# MECH 3340 - Assignment #4

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## Exercise 1

### 1.1 Part A

Given,

$$A_c = 20m^2, \quad \rho = 1000kg/m^3, \quad h_0 = 5m, \quad R = 24.525m^{-1}s^{-1}.$$

We can form the system using conservation laws,

$$\frac{dm}{dt} = q_{mi} - q_{mo} \tag{1.1.1}$$

where,

$$q_{mo} = \frac{\rho g}{R}h, \quad \frac{dm}{dt} = \rho A\dot{h} \tag{1.1.2}$$

This gives us,

$$\rho A\dot{h} = q_{mi} - \frac{\rho g}{R}h \tag{1.1.3}$$

$$RC\dot{h} + h = \frac{R}{\rho g}q_{mi} \tag{1.1.4}$$

where,

$$C = \frac{A}{g} \tag{1.1.5}$$

Finally, we can calculate the time constant,

$$\tau = RC = 24.525 \times \frac{20}{9.8} = 50.051 \text{ seconds} \tag{1.1.6}$$

## 1.2 Part B

Find the height after 98% of the water has left,
General form,

$$\tau \dot{x} + x = kf(t) \tag{1.2.1}$$

where,

$$f(t) = 0, \quad h(0) = 5m \tag{1.2.2}$$

For free response, we have,

$$h(t) = h_0 e^{-\frac{1}{\tau}t} \tag{1.2.3}$$

Clearly, $h_\infty = 0$ this means that,

$$\Delta h = h_0 - h_\infty = 5m \tag{1.2.4}$$

$$h_0 - 0.98\Delta h = 0.1m \tag{1.2.5}$$

$h = 0.1m$ when $t = 195.801 = 7.95\tau$. Therefore, 98% of the water will have left the tank after 7.95 time constants.

## 1.3 Part C

We have,

$$RC\dot{h} + h = \frac{R}{\rho g} q_{mi} \tag{1.3.1}$$

If water steadily flows in at a rate of 3000 kg/s, this means that the system is being subjected to a unit step response times 3000. Therefore, the steady state value will be

$$h_{ss} = 3000 * \frac{R}{\rho g} = 7.507m \tag{1.3.2}$$

# Exercise 2

## 2.1 Part A

We determine the system to be,

$$\begin{aligned} \rho A_1 \dot{h}_1 &= q_{mi} - \frac{\rho g}{R_1}(h_1 - h_2) \\ \rho A_2 \dot{h}_2 &= \frac{\rho g}{R_1}(h_1 - h_2) - \frac{\rho g}{R_2} h_2 \end{aligned} \tag{2.1.1}$$

Move constants to RHS

$$\begin{aligned} \dot{h}_1 &= \frac{1}{\rho A_1} q_{mi} - \frac{g}{R_1 A_1}(h_1 - h_2) \\ \dot{h}_2 &= \frac{g}{R_1 A_2}(h_1 - h_2) - \frac{g}{R_2 A_2} h_2 \end{aligned} \tag{2.1.2}$$

We can write the system in state space form, where $h_1$, $h_2$ are the state variables.

$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \begin{bmatrix} -\frac{g}{R_1 A_1} & \frac{g}{R_1 A_1} \\ \frac{g}{R_1 A_2} & -\left(\frac{g}{R_1 A_2} + \frac{g}{R_2 A_2}\right) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{\rho A_1} \\ 0 \end{bmatrix} q_{mi} \tag{2.1.3}$$

$$y = \begin{bmatrix} 0 & 1 \\ 0 & \frac{\rho g}{R_2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \tag{2.1.4}$$

## 2.2 Part B

Substitue given values for $R_1, R_2, A_1, A_2, \rho, g$.

$$\dot{h} = \begin{bmatrix} -1 & 1 \\ \frac{1}{4} & \frac{1}{3} \end{bmatrix} h + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_{mi} \tag{2.2.1}$$

$$y = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{3} \end{bmatrix} \tag{2.2.2}$$

The transfer function matrix, $G$, can be found with

$$G = C(sI - A)^{-1} B + D \tag{2.2.3}$$

Note, D is a 2 x 1 matrix of zeros.

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+1/3) - 1/4} \begin{bmatrix} s+1/3 & 1 \\ 1/4 & s+1 \end{bmatrix} \tag{2.2.4}$$

Multiplying by C and B gives us

$$G(s) = \frac{1}{s^2 + 4/3s + 1/12} \begin{bmatrix} 1/4 \\ 1/12 \end{bmatrix} \tag{2.2.5}$$

# Exercise 3

## 3.1 Part A

The conservation of energy for each part of the system is

$$\frac{dU_1}{dt} = q_i - q_1 \tag{3.1.1}$$

$$\frac{dU_2}{dt} = q_1 - q_o \tag{3.1.2}$$

where,

$$\frac{dU_i}{dt} = C_i \dot{T}_i \tag{3.1.3}$$

$$q_1 = \frac{1}{R_1}(T_1 - T_2) \tag{3.1.4}$$

$$q_o = \frac{1}{R_2}(T_2 - T_o) \tag{3.1.5}$$

This gives us the system,

$$\begin{aligned} C_1\dot{T}_1 &= q_i - \frac{1}{R_1}(T_1 - T_2) \\ C_2\dot{T}_2 &= \frac{1}{R_1}(T_1 - T_2) - \frac{1}{R_2}(T_2 - T_o) \end{aligned} \tag{3.1.6}$$

Which we can simplify,

$$\begin{aligned}\dot{T}_1 &= \frac{1}{C_1}q_i - \frac{1}{R_1C_1}T_1 + \frac{1}{R_1C_1}T_2 \\ \dot{T}_2 &= \frac{1}{R_1C_2}T_1 - \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)T_2 + \frac{1}{R_2C_2}T_o\end{aligned} \tag{3.1.7}$$

State space form,

$$\dot{T} = \begin{bmatrix} -\frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1C_2} & \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right) \end{bmatrix} T + \begin{bmatrix} \frac{1}{C_1} & 0 \\ 0 & \frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} q_i \\ T_o \end{bmatrix} \tag{3.1.8}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} T \tag{3.1.9}$$

## 3.2 Part B

Simplified state space model

$$\dot{T} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} T + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_i \\ T_o \end{bmatrix} \tag{3.2.1}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} T \tag{3.2.2}$$

Listing 1 shows the matlab script used to find the steady state values as functions of the input variables.

```
G =

 - T_0/(- s^2 + s + 3) - (q*(s - 2))/(- s^2 + s + 3)
 - q/(- s^2 + s + 3) - (T_0*(s + 1))/(- s^2 + s + 3)


final_values =

 0
 0
```

Figure 1: Output of Listing 1

Listing 1: Script for Problem 3, Steady state value as a function of input variable

```
% Define system as a function of input variables
syms s q T_0

u = [q; T_0];

A = [-1 1; 1 2];
B = eye(2) * u;
C = eye(2);
D = 0;

% Calculate transfer function
G = C*inv((s*eye(2) - A))*B

% Final Value
final_values = limit(s*G,s,0)
```

# Exercise 4

## 4.1 Part A

Given,

$$r = 0.01m, \quad h = 350W/m^2C, \quad \rho = 7850, \quad c_p = 440J/kgC.$$

$$k = 43W/m^2C, \quad T_f = 100C, \quad T_0 = 25C.$$

Calculate Biot Criterion,

$$Bi = \frac{R_{cond}}{R_{conv}} = \frac{hL_c}{k} \tag{4.1.1}$$

where,

$$L_c = \frac{V}{A} = \frac{r}{3} = 0.0033333 \tag{4.1.2}$$

The Biot Criterion is less than 0.1, therefore the temperature is spatially uniform.

## 4.2 Part B

Conservation of energy,

$$\frac{dU}{dt} = q_{in} - q_{out} \tag{4.2.1}$$

where,

$$q_{in} = 0, \quad q_{out} = \frac{1}{R}(T - T_f), \quad \frac{dU}{dt} = \rho V c_p \dot{T} \tag{4.2.2}$$

We can substitute these values to get the differential equation relating the temperature $T$ to the temperature of the water $T_f$.

$$mc_p\dot{T} = -\frac{1}{R}(T - T_f) \tag{4.2.3}$$

which can be rewritten as,

$$R\rho V c_p \dot{T} + T = T_f \tag{4.2.4}$$

### 4.3 Part C

The time constant $\tau$ is equal to,

$$\tau = R\rho V c_p = \frac{\rho V c_p}{hA_s} = \frac{7850 \cdot 440}{350} \cdot \frac{0.01}{3} = 32.89s \tag{4.3.1}$$

### 4.4 Part D

Treat the heating process as a step response. Take the laplace transform of both sides.

$$\frac{\rho V c_p}{hA_s} sT(s) + T(s) = \frac{T_f}{s} \tag{4.4.1}$$

$$T(s) = \frac{T_f}{s\left(\frac{\rho V c_p}{hA_s} s + 1\right)} \tag{4.4.2}$$

$$T(s) = \frac{T_f}{s} - \frac{T_f}{s + \frac{hA_s}{\rho V c_p}} \tag{4.4.3}$$

Take the inverse laplace transform

$$T(t) = T_f \left(1 - e^{-\dfrac{hA_s}{\rho V c_p} t}\right) \tag{4.4.4}$$

### 4.5 Part E

The result obtained in part b is modeled in simulink. The block diagram is shown in Figure 2 and the scope output is shown in Figure 3. As expected, we can see the of the sphere rise to the temperature of the water.

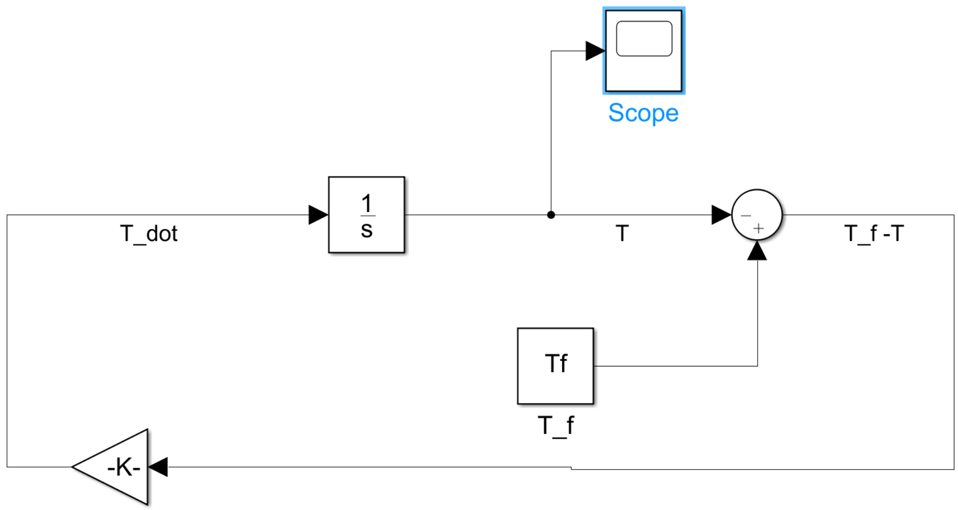


Figure 2: Block diagram of result obtained in part b

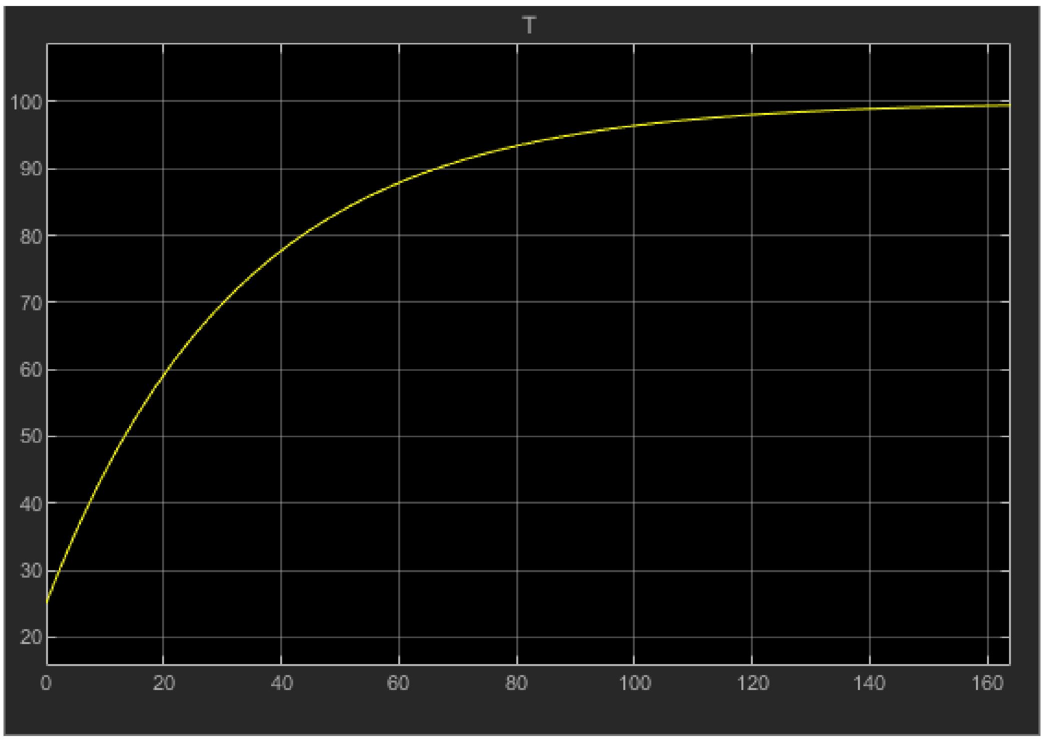


Figure 3: Temperature as a function of time. Simulated for 164 seconds or 5 time constants.