))^q \le (1+\tilde{c}(\lambda))\gamma(h)^q+\tilde{c}(\lambda),$$

where $\tilde{c}(\lambda) \in \mathrm{Lip}_0(1)$ is

$$\tilde{c} = \max\left\{(1+c)^q(1+2^{q-1}(q-1)c)-1, (2^{q-1}c^q+1)(q-1)c\right\},$$

since we have that $c(\lambda)$ is in $\mathrm{Lip}_0(1)$.

For statement (ii), we proceed similarly. If $|h-\Gamma| > \Im\Gamma/2$, then

$$\begin{aligned}\frac{\gamma(\mu h+v)}{\gamma(h)} &= \frac{\Im h}{\mu\Im h+\Im v}\left(\frac{|\mu(h-\Gamma)+v-(1-\mu)\Gamma|}{|h-\Gamma|}\right)^2\\ &\le \frac{1}{\mu}\left(\mu+\frac{|v|+|1-\mu||\Gamma|}{\Im\Gamma/2}\right)^2\\ &\le 2\mu+16\frac{|v|^2+|1-\mu|^2|\Gamma|^2}{\mu(\Im\Gamma)^2}.\end{aligned}$$

Where we used $(a+b)^2 \le 2(a^2+b^2)$. Otherwise, we have $|h-\Gamma| \le \Im\Gamma/2$ and $\Im h \ge \Im\Gamma/2$,

$$\begin{aligned}\gamma(\mu h+v) &\le \frac{(\mu|h-\Gamma|+|v|+|1-\mu||\Gamma|)^2}{(\mu\Im h+\Im v)\Im\Gamma}\\ &\le 2\frac{\mu^2|h-\Gamma|^2}{\mu\Im h\Im\Gamma}+2\frac{(|v|+|1-\mu||\Gamma|)^2}{\mu(\Im\Gamma)^2/2}\\ &\le 2\mu\gamma(h)+8\frac{|v|^2+|1-\mu|^2|\Gamma|^2}{\mu(\Im\Gamma)^2}.\end{aligned}$$

We get for some constant C depending on E,

$$\gamma\big(\mu h+v\big) \le C(1+|v|)^2\left(\mu+\frac{1}{\mu}\right)(\gamma(h)+1).$$

Using $(a+b)^p \le 2^{p-1}(a^p+b^p)$, this gives (ii). □

Asymmetric contraction

In this subsection we derive our main technical tool. Although we noticed that $h \mapsto -(z+h)^{-1}$ is a contraction in $\mathbb{H}$, it tends to an isometry as z approaches the real axis and thus $\gamma\left((z+h)^{-1}\right) \simeq \gamma(h)$. This prevent us to obtain contraction (2.15) for $\phi(h) = \hat{h}$, and this is precisely why the stability of absolutely continuous spectrum fails in one dimensional graphs such as $\mathbb{Z}$. We overcome this issue with the following observation. If instead we consider two points $h_r, h_l \in \mathbb{H}$, then for some $\varepsilon = \varepsilon(E,p) > 0$ we show that

$$\gamma^p\left(\frac{h_r + h_l}{2}\right) + \gamma^p\left(\frac{\hat{h}_r + h_l}{2}\right) \leq (1-\varepsilon)\left(\gamma^p(h_r) + \gamma^p(h_l)\right),$$

as soon as h_r and h_l are far enough from Γ for $\mathrm{d}_{\mathbb{H}}$. The terms asymmetric is motivated from the fact that h_l is fixed while the terms with subscripts r varies on the left-hand side.

More precisely, let $\lambda \in [0,1)$ and let $n \geq 1$ be an integer. For $\boldsymbol{h} = (h_1, \dots, h_n) \in \mathbb{H}^n$, $u, v \in \mathbb{R} \times \mathbb{H}$ with $|u|, |v| \leq \lambda$, we define

$$\hat{h} = \hat{h}(\boldsymbol{h}, u, v, n) := -\left(z + \frac{1+u}{n}\sum_{i=1}^{n} h_i + v\right)^{-1} \in \mathbb{H}. \tag{2.17}$$

We define the random variable $h_r \in \mathbb{H}$ with parameter $(\boldsymbol{h}, u, v, n)$ as follows:

$$h_r = \begin{cases} h_i & \text{with } \mathbb{P}(h_r = h_i) = 1/(2n),\ i = 1, \dots, n, \\ \hat{h} & \text{with } \mathbb{P}(h_r = \hat{h}) = 1/2, \end{cases} \tag{2.18}$$

and fix a deterministic variable $h_l \in \mathbb{H}$.

Proposition 14 (Asymmetric uniform contraction)**.** *Let $0 < E < 2$ and $p \geq 2$. There exist constants ε, λ_0 in $(0,1)$ and a function $R \in \mathrm{Lip}_0(\lambda_0)$ depending only on (E,p) such that, if $\lambda \leq \lambda_0$,*

$$\mathbb{E}\left[\gamma^p\left(\frac{h_l + h_r}{2}\right)\right] \leq (1-\varepsilon)\left(\frac{\gamma^p(h_l) + \frac{1}{n}\sum_{i=1}^{n}\gamma^p(h_i)}{2}\right) + R(\lambda),$$

where the expectation is over the law of h_r defined in (2.18). This holds uniformly in $n \geq 1$, $(\boldsymbol{h}, h_l) \in \mathbb{H}^{n+1}$, $u, v \in \mathbb{R} \times \mathbb{H}$ with $|u|, |v| \leq \lambda$ and $z \in \mathbb{H}_E$.

Before going into the proof, we introduce some notation. For $f \in \mathbb{C}^n$, let $\mathbb{E}_s f$ and $\mathrm{Var}_s f$ denote its average and variance, i.e

$$\mathbb{E}_s f = \frac{1}{n}\sum_{i=1}^{n} f_i \quad \text{and} \quad \mathrm{Var}_s f = \frac{1}{n}\sum_{i=1}^{n} |f_i - \mathbb{E}_s f|^2.$$

We set $h_s = \mathbb{E}_s \boldsymbol{h}$. We simply write γ_x for $\gamma(h_x)$ where $x \in \{1, \dots, n\} \cup \{r, l, s\}$ and $\hat{\gamma} = \gamma(\hat{h})$. For example, with our notation, we have

$$\gamma_s = \gamma(h_s) = \gamma\left(\frac{1}{n}\sum_i h_i\right) \quad \text{and} \quad \mathbb{E}_s\gamma_i = \frac{1}{n}\sum_{i=1}^{n}\gamma_i = \frac{1}{n}\sum_{i=1}^{n}\gamma(h_i).$$

We highlight the fact that only h_r and γ_r are random. We also introduce the following quantities, with the convention $0/0 = 0$:

$$Q_s = \frac{\gamma_s^{p/2}}{\mathbb{E}_s \gamma_i^{p/2}}, \qquad Q_2 = \frac{\mathbb{E}_s \gamma_i^{p/2}}{\sqrt{\mathbb{E}_s \gamma_i^p}}, \qquad Q_l = \frac{2\sqrt{\gamma_l^p \mathbb{E}_s \gamma^p}}{\gamma_l^p + \mathbb{E}_s \gamma_i^p}.$$

It follows from convexity (see Lemma 12(ii)) that $Q_s, Q_2, Q_l \in [0,1]$. We also introduce two quantities β_l, β_2 related to the Q's that measure how $(\gamma_1, \ldots, \gamma_n)$ and γ_l concentrate around the same value.

Lemma 15. *Let $\beta_2 = \sqrt{1 - Q_2^2}$ and $\beta_l = \sqrt{1 - Q_l^2}$. We have*

$$\mathrm{Var}_s \gamma_i^{p/2} = \beta_2^2 \mathbb{E}_s \gamma_i^p. \tag{2.19}$$

Furthermore, for $a = \gamma_l^p$ or $a = \mathbb{E}_s \gamma_i^p$,

$$(1 - \beta_l)\frac{\gamma_l^p + \mathbb{E}_s \gamma_i^p}{2} \le a \le (1 + \beta_l)\frac{\gamma_l^p + \mathbb{E}_s \gamma_i^p}{2}. \tag{2.20}$$

Proof. Equation (2.19) is straightforward. To prove Equation (2.20), we note that Q_l is of the form $2\sqrt{ab}/(a+b)$. Hence,

$$Q_l^2 = \frac{(a+b)^2 - (a-b)^2}{(a+b)^2} = 1 - \left(\frac{a-b}{a+b}\right)^2.$$

We deduce that $|a-b| = (a+b)\sqrt{1 - Q_l^2}$ and thus

$$\max\{a,b\} = \frac{a+b}{2} + \frac{|a-b|}{2} = \frac{a+b}{2}\left(1 + \sqrt{1 - Q_l^2}\right) = (1 - \beta_l)\frac{a+b}{2}.$$

Similarly,

$$\min\{a,b\} = \frac{a+b}{2}\left(1 - \sqrt{1 - Q_l^2}\right) = (1 - \beta_l)\frac{a+b}{2},$$

as requested. $\square$

Our next lemma uses the convexity to express $\mathbb{E}\gamma^p((h_r + h_l)/2)$ in terms of the Q's.

Lemma 16 (Convexity)**.** *Let $0 < E < 2$, $p \ge 2$ and $c(\cdot) \in \mathrm{Lip}_0(1)$ be as in Lemma 13. We have*

$$\mathbb{E}\gamma^p\left(\frac{h_r + h_l}{2}\right) \le (1 + 2c(\lambda))\left(\frac{3 + Q_s Q_2 Q_l}{4}\right)\frac{\gamma_l^p + \mathbb{E}_s \gamma_i^p}{2} + c(\lambda).$$

Proof. From Lemma 12(i), we have $\gamma(\hat{h}) = \gamma\left(-1/(z + (1+u)h_s + v)\right) \le \gamma\left((1+u)h_s + v\right)$ which combined with Lemma 13(i) leads to $\hat{\gamma}^q \le (1 + c(\lambda))\gamma_s^q + c(\lambda)$, $q \in \{p/2, p\}$.

From the definition of the Q's we have

$$\gamma_s^{p/2} = Q_s Q_2 \sqrt{\mathbb{E}_s \gamma_i^p} \quad \text{ and } \quad \gamma_s^{p/2}\gamma_l^{p/2} = Q_s Q_2 Q_l \frac{\gamma_l^p + \mathbb{E}_s \gamma_i^p}{2}.$$

We get the two bounds:

$$\mathbb{E}\gamma_r^p \leq \frac{\mathbb{E}_s\gamma_i^p}{2} + \frac{(1+c(\lambda))Q_s^2 Q_2^2 \mathbb{E}_s \gamma_i^p + c(\lambda)}{2} \leq (1+c(\lambda))\mathbb{E}_s\gamma_i^p + \frac{c(\lambda)}{2} \tag{2.21}$$

and

$$\begin{aligned}\gamma_l^{p/2}\mathbb{E}\gamma_r^{p/2} &\leq \gamma_l^{p/2}\frac{\mathbb{E}_s\gamma_i^{p/2}}{2} + \gamma_l^{p/2}\frac{(1+c(\lambda))\gamma_s^{p/2} + c(\lambda)}{2} \\ &\leq (1+c(\lambda))\left(\frac{1+Q_sQ_2Q_l}{2}\right)\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2} + \frac{c(\lambda)(\gamma_l^p+1)}{4}, \end{aligned} \tag{2.22}$$

where we used the fact that the Q's are less than one, the AM-GM inequality and $2\gamma_l^{p/2} \leq \gamma_l^p + 1$. Now from the convexity of $x \mapsto x^{p/2}$, Lemma 12(ii), we have

$$\begin{aligned}&\mathbb{E}\gamma^p\left(\frac{h_r+h_l}{2}\right) \leq \mathbb{E}\left(\frac{\gamma_l^{p/2}+\gamma_r^{p/2}}{2}\right)^2 \leq \frac{1}{2}\frac{\gamma_l^p + \mathbb{E}\gamma_r^p}{2} + \frac{1}{2}\gamma_l^{p/2}\mathbb{E}\gamma_r^{p/2} \\ &\leq \frac{1}{2}\left((1+c(\lambda))\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2} + \frac{c(\lambda)}{4}\right) + \frac{1}{2}\left((1+c(\lambda))\left(\frac{1+Q_sQ_2Q_l}{2}\right)\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2} + \frac{c(\lambda)(\gamma_l^p+1)}{4}\right) \\ &\leq \left((1+c(\lambda))\left(\frac{3+Q_sQ_2Q_l}{4}\right) + \frac{c(\lambda)}{2}\right)\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2} + \frac{c(\lambda)}{4}.\end{aligned}$$

The conclusion follows. $\square$

Lemma 16 already implies that if $Q_sQ_2Q_l \leq 1-\varepsilon$ then the conclusion of Proposition 14 holds. We will now argue that if $Q_sQ_2Q_l$ is large then a contraction occurs provided that $(\gamma_l^p + \mathbb{E}_s\gamma_i^p)/2$ is large enough. This source of contraction will come from a default of alignment of $h_r - \Gamma$ and $h_l - \Gamma$ which we now explain precisely. With the convention that $\arg 0 = \pi/2$, let

$$\alpha_{rl} = \arg\,(h_r - \Gamma)\overline{(h_l-\Gamma)}.$$

For $\alpha \in \mathbb{S}^1$, we set $\cos^+\alpha = \max\{\cos\alpha, 0\}$.

Lemma 17 (Non-alignment)**.** *Let $0 < E < 2$, $p \geq 2$. There exist $\delta, \lambda_0 > 0$ and $R_0 \in \mathrm{Lip}_0(\lambda_0)$ depending only on (E,p) such that if $\lambda < \lambda_0$ the following holds:*

$$\textit{if } Q_sQ_2Q_l \geq 1/2 \textit{ and } \frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2} \geq R_0(\lambda) \textit{ then } \mathbb{E}\cos^+\alpha_{rl} \leq 1-\delta.$$

Proof. If $h_s = \Gamma$ or $h_l = \Gamma$ then from the convention that $\arg 0 = \pi/2$, $\alpha_{rl} = \pi/2$

with probability at least $1/2$ and the statement follows with $\delta \leq 1/2$. We thus assume $h_s \neq \Gamma$ and $h_l \neq \Gamma$. Let $1 \leq i \leq n$, $\alpha_i, \alpha_s, \hat{\alpha}$ be defined in the same way that α_{rl} with $h_i, h_s, \hat{h}$ instead of h_r. Using the fact that $-\hat{h}^{-1} + \Gamma^{-1} = (1+u)h_s + v - \Gamma$ together with the identity $\hat{h} - \Gamma = \hat{h}\Gamma(-\hat{h}^{-1} + \Gamma^{-1})$, we get,

$$\hat{h} - \Gamma = (1+u)\hat{h}\Gamma(h_s - \Gamma)\left(1 + \frac{v + u\Gamma}{(1+u)(h_s - \Gamma)}\right).$$

By setting $\theta = \arg \hat{h}\Gamma$ and $\alpha_e = \arg\{1 + (v+u\Gamma)((1+u)(h_s-\Gamma))^{-1}\}$, the above relation multiplied both sides by $\overline{(h_l - \Gamma)}$ translates into

$$\hat{\alpha} = \theta + \alpha_s + \alpha_e, \tag{2.23}$$

where the sum must be seen in $\mathbb{S}^1$. We take the argument in $(-\pi, \pi]$. Since $0 < E < 2$, there exists $0 < \theta_0 \leq \pi/2$ such that for all $z \in \mathbb{H}_E$,

$$\arg \Gamma(z) \in (\theta_0, \pi - \theta_0).$$

In particular, since $\hat{h} \in \mathbb{H}$, we find $|\theta| \geq \theta_0$. We will show that for some $\delta_s > 0$ to be fixed later on, under the additional hypothesis that $\mathbb{E}_s \cos^+ \alpha_i \geq 1 - \delta_s$, we have

$$\cos \alpha_s \geq \cos(\theta_0/4) \quad \text{and} \quad \cos \alpha_e \geq \cos(\theta_0/4). \tag{2.24}$$

This implies from (2.23) that $|\hat{\alpha}| \geq \theta - 2\theta_0/4 \geq \theta_0/2$. Considering the two cases $\mathbb{E}_s \cos^+ \alpha_i \geq 1 - \delta_s$ and $\mathbb{E}_s \cos^+ \alpha_i < 1 - \delta_s$, we obtain

$$\mathbb{E} \cos^+ \alpha_{rl} = \frac{1}{2}\cos^+ \hat{\alpha} + \frac{1}{2}\mathbb{E}_s \cos^+ \alpha_i \leq \frac{1}{2} + \frac{\max\{1 - \delta_s, \cos\theta_0/2\}}{2} = 1 - \delta.$$

We get the statement of Lemma 17 for this choice of δ. It thus remain to prove that (2.24) holds if $\mathbb{E}_s \cos^+ \alpha_i \geq 1 - \delta_s$.

Bound on α_s. We have

$$\cos \alpha_s = \frac{\Re\left[(h_s - \Gamma)\frac{\overline{(h_l - \Gamma)}}{|h_l - \Gamma|}\right]}{|h_s - \Gamma|} = \frac{1}{n}\sum_{i=1}^{n} \frac{|h_i - \Gamma| \cos \alpha_i}{|h_s - \Gamma|} = \frac{1}{n}\sum_{i=1}^{n} \frac{\sqrt{\gamma_i \Im h_i} \cos \alpha_i}{\sqrt{\gamma_s \Im h_s}} = \sqrt{\frac{n}{\gamma_s}}\mathbb{E}_s \sqrt{q_i \gamma_i} \cos \alpha_i,$$

where we set $q_i = \Im h_i / (\sum \Im h_j)$ already appeared in Lemma 12(iii). From this Lemma we get

$$\left(\sqrt{\frac{n}{\gamma_s}}\mathbb{E}_s \sqrt{q_i \gamma_i}\right)^2 \geq \frac{n}{\gamma_s}\frac{1}{n^2}\sum_{i,j} \cos \alpha_{ij} \sqrt{q_i q_j \gamma_i \gamma_j} = 1.$$

Let $c = \cos^2(\theta_0/8) = \frac{1+\cos(\theta_0/4)}{2} > \cos(\theta_0/4)$ and suppose $\mathbb{E}_s \cos^+ \alpha_l \geq 1 - \delta_s$. Using that $\mathbb{E}_s q_i = 1/n$ and Hölder inequality with q such that $\frac{1}{2} + \frac{1}{2p} + \frac{1}{q} = 1$, we get

$$\begin{aligned}
\sqrt{\frac{n}{\gamma_s}} \mathbb{E}_s \sqrt{q_i \gamma_i \mathbf{1}_{\{\cos\alpha_i < c\}}} &\leq \sqrt{\frac{n}{\gamma_s}} \left(\mathbb{E}_s q_i\right)^{1/2} \left(\mathbb{E}_s \gamma^p\right)^{1/2p} \mathbb{P}_s(\cos\alpha_i < c)^{1/q} \\
&= \left(\frac{\sqrt{\mathbb{E}_s \gamma^p}}{\gamma_s^{p/2}}\right)^{1/p} \mathbb{P}_s(\cos\alpha_i < c)^{1/q} \\
&= \frac{1}{(Q_s Q_2)^{1/p}} \mathbb{P}_s(1 - \cos^+\alpha_i > 1 - c)^{1/q} \\
&\leq 2^{1/p} \left(\frac{\delta_s}{\sin^2(\theta_0/8)}\right)^{1/q},
\end{aligned}$$

where at the last line, we have used Markov inequality and the assumption $Q_s Q_2 \geq Q_s Q_2 Q_l \geq 1/2$. We now use the bound $\cos\alpha_i \geq c - 2\mathbf{1}_{\{\cos\alpha_i < c\}}$ and obtain

$$\cos\alpha_s \geq \frac{1 + \cos(\theta_0/4)}{2} - 2^{1+1/p} \left(\frac{\delta_s}{\sin^2(\theta_0/8)}\right)^{1/q} \geq \cos(\theta_0/4),$$

provided that δ_s is small enough.

Bound on α_e. Using $\Re(1+z)/|1+z| \geq 1 - 2|z|$, $|\Gamma| \leq 1$ and $|u|, |v| \leq \lambda$, we have

$$\cos\alpha_e = \frac{\Re\left(1 + \frac{v+u\Gamma}{(1+u)(h_s-\Gamma)}\right)}{\left|1 + \frac{v+u\Gamma}{(1+u)(h_s-\Gamma)}\right|} \geq 1 - 2\left|\frac{v+u\Gamma}{(1+u)(h_s-\Gamma)}\right| \geq 1 - \frac{8\lambda}{|h_s - \Gamma|},$$

where we assumed without loss of generality that $\lambda \leq \lambda_0 \leq 1/2$ in the last inequality. We want to lower bound $|h_s - \Gamma|$ in terms of $\gamma_s = \gamma(h_s)$. This is easily done from the following observation. Let $r \geq 0$ and define

$$\psi(r) = \inf\{|h - \Gamma| : h \in \mathbb{H}, \gamma(h) \geq r\}.$$

A drawing shows that this infimum is attained at the unique h^* such that $\Re h^* = \Re\Gamma$, $0 < \Im h^* < \Im\Gamma$ and $\gamma(h^*) = r$. For this configuration, we have $\Im\Gamma = \Im h^* + |h^* - \Gamma|$. Since $\psi(r) = |h^* - \Gamma|$, we get

$$r = \gamma(h^*) = \frac{|h^* - \Gamma|}{\Im\Gamma \Im h^*} = \frac{\psi^2(r)}{\Im\Gamma(\Im\Gamma - \psi(r))}.$$

Thus ψ^{-1} is defined on $[0, \Im\Gamma)$ and $\psi^{-1}(t) = t^2/(\Im\Gamma(\Im\Gamma - t))$. As $Q_s Q_2 \geq 1/2$ and $Q_l \geq 1/2$, from (2.20) in Lemma 15, we have $1 - \beta_l \geq 1/8$ and

$$\gamma_s^p = (Q_s Q_2)^2 \mathbb{E}_s \gamma_i^p \geq \frac{1}{32} \frac{\gamma_l^p + \mathbb{E}_s \gamma_i^p}{2}.$$

Let $R_0(\lambda) = 32\left(\psi^{-1}\left(8\lambda/(1-\cos(\theta_0/4))\right)\right)^p$ which is defined for any $z \in \mathbb{H}_E$ as soon as

$$\lambda < \lambda_0 := \frac{1-\cos(\theta_0/4)}{8} \min_{z \in \mathbb{H}_E} \Im\Gamma(z).$$

We have $\cos\alpha_e \geq \cos(\theta_0/4)$ as soon as $(\gamma_l^p + \mathbb{E}_s\gamma_i^p)/2 \geq R_0(\lambda)$. Since $\psi^{-1}(t) \sim t^2$ as t goes to 0, we have $R_0 \in \mathrm{Lip}_0(\lambda_0)$. This concludes the proof. $\square$

The next lemma expresses the contraction in terms of $\cos^+\alpha_{rl}$. Recall the definition β_2, β_l from Lemma 15.

Lemma 18 (Contraction from non-alignment)**.** *Let $0 < E < 2$, $p \geq 2$ and $c(\cdot) \in \mathrm{Lip}_0(1)$ be as in Lemma 13. We have*

$$\mathbb{E}\gamma^p\left(\frac{h_r+h_l}{2}\right) \leq (1+2c(\lambda))(1+\beta_l)\left(\frac{3+\mathbb{E}\cos^+\alpha_{rl}}{4}+\beta_2\right)\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2}+c(\lambda).$$

Proof. Let $p_r = \Im h_r/(\Im h_r + \Im h_l)$ from Lemma 12(iii) and the AM-GM inequality, we have

$$\begin{aligned}\gamma\left(\frac{h_r+h_l}{2}\right) &= \frac{1}{2}p_r\gamma_r + \frac{1}{2}(1-p_r)\gamma_l + \cos\alpha_{rl}\sqrt{p_r(1-p_r)}\sqrt{\gamma_r\gamma_l}\\ &\leq \frac{1}{2}p_r\gamma_r + \frac{1}{2}(1-p_r)\gamma_l + \frac{1}{4}\cos^+(\alpha_{rl})(\gamma_r+\gamma_l).\end{aligned}$$

Using the convexity of $x^{p/2}$ and that $\cos^{+\,p/2} \leq \cos^+$ we get

$$\gamma^{p/2}\left(\frac{h_r+h_l}{2}\right) \leq \frac{1}{2}p_r\gamma_r^{p/2} + \frac{1}{2}(1-p_r)\gamma_l^{p/2} + \frac{1}{4}\cos^+(\alpha_{rl})(\gamma_r^{p/2}+\gamma_l^{p/2}).$$

We treat the two events $\{h_r = h_i \text{ for some } i \in \{1,\dots,n\}\}$ and $\{h_r = \hat{h}\}$ separately. For the first one, we write, with $p_i = \Im h_i/(\Im h_r + \Im h_l)$ and α_i as in Lemma 17,

$$\begin{aligned}\mathbb{E}_s\gamma^{p/2}\left(\frac{h_i+h_l}{2}\right) &\leq \frac{1}{2}\mathbb{E}_s[p_i\gamma_i^{p/2}] + \frac{1}{2}(1-\mathbb{E}_s[p_i])\gamma_l^{p/2} + \frac{1}{4}\mathbb{E}_s[\cos^+(\alpha_i)(\gamma_i^{p/2}+\gamma_l^{p/2})]\\ &\leq \frac{1}{2}\mathbb{E}_s[p_i]\mathbb{E}_s[\gamma_i^{p/2}] + \frac{1}{2}(1-\mathbb{E}_s[p_i])\gamma_l^{p/2} + \frac{1}{2}\mathbb{E}_s[\cos^+\alpha_i]\left(\frac{\mathbb{E}_s[\gamma_i^{p/2}]+\gamma_l^{p/2}}{2}\right)\\ &\qquad + (\frac{1}{2}+\frac{1}{4})\sqrt{\mathrm{Var}_s\gamma_i^{p/2}}\\ &\leq \left(\frac{1+\mathbb{E}_s\cos^+\alpha_i}{2}+\beta_2\right)\sqrt{1+\beta_l}\sqrt{\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2}},\end{aligned}$$

where we used Jensen inequality, (2.19)-(2.20) from Lemma 15 together with $\mathbb{E}_s[f_i\gamma_i^{p/2}] \leq \mathbb{E}_s[f_i]\mathbb{E}_s[\gamma_i^{p/2}] + \sqrt{\mathrm{Var}_s\gamma_i^{p/2}}$ if $0 \leq f_i \leq 1$ (for $f_i = \cos^+\alpha_i$ and $f_i = p_i$).

Now, we deal with event $\{h_r = \hat{h}\}$. From Lemma 16(i) and Lemma 13(i), we have $\hat{\gamma}^{p/2} \leq (1+c(\lambda))\mathbb{E}_s\gamma_i^{p/2} + c(\lambda)$. We deduce, with $\hat{p} = \Im\hat{h}/(\Im\hat{h} + \Im h_l)$,

$$\begin{aligned}
\gamma^{p/2}\left(\frac{h_r+h_l}{2}\right) &\leq \frac{1}{2}\hat{p}\hat{\gamma}^{p/2} + \frac{1}{2}(1-\hat{p})\gamma_l^{p/2} + \frac{1}{4}\cos^+\hat{\alpha}(\hat{\gamma}^{p/2} + \gamma_l^{p/2}) \\
&\leq (1+c(\lambda))\left(\frac{1}{2}\hat{p}\mathbb{E}_s\gamma_i^{p/2} + \frac{1}{2}(1-\hat{p})\gamma_l^{p/2} + \frac{\cos^+\hat{\alpha}}{2}\frac{\mathbb{E}_s\gamma_i^{p/2} + \gamma_l^{p/2}}{2}\right) + (\frac{1}{2}+\frac{1}{4})c(\lambda) \\
&\leq (1+c(\lambda))\left(\frac{1+\cos^+\hat{\alpha}}{2}\right)\sqrt{1+\beta_l}\sqrt{\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2}} + c(\lambda),
\end{aligned}$$

where have used Jensen inequality and Lemma 15. Combining the two events and multiplying by $\gamma_l^{p/2}$ we get from a new application of Lemma 15 and Jensen inequality,

$$\begin{aligned}
\gamma_l^{p/2}\mathbb{E}\gamma^{p/2}\left(\frac{h_r+h_l}{2}\right) &\leq (1+c(\lambda))\left(\frac{1+\mathbb{E}\cos^+\alpha_{rl}}{2} + \beta_2\right)(1+\beta_l)\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2} + \frac{c(\lambda)}{2}\gamma_l^{p/2} \\
&\leq (1+2c(\lambda))\left(\frac{1+\mathbb{E}\cos^+\alpha_{rl}}{2} + \beta_2\right)(1+\beta_l)\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2} + \frac{c(\lambda)}{4}.
\end{aligned}$$

Moreover, from (2.21)-(2.22) we obtain

$$\begin{aligned}
\mathbb{E}\gamma_r^p &\leq (1+c(\lambda))\mathbb{E}_s\gamma_i^p + \frac{c(\lambda)}{2} \leq (1+c(\lambda))(1+\beta_l)\left(\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2}\right) + \frac{c(\lambda)}{2} \\
\gamma_l^{p/2}\mathbb{E}\gamma_r^{p/2} &\leq (1+c(\lambda))\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2} + \frac{c(\lambda)(\gamma_l^p+1)}{4} \leq (1+2c(\lambda))(1+\beta_l)\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2} + c(\lambda)/4.
\end{aligned}$$

Finally, using Lemma 12(ii) at the first line and then applying the above bounds, we find

$$\begin{aligned}
\mathbb{E}\gamma^p\left(\frac{h_r+h_l}{2}\right) &\leq \mathbb{E}\left[\gamma^{p/2}\left(\frac{h_r+h_l}{2}\right)\left(\frac{\gamma^{p/2}(h_r) + \gamma^{p/2}(h_l)}{2}\right)\right] \\
&\leq \frac{1}{4}\mathbb{E}\gamma_r^p + \frac{1}{4}\mathbb{E}[\gamma_r^{p/2}]\gamma_l^{p/2} + \frac{\gamma_l^{p/2}}{2}\mathbb{E}\gamma^{p/2}\left(\frac{h_r+h_l}{2}\right) \\
&\leq (1+2c(\lambda))(1+\beta_l)\left[\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\left(\frac{1+\mathbb{E}\cos^+\alpha_{rl}}{2} + \beta_2\right)\right]\left(\frac{\gamma_l^p + \mathbb{E}_s\gamma_i^p}{2}\right) + c(\lambda).
\end{aligned}$$

This concludes the proof. □

All ingredients are now gathered to establish the asymmetric uniform contraction.

Proof of Proposition 14. Let $\delta, \lambda_0, R_0(.)$ from Lemma 17. There exists $\varepsilon_0 = \varepsilon_0(E,p) >$

0 such that

$$Q_sQ_2Q_l \geq 1-\varepsilon_0 \implies (1+\beta_l)\left(1-\frac{\delta}{4}+\beta_2\right) \leq 1-\frac{\delta}{5}.$$

Without loss of generality, we suppose that $\varepsilon_0 \leq 1/2$. We may also assume that λ_0 is small enough so that

$$(1+2c(\lambda_0))\max\left\{1-\frac{\varepsilon_0}{4}, 1-\frac{\delta}{5}\right\} = 1-\varepsilon,$$

for some $\varepsilon = \varepsilon(E,p) > 0$. Now if $(\gamma_l^p + \mathbb{E}_s\gamma_i^p)/2 \leq R_0(\lambda)$, then by Lemma 16 we have

$$\mathbb{E}\gamma^p\left(\frac{h_r+h_l}{2}\right) \leq (1+2c(\lambda))R_0(\lambda)+c(\lambda) := R(\lambda).$$

Note that by definition $R \in \mathrm{Lip}_0(\lambda_0)$. If $Q_sQ_2Q_l \leq 1-\varepsilon_0$ then by Lemma 16 again we have

$$\mathbb{E}\gamma^p\left(\frac{h_r+h_l}{2}\right) \leq (1+2c(\lambda))\left(1-\frac{\varepsilon_0}{4}\right)\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2}+c(\lambda).$$

Otherwise $(\gamma_l^p+\mathbb{E}_s\gamma_i^p)/2 \geq R_0(\lambda)$ and $Q_sQ_2Q_l \geq 1-\varepsilon_0 \geq 1/2$. We may use Lemma 17 together with Lemma 18 and get

$$\mathbb{E}\gamma^p\left(\frac{h_r+h_l}{2}\right) \leq (1+2c(\lambda))\left(1-\frac{\delta}{5}\right)\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2}+c(\lambda).$$

From our choice of $\varepsilon > 0$ and $R(\lambda)$, we find in that all cases

$$\mathbb{E}\gamma^p\left(\frac{h_r+h_l}{2}\right) \leq (1-\varepsilon)\frac{\gamma_l^p+\mathbb{E}_s\gamma_i^p}{2}+R(\lambda),$$

which concludes the proof. □

Proof of Theorem 7

Step 1: rough bound on $\gamma(g)$. We assume that N has finite $2p$-moments. We first check that $\mathbb{E}\gamma^p(g)$ is finite for any $z \in \mathbb{H}_E$ (recall that $\gamma(g) = \gamma(g,\Gamma(z))$ depends implicitly on z). From (2.6),

$$\Im g \overset{d}{=} \frac{\Im(z+\frac{1}{d}\sum_{i=1}^N g_i + v)}{|z+\frac{1}{d}\sum_{i=1}^N g_i+v|^2} \geq \frac{\Im z}{|z+\frac{1}{d}\sum_{i=1}^N g_i+v|^2} \geq \frac{\Im z}{\left(|z|+\frac{N}{d\Im z}+|v|\right)^2}.$$

Moreover $|g|, |g_i|, |\Gamma| \leq 1/\Im z$. We get

$$\mathbb{E}\gamma^p(g) = \mathbb{E}\frac{|g-\Gamma|^{2p}}{(\Im\Gamma\Im g)^p} \leq \frac{1}{(\Im\Gamma)^p}\left(\frac{2}{\Im z}\right)^{2p}\frac{1}{(\Im z)^p}\mathbb{E}\left(|z|+\frac{N}{d\Im z}+|v|\right)^{2p} < \infty,$$

since by assumption both N and v admit $2p$-moments. Now as the expectation is finite, we will show that $\mathbb{E}\gamma^p(g) \leq (1-\varepsilon)\mathbb{E}\gamma^p(g) + R$ with ε, R independent of $z \in \mathbb{H}_E$, to obtain the uniform bound $\mathbb{E}\gamma^p(g) \leq R/\varepsilon$.

Step 2: averaging with two copies. We now upper bound $\mathbb{E}\gamma^p(g)$ in terms of $\mathbb{E}\gamma^p((g_r+g_l)/2)$ where g_r, g_l are two independents copies of g. On the event $\mathcal{E}_\lambda$ defined in (2.10) we have $N \geq 2$. We use Lemma 12 and Lemma 13(i) and obtain

$$\begin{aligned}\mathbf{1}_{\mathcal{E}_\lambda}\gamma^p(g) &\leq \mathbf{1}_{\mathcal{E}_\lambda}\gamma^p\left(\frac{N}{d}\frac{1}{N}\sum_{i=1}^N g_i + v\right) \leq \mathbf{1}_{\mathcal{E}_\lambda}(1+c(\lambda))\gamma^p\left(\frac{1}{N}\sum_{i=1}^N g_i\right) + \mathbf{1}_{\mathcal{E}_\lambda}c(\lambda)\\ &\leq (1+c(\lambda))\frac{1}{N(N-1)}\sum_{1\leq i\neq j\leq N}\gamma^p\left(\frac{g_i+g_j}{2}\right) + c(\lambda).\end{aligned}$$

By taking expectation over the g_i's and use that given (N, v), the g_i's are iid copies of g, we get

$$\mathbb{E}\mathbf{1}_{\mathcal{E}_\lambda}\gamma^p(g) \leq (1+c(\lambda))\mathbb{E}\gamma^p\left(\frac{g_r+g_l}{2}\right) + c(\lambda).$$

On the complementary event $\mathcal{E}_\lambda^c$, we use Lemma 13(iii):

$$\begin{aligned}\mathbf{1}_{\mathcal{E}_\lambda^c}\gamma^p(g_o) &\leq \mathbf{1}_{\mathcal{E}_\lambda^c}\gamma^p\left(\frac{N}{d}\frac{1}{N}\sum_{i=1}^N g_i + v\right) \leq \mathbf{1}_{\mathcal{E}_\lambda^c}C(1+|v|)^{2p}\left(\frac{N}{d}+\frac{d}{N}\right)^p\left(\gamma^p\left(\frac{1}{N}\sum_{i=1}^N g_i\right)+1\right)\\ &\leq \mathbf{1}_{\mathcal{E}_\lambda^c}C(1+|v|)^{2p}\left(\frac{N}{d}+\frac{d}{N}\right)^p\frac{1}{N}\sum_{i=1}^N\left(\gamma^p(g_i)+1\right).\end{aligned}$$

We average first over the g_i's and obtain

$$\mathbb{E}\mathbf{1}_{\mathcal{E}_\lambda^c}\gamma^p(g) \leq C\mathbb{E}\left[\mathbf{1}_{\mathcal{E}_\lambda^c}(1+|v|)^{2p}\left(\frac{N}{d}+\frac{d}{N}\right)^p\right]\left(\mathbb{E}\gamma^p(g)+1\right) = C\alpha_p(\lambda)\left(\mathbb{E}\gamma^p(g)+1\right).$$

From the identity $1 = \mathbf{1}_{\mathcal{E}_\lambda} + \mathbf{1}_{\mathcal{E}_\lambda^c}$, we obtain:

$$\mathbb{E}\gamma^p(g) \leq (1+c(\lambda))\mathbb{E}\gamma^p\left(\frac{g_r+g_l}{2}\right) + c(\lambda) + C\alpha_p(\lambda)\left(\mathbb{E}\gamma^p(g)+1\right). \tag{2.25}$$

Step 3: asymmetric uniform contraction. We now bound $\mathbb{E}\gamma^p((g_r+g_l)/2)$ in terms of $\mathbb{E}\gamma^p(g)$. There exists an enlarged probability space for the variables (g_r, g_l) such that

$$g_r = -\left(z + \frac{1}{d}\sum_{i=1}^N h_i + v\right)^{-1},$$

where $(h_i)_{i\geq 1}$ are iid with law $g_r \stackrel{d}{=} g$ and independent of (N, v). For integer $n \geq 1$ and $\lambda \in [0,1)$, let $\mathcal{E}_{\lambda,n} = \{|v| \leq \lambda, N = n\}$. We have $\mathcal{E}_\lambda = \bigcup_n \mathcal{E}_{\lambda,n}$ where the union is over all $n \geq 2$ such that $n/d \in (1-\lambda, 1+\lambda)$. On the event $\mathcal{E}_{\lambda,n}$, we set $u_n = n/d - 1 \in (-\lambda, \lambda)$ and $\boldsymbol{h} = (h_1, \dots, h_n)$. Hence, on the event $\mathcal{E}_{\lambda,n}$, we have

$$g_r = -\left(z + \frac{1+u_n}{d}\sum_{i=1}^{n} h_i + v\right)^{-1} = \hat{h}(\boldsymbol{h}, u_n, v, n),$$

where $\hat{h}$ was defined in (2.17). Let v_n be a random variable whose distribution is v given $\mathcal{E}_{\lambda,n}$ and let $h_r = h_r(\boldsymbol{h}, u, v, n)$ be the random variable defined in (2.18). Using Proposition 14, we deduce that if $\lambda \leq \lambda_0$,

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\gamma^p\left(\frac{g_r+g_l}{2}\right)\right] + \frac{1}{2}\mathbb{E}\left[\gamma^p\left(\frac{g_r+g_l}{2}\right)\middle|\mathcal{E}_{\lambda,n}\right] \\
&= \mathbb{E}\left[\frac{1}{2n}\sum_{i=1}^{n}\gamma^p\left(\frac{h_i+g_l}{2}\right) + \frac{1}{2}\gamma^p\left(\frac{\hat{h}(\boldsymbol{h}, u_n, v_n, n)+g_l}{2}\right)\right] \\
&= \mathbb{E}\left[\gamma^p\left(\frac{h_r(\boldsymbol{h}, u_n, v_n, n)+g_l}{2}\right)\right] \\
&\leq \mathbb{E}\left[(1-\varepsilon)\left(\frac{1}{2}\gamma^p(g_l) + \frac{1}{2n}\sum_{i=1}^{n}\gamma^p(h_i)\right) + R(\lambda)\right] \\
&= (1-\varepsilon)\mathbb{E}[\gamma^p(g)] + R(\lambda).
\end{aligned}$$

We now use the identity

$$1 = \frac{\mathbb{P}(\mathcal{E}_\lambda^c) + \mathbf{1}_{\mathcal{E}_\lambda^c}}{2} + \sum_n \frac{\mathbb{P}(\mathcal{E}_{\lambda,n}) + \mathbf{1}_{\mathcal{E}_{\lambda,n}}}{2},$$

where the sum is over all $n \geq 2$ such that $n/d \in (1-\lambda, 1+\lambda)$. The first term is bounded using the convexity of γ^p, Lemma 12(ii), and Lemma 13(iii),

$$\begin{aligned}
&\mathbb{E}\left[\gamma^p\left(\frac{g_r+g_l}{2}\right)\frac{\mathbb{P}(\mathcal{E}_\lambda^c) + \mathbf{1}_{\mathcal{E}_\lambda^c}}{2}\right] \leq \mathbb{E}\left[\frac{\gamma^p(g_r)+\gamma^p(g_l)}{2}\frac{\mathbb{P}(\mathcal{E}_\lambda^c) + \mathbf{1}_{\mathcal{E}_\lambda^c}}{2}\right] \\
&\leq \frac{3}{4}\mathbb{P}(\mathcal{E}_\lambda^c)\mathbb{E}\gamma^p(g) + \frac{C}{4}\mathbb{E}\left[\mathbf{1}_{\mathcal{E}_\lambda^c}(1+|v|)^{2p}\left(\frac{N}{d}+\frac{d}{N}\right)^p\right](\mathbb{E}\gamma^p(g)+1) \\
&\leq \frac{C+3}{4}\alpha_p(\lambda)\left(\mathbb{E}\gamma^p(g)+1\right),
\end{aligned}$$

where we used the fact that $\mathbb{P}(\mathcal{E}_\lambda^c) \leq \alpha_p(\lambda)$. Finally we obtain

$$\mathbb{E}\gamma^p\left(\frac{g_r+g_l}{2}\right) \leq (1-\varepsilon)\mathbb{E}\gamma^p(g) + R(\lambda) + \frac{C+3}{4}\alpha_p(\lambda)\left(\mathbb{E}\gamma^p(g)+1\right). \tag{2.26}$$

Step 4: end of proof. If λ is small enough so that $1 + c(\lambda) \leq 2$, combining

(2.25)-(2.26), we get

$$\mathbb{E}\gamma^p(g) \le (1-\varepsilon)(1+c(\lambda))\mathbb{E}\gamma^p(g) + 2R(\lambda) + c(\lambda) + (2C+1)\alpha_p(\lambda)\left(\mathbb{E}\gamma^p(g)+1\right).$$

We now assume that $\lambda + \alpha_p(\lambda)$ is small enough, say λ_1, so that $(1-\varepsilon)(1+c(\lambda)) + (2C+1)\alpha_p(\lambda) \le 1-\varepsilon/2$. Then

$$\mathbb{E}\gamma^p(g) \le \left(1-\frac{\varepsilon}{2}\right)\mathbb{E}\gamma^p(g) + 2R(\lambda) + c(\lambda) + (2C+1)\alpha_p(\lambda).$$

The above equation is of the form

$$\mathbb{E}\gamma^p(g) \le (1-\varepsilon_1)\mathbb{E}\gamma^p(g) + R_1(\lambda+\alpha_p(\lambda)),$$

where $\varepsilon_1 = \varepsilon/2 > 0$ and $R_1 \in \mathrm{Lip}_0(\lambda_1)$. We deduce that for all $\lambda \le \lambda_1$,

$$\mathbb{E}\gamma^p(g) \le \frac{R_1(\lambda+\alpha_p(\lambda))}{\varepsilon_1}. \tag{2.27}$$

Finally, we use Lemma 11 and Cauchy-Schwarz inequality,

$$\mathbb{E}|g-\Gamma|^p \le |\Gamma|^p 2^{p/2-1}\left(4^{p/2}\mathbb{E}\gamma^p(g) + 2^{p/2}\mathbb{E}\gamma(g)^{p/2}\right) \le |\Gamma|^p 2^{2p}\left(\mathbb{E}\gamma^p(g) + \sqrt{\mathbb{E}\gamma^p(g)}\right),$$

$$\mathbb{E}(\Im g)^{-p} \le (\Im\Gamma)^{-p}\mathbb{E}(4\gamma(g,h)+2)^p < \infty.$$

As $|\Gamma|, (\Im\Gamma)^{-1}$ are uniformly bounded in $\mathbb{H}_E$, this concludes the proof of Theorem 7 for suitable constants $\delta = \lambda_1/2$ and $C > 0$ when N has finite $2p$-moments.

Step 5: truncation. It remains to check that we can remove the assumption that $\mathbb{E}N^{2p}$ is finite if v is real-valued. This can be done easily by truncation. From (2.4), $g(z)$ has law $g_1(z) = \langle \delta_1, (H-z)^{-1}\delta_1\rangle$. For integer $n \ge 1$ and $v \in \mathcal{V}$, let $N_v^{(n)} = \min(N_v, n)$ and let $\mathcal{T}_n$ be the corresponding random rooted tree of descendants of 1. The tree $\mathcal{T}_n$ has law $\mathrm{GW}(P_n)$ where P_n is the law of $N^{(n)} = \min(N,n)$ with $N \stackrel{d}{=} P$. By monotone convergence, $d_n = \mathbb{E}N^{(n)}$ converges to $d = \mathbb{E}N$ as n goes to infinity. Let H_n be the corresponding operator and $g_1^{(n)}(z) = \langle \delta_1, (H_n-z)^{-1}\delta_1\rangle$. Then by [RS78, Theorem VIII.25(a)], for any fixed $z \in \mathbb{H}$, with probability one,

$$\lim_{n\to\infty} g_1^{(n)}(z) = g_1(z),$$

(note however that the convergence is a priori not uniform in $z \in \mathbb{H}$). Since $|g_1^{(n)}(z)| \le 1/\Im(z)$, the convergence also holds in L^q for any $q \ge 1$. Note also that by monotone convergence, for any $\lambda > 0$, $\alpha_p^{(n)}(\lambda)$ converges to $\alpha_p(\lambda)$ where $\alpha_p^{(n)}(\lambda)$ is defined as in (2.11) for P_n and d_n. It thus remains to apply Theorem 7 to P_n and let n go to infinity. □

Proof of Corollary 8

Set $N_o = N_\star$ and $V_o = v_\star$. From (2.4), we have

$$g_o = -\left(z + \frac{1}{d}\sum_{i=1}^{N_\star} g_i + v_*\right)^{-1},$$

where $(g_i)_{i\geq 1}$ are independent iid copies of g which satisfies the equation in distribution (2.6). From (2.8), we get

$$\gamma(g_o, \Gamma_\star) = \gamma\left(-\frac{1}{g_o}, -\frac{1}{\Gamma_\star}\right) \leq \gamma\left(\frac{1}{d}\sum_{i=1}^{N_\star} g_i + v_*, \rho\Gamma\right) = \gamma\left(\frac{N_\star}{d_\star}\frac{1}{N_\star}\sum_{i=1}^{N_\star} g_i + \frac{v_*}{\rho}, \Gamma\right),$$

where we have used $\gamma(ta, tb) = \gamma(a, b)$ for $t > 0$ and $\rho = d_\star/d$. From Lemma 12(ii) and Lemma 13(ii), we deduce that

$$\mathbf{1}_{\mathcal{F}_\lambda^c}\gamma^p(g_o, \Gamma_\star) \leq C\mathbf{1}_{\mathcal{F}_\lambda^c}\left(1 + \left|\frac{v_\star}{\rho}\right|\right)^{2p}\left(\frac{N_\star}{d_\star} + \frac{d_\star}{N_\star}\right)^p\left(\frac{1}{N_\star}\sum_{i=1}^{N_\star}\gamma^p(g_i) + 1\right),$$

$$\mathbb{E}\mathbf{1}_{\mathcal{F}_\lambda^c}\gamma^p(g_o, \Gamma_\star) \leq C\beta_p(\lambda)\left(\mathbb{E}\gamma^p(g) + 1\right).$$

Similarly, from Lemma 12(ii) and Lemma 13(i),

$$\mathbf{1}_{\mathcal{F}_\lambda}\gamma^p(g_o, \Gamma_\star) \leq (1 + c(\lambda))\frac{1}{N_\star}\sum_{i=1}^{N_\star}\gamma^p(g_i) + c(\lambda),$$

$$\mathbb{E}\mathbf{1}_{\mathcal{F}_\lambda}\gamma^p(g_o, \Gamma_\star) \leq (1 + c(\lambda))\mathbb{E}\gamma^p(g) + c(\lambda)$$

Taking the expectation, we deduce that if $\lambda + \alpha_p(\lambda) \leq \delta$,

$$\begin{aligned}
\mathbb{E}\gamma^p(g_o, \Gamma_\star) &\leq (1 + c(\lambda) + C\beta_p(\lambda))\mathbb{E}\gamma^p(g) + C\beta_p(\lambda) + c(\lambda)\\
&\leq C_1(1 + \lambda + \beta_p(\lambda))(\lambda + \alpha_p(\lambda)) + C_1(\beta_p(\lambda) + \lambda)\\
&\leq C_1\left((1 + \lambda + \beta_p(\lambda))(1 + \lambda + \alpha_p(\lambda)) - 1\right)
\end{aligned}$$

for some $C_1 > 0$, where we have used (2.27) and the fact that $c(\lambda) \in \mathrm{Lip}_0(\delta)$. We have

$$|\Gamma_\star| \leq \frac{1}{\Im(z + \rho\Gamma)} \leq \frac{1}{\rho\Im(\Gamma)} \leq \frac{C_2}{\rho},$$

$$\Im\Gamma_\star = \frac{\Im z + \rho\Im\Gamma}{|z + \rho\Gamma|^2} \geq \frac{1}{2}\frac{\rho\Im\Gamma}{|z|^2 + \rho^2|\Gamma|} \geq C_2\left(\rho + \frac{1}{\rho}\right)^{-1},$$

uniformly in $z \in \mathbb{H}_E$, for some constant C_2. It remains to use Lemma 11 and adjust the constants. □

2.1.4 Random trees with leaves

Proof of Lemma 9

Let $v \neq o \in \mathcal{V}$. Let $\bar{N}^s$ and $\bar{N}^e$ be the number of offspring of v in $\mathcal{S}$ and not in $\mathcal{S}$ (that is such that their subtree is infinite or finite). The pair $(\bar{N}^s, \bar{N}^e)$ has the same distribution than $(\sum_{i=1}^N (1-\varepsilon_i), \sum_{i=1}^N \varepsilon_i)$ where N has distribution P and is independent of the $(\varepsilon_i)_{i\geq 1}$ an iid sequence of Bernoulli variables with $\mathbb{P}(\varepsilon_i = 1) = \pi_e = 1 - \mathbb{P}(\varepsilon_i = 0)$. Moreover, conditioned on the root is in $\mathcal{S}$, $(\bar{N}^s, \bar{N}^e)$ is conditioned on $\bar{N}^s \geq 1$. As in Lemma 9, let (N^s, N^e) denote a pair of random variables with distribution $(\bar{N}^s, \bar{N}^e)$ conditioned on $\bar{N}^s \geq 1$. In particular, the moment generating function of (N^s, N^e) is given by

$$\varphi^{s,e}(x,y) = \mathbb{E}\left[x^{N^s} y^{N^e}\right] = \mathbb{E}\left[x^{\bar{N}^s} y^{\bar{N}^e} | \bar{N}^s \geq 1\right] = \frac{\varphi((1-\pi_e)x + \pi_e y) - \varphi(\pi_e y)}{1-\pi_e}. \tag{2.28}$$

Similarly, given $v \notin \mathcal{S}$, (N^s, N^e) is conditioned on $N^s = 0$. Then, we find that the moment generating function of N^e given $o \notin \mathcal{S}$ is

$$\varphi^e(x) = \frac{\varphi(\pi_e x)}{\pi_e}. \tag{2.29}$$

For more details see Athreya and Ney [AN72, Section I.12] or Durrett [Dur10, Section 2.1].

Similarly, let $\bar{N}^s_\star$ and $\bar{N}^e_\star$ be the number of offspring of o in $\mathcal{S}$ and not in $\mathcal{S}$. Let $(N^s_\star, N^e_\star)$ be the law of $(\bar{N}^s_\star, \bar{N}^e_\star)$ given $\bar{N}^s_\star \geq 1$. The moment generating function of $(N^s_\star, N^e_\star)$ is

$$\varphi^{s,e}_\star(x,y) = \mathbb{E}\left[x^{N^s_\star} y^{N^e_\star}\right] = \frac{\varphi_\star((1-\pi_e)x + \pi_e y) - \varphi_\star(\pi_e y)}{1-\varphi_\star(\pi_e)}.$$

We are ready to prove Lemma 9.

Proof of Lemma 9. Recall that $H = A/\sqrt{d_s}$ where $d_s = \mathbb{E}N^s$ and $g_v(z) = \langle \delta_v, (H_v - z)^{-1}\delta_v \rangle$. For $v \neq o$, let $g^s(z)$ and $g^e(z)$ be the law of $g_v(z)$ conditioned on $v \in \mathcal{S}$ and $v \notin \mathcal{S}$. We deduce from (2.4) that the variable $g^s(z)$ satisfies the recursive distribution equation,

$$g^s(z) \stackrel{d}{=} -\left(z + \frac{1}{d_s}\sum_{i=1}^{N^s} g_i^s(z) + v(z)\right)^{-1},$$

where g_i^s are iid copies of g^s, independent of $(N^s, v(z))$ defined by

$$v(z) = \frac{1}{d_s}\sum_{i=1}^{N^e} g_i^e(z), \tag{2.30}$$

and g_i^e are iid copies of g^e and are independent of (N^s, N^e) with moment generating function given by (2.28). Similarly, conditioned on $o \in \mathcal{S}$, we have

$$g_o(z) \stackrel{d}{=} -\left(z + \frac{1}{d_s}\sum_{i=1}^{N_\star^s} g_i^s(z) + \frac{1}{d_s}\sum_{i=1}^{N_\star^e} g_i^e(z)\right)^{-1},$$

where g_i^s are iid copies of g^s, independent of $(N_\star^s, N_\star^e)$. It concludes the proof of Lemma 9. □

Remark 1. *As we defined it, the skeleton tree $\mathcal{S}$ is not a unimodular tree. To obtain a unimodular Galton-Watson tree, we could have modified the rule for the root to be in $\mathcal{S}$: the root is in $\mathcal{S}$ if it has at least two offsprings with infinite subtrees.*

Proof of Lemma 10

We start with a rough upper bound on the extinction probability π_e.

Lemma 19 (Extinction probability)**.** *We have*

$$\pi_e \leq 2\mathbb{P}(N \leq 1).$$

Proof. We note that if $x \in [0,1]$ satisfies $\varphi(x) \leq x$ then $\pi_e \leq x$. Set $\pi_1 = \mathbb{P}(N \leq 1)$. For $x \in [0,1]$, we have $\varphi(x) \leq \pi_1 + (1-\pi_1)x^2$. In particular, since $\pi_1(1-\pi_1) \leq 1/4$, $\varphi(2\pi_1) \leq \pi_1(1 + 4(1-\pi_1)\pi_1) \leq 2\pi_1$. We conclude that $\pi_e \leq 2\pi_1$. □

We will also use [Bor15, Lemma 23] on the size of the sub-critical Galton-Watson trees.

Lemma 20 (Total progeny of subcritical Galton-Watson tree)**.** *Let Q be a probability measure on non-negative integers whose moment generating function ψ satisfies $\psi(\rho) \leq \rho$ for some $\rho \geq 1$. Let Z be the total number of vertices in a $\mathrm{GW}(Q)$ tree. We have for any integer $k \geq 1$,*

$$\mathbb{P}(Z \geq k) \leq \rho\left(\frac{\psi(\rho)}{\rho}\right)^k.$$

As a corollary, we deduce that the random tree $\mathcal{T}$ conditioned on extinction is likely to have very few vertices.

Corollary 21. *Let $n \geq 2$, $q = \mathbb{P}(N < n)$ and let Z be the total number of vertices in $\mathcal{T}$ conditioned on extinction. If $q \leq 2^{-n/(n-1)}$, we have for any integer $k \geq 1$,*

$$\mathbb{P}(Z \geq k) \leq \frac{q^{1/n}}{\pi_e}\left(2q^{1-1/n}\right)^k.$$

Proof. We have $\varphi(q^{1/n}) \leq q + (q^{1/n})^n(1-q) \leq 2q \leq q^{1/n}\delta$ with $\delta = 2q^{1-1/n}$. By assumption $\delta \leq 1$. Let $\rho = q^{1/n}/\pi_e$. From (2.29), we deduce that

$$\varphi_e(\rho) = \frac{\varphi(q^{1/n})}{\pi_e} \leq \delta\rho.$$

Note that since $\varphi_e'(1) = \varphi'(\pi_e) < 1$ and φ_e convex, we have $\varphi_e(x) > x$ on $[0,1)$. It follows that $\rho \geq 1$. We then conclude by applying Lemma 20. □

We are now ready to prove Lemma 10.

Proof of Lemma 10. For integer $k \geq 1$, let $\Lambda_k \subset \mathbb{R}$ be the set of λ such that there exists a tree with k vertices and λ is an eigenvalue of the adjacency matrix of this tree. We note that $\cup_k \Lambda_k$ is the set of totally-real algebraic integers by [Sal15]. Obviously, $|\Lambda_k|$ is bounded by k times the number of unlabeled trees with k vertices. In particular, for some $C > 1$,

$$|\Lambda_k| \leq C^k, \tag{2.31}$$

see Flajolet and Sedgewick [FS09, Section VII.5]. We define $B_{k,\varepsilon} = \{x \in \mathbb{R} : \exists \lambda \in \Lambda_k, |\lambda/\sqrt{d_s} - x| \leq \varepsilon 2^{-k-1}/|\Lambda_k|\}$ and $B = \mathbb{R}\backslash \cup_{k\geq 1} B_{k,\varepsilon}$. By construction,

$$\ell(B^c) \leq \sum_{k\geq 1} |\Lambda_k| \frac{\varepsilon 2^{-k}}{|\Lambda_k|} = \varepsilon.$$

Note that if $|\mathcal{T}| = k$ and $z \in \mathbb{H}_B$ then $|g_o(z)| \leq 2^{k+1}|\Lambda_k|/\varepsilon$ (since $H = A/\sqrt{d_s}$ and the Cauchy-Stieltjes transform of a measure is bounded by one over the distance to the support). Let $g^e(z)$ be the law of $g_o(z)$ given $o \notin \mathcal{S}$. We deduce from (2.31) and Corollary 21 with $n = 2$ and $q = \pi_1$ that for $z \in \mathbb{H}_B$,

$$\begin{aligned}
\mathbb{E}[|g^e(z)|^m] &= \sum_{k=1}^{\infty} \mathbb{E}[|g^e(z)|^m \,|\, |\mathcal{T}| = k]\mathbb{P}(|T| = k) \\
&\leq \sum_{k=1}^{\infty} \frac{\sqrt{\pi}_1}{\pi_e} \left(2\sqrt{\pi}_1\right)^k \left(\frac{2^{k+1}|\Lambda_k|}{\varepsilon}\right)^m \\
&\leq \frac{2^m\sqrt{\pi_1}}{\pi_e \varepsilon^m} \sum_{k=1}^{\infty} \left(2^{m+1}C^m\sqrt{\pi_1}\right)^k \\
&\leq \frac{2^{2m+2}C^m}{\pi_e\varepsilon^m}\pi_1,
\end{aligned}$$

where at the last line, we have assumed that $\sqrt{\pi}_1 \leq 2^{-m-2}C^{-m}$ and used that $\sum_{k\geq 1} a^k \leq 2a$ if $0 \leq a \leq 1/2$. Note that this last condition is satisfied if $\pi_1 \leq c_0^m$ with $1/c_0 = 2^4C^2$. We deduce that for some new constant $C > 1$, we have for any integer $m \geq 1$ such that $\pi_1 \leq c_0^m$,

$$\mathbb{E}[|g^e(z)|^m] \leq \left(\frac{C}{\varepsilon}\right)^m \frac{\pi_1}{\pi_e}.$$

In particular, from (2.30) and Hölder inequality,

$$\mathbb{E}\left[|v(z)|^m\right] = \frac{1}{d_s^m}\mathbb{E}\left|\sum_{i=1}^{N^e} g_i^e(z)\right|^m \leq \frac{1}{d_s^m}\mathbb{E}\left[(N^e)^{m-1}\sum_{i=1}^{N^e}|g_i^e(z)|^m\right] \leq \mathbb{E}[(N^e)^m]\left(\frac{C}{\varepsilon d_s}\right)^m \frac{\pi_1}{\pi_e}.$$

It remains to estimate $\mathbb{E}[(N_e)^m]$. We write

$$\mathbb{E}[(N_e)^m] = \mathbb{E}[\bar{N}_e^m|\bar{N}_s \geq 1] \leq \mathbb{E}[(\bar{N}^e)^m]/(1-\pi_e) \leq 2\mathbb{E}[(\bar{N}^e)^m],$$

indeed, from Lemma 19, $\pi_e \leq 2\pi_1 \leq 2c_0 \leq 1/2$. We have seen that $\bar{N}^e \stackrel{d}{=} \sum_{i=1}^N \varepsilon_i$ where N has distribution P, independent of $(\varepsilon_i)_{i\geq 1}$ an iid sequence of Bernoulli variables with $\mathbb{P}(\varepsilon_i = 1) = \pi_e = 1 - \mathbb{P}(\varepsilon_i = 0)$. We find

$$\mathbb{E}[(\bar{N}^e)^m] = \mathbb{E}\left(\sum_{i=1}^N \varepsilon_i\right)^m \leq \sum_{k=1}^m \mathbb{E}[N^k]\pi_e^k \leq \frac{\mathbb{E}[N^m]}{d^m}\sum_{k=1}^m (d\pi_e)^k \leq 2\frac{\mathbb{E}[N^m]}{d^m}d\pi_e,$$

where, in the first sum, we have used that $\mathbb{E}\varepsilon_i^l = \pi_e$ for any $l \geq 1$ and decomposed the sum over the number k of distinct indices in a m-tuple. At the next step, we have used that $\mathbb{E}N^k/d^k$ is non-decreasing in k from Jensen inequality. In the final inequality, we used Lemma 19 and the assumption $d\pi_e \leq \mathbb{E}N^2\pi_e/d \leq 1/2$. We thus have proved that

$$\mathbb{E}\left[|v(z)|^m\right] \leq 4\left(\frac{C}{\varepsilon d_s}\right)^m \frac{\mathbb{E}[N^m]}{d^m}d\pi_1.$$

Finally, from (2.28) and the convexity of φ', we have

$$d_s = \partial_1\varphi^{s,e}(1,1) = \varphi'(1-\pi_e) \geq \varphi'(1) - \pi_e\varphi''(1) = d - \pi_e\mathbb{E}[N(N-1)] \geq d/2.$$

The claim of the lemma for $v(z)$ follows by adjusting the constant C. Replacing $N_\star$ by N, the claim on $v_\star(z)$ follows. □

2.1.5 From resolvent to spectral measures

Absolutely continuous part of a random measure

In this final section, we prove the main results in introduction, Theorem 4, Theorem 5 and Theorem 6. This is done by standard tools connecting the Cauchy-Stieltjes transform of a measure and its absolutely continuous part.

We will use the following lemma. Recall the definition of $\mathbb{H}_B$ in (2.14).

Lemma 22. *Let μ_1 and μ_2 be two random probability measures on $\mathbb{R}$ with Cauchy-Stieltjes transforms g_1 and g_2 respectively. Assume that for some deterministic Borel set $B \subset \mathbb{R}$, $C > 0$ and $p > 0$ we have*

$$\liminf_{\eta \downarrow 0} \int_B \mathbb{E}[|g_1(\lambda + i\eta) - g_2(\lambda + i\eta)|^p] \mathrm{d}\lambda \leq C.$$

Then, if f_1 and f_2 are the densities of the absolutely continuous parts of μ_1 and μ_2, we have

$$\mathbb{E} \int_B |f_1(\lambda) - f_2(\lambda)|^p \mathrm{d}\lambda \leq \frac{C}{\pi^p}.$$

Proof. For $\eta > 0$ and real λ, we set $f_i^\eta(\lambda) = \Im g_i(\lambda + i\eta)/\pi$. For $i = 1, 2$, almost everywhere $\lim_{\eta \to 0} f_i^\eta(\lambda) = f_i(\lambda)$, see for example Simon [Sim05, Chapter 11]. We consider the measure on $\mathbb{C} \times B$: $M = \mathbb{P} \otimes \ell(\cdot)$. By assumption, from Fubini's Theorem, for any $\eta > 0$,

$$\liminf_{\eta \downarrow 0} \int |f_1^\eta - f_2^\eta|^p \mathrm{d}M = \liminf_{\eta \downarrow 0} \int_B \mathbb{E}[|f_1^\eta(\lambda) - f_2^\eta(\lambda)|^p] \mathrm{d}\lambda \leq \frac{C}{\pi^p}.$$

Moreover, from what precedes, M-almost everywhere, $|f_1^\eta - f_2^\eta|^p$ converges to $|f_1 - f_2|^p$. The conclusion is then a consequence of Fatou's lemma. □

The following theorem due to Simon [Sim95, Theorem 2.1] generalizes the case $p = 2$ due to Klein [Kle98, Theorem 4.1] (see also (4.46) there). It is a useful criterion for a random probability measure to be purely absolutely continuous.

Lemma 23 ([Sim95; Kle98])**.** *Let $p > 1$ and μ be a random probability measure on $\mathbb{R}$ with Cauchy-Stieltjes transform g. For any $E > 0$, if*

$$\liminf_{\eta \downarrow 0} \int_{-E}^{E} \mathbb{E}[|g(\lambda + i\eta)|^p] \mathrm{d}\lambda < \infty$$

then, with probability one, μ is absolutely continuous on $(-E, E)$.

Proof of Theorem 6

Step 1: absolute continuity. Let $H = A/\sqrt{d-1} + \lambda V/\sqrt{d-1}$ and $g_o(z) = \langle \delta_o, (H - z)^{-1} \delta_o \rangle$. By construction g_o is the Cauchy-Stieltjes transform of $\tilde{\mu}_o = \mu_o(\cdot/\sqrt{d-1})$. We apply Corollary 8 with $N_\star = d$, $N = d-1$, independent of $v_\star \stackrel{d}{=} v \stackrel{d}{=} \lambda V_o/\sqrt{d-1}$. With $\rho = d/(d-1) \geq 1$ and α, β as in Corollary 8 with $p = 2$, we find

$$\alpha(t) = 4\mathbb{E}\left[\mathbf{1}_{|v|>t}(1 + |v|)^4\right] \quad \text{and} \quad \beta(t) = 4\mathbb{E}\left[\mathbf{1}_{|v|>t\rho}\left(1 + \frac{|v|}{\rho}\right)^4\right].$$

Note that $\beta(t) \leq \alpha(t)$. Also, by assumption $\mathbb{E}|V_o|^4 < \infty$. Since $(a+b)^n \leq 2^{n-1}(|a|^n + |b|^n)$,

$$\alpha(t) \leq 2^6 \left(\mathbb{P}\left(|V_o| > \frac{t\sqrt{d-1}}{\lambda} \right) + \frac{\lambda^4}{(d-1)^2} \mathbb{E}\left[\mathbf{1}_{|V_o| > \frac{t\sqrt{d-1}}{\lambda}} |V_o|^4 \right] \right) \leq 2^6 \left(1 + \frac{1}{t^4} \right) \frac{\lambda^4}{(d-1)^2} \mathbb{E}|V_o|^4.$$

It follows that for each $t, \kappa > 0$, there exists $\lambda_1 > 0$ such that $\lambda \leq \lambda_1 \sqrt{d}$ implies $\alpha(t), \beta(t) \leq \kappa$. Also $1 \leq \rho \leq 3/2$ (since $d \geq 3$, $d-1 \geq (2/3)d$). Now, we fix $E > 0$, we apply this observation to $t = \delta/2$ and $\kappa = \delta/2$ with δ as in Corollary 8. We deduce that for $\lambda \leq \lambda_1 \sqrt{d}$, we have $\mathbb{E}[|g_o(z)|^2] \leq C$ for some C. From Lemma 23, we deduce that with probability one, μ_o is absolutely continuous on $(-E\sqrt{d-1}, E\sqrt{d-1})$.
Step 2: positive density. We now prove that μ_0 has positive density on $(-E\sqrt{d-1}, E\sqrt{d-1})$ or equivalently that $\tilde{\mu}_0$ has positive density on $(-E, E)$. Corollary 8 implies that, for some $C > 0$, for all $z \in \mathbb{H}_E$,

$$\mathbb{E}(\Im g_o(z))^{-2} \leq C.$$

Let f be the density of the absolutely continuous part of $\tilde{\mu}_o$. Almost-everywhere, we have $\lim_{\eta \to 0} \Im g_o(\lambda + i\eta) = f(\lambda)/\pi$. We deduce from Fatou's Lemma that

$$\mathbb{E} \int_{-E}^{E} \frac{1}{f(\lambda)^2} \mathrm{d}\lambda < \infty.$$

In particular, with probability one, $\int_{-E}^{E} 1/f(\lambda)^2 \mathrm{d}\lambda < \infty$ and $f(\lambda) > 0$ almost everywhere.
Step 3: mass of absolutely continuous part. It remains to prove that for any $\varepsilon > 0$, there exists λ_0 such that $\mathbb{E}\mu_o^{\mathrm{ac}}(\mathbb{R}) \geq 1 - \varepsilon$. There exists $E < 2$ such that $\mu_\star((-E, E)) \geq 1 - \varepsilon/2$ uniformly in $d \geq 3$. As explained above, for each $t, \kappa > 0$, there exists $\lambda_1 > 0$ such that $\lambda \leq \lambda_1 \sqrt{d}$ implies $\alpha(t), \beta(t) \leq \kappa$. By adjusting the value of $t = \kappa$, by Lemma 22, we deduce that if $\lambda \leq \lambda_0(\varepsilon)\sqrt{d}$, we have

$$\mathbb{E} \int_{-E}^{E} |f(\lambda) - f_\star(\lambda)| d\lambda \leq \frac{\varepsilon}{2},$$

where f is the density of the absolutely continuous part of $\tilde{\mu}_o$ and $f_\star$ the density of $\mu_\star$. Therefore, from the triangle inequality, we find

$$\mathbb{E}\mu_o^{\mathrm{ac}}(\mathbb{R}) \geq \mathbb{E} \int_{-E}^{E} f(\lambda) d\lambda \geq \int_{-E}^{E} f_\star(\lambda) d\lambda - \frac{\varepsilon}{2} \geq 1 - \varepsilon,$$

as requested. □

Proof of Theorem 5

We treat the case of $P_\star = Q_d$ is $\mathrm{Bin}(n, d/n)$ conditioned on being at least 2. The Poisson case is essentially the same (take n to infinity in the argument below). Let $N_\star$ be a random variable with law Q_d and N with law $P = \widehat{Q}_d$. The moment generating function of $P_\star$ is

$$\varphi_\star(x) = \mathbb{E}[x^{N_\star}] = \left(\left(1 + \frac{d}{n}(x-1)\right)^n - q_0 - q_1 x \right) / (1 - q_0 - q_1),$$

where $q_0 = (1-d/n)^n$ and $q_1 = d(1-d/n)^{n-1}$ are the probabilities that $\mathrm{Bin}(n, d/n)$ is 0 and 1. In particular, the moment generating function of P is

$$\varphi(x) = \mathbb{E}[x^N] = \frac{\varphi_\star'(x)}{\varphi_\star'(1)} = \left(\left(1 + \frac{d}{n}(x-1)\right)^{n-1} - p_0 \right) / (1 - p_0),$$

where $p_0 = q_1/d = (1-d/n)^{n-1}$. Therefore, P is $\mathrm{Bin}(n-1, d/n)$ conditioned on being at least 1.

It is straightforward to check that q_0, q_1 and p_0 converge to 0 as d goes to infinity, uniformly in $n \geq d$. More is true: from Bennett's inequality, if $Z = Z_n \stackrel{d}{=} \mathrm{Bin}(n, d/n)$, for any $\lambda > 0$,

$$\mathbb{P}(|Z - d| \geq \lambda d) \leq 2\exp(-dh(\lambda)), \tag{2.32}$$

with $h(u) = (1+u)\ln(1+u) - u$. We deduce that for $m \geq 1$,

$$\left| \left(\frac{\mathbb{E}Z^m}{d^m}\right)^{1/m} - 1 \right|^m \leq \mathbb{E}\left|\frac{Z}{d} - 1\right|^m \leq m \int_0^\infty \lambda^{m-1} \mathbb{P}(|Z/d - 1| \geq \lambda) \mathrm{d}\lambda$$

goes to 0 as d goes to infinity, uniformly in $n \geq d$. As a consequence, since $\mathbb{E}Z_{n-1}^m \leq \mathbb{E}N^m \leq \mathbb{E}Z_{n-1}^m/(1-p_0)$ and $\mathbb{E}Z_n^m \leq \mathbb{E}N_\star^m \leq \mathbb{E}Z_n^m/(1-q_0-q_1)$, for any integer $m \geq 1$, $\mathbb{E}(N/d)^m$ and $\mathbb{E}(N_\star/d)^m$ go to 1 as d goes to infinity, uniformly in $n \geq d$.

Similarly, from (2.32), we write for $0 < \lambda < 1$,

$$\mathbb{E}\left(\frac{d}{\max(Z,1)}\right)^m \leq d^m \mathbb{P}(Z \leq (1-\lambda)d) + (1-\lambda)^{-m} \leq 2d^m e^{-dh(\lambda)} + (1-\lambda)^{-m}.$$

We deduce that $\mathbb{E}(d/N)^m$ and $\mathbb{E}(d/N_\star)^m$ go to 1 as d goes to infinity, uniformly in $n \geq d$.

The rest of the proof is the same than the proof of Theorem 6: we apply Corollary 8 with $p = 2$ to $H = A/\sqrt{\mathbb{E}N}$. The above analysis shows that, for any $\lambda > 0$, $\alpha_2(\lambda)$ and $\beta_2(\lambda)$ goes to 0 as d goes to infinity, uniformly in $n \geq d$. □

Proof of Theorem 4

Again, we only treat the binomial case, the Poisson case being essentially the same. Let π_e be the extinction probability. Let $N^s_\star$ and N^s be the offspring distribution of the root and $v \neq o$ conditioned on being in $\mathcal{S}$. The moment generating functions of $N^s_\star$ and N^s are given by (2.13). We set $d_s = \mathbb{E}N^s$. Arguing as in the proof of Theorem 5, we find that $N^s_\star$ has law $\mathrm{Bin}(n,(1-\pi_e)d/n)$ conditioned on being at least 1 and that N^s has law $\mathrm{Bin}(n-1,(1-\pi_e)d/n)$ conditioned on being at least 1. By Lemma 19, π_e goes to 0 as d goes to infinity, uniformly in $n \geq d$. Also, from the proof of Theorem 5, we deduce that, for any integer $m \geq 1$,

$$\mathbb{E}\left(\frac{N^s}{d}\right)^m, \ \mathbb{E}\left(\frac{d}{N^s}\right)^m, \ \mathbb{E}\left(\frac{N^s_\star}{d}\right)^m, \ \mathbb{E}\left(\frac{d}{N^s_\star}\right)^m$$

go to 1 as d goes to infinity, uniformly in $n \geq d$. Also, if $d \geq 2$, from Bennett's inequality (2.32),

$$\pi_1 = \mathbb{P}(N \leq 1) \leq \mathbb{P}(N \leq d/2) \leq 2\exp(-dh(1/2)) = o(1/d).$$

Now, let $\varepsilon_0 > 0$, $B = B(\varepsilon_0)$ be as in Lemma 10. If $v(z)$, $v_\star(z)$ are as in Lemma 9, we deduce from what precedes that, by Lemma 10, $\mathbb{E}|v(z)|^m$ and $\mathbb{E}|v_\star(z)|^m$ go to 0 as d goes to infinity, uniformly in $n \geq d$ and $z \in \mathbb{H}_B$.

We set $H = A/\sqrt{d_s}$ and let $0 < E < 2$. We denote by $\mathbb{E}^s$ the conditional expectation given $\mathcal{T}$ infinite. By Lemma 9, Corollary 8 and the above analysis with $m = 12$ (see (2.12)), we deduce that for any $\kappa > 0$, for all $d \geq d_0(\varepsilon_0, E, \kappa)$ large enough and for all $z \in \mathbb{H}_E \cap \mathbb{H}_B$,

$$\mathbb{E}^s|g_o(z) - \Gamma_\star(z)|^2 \leq \kappa/\rho^2 \quad \mathbb{E}^s(\Im g_o(z))^{-2} \leq C_\star\left(\rho + \frac{1}{\rho}\right)^2,$$

where $g_o(z) = \langle \delta_o, (H-z)^{-1}\delta_o\rangle$ and $\rho = \mathbb{E}N^s_\star/\mathbb{E}N^s$ can be taken to be approximately close to 1 for d large enough.

Let $\tilde{\mu}_o = \mu_o(\cdot/\sqrt{d_s})$ and let $f(\lambda)$ be the density of the absolutely continuous part of $\tilde{\mu}_o$. We fix $\kappa > 0$ and $d \geq d_0(\varepsilon_0, \kappa)$. The argument in step 2 of the proof of Theorem 6 proves that

$$\mathbb{E}^s \int_{B\cap(-E,E)} \frac{1}{f(\lambda)^2} < \infty.$$

In particular, conditioned on non-extinction, with probability one, $f(\lambda) > 0$ almost-everywhere on $B \cap (-E, E)$. This proves the first claim of the theorem.

For the second claim, we fix $\varepsilon > 0$. We have $\mathbb{E}\mu^{\mathrm{ac}}(\mathbb{R}) \geq (1-\pi_e)\mathbb{E}^s\mu^{\mathrm{ac}}(\mathbb{R})$ and π_e goes to 0 as d goes to infinity, uniformly in $n \geq 2$. Hence, up to adjusting the value of ε, it suffices to prove that $\mathbb{E}^s\mu^{\mathrm{ac}}(\mathbb{R}) \geq 1-\varepsilon$. Moreover, since $\ell(B^c) \leq \varepsilon_0$, there exists (ε_0, E) such that for all d large enough, $\mu_\star(B \cap (-E,E)) \geq 1 - \varepsilon/2$. We may then repeat step 3 in the proof of Theorem 6 and conclude that $\mathbb{E}^s \int_{B\cap(-E,E)} f(\lambda)d\lambda \geq$

$1 - \varepsilon$ for all d large enough, uniformly in n.

* * * * * * * * *

2.2 Discussion

To conclude this chapter, we make some remarks on possible future work and the limitations of the method. Once the absolute continuity of the spectral measure is proven, a natural question to address is the regularity of the density. With this in mind, one can hope to obtain an estimate on the two-point correlation for the resolvent at the root of the tree. For instance, an inequality of the type

$$\mathbb{E}|g(z_2) - g(z_1)|^p \leq C|z_2 - z_1|^{1+q+\epsilon},$$

for some positive constants $C, p, q, \epsilon > 0$ that is uniform in the complex parameters z_2, z_1 above some interval $(-E, E) \subset [-2, 2]$. This would imply the q/p-Holder continuity of the spectral measure in the interval, by an adaptation of the Kolmogorov's Continuity Theorem [Dur19, Theorem 8.1.3]. Unfortunately, we could not achieve such a result by adapting the above approach. The main obstacle is that one has to deal with non-trivial correlation when using fixed point recursion. We also emphasise that the presence of an absolutely continuous spectrum for a Galton-Watson tree with both leaves and random potential is still open. In this case, there is no prior information on the location of the eigenvalues of finite pending trees. This implies that no statement (as in lemma 10) about the moments of their Stieljes transform is possible.

3 Correspondence between Random Schrödinger operator and wreath product of groups

3.1 Introduction

We start this chapter by recalling the well known correspondence for random Schrödinger operators with binary valued potential $\pm\lambda$, for a fixed parameter $\lambda > 0$.

Schrödinger with Rademacher distribution. We consider for each site $x \in \mathcal{G}$ a value $v_x \in \{\pm\lambda\}$ of the potential, and let $H = A + V$ denote the Schrödinger operator where A is the adjacency operator and $V = \mathrm{Diag}(\{v_x\}_{x\in\Gamma})$. If we draw the potential uniformly at random on probability space $\Omega = \{\pm 1\}^{\mathcal{G}}$, we obtain the operator valued random variable $H(\omega)$:

$$H(\omega)\delta_x = \sum_{y\sim x} \delta_y + v_x(\omega)\delta_x, \qquad x \in \mathcal{G}, \omega \in \Omega. \tag{3.1}$$

The lamplighter graph $\mathcal{L}$. Consider now that at each site $x \in \mathcal{G}$ is attached a lamp whose state can be either off or on. A lamp configuration is a finite subset $l \subset \mathcal{G}$ of vertices, we denote $l\Delta\{y\}$ the configuration obtained from M by switching the lamp at y. We consider a lamplighter on the graph, that can either move in $\mathcal{G}$ or change the states of the lamp at his current position. The process is encoded in the following graph $\mathcal{L}$. The vertices is the set of (finite) lamp configurations together with the position of the lamplighter

$$\left\{(l,x) \in 2^{\mathcal{G}} \times \mathcal{G} \,\middle|\, |l| < \infty\right\}.$$

There is an edge between two vertices (l,x) and (l',y) if either $x \sim y$ in $\mathcal{G}$ and $l = l'$, or if $x = y$ and $l\Delta l' = \{x\}$. It is useful to interpret $\mathcal{L}$ as a lift of the graph obtained from $\mathcal{G}$ by adding a self-loop at each vertex. In this view, the application $\pi\colon (l,g) \mapsto g$ is the covering map. Let M be the weighted adjacency operator of the graph $\mathcal{L}$ such that: we assign a weight 1 for displacement of the lamplighter, and weight λ for each change of lamp configuration. The action of M on $l^2(\mathcal{L})$ is given by

$$M\delta_{(l,x)} = \sum_{y\sim x} \delta_{(l,y)} + \lambda\delta_{(l\Delta\{x\},x)}. \tag{3.2}$$

We recall the definition of the density of states measure $\bar{\mu}^x_H = \mathbb{E}\mu^x_{H(\omega)}(\cdot) = \mathbb{E}\langle\delta_x, E_{H(\omega)}(\cdot)\delta_x\rangle$ at a given site $x \in \mathcal{G}$, and denote the spectral measures $\mu^{(l,x)}_M(\cdot) = \langle\delta_{(l,x)}, E_M(\cdot)\delta_{(l,x)}\rangle$, which do not depend on l by transitivity. The usual correspondence relates these two quantities.

Proposition 24. *Let $\mathcal{G}$ be a bounded degree graph and consider the two operators defined in (3.1) and (3.2). For any site $x \in \mathcal{G}$ one has*

$$\bar{\mu}^x_H = \mathbb{E}\mu^x_{H(\omega)} = \mu^{(\emptyset,x)}_M, \tag{3.3}$$

where $\mathbb{E}$ is the expectation over uniformly chosen signs of the potential.

Proof of proposition 24. The operators involved are uniformly bounded in norm by the maximum degree of $\mathcal{G}$ plus λ. In particular, all spectral measures are determined by their moments, it suffices by a walk counting argument to obtain the identity

$$\mathbb{E}\langle\delta_x, H^n(\omega)\delta_x\rangle = \langle\delta_{(\emptyset,x)}, M^n\delta_{(\emptyset,x)}\rangle, \quad n \in \mathbb{N}. \tag{3.4}$$

We consider sequences $u = x_0x_1\cdots x_n$ of vertices in $\mathcal{G}$ such that either $x_{i-1} \sim x_i$ or $x_{i-1} = x_i$, that we interpret as a walk. In the latter case we say that i is a lazy step. We denote W_n the set of all such walks, starting at $x_0 = x$ of given length n. The matrix expansion of both $H^n\delta_x$ and $M^n\delta_{(\emptyset,x)}$ can be expressed as a summation over W_n. Each lazy step $x_{i-1} = x_i$ corresponds either to the multiplication by the potential $v_{x_i} = \pm\lambda$, or to a switch of the lamp at site x_i. Therefore one can write

$$H(\omega)^n\delta_x = \sum_{u\in W_n} \pm\lambda^{k(u)}\delta_{x_n}, \qquad M^n\delta_{(\emptyset,x)} = \sum_{u\in W_n} \lambda^{k(u)}\delta_{(l(u),x_n)},$$

with $k(u)$ being the total number of lazy steps and $l(u)$ some final lamp configuration. For the walk to be closed, we must have $x_n = x$. In $\mathcal{L}$, each lamp has to be switched an even number of time, to obtain $l(u) = \emptyset$. These are precisely the only terms δ_x that will remain in the expansion $\mathbb{E}H^n\delta_x$, as each potential v_y must appear with even multiplicity. This gives (3.4). □

In the rest of this chapter, we will discuss improvements of this elementary correspondence. In the next section, we will put this result into the more general framework of convolution operator on wreath product, to include arbitrary potential distribution. We will then show how this correspondence can be improved into a unitary equivalence. As a corollary, we will obtain for the random Schrödinger operator a new formula for the second moment of Green functions.

3.2 General correspondence for density of states

Since we believe that this is the only interesting case, we will assume that the base graph has an underlying group structure. For a symmetric generating subset S of a group Γ, we consider the Cayley graph $\mathcal{G} = \mathrm{Cay}(\Gamma, S)$. More generally, we may assume that the operator A is a symmetric convolution on Γ [1]. We summarize the correspondence as follows. Consider an i.i.d. diagonal perturbation of the form $H = A + V$, and assume that the distribution of the potential V agrees with the spectral measure of another convolution operator B on group Λ, called the group of lamps. Then there is a larger but deterministic convolution operator M on the wreath product $\Lambda \wr \Gamma$, such that the density of states of H coincides with the spectral measure M. For the sake of simplicity, we assume that both Λ, Γ are finitely generated groups, in particular they are discrete and countable. In order to set the notation, we recall some basic notion of convolution operator on groups [Fol16].

Convolution operator Given a countable group G, we denote $\delta_x, x \in G$ the canonical basis of the Hilbert space $l^2(G)$. The Banach space $l^1(G)$ of complex summable functions will be interpreted as the set of finite complex measures and denoted $\mathcal{M}(G)$. The (left regular) representation $L\colon G \to \mathcal{B}(l^2(G))$ associates to each group element g the unitary operator L_g defined by $L_g\delta_x = \delta_{gx}$. One can extend this definition to any measure by setting:

$$L_m = \int_G L_x \mathrm{d}m(x) := \sum_{x\in G} m(x) L_x, \qquad m \in \mathcal{M}(G).$$

Any operator of this form is called a convolution operator. Since L_m is a bounded operator, it is self-adjoint if and only if it is symmetric. This turns out to be equivalent to the condition $m(g) = \overline{m(g^{-1})} := m^*(g)$ for all group elements. A measure will therefore be called symmetric if $m = m^*$.

Wreath product The Wreath product of two given groups Γ, Λ is constructed as follows. As a set one has

$$\Lambda \wr \Gamma =_{set} \bigoplus_{\Gamma} \Lambda \times \Gamma := \left\{ (l, g) \in \Lambda^{\Gamma} \times \Gamma \,\middle|\, |\operatorname{supp} l| < \infty \right\},$$

where we recall that the direct sum $\bigoplus_\Gamma \Lambda$ is the subgroup of sequence $l = (l_g)_{g\in\Gamma} \in \prod_\Gamma \Lambda$ whose components differ from the identity e_Λ of Λ for finitely many factors. The multiplication is given by

$$(l', g')(l, g) = ((l'_{gx} l_x)_x, g'g).$$

1. This assumption can be removed from most results in this chapter.

This is a particular type of semi-direct product $\bigoplus_\Gamma \Lambda \rtimes \Gamma$, where the group Γ acts on $\bigoplus_\Gamma \Lambda$ by reindexing the factors. It has a nice geometric interpretation as acting on the two levels rooted tree $\mathcal{T}_{\Gamma,\Lambda}$ constructed as follows. The root o is connected to each element g of Γ and each g is attached to a copy of Λ, denoted $gl, l \in \Lambda$.

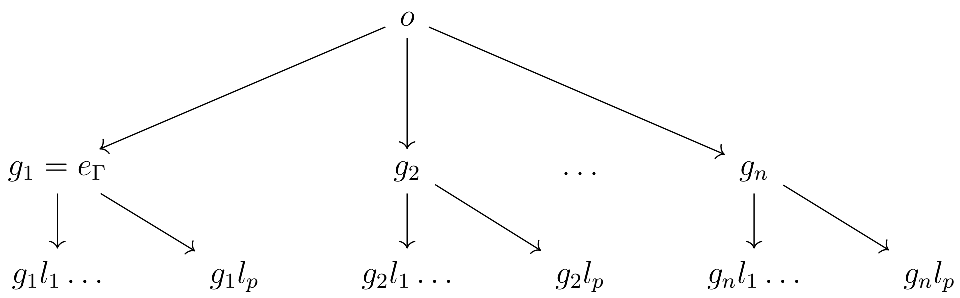

Figure 3.1 – The tree $\mathcal{T}_{\Gamma,\Lambda}$ with Γ, Λ of order n, p.

An automorphism of $\mathcal{T}_{\Gamma,\Lambda}$ is a permutation of the vertices that preserves the adjacency relation. In particular, it must fix the root. Given an element $h = ((t_x), s) \in \Lambda \wr \Gamma$, one associates the automorphism φ_h sending g to $\varphi_h(g) = sg$ and gl to $\varphi_h(gl) = sgt_g l$. It is easy to see that $\varphi_{h'} \circ \varphi_h = \varphi_{h'h}$ and we can therefore view $\Lambda \wr \Gamma$ as a subgroup of $\mathrm{Aut}(\mathcal{T}_{\Gamma,\Lambda})$.

The wreath product contains the base group and the group of lamps as subgroups. This corresponds on figure 3.1 to the action of permuting the first level, and the action of permuting leaves of the left branch of $\mathcal{T}_{\Gamma,\Lambda}$. Both embeddings $\Gamma, \Lambda \hookrightarrow \Lambda \wr \Gamma$ will be denoted by a hat, and are defined as follows:

$$s \in \Gamma \hookrightarrow \hat{s} = ((e_\Gamma), s) \in \Lambda \wr \Gamma, \tag{3.5}$$

$$t \in \Lambda \hookrightarrow \hat{t} = ((l_x), e_\Gamma) \in \Lambda \wr \Gamma \text{ with } l_{e_\Gamma} = t \text{ and } l_x = e_\Lambda, x \neq e_\Gamma. \tag{3.6}$$

We embed measures $\mathcal{M}(\Gamma), \mathcal{M}(\Lambda) \hookrightarrow \mathcal{M}(\Lambda \wr \Gamma)$ similarly. The correspondence concerns the very special type of convolution operator such that the measure $m \in \mathcal{M}(\Lambda \wr \Gamma)$ decomposes as a sum $\hat{m}_\Gamma + \hat{m}_\Lambda$, where m_Γ and m_Λ are measures on the base group and the group of lamps respectively. When $\Lambda = \mathbb{Z}/2\mathbb{Z}$, we recover the situation of the introduction. The decomposability condition means that the lamplighter can only change the state of the lamp at its current position.

Theorem 25. *Let $A = L_{m_\Gamma}$, $B = L_{m_\Lambda}$ be two self-adjoint convolution operators on finitely generated groups Γ, Λ. Consider the convolution operator $M = L_{m_{\Lambda\wr\Gamma}}$ on $\Lambda \wr \Gamma$, where $m_{\Lambda\wr\Gamma} = \hat{m}_\Gamma + \hat{m}_\Lambda$, and the random Schrödinger operator on Γ of the form $H(\omega) = A + \mathrm{Diag}(v_x)_{x\in\Gamma}$, such that the random potential (v_x) is i.i.d., following the spectral measure $\mu_B^{e_\Lambda}$. Then one has*

$$\bar{\mu}_H^{e_\Gamma} := \mathbb{E}\mu_{H(\omega)}^{e_\Gamma} = \mu_M^{e_{\Lambda\wr\Gamma}}. \tag{3.7}$$

Proof. All operators are bounded in norm by the sum of the absolute mass of the measures m_Γ and m_λ. It suffices to show

$$\mathbb{E}\langle \delta_{e_\Gamma}, H(\omega)^n \delta_{e_\Gamma}\rangle = \langle \delta_{e_{\Lambda\wr\Gamma}}, M^n \delta_{e_{\Lambda\wr\Gamma}}\rangle, \quad n \in \mathbb{N}. \tag{3.8}$$

To do this, we introduce a new formal variable v, and consider the set W_n of all words of length n in the alphabet $\Gamma \sqcup \{v\}$, which reduces to the identity element e_Γ:

$$W_n = \left\{ u = u_1 u_2 \cdots u_n \in (\Gamma \sqcup \{v\})^n \,\middle|\, \prod_{\substack{i=1\\ u_i \neq v}}^{n} u_i = e_\Gamma \right\}.$$

We now express both sides of (3.8) in terms of a summation over W_n. It is clear from the group multiplication that any word u gives rise to a closed walk $x_0, x_1, \dots x_n$ in Γ. Starting from $x_o = e_\Gamma$, one sets $x_i = u_i x_{i-1}$ if $u_i \in \Gamma$, and $x_i = x_{i-1}$ if $u_i = v$. To obtain the expansion $\langle \delta_{e_\Gamma}, H(\omega)^n \delta_{e_\Gamma}\rangle$, one must keep track of the weights of the walk. We assign $m_\Gamma(u_i)$ if $u_i \in \Gamma$, otherwise $u_i = v$ and we assign the value v_{x_i} of the potential at the current position $x_i = x_{i-1}$ of the walk. This gives

$$\langle \delta_{e_\Gamma}, H(\omega)^n \delta_{e_\Gamma}\rangle = \sum_{u \in W_n} \prod_{\substack{i=1\\ u_i \in \Gamma}}^{n} m_\Gamma(u_i) \prod_{\substack{j=1\\ u_j = v}}^{n} v_{x_j}.$$

In order to use the independence of the $\{v_x\}_{x\in\Gamma}$, it is convenient to reorganise the sum. For each $y \in \Gamma$, let $k(u)$ denote the number of occurrences of v_y in the last product:

$$k_y(u) = \#\left\{ i \in \{1, \dots, n\} | u_i = v \text{ and } x_i = y \right\}, \qquad y \in \Gamma.$$

Note that for most y one has simply $k_y(u) = 0$. We obtain

$$\mathbb{E}\langle \delta_{e_\Gamma}, H(\omega)^n \delta_{e_\Gamma}\rangle = \sum_{u \in W_n} \prod_{\substack{i=1\\ u_i \in \Gamma}}^{n} m_\Gamma(u_i) \prod_{y\in\Gamma} \mathbb{E}[v_y^{k_y(u)}].$$

We now express the l.h.s of (3.8). By construction, the action of M on any element $(l, g) \in \Lambda \wr \Gamma$ is given by

$$M\delta_{(l,g)} = \sum_{s\in\Gamma} m_\Gamma(s)\delta_{\hat{s}(l,g)} + \sum_{t\in\Lambda} m_\Lambda(t)\delta_{\hat{t}(l,g)},$$

We follow the lamplighter interpretation: the first sum corresponds to displacements $g \mapsto sg$ of the lamplighter, while the second sum corresponds to a switch $l_g \mapsto tl_g$ of the lamp at the position g of the lamplighter. A given word $u \in W_k$ already prescribes the displacement of the lamplighter. A letter $u_i = v$ with $x_i = y$ will correspond to an update of the lamp y. In total, there are $k_y(u)$ updates of the lamp at y. One must therefore choose a sequence $t_y = t_1^y \cdots t_{k_y(u)}^y \in \Lambda^{k(u)}$ to obtain a walk from $e_{\Lambda\wr\Gamma}$ with n steps. For the walk in $\Lambda \wr \Gamma$ to be closed, each lamp must be in the state e_Λ at the end of the walk. We introduce T_k, the subset of Λ^k which reduces to the identity e_Λ by the group multiplication:

$$T_k = \left\{ t = t_1 \cdots t_k \in \Lambda^k \,\middle|\, \prod_{i=1}^{k} t_i = e_\Lambda \right\}.$$

Keeping track of the weight, we obtain:

$$\langle \delta_{e_{\Lambda\wr\Gamma}}, M^n \delta_{e_{\Lambda\wr\Gamma}} \rangle = \sum_{u \in W_n,} \prod_{\substack{i=1 \\ u_i \in \Gamma}}^{n} m_\Gamma(u_i) \prod_{y \in \Gamma} \left(\sum_{t^y \in T_{k_y(u)}} \prod_{j=1}^{k_y(u)} m_\Lambda(t_j^y) \right).$$

To conclude, we recall that the one-site potential distribution follows the spectral measure of the convolution operator $B = L_{m_\Lambda}$. Since $\int v^k \mathrm{d}\mu_B(v) = \langle \delta_{e_\Lambda}, B^k \delta_{e_\Lambda} \rangle = \sum_{t \in T_k} \prod_{j=1}^{k} m_\Lambda(t_j)$, it follows

$$\langle \delta_{e_{\Lambda\wr\Gamma}}, M^n \delta_{e_{\Lambda\wr\Gamma}} \rangle = \sum_{u \in W_n,} \prod_{\substack{i=1 \\ u_i \in \Gamma}}^{n} m_\Gamma(u_i) \prod_{y \in \Gamma} \left(\mathbb{E}[v_y^{k_y(u)}] \right) = \mathbb{E} \langle \delta_{e_\Gamma}, H(\omega)^n \delta_{e_\Gamma} \rangle.$$

□

There are several simple consequences from theorem 25. For example, it is well known that any random Schrödinger operator $H = A + V$ on an infinite transitive graphs with potential of the form distribution ν, the spectrum of H is deterministic. More precisely:

$$\Sigma(H(\omega)) =_{a.s} \Sigma(A) + \operatorname{Supp} \nu := \{ a + v \,|\, a \in \Sigma(A),\, v \in \operatorname{Supp}(\nu) \}.$$

which directly implies $\Sigma(M) = \Sigma(A) + \Sigma(B)$. A less straightforward consequence is the following.

Corollary 26. *Consider a convolution operator $M = L_{m_{\Lambda\wr\Gamma}}$ on the wreath product $\Lambda \wr \Gamma$. Suppose that the measure $m_{\Lambda\wr\Gamma}$ decomposes, as in theorem 25, into a sum $\hat{m}_\Lambda + \hat{m}_\Gamma$. If the spectral measure $\mu_B^{e_\Lambda}$ is absolutely continuous with bounded density $\rho(v)\mathrm{d}v = \mathrm{d}\mu_B^{e_\Lambda}$, then the same holds for the spectral measure $\mu_M^{e_{\Lambda\wr\Gamma}}$. In particular, M is purely absolutely continuous.*

Proof. This follows from the correspondence (3.7) together with a regularity estimate of $\bar{\mu}_H^{e_\Gamma} := \mathbb{E}\mu_{H(\omega)}^{e_\Gamma}$ (Wegner estimate, e.g [AW15, Chapter 4]). Consider the random Schrödinger operator $H(\omega) = A + \operatorname{Diag}(v_x)_{x\in\Gamma}$, conditionally to the value of potentials $\{v_y\}_{x\neq e_\Gamma}$, we obtain the rank one perturbation family:

$$H_0 = A + \sum_{y\in\Gamma\setminus\{e_\Gamma\}} v_y\delta_y\delta_y^*, \qquad H_v = H_0 + v\delta_{e_\Gamma}\delta_{e_\Gamma}^*, \quad v\in\mathbb{R}$$

Given $z\in\mathbb{H}$, the diagonal element $g_v(z) = \langle\delta_{e_\Gamma}, (H_v - z)^{-1}\delta_{e_\Gamma}\rangle$ of the resolvent can be expressed explicitly in terms of g_0, the second resolvent identity

$$(H_v - z)^{-1} - (H_0 - z)^{-1} = -(H_v - z)^{-1}v\delta_{e_\Gamma}\delta_{e_\Gamma}^*(H_0 - z)^{-1}$$

implies $g_v(z) = (g_0(z)^{-1} + v)^{-1}$. It remains to average over the distribution $\rho(v)\mathrm{d}v$

$$\frac{1}{\pi}\int \Im g_v(z)\rho(v)\mathrm{d}v \le \frac{\|\rho\|_\infty}{\pi}\int_{\mathbb{R}} \Im g_v(z)\mathrm{d}v = \frac{\|\rho\|_\infty}{\pi}\int_{\mathbb{R}} \Im\frac{1}{g_0(z)^{-1}+v}\mathrm{d}v = \|\rho\|_\infty\,.$$

Since this bound holds independently of $\{v_y\}_{x\neq e_\Gamma}$, we obtain

$$\frac{1}{\pi}\mathbb{E}\Im\langle\delta_{e_\Gamma}, (H(\omega)-z)^{-1}\delta_{e_\Gamma}\rangle \le \|\rho\|_\infty\,.$$

Recalling that the imaginary part of the Stieltjes transformation weakly converges to the measure as one approaches the real axis, we obtain that $\bar{\mu}_H^{e_\Gamma} := \mathbb{E}\mu_{H(\omega)}^{e_\Gamma}$ is absolutely continuous with density bounded by $\|\rho\|_\infty$. □

3.3 Direct integrals and unitary equivalence for Abelian groups of lamps

Concerning the random Schrödinger operator, one is often more interested in the almost sure properties of a given realization $H(\omega)$ rather than averaged quantity such as the density of states measure [2]. Going back to a binary potential, i.e. $\Lambda = \mathbb{Z}/2\mathbb{Z}$ and for a base group Γ of finite order, then the wreath product is of order $|\Gamma|2^{|\Gamma|}$. This matches the dimension of the direct sum $\oplus_\omega H(\omega)$, where ω ranges over all the 2^n sign assignments. It is then clear from theorem 25 that M is unitary equivalent to the direct sum $\oplus_\omega H(\omega)$. This suggests that the correspondence can be extended to eigenvectors. Also one should be able to reverse the identity (3.7) to express a given realisation $H(\omega)$ in terms of some *average* over the Lamplighter operators M. This is the main objective of this section. We gather some basic facts about the concept of direct integral, which generalize the direct sum. We refer to [Dix69] for a detailed treatment.

2. Nevertheless, note that any proof of the almost sure localization via multi-scale analysis starts with a prior estimate of the density of states, e.g. [KG12; Ves08]

Direct integral. We will only be concerned with direct integrals of a constant Hilbert field, which allows us to avoid many technicalities concerning measurability. Given a Hilbert space $\mathcal{H}$ and a standard Borel space (X, μ), we denote $\mathcal{H}_X$ the direct integral of $\mathcal{H}$ over (X, μ),

$$\mathcal{H}_X = \int_X^{\oplus} \mathcal{H} \mathrm{d}\mu(s) := L^2(X, \mu, \mathcal{H}) \simeq L^2(X, \mu) \otimes \mathcal{H}.$$

as the set of $\mathcal{H}$ valued (equivalent classes of) weakly measurable maps from X to $\mathcal{H}$, verifying the summability condition

$$\|\varphi\|_{\mathcal{H}_X}^2 = \int_X \|\varphi(s)\|_{\mathcal{H}}^2 \, \mathrm{d}\mu(s).$$

The aim of this construction is to introduce the direct integral of an operator field denoted

$$H_X := \int_X^{\oplus} H(s) \mathrm{d}\mu(s),$$

which is defined by the point-wise action $(H_X\varphi)(s) = H(s)\varphi(s)$. It is well defined for weakly measurable and uniformly bounded maps $H\colon X \to \mathcal{B}(\mathcal{H})$.

Remark 2. *For non-constant fields $s \mapsto \mathcal{H}_s$, the construction of $\mathcal{H}_X = \int_X^{\oplus} \mathcal{H}_s \mathrm{d}\mu(s)$ is similar but depends on a choice of a basis for each space $\mathcal{H}_s$ that is compatible with the structure (X, μ), see [Dix69]. A third formulation of the spectral theorem, that we omitted in the introduction, says that any self-adjoint operator is unitary equivalent to the direct integral of a real multiple of the identity operator: for some Borel space (X, μ), there is a measurable field of Hilbert space $(\mathcal{H}_s)$ together with a real map $\lambda(s)$ such that $H \simeq \int^{\oplus} \lambda(s) I_{\mathcal{H}_s} \mathrm{d}\mu(s)$. We refer to [Hal13, Theorem 7.19] for more details.*

We are now ready to construct the unitary equivalence between the Lamplighter operator and the direct integral of Schrödinger operator. For the sake of simplicity, we restrict to the case $\Lambda = \mathbb{Z}$. We highlight that the same result applies if Λ is a finitely generated Abelian group, and might also hold for any countable lamp groups Λ on which there is a suitable notion of Plancherel identity [Bek21]. Keeping the notation of Theorem 25 with $\Lambda = \mathbb{Z}$, we fix two finite symmetric measures m_Γ and $m_\mathbb{Z}$. We then consider the three operators $A = L_{m_\Gamma}$, $B = L_{m_\mathbb{Z}}$, $M = L_{\hat{m}_\Gamma + \hat{m}_\mathbb{Z}}$ acting respectively on $l^2(\Gamma), l^2(\mathbb{Z})$ and $l^2(\mathbb{Z} \wr \Gamma)$.

Lemma 27. *Let $V(\cdot) = \sum_n m_\mathbb{Z}(n) e^{2\pi i \cdot} \in L^2(\mathbb{R}/\mathbb{Z})$ be the Fourier transform of the symmetric measure $m_\mathbb{Z} \in \mathcal{M}(\mathbb{Z})$. If ν follows the uniform distribution on $[0, 1]$, then $V(\nu)$ is distributed according to the spectral measure $\mu_B^{e_\mathbb{Z}}$.*

Proof. Since $m_\mathbb{Z}$ is symmetric and finite, $V(\cdot) = \sum_n m_\mathbb{Z}(n) e^{2\pi i \cdot}$ is a well-defined real and bounded function. The fact that $V(\mathcal{U}([0, 1]))$ is distributed following $\mu_B^{e_\mathbb{Z}}$ follows from $\int_{[0,1]} V(\nu)^k \mathrm{d}\nu = \langle \delta_{e_\mathbb{Z}}, B^k \delta_{e_\mathbb{Z}} \rangle$, $k \in \mathbb{N}$. $\square$

From the above lemma, we construct $H(\omega)$ as the operator valued random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = [0,1]^\Gamma$ (endowed with the cylinder algebra and the uniform product measure) as follows

$$H(\omega) = A + \mathrm{Diag}(v_x)_{x\in\Gamma}, \quad v_x(\omega) = V(\omega_x), \quad \omega = \{\omega_x\} \in [0,1]^\Gamma. \tag{3.9}$$

With this construction, the direct integral of $H(\omega)$ over the probability space $(\Omega, \mathbb{P})$ will be unitary equivalent to the lamplighter operator. We recall that the Fourier transform from $\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$ extends to the countable product over Γ (e.g [Fol16, chapter 4]). We denote $\mathcal{F}$ the induced unitary map

$$\mathcal{F} \colon l^2\Big(\bigoplus_\Gamma \mathbb{Z}\Big) \to L^2(\Omega, \mathbb{P}), \tag{3.10}$$

which sends any basis element δ_l to the character function χ_l defined as the finite product $\chi_l(\omega) = e^{2\pi i l\cdot\omega} = \prod_{x\in\Gamma} e^{2\pi i l_w \omega_x}$.

Theorem 28 (Unitary equivalence)**.** *Let $M = L_{m_{\mathbb{Z}\wr\Gamma}}$ with $m_{\mathbb{Z}\wr\Gamma} = \hat{m}_\Gamma + \hat{m}_\mathbb{Z}$ as in theorem 25, and let $H(\omega)$ be the operator valued random variable on the probability space $([0,1]^\Gamma, \mathcal{F}, \mathbb{P})$ as in (3.9). Then M is unitary equivalent to the direct integral of $H(\omega)$ over $(\Omega, \mathbb{P})$. More precisely if $U = \mathcal{F} \otimes I_{l^2(\Gamma)}$ with $\mathcal{F}$ as in (3.10), one has*

$$H_\Omega = \int_\Omega^\oplus H(\omega) \mathrm{d}\mathbb{P}(\omega) = UMU^*. \tag{3.11}$$

Proof. This follows from a simple computation on the basis elements. By construction, U maps the canonical basis $\delta_{(l,g)}$ to the unit element of $L^2(\Omega, \mathbb{P}, l^2(\Gamma))$ defined as $\chi_l\delta_g \colon \omega \mapsto \chi_l(\omega)\delta_g \in l^2(\Gamma)$. On the one hand we have

$$\begin{aligned}\langle \delta_{(l',g')}, M\delta_{(l,g)}\rangle &= \sum_{s\in\Gamma} m_\Gamma(s)\langle \delta_{(l',g')}, \delta_{\hat{s}(l,g)}\rangle + \sum_{t\in\mathbb{Z}} m_\Lambda(t)\langle \delta_{\hat{t}(l',g')}, \delta_{\hat{s}(l,g)}\rangle \\ &= 1_{l=l'}\langle \delta_g' | A\delta_g\rangle + 1_{\{g=g'\}}\left(\prod_{\substack{x\in\Gamma\\ x\neq g}} 1_{\{l_x = l'_x\}}\right)\langle \delta_{l'_g}, B\delta_{l_g}\rangle,\end{aligned}$$

and on the other hand we have the direct integral

$$\begin{aligned}\langle \chi_{l'}\delta_{g'}, H_\Omega \chi_l \delta_g\rangle &= \int_\Omega \langle \chi_{l'}(\omega)\delta_{g'}, (H\chi_l\delta_g)(\omega)\rangle \mathrm{d}\mathbb{P}(\omega)\\ &= \int_\Omega \langle \delta_{g'}, A\delta_g + v_g(\omega)\delta_g\rangle \overline{\chi_{l'}}(\omega)\chi_l(\omega)\mathrm{d}\mathbb{P}(\omega)\\ &= 1_{l=l'}\langle \delta_g' | A\delta_g\rangle + 1_{\{g=g'\}}\int_\Omega V(\omega_x)e^{2\pi i(l-l')\cdot\omega}\mathrm{d}\mathbb{P}(\omega)\\ &= 1_{l=l'}\langle \delta_g' | A\delta_g\rangle + 1_{\{g=g'\}}\left(\prod_{\substack{x\in\Gamma\\ x\neq g}} 1_{\{l_x=l'_x\}}\right)\langle \delta_{l'_g}, B\delta_{l_g}\rangle.\end{aligned}$$

□

The benefit from the direct integral representation is that it is now possible to reverse the identity (3.1), to express $H(\omega)$ in terms of a weighted average over M.

Corollary 29 (Reverse correspondence)**.** *For M and H as in theorem 28 and any smooth bounded function $f\colon \Sigma(M)\to\mathbb{C}$. Then one can compute $f(H(\omega))$ as a Fourier series in terms of $f(M)$. More precisely, the following identity holds in $L^2(\Omega,\mathbb{P})$*

$$\langle \delta_e, f(H(\omega))\delta_x\rangle = \sum_{l\in\bigoplus_\Gamma \mathbb{Z}} \langle \delta_{e_{\mathbb{Z}\wr\Gamma}}, f(M)\delta_{(l,x)}\rangle e^{2\pi i l\cdot\omega}. \tag{3.12}$$

In particular, from Parseval identity one has

$$\mathbb{E}|\langle \delta_{e_\Gamma}, f(H(\omega))\delta_x\rangle|^2 = \sum_{l\in\bigoplus_\Gamma \mathbb{Z}} |\langle \delta_{e_{\mathbb{Z}\wr\Gamma}}, f(M)\delta_{(l,x)}\rangle|^2. \tag{3.13}$$

Proof. It suffices to compute the Fourier coefficient of the map $\omega\mapsto\langle\delta_e, f(H(\omega))\delta_x\rangle$, seen as an element of the Hilbert space $L^2(\Omega,\mathbb{P})$. From the basic theory of direct integral [Dix69], one has for a smooth function f

$$f\left(\int_\Omega^\oplus H(\omega)\mathrm{d}\mathbb{P}(\omega)\right) = \int_\Omega^\oplus f(H(\omega))\mathrm{d}\mathbb{P}(\omega),$$

and theorem 28 implies that

$$\int_\Omega \langle \delta_e, f(H(\omega))\delta_x\rangle \chi_l(\omega)\mathrm{d}\mathbb{P}(\omega) = \langle \chi_0\delta_{e_\Gamma}, \left(\int_\Omega^\oplus f(H(\omega))\mathrm{d}\mathbb{P}(\omega)\right)\chi_l\delta_x\rangle = \langle \delta_{e_{\mathbb{Z}\wr\Gamma}}, f(M)\delta_{(l,x)}\rangle.$$

□

For z a given complex number in $\mathbb{C}\setminus\Sigma(M)$, equation (3.13) for the rational map

$f(\cdot) = (\cdot - z)^{-1}$ gives an exact formula for the L^2 moment of the green function:

$$\mathbb{E}|\langle \delta_{e_\Gamma}(H(\omega) - z)^{-1}\delta_{e_\Gamma}\rangle|^2 = \sum_{l \in \bigoplus_\Gamma \Lambda} |\langle \delta_{e_{\Lambda \wr \Gamma}}, (M - z)^{-1}\delta_{(l,e_\Gamma))}\rangle|^2. \tag{3.14}$$

For the lamplighter graphs discussed in the introduction, we recall that the map $\pi\colon (l, g) \in \Lambda \wr \Gamma \mapsto g \in \Gamma$ can be interpreted as a covering map. In this view, the almost sure delocalization for the operator $H(\omega)$ will occur if the eigenfunctions of M decays fast enough over the fiber $\pi^{-1}(e_\Gamma) = \{(l, e_\Gamma)| l \in \bigoplus_\Gamma \Lambda\}$.

3.4 Discussions

3.4.1 A remark on generalized lamplighter

Throughout this chapter, the central idea was to study the random Schrödinger operators in terms of the direct integral

$$H_\Omega = \int_\Omega^\oplus H(\omega)\mathrm{d}\mathbb{P}(\omega).$$

If $\delta_x(\cdot)$ denotes the constant function $\omega \mapsto \delta_x$ in $L^2(\Omega, \mathbb{P}, l^2(\Gamma))$, one recovers the density of states measure by $\mu_{H_\Omega}^{\delta_x(\cdot)} = \mathbb{E}\mu_{H(\omega)}^x$. To recover almost sure properties of $H(\omega)$, one has to consider off-diagonal elements of H_Ω. Let us briefly mention a similar approach, where one can recover similar properties by solely considering diagonal elements and which leads to a natural generalisation of lamplighters with two walkers. Instead of H_Ω, consider the operator J_Ω as the direct integral

$$J_\Omega = \int_\Omega (H(\omega) \otimes I_{l^2(\Gamma)} - I_{l^2(\Gamma)} \otimes H(\omega))\mathrm{d}\mathbb{P}(\omega),$$

acting on $L^2(\Omega, \mathbb{P}, l^2(\Gamma \times \Gamma))$. Here the integrand is the Kronecker sum of $H(\omega)$ with $-H(\omega)$. If now $\delta_{xx}(\cdot)$ denotes the constant function $\omega \mapsto \delta_x \otimes \delta_x$, it is not hard to see that one has

$$\mu_{J_\Omega}^{xx} := \mu_{J_\Omega}^{\delta_{xx}(\cdot)} = \mathbb{E}[\mu_{H(\omega)}^x(\cdot) * \mu_{H(\omega)}^x(-\cdot)].$$

where one recognizes the auto-correlation of the measure $\mu_{H(\omega)}^x$. Therefore the size of the kernel of J_Ω measures the expected atomic mass at δ_x of the operator $H(\omega)$. For example, one has:

$$\mu_{J_\Omega}^{xx}(\{0\}) = \mathbb{E}\left[\int_\lambda \mu_{H(\omega)}^x(\lambda)\mathrm{d}\mu_{H(\omega)}^x(-\lambda)\right] = \mathbb{E}\left[\sum_\lambda (\mu_{H(\omega)}^x(\{\lambda\}))^2\right].$$

We discuss the lamplighter interpretation for the Bernoulli Schrödinger operator $H(\omega) = A + \lambda V$ described in the introduction. One can therefore use the probability space $\Omega = \{0,1\}^{\Gamma}$. By a similar Fourier argument as used previously, the operator J_{Ω} is unitary equivalent to the weighted adjacency operator N described as follows. The space on which the operator N acts is given by a lamp configuration $l \in \bigoplus_{\Gamma} \mathbb{Z}/2\mathbb{Z}$, together with the position $x, y \in \Gamma$ of two walkers. At each step, either one of the walker moves, or changes the lamp at his current position, with the following weights:

$$N\delta_{(l,x,y)} = \sum_{x' \sim x} \delta_{(l,x',y)} - \sum_{y' \sim y} \delta_{(l,x,y')} + \lambda(\delta_{(l\Delta\{x\},x,y)} - \delta_{(l\Delta\{y\},x,y)}).$$

Note that the underlying graphs on which N acts is not a Cayley graph.

3.4.2 Conclusions

The mere fact that the density of states measure can be expressed as the spectral measure of a convolution operator is not especially relevant. In fact, from Lemma 27 above, any measure can be expressed as the spectral measure of the convolution operator on $\mathbb{Z}$. The main interest in representing random Schrödinger operator as a wreath product is that the correspondence extends beyond the DOS measure. At least in the case of an Abelian group of lamps, it is possible to reconstruct a given realisation $H(\omega)$ from the operator M. This makes it possible to transfer the analysis of the Anderson model from a probabilistic point of view to a geometric point of view. Unfortunately, wreath products are sophisticated structures. For example finding the shortest path on a lamplighter graph is equivalent to the traveling salesman problem in its associated base graph [EZ22]. Nevertheless, this correspondence motivated the next chapter, in which we study a possible approach to obtain spectral information about convolution operators on groups.

4 On commutators methods

4.1 Introduction

This chapter discusses ongoing work about commutator methods. We start with the following observation. Let H be a self-adjoint operator on an Hilbert space $\mathcal{H}$ and suppose that there is another symmetric operator A, such that the following identity holds

$$[H, iA] = i(HA - AH) > 0,$$

meaning that $\langle \varphi [H, iA] \varphi \rangle > 0$ for any nonzero vector $\varphi \in \mathcal{H}$. We will say that A is a conjugate operator. It is clear that such operator cannot exist for finite-dimensional Hilbert spaces, since the trace of a commutator is equal to zero. Moreover, a formal argument suggests that H has no point spectrum, as if $H\varphi = \lambda\varphi$

$$\begin{aligned} \langle \varphi, [H, iA]\varphi \rangle &= \langle \varphi, i[H, A]\varphi \rangle = \langle \varphi, iHA\varphi \rangle - \langle \varphi, iAH\varphi \rangle \\ &= \langle H\varphi, iA\varphi \rangle - \langle \varphi, iAH\varphi \rangle = i(\bar{\lambda} - \lambda)\langle \varphi, A\varphi \rangle = 0, \end{aligned}$$

and therefore $\ker(H - \lambda) = \{0\}$. However, one must be careful with the definition of $[H, iA]$, since in practice A has to be unbounded (see remark 3). Conjugate operators arise naturally in quantum physics.

Uncertainty principle. In quantum mechanics each observable m is associated with a self-adjoint operator $M = \int \lambda \mathrm{d}E_M(\lambda)$. The Born's rule states that each observation is a realisation of a random variable. The outcome follows the distribution $P_\varphi(m \in I) = \|E_M(I)\varphi\|^2$, where φ is the wave function describing the system. The measurement m is therefore distributed according to the spectral measure μ_M^φ. The uncertainty, in terms of standard deviation, of the observable is $\sigma_M = \sqrt{\langle \varphi, M^2\varphi \rangle - \langle \varphi, M\varphi \rangle^2}$. As in classical probability, one has the variational formula $\sigma_M^2 = \min_{\lambda \in \mathbb{R}} \|(M - \lambda)\varphi\|^2$ implying that the observable is deterministic if and only if the state φ is an eigenfunction of M. Two observables are said to be incompatible if they cannot be simultaneously measured. This turns out to be equivalent to the non-commutativity of their corresponding operators.

Proposition 30 (Robertson–Schrödinger uncertainty relations)**.** *Let P, Q be self-adjoint. For any φ such that $\varphi, P\varphi, Q\varphi \in D(P) \cap D(Q)$ and $\lambda, \mu \in \mathbb{R}$:*

$$\|(P - \lambda)\varphi\| \, \|(Q - \mu)\varphi\| \geq \frac{1}{2} |\langle \varphi, [P, Q]\varphi \rangle|.$$

Proof. The proof is elementary, since

$$\begin{aligned}\langle\varphi,[P,Q]\varphi\rangle &= \langle\varphi,[P-\lambda,Q-\mu]\varphi\rangle\\ &= \langle\varphi,(P-\lambda)(Q-\mu)\varphi\rangle - \langle\varphi,(Q-\mu)(P-\lambda)\varphi\rangle\\ &= \langle(P-\lambda)\varphi,(Q-\mu)\varphi\rangle - \langle(Q-\mu)\varphi,(P-\lambda)\varphi\rangle\\ &= 2i\,\mathrm{Im}\left(\langle(P-\lambda)\varphi,(Q-\mu)\varphi\rangle\right),\end{aligned}$$

one has $|\langle\varphi,[P,Q]\varphi\rangle| \le 2|\langle(P-\lambda)\varphi,(Q-\mu)\varphi\rangle| \le 2\,\|(P-\lambda)\varphi\|\,\|(Q-\mu)\varphi\|$. □

As an illustration, we derive the uncertainty principles on the real line $\mathcal{H} = L^2(\mathbb{R})$. The position and the momentum operator are respectively $Q = x\cdot$ and $P = -i\hbar\partial_x\cdot$. The Schwartz space $\mathcal{S}(\mathbb{R})$, of all functions whose derivatives are rapidly decreasing, is invariant under both P and Q [1]. On this subspace one expands the commutator $[Q,P]\varphi = x(-i\hbar\partial_x(\varphi)) + i\hbar\partial(x\varphi) = i\hbar\varphi$ and obtains the canonical commutation relation

$$[Q,P] = i\hbar I. \tag{CCR}$$

From proposition 30 we immediately obtain the Heisenberg's uncertainty inequality

$$\sigma_Q\sigma_P \ge \hbar/2.$$

An elementary version. The first result on the regularity of the spectrum of an operator by the help of a positive commutator originally goes back to Putnam [Put12].

Theorem 31 (Putnam's theorem). *Assume $H, A \in B(\mathcal{H})$ are two bounded self-adjoint operators such that $[H, iA] > 0$. Then H is purely absolutely continuous.*

The assumption that both H and A are bounded is in practice very restrictive. In fact, we did not find any interesting cases in the context of graph theory to illustrate Putnam's theorem. Nevertheless, the proof being elementary and enlightening, we decided to include it here.

Proof of theorem 31. We follow [Cyc+09, section 4.1]. By hypothesis, one can write $[H, iA] = C^*C$ for some bounded operator with $\ker C = \{0\}$, e.g. $C = \sqrt{[H, iA]}$. We consider the resolvent $G = G(z) = (H - z)^{-1}$. Since $G = f(H)$ with $f(\lambda) = (\lambda - z)^{-1}$, it follows by functional calculus that $\mathrm{Im}\, G = \eta^{-1}GG^* = \eta^{-1}G^*G$, with

1. It can be shown that $\mathcal{S}(\mathbb{R})$ is the largest domain with this property.

$\eta = \Im(z) > 0$. From this,

$$\begin{aligned}
\eta^{-1} \|C \operatorname{Im} GC^*\| &= \|CGG^*C^*\| = \|CG\|^2 \\
&= \|G^*C^*CG(z)\| = \|G^*[H, iA]G\| = \|G^*[H - \bar{z}, iA]G(z)\| \\
&= \|G^*(H - \bar{z})iAG - G^*iA(H - \bar{z})G\| \\
&= \|iAG - G^*iA(H - z + 2i\eta)G\| \\
&= \|iAG - G^*iA - G^*iA2i\eta G\| \\
&\leq \|AG\| + \|G^*A\| + 2\eta \|G^*AG\| \\
&\leq 4\eta^{-1} \|A\| .
\end{aligned}$$

For any $\psi = C^*\varphi$, we obtain the bound $\Im g_H^\psi(z) = \langle \psi, \operatorname{Im} G(z)\psi \rangle \leq 4 \|A\| \|\varphi\|$. The absolute continuity follows by the limiting absorption principle. We recall that the imaginary part of the resolvent is the convolution of the spectral measure with the Poisson kernel $C_\eta(x) \propto \eta(\eta^2 + x^2)^{-1}$, one has $\pi^{-1}\Im g_H^\psi(E + i\eta)\mathrm{d}E = (\mu_H^\psi * C_\eta)(\mathrm{d}E)$. By sending η to zero, we obtain that μ_H^ψ is purely *a.c.* with a bounded density. Since $\ker C = \{0\}$, the range of C^* is dense and H has a purely absolutely continuous spectrum. □

Remark 3. *In particular, we have shown that a strict positivity $[H, iA] > \alpha I$ of the commutator, as in (CCR), cannot hold if both H, A are bounded operators. Otherwise we would obtain a resolvent bound uniform in both z and ψ, which would imply that the spectrum of the self-adjoint operator H is empty.*

For the sake of simplicity, we will consider H to be a bounded operator, denoted $H \in B(\mathcal{H})$. We will review two results concerning commutator methods given an unbouded conjugate operator $(A, D(A))$. With some minor changes, both statements also apply if H is also unbounded.

- First order commutator: if the first order commutator is bounded and positive, then H has no point spectrum. This is called the Virial theorem [AMG+96, proposition 7.2.10.], detailed discussion can be found in [GG99].
- Second order commutator: if furthermore the operator $[[H, iA], iA]$ is bounded in suitable interpolating spaces, then H is purely absolutely continuous with explicit bound on the resolvent on a dense subspace. This is called the method of weakly conjugate operator (WCO) in the literature.

In [JMP84], it has been shown that one can obtain the C^k regularity of the spectral measure by considering higher order commutators. To conclude, we mention the Mourre's method [Mou+81] that allows to localize the above result in a given spectral interval $I \subset \Sigma(H)$. It requires a strict lower bound on the commutator of the form $E_H(I)SE_H(I) > cE_H(I)$, where $c > 0$. The conclusion is that H is purely absolutely continuous inside the interval I regardless of the type of spectrum in $\mathbb{R} \setminus I$. In principle, this version may directly apply to the Anderson model, where one expects the persistence of an absolutely continuous part in the bulk of the spectrum despite the presence of eigenvalues near spectral edges

[FS83; Frö+85]. This would require the construction of coupled pairs $(H(\omega), A(\omega))$ of conjugate operators. The previous chapter motivates another approach. It might be simpler to exhibit an operator conjugate to the deterministic and more symmetric operator $M \simeq \int^{\oplus} H(\omega) \mathrm{d}\mathbb{P}(\omega)$. We note that in [MRD08; MD07], the WCO method has successfully been applied in the context of Cayley graphs and convolution operators on groups.

4.2 Absence of point spectrum

In order to define products of unbounded operators, we will need some basics on the theory of strongly continuous groups of linear operators. We will restrict ourselves to bounded operators acting on Hilbert spaces, although most statements apply in the more general context of Banach spaces. We refer to [ENB00] for an elementary introduction and to [AMG+96] for an advanced treatment of this topic.

Definition 32. *Let $\mathcal{H}$ be a Hilbert space. A strongly continuous group, or C_0-group, is a map $W\colon \mathbb{R} \to B(\mathcal{H})$ such that:*
- *for all $t, s \in \mathbb{R}$ one has $W(t+s) = W(t)W(s)$, i.e. W is a representation of the additive group $\mathbb{R}$,*
- *for all $\varphi \in \mathcal{H}$, the map $t \mapsto W(t)\varphi$ is continuous, i.e. W is continuous for the strong operator topology.*

Any C_0-group uniquely defines a closed and densely defined operator $(A, D(A))$ called the infinitesimal generator of W (e.g [ENB00, Theorem 1.4]). One has

$$D(A) := \{\varphi \in \mathcal{H} | \lim_{t\downarrow 0} t^{-1}(W(t)\varphi - \varphi) \text{ exists } \}, \tag{4.1}$$

$$A\varphi := \lim_{t\downarrow 0} (it)^{-1}(W(t)\varphi - \varphi), \quad \varphi \in D(A). \tag{4.2}$$

Note that our definition of the infinitesimal generator differs from [ENB00] by a factor i^{-1}. We will mostly (but no only) deal with unitary C_0-groups, for which one has the additional property that for each t the operator $W(t)$ is unitary. In this case, the infinitesimal generator will be self-adjoint. As a useful regularized approximation of A, we introduce the bounded operator $A_t = (it)^{-1}(W(t) - I)$, $t \neq 0$.

4.2.1 Definition of first order commutators

The theory of strongly continuous groups allows to replace the unbounded operator A with the one parameter family $W(t)$ of bounded operators, for which the product $e^{-itA}He^{itA}$ is well-defined. On can then consider the derivative at $t = 0$ as an extension of the definition of the commutator $[H, iA]$. The natural condition for which this procedure is well-defined is gathered in the next proposition.

Proposition 33 (Lemma 6.2.9 in [AMG+96])**.** *Let W be a C_0-group with infinitesimal self-adjoint generator $(A, D(A))$ and $H \in B(\mathcal{H})$ be a bounded operator. The following conditions are equivalent:*

(i) the map $t \mapsto W(-t)HW(t) \in \mathcal{B}(\mathcal{H})$ is strongly C^1,

(ii) the map $D(A) \times D(A) \ni (\psi, \varphi) \mapsto i\langle Hf|Ag\rangle - i\langle Af|Hg\rangle$ extends to a continuous quadratic form on $\mathcal{H}$.

Definition 34 (First order commutator)**.** *When the conditions of proposition 33 hold, we say that H is of class $C^1(A)$. If $H \in C^1(A)$, we denote $[H, iA]$ the bounded operator defined either as the completion of the quadratic form or the strong derivative of $W(-t)HW(t)$ at $t = 0$.*

From the proof of proposition 33 below, we obtain an important description of the commutator. We denote $s - \lim$ the operator limit induced by the strong operator topology. Recalling that $A_t = (it)^{-1}(W(t) - I)$, one has

$$[H, iA] = s - \lim_{t \to 0}[H, iA_t] \in \mathcal{B}(\mathcal{H}). \tag{4.3}$$

Sketch of proof of proposition 33. It follows from (4.1) that for fixed $\psi, \varphi \in D(A)$ the map $t \mapsto \langle \psi, W(-t)TW(t)\varphi\rangle$ is C^1. We have therefore the representation

$$\begin{aligned}\langle \psi, \left(\frac{W(-t)HW(t) - H}{t}\right) \varphi\rangle &= \frac{1}{t}\int_0^t (\langle \psi, -iAW(-\tau)HW(\tau)\varphi\rangle + \langle \psi, W(-\tau)HW(\tau)iA\varphi\rangle)\,\mathrm{d}\tau \\ &= \frac{1}{t}\int_0^t (i\langle HW(\tau)\psi, AW(\tau)\varphi\rangle - i\langle AW(\tau)\psi, HAW(\tau)\varphi\rangle)\,\mathrm{d}\tau.\end{aligned}$$

If (i) is true, it follows from the uniform boundedness principle that $t^{-1}\|W(-t)HW(t) - H\|$ is bounded around zero and one gets (ii) from above with $t \to 0$. Conversely, if (ii) is true we denote S the extended quadratic form. One obtains from dominated convergence and the density of $D(A)$ that for all $\varphi, \psi \in \mathcal{H}$ one has

$$\langle \psi, \left(\frac{W(-t)HW(t) - H}{t}\right) \varphi\rangle = \frac{1}{t}\int_0^t \langle W(\tau)\psi, SW(\tau)\varphi\rangle \mathrm{d}\tau.$$

Letting $t \to 0$, we get that $W(-t)HW(t)$ is weakly differentiable at $t = 0$ and therefore any t by the group property, which is $W(-t)SW(t)$. But since this weak derivative is strongly C^0, the map itself is strongly C^1. □

4.2.2 The Virial theorem

We are now ready to make rigorous the formal derivation detailed in the introduction.

Theorem 35 (Virial theorem). *Consider two self-adjoint operator $H = \int \lambda \mathrm{d}E_H(\lambda)$ and A. If H is bounded and $H \in C^1(A)$. Then for any $\lambda \in \mathbb{R}$ one has*

$$E_H(\{\lambda\})[H, iA]E_H(\{\lambda\}) = 0.$$

In particular, if $[H, iA] > 0$, then H has no eigenvalues

Proof of theorem 35. Let φ, ψ be in the range of $E(\{\lambda\})$, since $[H, iA] = s - \lim_t [H, iA_t]$ one has

$$\begin{aligned}\langle \varphi | [H, iA]\psi \rangle &= \lim_t \langle \varphi | [H, iA_t]\psi \rangle = \lim_t \langle \varphi | HiA_t - iA_t H\psi \rangle \\ &= \lim_t (\langle H\varphi | iA_t\psi \rangle - \langle \varphi | iA_t H\psi \rangle) \\ &= \lim_t (\bar{\lambda} \langle \varphi | iA_t\psi \rangle - \lambda \langle \varphi | iA_t\psi \rangle) = 0.\end{aligned}$$

□

As already mentioned in the introduction, the commutator method cannot directly apply in finite dimension since any commutator has vanishing trace. This makes interpretation of the method difficult and to our knowledge no delocalization results have been obtained in the finite dimensional setting. We propose the following elementary result as a finite analogue.

Theorem 36 (Finite dimensional Virial theorem). *Let two Hermitian matrices $H, A \in \mathbb{C}^{n\times n}$ and $\mathcal{E}$ a subspace of $\mathbb{C}^n$ such that*

$$[H, iA]|_{\mathcal{E}} > 0.$$

Then any eigenvalue $\lambda \in \mathbb{R}$ of H has a multiplicity smaller than the co-dimension of $\mathcal{E}$, i.e $\dim\ker(H - \lambda) \leq n - \dim(\mathcal{E})$.

Proof. Let φ be in $\mathcal{E} \cap \mathcal{K}$, with $\mathcal{K} = \ker(H - \lambda)$.

$$\langle \varphi | [H, iA]\varphi \rangle = i\langle H\varphi | A\varphi \rangle - i\langle \varphi | AH\varphi \rangle = i(\bar{\lambda} - \lambda)\langle \varphi | A\varphi \rangle = 0 \implies \varphi = 0$$

Since $\dim \mathcal{K} = \dim(\mathcal{E} + \mathcal{K}) + \dim(\mathcal{E} \cap \mathcal{K}) - \dim \mathcal{E}+$, the result follows. □

In [GP21], the following elementary local criterion has been derived for Schrödinger operator of the form $H = A + V$, where A is the adjacency operator of some graph $\mathcal{G}$. Suppose that there is a labelling on the vertices $h\colon \mathcal{G} \to \mathbb{Z}$ which is 1-Lipschitz, and such that in the neighbourhood of any vertex y, there is a unique x and at least one z verifying $h(x) + 1 = d(y) = h(z) - 1$. Then any finitely supported solution of the Laplace type equation $(H - \lambda)\varphi = 0$ must be the zero function. This is done by removing recursively each vertex of the support $\operatorname{Supp}\varphi$. It would be interesting in a future work to study whether a positive commutator $[H, iA]$ can be constructed from the labeling function h. A similar criterion in the context of unimodular random trees is given in [BSV17].

4.3 Absence of singular continuous spectrum

We keep the assumption of the former section, both A, H are self-adjoint and H is bounded of class $C^1(A)$. We further assume that the first order commutator is positive, we will denote it $S := [H, iA] > 0$. From this positive operator S, we construct two Hilbert spaces. The space $\mathcal{S}$ as the completion of $\mathcal{H}$ for $\|\varphi\|_{\mathcal{S}} = \sqrt{\langle\varphi, S\varphi\rangle} = \left\|S^{1/2}\varphi\right\|_{\mathcal{H}}$, and the space $\mathcal{S}^*$ given by the completion of range $S(\mathcal{H})$ for the norm $\|\psi\|_{\mathcal{S}^*} = \sqrt{\langle\psi|S^{-1}\psi\rangle} = \left\|S^{-1/2}\psi\right\|_{\mathcal{H}}$. Somehow the space $\mathcal{H}$ lies in between $\mathcal{S}^*$ and $\mathcal{S}$ since we have the following structure, where arrows indicate dense and continuous embedding:

$$\left(\mathcal{S}^*, \left\|S^{-1/2}\cdot\right\|_{\mathcal{H}}\right) \hookrightarrow (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \hookrightarrow \left(\mathcal{S}, \left\|S^{-1/2}\cdot\right\|_{\mathcal{H}}\right).$$

By continuous linear extension, e.g. [RS+80, Theorem I.7], the scalar product on $\mathcal{H}$ extends from $\mathcal{H} \times (\mathcal{H} \cap \mathcal{S}^*)$ to an hermitian form on the pair $(\mathcal{S}, \mathcal{S}^*)$:

$$\mathcal{S} \times \mathcal{S}^* \ni (\varphi|\psi) \to (\varphi|\psi)_{\mathcal{S}\times\mathcal{S}^*} \in \mathbb{C},$$

and we have therefore

$$\|\varphi\|_{\mathcal{S}} = \sup_{\|\psi\|_{\mathcal{S}^*}=1} (\varphi|\psi)_{\mathcal{S}\times\mathcal{S}^*}, \quad \|\psi\|_{\mathcal{S}^*} = \sup_{\|\varphi\|_{\mathcal{S}}=1} (\varphi|\psi)_{\mathcal{S}\times\mathcal{S}^*}. \tag{4.4}$$

One says that the spaces $\mathcal{S}$ and $\mathcal{S}^*$ are in duality for the scalar product of $\mathcal{H}$. As a consequence, one has $B(\mathcal{H}) \subset B(\mathcal{S}^*, \mathcal{S})$. The operator S as an element of $B(\mathcal{S}, \mathcal{S}^*)$ is by construction unitary.

Remark 4. *The construction above is classical in interpolation theory. Given a self-adjoint operator $(T, D(T))$ on a Hilbert space $\mathcal{H}$. Its form domain $Q(T)$ coincides with the domain of the operator $|T|^{1/2}$. One can equip the subspaces $D(T)$, $D(|T|^{1/2})$ with their associated graphs norm and one has :*

$$D(T) \hookrightarrow Q(T) = D(|T|^{1/2}) \hookrightarrow \mathcal{H} \simeq \mathcal{H}^* \hookrightarrow D(|T|^{-1/2})^* \hookrightarrow D(|T|^{-1})^*.$$

For example if T is the Laplacian on $L^2(\mathbb{R}^d)$, one recovers the Sobolev spaces [AMG+96, section 2.8].

4.3.1 Definition of the second order commutator

We need to extend the unbounded operator A, as operators on both $\mathcal{S}$ and $\mathcal{S}^*$. This will be done as infinitesimal generators of corresponding C_0-groups. We assume that for all $t \in \mathbb{R}$, one has $W(t)\mathcal{S}^* \subset \mathcal{S}^*$, i.e. W leaves invariant the subspace $\mathcal{S}^* \subset \mathcal{H}$. As shown in [AMG+96, proposition 3.2.5] , this stability condition already implies that the restriction $W_{\mathcal{S}^*}(t) = W(t)|_{\mathcal{S}^*}$ defines a C_0-group in $\mathcal{S}^*$. Note that

$W_{\mathcal{S}^*}$ is no longer unitary. By duality, we obtain a C_0-groups in $\mathcal{S}$ via

$$(W_{\mathcal{S}}(t)\varphi|\psi)_{\mathcal{S}\times\mathcal{S}^*} := (\varphi|W(-t)\psi)_{\mathcal{S}\times\mathcal{S}^*},$$

see [AMG+96, proposition 6.3.1]. We will still denote by W these two new C_0-groups and by A their corresponding generators. We denote $D(A,\mathcal{K})$ for $\mathcal{K}=\mathcal{S}^*$, $\mathcal{H}$ or $\mathcal{S}$ their corresponding domains.

Remark 5. *We have first defined W on $\mathcal{S}^*$ by restriction, and then on $\mathcal{S}$ by duality. We can proceed in the opposite direction, defining W on $\mathcal{S}$ by continuous linear extension (BLT theorem, e.g. [RS+80, Theorem I.7]), and then on $\mathcal{S}^*$ by duality. To do this, one must suppose that $W(t)$ is a bounded densely defined operator. It turns out that the two constructions are equivalent. We refer to [AMG+96, Section 6.3] for details.*

The second order commutator $[[H,iA],iA]$ is defined in $\mathcal{B}(\mathcal{S},\mathcal{S}^*)$ similarly to definition 34.

Proposition 37. *Consider the above construction. The following conditions are equivalent:*
- (*i*) *the map $t\mapsto W(-t)SW(t)\in\mathcal{B}(\mathcal{S},\mathcal{S}^*)$ is strongly C^1,*
- (*ii*) *the map $D(A,\mathcal{S})\times D(A,\mathcal{S})\ni(\varphi,\psi)\mapsto i(\varphi|SA\psi)_{\mathcal{S}\times\mathcal{S}^*}-i(A\varphi|S\psi)_{\mathcal{S}\times\mathcal{S}^*}$ extends to a continuous quadratic form on $\mathcal{S}$.*

Definition 38 (Second order commutator)**.** *When the conditions of proposition 37 hold, we say that $S\in C^1(A,\mathcal{S},\mathcal{S}^*)$. We define $[S,iA]\in\mathcal{B}(\mathcal{S},\mathcal{S}^*)$ as the strong derivative of $W(-t)SW(t)$ at $t=0$.*

The above definition allows to consider $[S,iA]\colon\mathcal{S}\to\mathcal{S}^*$ as a bounded operator. Similarly to (4.3), we have the characterisation of $[S,iA]$ in terms of strong operator topology

$$[S,iA]=s-\lim_t[S,iA_t]\in\mathcal{B}(\mathcal{S},\mathcal{S}^*). \tag{4.5}$$

4.3.2 The method of the weakly conjugate operator

We are now ready to state the weakly conjugate operator (WCO) method.

Theorem 39 (Theorem 2.1 in [MM97])**.** *Assume that $H\in C^1(A)$, $[H,iA]:=S>0$ and that W leaves invariant S^*. If $S\in C^1(A,\mathcal{S},\mathcal{S}^*)$, then the operator H is purely absolutely continuous. Furthermore, there is a universal constant $C>0$ such that if the above holds, one has for any $\psi\in D(A,\mathcal{S}^*)$*

$$|\langle\psi,(H-z)^{-1}\psi\rangle|\le C(1+\|[S,iA]\|^2_{\mathcal{S}\to\mathcal{S}^*})(\|\psi\|^2_{\mathcal{S}^*}+\|A\psi\|^2_{\mathcal{S}^*}),\quad\forall z\in\mathbb{C}\setminus\mathbb{R}. \tag{4.6}$$

Remark 6. *We have stated the result in a simpler form for the sake of simplicity. The original theorem in [MM97] is stronger, as the resolvent bound concerns a* *larger subspace $\mathcal{E}$ verifying $D(A,\mathcal{S}^*) \hookrightarrow \mathcal{E} \hookrightarrow \mathcal{S}^*$, while in theorem 39 we use the norm $\|\cdot\|^2_{D(A,\mathcal{S}^*)} \propto \|\psi\|^2_{\mathcal{S}^*} + \|A\psi\|^2_{\mathcal{S}^*}$.*

The proof proceeds by a Grönwall type argument on the modified resolvent $G_\epsilon(z) = (H - z - i\epsilon S)^{-1}$. We compute first a suitable expression for its weak derivative. In the sequel, we fix a vector $\psi \in D(A,\mathcal{S}^*)$, a complex number $z = E + i\eta \in \mathbb{H}$, and set

$$y(\epsilon) = \langle \psi, (H - z - i\epsilon S)^{-1}\psi\rangle = \langle \psi, G_\epsilon \psi\rangle,$$

for $\epsilon \geq 0$.

Lemma 40. *For $\epsilon > 0$, we have*

$$y'(\epsilon) = -\langle A\psi, G_\epsilon\psi\rangle + \langle G_\epsilon^*\psi, A\psi\rangle - \epsilon\langle G_\epsilon^*\psi, [S, iA]G_\epsilon\psi\rangle. \tag{4.7}$$

Proof of lemma 40. From the second resolvent identity, one has $\frac{\mathrm{d}}{\mathrm{d}\epsilon}G_\epsilon = iG_\epsilon S G_\epsilon$ in operator norm topology, and as S is the strong limit of $[H, iA_t]$ we obtain $y'(\epsilon) = \lim_t i\langle \psi, G_\epsilon[H, iA_t]G_\epsilon\psi\rangle$. By elementary manipulations on bounded operators, we have

$$G_\epsilon[H, iA_t]G_\epsilon = G_\epsilon[H - z - i\epsilon S + i\epsilon S, iA_t]G_\epsilon = iA_tG_\epsilon - iG_\epsilon A_t + i\epsilon G_\epsilon[S, iA_t]G_\epsilon,$$

therefore,

$$i\langle \psi, G_\epsilon[H, iA_t]G_\epsilon\psi\rangle = -\langle A_{-t}\psi, G_\epsilon\psi\rangle + \langle G_\epsilon^*\psi, A_t\psi\rangle - \epsilon\langle G_\epsilon^*\psi, [S, iA_t]G_\epsilon\psi\rangle.$$

Using $\psi \in D(A,\mathcal{S}^*)$, one can let $t \to 0$ to obtain (4.7) $\square$

Proof of theorem 39. Consider the bounded operator $(H - z - i\epsilon S) \in B(\mathcal{H})$. For $\varphi \in \mathcal{H}$ one has

$$-\operatorname{Im}\langle \varphi, (H - z - i\epsilon S)\varphi\rangle = \eta\|\varphi\|^2 + \epsilon\|\varphi\|^2_{\mathcal{S}},$$

and therefore

$$\|\varphi\| \leq \eta^{-1}\|(H - z - i\epsilon S)\varphi\|, \qquad \|\varphi\|^2_{\mathcal{S}} \leq \epsilon^{-1}|\langle \varphi, (H - z - i\epsilon S)\varphi\rangle|.$$

Setting $\varphi = G_\epsilon\psi$, we obtain

$$\|G_\epsilon\| \leq \frac{1}{\eta}, \quad \|G_\epsilon\psi\|_{\mathcal{S}} \leq \frac{1}{\sqrt{\epsilon}}|\langle G_\epsilon\psi|\psi\rangle|^{1/2}, \quad \|G_\epsilon\|_{\mathcal{S}^*\to\mathcal{S}} \leq \frac{1}{\epsilon}. \tag{4.8}$$

Note that similar inequalities hold for $G_\epsilon^* = (H - E + i\eta + i\epsilon S)^{-1}$. Since $|y(\epsilon)| \leq \|G_\epsilon\|_{\mathcal{S}^*\to\mathcal{S}}\|\psi\|^2_{\mathcal{S}^*}$, one has $|y(1)| \leq \|\psi\|^2_{\mathcal{S}^*} \leq \|\psi\|^2_{\mathcal{S}^*} + \|A\psi\|^2_{\mathcal{S}^*}$. We will show that this bound remains valid for $y(0)$ up to some constant. From Lemma 40

$$|y'(\epsilon)| \leq \|A\psi\|_{\mathcal{S}^*}(\|G_\epsilon\psi\|_{\mathcal{S}} + \|G_\epsilon^*\psi\|_{\mathcal{S}}) + \epsilon\|G_\epsilon^*\psi\|_{\mathcal{S}}\|[S, iA]\|_{\mathcal{S}\to\mathcal{S}^*}\|G_\epsilon\psi\|_{\mathcal{S}}.$$

and by (4.8), we get

$$|y'(\epsilon)| \leq 2 \|A\psi\|_{\mathcal{S}^*} \frac{|y(\epsilon)|^{1/2}}{\sqrt{\epsilon}} + \|[S, iA]\|_{\mathcal{S}\to\mathcal{S}^*} |y(\epsilon)|. \tag{4.9}$$

Plugging in the prior bound $|y(\epsilon)| \leq \|G_\epsilon\|_{\mathcal{S}^*\to\mathcal{S}} \|\psi\|^2_{\mathcal{S}^*} \leq \epsilon^{-1} \|\psi\|^2_{\mathcal{S}^*}$ in (4.9) gives

$$\begin{aligned}|y(\epsilon)| = |y(1) + \int_\epsilon^1 y'(s)\mathrm{d}s| &\leq \|\psi\|^2_{\mathcal{S}^*} + \big(2\|A\psi\|_{\mathcal{S}^*}\|\psi\|_{\mathcal{S}^*} + \|[S, iA]\|_{\mathcal{S}\to\mathcal{S}^*}\|\psi\|^2_{\mathcal{S}^*}\big)\,|\ln\epsilon| \\ &\lesssim (1 + \|[S, iA]\|_{\mathcal{S}\to\mathcal{S}^*})(\|\psi\|^2_{\mathcal{S}^*} + \|A\psi\|^2_{\mathcal{S}^*}))(1 + |\ln\epsilon|),\end{aligned}$$

where $a \lesssim b$ stands for $a \leq Cb$ for some universal constant $C > 0$. Using this new bound in (4.9) again

$$\begin{aligned}|y'(\epsilon)| \lesssim& 2\|A\psi\|_{\mathcal{S}^*}\sqrt{(\|\psi\|^2_{\mathcal{S}^*} + \|A\psi\|^2_{\mathcal{S}^*})(1 + \|[S, iA]\|_{\mathcal{S}\to\mathcal{S}^*})}\frac{\sqrt{1 + |\ln\epsilon|}}{\sqrt{\epsilon}} \\ &+ \|[S, iA]\|_{\mathcal{S}\to\mathcal{S}^*}(\|\psi\|^2_{\mathcal{S}^*} + \|A\psi\|^2_{\mathcal{S}^*})(1 + \|[S, iA]\|_{\mathcal{S}\to\mathcal{S}^*})(1 + |\ln\epsilon|) \\ \lesssim& (\|\psi\|^2_{\mathcal{S}^*} + \|A\psi\|^2_{\mathcal{S}^*})(1 + \|[S, iA]\|^2_{\mathcal{S}\to\mathcal{S}^*})(\frac{\sqrt{1 + |\ln\epsilon|}}{\sqrt{\epsilon}} + 1 + |\ln(\epsilon)|).\end{aligned}$$

The term $\sqrt{(1 + |\ln(\epsilon)|)/\epsilon} + |\ln(\epsilon)|$ is integrable therefore $y(0) = \lim_\epsilon y(\epsilon)$ exists. As a conclusion, we have shown that there is a universal constant $C > 0$, such that for all $\psi \in D(A, \mathcal{S}^*)$, one has

$$|\langle\psi, (H - z)^{-1}\psi\rangle| \lesssim (1 + \|[S, iA]\|^2_{\mathcal{S}\to\mathcal{S}^*})(\|\psi\|^2_{\mathcal{S}^*} + \|A\psi\|^2_{\mathcal{S}^*}).$$

As $D(A, \mathcal{S}^*)$ is dense in $\mathcal{H}$, the bounded (4.6) implies that H is purely absolutely continuous by the same limiting absorption principle detailed in the proof of the Putnam's theorem 31. □

4.3.3 An illustration of the method

Let us briefly illustrate this abstract method in the simple case of a diagonal operator on the space of periodic functions $\mathcal{H} = L^2(\mathbb{T})$. Given a measurable function $f\colon \mathbb{T} \to \mathbb{C}$, we denote M_f the associated multiplication operator. We also consider the momentum operator $P = -i\partial_x(\cdot)$ with initial domain $C^\infty(\mathbb{T})$, the space of smooth periodic functions. It extends to a self-adjoint operator on $D(P) = \{\varphi \in L^2 | \sum_k |k\hat{\varphi}_k|^2 < \infty\}$.

Lemma 41. *Assume that $f \in C^\infty(\mathbb{T})$. Let A be the densely defined operator on $C^\infty(\mathbb{T})$ by*

$$A = -(PM_{f'} + M_{f'}P)/2. \tag{4.10}$$

Then $M_f \in C^1(A)$ *in the sense of definition 34, and one has*

$$[M_f, iA] = M_{f'^2} \tag{4.11}$$

The fact that A is essentially self-adjoint follows from the general theory [AMG+96]. For a smooth function $C^\infty(\mathbb{T})$

$$\begin{aligned} 2[M_f, iA]\varphi &= -[f, \partial(f'\cdot) + f'i\partial\cdot]\varphi = -f\partial(f'\varphi) - ff'\partial\varphi + \partial(f'f\varphi) + f'\partial(f\varphi) \\ &= -ff''\varphi - ff'\varphi' - ff'\varphi' + ff''\varphi + f'^2\varphi + ff'\varphi' + f'^2\varphi + ff'\varphi' \\ &= 2f'^2\varphi. \end{aligned}$$

Since $C^\infty(\mathbb{T})$ is a core of A (or essential domain), the above identity densely extends to $D(A)$ and the condition $M_f \in C^1(A)$ of proposition 33 is fulfilled. The Virial theorem implies that any eigenfunction φ of H must verify $\|\varphi f'\| = 0$. In particular the operator M_f has no point spectrum as soon as the zero set $f'^{-1}(\{0\})$ has a vanishing Lebesgue measure.

We now illustrate the WCO method for $H = M_{\cos(\cdot)}$. Since $S = M_{\sin^2(\cdot)}$, the interpolating spaces are

$$\mathcal{S}^* = L^2(\mathbb{T}, |\sin(\theta)|^{-2}\mathrm{d}\theta) = \left\{\varphi\colon \mathbb{T} \to \mathbb{C} \middle| \int_{\mathbb{T}} |\varphi(\theta)|^2 \sin^{-2}(\theta)\mathrm{d}\theta < \infty \right\},$$

and

$$\mathcal{S} = L^2(\mathbb{T}, \sin^2(\theta)\mathrm{d}\theta) = \left\{\varphi\colon \mathbb{T} \to \mathbb{C} \middle| \int_{\mathbb{T}} |\varphi(\theta)|^2 \sin^{2}(\theta)\mathrm{d}\theta < \infty \right\}.$$

To apply the WCO method one needs to define A as an operator on $\mathcal{S}^*$ and $\mathcal{S}$. Following remark 5, we must verify that $W(t)$ continuously extends as a bounded operator on $\mathcal{S}$. For a smooth function, the following formal observation becomes rigorous. If follows from $\cos^2 + \sin^2 = 1$ that $S = I - H^2$ and $[S, iA] = [I - H^2, iA] = [-H^2, iA] = -(H[H, iA] + [H, iA]H) = -(HS + SH)$, which is a bounded operator of $\mathcal{H}$. On the dense subset $\varphi \in C^\infty$, one has

$$\begin{aligned} \|W(t)\varphi\|_{\mathcal{S}}^2 &= \left\|S^{1/2}W(t)\varphi\right\|_{\mathcal{H}}^2 = \left\|S^{1/2}\varphi\right\|_{\mathcal{H}}^2 + \int_0^t \langle W(s)\varphi, [S, iA]W(s)\varphi\rangle_{\mathcal{H}}\mathrm{d}s \\ &\leq \|\varphi\|_{\mathcal{S}}^2 + \int_0^t C\,\|W(t)\varphi\|_{\mathcal{H}}^2 \leq \|\varphi\|_{\mathcal{S}}^2 + C\int_0^t \|W(t)\varphi\|_{\mathcal{S}}^2, \end{aligned}$$

implying $\|W(t)\varphi\|_{\mathcal{S}}^2 \leq e^{C|t|}\,\|\varphi\|_{\mathcal{S}}$. Therefore $W(t)$ extends as a C_0-group on both $\mathcal{S}$ and $\mathcal{S}^*$. It remains to check that $S \in C^1(A, \mathcal{S}, \mathcal{S}^*)$. But the formal calculation $[S, iA] = -(HS + SH)$ directly shows that $[S, iA] \in B(\mathcal{H}) \subset B(\mathcal{S}, \mathcal{S}^*)$, all the conditions of theorem 39 are fulfilled. We recover the fact that $H = M_{\cos(\cdot)}$ is absolutely continuous, and obtain the resolvent bound uniform in z:

$$|\langle\psi,(H-z)^{-1}\psi\rangle| \leq C(\|\psi\|^2_{\mathcal{S}^*}+\|A\psi\|^2_{\mathcal{S}^*})$$

4.4 Applications

The commutator method has been applied in the context of convolution operators on locally compact groups [MD07; MRD08]. We discuss how one can obtain conjugate operators from this approach. We will keep the notation of chapter III, and also denote the convolution by $\delta_x * \delta_y = \delta_{xy}$. We consider $H = L_m$ a convolution operator on a finitely generated group Γ. To avoid technicalities, we assume that the symmetric measure $m \in \mathcal{M}(\Gamma)$ is finitely supported. Consider an homomorphism into the additive group $\Psi\colon \Gamma \to \mathbb{R}$, and denote the associated multiplication operator $D_\Psi = \mathrm{Diag}(\Psi)$, self adjoint with domain $D(D_\Psi) = \{\varphi \in l^2(\Gamma) | \sum_x |\Psi(x)\varphi(x)|^2 < \infty\}$. The crucial observation is that for finitely supported functions $f, g \in l^2(\Gamma)$, one has

$$D_\Psi(f*g) = (D_\Psi f)*g + f*(D_\Psi g).$$

Therefore D_Ψ acts as the derivation $-i\partial_x$ in lemma 41. The analogy suggests the following correspondence.

$L^2(\mathbb{T})$	$l^2(\Gamma)$
f	m
$P = -i\partial_x$	D_Ψ
f'	$D_\psi m = i\Psi.m$
M_f	L_m
$M_{f'}$	$L_{i\Psi.m} = iL_{\Psi.m}$
$A = -(PM_{f'} + M_{f'}P)/2$	$A = i(D_\psi L_{\Psi.m} + L_{\Psi.m} D_\psi)/2$
$[M_f, iA] = M_{f'^2}$	$[L_m, iA] = L_{(\Psi.m)*(\Psi.m)} = (L_{\Psi.m})^2$

The product $\Psi.m$ is defined point-wise, this is a finitely supported measure with mass $\Psi(x)m(x)$. In this analogy the product $f.g$ of two functions in $\mathbb{T}$ has been replaced by the non-commutative product of convolution $f*g$ on Γ. To obtain the identity $[L_m, iA] = M_{(\Psi.m)*(\Psi.m)}$, one must therefore impose the condition $(\Psi.m)*m = m*(\Psi.m)$, see [MD07; MRD08]. In this case the Virial theorem applies and any eigenvalues of L_m must vanish under the convolution by $\Psi.m$, therefore $\mathcal{H}_{\mathrm{pp}}(H) \subset \ker L_{\Psi.m}$. Since the WCO method involves second order commutators, one needs a supplementary commutation condition in order to exclude singular continuous spectra. We summarize the result in the following theorem, and refer to [MD07] for the proof of a more general statement.

Theorem 42 (Theorem 2.2 in [MD07])**.** *Consider a convolution operator $H = L_m$ by a finitely supported symmetric measure m on a finitely generated group Γ. Let $\Psi\colon \Gamma \to \mathbb{R}$ be any additive morphism.*

— *If* $(\Psi.m) * m = m * (\Psi.m)$, *then* $\mathcal{H}_{\mathrm{pp}}(H) \subset \ker L_{\Psi.m}$.
— *If furthermore* $(\Psi^2.m) * (\Psi.m) = (\Psi.m) * (\Psi^2.m)$, *then* $\mathcal{H}_{\mathrm{sc}}(H) \subset \ker L_{\Psi.m}$.

In order to conclude that the operator $H = L_m$ is absolutely continuous, one must therefore show that the kernel of $L_{\Psi.\mu}$ is reduced to zero. In general, this is a difficult task. As pointed out by Christophe Pittet (personal communication, April 11, 2023), there are natural conditions under which this is always satisfied. The Kaplansky's zero-divisor conjecture asserts that if Γ is torsion free (i.e. contains no elements of finite order other than the identity), then the group algebra has no zero divisor. With our notation, this means that there is no non-trivial solution of $f * g = 0$ for finitely supported functions $f, g \in l_c(\Gamma)$. This conjecture is known to hold for all virtually solvable groups, and more generally also for all residually torsion-free solvable groups. Under amenability, this result extends if one of the two elements is infinitely supported, as shown in [Ele03].

Theorem 43 (Theorem 1 in [Ele03])**.** *Let* Γ *be a finitely generated amenable group and* $f \in l_c(\Gamma)$. *If* $f * g = 0$ *for some* $g \in l^2(\Gamma)$, *then there is* $\tilde{g} \in l_c(\Gamma)$ *such that* $f * \tilde{g} = 0$.

By a combination of the two above theorems, we obtain the following result.

Theorem 44. *Let* Γ *be a finitely generated amenable groups with the zero-divisor properties. Let* m *be a finitely supported measure on* Γ *such that there is an additive morphism* Ψ *verifying the two commutation relations* $(\Psi.m) * m = m * (\Psi.m)$ *and* $(\Psi^2.m) * (\Psi.m) = (\Psi.m) * (\Psi^2.m)$. *Then* L_m *is a purely absolutely continuous operator.*

Proof. Since Γ verifies the zero divisors conjecture, then by assumption L_m has no finitely supported eigenfunctions. If furthermore Γ is amenable, by theorem 43, L_m has no point spectrum. Now suppose that there is an additive homorphism verifying the commutation condition of theorem 42. Then L_m is purely absolutely continuous. Indeed by theorem 42 one has $\mathcal{H}_{\mathrm{sc}}(L_m) \subset \ker L_{\Psi.m}$, but by theorem 43 again one has $\ker L_{\Psi.m} = \{0\}$. □

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