MECH 3340 - Assignment #2

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Exercise 1

Given,

$$x(t) = e^{-2t} \cos 3t.$$

The laplace transform of $x(t)$ can be found via lookup table.

$$\mathcal{L}\{x(t)\} = \frac{s+2}{(s+2)^2 + 9} \tag{1.1}$$

Exercise 2

Given,

$$\ddot{x} + \omega^2 x = 0; \quad x(0^-) = 1, \dot{x}(0^-) = 0.$$

We can solve the IVP with laplace transforms.

$$s^2 X(s) - sx(0) + \omega^2 X(s) = 0 \tag{2.1}$$

$$\left(s^2 + \omega^2\right) X(s) = s \tag{2.2}$$

$$X(s) = \frac{s}{s^2 + \omega^2} \Rightarrow x(t) = \cos \omega t \tag{2.3}$$

We can verify our solution with the homogeneous solution. The characteristic equation is,

$$\lambda^2 e^{\lambda t} + \omega^2 e^{\lambda t} = 0 \tag{2.4}$$

$$\lambda^2 + \omega^2 = 0 \tag{2.5}$$

Which gives us,

$$\lambda = j\omega, \quad (\mathrm{Re}(\lambda) = 0) \tag{2.6}$$

We can write $x_h(t)$ as,

$$x_h(t) = C_i e^{j\omega} + C_{i+1} e^0 \tag{2.7}$$

$$x_h(t) = C_i' \cos \omega t + C_{i+1}' \sin 0 \tag{2.8}$$

Clearly, x_h and $x(t)$ have the same form, which suggests our solution is correct.

Exercise 3

Given,

$$F(s) = \frac{1}{s^2(s^2 + 3s + 2)}.$$

Using the final value theorem, we can find that

$$\lim_{s \to \infty} sF(s) = \frac{1}{\infty} = 0 \tag{3.1}$$

For the initial value theorem, we can find that

$$\lim_{s \to 0^-} sF(s) = -\infty \tag{3.2}$$

$$\lim_{s \to 0^+} sF(s) = \infty \tag{3.3}$$

This means that the initial value of $F(s)$ does not exist. We can confirm this by taking observing the time domain behavior. The partial fraction expansion of $F(s)$ is,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \tag{3.4}$$

where,

$$B = 1/2 \tag{3.5}$$
$$C = 1 \tag{3.6}$$
$$D = -1/4 \tag{3.7}$$

solve for A by substituting numerical values for constants and comparing similar powers.

$$1 = s^3\left(A + \frac{3}{4}\right) + s^2(\dots) + s(\dots) + 1 \tag{3.8}$$

$$A = -3/4 \tag{3.9}$$

Note, $A = -3/4$ regardless of which powers you compare. Therefore, $F(s)$ is equal to,

$$F(s) = -\frac{3}{4s} + \frac{1}{2s^2} + \frac{1}{s+1} - \frac{1}{4(s+2)} \tag{3.10}$$

and $f(t)$ is equal to,

$$f(t) = -\frac{3}{4} + \frac{1}{2}t + e^{-t} + \frac{1}{4}e^{-2t} \tag{3.11}$$

Our above answer checks out. The value of $f(t)$ approaches zero as t approaches ∞. Additionally, whether t is a very small negative or a very small positive affects the value of $f(0)$. $(-\infty,\ \text{or}\ \infty.)$

Exercise 4

Given,

$$F(s) = \frac{2s+1}{s^2+4s+3} + \frac{1}{s^2}.$$

We can find $\mathcal{L}^{-1}\{F(s)\}$ with a partial fraction expansion.

$$\frac{2s+1}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1} \tag{4.1}$$

Using the cover up method, we get that $A = B = 5/2$. Therefore,

$$\mathcal{L}^{-1}\{F(s)\} = \frac{5}{2}\left(e^{3t} + e^{-t}\right) + t \tag{4.2}$$

Exercise 5

Given,

$$F(s) = \frac{1}{s(s^2+2s+2)}.$$

Do a parital fraction expansion of $F(s)$.

$$\frac{A}{s} + \frac{Bs+C}{s^2+2s+2} = \frac{A(s^2+2s+2)+s(Bs+C)}{s(s^2+2s+2)} \tag{5.1}$$

$$\frac{(A+B)s^2+(2A+C)s+2A}{s(s^2+2s+2)} = \frac{1}{s(s^2+2s+2)} \tag{5.2}$$

$$\begin{aligned} 2A &= 1 \\ A+B &= 0 \\ 2A+C &= 0 \end{aligned}$$

This gives us $A = 1/2, B = -1/2, C = -1$.
Finally,

$$F(s) = \frac{1}{2}\cdot\frac{1}{s} + \frac{1/2(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1} \tag{5.3}$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2} + \frac{1}{2}e^{-t}\cos t - e^{-t}\sin t \tag{5.4}$$

Exercise 6

6.1 $5\dot{x} + 7x = f(t)$

Take the laplace transform of both sides,

$$(5s+7)X(s) = F(s) \tag{6.1}$$

$$\frac{X(s)}{F(s)} = \frac{1}{5s+7} \Rightarrow p = -7/5 \tag{6.2}$$

6.2 $\ddot{x} + 10\dot{x} + 21x = 4f(t)$

Take the laplace transform of both sides,

$$(s^2 + 10s + 21)X(s) = 4F(s) \tag{6.3}$$

$$\frac{X(s)}{F(s)} = \frac{4}{(s+3)(s+7)} \Rightarrow p = \{-3, -7\} \tag{6.4}$$

6.3 $\ddot{x} + 14\dot{x} + 58x = 6\dot{f}(t) + 4f(t)$

Take the laplace transform of both sides,

$$(s^2 + 14s + 58)X(s) = (6s + 4)F(s) \tag{6.5}$$

$$\frac{X(s)}{F(s)} = \frac{6s + 4}{s^2 + 14s + 58} \tag{6.6}$$

Compute p with the quadratic formula.

$$\frac{-14 \pm \sqrt{196 - 232}}{2} = \frac{-14 \pm j6}{2} \tag{6.7}$$

$$p = -7 \pm j3 \tag{6.8}$$

Exercise 7

Given,

$$3\dot{x} = y.$$

$$\dot{y} = f(t) - 3y - 15x.$$

Take the laplace transform of both sides of each equation,

$$\begin{aligned} 3sX(s) &= Y(s) \\ (s+3)Y(s) &= F(s) - 15X(s) \end{aligned} \tag{7.1}$$

Substitute $3sX(s)$ for $Y(s)$ on the bottom equation

$$(s+3)3sX(s) = F(s) - 15X(s) \tag{7.2}$$

$$(3s^2 + 9s + 15)X(s) = F(s) \tag{7.3}$$

$$\frac{X(s)}{F(s)} = \frac{1}{3s^2 + 9s + 15} \tag{7.4}$$

We can compute $Y(s)/F(s)$ by substituting $1/3sX(s)$ for $Y(s)$.

$$\frac{Y(s)}{F(s)} = \frac{1}{3s(3s^2 + 9s + 15)} \tag{7.5}$$

Exercise 8

Given,

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)}.$$

Compute the partial fraction expansion.

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+3} \tag{8.1}$$

using the cover up method, we can find that

$$B = 10/3 \tag{8.2}$$
$$C = 5/2 \tag{8.3}$$
$$D = 5/18 \tag{8.4}$$

The value of A can be found by equating the top two equations.

$$5s+2 = As(s+1)(s+3) + \frac{10}{3}(s+1)(s+3) + \frac{5}{2}s^2(s+3) + \frac{5}{18}s^2(s+1) \tag{8.5}$$

Group the powers of s together and equate their corresponding powers

$$5s+2 = s^3(\ldots) + s^2(\ldots) + s(3A + \frac{40}{3}) + 10 \tag{8.6}$$

$$A = -25/9 \tag{8.7}$$

Therefore,

$$F(s) = \frac{-25}{9s} + \frac{10}{3s^2} + \frac{5}{2(s+1)} + \frac{5}{18(s+3)} \tag{8.8}$$

Figure 1 shows the confirmation of partial fraction constant terms.
Finally, the inverse laplace transform is,

$$f(t) = \frac{-25}{9} + \frac{10}{3}t + \frac{5}{2}e^{-t} + \frac{5}{18}e^{-3t} \tag{8.9}$$

```
>> num = [5 10]

num =

     5    10

>> den = [1 4 3 0 0]

den =

     1     4     3     0     0

>> [r,p,k] = residue(num,den)

r =

    0.2778
    2.5000
   -2.7778
    3.3333


p =

    -3
    -1
     0
     0


k =

     []

>>
```

Figure 1: Output of residue(num, den), the r values are equivalent to A, B, C, D.

Exercise 9

Given,

$$F(s) = \frac{3s+1}{s^4+3s^3+2s^2}.$$

Factor the denominator to get the poles and zeroes,

$$F(s) = \frac{3s+1}{s^2(s+2)(s+1)} \tag{9.1}$$

$$\begin{aligned} p &= 0, 0, -2, -1 \\ z &= -1/3 \end{aligned}$$

Figure 2 shows the matlab commands used to check the values for p, z.

```
>> num = [3 1]

num =

     3     1

>> den = [1 3 2 0 0]

den =

     1     3     2     0     0

>> z = roots(num)

z =

   -0.3333

>> p = roots(den)

p =

     0
     0
    -2
    -1

>>
```

Figure 2: Matlab commands confirming values for p, z.