MECH 3340 - Assignment #4

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Exercise 1

1.1 Part A

Given,

$$A_c = 20m^2, \quad \rho = 1000kg/m^3, \quad h_0 = 5m, \quad R = 24.525m^{-1}s^{-1}.$$

We can form the system using conservation laws,

$$\frac{dm}{dt} = q_{mi} - q_{mo} \tag{1.1.1}$$

where,

$$q_{mo} = \frac{\rho g}{R}h, \quad \frac{dm}{dt} = \rho A\dot{h} \tag{1.1.2}$$

This gives us,

$$\rho A\dot{h} = q_{mi} - \frac{\rho g}{R}h \tag{1.1.3}$$

$$RC\dot{h} + h = \frac{R}{\rho g}q_{mi} \tag{1.1.4}$$

where,

$$C = \frac{A}{g} \tag{1.1.5}$$

Finally, we can calculate the time constant,

$$\tau = RC = 24.525 \times \frac{20}{9.8} = 50.051 \text{ seconds} \tag{1.1.6}$$

1.2 Part B

Find the height after 98% of the water has left,
General form,

$$\tau \dot{x} + x = kf(t) \tag{1.2.1}$$

where,

$$f(t) = 0, \quad h(0) = 5m \tag{1.2.2}$$

For free response, we have,

$$h(t) = h_0 e^{-\frac{1}{\tau}t} \tag{1.2.3}$$

Clearly, $h_\infty = 0$ this means that,

$$\Delta h = h_0 - h_\infty = 5m \tag{1.2.4}$$

$$h_0 - 0.98\Delta h = 0.1m \tag{1.2.5}$$

$h = 0.1m$ when $t = 195.801 = 7.95\tau$. Therefore, 98% of the water will have left the tank after 7.95 time constants.

1.3 Part C

We have,

$$RC\dot{h} + h = \frac{R}{\rho g} q_{mi} \tag{1.3.1}$$

If water steadily flows in at a rate of 3000 kg/s, this means that the system is being subjected to a unit step response times 3000. Therefore, the steady state value will be

$$h_{ss} = 3000 * \frac{R}{\rho g} = 7.507m \tag{1.3.2}$$

Exercise 2

2.1 Part A

We determine the system to be,

$$\begin{aligned}\rho A_1\dot{h}_1 &= q_{mi} - \frac{\rho g}{R_1}(h_1 - h_2)\\ \rho A_2\dot{h}_2 &= \frac{\rho g}{R_1}(h_1 - h_2) - \frac{\rho g}{R_2}h_2\end{aligned} \tag{2.1.1}$$

Move constants to RHS

$$\begin{aligned}\dot{h}_1 &= \frac{1}{\rho A_1}q_{mi} - \frac{g}{R_1A_1}(h_1 - h_2)\\ \dot{h}_2 &= \frac{g}{R_1A_2}(h_1 - h_2) - \frac{g}{R_2A_2}h_2\end{aligned} \tag{2.1.2}$$

We can write the system in state space form, where h_1, h_2 are the state variables.

$$\begin{bmatrix}\dot{h}_1\\ \dot{h}_2\end{bmatrix} = \begin{bmatrix}-\frac{g}{R_1A_1} & \frac{g}{R_1A_1}\\ \frac{g}{R_1A_2} & -\left(\frac{g}{R_1A_2} + \frac{g}{R_2A_2}\right)\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix} + \begin{bmatrix}\frac{1}{\rho A_1}\\ 0\end{bmatrix}q_{mi} \tag{2.1.3}$$

$$y = \begin{bmatrix}0 & 1\\ 0 & \frac{\rho g}{R_2}\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix} \tag{2.1.4}$$

2.2 Part B

Substitue given values for $R_1, R_2, A_1, A_2, \rho, g$.

$$\dot{h} = \begin{bmatrix}-1 & 1\\ \frac{1}{4} & \frac{1}{3}\end{bmatrix}h + \begin{bmatrix}1\\ 0\end{bmatrix}q_{mi} \tag{2.2.1}$$

$$y = \begin{bmatrix}0 & 1\\ 0 & \frac{1}{3}\end{bmatrix} \tag{2.2.2}$$

The transfer function matrix, G, can be found with

$$G = C(sI - A)^{-1}B + D \tag{2.2.3}$$

Note, D is a 2 x 1 matrix of zeros.

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+1/3) - 1/4} \begin{bmatrix} s+1/3 & 1 \\ 1/4 & s+1 \end{bmatrix} \tag{2.2.4}$$

Multiplying by C and B gives us

$$G(s) = \frac{1}{s^2 + 4/3s + 1/12} \begin{bmatrix} 1/4 \\ 1/12 \end{bmatrix} \tag{2.2.5}$$

Exercise 3

3.1 Part A

The conservation of energy for each part of the system is

$$\frac{dU_1}{dt} = q_i - q_1 \tag{3.1.1}$$

$$\frac{dU_2}{dt} = q_1 - q_o \tag{3.1.2}$$

where,

$$\frac{dU_i}{dt} = C_i \dot{T}_i \tag{3.1.3}$$

$$q_1 = \frac{1}{R_1}(T_1 - T_2) \tag{3.1.4}$$

$$q_o = \frac{1}{R_2}(T_2 - T_o) \tag{3.1.5}$$

This gives us the system,

$$\begin{aligned} C_1 \dot{T}_1 &= q_i - \frac{1}{R_1}(T_1 - T_2) \\ C_2 \dot{T}_2 &= \frac{1}{R_1}(T_1 - T_2) - \frac{1}{R_2}(T_2 - T_o) \end{aligned} \tag{3.1.6}$$

Which we can simplify,

$$
\begin{aligned}
\dot{T}_1 &= \frac{1}{C_1}q_i - \frac{1}{R_1C_1}T_1 + \frac{1}{R_1C_1}T_2 \\
\dot{T}_2 &= \frac{1}{R_1C_2}T_1 - \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)T_2 + \frac{1}{R_2C_2}T_o
\end{aligned} \tag{3.1.7}
$$

State space form,

$$
\dot{T} = \begin{bmatrix} -\frac{1}{R_1C_1} & \frac{1}{R_1C_1} \\ \frac{1}{R_1C_2} & \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right) \end{bmatrix} T + \begin{bmatrix} \frac{1}{C_1} & 0 \\ 0 & \frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} q_i \\ T_o \end{bmatrix} \tag{3.1.8}
$$

$$
y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} T \tag{3.1.9}
$$

3.2 Part B

Simplified state space model

$$
\dot{T} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} T + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_i \\ T_o \end{bmatrix} \tag{3.2.1}
$$

$$
y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} T \tag{3.2.2}
$$