# ECE358: Foundations of Computing

Aman Bhargava

September-December 2020

# Contents

## 0.1 Introduction and Course Information

This document offers an overview of the ECE358 course. They comprise my condensed course notes for the course. No promises are made relating to the correctness or completeness of the course notes. These notes are meant to highlight difficult concepts and explain them simply, not to comprehensively review the entire course.

**Course Information**

- Professors: Prof. Andreas Veneris and Prof. Zissis Poulos

- Course: Engineering Science, Machine Intelligence Option

- Term: 2020 Fall

# Chapter 1

# Math Review

## 1.1 Logarithms

**Logarithm Rules:**

- **Definition:** $a = b^c$ iff $log_b a = c$.

- $a = b^{\log_b a}$.

- $\log_c(ab) = \log_c a + log_c b$.

- $\log_c(a^n) = n \log_c(a)$.

- $\log_b(a) = \log_a(b)$.

- $\log(1/a) = -\log a$.

- $\log(a/c) = \log a - \log c$.

- $a^{\log n} = n^{\log a}$.

**Variations on Logarithm**

$$\log^{(i)} n = \begin{cases} n & \text{iff } i = 0 \\ \log(\log^{(i)} n) & \text{otherwise} \end{cases} \tag{1.1}$$

$$\log^* n = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + log^*(\log n) & \text{otherwise} \end{cases} \tag{1.2}$$

## 1.2 Sequences and Series

**Theorem 1** *Fibonnaci Definition:*

$$F_i = F_{i-1} + F_{i-2} \tag{1.3}$$

*Where $F_0 = 0$, $F_1 = 1$.*

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \tag{1.4}$$

*Where $\phi = \frac{1+\sqrt{5}}{2}$, $\hat{\phi} = \frac{1-\sqrt{5}}{2}$*

**Theorem 2** *Arithmetic Series:*

$$\sum_{i=1}^{n} = \frac{n(n+1)}{2} = \Theta(n^2) \tag{1.5}$$

**Theorem 3** *Geometric Series:*

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} \tag{1.6}$$

**Theorem 4** *Infinite Series:*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \;\; iff \; |x| < 1 \tag{1.7}$$

**Theorem 5** *Telescoping Series:*

$$\sum_{i=1}^{n} a_i - a_{i-1} = a_n - a_0 \sum_{i=1}^{n} a_i - a_{i+1} = a_0 - a_n \tag{1.8}$$

## 1.3 Combinatorics

**Theorem 6** *Binomial Coefficient:*

$$(x + y)^r = \sum_{i=0}^{r} \binom{r}{i} x^i y^{r-i} \tag{1.9}$$

*Where $\binom{r}{i}$ is the binomial coefficient which equals*

$$\frac{r!}{i!(r - i)!}$$

# Chapter 2

# Asymptotics

**What are Asymptotics?** Analysis method for performance and complexity of an algorithm (space/memory, time/clock cycle complexity).

- Tight Bound: $\Theta()$.

- Worst Case: $O()$.

- Best Case: $\Omega()$

- Average & Expected Case: *Randomized + probabilistic analysis. Not a focus for the course.*

- Amortized Case: Discused later on. "Average worst case".

**Useful Facts**

- $\Theta$ bound is not always possible to find.

- **Transitivity:** if $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $f(n) = \Theta(h(n))$.

- **Symmetry:** $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

- **Transpose:** $f(n) = O(g(n))$ if and only if $g(n)\Omega(f(n))$.

- $n^a \in O(n^b)$ iff $a \leq b$.

- $\log_a(n) \in O(\log_b(n)) \,\forall a, b$.

- $c^n \in O(d^n)$ iff $c \leq d$.

- If $f(n) \in O(f'(n))$ and $g(n) \in O(g'(n))$ then

$$f(n) \cdot g(n) \in O(f'(n) \cdot g'(n))$$

$$f(n) + g(n) \in O(\max(f'(n), g'(n))$$

**Theorem 7** *Big O Definition: $f(n) = O(g(n))$ if and only if: There exists $c > 0$ and $n_0 > 0$ such that*

$$0 \le f(n) \le cg(n) \, \forall n \ge n_0 \tag{2.1}$$

**Theorem 8** *Big $\Omega$ Definition:*

**"Lower Bound" Big $\Omega$ Definition:** $f(n) = \Omega(h(n))$ *if and only if: There exists $c_1, n_1 > 0$ such that*

$$0 \le c_1 h(n) \le f(n) \, \forall n \ge n_1 \tag{2.2}$$

**Theorem 9** *Big $\Theta$ Definition:*

**"Tight Bound" Big $\Theta$ Definition:** $f(n) = \Theta(g(n))$ *if and only if: There exists $c_1, c_2, n_0 > 0$ such that*

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n) \, \forall n > n_0 \tag{2.3}$$

# Chapter 3

# Graphs

## 3.1 Definitions

**Theorem 10** *Graph Definition: A **graph** is a collection of **vertices** and **edges** represented by*

$$G = (V, E) \tag{3.1}$$

*Graphs can be:*

- *Directed or Undirected.*

- *Weighted or Unweighted.*

**Path:** A sequence of **edges** between adjacent vertices.

- **Simple Path:** No vertex is repeated in the path.

- **Cycle:** A path that starts and ends with the same vertex.

- A **connected** graph has a path between any two vertices (else it is **disconnected**)

**Bipartite Graphs:** A graph is **bipartite** if and only if vertices $V$ can be divided into $V_1, V_2$ such that:

1. $V_1 \cap V_2 = \emptyset$.

2. $V_1 \cup V_2 = V$.

3. Adjacencies exist only between elements and $V_1$ and $V_2$ (i.e., vertices within each set are disconnected from eachother, but are connected to elements from the other set).

**Vertex Degree**   is the number of **edges** adjacent to a vertex (includes incoming and outgoing edges).

- **In-degree** is the number of incoming edges.

- **Out-degree** is the number of outgoing edges.

**Clique:**   For all $v_1, v_2 \in V$, there exists an edge connecting them (i.e. a fully-connected set of vertices).

**Representations for Graphs:**   We either use an **adjacency matrix** or an **adjacency matrix**.

- **Adjacency List:**   Each element in the list represents a vertex and "has" a collection of **pointers** connecting them to other elements of the list.

  - **Time Complexity:**   $O(n)$ – Determining if $E_{v_1,v_2}$ exists would, at worst, require you to check $n$ pointers arising from vertex $v_1$ and/or $v_2$.
  - **Space Complexity:**   $O(|E|)$ – with worst case that $|E| = n^2$ for clique.

- **Adjacency Matrix:**   Matrix contains weights connecting $v_i$ to $v_j$ at index $M_{i,j}$.

  - **Time Complexity:**   $O(1)$ – Just check address $M_{i,j}$ and $M_{j,i}$.
  - **Space Complexity:**   $O(n^2$ every time.

## 3.2   Graph Traversals

*How do we choose an order to traverse all nodes in a graph?*

### 3.2.1   Breadth First Search

**Key Ideas:**

- From a given point, we traverse all immediate neighbours.

- Only then do we go to second-order neighbours of the original point.

- Implemented with a **Queue** (first-in-first-out).

**Breadth First Search Algorithm:**   Given $G = (V, E, s)$ where $s$ is the starting vertex,

1. Initialize $d[u] = \infty \; \forall u \in V$. This is the **distance** for each vertex.

2. Let $Q$ be an empty queue, and **enqueue** vertex $s$.

3. While $Q$ is not empty:

   (a) $u = \text{dequeue}(Q)$.

   (b) For $v \in adj(u)$:

      i. if $d[v] = \infty$, let $d[v] = d[u] + 1$ and **enqueue** $v \to Q$.

**Performance:**   Time complexity is $O(|V| + |E|)$.

## 3.2.2   Depth First Search

**Key Ideas:**

- **Input:**   $G = (V, E)$ (no source vertex this time).

- **Output:**

  1. $d[v]$: Time taken to **discover** vertex $v$ (in number of algorithm "steps").

  2. $f[v]$: Time taken to **finish** with vertex $v$ (meaning to exhaust all 'down stream' vertices).

  3. $\Pi[v]$: Predecessor of $v$ (i.e. the vertex immediately before it in the search).

**Color Coding:**

- **White:**   Node is as of yet undiscovered.

- **Gray:**   Node has been discovered, but the descendants have not all been discovered (node is not "exhausted").

- **Black:**   The algorithm has traversed the node AND all its descendants.

---

**Algorithm 1:** Depth First Search Algorithm.

---

    **Input**   : $G = (V, E)$

    **Output:** List of discovery times, finishing times, and predecessors
                 for each $v \in V$.

    color$[u]$ = White $\forall u \in V$;

    time = 0;

    **for** *each $u \in V$* **do**

        |   **if** *color$[u] = White$* **then**

        |   |   DFS-Visit$(u)$;

        |   **end**

    **end**

---

---

**Algorithm 2:** DFS-Visit Algorithm (subroutine for DFS Search)

---

    **Input**   : Vertex $u$

    **Output:** Recursively traverses nodes depth-first.

    color$[u] \leftarrow$ Gray;

    time+ = 1;

    $d[u]$ = time;

    **for** *each $v \in adj(u)$* **do**

        |   **if** *$v = White$* **then**

        |   |   DFS-Visit$(v)$;

        |   **end**

    **end**

    color$[u] \leftarrow$ Black;

    time+ = 1;

    $f[u] \leftarrow$ time;

---

**Edge Classification:**    **DFS** Produces one or more **trees**, leading to the following classification.

1. **Tree Edges:** Edges found during DFS exploration.

2. **Back Edge:** $(u, v)$ when $u$ is a descendant of $v$.

3. **Forward Edge:** $(u, v)$ when $v$ is a descendant of $u$ **but** $(u, v)$ is not a tree edge.

4. **Cross Edge:** Any other type of edge.

**Theorem 11** *White Path Theorem: $v$ is a descendant of $u$ if, and only if: At time $d[u]$, there exists a **white-only** path from $v \to u$.*

11

**Theorem 12** *Parenthesis Theorem:  For all $u, v \in V$: Exactly **one** of the following holds:*

1. *$d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$.  That is, **neither** $u, v$ **are descended from eachother**.*

2. *$d[u] < d[v] < f[v] < f[u]$. That is, $v$ is a descendant of $u$.*

3. *$d[v] < d[u] < f[u] < f[v]$. That is, $u$ is a descendant of $v$.*

**Theorem 13** *Depth First Search Edge Types:  The depth-first search of **undirected graph** $G$ yields only tree edges and back edges .*

### 3.2.3  Topological Sort

**Key Idea:**  All DAG (directed, acyclic graphs) have *some* amount of **implied order**. Think of it like a dependency graph. We use topological sort to turn **partial order** $\rightarrow$ **total order** (that is, an absolute list).

---
**Algorithm 3:** Topological Sorting Algorithm.

---
**Input**  : DAG $G = (V, E)$
**Output:** Total ordering of elements of graph.

Call Depth First Traversal($V, E$);
Return the vertices in **decreasing order** of **finish time**;

---

## 3.3   Minimum Spanning Trees

**Key Idea:**  Given a weighted graph $G = (V, E)$, we want a tree $G' = (E', V)$ where $E' \subseteq E$ such that there is a path between all vertices **and** $\sum_{e \in E'} e$ is minimized.

- $|E'| = |V| - 1$.

- There are **no cycles** (hence "tree").

- It may or may not be a **unique** solution.

### 3.3.1   Generic Minimum Spanning Tree Approach

**High-Level Description:**

- Set $A$ is instantiated as empty.

- Edges from the graph are added to $A$, ensuring **loop invariant** $A \subset$ Some Minimum Spanning Tree.

- Edge $(u, v)$ is **safe** iff $A \cup \{(u, v)\}$ is also a subset of some minimum spanning tree.

**Generic MST Algorithm:**   Given graph $G$ with weights $w$,

1. Let $A$ be an empty set.

2. While $A$ is not a spanning tree:

   (a) Find a **safe** edge $(u, v)$.
   (b) $A \leftarrow A \cup (u, v)$.

**Terminology:**   Relating to how to pick a **safe** edge.

- Let $S$ represent the set of vertices currently "covered" by edges in $A$, our current working minimum spanning tree.

- We can partition $V$ into set $S$ and $V - S$.

- Edge $(u, v)$ **crosses** the **cut** $(S, V - S)$ if one of $u, v$ is in $S$ and the other is in $V - S$.

- A cut **respects** $A$ iff no edge in $A$ **crosses** the cut.

- The **light edge** with respect to cut $(S, V - S)$ is the lowest weight edge that **crosses** the cut.

- **The light edge is the safe edge**.

### 3.3.2 Prims Algorithm

---

**Algorithm 4:** Prim's Algorithm

---

  **Input**   : Graph $G = (V, E)$, weights $w$, root node $r$.
  **Output:** Minimum spanning tree of $G$.

  $Q \leftarrow \{\emptyset\}$;
  **for** *each $u \in V$* **do**
     |  key$[u] \leftarrow \infty$;
     |  $\Pi[u] \leftarrow$ NIL;
     |  Insert$u \rightarrow Q$;
  **end**
  Set key of root node $r$ to 0 in $Q$.
  **while** $Q \neq \{\emptyset\}$ **do**
     |  $u \leftarrow$ Pop-Minimum$(Q)$;
     |  **for** *each $v \in Adj(u)$* **do**
     |     |  **if** *$v \in Q$ and $w(u, v) \leq key(v)$* **then**
     |     |     |  $\Pi[v] \leftarrow u$;
     |     |     |  Set the key of $v$ in $Q$ to $w(u, v)$;
     |     |  **end**
     |  **end**
  **end**

---

**Complexity:** Time complexity is $O(|E| + |V| \log |V|)$ when using Fibonnaci heaps.

## 3.4 Shortest Paths

**Goal:** Find path of minimum total weight between two vertices in a graph.

**Variants of Shortest Paths Problem:**

1. Single Source Shortest Paths (SSSP).

2. Single Destination Shortest Paths.

3. Single Pair – *most efficient method is to just run SSSP*.

4. All Pairs Shortest Paths.

**Key Algorithms:**

1. Dijkstra's – for no negative weights.

2. Bellman-Ford – when there are negative weights.

3. Difference Constraints – a surprising application of shortest paths.

# Chapter 4

# Trees

## 4.1 Definitions

**Theorem 14** *Tree Definition: A **tree** is a **connected, acyclic, undirected** graph.*

- ***Root node** is usually defined.*

- ***Parent node** is the 'next node up' going toward the root.*

- ***Child node** is one of the 'next node(s) down' going away from the root.*

- ***Binary tree:*** *$\leq 2$ children per node.*

- ***Depth of node:*** *Length of path from **root** $\rightarrow$ **node**.*

- ***Height of node:*** *Number of edges on the **longest path** from the node $\rightarrow$ leaf.*

- ***Complete** $k$-**ary tree:*** *Every internal node has $k$ children and all leaves are at the same depth.*

**Theorem 15** *One-for-all Tree Theorem: **If one of these is true, they all are:***

- *$G$ is a tree.*

- *Every pair of vertices $v_1, v_2 \in G$ is connected by a **unique, simple path**.*

- *$G$ is disconnected, but becomes disconnected when one edge is removed.*

- *G is connected with $|E| = |V| - 1$.*

- *G is acyclic.*

- *If an edge is added G **becomes cyclic**.*

## 4.2   Binary Search Trees

**Theorem 16** *Binary Search Tree Definition: For any node x of a binary search tree:*

- *If $y \in$ left sub-tree: $y.key \leq x.key$.*

- *If $y \in$ right sub-tree: $y.key \geq x.key$.*

**Traversals:**

1. **In-Order:**   LNR – Actually prints them from least to greatest.

2. **Pre-Order:**   NLR

3. **Post-Order:**   LRN

### 4.2.1   Querying Binary Search Trees

We can perform minimum, maximum, successor, predecessor in $O(h)$ time where $h$ is the height of the tree.

**Tree Search:**     Given a node $x$ and value we are looking for $k$:

1. If $x$.key is $k$ or $x$ is NIL: return $x$. *We either found it or it wasn't there.*

2. If $x$.key $< k$: Run **Tree Search** on the left sub-tree.

3. Otherwise, run **Tree Search** on the right sub-tree.

**Iterative Tree Search:**     Same input/output as above: **Repeat while $x$.key is not $k$ and NIL**

1. If $x$.key $< k$: $x = x$.left.

2. Else, $x = x$.right.

3. After break condition above, return $x$.

**Minimum Tree Search:** To find the minimum, keep setting $x = x$.left while $x$.left isn't NIL.

**Tree Successor:** To find the successor of node $x$:

1. If $x$ has a right child: Call **Tree Minimum** on $x$.right.

2. Else:

    (a) $y = x$.parent
    (b) While $y \neq$ NIL and $x == y$.right:
        i. $x = y$
        ii. $y = y.parent$
    (c) return $y$

**Tree Insert:** To insert $z$ into BST $x$, repeatedly compare $z$ to nodes of $x$, moving right/left according to result of comparison. Eventually, you will add $z$ as a leaf node.

**Tree Deletion:** To delete node from $z$ from BST:

1. $z$ has **no children** $\rightarrow$ modify $z$.parent.child = NIL.

2. $z$ has **one child** $\rightarrow$ modify z.parent.child $\leftarrow$ z.child.

3. $z$ has **two children**:

    (a) Let $y =$ Tree Successor($z$) [must be from right sub-tree of $z$]
    (b) Convenient case: $y = z$.right. In this case, just replace $z$ with $y$.
    (c) Replace $z \leftarrow y$, ensuring $y$ keeps its right sub-tree.
    (d) $z$.right $\rightarrow y$.right.
    (e) $z$.left $\rightarrow y$.left.

## 4.3   Red-Black Trees

**Theorem 17** *Red Black Tree Definition: A red-black tree is a **binary search tree** where each node has a "color" of either **red or black**.Each BST operation has additional rules that relate to color, with the end results being:*

- *No path from the **root** to a **leaf** is more than **two times the shortest path**.*

- *The tree is approximately balanced and therefore is efficient.*

**Properties of Red-Black Tree**

1. All nodes are either **red or black**.

2. The root node is black.

3. All leaves are black.

4. Red nodes have black children.

5. All simple paths from a node to descendant leaves have **the same numberof black nodes**.

**Resultant Properties**

1. $bh(x)$ is the **"black height"** of node $x$ (numberof black nodes from (but not including) $x$ to a leaf.

2. Red-black tree with $n$ internal nodes has height $\leq 2\log(n+1)$.

3. All leaves are set to $T.nil$ values instead of regular NIL.

**Easy Red-black Tree Operations:** These are exactly the same as BST operations. They all run in $O(\log n)$ time, too.

1. Search.

2. Minimum.

3. Maximum.

4. Successor.

5. Predecessor.

**Rotations:** Rotations swap two nodes while maintaining the BST properties. You can visualize or "feel" it as taking the edge connecting two nodes and physically **twisting** it either right or left for their respective rotation.

**Red-Black Insert Fixup:** This algorithm reinstates the red-black properties of the tree around a given recently-inserted node $z$.

**Red-Black Insertion:**     To insert node $z$...

1. Perform conventional BST insertion of $z$, setting the new node's children to $T$.nil instead of NIL.

2. Set $z$'s color to RED.

3. Call **Red-Black Insert Fixup**$(T, z)$.

**Red-Black Transplant:**     Same function as before, just replace 'NIL' with $T$.nil.

**Red-Black Deletion:**     Deletion is substantially more involved and uses a particular **Red-Black Delete Fixup** function.

# Chapter 5

# Proof Methods

## 5.1   Induction

**Basic Idea:**   To prove by induction, we show that the statement holds for some base case $n = 1$, then show that the statement holding for aritrary $n$ implies that the statement holds for $n + 1$. That concludes the proof.

**Basis:**   We prove that the statement holds for some set values of $n$ (usually $n = 1$ or $n = 0, 1, ..., 4$).

**Hypothesis:**   We hypothesize that the thing we are trying to prove is true.

**Inductive Step:**   We "plug in" the $n + 1$ case to the hypothesis and show that it boils down to the $n$ case and some algebraic equivalence. Then you can make the claim that it holds for all $n$.

## 5.2   Contradiction

**Basic Idea:**   Given a true or false proposition $P$, we assume that $\neg P$ ("not $P$") holds. After some clever algebra and symbol-shunting, we get a contradiction. This implies that $P$ must hold instead.

**Illustrative Example:**   $P$: If $x^2 - 5x + 4 < 0$, then $x > 0$.

- To prove $P$, we **assume towards a contradiction** (ATaC) that $\neg P$ holds. That is, we assume that if $x^2 - 5x + 4 < 0$ then $x \leq 0$.

- But then:

$$x^2 < 5x - 4$$
$$x^2 < 0 \qquad (5.1)$$

Results in a contradiction! Therefore $P$ must hold.

## 5.3   Master Theorem

This provides a "hammer" to prove the time complexity of recurrent functions.

**Theorem 18** *The Master Theorem: Let $a \geq 1$ and $b \geq 1$ and $f(n)$ be some function.*

$$T(n) = aT(\frac{n}{b}) + f(n) \qquad (5.2)$$

**Case 1:**    $f(n) = O(n^{\log_b a - \epsilon})$ *for some $\epsilon > 0$. Then*

$$T(n) = \Theta(n^{\log_b a})$$

**Case 2:**    $f(n) = \Theta(n^{\log_b a})$. *Then*

$$T(n) = \Theta(n^{\log_b a} \log n)$$

**Case 3:**    $f(n) = \Omega(n^{\log_b a + \epsilon})$ *for some $\epsilon > 0$ **AND** $af(n/b) \leq cf(n)$ for $0 < c < 1$. Then*

$$T(n) = \Theta(f(n))$$

.

## 5.4   Substitution

**Substitution** is a method for determining the **closed from** runtime of an algorithm via induction.

**Example – Mergesort:**    The runtime for mergesort is recursively defined as $T(n) = 2T(\lceil n/2 \rceil) + n$.

1. We start by **guessing** $T(n) = O(n \log n)$.

2. We **hypothesize** that $T(n) = O(n \log n)$ for all cases $\leq n$, meaning that
$$T(n/2) \leq c\lfloor n/2 \rfloor \log \lfloor n/2 \rfloor$$

3. **Inductive step:** We prove that
$$T(n) \leq cn \log n$$

Which is pretty obvious from there.

# Chapter 6

# Sorting Algorithms

## 6.1 Mergesort

df

## 6.2 Heapsort

df

### 6.2.1 Heaps

**Theorem 19** *Heap Definition: A heap is an **array** representing a **binary tree** with the following properties:*

- *For **max heap**: $A[parent(i)] \geq A[i]$.*

- *For **min heap**: $A[parent(i)] \leq A[i]$.*

- ***Height** of tree is $\Theta(\log n)$ (i.e. number of edges to get from root to leaf node).*

*Parent-child relationships are as follows:*

- *$Parent(i) = \lfloor i/2 \rfloor$ (Parent node of $i$).*

- *$Left(i) = 2i$ (Left child).*

- *$Right(i) = 2i + 1$ (Right child).*

## 6.2.2   Heap Algorithms

**Max-Heapify** is an algorithm that sorts array $A$ into a heap at index $i$, assuming that right and left sub-trees satisfy the heap property at $i$.

---
**Algorithm 5:** Max Heapify

    **Input**   : Array $A$, index $i$. Left and right sub-trees of $i$ should already satisfy heap property.

    **Output:** $A$ is sorted in place to satisfy heap property.

    $l = \text{left}(i)$;
    $r = \text{right}(i)$;
    Let largest $= \max(l, r, i)$.
    **if** $i \neq largest$ **then**
        Exchange $i$, largest;
        MaxHeapify($A$, largest);
    **else**
    **end**

---

**Algorithm Complexity:**   $O(1)$ Space complexity (sorts in place), $O(\log n)$ time complexity.

**Build Heap**   is an algorithm that sorts a completely unsorted array $A$ into a heap.

---
**Algorithm 6:** Build Heap

    **Input**   : Unsorted list $A$

    **Output:** Heap $A$

    **for** $i = \lfloor |A|/2 \rfloor : 1$ **do**
        MaxHeapify($A, i$);
    **end**

---

**Algorithm Complexity:**   $O(1)$ Space Complexity, $O(n)$ time complexity.

**Heap Sort**    is a sorting algorithm that runs in $O(n \log n)$ time.

---
**Algorithm 7:** Heap Sort Algorithm

---
**Input**   : Unsorted array $A$
**Output:** Sorted array $A$

**for** $i = |A| : 2$ **do**
   | exchange $A[i]$, $A[1]$;
   | $A$.heapsize -= 1;
   | MaxHeapify($A, 1$);
**end**

---

**Algorithm Complexity:**    $O(1)$ Space Complexity, $O(n)$ time complexity.

## 6.3  Quicksort

**Key Facts:**    Quick sort sorts in $\Theta(n^2)$ in the worst case and $\Theta(n \lg n)$ in the average case. The factors on the average case are small, so it ends up being very practical.

**Algorithm Description:**    Given algorithm $A$, we:

1. Divide $A$ into two **partitions**. We usually select the last element as the partition.

2. We call the partition function to put all values less than or equal to the pivot before the pivot and all values greater than the pivot after the pivot in $\Theta(n)$ time.

3. Then we call the partition function on the two remaining partitions recursively.

4. We end with a sorted list.

---

**Algorithm 8:** Partition Subroutine for Quicksort ($\Theta(n)$)

**Input** : Array $A$, partition start index $r$, partition end index $r$.
**Output:** $A$ partitioned about $A[r]$, index of partition returned.

$x = A[r]$ $i = p - 1$ **for** $j = p : r - 1$ **do**
   | **if** $A[j] \leq x$ **then**
   |   | i = i+1 exchange $A[i], A[j]$
   | **end**
**end**
exchange $A[i + 1], A[r]$ return i+1

---

**Algorithm 9:** Quicksort Algorithm.

**Input** : Unsorted $A$, partition start index $p$, end index $r$
**Output:** Sorted $A$

**if** $p < r$ **then**
   | q = Partition($A, p, r$);
   | Quicksort($A, p, q - 1$);
   | Quicksort($A, q + 1, r$);
**end**

---

**Algorithm Complexity:** $\Theta(n^2)$ worst case, $O(n \log n)$ average case with small coefficients.

## 6.4   Linear Time Sort

### 6.4.1   Counting Sort

**Key Idea:** If we know that the input are integers and can be used as the address in an array, we can go through one-by-one and count the number of each possible element. This requires a lot of space, though, if the range of

integers is high.

---
**Algorithm 10:** Counting Sort Algorithm.

---
**Input** : Input array $A$, output array $B$, upper bound on integers $k$
**Output:**

$C$ = array of zeros of size $k + 1$;
**for** $j = 1 : |A|$ **do**
$\quad|\quad C[A[j]]$++;
**end**
**for** $i = 1 : k$ **do**
$\quad|\quad C[i]$ += $C[i] + C[i - 1]$;
**end**
**for** $j = |A| : 1$ **do**
$\quad|\quad B[C[A_j]] = A[j]$;
$\quad|\quad C[A_j]$–;
**end**

---

**Time Complexity:** $\Theta(k + n)$.

**Stability:** The order of the entries with the same values is maintained from input to output. Relevant for **radix sort**, and for when "satellite" data is carried with "keys" used in sort.

## 6.4.2 Radix Sort

**Algorithm Idea:** Start by sorting based on least significant digit then the most. This, again, only works for integer sorting or similar. We make use of a **stable sort** (usually **counting sort**).

---
**Algorithm 11:** Radix Sorting Algorithm.

---
**Input** : Integer array $A$, number of digits in each number $d$
**Output:** Sorted array $A$

**for** $i = 1 : d$ **do**
$\quad|\quad$ StableSort $A$ based on digit $i$;
**end**

---

**Algorithm Complexity:** $O(d(n + k))$ time complexity, assuming that StableSort is $O(n + k)$.

For $b$-bit numbers and any $r \in [0, b]$:

$$\text{Radix Sort} = \Theta((\frac{b}{r}(n + 2^r)))$$

## 6.5 Order Statistics

**Theorem 20** *Order Statistic Definition: The **Order Statistic** of set $A$ is the $i$th smallest element of the set. So,*

- *$\max A$ is the nth order statistic of $A$.*

- *$\min A$ is the 1st order statistic of $A$.*

- *Median $A$ is the $\lfloor \frac{n+1}{2} \rfloor$ order statistic.*

### 6.5.1 Minimum and Maximum Algorithms

Minimum and maximum algorithms just iterate through to find min or max.

**Finding minimum and maximum efficiently:** Iterate through, but compare **next two elements** at a time. Compare the larger element to the current maximum and the smaller of the two to the current minimum.

- **Even length array:** $3n/2 - 2$ comparisons.

- **Odd length array:** $3\lfloor n/2 \rfloor$ comparisons.

## 6.5.2 Selection in Average Linear Time

**Outcome:** It is possible to find the $i$th smallest element in average case $O(n)$ time (worst case $\Theta(n^2)$).

---

**Algorithm 12:** Randomized Select Algorithm

**Input** : Array $A$, partition bounds $p, r$, desired order statistic $i$
**Output:** The $i$th order statistic of $A$

**if** $p = r$ **then**
  |   return $A[p]$;
**end**
$q = \text{RandomizedPartition}(A, p, r)$;
$k = q - p + 1$;
**if** $i = k$ **then**
  |   return $A[q]$;
**else**
  |   $i < k$
**end**
return $\text{RandomizedSelect}(A, p, q - 1, i)$;
return $\text{RandomizedSelect}(A, q + 1, r, i - k)$;

---

## 6.5.3 Selection in Worst Case Linear Time

**Outcome:** This algorithm can select the $i$th order statistic in $O(n)$ time.

**Algorithm:**

1. Partition $A$ into $\lfloor n/5 \rfloor$ groups of 5 and one "extras" group.

2. Find median of each group of find using **insertion sort**.

3. Let $x = \textbf{Select}(\text{medians}, \lfloor number of groups/2 \rfloor)$.

4. Let $k = \textbf{Partition}(A, 0, |A|, \text{pivot} = x)$. $k$ is now the number of elements larger than $x$.

5. If $i = k$ then return $x$. Else, if $i < k$ then return $\textbf{Select}(A[0 : k], i)$. Otherwise, return $\textbf{Select}(A[k:], i-k)$.

By substitution, we can show that the algorithm runs in $O(n)$.

# Chapter 7

# Hash Tables

Perform well for tasks like **insert, search, and delete**. $\Theta(n)$ worst case for search, $O(1)$ is the practical average.

## 7.1 Direct Addressing

- Each element we want to store has a **key** k in **universe** $U$.

- $|U| = m$, $U = \{0, 1, ..., m - 1\}$.

- $T[0, ..., m - 1]$ is the **direct address table**.

**Direct Address Search**  $(T, k)$: return $T[k]$.

**Direct Address Insert**  $(T, x)$: $T[x.\text{key}] = x$.

**Direct Address Delete**  $(T, x)$: $T[x.\text{key}] = \text{NIL}$.
   The problem arises when the universe $U$ gets large. This happens when you have a ton of unique elements.

## 7.2 Hash tables

**Theorem 21** *Hash Function Definition: A hash function h maps the latent universe of keys $U$ to a universe of some specific size m. That is,*

$$h : U \rightarrow \{0, ..., m - 1\} \tag{7.1}$$

**Collision Resoution with Chaining:**     Just add to a linked list.

1. **Insertion:**   $x$ is added to the **head** of the linked list.

2. **Search:**   Search for element in the linked list that it hashes to.

3. **Delete:**   Delete from linked list as usual.

**Performance:**     If the number of elements $n = O(m)$, you'll get, on average, $O(1)$ performance.

**Simple Uniform Hashing:**     Each element has same probability of hashing to given value.

# 7.3   Hash Functions

Here are a few types of commonly-used hash functions. Note that most datatypes (strings, characters, etc.) can be converted to natural numbers by place value. They just might be very large.

## 7.3.1   Divisision Method

To map key $k$ to one of $m$ slots, we can let

$$h(k) = k\%m \tag{7.2}$$

**Considerations:**

- $m$ should not be $2^n$ because then it'd just extract bottom $n$ bits.

- Usually good to pick prime number $m$.

## 7.3.2   Multiplication Method

We multiply $k$ by some $A \in [0, 1]$ then use division method.

$$h(k) = \lfloor m(kA\%1) \rfloor \tag{7.3}$$

**Considerations:**

- Value of $m$ does not matter any more ($2^n$ makes for easy hardware implementation.

- $A \approx \frac{\sqrt{5}-1}{2}$ tends to work well.

### 7.3.3   Open Addressing

The key idea is to let $h()$ take in a "probe number" $i$. We increment this number to make sure hash table gets filled without doubling up on a single slot.

- **Probe sequence** of $k$ is $\langle h(k, 0), ..., h(k, m-1)\rangle$.

- The probe sequence for each $k$ is a **permutation of** $\langle 0, ..., m-1\rangle$.

- Because of this property, we can always fill up the table perfectly.

**Insertion:**   Probe table, incrementing $i$ until you reach an empty slot. Add the element there.

**Search:**   Keep probing until you either find the element **or you find NIL**.

**Deletion:**   When you find value, set it to DELETED – NIL would stop searches that would have to traverse.

**Performance of Open Addressing:**

- If $n/m \leq 1$: The expected number of probes for an **unsuccessful** search is

$$\gamma \leq \frac{1}{1 - \frac{n}{m}}$$

- Inserting to open address table also takes $\gamma$ probes on average.

- Number of probes for a successful search is

$$\leq \frac{1}{\alpha} \ln \frac{1}{1 - \alpha}$$

  Where $\alpha = n/m \leq 1$.

**Linear Probing:**   If $h'(k)$ is single-variable hash function, we can make a probing hash function via:

$$h_{lin}(k, i) = (h'(k) + i)\% m \tag{7.4}$$

**Drawback:** Long strings of filled slots build up one after another.

**Quadratic Probing:** $h'(k)$ is a single-variable hash function:

$$h(k, i) = (h'(k) + c_1 i + c_2 i^2)\% m \tag{7.5}$$

*Where $c_1, c_2 \geq 0$, $i \in [m - 1]$.*

- Better performance than linear.

- We still struggle with **secondary clustering** – if $h'(k_1) = h'(k_2)$, we still have the same hash sequences.

**Double Hashing**

$$h(k, i) = (h_1(k) + i h_2(k))\% m) \tag{7.6}$$

Now probe sequences are highly unlikely to overlap for different keys. **CONDITIONS:**

- $h_2(k)$ should be "relatively prime" to $m$. The entire should be searched by multiples of each $h_2$.

- **Method 1:** $m = 2^x$, make sure $h_2(k)$ maps to an odd number.

- **Method 2:** $m$ is prime, make sure $h_2(k)$ maps to positive inteer less than $m$.

# Chapter 8

# Dynamic Programming

**Motivation:** We often solve the same sub-problem many times, particularly during recursive optimization algorithms.

**Solution:** Memorize the intermediate optimization answers.

## 8.1 Elements of Dynamic Programming

1. Optimal Substructure.

2. Overlapping sub-problems.

**Optimal sub-structure:** Optimal solution is $f(\text{input}, \text{optimal solutions to sub-problem})$. A common pattern is:

1. Make some choice (hopefully optimal choice).

2. Realize you need to solve basically the same problem (but smaller).

3. Observer recurrence.

4. Prove **by contradiction** that optimal sub-solution leads to the optimal solution.

**Tips and Tricks**

- Keep sub-problem space as simple as possible.

- Think through sequential alternatives you might try to make things countable.

- You can try to create a sub-problem graph, too.

- **PRACTICE**.

## 8.2 Examples

*High-level descriptions of dynamic programming algorithms.* It seems that most problems are variations on a smaller set of dynamic programming problems.

### 8.2.1 Rod Cutting

**Given:** A rod of length $n$ and a vector $p = [15, 20, 31, ..., 100]$ representing selling price for different lengths of rod.

**Wanted:** Maximum price to sell the rod for given optimal cutting.

**Dynamic Programming Solution:**

- The first step is to determine where to place your first cut.

- To decide between a 1-unit long cut and a $p[10]$ unit long cut, you would want to know the reward from each possible cut AND the maximum possible reward for what remains.

- Therefore,

$$\text{optim}(n, p) = \max\{p_n, p_{n-1}+\text{optim}(1, p), ..., p_1+\text{optim}(n-1, p)\} \quad (8.1)$$

- With a general recursive strategy, we would end up calculating the same $\text{optim}(n, p)$ many time.

- Therefore, as soon as we calculate some $\text{optim}(n, p)$, we store it in a hash table (called **memoization**.

### 8.2.2 Matrix Chain Multiplication

### 8.2.3 Longest Common Subsequence

### 8.2.4 Knapsack Problem

### 8.2.5 Huffman Codes

# Chapter 9

# Greedy Algorithms

## 9.1 Introduction

The essence of a greedy algorithm is to select the **locally optimal** decision at every opportunity. It is generally efficient, and when it is optimal, is a very convenient choice. Greedy algorithms approximate NP-complete problems well.

## 9.2 Examples

### 9.2.1 Class Scheduling

# Chapter 10

# Amortized Analysis

**What is amortized analysis?**

- We are often interested in the **aggregate** cost over a sequence of operations.

- There are often connections between the number of one operation and another.

  - For instance, you can only pop as many items as you have pushed to a stack.

- Amortized analysis guarantees the **average performance for the worst case** via *deterministic analysis*.

## 10.1  Aggregate Method

1. Define an $O(1)$ operation for the problem.

2. Observe the worst-case number of operations for $n$ executions of a process.

3. Divide the cost for $n$ executions by $n$ to get amortized cost.

**Example: Stack**

- Push and pop are $O(1)$.

- Multipop is $O(n)$ – this is when you pop every element.

- Therefore, for $n$ operations, we have at most $n \cdot O(n)$, or $n$ multipops.

- **Aggregate:**

  - We cannot pop more items than have been pushed.
  - The maximum number of items pushed is $n$.
  - Therefore the average worst case cost for $n$ operations is $O(n)$, meaning that we have average $O(1)$ for each operation.

## 10.2   Accounting Method

**Central Ideas:**

- We charge some number of **dollars** for each operation. This is the **amortized cost**.

- The datastructure gets those **dollars** to complete the operation.

- We define some "real cost" of each operation. The exact dollar value doesn't matter, the important thing is **whether it is a function of $n$ or just constant**.

- If our imposed **amortized cost** is more than the "real cost", the data structure gets **credit**.

  - Else, the data structure **uses credit** it has stored to complete the operation.
  - **NEGATIVE BALANCE MEANS A WRONG CREDIT ALLOCATION**.

## 10.3   Splay Trees

**Motivation:**     We need efficient insertion, deletion, search, and sorting.

- If key access frequencies are known, we can use **dynamic programming**.

- if the key frequencies are unknown, we use **splay trees**.

**Splay trees are theoretically optimal** in the amortized case.

**Key Idea:** We simply use a conventional BST, but introduce a **splay** operation.

- An unbalanced tree is **credit rich**.

- A balanced tree is not credit rich.

- Credit is spent to make the tree more balanced, generally.

---

**Algorithm 13:** Splay Algorithm

**Input** : Node $x$ of splay tree.
**Output:** Tree is splayed (re-balanced) in place.

**while** $x \neq root$ **do**
    **if** $P(x) = root$ **then**
        rotate $P(x)$;
    **end**
    **if** $P(x)$ *and* $x$ *are both (left or right) children* **then**
        rotate $P(P(x))$;
        rotate $P(x)$;
    **else**
        rotate $P(x)$;
        rotate new $P(x)$;
    **end**
**end**

---

**Cost of Splay:**

- $wt(x)$ is defined as the **number of nodes** in the sub-tree rooted at $x$ (descendants of $x$ including $x$).

- $\text{rank}(x) = \lceil \log wt(x) \rceil$ is the **height** of sub-tree rooted at $x$.

- **Credit Invariant:** Node $x$ has \$ $\text{rank}(x)$ in credit.

- Therefore, each {zig, zig-zig, zig-zag} costs

$$3[\text{new rank}(x) - \text{old rank}(x)]$$

- The amortized cost of **splay** is therefore

$$O(\log n)$$

**Splay Tree Operations:**

1. **Search:**

   (a) BST Search for $x$.

   (b) **Splay**$(x)$.

   (c) *Cost:* $O(\log n)$

2. **Insert:**

   (a) BST Insert $x$ ($x$ is now a leaf).

   (b) **Splay**$(x)$.

   (c) *Cost:* $O(\log n)$

3. **Delete**

   (a) BST Delete $x$.

   (b) **Splay**$(P(x))$.

   (c) *Cost:* $O(\log n)$

4. **Split:** If you want to create two new sub-BST's, one with all values less than $x$ and one with values greater than $x$,

   (a) Perform **Splay Search**$(x)$ – $x$ is now the root.

   (b) Split off the right and left trees.

   (c) *Cost:* $O(\log n)$

5. **Join:** Basically the reverse operation of **split**. Given two splay trees, one with values $\leq x$ and one with values $> x$,

   (a) Make the $\leq x$ tree the left child of $x$, the $> x$ the right child of $x$.

   (b) **Credit the tree** $O(\log n)$ dollars so that $x$ satisfies the credit invariant.

   (c) *Cost:* $O(\log n)$