MECH 4110 - Pre-lab #2

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1 Linearization

Linearize Prelab 2 Equation (1.3.16) about a general operating point ($\bar{h}_1$, $\bar{v}_p$). First, write Prelab 2 Equation (1.3.16) in the form of $\dot{h}_1 = f(h_1, v_p)$.

$$\dot{h}_1 = \frac{K_p}{A_c}(v_p - v_p^{min}) - \frac{C_d A_o}{A_c}\sqrt{2gh_1} \tag{1.1}$$

We can linearize the above equation with a Taylor Series Expansion. Evaluate the derivatives at the operating point ($\bar{h}_1$, $\bar{v}_p$).

$$\Delta\dot{h}_1 = \frac{df}{dh_1}\Delta h_1 + \frac{df}{dv_p}\Delta v_p \tag{1.2}$$

where,

$$\left.\frac{df}{dh_1}\right|_{\bar{h}_1,\bar{v}_p} = -\frac{C_d A_o g}{A_c\sqrt{2g\bar{h}_1}} \tag{1.3}$$

$$\left.\frac{df}{dv_p}\right|_{\bar{h}_1,\,\bar{v}_p} = \frac{K_p}{A_c} \tag{1.4}$$

Therefore, the linearized system is,

$$\Delta\dot{h} = -\frac{C_d A_o g}{A_c\sqrt{2g\bar{h}_1}}\Delta h_1 + \frac{K_p}{A_c}\Delta v_p \tag{1.5}$$

2 Open Loop Transfer Function

Using Equation (1.5) we can find the open loop transfer function. We have,

$$\dot{y} + \frac{C_d A_o g}{A_c \sqrt{2g\bar{y}}} y = \frac{K_p}{A_c} u \tag{2.1}$$

$$A_c \sqrt{2g\bar{y}} \dot{y} + C_d A_o g y = K_p \sqrt{2g\bar{y}} u \tag{2.2}$$

$$\frac{A_c \sqrt{2g\bar{y}}}{C_d A_o g} \dot{y} + y = \frac{K_p}{C_d A_o g} \sqrt{2g\bar{y}} u \tag{2.3}$$

Take the laplace transform of both sides

$$Y(s) \left(\frac{A_c \sqrt{2g\bar{y}}}{C_d A_o g} + 1 \right) = \frac{K_p}{C_d A_o g} \sqrt{2g\bar{y}} U(s) \tag{2.4}$$

Therefore,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{K_p}{C_d A_o g} \sqrt{2g\bar{y}}}{\frac{A_c \sqrt{2g\bar{y}}}{C_d A_o g} + 1} \tag{2.5}$$

Compare Equation (2.5) to Prelab 2 Equation (1.3.6). Clearly,

$$\tau = \frac{A_c \sqrt{2g\bar{y}}}{C_d A_o g} \tag{2.6}$$

$$K_{DC} = \frac{K_p}{C_d A_o g} \sqrt{2g\bar{y}} \tag{2.7}$$

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ans =
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$$\frac{K_{\mathrm{DC}}\ (k_i + k_p\, s)}{s + K_{\mathrm{DC}}\, k_i + s^2\, \tau + K_{\mathrm{DC}}\, k_p\, s}$$

Figure 1: Output of Listing 1

3 Closed Loop Transfer Function

Let $G(s) = \frac{K_{DC}}{\tau s+1}$, $C(s) = \frac{k_p s+k_i}{s}$. G is the open loop transfer function derived in section 2, C is a PI controller. We can find the closed loop transfer function with

$$\frac{Y}{R} = \frac{GC}{1+GC} \tag{3.1}$$

Using MATLAB to compute the expression, we get

$$\frac{Y}{R} = \frac{K_{DC}(k_i + k_p s)}{\tau s^2 + (K_{DC}k_p + 1)s + K_{DC}k_i} \tag{3.2}$$

Listing 1: Determination of Closed Loop Transfer Function

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syms k_p k_i K_DC s tau

C = (k_p*s + k_i)/s;
G = (K_DC) / (tau * s + 1);

closed = C*G / (1 + C*G);

simplify(closed)
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4 Determination of k_i, k_p.

Compare the denominator of Equation (3.2) to Prelab 2 Equation (1.3.10). We find that,

$$\omega_n^2 = \frac{K_{DC}k_i}{\tau} \Rightarrow k_i = \frac{\omega_n^2}{K_{DC}}\tau \tag{4.1}$$

$$2\zeta\omega_n = \frac{K_{DC}k_p + 1}{\tau} \Rightarrow k_p = \frac{2\tau\zeta\omega_n - 1}{K_D C} \tag{4.2}$$