Review of Q-Type Polynomials: A New Asphere Representation

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ABSTRACT

With the increasing ubiquity of aspheric surfaces in modern optical designs, updated mathematical representations are needed to address the needs of the designer and manufacturer. The Forbes Q-Type polynomials — $Q^{bfs}$and Q^{con}representations — address specific concerns with mathematical robustness, generality, and interpretability. These specific characteristics give the designer more degrees of freedom to understand each surface contribution and to optimize for cost-effective specifications. The Q-Type polynomial also gives the optical manufacturer more information on assessing generation, polishing, and metrology difficulty. Additional research has been conducted into sensitivity and yield assessment for Q-Type aspheres. This paper provides a brief introduction into the characteristics of Q-Type polynomials at the intersection of design and manufacture.

Keywords: Aspherics, Optical fabrication, Systems design

Optical design stems from the nature of light bending at interfaces of differing indices of refraction. The most basic surface shape used is the sphere. It has the unique property of being rotationally symmetric and maintaining a constant radius of curvature at all points of the element's radius. When factoring in manufacturability, determining the sag yields useful insight. Equation 1 shows the sag of a conic with curvature ($\mathcal{C}$) and conic coefficient (κ) as a function of element radius (ϱ).[1] When dealing with spherical surfaces ($\kappa = 0$), this equation simply plots the surface of a sphere.

$$z(\varrho) = \frac{\mathcal{C}\varrho^2}{1+\sqrt{1-(\kappa+1)\mathcal{C}^2\varrho^2}} \tag{1}$$

Spherical surfaces only allow the optical designer one degree of freedom to balance aberrations and constrain the first-order properties of the system—curvature. Since the sphere is a special case of conic with $\kappa = 0$, it is easy to extend the surface description to include all conic sections. However, optical design necessitated more degrees of freedom for each surface. Since Equation 1 already returns the departure of a surface, it is simple enough to expand the equation in a power series where a series of monomials describe departure from a conic. This derives the even-asphere, Equation 2, which has become the defacto representation for rotationally-symmetric optical surfaces.[2,3]

$$z(\varrho) = \frac{\mathcal{C}\varrho^2}{1+\sqrt{1-(\kappa+1)\mathcal{C}^2\varrho^2}} + \sum_{n=0}^{\infty} A_{2n+4}\varrho^{2n+4} \tag{2}$$

If we examine the mathematics behind Equation 2, a few peculiarities become evident. Since this equation consists of an infinite power series of monomials, any surface profile can be constructed. This reinforces the generality of this equation. However, due to rotational symmetry, the odd monomial terms cancel out. Additionally, this leads to a lack of robustness in normal use. When Ref. 4 analyzed the contributions of each coefficient, it was found that there was significant cancellation. Monomial power series are not intrinsically orthogonal which leads to a problem of degeneracy—when high order coefficients impact lower order coefficients resulting in arbitrary cancellations. This directly relates to the decreased efficacy of the representation since each coefficient is not related to a physical quantity and is not naturally convergent.[2] Forbes demonstrated degeneracy of a simplistic

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case where the coefficients were fit to an arbitrary function.[5] Issues arose when, "using double-precision arithmetic, the associated Gram matrix...is so ill-conditioned that this simplistic process fails when more than about ten terms are used".[6] When the problem is generalized, there is little order with the use of coefficients. Furthermore, as the number of coefficients increases, there is a decrease in precision.[7] At this point, any performance gains are effectively nullified by the limits of numerical precision. It is evident that another method of expressing aspheres could rectify some of these issues.

Indeed, a better representation for optical surfaces was needed to counteract the limitations of the traditional monomial expansion. Greg Forbes formulated a new representation that solved some of the intrinsic challenges mentioned prior. His goal was to alleviate some of the challenges for the designer and manufacturer. An ideal surface representation would be mathematically efficient, robust, and general enough for a wide usage case.[3] The surface would also draw connections to the manufacturability and testability which would justify its production costs.[8] In order to meet all of these criteria, Forbes focused his efforts on describing the local surface sag and surface slope. This alternative representation "would avoid degeneracy and ideally also facilitate the determination of appropriate metrics of slope".[9]

Forbes started with the first monomial from the power series expansion. For each subsequent order, the basis was orthogonalized. After the basis was defined, it was normalized over the surface radius. These derivations yield the $Q^{bfs}$and Q^{con}surface types representing a slope dependent and sag dependent polynomial. Since the explicit derivations are non-trivial and beyond the scope of this paper, they are shown in depth in Ref. 9–15. We can examine the efficacy of these surface profiles. These definitions are an adaptation of the Jacobi polynomials and thus inherit the same functional form. This vector space was orthogonalized using the Gram-Schmidt Orthonormalization process.[16] As such, each coefficient is intrinsically independent from one another. Any linear combination of orders m will avoid the pitfalls of degeneracy. This prevents the lack of numerical precision and ensures that the minimum number of integers are necessary for each coefficient. Moreover, a decomposition of these terms yield a spatial frequency spectrum where each coefficient has a direct meaning on its own.[9] Units can be assigned to each order m giving physical significance to the mathematical representation.[17] The first six orders of the $Q^{bfs}$and Q^{con}polynomials are shown in Figure 1 and Figure 2 respectively.[10] The relative frequency distribution and local slope of each surface can be seen.

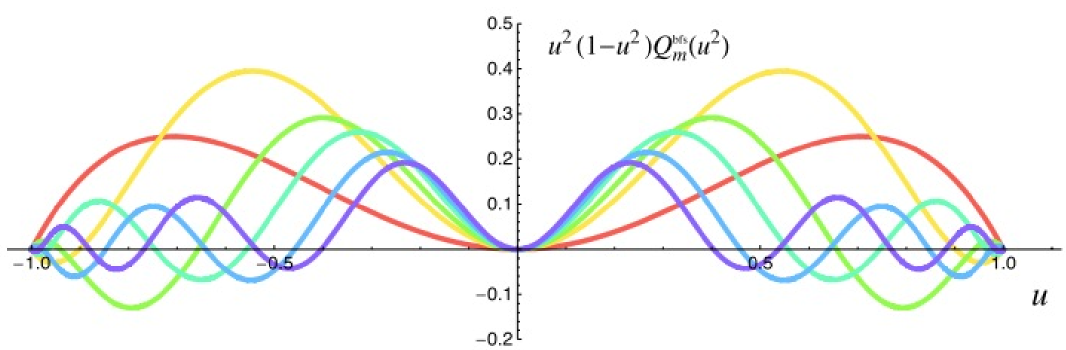

Figure 1. Q^{bfs}

While the derivation is useful, it is important to explore the practical applications for Q-type aspheres in optical design. Aspheres tend to be used for aberration control. Their localized surface profile allow them to correct aberrations across the ray bundle. For aspheric surfaces near apertures, low-order aspheres can correct for 3^{rd} and 5^{th} order spherical aberration (SA). For aspheres near image locations, higher order aspheres can locally impact the chief ray affecting Petzval and flattening the field. Using the ray aberration function from Ref. 18 and its derivation into primary transverse ray aberrations from Ref. 19, we can examine the surface contribution for third order spherical aberration in Equation 3.

$$a_{1j} = \frac{1}{2} n_j \left(\frac{n_j}{n'_j} - 1 \right) y_a i_{aj}^2 (i_{aj} + u'_{aj}) \tag{3}$$

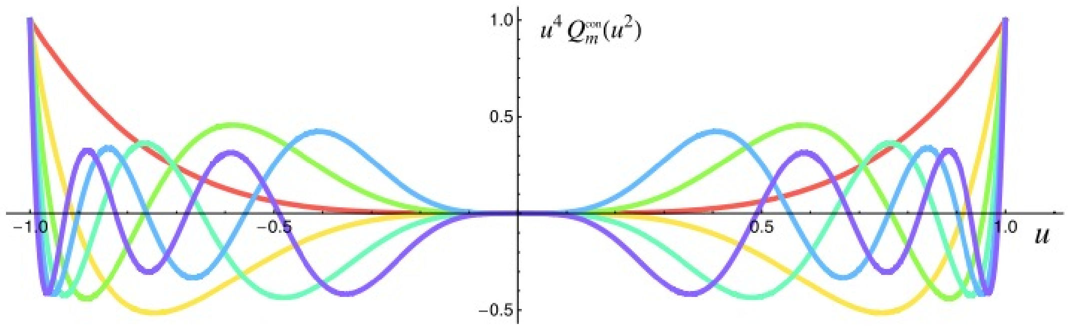

Figure 2. Q^{con}

One case when the third order SA is minimized is when the angle of the marginal ray approaches zero ($i_{aj} \to 0$). Thus, local surface slopes can be matched to the AOI of the marginal ray while maintaining the ideal best fit sphere. The Q^{bfs} and Q^{con} polynomials accomplish the goals of aberration correction using the properties described earlier.

In addition to correcting aberrations, aspheres must be manufacturable and testable if they are to be effective. This is where the Q^{bfs} polynomial differs from the Q^{con}. While both representations can yield the same surfaces, the Q^{bfs} surface is preferred due to its local slope metric. The local RMS slope of the Q^{bfs} surface can be computed as the mean square gradient of the coefficients.[20] This is a critical parameter since it determines the local fringe density for any form of non-contact interferometric testing. This method is applicable for both full-aperture interferometry and sub-aperture stitching interferometry.[21–23] As long as the local fringe count is less than the Nyquist frequency of the detector, direct spherical testing methods can be used. This eliminates the need for CGH's or null lenses.[24] Since the RMS slope is generalizable for all aspheres, constraints can be applied during the optimization process that limit aspheres to the testing capability of particular manufacturers. Both Synopsys CodeV[25] and Zemax OpticStudio[26] have operands that can be used to control these surfaces. In CodeV, the (QSL) operand can control the maximum permissible RMS slope departure for Q^{bfs} surfaces. The (QSG) operand can control the maximum permissible RMS sag departure for Q^{con} surfaces. In Zemax, the RSS of the coefficients can be calculated by using the operands shown in Figure 3. With this, mild aspheres can be designed. The implementation of these operands gives the optical designer the ability to design for manufacture long before tolerancing and sensitivity analysis. Additionally, surfaces with strong aspheric departure can be optimized. Ref. 23 derived a user defined macro for CodeV which allows the user to optimize within the capabilities of QED's SSI family. When it comes time for production, the latest iteration of ISO 10110:12 (Ref. 27) includes explicit procedures for specifying Q-type aspheres. This means that that surfaces will not need to be remapped to even-aspheres or any other definition. The standard simply helps the optical designer convey the correct information to the manufacturing shop.

Oper #	Type	Surf	Param		Target	Weight	Value
1 PMVA	PMVA	1	3		0.000000	0.000000	0.000000
2 PMVA	PMVA	1	4		0.000000	0.000000	0.000000
3 PMVA	PMVA	1	5		0.000000	0.000000	0.000000
4 PMVA	PMVA	1	6		0.000000	0.000000	0.000000
5 PMVA	PMVA	1	7		0.000000	0.000000	0.000000
6 PMVA	PMVA	1	8		0.000000	0.000000	0.000000
7 QSUM	QSUM	1	6		0.000000	0.000000	0.000000
8 OPLT	OPLT	7			3.675000E-003	1.000000E+008	3.675000E-003

Figure 3. Q^{bfs} constraints for Zemax

Recent research has been comparing the efficacy of different aspheric surface profiles. This paper has shown the benefits over the even-asphere representation, but comparisons have been done with Zernike representations. The FRINGE Zernike set can be used to describe aspheric surfaces in a manner similar to that explained by Forbes. However, in its parameterization, it does not begin with a ϱ^4 dependency. Thus, the radius of the best fit sphere and axial intercept vary with changes in the coefficients.[10] This means that the first order properties of the element will change with the coefficients. Since first order properties are essential to specifying a system, this surface profile is limited in its application. A more applicable application for Zernikes is when decomposing surface figure. Ref. 3, 22 examined the error function for various systems with differing aspheric representations. In each design example, the authors found that the Q^{bfs}polynomial achieved the lowest error function with all other system factors held constant. They speculate that the orthogonalized basis mentioned above may improve the convergence properties of the optimizer. They conclude by remarking that "as an unexpected bonus, gains in raw optical performance may also be won".[3] Other studies have examined additional benefits of using Q-type polynomials in the design process. In one paper, the authors examined a lithographic system. By converting all of their aspheric surfaces to Q^{bfs}representations and constraining the local slope, they were able to reduce the assembly tolerances by a factor of 3.[21] This system maintains equivalent performance, but is significantly more testable and has lower assembly residuals. Another study investigated the design of a mobile phone camera with Q^{bfs}aspheres. They found an approximate 10x reduction to manufacturing sensitivities.[17] With the cell phone lens, the constrained slope was able to redistribute excessive departures from a best-fit sphere on one asphere to multiple surfaces. This redistribution of aspheric power helped their design converge to a solution faster, reduce manufacturing tolerances, make testing easier, and desensitize the assembly tolerances.

As the optics community continues to innovate, the implementation of Q-type polynomials will likely increase. With the benefits outlined in this overview, the theoretical gains have been demonstrated. With the International Organization for Standardization including Q-type polynomials in their standards, most optics shops should fully implement the surfaces in the near future. As sub-aperture interferometry and polishing advance, it is likely that optical manufacturers will be able to tailor Forbes coefficients to reduce the costs of manufacturing. As further research is conducted in 2D non-rotationally symmetric freeform surfaces, the mathematical intuition behind orthonormalized polynomials will likely lead to new discoveries.

Forbes Q-type polynomials represent a major advance in aspherical surface representations. Advances in the underlying mathematics has enabled a more robust and general surface that has demonstrable performance gains over other equivalent representations. Gains in convergence, manufacturability, and assembly tolerances make the $Q^{bfs}$and Q^{con}surfaces enticing for designers and manufacturers. As the number of aspheric coefficients and surfaces continue to increase, the ubiquity of Forbes polynomials will lead to the next generation of optical systems.

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