MECH 4110 - Pre-lab #1

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1 Equation of Motion Derivation

1.1 Configuration 1

Configuration 1 consists of the pump and tank 1. Based on this, we know that the input and output flowrates are given by,

$$q_{m,\,pump} = \rho K_p v_p \tag{1.1.1}$$

$$q_m = C_d A_o \sqrt{2\rho \left(P_1 - P_2\right)} \tag{1.1.2}$$

where,

$$P_1 = P_a + \rho g h \tag{1.1.3}$$
$$P_2 = P_a \tag{1.1.4}$$

Finally, we also know that,

$$\frac{dm}{dt} = \rho A_c \dot{h} \tag{1.1.5}$$

The change in mass with respect to time is given by conservation laws, we can develop the equation of motion by substituting Equations (1.1.1) - (1.1.5) and simplifying.

$$\frac{dm}{dt} = \sum q_{m,\,in} - \sum q_{m,\,out} \tag{1.1.6}$$

$$\rho A_c \dot{h} = \rho K_p v_p - C_d A_o \sqrt{2\rho \left(P_a + \rho g h - P_a\right)} \tag{1.1.7}$$

$$\rho A_c \dot{h} = \rho K_p v_p - C_d A_o \sqrt{2\rho^2 g h} \tag{1.1.8}$$

$$A_{c,1} \dot{h}_1 + C_{d,1} A_{o,1} \sqrt{2 g h_1} = K_p v_p \tag{1.1.9}$$

Equation (1.1.9) is a non-linear ODE relating the height to the input voltage. We have added a subscripts of 1 to $h, A_c, C_d,$ and A_o so that we can differentiate between the two tanks further down the line.

1.2 Configuration 2

Configuration 2 consists of both tanks, with only 1 output from the pump. The flow into Tank 2 is the flow out of Tank 1.

$$q_{m,in} = \rho C_{d,1} A_{o,1} \sqrt{2 g h_1} \tag{1.2.1}$$

$$q_{m,out} = \rho C_{d,2} A_{o,2} \sqrt{2 g h_2} \tag{1.2.2}$$

Using the same conservation law, we can find the equation of motion for Tank 2.

$$\frac{dm}{dt} = \sum q_{m,in} - \sum q_{m,out} \tag{1.2.3}$$

$$\rho A_{c,2} \dot{h}_2 = \rho C_{d,1} A_{o,1} \sqrt{2 g h_1} - \rho C_{d,2} A_{o,2} \sqrt{2 g h_2} \tag{1.2.4}$$

$$A_{c,2} \dot{h}_2 + C_{d,2} A_{o,2} \sqrt{2 g h_2} = C_{d,1} A_{o,1} \sqrt{2 g h_1} \tag{1.2.5}$$

Equation (1.1.9) coupled with (1.2.5) model the coupled tanks of configuration two.

$$\begin{aligned} A_{c,1} \dot{h}_1 + C_{d,1} A_{o,1} \sqrt{2 g h_1} &= K_p v_p \\ A_{c,2} \dot{h}_2 + C_{d,2} A_{o,2} \sqrt{2 g h_2} &= C_{d,1} A_{o,1} \sqrt{2 g h_1} \end{aligned} \tag{1.2.6}$$

1.3 Configuration 3

The difference between configuration 2 and configuration 3 is that both tanks receive flow from the pump. This means that tank one is the same as before and the flow of tank two is given by,

$$q_{m,\,out} = \rho C_{d,\,2} A_{o,\,2} \sqrt{2gh_2} \tag{1.3.1}$$

$$q_{m,\,in} = \rho C_{d,\,1} A_{o,\,1} \sqrt{2gh_1} + K_p v_p \tag{1.3.2}$$

By conservations laws we get that,

$$\rho A_{c,\,2} \dot{h}_2 = \rho C_{d,\,1} A_{o,\,1} \sqrt{2gh_1} + K_p v_p - \rho C_{d,\,2} A_{o,\,2} \sqrt{2gh_2} \tag{1.3.3}$$

which simplifies to

$$A_{c,\,2} \dot{h}_2 + C_{d,\,2} A_{o,\,2} \sqrt{2gh_2} = C_{d,\,1} A_{o,\,1} \sqrt{2gh_1} + K_p v_p \tag{1.3.4}$$

Finally, configuation 3 is modeled by the coupled nonlinear ODEs.

$$\begin{aligned} A_{c,\,1} \dot{h}_1 + C_{d,\,1} A_{o,\,1} \sqrt{2gh_1} &= K_p v_p \\ A_{c,\,2} \dot{h}_2 + C_{d,\,2} A_{o,\,2} \sqrt{2gh_2} &= C_{d,\,1} A_{o,\,1} \sqrt{2gh_1} + K_p v_p \end{aligned} \tag{1.3.5}$$

2 Unknown System Parameters

The EOM for configuration one has 5 parameters that we must account for.

1. A_c, A_o - the cross sectional areas of the tank and the orifice can be measured

2. g - the acceleration due to gravity is a well known constant that can be looked up

3. K_p - the pump constant must be determined numerically, as there may be small differences in each pump

4. C_d - the discharge coefficient must be determined numerically, as it depends on the tank setup

3 Pump Constant Calculation, K_p

Given Equation (1.1.9) and steady state operating condition ($\dot{h} = 0$), we have

$$C_{d,1} A_{o,1} \sqrt{2gh_1} = K_p v_p \tag{3.0.1}$$

which can be rearranged to determine an expression for K_p.

$$K_p = \frac{C_{d,1} A_{o,1}}{v_p} \sqrt{2gh_1} \tag{3.0.2}$$