Mathematical Interpretation of Urban Growth Model

LaTeX

Zhengzi Zhou

School of Resource and Environmental Sciences
Wuhan University

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Outline

Model hypothesis

Bettencourt, L. M.Growth, innovation, scaling, and the pace of life in cities.

Scaling Power Law

$$Y_t = Y_0 N_t^{\beta}$$

$$y_t = C x_t^{\beta} \tag{1}$$

$$y_0 = C x_0^{\beta} \tag{2}$$

$$C = \frac{y_0}{x_0^{\beta}} \tag{3}$$

- Y_t, y_t:Material Resources at time t1
- Y_0, c:Nomarlization constant at time t0
- N_t, x_t:Population at time t1

Model Hypothesis

Urban Growth Equation

$$Y = RN + E\frac{dN}{dt}$$

$$y_t = Rx_t + E\frac{dx_t}{dt} \tag{4}$$

- **Interpretion**: Y are used for ***maintence and growth***
- **R**:per unit time to maintain an individual *on average*
- **E**:quantity consumed by a new onee

Mathematical Deduction

$$
\begin{aligned}
Rx_t + E\frac{dx_t}{d_t} &= Cx_t^{\beta} && (5) \\
x^{-\beta}\frac{dx}{dt} + \frac{R}{E}x^{1-\beta} &= \frac{C}{E} && (6) \\
z &= x^{1-\beta} && (7) \\
\frac{dz}{dt} &= (1-\beta)x^{-\beta}\frac{dx}{dt} && (8) \\
\frac{1}{1-\beta}\frac{dz}{dt} + \frac{R}{E}z &= \frac{C}{E} && (9) \\
Let \quad K = \frac{R(1-\beta)}{E} \quad &, \quad B = \frac{C(1-\beta)}{E} && (10) \\
z' + Kz &= B && (11)
\end{aligned}
$$

Mathematical Deduction

$$
\begin{aligned}
(ze^{Kt})' &= z'e^{kt} + Kze^{Kt} = Be^{Kt} && (12)\\
ze^{Kt} &= \int Be^{kt}dt = \frac{B}{K}e^{Kt} + S && (13)\\
x^{1-\beta} = z &= \frac{B}{K} + \frac{S}{e^{Kt}} = \frac{C}{R} + S \times e^{-Kt} && (14)\\
S &= (x_0^{1-\beta} - \frac{C}{R})e^{Kt_0} && (15)\\
x^{1-\beta} &= \frac{C}{R} + (x_0^{1-\beta} - \frac{C}{R})e^{-\frac{R(1-\beta)}{E}(t-t_0)} && (16)
\end{aligned}
$$

Solution

$$X_t = \left[\frac{Y_0}{R} + \left(X_0^{1-\beta} - \frac{Y_0}{R}\right) exp\left[-\frac{R}{E}(1-\beta)\right] t\right]^{\frac{1}{1-\beta}}$$

Case : $\beta = 1$

when β=1, the solution reduce to a exponential:

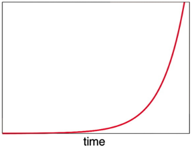
$$y_t = Rx_t + E\frac{dx_t}{dt} = Cx_t^{\beta} \tag{17}$$

$$\frac{dx_t}{x_t} = \frac{C-R}{E}dt \tag{18}$$

$$lnx_t = \frac{C-R}{E}t + s \tag{19}$$

$$x_t = x_0 e^{\frac{C-R}{E}t} \tag{20}$$

and the curve is:



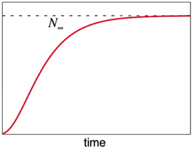
Case : $\beta < 1$

when $\beta < 1$,it leads to a sigmoidal curve :

$$x^{1-\beta} = \frac{C}{R} + (x_0^{1-\beta} - \frac{C}{R})e^{-\frac{R(1-\beta)}{E}(t-t_0)} \tag{21}$$

$$x_0^{1-\beta} - \frac{C}{R} = (x_0 - \frac{y_0}{R})x_0^{-\beta} < 0 \tag{22}$$

$$t \to \infty \quad , \quad x = (\frac{C}{R})^{\frac{1}{1-\beta}} \tag{23}$$



Thus, cities and social organizations that are driven by economics of scale are destined to eventually stop growth.

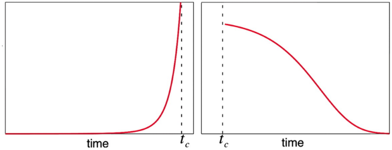
Case : $\beta > 1$

when $\beta > 1$

$$\frac{dx_t}{dt} = \frac{y_0}{Ex_o^{\beta}} x_t^{\beta} - \frac{R}{E} x_t \quad (24)$$

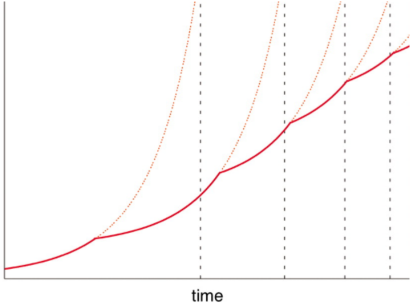
$$x_0 = (\frac{R}{Y_0})^{\frac{1}{\beta - 1}} \quad (25)$$

$$t_c = -\frac{E}{(\beta - 1)R} ln[1 - \frac{R}{C} x_0^{1-\beta}] \quad (26)$$



E_i/R_i and t_c

- E_i/R_i: the time needed for an average individual to reach productive maturity
- $t_i \propto t_c \approx \frac{1}{x_0^{\beta-1}}$:inovations arise at an accelerated rate



Discussion

1. The technique of **introducing the variable t**, and **mathematical function analysis**
2. Rationality analysis
   1. regard *population growth rate* as the constraints to resources Y
   2. ignore the **difference of consumption patterns** between resources
   3. give the conclusion that technological change *slows* population growth

Discussion

x	$0.5x^{1.15}$	
100	100	$100 \times 1 + 0 \times 1$
200	221	$200 \times 1 + 21 \times 1$
221	250	29
250	286	36
286	334	48
334	400	$334 \times 1 + 66 \times 1$
400	492	$400 \times 1.2 + 10 \times 1.2$
410	505	$410 \times 1.2 + 11 \times 1.2$
421	521	13

Discussion

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500	642	$500 \times 1.3 - 12 \times 1.3$
493	624	$494 \times 1.3 - 13 \times 1.3$