ECE259 Abridged

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Chapter 1

Electrostatics

1.1 Coulomb's Law and Electric Field

$$\vec{F}_e = k\frac{Q_1 Q_2}{|\vec{R}^2|}(\vec{R}_2 - \vec{R}_1)$$

1.2 Electric Field Intensity $\vec{E}$

- **Source** charges 'make' the charge.
- **Test** charges 'feel' the charge.
- Field lines point in the direction that a **positive test charge** would be pushed if it were placed at a given point in space.

Electric Field at R_2 due to Q_1 at R_1:

$$\vec{E}_1 = \lim_{Q_2 \to 0} \frac{\vec{F}_{12}}{Q_2} = \frac{Q_1}{4\pi\varepsilon_0 |\vec{R}_2 - \vec{R}_1|^3}(\vec{R}_2 - \vec{R}_1)$$

Note: F_{12} is the force acting on 1 due to charge 2.

Properties of $\vec{E}$

- Points away from positive charges.
- Points toward negative charges.
- Points along line between source and point of measurement.
- Linear such that $\vec{E}_{tot} = \sum \vec{E}_i$

1.3 $\vec{E}$ from Continuous Charge Distribution

$$\vec{E}_{tot} = \int d\vec{E}' = \int \frac{dQ'(\vec{R} - \vec{R}')}{4\pi\varepsilon_0|\vec{R} - \vec{R}'|^3}$$

Possible charge distribution types consist of **linear, surface, and volumes**. Arbitrary surfaces will not be tested - they generally fall into **disk, cone, cylinder, sphere, and cube**.

Steps for Finding $\vec{E}_{tot}$:

1. Write down Coulomb's Law:

$$\vec{E} = \int d\vec{E}' = \int \frac{dQ'}{4\pi\varepsilon_0|\vec{R} - \vec{R}'|^3}(\vec{R} - \vec{R}')$$

2. State dQ', $\vec{R}$, and $\vec{R}'$.

$$dQ' = \rho_s dS$$

3. Integrate:

   - Determine $d\vec{E}' = \frac{dQ'(\vec{R}-\vec{R}')}{4\pi\varepsilon_0|\vec{R}-\vec{R}'|^3}$
   - Ensure that position vectors do not vary with spatial coordinates (if they do, convert them to cartesian or otherwise).
   - Cancel any unit vectors you can using symmetry.
   - Perform the integration.

1.4 Gauss's Law

1.4.1 Fundamental Postulates of Electrostatics

$$\nabla \times \vec{E} = 0 \;\rightarrow\; \int_C \vec{E} \cdot d\vec{l} = 0$$

$$\nabla \cdot \vec{D} = \rho_S \;\rightarrow\; \iint_C \vec{D} \cdot d\vec{S} = Q_{enc} = \iiint \rho_v dV$$

The second line is Gauss's law. The first is simply stating that electric fields are conservative/irrotational/etc.

What is $\vec{D}$? $\vec{D} = \varepsilon\vec{E} = \varepsilon_r\varepsilon_o\vec{E}$. $\vec{D}$ is electric **flux density** while $\vec{E}$ is the electric **field density**.

1.4.2 Why is Gauss's law true?

Why should it be the case that local charge density is given by the divergence of electric flux density? Let's think about the properties of electric fields. We know that electric field lines (and thus electric flux lines) point away from positive charges. **Divergence is how much the vector field points away** (or to) a given point. For that reason, it makes a lot of sense for the divergence to give you the magnitude of local charge density. If you have a positive charge, your field lines will point away from that charge. Therefore, the divergence of $\vec{D}$ will be correlated (in this case, equal) to the charge density!

1.4.3 Problem Solving with Gauss's Law

Using Gauss's law, we can greatly simplify the process of finding $\vec{E}$ if a few conditions are met. We need to pick a surface that surrounds a given charge mass that has the following properties:

Gaussian Surface Properties:

1. Closed.
2. $\vec{E}$ is either perpenndicular or parallel to $d\vec{S}$.
3. $|\vec{E}|$ is constant over the surface.

Steps to getting $\vec{E}$ with Gauss's Law:

1. Determine the charge distribution from the given source.
2. Choose a corresponding Gaussian surface.
3. $\iint \vec{E} \cdot d\vec{S} \;\; \rightarrow \;\; E_R \iint d\vec{S} = E_r \cdot SA$. From here, we can solve for E_R, or the electric field at a given point on the surface (which is constant). The direction of that field can be found via common sense...

You can also solve things with the non-differential form of Gauss's law by knowing that $\nabla \cdot \vec{D} = \rho_v$.

1.5 Electric Scalar Potential

We have the following methods of finding $\vec{E}$ from a charge distribution:

1. Coulomb's law with integral superposition.
2. Gauss's Law.
3. **Potential Theory**
4. Getting a computer to do it for us.

We define electric potential ΔV as the amount of work needed to move a 1-Coulomb charge from one place to another in an electric field:

$$\Delta V = -\int_{p_1}^{p_2} \vec{E} \cdot dl = \frac{W_{ext}}{Q}$$

$$\nabla V = -\vec{E}$$

$$V = \frac{1}{4\pi\varepsilon} \int_{v,s,l} \frac{dQ'}{|\vec{R} - \vec{R}'|}$$

That second one is Maxwell's 2nd equation.

Fortunately, this is conservative and linear. If you know that ΔV from point a to point b and the ΔV from point b to point c, you can easily get the ΔV from point a to point c.

Absolute V: If we take our reference point as one that is infinite distance away, we define **absolute** V as the energy required to move a 1-Coulomb test charge from infinite distance to a given point. If it's not otherwise stated, this is the voltage value you are being given.

Equipotential surfaces: These surfaces have the same V value. Remember how $\nabla V = -\vec{E}$? Well thanks to our knowledge of gradients, we know that these are always going to be perpendicular to $\vec{E}$ lines. Also, we now know that $\vec{E}$ points from **high to low** V thanks to our awesome vector calculus prowess. Very cool.

Steps for Problem Solving:

- Write $\Delta V = -\int_{p_1}^{p_2} \vec{E}_a \cdot dl$.
- Evaluate $V_{origin \to p_1}$ and $V_{origin \to p_2}$.
- Subtract.

Electric Potential of a Point Charge: Using Coulomb's law, we get:

$$V = \frac{Q}{4\pi\varepsilon_0|\vec{R} - \vec{R}'|}$$

For larger bodies, V is a **simple scalar summation** of the voltage due to individual point charges.

Note that the formulae for the **gradient operator** ∇ in various coordinate systems are on the aid sheet for the sheet.

1.6 Dielectrics

If you put a material that's not conductive in an electric field, the electron clouds will stretch, and you'll get some surface-bound net charges ρ_{sb}. You don't actually have any volume charges, because in the middle of the material, the net charges still cancel out with adjacent atoms since you don't have any net flow of electrons away from their nucleii.

In the case of a capacitor, the original electric field is due entirely to the surface charge on the plates of the capacitor $\pm\rho_s$. This is the sole factor for generating the actual field within the capacitor $E_0 = \frac{\rho_s}{\varepsilon_0}$. Similarly, there is an electric field due to polarization within the material due to the **bound charge density** ρ_{sb}:

$$\vec{E}_p = \frac{\rho_{sb}}{\varepsilon_0}; \;\; E_{tot} = E_0 + E_p = \frac{\vec{E}_0}{\varepsilon_r}$$

After some fancy algebra, we also get:

$$\vec{P} = \rho_{sb}\hat{n}; \;\; \vec{D} = \rho_s\hat{n} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_r\varepsilon_0\vec{E}$$

This is super cool, because it shows that D doesn't really care about dielectrics or anything that gets in its way – it just cares about the original sources of electric charge: **free charge** densities.

We also have these super important relationships for when you don't actually have any field causing the polarization and you want to know stuff about the polarization vector or whatever.

$$\rho_{p,v} = -\nabla \cdot \vec{P}; \;\; \rho_{p,s} = -\hat{a}_n \cdot (\vec{P}_2 - \vec{P}_1)$$

Where $\rho_{p,v}$ is the volume charge density caused by the polarization, etc.

1.6.1 Dielectric Strength

Dielectrics have a maximum $\vec{E}_b$ at which they turn into conductors because the electrons are ripped into the conduction band.

1.6.2 Dielectric Boundary Conditions

At the interface between two materials, the magnitude and direction of the electric field changes. There are **two boundary conditions** that are able to effectively describe that change, derived from Maxwell's Equations.

- $E_{t1} = E_{t2}$. *Tangential $\vec{E}$ is continuous across boundary.*
- $D_{n1} - D_{n2} = \rho_s$. *Normal $\vec{D}$ is discontinuous by difference of ρ_s across boundary.*

These both apply in static AND time-varying situations.

Resultant properties of boundary conditions:

- $\vec{E} = 0$ for all conductors.
- **Current densities** at imperfect conductor boundaries are:

$$\frac{J_{t1}}{\sigma_1} = \frac{J_{t2}}{\sigma_2}; J_{n1} = J_{n2} = J_n$$

- At imperfect boundaries, the surface charge density is:

$$\rho_s = J_n\{\frac{\varepsilon_{r1}\varepsilon_0}{\sigma_1} - \frac{\varepsilon_{r2}\varepsilon_0}{\sigma_2}\}$$

Do note that all these rules only apply at interfaces, not in free spaces.

1.6.3 Capacitance

$$Q = C\Delta V; \quad C = \frac{Q}{V}$$

Capacitance is the proportionality constant for a given setup between the voltage applied and the charge accumulated. Converting the above statement into integrals, we get:

$$C = \frac{Q}{V} = \frac{\iint_s \vec{D} \cdot d\vec{S}}{|-\int \vec{E} \cdot dl|}$$

Capacitors: Pretty wacky, can usually be solved with a combination of Gauss's law and knowing your material properties and how to get voltage from an electric field or electric flux density distribution.

- **Compound capacitors** have multiple dielectrics. BOUNDARY CONDITIONS MUST BE APPLIED when solving these.
- **DIELECTRIC RESISTANCE** is given by:

$$RC = \frac{\varepsilon_r \varepsilon_0}{\sigma}$$

- Supercapacitors use nanoengineering to make high capacitance capacitors.

1.7 Electrostatic Energy

1.7.1 Parallel Plate Capacitor

$$Q = CV_0; |\vec{D}| = \rho_s = \frac{Q}{S}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_r \varepsilon_0} = \frac{\rho_s}{\varepsilon_R \varepsilon_0} = \frac{V_0}{d}$$

Energy in an electrostatic system is defined as **work** is required to go from **infinite disperasl of particles** to the given configuration of charges.

$$W_e = \frac{1}{2} \sum_{i=1}^{n} Q_i V_i$$

$$W_e = \frac{1}{2} \int_l \rho_l V dl = \frac{1}{2} \iint_s \rho_s V dS = \frac{1}{2} \iiint \rho_v V dv$$

$$W_e = \frac{1}{2} \iiint \vec{D} \cdot \vec{E} dv = \frac{1}{2} \iiint \frac{|\vec{D}^2|}{\varepsilon_r \varepsilon_0} dv$$

1. V_i is the voltage 'seen' by particle i.
2. Q_i is the charge of particle i.
3. $\frac{1}{2}$ is because we technically count each charge/associated energy twice.

Energy density: $w_e = \frac{1}{2} \vec{D} \cdot \vec{E}$

Two Paths to Energy W_e Pathway I:

1. $\nabla V = \vec{E}$
2. $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$
3. $\rho_s = |\vec{D}|$
4. $Q = \iiint \rho_v dv$
5. $C = \frac{Q}{V_0}$

$$W_e = \frac{1}{2} C V_0^2$$

Pathway II:

$$W_e = \frac{1}{2} \iiint_v \varepsilon_r \varepsilon_0 |\vec{E}|^2 dv = \frac{1}{2} C V_0^2$$

1.8 LaPlace and Poisson's Equations

These are techniques for going from **charge density** to $\vec{E}$ and are generally used in most practical situations.

Poisson's Equation:

$$\nabla \cdot (\varepsilon_r \varepsilon_0 \nabla V) = -\rho_v$$

Laplace's Equation:

$$\nabla \cdot (\varepsilon_r \varepsilon_0 \nabla V) = 0$$

Some vocabulary:

- $\nabla \nabla V = \frac{-\rho_v}{\varepsilon_r \varepsilon_0} \to \vec{\nabla}^2 V$ is the **Laplacian**
- Therefore $\vec{\nabla}^2 V = 0$ is the true version of Laplace's equation.
- Under the same notation, $\vec{\nabla}^2 V = -\frac{\rho_s}{\varepsilon_r \varepsilon_0}$.

This leaves us with a **boundary value problem**. If we know the voltage at the bounds (or we can infer it from something else), we can then use the fact that $\nabla \vec{V}^2 = 0$ wherever there is no surface charge to solve the rest!

Notes on solving the BVP:

- Even though the dielectric can be polarized in a question, it doesn't count because it's not a **FREE** charge.
- Since you rarely know the actual ρ_s (you would likely know the V_0), this is a far more utile problem solving strategy.
- A reliable **technique to solve the boundary value problem** is to *integrate the equation twice.*

Procedure for BVP's:

- Solve **LaPlace or Poisson's** equation via *direct integration* or *separation of variables.*
- Apply *boundary conditions* to solve for arbitrary constants to get to a *particular solution.*
- Use the following formulae to transform to any other variable of interest after solving for the voltage distribution:

$$\vec{E} = -\nabla V; \quad \vec{D} = \varepsilon_r \varepsilon_0 \vec{E}; \quad \vec{J} = \sigma \vec{E}$$

$$R = \frac{V_0}{\iint \vec{J} \cdot dS} = \frac{\varepsilon_r \varepsilon_0}{\sigma C}$$

Electric Shielding A conducting material can be used to 'shield' the inside from external $\vec{E}$. This is because you can't have a ΔV across a conductor – it just conducts the difference in charge. Since we know that $V_{in} = V_{out}$, we can solve for

$$\vec{\nabla} V = 0 \rightarrow V(x, y, z) = V_0$$

By the uniquess principle, we know that this HAS to be the only solution. That's how **Faraday cages** work!

1.9 Resistance and Joule's Law

Recall that

- $\vec{\nabla} \cdot (\varepsilon_r \varepsilon_0 \vec{\nabla} V) = -\rho_v$
- $D_{n1} - D_{n2} = \rho_s$
- $E_{t1} = E_{t2}$

1.9.1 Volume Current Density

$\vec{J}$ is the vector field that represents microscopic current flow.

$$\vec{J} = \sigma \vec{E}$$

$$I = \iint_S \vec{J} \cdot dS$$

$$\mu_e = \frac{e\tau}{m_e}$$

$$\sigma = \frac{N_e e^2 \tau}{m_e}$$

Resistance: plays into the situation as follows.

$$R = \frac{V_0}{I} = \frac{|-\int \vec{E} \cdot dl|}{\iint_s \vec{J} \cdot dS} = \frac{|-\int \vec{E} \cdot dl|}{|\iint_s \sigma \vec{E} \cdot d\vec{S}|}$$

And if S **is uniform over** L:

$$R = \frac{L}{\sigma S}$$

1.9.2 Power

$$P = \iiint_V \vec{E} \cdot \vec{J} dV$$

This gives the power loss due to resistance leading to heat loss.

Chapter 2

Magnetostatics

2.1 Biot-Savart Law

$\vec{H}$ is the magnetic field intensity and $\vec{B}$ is the magnetic flux density. $\vec{B}$ is kind of like $\vec{E}$ and $\vec{H}$ is like $\vec{D}$.

All magnetic fields are due to moving charges of one sort or another. In general,

$$d\vec{B} = \frac{\mu_0 I d\vec{l} \times (\vec{R} - \vec{R}')}{4\pi |\vec{R} - \vec{R}'|^3}$$

- Point charge current:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{Q\vec{u} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}$$

- Line current:

$$B = \frac{\mu_0 I}{4\pi} \int_C \frac{dl' \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}$$

- Surface current:

$$B = \frac{-\mu_0}{4\pi} \iint_S \frac{\vec{J}_s \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3}$$

- Volume current:

$$B = \frac{\mu_0}{4\pi} \iiint_{vol} \frac{\vec{J} \times (\vec{R} - \vec{R}')}{|\vec{R} - \vec{R}'|^3} dV'$$

μ is the new version of the electric permeability:

$$\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}$$

2.2 Boundary Conditions

$$B_{n1} = B_{n2}; H_{t1} - H_{t_2} = J_s$$

2.3 Energy

$$w_m = \frac{1}{2}\vec{B} \cdot \vec{H}$$

$$W_m = \frac{1}{2}\iiint_{vol} \vec{B} \cdot \vec{H} dV$$

2.4 Lorentz Force

This relates the movement of a charge in a magnetic field to the force it experiences.

$$F_m = q\vec{u} \times \vec{B} = \vec{I}l \times \vec{B}$$

You use the right-hand-rule for this relationship: Your fingers represent the direction of motion $\vec{u}$, your palm is the field $\vec{B}$, and then your thumb is F_m.

$$F_{tot} = F_e + F_m = q\vec{E} + q\vec{u} + \vec{B}$$

2.5 Ampere's Law

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

$$\int_C \vec{H} \cdot d\vec{l} = \iint_S \vec{J} \cdot d\vec{S} = I_{enc}$$

You relate the current with $\vec{H}$ via the right hand rule. Your fingers wrap around the wire in the direction of $\vec{H}$ while your thumb points in the direction of the net current.

This is basically Gauss's law. Make sure that your $\vec{H}$ is going to be constant and either be perpendicular or parallel to the closed path you draw, and then use the fact that:

$$\int_C \vec{H} \cdot d\vec{l} = |\vec{H}| \cdot \text{circumference}$$

2.6 Magnetization

Magnetic Dipole: A closed loop of current. $\vec{m} = IS\hat{n}$, where I is the current, S is the surface area of the loop, and $\hat{n}$ is given by the right hand rule.

Torque: Is given by

$$\vec{T} = \vec{m} \times \vec{B}$$

In other worse, $\vec{m}$ will rotate until it conforms to $\vec{B}$ so that torque is minimized. This leads to an internal magnetic field

$$\vec{B}_{material} > \vec{B}_{external}$$

Bound current density: $\vec{J}_{ms} = \vec{M} \times \hat{n}$
Magnetic field intensity: $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$
Magnetic susceptibility: $\vec{M} = \chi_m \vec{H}$
Relative permeability: $\mu_r = \chi_m + 1$
Much like with electrostatics, $\vec{H}$ only arises from **free currents**. $\vec{B}$ is the one that deals with both free and bound currents.

2.7 Material Properties: Atomic Magnetic Dipoles

1. **Non-zero moment** materials will have $\vec{M}$ align with $\vec{B}$. If strong alignment is achieved, then the field is enhanced (*ferromagnetism*. If weak alignment, then there will be moderate enhancement (*paramagnetic*).

2. **Net-zero moment** materials will have a small *reduction* in $\vec{B}$, leading to *diamagnetism*.

- **Diamagnetic:** $\mu_r \approx\leq 1$ (values like 0.999). $B_{internal} < B_{applied}$.
- **Paramagnetic:** $\mu_r \approx\geq 1$ (values like 1.01). $B_{internal} > B_{applied}$.
- **Ferromagnetic:** $\mu_r >>> 1$. $B_{internal} >>> B_{applied}$.
- **Ferrimagnetic:** $\mu_r > 1$. $B_{internal} > B_{applied}$.

2.8 Modeling Transformer Circuits

Electric	**Magnetic**
Voltages	$V_m = NI_0$
Current	Φ
Resistance	$R = \frac{L}{\mu S}$
Conductivity	Permeability $\mu = \mu_r \mu_0$

2.8.1 Mesh Analysis

2.9 Inductance