Systems Dynamics Modeling and Analysis

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January 19, 2021

Part I

Week 1

LECTURE

1

INTRODUCTION

1.1 Syllabus and Textbook

Instructor Information:

1. Professor Email: Justin.Koeln@UTDallas.edu
2. Office hours by appointment
3. TA Email Sahand.HadizadehKafash@UTdallas.edu
4. Office hours Friday 1:00 pm - 3:00 and by appointment

Textbook: System Dynamics, William Palm 3rd Edition. Chapters 1-9 (not in that order), balance of math, theory, modeling, and application.

1.2 Homework

Approximately one assignment per week, due Tuesday before class. No late credit for homework and the lowest score will be dropped. Points will be assigned based on the rubric. Each lecture will have a quiz associated with it, due before the start of next class. The quiz is meant to test your understanding of the material.

Table 1.1: Grading Criteria

Homework	30%
Participation	10%
Midterm	15% each
Final	30%

1.3 Exam Schedule

- Exam 1: Week of Thursday, Feb. 25
- Exam 2: Week of Thursday, Apr. 8
- Final Exam: Week of Monday, May 10

Open book, open notes exams with a time limit. A calculator is allowed and a formula sheet is provided. Additionally, a sheet of equations that we are expected to know if provided as well.

1.4 Quiz 1

1. What year are you in school?
2. Have you taken systems and controls?
3. Do you have access to Matlab/Simulink
4. What is the most important thing you want to remember from this lecture?
5. What are your goals for the course? ..etc

1.5 Systems

A system is a combination of elements intended to act togehter to accomplish an objective. A systems point of view focuses on how the connections between elements influence overall behavior. We can accept a less-detailed description of individual elements to understand the entire system. We want to look at the general behavior of the system as it evolves over time.

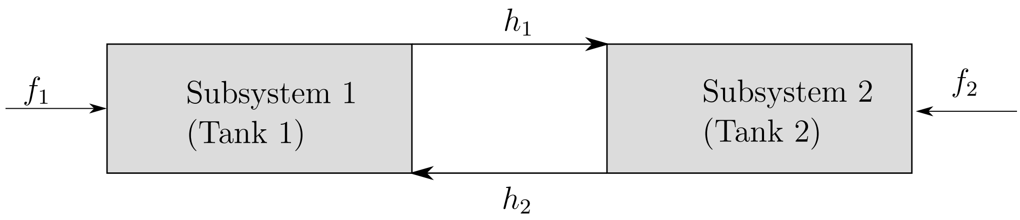

Figure 1.1: Subsystem Interaction

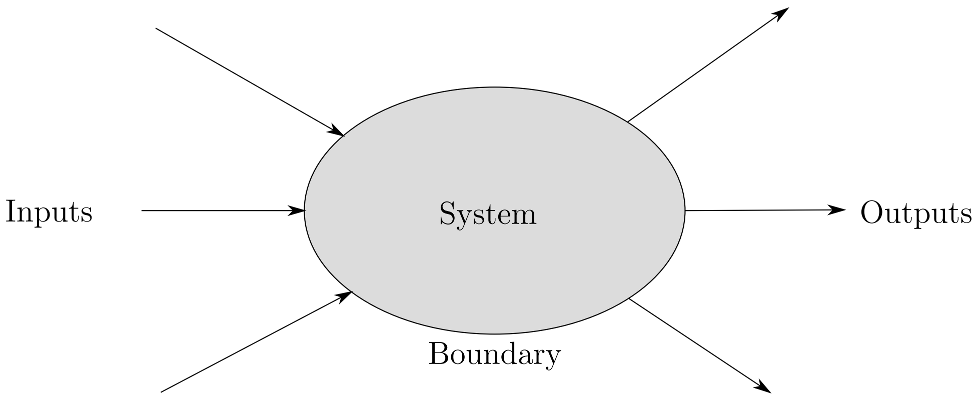

Figure 1.2: Input-Output Relationship

1.6 Inputs and Outputs

1. Input is a cause u.
2. Output is an effect y.
3. System dynamics are goverened by input-output relationships $y = f(t, u)$.

What is $y = f(t, u)$. and how can we approximate it?

1.7 Dynamics

Mechaincal Engineers study how things behave as a function of location with PDE's.

$$\frac{\partial g(x)}{\partial x} = f(x).$$

We are more interested in how a system behaves as a function of time, which gives us ODEs $\frac{dx}{dt} = f(x)$. A static relationship means that the current output only depends on the current input (no dynamics, algebraic relationsship) $y = f(u(t))$. A dynamic relationship means that the current output depends on past inputs. Check out 3b1b video on differential equations.

$$\frac{dx}{dt}(t) = f(x(t), u(t)), \quad y(t) = g(x(t)).$$

1.8 Modeling

A model is a mathematical description of a systems behavior as a function of time. Modeling is to understand the problem, apply simplifying assumptions, and apply appropriate fundamental principles. We need to be able to solve that model for the behavior of that system. This can be done by hand for simple systems. If the model is too complex, a numerical approach may be required. Our goal is to make the model simple enough to work with but realistc enough to trust.

1.8.1 Potato Example

We want to be able to heat a potato in 2 minutes with a microwave. We could use a complex model.

1. Capture unique shape
2. Exact thermal properties
3. FEA Analysis for temperature distribution

We could also use a simple model.

1. Potatoe is a sphere with thermal properties of water
2. We can solve this problem really quickly using Ch.7

The complex model could take days or weeks, but the simple model could take a few minutes. The results could be within 10% of each other!

1.9 Modeling Ethics

We will also discuss modeling ethics. A lot of engineering decisions are made from models. We will focus on understanding the model's limits, what they cannot predict, and what how "simulations are doomed to succeed".

LECTURE

2

MATHMATICAL FOUNDATION

2.1 Objectives

1. Review Complex numbers
2. Understand ODEs
3. Laplace Transforms

2.2 Complex Numbers

$\sqrt{-1} \equiv i, or j$. We will use j. In systems, i is used for current, j is for the complex number. Numbers of the form

$$z = x + iy.$$

are complex numbers in rectangular form. x is the real part, y is the imaginary part. Put the figure of the complex plane here.

NOTE: the imaginary part is y. not jy.
Some properties of complex numbers are the magnitude and argument.

$$|z| = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

The complex conjugate is the reflection of a complex number through the real axis.

$$\bar{z} = x = jy.$$

Properties:

$$z + \bar{z} = 2Re(z).$$

2.2.1 Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

This leads to,

$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{2.1}$$

$$\cos\theta = \frac{e^{j\theta} - e^{j\theta}}{2} \tag{2.2}$$

2.2.2 Polar Form

Define magnitude

$$r = |z| = \sqrt{x^2 + y^2}.$$

$$\theta = \arctan\frac{y}{x}.$$

$$x = r\cos\theta, y = r\sin\theta.$$

2.2.3 Complex Algebra

Addition and subtraction are easily done in rectangular form.

$$z_1 \pm z_2 = (x_1 + jy_1) \pm (x_2 + jy_2).$$

Multiplication and division are easily done in polar form.

2.2.4 Examples

Rectangular to Polar. Remember to think about quadrants!

$$z = 1 - j \tag{2.3}$$

$$|z| = \sqrt{1^2 + (-1^2)} \tag{2.4}$$

$$\theta = \arctan\frac{-1}{1} = 45^\circ z \qquad = \sqrt{2}e^{-j\frac{\pi}{4}} \tag{2.5}$$

Division, can be done in rectangular form by multiplying by the complex conjugate. Can also be done in polar form.

$$\frac{3+2j}{1-3j} = \frac{3+2j}{1-3j} * \frac{3+2j}{1+3j} \tag{2.6}$$

$$\frac{3-6+2j+9j}{1+9} = \frac{-3+11j}{10} \tag{2.7}$$

Powers, $z = 1 - j, z^2 = ?$.

$$z = \sqrt{2}e^{-j\frac{\pi}{4}} \tag{2.8}$$

$$\tag{2.9}$$

In fact,

$$z^n = re^{j\theta}n = r^n e^{jn\theta}.$$

2.2.5 Complex Functions

$$z_1 \to G(z_1) \to z_2.$$

Example,

$$G(z_1) = \frac{z_1 + 2}{z_1^2 + z_1 + 1}.$$

let $z_1 = j2$. We can evaulate the function by plugging in $j2$. for z_1.

2.3 ODEs - Basics

Independent variable is time, t. The dependent variable is usually denoted as x.
Derivatives,

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}, \quad \dots \quad x^{(n)} = \frac{d^nx}{dt^n}.$$

Standard form is to have all functions of the dependent variable on the LHS, all isolated constants and functions of time on the RHS. The RHS is considered the input or the forcing function. If the RHS = 0 the equation is homogeneous. If not, it is nonhomogeneous.
The goal is to solve for $x(t)$. This gives us the solution or the response.

2.3.1 Examples of Standard Form

$$\ddot{x} + 3\dot{x} + 5x = 0 \tag{2.10}$$

$$\ddot{x} + 3\dot{x}x + 5x^2 = 5\sin 2t \tag{2.11}$$

$$\dot{x} = 6 + 5\sin t \tag{2.12}$$

2.3.2 Initial COnditions

When solving an ODE, there is a family of solutions. Example:

$$2\dot{x} + 6x = 3 \quad x(t) = Ce^{-3t} + 0.5.$$

you could choose many values for C and still have a solution to the equation. No matter the value of C, $x(t)$. still approaches 0.5. Include the graph here. Without loss of generality, we assume that the system starts at $t = 0$. Determining C requires knowledge of x at some time t. Commonly, the initial condition is given as $x(t_0) = x_0$.

2.3.3 Classification of ODEs

Linearity generally means that $f(x+y) = f(x)+f(y)$. Linear ODEs contain only linear functions of the dependent variable and its derivatives. Nonlinear functions of the independent variable do not make an ODE nonlinear. The order is the highest derivative of the dependent variable (double dot, 2nd order). A coupled set of ODEs can be reducded to a single ODE whose order is the sum of the oder of the individual ODEs.
Example:

$$3\dot{x_1} + 5\dot{x_2} - 7x_2 = 5 \tag{2.13}$$

$$\dot{x}_2 + 4x_1 + 6x_2 = 0 \tag{2.14}$$

Add the sum of the orders of the two dependent variables. 1 order from x1 and 1 order from x2. This is a second order system, which could be represented by a single 2nd order ODE.

2.3.4 Combining Coupled ODEs

We have two equations.

$$\dot{x}_1 + 4x_2 = 3 \tag{2.15}$$

$$2\dot{x}_2 + x_1 = 5 \tag{2.16}$$

Solution,

$$\dot{x}_1 + 4x_2 = 3 \Rightarrow \ddot{x}_1 + 4x_2 = 0.$$

Rearrange the second equation for x2 and substitute.

2.4 Solution

2.4.1 Direct Integration

With some first order ODEs, we can rearrange as

$$\frac{dx}{dt} = f(t).$$

and integrate both sides.

$$\int dx = \int f(t)dt.$$

$$x(t) = x(t_0) + \int f(t)dt.$$

Example $\dot{x} = 5$.

2.4.2 Seperation of Variables

Doesn't show up that often in this class.

$$\frac{dx}{dt} = g(t)f(t).$$

2.4.3 Trial-Solution Method

For any nth order linear ODE, there is a response $x(t) = x_h + x_p$. The response is the sum of the homogeneous solution and the particular solution. The sum of the internal behavior and the external behavior.
Assume the solution is of the form $x(t) = e^{\lambda t}$. Take the derivative of $x(t)$. n times and substitute into the ODE. Note that $e^{\lambda t} \neq 0$. This gives us the characteristic equation.

$$(\lambda - \lambda_1)(\ldots) = 0.$$

Part II

Week 2

LECTURE

3

ODES AND LAPLACE TRANSFORMS

3.1 Trial Solution

3.1.1 Characteristic Equation

Plug in the form of the solution

$$x(t) = e^{\lambda t}.$$

3.1.2 Example

$$\ddot{x} + 10\dot{x} + 9x = 5\sin 2t \tag{3.1}$$
$$\ddot{x} + 10\dot{x} + 9x = 0 \tag{3.2}$$
$$\lambda^2 + 10\lambda + 9 = 0 \tag{3.3}$$
$$(\lambda + 9)(\lambda + 1) = 0 \tag{3.4}$$

The roots are distinct. -9, -1. The roots can be repeated, or complex as well!

3.1.3 Types of Roots

If all roots are distinct, there will be n different coefficients.

Table 3.1: Solution Forms

Term in $f(t)$.	Form of solution $x_p(t)$.
$A_n^t + \ldots + A_1 t + A_0$.	$C_n^t + \ldots + C_1 t + C_0$.
Ae^{at}.	Ce^{at}.
$A \sin \omega t$	$C \sin \omega t$.
$Ae^{at} \sin \omega t$.	$Ce^{at} \sin \omega t$.

$$\dot{x} + ax = 0, \quad x(t_0 = 0) = 0.$$
$$(\lambda + a)e^{\lambda t} = 0, \quad \lambda = -a.$$

$$x_0 = Ce^{-a*0} = C, \quad x(t) = x_0 e^{-at}.$$

If we have repeated roots, the coefficients gain increasing powers of time.

$$x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} \ldots.$$

Complex roots can be handled using Euler's formula.

$$\lambda_{i,i+1} = \sigma \pm j\omega.$$

$$x(t) = \ldots + C_i e^{\sigma + j\omega} + C_{i+1} e^{\sigma - j\omega} + \ldots.$$

Using eulers formula,

$$x(t) = \ldots + \left(C_i' \cos \omega t + C_{i+1}' \sin \omega t\right) e^{\sigma t} + \ldots.$$

The prime means that the coefficient is different from the line above. This solution is going to be oscillating.

3.2 Forced Response

Just discussed how to obtain x_h. Use Method of Undetermined Coefficients to get x_p. Choose a solution form based on the term in $f(t)$.
Note, the form of the forcing function always shows up i nthe form of the solution (property of linear ODEs) See Table 3.1
Substitute assumed forms of x_p. int othe ODe and determine the coefficients, then go back and solve the homogeneous problem seperately. Add the two solutions with linear superpositions. Finally, find the coefficients to the homogeneous solution.

3.3 Laplace Transforms

Finding solutions in the time domain is hard. While using laplace transforms takes longer, the solution is generally easier to find (algebra over calculus). The laplace transform is a linear operator, so it is useful for

1. Nonhomogeneous equations with a forcing function of time
2. Sets/systems of equations

Linear ODEs become algebraic relationships, and convolution becomes multiplication.

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0-}^{\infty} e^{st} f(t)dt \tag{3.5}$$

Assume $f(t) = 0, t < 0$. We tend to start caring about what happens at time $t = 0$. However, interesting "stuff" happens at time $t = 0$.

1. "stuff" - step changes, impulses
2. We need a way to denote before and after this stuff happens

Let ϵ, be an infitessimally small unit of time. $0-, 0+$ occur ϵ seconds before and after time $t = 0$. Unless stated explicitly, we assume $f(t) = 0 \forall t < 0$.

3.3.1 Inverse Laplace Transform

The definition of inverse laplace tranforms is complex, so we use lookup tables in the book.
Some common examples,

$$f(t) = e^{-at} \iff F(s) = \frac{1}{s+a}, \quad a > 0.$$

3.3.2 Properties

Laplace Transforms are linear. Inverse Laplace transforms too.

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

This property does NOT hold for multiplication. Instead, multiplication corresponds to a convolution integral in the time domain.

$$F(s)G(s) = \mathcal{L}\left\{\int_0^t f(t)g(t-\tau)d\tau\right\}.$$

Derivatives,

$$\mathcal{L}\{\dot{f(t)}\} = sF(s) - f(0) \tag{3.6}$$

$$\mathcal{L}\{\ddot{f(t)}\} = s^2F(s) - sf(0) - \dot{f(0)} \tag{3.7}$$

Integrals,

$$\mathcal{L}\left\{\int f(\tau)d\tau\right\} = \frac{1}{s}F(s).$$

LECTURE

4

LAPLACE TRANSFORMS

4.1 Laplace Transforms of Special Functions

1. Unit Step $\mathcal{L}\{u_s\} = \frac{1}{s}$.
2. Unit Ramp $\mathcal{L}\{u_r\} = \frac{1}{s^2}$.
3. Unit Pulse $\mathcal{L}\{u_p\} = \frac{1}{sD}\left(1 - e^{sD}\right)$.
4. As D approaches zero, we get the unit impulse function
5. $\mathcal{L}\{\delta(t)\} = 1$.

4.2 Properties of Laplace Transforms

4.2.1 Time Shift

Many time shifts occur due to distance traveled. E.g. Flow through a pipe. A time shift results in a multiplication by an exponential in the frequency domain.

$$\mathcal{L}\{f(t-\alpha)\} = e^{-\alpha s}F(s).$$

4.2.2 Multiplication by Exponential

Multiplication by an exp in the time domain results in a shift in the frequency domain.

$$\mathcal{L}\{e^{-\alpha t} f(t)\} = F(s+\alpha).$$

4.3 Initial Value Theorem

Sometimes we would like to find the value of $x(t)$ at $t = 0^+$. Given the laplace transform of the function we have the initial value theorem (IVT)

$$x(0^+) = \lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s) \tag{4.1}$$

This is only applicable if the limit exists, and the transforms and derivatives of $x(t)$ exist.

4.4 Final Value Theorem

Sometimes we would like to know the limit of a function as time goes to infinity. If the limit exists, and transforms/derivatives of $x(t)$ exist we have the final value theorem (FVT).

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{4.2}$$

4.5 Example - Solving ODE with Laplace

We have the EOM,

$$m\ddot{x} + c\dot{x} + kx = 0.$$

laplace transform and initial conditions gives us

$$m(s^2X(s) - s) + c(sX(s) - 1) + kX(s) = 0.$$

rearrange,

$$(ms^2 + cs + k)X(s) = ms + c.$$

$$X(s) = \frac{ms + c}{ms^2 + cs + k}.$$

4.5.1 Partial Fraction Expansion

we can find $x(t)$ with inverse laplace transforms. Our transfer function is a ration of polynomials.

$$X(s) = \frac{N(s)}{D(s)}.$$

Our goal is to express $X(s)$ as a suitable sum of fractions by dividing $D(s)$ into 1st and 2nd order roots. Then we can preform the inverse laplace transform.

Case 1: Distinct Roots

If $D(s)$ contains $(s + p_i)$. The corresponding factor is,

$$\frac{A_i}{s + p_i}.$$

where A_i is the called the "residue".

Case 2: Repeated Roots

If $D(s)$ contains $(s + p_i)^{r_i}$. The corresponding factor is,

$$\frac{A_{r_i}}{(s + p_i)^{r_i}} + \frac{A_{r_i - 1}}{(s + p_i)^{r_i - 1}} + \dots$$

from the table,

$$\mathcal{L}^{-1}\left\{\frac{1}{(s + a)^n}\right\} = \frac{1}{(n - 1)!} t^{n-1} e^{-at} \tag{4.3}$$

increase the power of t as you have more repetition!

Case 3: Complex Roots

If $D(s)$ contains $s^2 + as + b$. The corresponding factor is,

$$\frac{Bs + C}{s^2 + as + b}.$$

Before taking the inverse Laplace transform, we need to complete the square

$$s^2 + as + b = (s + \sigma)^2 + \omega^2.$$

eg,

$$s^2 + 2s + 2 = (s^2 + 2s + 1) + 1 = (s + 1)^2 + 1.$$

The goal is to write in terms of sine and cosine. This introduces oscillation in our signal.

$$= A_1\mathcal{L}^{-1}\left\{\frac{s+\sigma}{(s+\sigma)^2+\omega^2}\right\} + A_2\mathcal{L}^{-1}\left\{\frac{\omega}{(s+\sigma)^2+\omega^2}\right\}.$$

4.6 Parts of the Response

The **Transient Reponse** is the part that disappears with time. The parts that remain with time are the **Steady-state Response**. The **Free Response** is the part that depends on the ICs, and the **Forced Response** is the part due to the forcing function. We can write the total response as a sum of the transient and steady state or the sum of the free and forced response! Note that we will not always have two terms.

4.7 Transfer Functions

Transfer Functions capture the dynamics of the system **independent of ICs and input** (forcing functions)

$$T(s) = \frac{X(s)}{F(s)} \tag{4.4}$$

example,

$$\dot{x} + ax = f(t) \Rightarrow (s+a)X(s) = F(s).$$

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{s+a}.$$

Denominator of transfer function is the caracteristic equation, it can tell us about the general behavior of the system.

$$T(S) = \frac{1}{s^2+7s+10}, \quad r = -2, -5.$$

- Roots are real - no oscillation in response
- Roots are negative - model is stable and free response converges to zero

Transfer function is internal to the system, and is a mapping between the input and the output. We can also go back and forth between ODE and Transfer Function.

$$G : X \to F.$$

The transfer function is a ratio of polynomials. The system zeroes are the roots of the numerator and the poles are the roots of the denominator.
Example,

$$G(s) = \frac{s+1}{s+2}.$$

has a zero at -1 and a pole at -2. Zeroes are given by models with derivatives of the input. The numerator dynamics can significantly alter the response but the analysis is the same.
Note,

$$g(t) = u_s(t) \Rightarrow \dot{g}(t) = \delta(t).$$

4.8 Transfer Function and Convolution

For LTI systems, teh system response is the convolution of the input function and the system's impulse response. From the definition of a transfer function, the response of our system is,

$$X(s) = G(s)F(s) \tag{4.5}$$

We can also think about $G(s)$ as the impulse response of our system.

$$f(t) = \delta(t) \Rightarrow F(s) = 1 \Rightarrow X(s) = G(s)F(s) = G(s) \cdot 1.$$

4.9 section name

Part III

Week 3

LECTURE

5

TRANSFER FUNCTIONS

5.1 Quiz Review

- L4Q3 - Remember that the transfer function is independent of initial conditions!

5.2 Value of Transfer Functions

Transfer functions are useful when working with matlab and simulink. They are the basis for block diagrams. Simulink is how you program block diagrams. Transfer functions can scale with multiple inputs and outputs as well. We can determine a transfer function for each IO pair.

$$T_{ab}(S) = \frac{X_a(S)}{U_b(S)}.$$

5.2.1 Multiple Input Eample

We can derivate a transfer function for each input by temporarily assuming all other inputs are zero.

$$5\ddot{x} + 30\dot{x} + 40x = 6f(t) - 20g(t).$$

$$T_1(S) = \frac{X(S)}{F(S)} = \frac{6}{5s^2 + 30s + 40}$$
$$T_2(S) = \frac{X(S)}{G(S)} = \frac{-20}{5s^2 + 30s + 40}$$

Note that the denominators are the same. By **superposition**, the total response is,

$$X(S) = T_1(S)F(S) + T_2(S)G(S).$$

5.2.2 Multiple Ouputs

Systems of equations may have more than one dependent variable. We can derive transfer functions for each as well.
Example:

$$\dot{x} = -3x + 2y$$
$$\dot{y} = -9y - 4x + 3v(t)$$

Transform Equations:

$$sX(s) = -3X(s) + 2Y(s)$$
$$sY(S) = -9Y(s) - 4X(S) + 3V(S)$$

Eliminate all variables except for desired output and input.

$$X(S) = \frac{2}{s+3}Y(S), \quad Y(S) = \frac{1}{s+9}\left(-4X(s) + 3V(s)\right).$$

$$X(S) = \frac{2}{s+3}\left(\frac{1}{s+9}(-4X(S) + 3V(s))\right).$$
$$\frac{X(s)}{V(S)} = \frac{6}{s^2 + 12s + 35}.$$
$$\frac{Y(S)}{V(S)} = \frac{3(s+3)}{s^2 + 12s + 35}.$$

Note again that they have the same denominator! That is because the two equations represent one system.

5.2.3 Fluid Tank

For one tank, we will learn how to derive,

$$\frac{Q_{out}(S)}{Q_{in}(S)} = \frac{1}{RCs+1}.$$

What if we had two tanks?

$$\frac{Q_{12}(s)}{Q_{in}(s)} = \frac{1}{R_1C_1s+1}.$$

$$\frac{Q_{out}(s)}{Q_{12}(s)} = \frac{1}{R_2C_2s+1}.$$

$$\frac{Q_{out}(s)}{Q_{in}(s)} = \frac{Q_{out}(s)Q_{12}(s)}{Q_{12}(s)Q_{in}(s)}.$$

SLIDE 13

5.3 Block Diagrams

Can use transfer functions of a system/model to construct a visual representation of the dynamics - **Block Diagram**. They show the physical connections of the system, and the cause and effect relationships between components. Block diagrams can be directly used to develop simulation diagrams in Simulink.

5.3.1 Basic Symbols

- Arrow - represents a variable/signal with direction showing cause and effect relationship. Can be labeled with the signal name in the laplace domain.
- Block - input-output relation of a transfer function.
- Circle (summation) - Addition or subtraction of signals.
- Takeoff point - obtain value of variable for use elsewhere. Note that you cannot branch a signal into another.
- Multiplier (Gain) $X(s) = KF(s)$.
- Integration $X(s) = \frac{1}{s}F(s) = x(t) = \int f(t)dt$. This is used to represent $\dot{x} = f(t)$.

5.3.2 Equivalent Block Diagrams

- Block diagrams are not unique
- More than one diagram can represent the same model

Example:

$$\dot{x} + 7x = f(t) \Rightarrow \frac{X(s)}{F(s)} = \frac{1}{s+7}.$$

Can also be written as,

$$X(s) = \frac{1}{s}\left(F(s) - 7X(s)\right).$$

The above equation can be written with a transfer function block, and the below equation can be written with a negative feedback loop.

5.3.3 Block Algebra

Blocks in series can be multiplied. For blocks $T_1, \ldots T_n$. in series,

$$X(s) = \prod_{n=1}^{k} T_n.$$

Blocks in parallel are added together. For blocks $T_1, \ldots T_n$. in parallel,

$$X(s) = \sum_{n=1}^{k} T_n.$$

Add in the closed loop feedback

LECTURE

6

BLOCK DIAGRAMS

6.1 Block Diagram from System Model

$$3\dot{y} + y = 2u.$$

Take laplace transform and arrange highest power of s on the left.

$$3sY(s) + Y(s) = 2U(s) \tag{6.1}$$

$$sY(s) = \frac{1}{3}\left(2U(s) - Y(s)\right) \tag{6.2}$$

- Draw output signal going to the right
- Draw string of integrators
  - # of integrators = highest power of s
  - In this case just 1 integrator
- Draw a gain block corresponding to the common coefficient on the RHS
- Add summation block
- Use existing signals to build up the sum of terms on the RHS

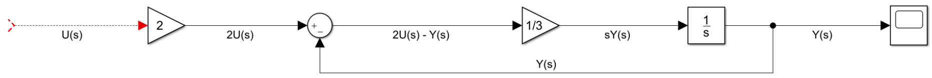

Figure 6.1: Example Block Diagram

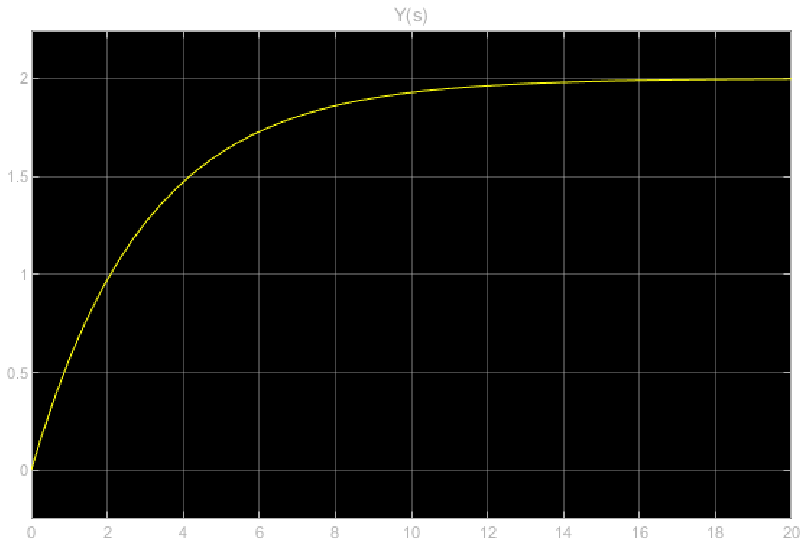

Figure 6.2: Scope output of 6.1

6.1.1 Alternative Approach

This can be done in the time domain as well.

$$\ddot{x} + 2\dot{x} + 3x = f(t).$$

Isolate highest derivative,

$$\ddot{x} = f(t) - 2\dot{x} - 3x.$$

String integrators to get x, and add terms to compute the highest derivative.

6.2 Transfer Function and the IO equation

Can go backwards from tansfer function to ODE.

$$G(s) = \frac{s+1}{s^2 + 3s + 5}.$$

$$(s^2 + 3s + 5)Y(s) = (s+1)U(s).$$

$$\ddot{y} + 3\dot{y} + 5y = \dot{u} + u.$$

6.3 Transfer Function to State-Space

For MIMO systems, we have see that the transfer function looks like

$$Y(s) = G(s)U(s) \tag{6.3}$$

Where Y, G, and U are matricies. Each transfer function in G has the same denominator. However, this is not always the most convient representation of the system. **State-space form** is the transform of system of ODEs of various orders into a larger system of 1st order ODEs.

6.3.1 State Variables

State variables form the smallest set of independent variables that completely describe the state of a system. If you know the initial values and input trajectories you can predict the state/output trajectories for the system. **State variables are independent**, and cannot be expressed as algebraic functions of another and the system inputs. Additionally, **State variables are not unique** and the number of state variables is equal to the number of initial conditions.

Example: Mass-Spring-Damper

Given MSD system with $m = 1, k = 2, c = 1$.

$$\ddot{x} + \dot{x} + 2x = f(t), \Rightarrow \frac{X(s)}{F(s)} = \frac{1}{s^2 + s + 2}.$$

This is a second order system, so we need two initial conditions. But what are the states?

1. Position $x_1 = x$.
2. Velocity $x_2 = \dot{x}$.

Initial conditions are only needed for state variables $x(0)\dot{x}(0)$.

6.3.2 State-Variable Equations

There are as many state-variable equations as there are state variables. Each state-variable equation is a **1st order ODE**.

1. Left side - derivative of a state variable
2. Right side - only a function of state variables, system inputs, maybe time t.

General form,

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_1, \ldots, u_n) \tag{6.4}$$

Example Cont

The original ODE can be rearrange in terms of the state variables,

$$\ddot{x} + \dot{x} + 2x = f(t) \Rightarrow \dot{x}_2 + x_2 + 2x_1 = f(t).$$

Where,

$$x_1 = x, x_2 = \dot{x} \Rightarrow \dot{x}_1 = x_2, \dot{x}_2 = \ddot{x}.$$

This gives us,

$$\dot{x}_2 = -2x_1 - x_2 + f(t), \dot{x}_1 = x_2.$$

6.4 State equation - Nonlinear

In general, the nonlinear state space form is,

6.5 State Equation - Linear

If all elements are linear, then the state vector are a linear combination of the states.

$$\dot{x} = Ax + Bu \tag{6.5}$$

where,

- State vector $\dot{x}$.
- Dynamics matrix A.

6.5.1 Output Equation

Often a system will have many more states than we can measure (or care about knowing). Outputs are things we can measure, or things we want to know.
Assume that the system has p outputs (measured outputs), if the outputs are linear we have

$$y = Cx + Du \tag{6.6}$$

where,

- output matrix C.
- direct transmission matrix D.

Note: Outputs are based on what you want to see!

6.6 Transfer Functions to State Space

Sometimes it is necessary to go from an input-output form to a state-space form. There is a formal procedure for doing this but it is beyond the scope of the class. We will use MATLAB to perform this conversion whenever necessary. Works for both SISO and MIMO.

Part IV

Week 4

LECTURE

7

MATLAB AND FLUIDS

7.1 Eigenvalues

For a transfer function, we defined the poles as the roots of the denominator. Same as the roots of the characteristic polynomial.

$$G(s) = \frac{N(s)}{D(s)} \rightarrow r = \text{roots}(D(s)).$$

For a state-space model, the polse are equal to the eigenvalues of the state matrix. A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
Note: For MIMO systems, the same is true for state space and transfer functions, each transfer function has the same denominator and thus the same poles (a system has a single set of poles.)

7.2 MATLAB - LTI Systems

We can define the transfer function in matlab below.

$$G(s) = \frac{1}{5s^2 + 7s + 4}.$$

Matlab will store the transfer function as a 1x1 tf. (1 input - 1 output transfer function)

```
sys = tf(1, [5,7,4])
```

7.2.1 State Space

For state space equations,

$$\dot{x}_1 = x_2.$$

$$\dot{x}_2 = \frac{1}{5}f(t) - \frac{4}{5}x_1 - \frac{7}{5}x_2.$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{-4}{5} & \frac{-7}{5} \end{bmatrix}.$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{5} \end{bmatrix}.$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We can define the system with the following,

```
A = [...]
B = [..]
C = [..]
D = [...]
sys = ss(A, B, C ,D)
```

It is also an LTI object.

7.2.2 TF to SS

To convert from transfer function to state-space,

```
num = 1
den = [5,7,4]
sys = tf(num,den)

sys_ss = ss(sys)
```

Accessing Matricies

All at once,

```
[A,B,C,D] = ssdata(sys_ss)
```

One at a time,

```
A = sys_ss.A
```

7.2.3 SS to TF

```
sys = ss(A,B,C,D)
```

```
sys_tf = tf(sys)
```

Access Num and Den

All at once,

```
[num,den] = tfdata(sys_tf, 'v')
```

One at a time,

```
num = cell2mat(sys_tf.num)
```

7.3 MATLAB - Block Diagram Algebra

For blocks in series, if sys1 and sys2 are transfer functions in series they can be combined with

```
sys = series(sys1, sys2)
```

For a negative feedback loop, the closed-loop transfer function is

```
sys3 = feedback(sys1, sys2)
```

For a positive feed back loop,

```
sys3 = feedback(sys1, sys2, +1)
```

7.4 MATLAB - Roots

When converting from state space to transfer function, the denominator comes from $(sI - A)^{-1}$.

$$G(s) = C(sI - A)^{-1}B + D.$$

We can get the characteristic polynomial from A.

```
A = [0 1; 6 5]
poly(A)
```

We can also get the roots,

```
roots(poly(A))
```

7.5 Basic Input Responses

Unit impulse response (assumes zeroes for ICs)

```
num = 1;
den = [5, 7, 4];
sys = tf(num, den);

impulse(sys);

# Alternatively,

[y,t] = impulse(sys);
figure;plot(t,y)
```

7.6 Arbitrary Input Responses

Plot the response of the system to an input you define for $t \in [0, 2]$,

$$\ddot{x} + 3\dot{x} + 5x = 10f(t), \quad f(t) = 1.5t.$$

```
# Define time vector
t = linspace(0,2, 300);
# 300 values between 0 and 2

# Define the input
f = 1.5*t;

# Define the system
sys = tf(10, [1, 3 ,5]);

# Calculate the response
[y,t] = lsim(sys,f,t);

figure;plot(t,f,t,y)
xlabel('t')
ylabel('x(t) and f(t)')
xlim([0 2])
legend('(t), x(t)')
```

Lsim is great for any generic linear input!

7.7 Fluid and Thermal Systems

Most systems are goverened by basic conservation laws, a great way to start modeling a system!
Two important ones are conservation of mass and conservation of energy,

$$\sum m_{in} - \sum m_{out} = m_{stored} \tag{7.1}$$

(as a rate)

$$\sum q_{in} - \sum q_{out} = \frac{dm}{dt} = \dot{m} \tag{7.2}$$

$$\sum E_{in} - \sum E_{out} = E_{stored} \tag{7.3}$$

7.7.1 Assumptions

Flow through a restriction/resistance (e.g. valve) is proportional to the pressure across it. As resistance increases, flow decreases.

$$q = \frac{1}{R}(P_1 - P_2) \tag{7.4}$$

Pressure at the bottom of the tank is proportional to the height of the liquid (hydrostatic pressure)

$$P = \rho g h + P_a \tag{7.5}$$

Constant cross-sectional area (hydraulic capacitance)

$$C = \frac{A}{g} \tag{7.6}$$

7.7.2 Hydraulic System Modeling

Use the basic assumptions to model the system,

$$\frac{dm}{dt} = q_{mi} - q_{mo} \text{ (Conservation of mass).}$$

$$q_{mo} = \frac{1}{R}\left(P_{bottom} - P_a\right) \text{ (Resistance).}$$

$$P_{bottom} = \rho g h + P_a \text{ (Hydrostatic pressure).}$$

This gives us,

$$q_{mo} = \frac{1}{R}\left(\rho g h + P_a - P_a\right) = \frac{\rho g}{R} h.$$

We want to have the model in terms of the height of the tank, so

$$m = \rho V = \rho A h \Rightarrow \frac{dm}{dt} = \rho A \dot{h}.$$

Combining these equations, we get that

$$RC\dot{h} + h = \frac{R}{\rho g} q_{mi}.$$

LECTURE

8

FLUID THERMAL ANALYSIS

8.1 Fluid Resistance

We assumed a linear relationship for flow resistance.
In general,

$$R = \frac{d\Delta P}{dq}.$$

Common, nonlinearity

$$CdAo\sqrt{\rho(P_1 - P_2)}.$$

8.2 Combination of Fluid REsistance

- Series - $R = R_1 + R_2$.
- Parallel - $1/R = 1/R_1 + 1/R_2$.

8.3 Example - LL System With PRessure Source

Given an inlet pressure source, model the height of fluid.
Inlet flow,

$$q_{mi} = \frac{1}{R_1}\left[(\hat{p}_s + p_a) - p_{bottom}\right].$$

Outlet flow,

$$q_{mo} = \frac{1}{R_2}[p_{bottom} - p_a].$$

Combine with conservation equation, mass as a function of height, and hydrostatic pressure.

$$\frac{AR_1R_2}{g(R_1 + R_2)}.$$

is the **time constant**

8.4 Thermal Systems

While thermal systems are governed by conservation of mass, thermal systems are goverened by conservation of energy.

$$\frac{dU}{dt} = \dot{Q}_{in} - \dot{Q}_{out}.$$

Temperature is an indicator of stored thermal energy.

$$U = mc_pT.$$

Thermal capacitance

$$C = \frac{dU}{dT} \Rightarrow C = mc_p = \rho V c_p.$$

8.4.1 Thermal Resisitance

Similar to fluid resistance, relates heat transfer to temperature difference.

$$q = \frac{1}{R}(T_1 - T_2).$$

Assume that we know R, we can compute it based on the mode of heat transfer!
Conduction,

$$R = \frac{L}{kA_c}.$$

Convection,

$$R = \frac{1}{hA_s}.$$

Radiation,

$$R = \frac{1}{h_r A_s}.$$

8.5 Lumped-Parameter Assumption

Temperature is distributed spatially (function of location) and temporally (function of time). However, this results in dynamics governed by PDEs, which are difficult to control.
Often, we assume **lumped-parameters**. Ignore spatial variations and focus on temporal variations. We have the Biot Criterion,

$$Bi = \frac{R_{cond}}{R_{conv}} = \frac{q_{cond}}{q_{conv}} = \frac{hA_s}{kA_c/L} = \frac{hL_c}{k} \tag{8.1}$$

If $Bi < 0.1$, the lumped-parameter approximation is good.

8.5.1 Example - Large Bath

Quenching in constant temperature bath, if you are given dimensions and materials you can check the Biot Number, $Bi = 0.02 < 0.1$.
The energy of object can be represented by a single temp, T.
Conservation of energy,

$$\frac{dU}{dt} = \dot{Q}_{in} - \dot{Q}_{out}.$$

$$\dot{Q}_{in} = 0, \quad \dot{Q}_{out} = \frac{1}{R}(T - T_b).$$

$$mc_p\dot{T} = -\frac{1}{R}(T - T_b).$$

8.6 Thermocouple Example

We assume thermocouples measure the temperature exactly. However, thermocouples have their own dynamics.