MECH 3340 - Assignment #1

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Exercise 1

Given,

$$z = \frac{-1 + j\sqrt{3}}{\sqrt{3} + j}.$$

Multiply z by 1 and simplify.

$$\frac{-1 + j\sqrt{3}}{\sqrt{3} + j}\left(\frac{\sqrt{3} - 1}{\sqrt{3} - 1}\right) = \frac{-\sqrt{3} - j^2\sqrt{3} + j + 3j}{4} \tag{1.1}$$

$$\frac{-\sqrt{3} + \sqrt{3} + 4j}{4} = j \Rightarrow \mathrm{Re}(z) = 0, \quad \mathrm{Im}(z) = j \tag{1.2}$$

Magnitude and Phase,

$$r = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \sqrt{1^2} = 1 \tag{1.3}$$

$$\theta = \tan^{-1}\left(\frac{\mathrm{Re}(z)}{\mathrm{Im}(z)}\right) = \tan^{-1}\left(\frac{0}{1}\right) = \frac{\pi}{2} \tag{1.4}$$

Exercise 2

Given,

$$\bar{z} = \frac{5}{(2+j)}.$$

Mutiply z by -1.

$$\frac{5}{(2+j)}\left(\frac{2-j}{2-j}\right) = \frac{10-j5}{3} \tag{2.1}$$

This implies that,

$$z = \frac{10+j5}{3} \tag{2.2}$$

However, θ lies in the second quadrant.

$$\theta = \tan^{-1}\left(\frac{10/3}{5/3}\right) = \tan^{-1}(2) \tag{2.3}$$

Therefore, a complex number with conjugate $\bar{z}$ cannot exist in the fourth quadrant.

Exercise 3

Factor $z^3 + 27 = 0$ as a sum of two cubes.

$$z^3 + 27 = (z+3)(z^2 - 3z + 9) \tag{3.1}$$

Clearly, $z_1 = -3$. The other two roots can be found with the quadratic formula.

$$z_2, z_3 = \frac{3 \pm \sqrt{9-36}}{2} = \frac{3}{2} \pm j\frac{\sqrt{27}}{2} \tag{3.2}$$

z_1, z_2, z_3 are in rectangular form. In polar form,

$$r_1 = \sqrt{(-3)^2} = -3, \quad \theta_1 = \tan^{-1}\left(\frac{0}{3}\right) = 0 \tag{3.3}$$

$$r_2 = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{27}}{2}\right)^2} = 3, \quad \theta_2 = \tan^{-1}\left(\frac{\sqrt{27}/2}{3/2}\right) = 60^\circ \tag{3.4}$$

$$r_3 = 3, \quad \theta_3 = -60^\circ \tag{3.5}$$

Therefore,

$$z_1 = -3e^{j\cdot 0} = -3 \tag{3.6}$$

$$z_2 = 3e^{j\frac{\pi}{3}} \tag{3.7}$$

$$z_3 = 3e^{-j\frac{\pi}{3}} \tag{3.8}$$

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>> G = (j - 1) / (sqrt(2) + j)

G =

  -0.1381 + 0.8047i

>> magnitude = abs(G)

magnitude =

    0.8165

>> phase = angle(G)

phase =

    1.7407

>> |
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Figure 1: MATLAB Commands for Exercise 4

Exercise 4

Given that,

$$G(j) = \frac{j-1}{j+\sqrt{2}}.$$

The magnitude and phase of $G(j)$ are 0.8165 and 1.7407, respectively. Figure 1 shows the matlab commands used and their output.

Exercise 5

5.1 $\sin(A+B) = \sin A \cos B$

Proof. Let $\theta = A + B$. Using euler's formula, we know that

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{5.1}$$

Substitute $\theta = A + B$ into Equation (5.1).

$$\sin(A+B) = \frac{e^{jA}e^{jB} - e^{-jA}e^{-jB}}{2} \tag{5.2}$$

Substitute $e^{j\theta} = \cos\theta + j\sin\theta$ into the RHS of Equation (5.2) and simplify.

$$\frac{(\cos A + j\sin A)(\cos B + j\sin B) - (\cos A - j\sin A)(\cos B - j\sin B)}{2j} \tag{5.3}$$

$$\sin(A+B) = \frac{2j\cos A\sin B + \sin A\cos B}{2j} \tag{5.4}$$

$$\sin(A+B) = \cos A\sin B + \sin A\cos B$$ □

5.2 $\cos 2\theta = 1 - 2\sin^2\theta$

Proof. Let $\theta = A = B$. We know that,

$$2\cos\theta = e^{j\theta} + e^{-j\theta} \tag{5.5}$$

Substitute $\theta = A + B$.

$$2\cos(A+B) = e^{jA}e^{jB} + e^{-jA}e^{-jB} \tag{5.6}$$

$\theta = A = B$ implies that,

$$2\cos 2\theta = e^{j\theta}e^{j\theta} + e^{-j\theta}e^{-j\theta} \tag{5.7}$$

Examine the RHS and expand the two squared terms.

$$= (\cos\theta + j\sin\theta)^2 + (\cos\theta - j\sin\theta)^2 \tag{5.8}$$

$$= (\cos^2\theta + 2j\sin\theta\cos\theta - \sin^2\theta) + (\cos^2\theta - 2j\sin\theta\cos\theta - \sin^2\theta) \tag{5.9}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{5.10}$$

Using identity $\sin^2 + \cos^2 = 1$.

$$\cos 2\theta = 1 - \sin^2\theta - \sin^2\theta \tag{5.11}$$

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \square$$

Exercise 6

- $\dot{x} + x - e^x = 0$ is 1st order, nonlinear, and homogeneous
- $\ddot{x} + t\dot{x} + x = 0$ is 2nd order, linear, and homogeneous
- $\dddot{x} + \dot{x} + x^2 - \cos t = 0$ is 3rd order, nonlinear, and homogeneous

Exercise 7

Given,

$$\dot{x} + x = 2e^{-t}, \quad x(0^-) = -1.$$

7.1 Homogeneous Solution

The homogeneous equation is given by

$$\dot{x} + x = 0 \tag{7.1}$$

Substitute $x(t) = e^{\lambda t}$ into Equation (7.1).

$$\lambda e^{\lambda t} + e^{\lambda t} = 0 \tag{7.2}$$
$$e^{\lambda t}(1 + \lambda) = 0 \tag{7.3}$$

Clearly, $\lambda = -1$. Using ICs we can find the value of C.

$$x(t) = Ce^{-t} \tag{7.4}$$
$$-1 = Ce^{0} \tag{7.5}$$
$$C = -1 \tag{7.6}$$

Therefore, $x_h = -e^{-t}$.

7.2 Particular Solution

The form of the particular solution cannot be proportional to the form of the homogeneous solution.

$$f(t) = 2e^{-t} \Rightarrow x_p(t) = Cte^{-t}.$$

Substitute x_p into the original differential equation.

$$C\left(e^{-t} - te^{-t}\right) + Cte^{-t} = 2e^{-t} \tag{7.7}$$

$$Ce^{-t} = 2e^{-t} \tag{7.8}$$

$$C = -2 \Rightarrow x_p(t) = 2te^{-t} \tag{7.9}$$

The total response is the sum of the particular and homogeneous solutions.

$$x = 2te^{-t} - e^{-t} \tag{7.10}$$

Exercise 8

Find the family of solutions for,

$$\ddot{x} + 2\dot{x} + x = \sin 2t.$$

8.1 Homogeneous Solution

The characteristic equation is given by,

$$\left(\lambda^2 + 2\lambda + 1\right) = 0 \tag{8.1}$$

$$(\lambda + 1)^2 = 0 \tag{8.2}$$

$$\lambda = -1, 1.$$

This implies that,

$$x_h(t) = C_1e^{-t} + C_2te^{-t} \tag{8.3}$$

8.2 Particular Solution

$$f(t) = \sin 2t \Rightarrow x_p = C \sin 2t.$$

Substitute x_p.

$$-4C \sin 2t + 4C \cos 2t + c \sin 2t = \sin 2t \tag{8.4}$$
$$C(\sin 2t - 4\sin 2t + 4\cos 2t) = \sin 2t \tag{8.5}$$
$$C(-3\sin 2t + 4\cos 2t) = \sin 2t \tag{8.6}$$
$$\tag{8.7}$$

$$C = \frac{\sin 2t}{-3\sin 2t + 4\cos 2t} \tag{8.8}$$

Therefore,

$$x_p = \frac{\sin^2 2t}{-3\sin 2t + 4\cos 2t} \tag{8.9}$$

$$x(t) = C_1 e^{-t} + C_2 t e^{-t} + \frac{\sin^2 2t}{-3\sin 2t + 4\cos 2t} \tag{8.10}$$

Exercise 9

Given two coupled ODEs,

$$3\dot{x}_1 + 5x_1 - 7x_2 = 5 \tag{9.1}$$
$$\dot{x}_2 + 4x_1 + 6x_2 = 0 \tag{9.2}$$

Take the derivative of Equation (9.1).

$$3\ddot{x}_1 + 5\dot{x}_1 - 7x_2 = 0 \tag{9.3}$$

$$\dot{x}_2 = \frac{3}{7}\ddot{x}_1 + \frac{5}{7}\dot{x}_1 \tag{9.4}$$

We can use Equation (9.4) to rewrite Equations (9.1) and (9.2) as one ODE.

$$\frac{3}{7}\ddot{x}_1 + \frac{5}{7}\dot{x}_1 + 4x_1 + 6x_2 = 0 \tag{9.5}$$