Periods of hyperelliptic integrals expressed in terms of θ-constants by means of Thomae formulae

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ABSTRACT. Expressions for the periods of first and second kind integrals on hyperelliptic curves are given in terms of θ-constants. They are derived with the help of Thomae's classical formulae and Picard-Fuchs equations for complete integrals as functions of the parameters of the curves. The example of genus two is considered in detail.

1. Introduction

A large variety of mathematical and applied problems involving algebraic curves demands for an effective description of the periods of Abelian integrals. For instance, frequencies and actions of completely integrable systems are often expressible in terms of complete hyperelliptic integrals so that it is desirable to have rapidly convergent series for their determination. Another application is the integration of KdV- and KP-type equations where it is useful to express winding vectors in the Its-Matveev formula [**IM75**] in terms of θ-constants (see e.g. [**Dub81, Mum84**]). This approach was pioneered in the eighties by the Russian school, following Novikov's conjecture on the link between KP-type equations and the classical Schottky problem which led to its new solution on the basis of the concept of complete integrability. The periods of hyperelliptic integrals also appear in a wider context of modern research on integrable systems – the Riemann-Hilbert problem and associated Schlesinger equation (see, e. g.,[**DIAZ99**]), Picard-Fuchs (e. g. [**DRVW01**]), and Knizhnik-Zamolodchikov equations (e. g. [**Smi93**]). The interrelation of these problems has attracted much attention of modern research in theoretical physics and applied mathematics.

The effective computation of complete elliptic integrals has a long history which goes back to Gauss' computation of these integrals in terms of the arithmetic-geometric mean. The generalization of his procedure to the case of genus two was recently developed in [**BM89**] and, surprisingly, was shown to be impossible for genera higher than three [**DL99**]. On the surface, the question seems to have only practical relevance, but underlying there is obviously a deep mathematical background.

The present paper is based on two classical ingredients from this context. Thomae's formulae [**Tho70**] are one of them. In [**Mes91**] they were combined with the arithmetic-geometric mean to obtain criteria for the reducibility of ultraelliptic Jacobians. In [**BR87, Nak97**] they were generalized to the case of Z_N curves, and applied to conformal field theory. In [**Smi93**] the Thomae formulae are used to solve the Knizhnik-Zamolodchikov equation, and in [**KK98, Kor01, EG02**] to construct the τ-function of the Schlesinger equation associated with hyperelliptic and more general curves. This list is far from being complete.

The other classical source of our investigations is Picard-Fuchs type equations for the periods of Abelian integrals. Their use was pioneered by Fuchs [**Fuc71**], Schlesinger [**Sch95**], and Bolza [**Bol99**].

We shall be concerned with the expression of periods of Abelian integrals in terms of θ-constants of the associated curve. In this perspective, the modular variable τ is considered to be the principal parameter; all properties of the curve such as branching points and periods of first or second kind integrals are expressed as modular functions.

To outline our approach we recall a number of well known facts from the theory of elliptic functions. The periods $2\omega, 2\eta$ and $2\omega', 2\eta'$ of Abelian integrals,

$$
(1.1) \qquad
\begin{aligned}
\omega &= -\int_{e_3}^{e_2} \frac{\mathrm{d}x}{y}, & \omega' &= -\int_{e_2}^{e_1} \frac{\mathrm{d}x}{y}, & \text{(first kind)} \\
\eta &= \int_{e_3}^{e_2} \frac{x\mathrm{d}x}{y}, & \eta' &= \int_{e_2}^{e_1} \frac{x\mathrm{d}x}{y}, & \text{(second kind)}
\end{aligned}
$$

on the curve (Weierstrass cubic)

$$
(1.2) \qquad y^2 = 4(x - e_1)(x - e_2)(x - e_3), \qquad e_1 + e_2 + e_3 = 0,
$$

are expressible in terms of θ-constants (see e.g. [**Wei93, BE55**]). In particular, when e_i are real and $e_1 > e_2 > e_3$, the following formulae hold (p. 43 f. in [**Wei93**]):

$$
(1.3) \qquad 2\omega = \frac{\pi\,\vartheta_2^2}{\sqrt{e_2 - e_3}} \qquad = \frac{\pi\,\vartheta_3^2}{\sqrt{e_1 - e_3}} \qquad = \frac{\pi\,\vartheta_4^2}{\sqrt{e_1 - e_2}},
$$

$$
(1.4) \qquad 2\eta = -2\omega e_1 - \frac{1}{2\omega}\frac{\vartheta_2''}{\vartheta_2} \qquad = -2\omega e_2 - \frac{1}{2\omega}\frac{\vartheta_3''}{\vartheta_3} \qquad = -2\omega e_3 - \frac{1}{2\omega}\frac{\vartheta_4''}{\vartheta_4}.
$$

The θ-constants ϑ_2, ϑ_3, ϑ_4 are the values of the corresponding Jacobi θ-functions $\vartheta_i(v|\tau)$ at zero argument $v = 0$; their derivatives $\partial\vartheta_i(v|\tau)/\partial v$ and $\partial^2\vartheta_i(v|\tau)/\partial v^2$ at $v = 0$ are abbreviated as ϑ_i' and ϑ_i'', respectively. The set of eqs. (1.3) can be interpreted as a special case of the first Thomae formula, see (3.1). It can also be used to express the branch points in terms of θ-constants: the relations

$$
(1.5) \qquad \frac{e_2 - e_3}{e_1 - e_3} = \frac{\vartheta_2^4}{\vartheta_3^4}, \qquad \frac{e_1 - e_2}{e_1 - e_3} = \frac{\vartheta_4^4}{\vartheta_3^4},
$$

together with a normalization condition such as $e_1 + e_2 + e_3 = 0$, suffice to determine the e_i from the ϑ_i.

The set of eqs. (1.4) is derived from eqs. (1.3) using, in the general case, the Picard-Fuchs equations (4.3), Legendre's relation (2.4), and the "heat equation" (2.18).

In the case of genus one, the steps are as follows. First, take the derivative of the i-th eq. (1.3) with respect to e_i; for example,

$$(1.6) \qquad \frac{\partial \omega}{\partial e_1} = \frac{2\omega}{\vartheta_2} \frac{\partial \vartheta_2}{\partial e_1} = \frac{2\omega}{\vartheta_2} \frac{\partial \vartheta_2}{\partial \tau} \frac{\partial \tau}{\partial e_1}.$$

Next, use the first of the Picard-Fuchs equations ($i \neq j \neq k = 1, 2, 3$)

$$(1.7) \qquad \frac{\partial \omega}{\partial e_i} = -\frac{1}{2} \frac{\eta + e_i \omega}{(e_i - e_j)(e_i - e_k)},$$

$$(1.8) \qquad \frac{\partial \eta}{\partial e_i} = \frac{1}{2} \frac{e_i \eta - (e_i^2 + e_j e_k)\omega}{(e_i - e_j)(e_i - e_k)},$$

to eliminate $\partial \omega / \partial e_i$. These equations are obtained by taking the derivatives $\partial / \partial e_i$ of the two differentials $\mathrm{d}x/y$ and $x\mathrm{d}x/y$, respectively, using the fact that exact differentials do not contribute to the periods. For example, from

$$(1.9) \qquad 4(e_i - e_j)(e_i - e_k)\frac{\partial}{\partial e_i}\frac{1}{y} = 2\frac{x - e_i}{y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{x - e_i}$$

we find eq. (1.7) because the last term is a total derivative.

In the third step, we use the Picard-Fuchs equations (which hold, respectively, for ω' and η' as well) and the Legendre relation

$$(1.10) \qquad \eta\omega' - \omega\eta' = \frac{\pi\mathrm{i}}{2}$$

to compute

$$(1.11) \qquad \frac{\partial \tau}{\partial e_1} = \frac{\partial}{\partial e_1}\frac{\omega'}{\omega} = \frac{1}{4\omega^2}\frac{\pi\mathrm{i}}{(e_1 - e_2)(e_1 - e_3)}.$$

Finally, inserting this and the "heat equation"

$$(1.12) \qquad 4\pi\mathrm{i}\frac{\partial \vartheta_i(v|\tau)}{\partial \tau} = \frac{\partial^2 \vartheta_i(v|\tau)}{\partial v^2}$$

into (1.6), we obtain the first of eqs. (1.4). Adding all three equations and using $e_1 + e_2 + e_3 = 0$, we obtain

$$(1.13) \qquad \eta = -\frac{1}{12\omega}\Big(\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_4''}{\vartheta_4}\Big).$$

The second Thomae formula (3.4) allows for a generalization of Jacobi's derivative formula

$$(1.14) \qquad \vartheta_1' = \pi\vartheta_2\vartheta_3\vartheta_4$$

to higher genera, where it is called Riemann-Jacobi formula. Taking its derivative with respect to τ and using the heat equation again, one finds

$$(1.15) \qquad \frac{\vartheta_1'''}{\vartheta_1'} = \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_4''}{\vartheta_4},$$

which turns (1.13) into the remarkably simple relation

$$(1.16) \qquad \eta = -\frac{1}{12\omega}\frac{\vartheta_1'''}{\vartheta_1'}.$$

In what follows we shall generalize the above derivations to higher genera.
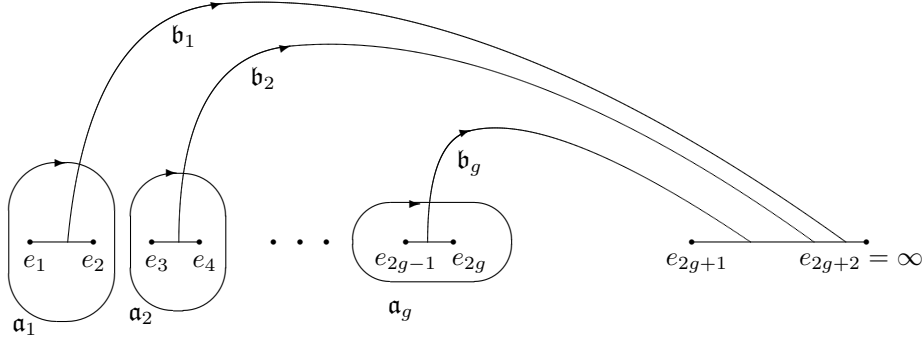
FIGURE 1. A homology basis on a Riemann surface of the hyperelliptic curve of genus g with real branching points $e_1, \ldots, e_{2g+2} = \infty$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i} for $i = 1, \ldots, g + 1$. The $\mathfrak{b}$-cycles are completed on the lower sheet (the picture on lower sheet is just flipped horizontally).

The paper is organized as follows. In section 2 we introduce canonical differentials and recall known facts about θ-functions and the theory of their characteristics. In section 3 we discuss the classical Thomae formulae (without proofs) and the Riemann-Jacobi relation. In section 4 we derive the appropriate Picard-Fuchs equations, and on their basis give expressions for the periods of meromorphic differentials in terms of θ-constants. In the last section, we consider the cases of genus 2 and 3 as examples.

2. Preliminaries

Let $V(x, y)$ be the hyperelliptic curve given by the equation

$$(2.1) \qquad y^2 = \sum_{i=0}^{2g+1} \lambda_i x^i = 4 \prod_{k=1}^{2g+1} (x - e_k) = R(x),$$

realized as a two sheeted covering over the Riemann sphere branched in the points $(e_k, 0)$, $k \in \mathcal{G} = \{1, \ldots, 2g + 1\}$, with $e_j \neq e_k$ for $j \neq k$, and at infinity, $e_{2g+2} = \infty$. Notice we do not require the e_k to be real. However, when they are real, we find it convenient to order them according to $e_1 < e_2 < \ldots < e_{2g+1}$, i.e., in the opposite way as compared to the Weierstrass ordering, see Fig. 1.

2.1. Canonical differentials. Given a canonical homology basis $\mathfrak{a}_1, \ldots, \mathfrak{a}_g$; $\mathfrak{b}_1, \ldots, \mathfrak{b}_g$ as shown in Fig. 1, choose canonical holomorphic differentials (first kind) $\mathrm{d}\boldsymbol{u}^t = (\mathrm{d}u_1, \ldots, \mathrm{d}u_g)$ and associated meromorphic differentials (second kind) $\mathrm{d}\boldsymbol{r}^t = (\mathrm{d}r_1, \ldots, \mathrm{d}r_g)$ in such the way that their periods

$$(2.2) \qquad 2\omega = \left(\oint_{\mathfrak{a}_k} \mathrm{d}u_i \right)_{i,k=1,\ldots,g} \qquad\qquad 2\omega' = \left(\oint_{\mathfrak{b}_k} \mathrm{d}u_i \right)_{i,k=1,\ldots,g}$$

$$(2.3) \qquad 2\eta = \left(-\oint_{\mathfrak{a}_k} \mathrm{d}r_i \right)_{i,k=1,\ldots,g} \qquad\qquad 2\eta' = \left(-\oint_{\mathfrak{b}_k} \mathrm{d}r_i \right)_{i,k=1,\ldots,g}$$

satisfy the generalized Legendre relation

$$(2.4) \qquad \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^t = -\frac{1}{2}\pi i \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}.$$

Such a basis of differentials can be realized as follows (see [**Bak95**], p. 195):

$$(2.5) \quad \mathrm{d}\boldsymbol{u}(x,y) = \frac{\boldsymbol{\mathcal{U}}(x)\mathrm{d}x}{y}, \quad \mathcal{U}_i(x) = x^{i-1}, \qquad i = 1\ldots, g,$$

$$(2.6) \quad \mathrm{d}\boldsymbol{r}(x,y) = \frac{\boldsymbol{\mathcal{R}}(x)\mathrm{d}x}{4y}, \quad \mathcal{R}_i(x) = \sum_{k=i}^{2g+1-i} (k+1-i)\lambda_{k+1+i}x^k, \qquad i = 1\ldots, g.$$

It is convenient to use normalized holomorphic differentials,

$$(2.7) \qquad \mathrm{d}\boldsymbol{v} = (2\omega)^{-1}\mathrm{d}\boldsymbol{u}.$$

DEFINITION 2.1. The Kleinian bi-differential $\mathrm{d}\omega(x,y;z,w)$ is defined on $V \times V$ by the following properties:

- it is symmetric

$$(2.8) \qquad \mathrm{d}\omega(x,y;z,w) = \mathrm{d}\omega(z,w;x,y)$$

- it has only one pole of second order along the diagonal $x = z$, in the vicinity of which

$$(2.9) \qquad \mathrm{d}\omega(x,y;z,w) = \left(\frac{1}{(\xi-\eta)^2} + O(1)\right)\mathrm{d}\xi\mathrm{d}\eta,$$

where ξ and η are local coordinates of the points (x,y) and (z,w) respectively, in the neighborhood of a point (X,Y).

It is known that this bi-differential can be realized as [**Kle86, Kle88**]

$$(2.10) \qquad \mathrm{d}\omega(x,y;z,w) = \frac{2yw + F(x,z)}{4(x-z)^2}\frac{\mathrm{d}x}{y}\frac{\mathrm{d}z}{w},$$

where

$$(2.11) \qquad F(x,z) = 2w^2 + 2(x-z)w\frac{\mathrm{d}w}{\mathrm{d}z} + (x-z)^2\,\boldsymbol{\mathcal{U}}^t(x)\boldsymbol{\mathcal{R}}(z).$$

Below we shall use the equivalence

$$(2.12) \qquad \frac{2yw + F(x,z)}{y(x-z)^2} = 2\frac{\partial}{\partial x}\frac{y+w}{2(z-x)} + \frac{\boldsymbol{\mathcal{U}}^t(z)\boldsymbol{\mathcal{R}}(x)}{y}.$$

We denote by $\mathrm{Jac}(V)$ the *Jacobian* of the curve V, i.e. the factor $\mathbb{C}^g/\Gamma$, where $\Gamma = 2\omega \oplus 2\omega'$ is the lattice generated by the periods of canonical holomorphic differentials.

Let $\mathcal{D}$ be a divisor of degree 0, $\mathcal{D} = \mathcal{X} - \mathcal{Z}$, with $\mathcal{X}$ and $\mathcal{Z}$, the effective divisors, of degree $\deg \mathcal{X} = \deg \mathcal{Z} = n$, given by $\mathcal{X} = \{(x_1,y_1),\ldots,(x_n,y_n)\} \in (V)^n$ and $\mathcal{Z} = \{(z_1,w_1),\ldots,(z_n,w_n)\} \in (V)^n$, where $(V)^n$ is the n-th symmetric power of V.

The *Abel map*

$$(2.13) \qquad \mathfrak{A} : (V)^n \to \mathrm{Jac}(V)$$

puts into correspondence the divisor $\mathcal{D}$, with fixed $\mathcal{Z}$, and the point $\boldsymbol{u} \in \mathrm{Jac}(V)$, according to

$$(2.14) \qquad \boldsymbol{u} = \int_{\mathcal{Z}}^{\mathcal{X}} \mathrm{d}\boldsymbol{u}, \qquad \text{or} \qquad u_i = \sum_{k=1}^{n}\int_{z_k}^{x_k}\mathrm{d}u_i, \qquad i = 1,\ldots,g.$$

(For notational convenience we write x_k instead of (x_k, y_k) etc. whenever there is no danger of confusion.)

2.2. θ-functions. Let $\mathcal{H}_g = \{\tau^t = \tau, \quad \mathrm{Im}\,\tau \geq 0\}$ be the Siegel half space of degree g, where $\tau = \omega^{-1}\omega'$ is the period matrix. The hyperelliptic θ–function, $\theta : \mathrm{Jac}(V) \times \mathcal{H}_g \to \mathbb{C}$, with characteristics

$$(2.15) \qquad [\varepsilon] = \left[\begin{array}{c} \boldsymbol{\varepsilon}'^t \\ \boldsymbol{\varepsilon}^t \end{array} \right] = \left[\begin{array}{ccc} \varepsilon_1' & \cdots & \varepsilon_g' \\ \varepsilon_1 & \cdots & \varepsilon_g \end{array} \right] \in \mathbb{R}^{2g}$$

is defined as the Fourier series

$$(2.16) \qquad \theta[\varepsilon](\boldsymbol{v}|\tau) = \sum_{\boldsymbol{m} \in \mathbb{Z}^g} \exp \pi \mathrm{i} \left\{ (\boldsymbol{m} + \boldsymbol{\varepsilon}')^t \tau (\boldsymbol{m} + \boldsymbol{\varepsilon}') + 2(\boldsymbol{v} + \boldsymbol{\varepsilon})^t (\boldsymbol{m} + \boldsymbol{\varepsilon}') \right\}$$

and has the periodicity properties

$$(2.17) \quad \theta[\varepsilon](\boldsymbol{v} + \boldsymbol{n} + \tau \boldsymbol{n}'|\tau) =$$
$$\exp\left\{-2\mathrm{i}\pi \boldsymbol{n}'^t (\boldsymbol{v} + \frac{1}{2}\tau \boldsymbol{n}')\right\} \exp\left\{2\mathrm{i}\pi(\boldsymbol{n}^t \boldsymbol{\varepsilon}' - \boldsymbol{n}'^t \boldsymbol{\varepsilon})\right\} \theta[\varepsilon](\boldsymbol{v}|\tau).$$

The θ-functions satisfy the "heat equation"

$$(2.18) \qquad \frac{\partial^2}{\partial v_k \partial v_l} \theta[\varepsilon](\boldsymbol{v}|\tau) = 2(1 + \delta_{kl}) \pi \mathrm{i} \frac{\partial}{\partial \tau_{kl}} \theta[\varepsilon](\boldsymbol{v}|\tau).$$

In all the following, the values ε_k, ε_k' will either be 0 or $\frac{1}{2}$. It is then typographically convenient to introduce the notation

$$(2.19) \qquad \theta[\varepsilon] \equiv \theta\left[\frac{1}{2} 2\varepsilon\right] =: \theta[2\varepsilon]_2.$$

The equality (2.17) implies that

$$(2.20) \qquad \theta[\varepsilon](-\boldsymbol{v}|\tau) = \mathrm{e}^{-4\pi \mathrm{i} \boldsymbol{\varepsilon}^t \boldsymbol{\varepsilon}'} \theta[\varepsilon](\boldsymbol{v}|\tau),$$

and therefore the function $\theta[\varepsilon](\boldsymbol{v}|\tau)$, with characteristics $[\varepsilon]$ of only half-integers, is even when $4\boldsymbol{\varepsilon}^t \boldsymbol{\varepsilon}'$ is an even integer, and odd otherwise. Correspondingly, $[\varepsilon]$ is called even or odd, and among the 4^g half-integer characteristics there are $\frac{1}{2}(4^g + 2^g)$ even and $\frac{1}{2}(4^g - 2^g)$ odd characteristics.

DEFINITION 2.2. The non-vanishing values of θ-functions with even characteristics, at $\boldsymbol{v} = \vec{0}$, are called θ-constants of the first kind,

$$(2.21) \qquad \theta[\varepsilon](\boldsymbol{0}|\tau) =: \theta[\varepsilon] \qquad \text{for even } [\varepsilon];$$

the non-vanishing values of the first derivatives of θ-functions with odd characteristics, at zero argument, are called non-singular θ-constants of the second kind,

$$(2.22) \qquad \frac{\partial}{\partial v_k} \theta[\delta](\boldsymbol{v}|\tau)\big|_{\boldsymbol{v}=0} =: \theta_k[\delta], \qquad k = 1, \ldots, g, \qquad \text{for odd } [\delta].$$

We stress that the derivatives are taken with respect to *normalized* variables v_i. This will turn out to be important for the validity of formula (3.4).

2.3. Characteristics. Identify each branching point e_j of the curve V with a vector

$$(2.23) \qquad \mathfrak{A}_j = \int_\infty^{e_j} \mathrm{d}\boldsymbol{v} =: \boldsymbol{\varepsilon}_j + \tau \boldsymbol{\varepsilon}'_j \in \mathrm{Jac}(V),$$

where $\mathrm{d}\boldsymbol{v}$ is the vector of normalized holomorphic differentials. Modulo 1, all components of the vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are either 0 or $\frac{1}{2}$. The $2 \times g$ matrices

$$(2.24) \qquad [\mathfrak{A}_j] = \left[\begin{array}{c} \boldsymbol{\varepsilon}'^t_j \\ \boldsymbol{\varepsilon}^t_j \end{array} \right] = \left[\begin{array}{ccc} \varepsilon'_{j_1} & \cdots & \varepsilon'_{j_g} \\ \varepsilon_{j_1} & \cdots & \varepsilon_{j_g} \end{array} \right]$$

will serve as a basis for the characteristics of the θ-functions to be discussed.

Let us identify the half periods $\mathfrak{A}_i$, $i = 1, \ldots, 2g+1$ (see e. g. [**FK80**], p. 303). Evidently, $[\mathfrak{A}_{2g+2}] = [0]$. Using the notation $\boldsymbol{f}_k = \frac{1}{2}(\delta_{1k}, \ldots, \delta_{gk})^t$ (where δ_{ik} is the Kronecker symbol) and $\boldsymbol{\tau}_k$ for the k-th column vector of the matrix τ, we find

$$\mathfrak{A}_{2g+1} = \mathfrak{A}_{2g+2} - \sum_{k=1}^{g} \int_{e_{2k-1}}^{e_{2k}} \mathrm{d}\boldsymbol{v} = \sum_{k=1}^{g} \boldsymbol{f}_k, \quad \rightarrow \quad [\mathfrak{A}_{2g+1}] = \frac{1}{2} \left[\begin{array}{ccccc} 0 & 0 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 1 & 1 \end{array} \right],$$

$$\mathfrak{A}_{2g} = \mathfrak{A}_{2g+1} - \int_{e_{2g+1}}^{e_{2g}} \mathrm{d}\boldsymbol{v} = \sum_{k=1}^{g} \boldsymbol{f}_k + \boldsymbol{\tau}_g, \quad \rightarrow \quad [\mathfrak{A}_{2g}] = \frac{1}{2} \left[\begin{array}{ccccc} 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 1 \end{array} \right],$$

$$\mathfrak{A}_{2g-1} = \mathfrak{A}_{2g} - \int_{e_{2g}}^{e_{2g-1}} \mathrm{d}\boldsymbol{v} = \sum_{k=1}^{g-1} \boldsymbol{f}_k + \boldsymbol{\tau}_g, \quad \rightarrow \quad [\mathfrak{A}_{2g-1}] = \frac{1}{2} \left[\begin{array}{ccccc} 0 & 0 & \ldots & 0 & 1 \\ 1 & 1 & \ldots & 1 & 0 \end{array} \right].$$

Continuing in the same manner, we get for arbitrary $k > 1$

$$[\mathfrak{A}_{2k+2}] = \frac{1}{2} \left[\begin{array}{ccccccccc} \overbrace{0 & 0 & \ldots & 0}^{k} & 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 & 1 & 0 & \ldots & 0 \end{array} \right],$$

$$[\mathfrak{A}_{2k+1}] = \frac{1}{2} \left[\begin{array}{ccccccccc} \overbrace{0 & 0 & \ldots & 0}^{k} & 1 & 0 & \ldots & 0 \\ 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \end{array} \right],$$

and finally

$$[\mathfrak{A}_2] = \frac{1}{2} \left[\begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{array} \right], \qquad [\mathfrak{A}_1] = \frac{1}{2} \left[\begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{array} \right].$$

The characteristics with even indices, corresponding to the branching points e_{2n}, $n = 1, \ldots, g$, are odd (except for $[\mathfrak{A}_{2g+2}]$ which is zero); the others are even. The vector of Riemann constants has the form

$$(2.25) \qquad \boldsymbol{K}_\infty = -\sum_{k=1}^{g} \int_\infty^{e_{2k}} \mathrm{d}\boldsymbol{v} = -\sum_{k=1}^{g} \mathfrak{A}_{2k}$$

(see e.g. [**FK80**], p. 305, for a proof).

The $2g+2$ characteristics $[\mathfrak{A}_i]$ serve as a basis for the construction of all 4^g possible half integer characteristics $[\varepsilon]$. There is a one-to-one correspondence between these $[\varepsilon]$ and partitions of the set $\bar{\mathcal{G}} = \{1, \ldots, 2g+2\}$ of indices of the branching

points ([**Fay73**], p. 13; [**Bak95**], p. 271). Notice the distinction between $\mathfrak{G}$ and $\bar{\mathfrak{G}} = \mathfrak{G} \cup \{\infty\}$. The partitions of interest are

$$(2.26) \quad \bar{\mathfrak{G}} = \mathfrak{I}_m \cup \mathfrak{J}_m, \qquad \mathfrak{I}_m = \{i_1, \dots, i_{g+1-2m}\}, \qquad \mathfrak{J}_m = \{j_1, \dots, j_{g+1+2m}\},$$

where m is any integer between 0 and $\left[\frac{g+1}{2}\right]$. The corresponding characteristic $[\varepsilon_m]$ is defined by the vector

$$(2.27) \qquad \boldsymbol{E}_m = \sum_{k=1}^{g+1-2m} \boldsymbol{\mathfrak{A}}_{i_k} + \boldsymbol{K}_\infty =: \boldsymbol{\varepsilon}_m + \tau \boldsymbol{\varepsilon}'_m.$$

It can be seen that characteristics with even m are even, and with odd m odd. There are $\frac{1}{2}\binom{2g+2}{g+1}$ different partitions with $m = 0$, $\binom{2g+2}{g-1}$ different with $m = 1$, and so on down to $\binom{2g+2}{1} = 2g+2$ for even g and $m = g/2$, or $\binom{2g+2}{0} = 1$ for odd g and $m = (g+1)/2$. (In the last case, the set $\mathfrak{I}_{(g+1)/2}$ is the empty set, and the only possible characteristic derives from $\boldsymbol{E}_{(g+1)/2} = \boldsymbol{K}_\infty$.) One may check that the total number of even (odd) characteristics is indeed $2^{2g-1} \pm 2^{g-1}$.

According to the Riemann theorem on the zeros of θ-functions, $\theta(\boldsymbol{E}_m + \boldsymbol{v})$ vanishes to order m at $\boldsymbol{v} = 0$. We shall call the characteristics with $m = 0$ *non-singular even characteristics* and those with $m = 1$ *non-singular odd characteristics*. The remaining characteristics, i.e., those with $m > 1$, are called *singular characteristics*. In the following we only deal with non-singular characteristics.

The correspondence between characteristics and partitions suggests the notation $\theta\{\mathfrak{I}_m\}(\boldsymbol{v}|\tau) \equiv \theta[\varepsilon_m](\boldsymbol{v}|\tau)$; we shall use whatever is more convenient.

Let us now take care of the fact that $e_{2g+2} = \infty$ is taken as the base point of the Abel map. It belongs to $\mathfrak{I}_m$ for even m and to $\mathfrak{J}_m$ for odd m. As we are only interested in $m = 0$ and $m = 1$, we decide to omit the point ∞ from consideration and define

$$(2.28) \qquad \mathfrak{I}_0 = \{i_1, \dots, i_g\} \in \mathfrak{G}, \qquad \mathfrak{I}_1 = \{i_1, \dots, i_{g-1}\} \in \mathfrak{G}.$$

That permits us to denote non-singular θ-constants of the first kind as $\theta\{i_1, \dots, i_g\}$, or $\theta\{\mathfrak{I}_0\}$, and non-singular θ-constants of the second kind as $\theta_k\{i_1, \dots, i_{g-1}\}$, or $\theta_k\{\mathfrak{I}_1\}$.

Let us put in correspondence with each partition $\mathfrak{I}_m \cup \mathfrak{J}_m$, $m = 0, 1$, the product of Vandermonde determinants

$$(2.29) \qquad \Delta(\mathfrak{I}_m) = \prod_{i_k < i_l \in \mathfrak{I}_m} (e_{i_k} - e_{i_l}) \prod_{j_k < j_l \in \mathfrak{J}_m} (e_{j_k} - e_{j_l})$$

and denote as $s_k(\mathfrak{I}_1)$ the elementary symmetric functions of degree k of elements e_i with $i \in \mathfrak{I}_1$, i.e.

$$(2.30) \qquad s_0(\mathfrak{I}_1) = 1, \quad s_1(\mathfrak{I}_1) = \sum_{i \in \mathfrak{I}_1} e_i, \quad \dots \quad s_{g-1}(\mathfrak{I}_1) = \prod_{i \in \mathfrak{I}_1} e_i.$$

3. Thomae formulae

In this section we exhibit classical results of Thomae [**Tho70**]. The Thomae formula which links branching points with nonsingular even θ–constants, i.e. θ–constants of the first kind, is well known and widely used. We shall also implement another Thomae formula, written for θ-constants of the second kind. We refer

to these formulae as first and second Thomae theorems. The important Riemann-Jacobi derivative relation which generalizes to higher genera Jacobi's relation (1.14) follows from the second Thomae theorem.

The θ-constants of the first kind are expressed in terms of branching points and periods of holomorphic integrals as follows:

THEOREM 3.1 (**First Thomae theorem**). *Let* $\mathcal{I}_0 \cup \mathcal{J}_0$ *be a partition of the set* $\mathcal{G} = \{1, \ldots, 2g+1\}$ *of indices of the finite branching points of the hyperelliptic curve* V. *Then the following formula is valid*

$$(3.1) \qquad \theta^4\{\mathcal{I}_0\} = \pm\frac{(\det 2\omega)^2}{(\mathrm{i}\pi)^{2g}}\Delta(\mathcal{I}_0).$$

The proof can be found in many places, see e. g. [**Tho70, Bol99, Fay73, Mum84**]. There are $\binom{2g+1}{g}$ different possibilities to choose the set $\mathcal{I}_0$. For $g = 1$ this gives the three equations (1.3), the case $g = 2$ is explicitly discussed in section 5.

Among various corollaries of the Thomae formula we shall single out the following two:

COROLLARY 3.2. *Let* $\mathcal{S} = \{i_1, \ldots, i_{g-1}\}$ *and* $\mathcal{T} = \{j_1, \ldots, j_{g-1}\}$ *be two disjoint sets of non-coinciding integers taken from the set* $\mathcal{G}$ *of indices of the finite branching points. Then for any two* $k \neq l$ *from the set* $\mathcal{G}\backslash(\mathcal{S} \cup \mathcal{T})$ *the following formula is valid*

$$(3.2) \qquad \frac{e_l - e_m}{e_k - e_m} = \epsilon\frac{\theta^2\{k, \mathcal{S}\}\theta^2[\{k, \mathcal{T}\}}{\theta^2\{l, \mathcal{S}\}\theta^2\{l, \mathcal{T}\}},$$

where m *is the remaining number when* $\mathcal{S}$, $\mathcal{T}$, k, l *are taken away from* $\mathcal{G}$, *and* $\epsilon^4 = 1$.

COROLLARY 3.3. *Let* $\mathcal{I}_0 = \{i_1, \ldots, i_g\}$ *and* $\mathcal{J}_0 = \{j_1, \ldots j_{g+1}\}$ *be the partition. Choose* $k, n \in \mathcal{I}_0$ *and* $i, j \in \mathcal{J}_0$. *Define the sets* $\mathcal{S}_k = \mathcal{I}_0\backslash\{k\}$, $\mathcal{S}_{k,n} = \mathcal{I}_0\backslash\{k, n\}$, $\mathcal{T}_{i,j} = \mathcal{J}_0\backslash\{i, j\}$. *Then*

$$(3.3) \qquad \frac{\prod\limits_{j_l \in \mathcal{J}_0}(e_k - e_{j_l})}{\prod\limits_{i_l \in \mathcal{I}_0, i_l \neq k}(e_k - e_{i_l})(e_k - e_n)^2} = \frac{\pm\theta^4\{i, \mathcal{S}_k\}\theta^4\{j, \mathcal{S}_k\}\theta^4\{n, \mathcal{T}_{i,j}\}}{\theta^4\{i, j, \mathcal{S}_{k,n}\}\theta^4\{i, \mathcal{T}_{i,j}\}\theta^4\{j, \mathcal{T}_{i,j}\}}.$$

Recent discussions concerning the proof of (3.2) can be found in [**Tak96, Koi97**] where the formula is called "generalization of Rosenhain's normal form". There is considerable freedom in the choice of combination of characters on the right hand side of formula (3.2). H. Farkas has used this to derive in an elegant and simple way Schottky's conditions for the case of hyperelliptic curves of genus 4 [**Far71**].

The signs $\pm$ and the values of ϵ should be determined in each particular case by some limiting procedure.

The Thomae paper [**Tho70**], see also [**KW15**], contains another set of formulae expressing the nonsingular θ-constants of the second kind in terms of branching points and periods of Abelian differentials:

THEOREM 3.4 (**Second Thomae theorem**). *Let* $\mathcal{I}_1 \cup \mathcal{J}_1$ *be a partition of the set* $\mathcal{G}$ *of indices of the finite branching points, and* $v_1, \ldots, v_g$ *the normalized holomorphic integrals. Then the* θ-*constants of the second kind are given by the formula*

$$(3.4) \qquad \frac{\partial}{\partial v_i}\theta\{\mathcal{I}_1\}(\boldsymbol{v}|\tau)\big|_{\boldsymbol{v}=0} =: \theta_j\{\mathcal{I}_1\} = 2\epsilon\sqrt{\frac{\det 2\omega}{\pi^g}}\Delta(\mathcal{I}_1)^{\frac{1}{4}}\sum_{i=1}^g \omega_{ij}s_{g-i}(\mathcal{I}_1),$$

where $s_l(\mathcal{I}_1)$ is the elementary symmetric function of degree l associated with the set $\mathcal{I}_1$ of indices of the branching points.

It is convenient to rewrite this Thomae theorem in matrix form. To do that we introduce for any set of nonsingular odd characteristics $[\delta_1], \ldots, [\delta_g]$ the Jacobi matrix

$$(3.5) \qquad D[\delta_1, \ldots, \delta_g] = \begin{pmatrix} \theta_1[\delta_1] & \theta_1[\delta_2] & \ldots & \theta_1[\delta_g] \\ \vdots & \vdots & \ldots & \vdots \\ \theta_g[\delta_1] & \theta_g[\delta_2] & \ldots & \theta_g[\delta_g] \end{pmatrix}.$$

THEOREM 3.5. *Let $\mathcal{I}_0 = \{i_1, \ldots, i_g\}$ and $\mathcal{J}_0 = \{j_1, \ldots j_{g+1}\}$ be the sets of a partition $\mathcal{I}_0 \cup \mathcal{J}_0 = \mathcal{G}$. Define the g sets $\mathcal{S}_k = \mathcal{I}_0 \backslash \{i_k\}$ and use the correspondence $[\delta_k] \Leftrightarrow \{\mathcal{S}_k\}$, $k = 1, \ldots, g$, for nonsingular odd characteristics. Then*

$$(3.6) \qquad D[\delta_1, \ldots, \delta_g] = \epsilon \sqrt{\frac{\det 2\omega}{\pi^g}} \, 2\omega^t \, S \, M,$$

where $\epsilon^8 = 1$; the matrices S and M are given as

$$(3.7) \qquad \begin{aligned} S &= \left(s_{g-i}(\mathcal{S}_k) \right)_{k,i=1,\ldots,g}, \\ M &= \mathrm{diag}\left(\sqrt[4]{\Delta(\mathcal{S}_1)}, \ldots, \sqrt[4]{\Delta(\mathcal{S}_g)} \right), \end{aligned}$$

where the $s_{g-i}(\mathcal{S}_k)$ are the symmetric functions (2.30).

Moreover, by choosing any $n \in \mathcal{S}_k$ and $i, j \in \mathcal{J}_0$, the formula (3.6) is transformed to

$$(3.8) \qquad D[\delta_1, \ldots, \delta_g] = \epsilon \, 2\omega^t \, S \, N,$$

with

$$N = \theta\{\mathcal{I}_0\} \times \mathrm{diag}\left(\ldots, \sqrt{e_k - e_n} \frac{\theta\{i, \mathcal{S}_k\}\theta\{j, \mathcal{S}_k\}\theta\{n, \mathcal{T}_{i,j}\}}{\theta\{i, j, \mathcal{S}_{k,n}\}\theta\{i, \mathcal{T}_{i,j}\}\theta\{j, \mathcal{T}_{i,j}\}}, \ldots \right)_{k=1,\ldots,g},$$

where we defined the sets $\mathcal{S}_{k,n} := \mathcal{I}_0 \backslash \{k, n\}$, $\mathcal{T}_{i,j} := \mathcal{J}_0 \backslash \{i, j\}$.

PROOF. The formula (3.6) is the matrix version of formula (3.4). To write (3.6) in the form (3.8), we use (3.1) to obtain for every $k = 1, \ldots, g$

$$\sqrt{\frac{\det 2\omega}{\pi^g}} \sqrt[4]{\Delta(\mathcal{S}_k)} = \epsilon \theta\{\mathcal{I}_0\} \sqrt[4]{\frac{\Delta(\mathcal{S}_k)}{\Delta(\mathcal{I}_0)}}.$$

The quotient under the sign of the fourth root is exactly the left hand side of the equality (3.3). $\qquad \square$

As an immediate corollary of the second Thomae theorem we obtain

THEOREM 3.6 (**Riemann-Jacobi formula**). *Fix g different positive integers $\{i_1, \ldots, i_g\} =: \mathcal{I}_0$ of the set $\mathcal{G}$, and let $\mathcal{J}_0 = \{j_1, \ldots, j_{g+1}\}$ be the complementary set. Define the g sets $\mathcal{S}_k = \mathcal{I}_0 \backslash \{i_k\}$ and use the correspondence $[\delta_k] \Leftrightarrow \{\mathcal{S}_k\}$, $k = 1, \ldots, g$, for nonsingular odd characteristics. Similarly, define $g + 2$ sets $\mathcal{T}_l$ where $\mathcal{T}_0 = \mathcal{I}_0$ and $\mathcal{T}_l = \mathcal{J}_0 \backslash \{j_l\}$ $l = 1, \ldots, g + 1$, and use the correspondence $[\varepsilon_l] \Leftrightarrow \{\mathcal{T}_l\}$ with the $g + 2$ nonsingular even characteristics. Then the following formula is valid*

$$(3.9) \qquad \det D[\delta_1, \ldots, \delta_g] = \pm \pi^g \theta[\varepsilon_0]\theta[\varepsilon_1] \cdots \theta[\varepsilon_{g+1}].$$

PROOF. Compute the determinant of both sides of the matrix equality (3.6)

$$\det D[\delta_1, \ldots, \delta_g] = \epsilon \, (\det 2\omega)^{\frac{g+2}{2}} \, \pi^{\frac{-g^2}{2}} \det M \det S.$$

One can see that the product

$$(3.10) \qquad \det M^4 \det S^4 = \prod_{l=0}^{g+1} \Delta(\mathcal{T}_l),$$

where the partitions $\mathcal{T}_l$ are given in the formulation of the theorem. The final formula follows immediately after expressing each $\Delta(\mathcal{T}_l)$ in terms of θ-constants $\theta\{\mathcal{T}_l\}$ by the formula (3.1). Our analysis enables us to give the exact value of ϵ from which it follows that the only remaining ambiguity in (3.9) is the $\pm$ sign, corresponding to the antisymmetry of the determinant. $\qquad \square$

Formula (3.9) was called generalized Riemann-Jacobi formula by Fay [**Fay79**]. Its general theory, including non-hyperelliptic curves, was developed in [**Igu79, Igu80, Igu82**]. In the elliptic case $g = 1$ it reduces to (1.14).

By inverting Eq. (3.8) we obtain the periods of the first kind and their inverse matrix ρ, i.e., the normalizing constants for the holomorphic differentials, see (2.7), in terms of θ-constants:

$$(3.11) \qquad \begin{aligned} 2\omega &= \epsilon (S^t)^{-1} N^{-1} D^t [\delta_1, \ldots, \delta_g], \\ \rho &:= (2\omega)^{-1} = \epsilon D[\delta_1, \ldots, \delta_g]^{-1} N S^t. \end{aligned}$$

4. Derivation of the periods of the second kind

Periods of the second kind can be obtained from the Picard-Fuchs equations for the derivatives with respect to the branching points e_i of the two sets of periods ω and η. Bolza described [**Bol99**] how to derive them by a variational procedure which goes back to Riemann, Thomae, and Fuchs; it was generalized in terms of Rauch's formula (see e.g. [**Rau59, Fay92**]) which in the case of the hyperelliptic curve (2.1) reads

$$(4.1) \qquad \frac{\partial}{\partial e_k} d\boldsymbol{v}(x, y) = -2\mathrm{Res}|_{z=e_k} \, d\omega_{\mathrm{norm}}(z, w; x, y) d\boldsymbol{v}(z, w)$$

where $d\boldsymbol{v}(x, y)$ is the vector of normalized holomorphic differentials and $d\omega_{\mathrm{norm}}(z, w; x, y)$ the normalized Kleinian bi-differential

$$d\omega_{\mathrm{norm}}(z, w; x, y) = d\omega(z, w; x, y) + d\boldsymbol{u}^t(z, w) \, \varkappa \, d\boldsymbol{u}(x, y),$$

$$(4.2) \qquad \varkappa^t = \varkappa = \eta(2\omega)^{-1}, \qquad \oint_{\mathfrak{a}_l} d\omega_{\mathrm{norm}}(z, w; x, y) = 0, \quad l = 1, \ldots, g,$$

which is often called the *Bergmann kernel*. In the proof below we use the same set of ideas using the Klein bi-differential (2.10) because our aim is to derive differential equations, with respect to the branching points, in the space of periods $\omega, \omega', \eta, \eta'$ of the non-normalized differentials (2.5), (2.6).

THEOREM 4.1. *For an arbitrary branching point e_l the following equations are valid*

$$(4.3) \qquad \frac{\partial}{\partial e_l} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} = \begin{pmatrix} \alpha_l & \beta_l \\ \gamma_l & -\alpha_l^t \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix},$$

where

$$(4.4) \qquad \alpha_l \;=\; -\frac{1}{2}\left\{\frac{1}{R'(e_l)}\mathbf{U}(e_l)\mathbf{\mathcal{R}}^t(e_l) - M_l\right\},$$

$$(4.5) \qquad \beta_l \;=\; -2\left\{\frac{1}{R'(e_l)}\mathbf{U}(e_l)\mathbf{U}^t(e_l)\right\},$$

$$(4.6) \qquad \gamma_l \;=\; \frac{1}{8}\left\{\frac{1}{R'(e_l)}\mathbf{\mathcal{R}}(e_l)\mathbf{\mathcal{R}}^t(e_l) - N_l\right\},$$

with

$$(4.7) \qquad M_l = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ e_l & 1 & 0 & \ldots & 0 & 0 \\ e_l^2 & e_l & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ e_l^{g-2} & e_l^{g-3} & \ldots & e_l & 1 & 0 \end{pmatrix}$$

and

$$(4.8) \qquad N_l = e_l(M_l Q_l + Q_l M_l^t) + Q_l, \qquad Q_l = \mathrm{diag}\left(\ldots, \frac{\mathcal{R}_k(e_l)}{\mathcal{U}_{k+1}(e_l)}, \ldots\right).$$

PROOF. First we consider the equation for $\partial\omega/\partial e_l$. It is obtained by integrating the following equivalence over cycles $\mathfrak{a}_l$:

$$(4.9) \quad R'(e_i)\frac{\partial}{\partial e_i}\frac{\mathcal{U}_m(x)}{y} = \frac{\mathcal{U}_m(e_i)}{2y}\left\{\mathbf{U}^t(e_i)\mathbf{\mathcal{R}}(x) - \mathbf{\mathcal{R}}^t(e_i)\mathbf{U}(x)\right\}$$

$$- \mathcal{U}_m(e_i)\frac{\partial}{\partial x}\frac{y}{x - e_i} + \frac{R'(e_i)}{2y}\sum_{j=1}^{m-1}\mathcal{U}_{m-j}(x)\mathcal{U}_j(e_i).$$

To prove this, substitute $(z, w) = (e_i, 0)$ in eq. (2.12) which leads to the relation

$$(4.10) \qquad -\frac{\partial}{\partial x}\frac{y}{(x - e_i)} + \frac{\mathbf{U}^t(e_i)\mathbf{\mathcal{R}}(x)}{2y} = \frac{F(x, e_i)}{2(x - e_i)^2 y};$$

insert

$$(4.11) \qquad F(x, e_i) = (x - e_i)R'(e_i) + (x - e_i)^2\,\mathbf{U}^t(x)\mathbf{\mathcal{R}}(e_i)$$

from eq. (2.11) to obtain the equality

$$(4.12) \quad -\frac{\partial}{\partial x}\frac{y}{(x - e_i)} + \frac{\mathbf{U}^t(e_i)\mathbf{\mathcal{R}}(x)}{2y} - \frac{\mathbf{\mathcal{R}}^t(e_i)\mathbf{U}(x)}{2y} = \frac{R'(e_i)}{2(x - e_i)y} \equiv R'(e_i)\frac{\partial}{\partial e_i}\frac{1}{y},$$

which is (4.9) for $m = 1$. The validity of (4.9) for $m = 2, \ldots$ follows from the equivalence

$$(4.13) \qquad \frac{\partial}{\partial e_i}\frac{x^{m-1}}{y} = e_i^{m-1}\frac{\partial}{\partial e_i}\frac{1}{y} + \frac{1}{2}\sum_{j=1}^{m-1}e_i^{j-1}\frac{x^{m-j-1}}{y},$$

which can be proved inductively.

Compute now the periods $2\omega_{m,l} = \oint_{\mathfrak{a}_l}(\mathcal{U}_m(x)/y)\,dx$ from (4.9):

$$(4.14) \quad \frac{\partial\omega_{m,l}}{\partial e_i} = -\frac{2\mathcal{U}_m(e_i)}{R'(e_i)}\sum_{k=1}^{g}\left\{\mathcal{U}_k(e_i)\eta_{k,l} + \frac{1}{4}\mathcal{R}_k(e_i)\omega_{k,l}\right\} + \frac{1}{2}\sum_{j=1}^{m-1}\omega_{m-j,l}\mathcal{U}_j(e_i).$$

The upper left block of (4.3) is nothing but this formula written in matrix form.

Next we derive the equation for $\partial\eta/\partial e_l$ in an analogous way, using the equivalence which may be checked by direct computing,

$$(4.15) \quad R'(e_i)\frac{\partial}{\partial e_i}\frac{\mathcal{R}_m(x)}{y} = \frac{\mathcal{R}_m(e_i)}{2y}\{\mathcal{U}^T(e_i)\mathcal{R}(x) - \mathcal{R}^T(e_i)\mathcal{U}(x)\}$$

$$- \mathcal{R}_m(e_i)\frac{\partial}{\partial x}\frac{y}{x-e_i} - \frac{R'(e_i)}{2y}\Big(\frac{\mathcal{R}_m(x)-\mathcal{R}_m(e_i)}{x-e_i} + 2\frac{\partial}{\partial e_i}\mathcal{R}_m(x)\Big).$$

The expression in the bracket of the last term can be written as

$$\frac{-1}{\mathcal{U}_{m+1}(e_i)}\Big(\sum_{k=1}^{m}\mathcal{U}_{m-k+1}(x)\mathcal{U}_k(e_i) + \sum_{k=m+1}^{g}(\mathcal{U}_k(x)\mathcal{R}_k(e_i) - \mathcal{R}_k(x)\mathcal{U}_k(e_i))\Big).$$

The periods $2\eta_{m,l} = -\oint_{\mathfrak{a}_l}(\mathcal{R}_m(x)/y)\,dx$ are obtained by integration:

$$(4.16) \quad \frac{\partial\eta_{m,l}}{\partial e_i} = \frac{\mathcal{R}_m(e_i)}{2R'(e_i)}\sum_{k=1}^{g}\Big(\mathcal{U}_k(e_i)\eta_{k,l} + \frac{1}{4}\mathcal{R}_k(e_i)\omega_{k,l}\Big) - \frac{1}{2}\sum_{k=m+1}^{g}\mathcal{U}_{k-m}(e_i)\eta_{k,l}$$

$$- \frac{1}{8}\Big(\sum_{k=m+1}^{g}\omega_{k,l}\frac{\mathcal{R}_k(e_i)}{\mathcal{U}_m(e_i)} + \sum_{k=1}^{m}\omega_{m-k+1,l}\mathcal{U}_{k-m}(e_i)\Big).$$

Written in matrix form, this is the lower left block of Eq. (4.3). The equations for the derivatives of ω' and η' are obtained in the same way. □

COROLLARY 4.2. *The following variational formula is valid:*

$$(4.17) \qquad\qquad \frac{\partial\tau}{\partial e_l} = \frac{i\pi}{2}\omega^{-1}\beta_l(\omega^t)^{-1}, \qquad e = 1,\ldots,2g+1,$$

where β_l is given in eq. (4.5).

This is a consequence of eqs. (4.3) and (4.4) for $\tau = \omega^{-1}\omega'$; it was derived in [**Tho70**] and has been proved again in many places. For the case of genus one the explicit formula is given in (1.11).

We are now in the position to give expressions for the second kind periods in terms of θ-constants.

THEOREM 4.3. *Choose any g different positive integers $\{i_1,\ldots,i_g\} =: \mathcal{I}_0$ from the set $\mathcal{G}$, and let $e_{i_1},\ldots,e_{i_g}$ be the corresponding branching points. Then the period matrix η is given as*

$$(4.18) \qquad\qquad B(\mathcal{I}_0)\eta = \sum_{i_l\in\mathcal{I}_0}\frac{\partial\omega}{\partial e_{i_l}} - A(\mathcal{I}_0)\omega,$$

where

$$B(\mathcal{I}_0) = \sum_{i_l\in\mathcal{I}_0}\beta(e_{i_l}), \qquad A(\mathcal{I}_0) = \sum_{i_l\in\mathcal{I}_0}\alpha(e_{i_l}),$$

and the matrix $B(\mathcal{I}_0)$ is invertible.

PROOF. Eq. (4.18) follows from (4.3), and it is straightforward to check that

$$\det B(\mathcal{I}_0) = (-2)^g\frac{\prod\limits_{i_l<i_k\in\mathcal{I}_0}(e_{i_l}-e_{i_k})^2}{\prod\limits_{i_n\in\mathcal{I}_0}R'(e_{i_n})} \neq 0.$$

$\square$

The derivative $\partial\omega/\partial e_l$ can be calculated with the help of formula (4.17) and the heat equation (2.18).

Formula (4.18) can be applied as follows. Let $\lambda_{2g} = \sum_k e_k = 0$. Define the matrices

$$(4.19) \qquad C(\mathfrak{I}_0) = \sum_{i_k \in \mathfrak{I}_0} e_{i_k} A(\mathfrak{I}_0)^{-1} B(\mathfrak{I}_0), \qquad D(\mathfrak{I}_0) = \sum_{i_k \in \mathfrak{I}_0} e_{i_k} A(\mathfrak{I}_0)^{-1}.$$

Then by taking a suitable sum of the $\binom{2g+1}{g}$ equations (4.18), we obtain

$$(4.20) \qquad \eta = \left(\sum_{\mathfrak{I}_0} C(\mathfrak{I}_0) \right)^{-1} \sum_{\mathfrak{I}_0} D(\mathfrak{I}_0) \sum_{i_k \in \mathfrak{I}_0} \frac{\partial\omega}{\partial e_{i_k}},$$

where the summation is over all subsets $\mathfrak{I}_0$ of the set $\mathcal{G}$ of indices. For genus one this formula reduces to (1.16).

5. Examples

5.1. Example $g = 1$. The case of genus was covered in the introduction.

5.2. Example $g = 2$. Consider the hyperelliptic curve V of genus two

$$(5.1) \qquad y^2 = 4(x - e_1)(x - e_2)(x - e_3)(x - e_4)(x - e_5).$$

The homology basis of the curve is fixed by defining the set of half-periods corresponding to the branching points, following the recipe of section 2.3:

$$[\mathfrak{A}_1] = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [\mathfrak{A}_2] = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad [\mathfrak{A}_3] = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$[\mathfrak{A}_4] = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_5] = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_6] = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For any $k \neq l \neq p, q, r$ from the set $\{1, \ldots, 5\}$ (there are 10 different possibilities) the following representation is valid (see theorem 3.5)

$$(5.2) \qquad 2\omega = \frac{T_{pqr}}{\theta\{k,l\}(e_k - e_l)^{\frac{3}{2}}} \begin{pmatrix} 1 & -1 \\ -e_l & e_k \end{pmatrix} \begin{pmatrix} \frac{1}{T_l} & 0 \\ 0 & \frac{1}{T_k} \end{pmatrix} \begin{pmatrix} \theta_1\{l\} & \theta_2\{l\} \\ \theta_1\{k\} & \theta_2\{k\} \end{pmatrix},$$

where

$$(5.3) \qquad \begin{aligned} T_k &= \theta\{p,k\}\theta\{q,k\}\theta\{r,k\}, \\ T_l &= \theta\{p,l\}\theta\{q,l\}\theta\{r,l\}, \\ T_{pqr} &= \theta\{p,q\}\theta\{p,r\}\theta\{q,r\}. \end{aligned}$$

Choosing $k = 2$ and $l = 4$, and normalizing the curve with $e_2 = 0$ and $e_4 = 1$, we obtain the explicit result

$$(5.4) \qquad 2\omega = \frac{-T_{135}}{T_2 T_4\, \theta\begin{bmatrix} 00 \\ 00 \end{bmatrix}} \begin{pmatrix} T_{135}\, \theta_1 \begin{bmatrix} 11 \\ 01 \end{bmatrix}_2 & T_{135}\, \theta_2 \begin{bmatrix} 11 \\ 01 \end{bmatrix}_2 \\ T_2\, \theta_1 \begin{bmatrix} 10 \\ 10 \end{bmatrix}_2 & T_2\, \theta_2 \begin{bmatrix} 10 \\ 10 \end{bmatrix}_2 \end{pmatrix},$$

where

$$(5.5) \qquad \begin{aligned} T_2 &= \theta \begin{bmatrix} 11 \\ 11 \end{bmatrix}_2 \theta \begin{bmatrix} 00 \\ 01 \end{bmatrix}_2 \theta \begin{bmatrix} 01 \\ 00 \end{bmatrix}_2, \\ T_4 &= \theta \begin{bmatrix} 00 \\ 10 \end{bmatrix}_2 \theta \begin{bmatrix} 11 \\ 00 \end{bmatrix}_2 \theta \begin{bmatrix} 10 \\ 01 \end{bmatrix}_2, \\ T_{135} &= \theta \begin{bmatrix} 00 \\ 11 \end{bmatrix}_2 \theta \begin{bmatrix} 01 \\ 10 \end{bmatrix}_2 \theta \begin{bmatrix} 10 \\ 00 \end{bmatrix}_2. \end{aligned}$$

In the derivation of this result we used the equality $(i = 1, 2)$

$$\theta_i \begin{bmatrix}01\\11\end{bmatrix} \theta \begin{bmatrix}11\\00\end{bmatrix} \theta \begin{bmatrix}00\\10\end{bmatrix} \theta \begin{bmatrix}10\\01\end{bmatrix} - \theta_i \begin{bmatrix}10\\10\end{bmatrix} \theta \begin{bmatrix}01\\00\end{bmatrix} \theta \begin{bmatrix}00\\01\end{bmatrix} \theta \begin{bmatrix}11\\11\end{bmatrix} = \theta_i \begin{bmatrix}11\\01\end{bmatrix} \theta \begin{bmatrix}00\\11\end{bmatrix} \theta \begin{bmatrix}01\\10\end{bmatrix} \theta \begin{bmatrix}10\\00\end{bmatrix},$$

which can be derived from addition theorems as given, e. g., on p. 342 in [**Bak95**]; a complete set of such relations can be found in [**For82**].

The entries to the inverse matrix $\rho = 2\omega^{-1}$ are the normalizing constants of the holomorphic differentials. Its columns represent the so-called winding vectors in the Its-Matveev formula [**IM75**] for the solution of the KdV equation. For the case of $g = 2$ the general formula (3.11) for ρ reduces to

$$(5.6) \qquad \rho = (2\omega)^{-1} = \frac{\sqrt{e_k - e_l}}{\pi^2 T_{pqr}^2} \begin{pmatrix} \theta_2\{k\} & -\theta_2\{l\} \\ -\theta_1\{k\} & \theta_1\{l\} \end{pmatrix} \begin{pmatrix} T_l & 0 \\ 0 & T_k \end{pmatrix} \begin{pmatrix} e_k & 1 \\ e_l & 1 \end{pmatrix}.$$

With the normalization $e_2 = 0$, $e_4 = 1$, this coincides with the formulae given in the Rosenhain memoir [**Ros51**], page 75:

$$(5.7) \qquad \rho = \frac{1}{\pi^2 T_{135}^2} \begin{pmatrix} T_2\, \theta_2 \begin{bmatrix}10\\10\end{bmatrix}_2 & -T_{135}\theta_2 \begin{bmatrix}11\\01\end{bmatrix}_2 \\ -T_2\, \theta_1 \begin{bmatrix}10\\10\end{bmatrix}_2 & T_{135}\theta_1 \begin{bmatrix}11\\01\end{bmatrix}_2 \end{pmatrix}.$$

The set of periods 2η can be computed with the help of (4.18).

The special case $g = 2$ of the Riemann-Jacobi formula (3.9) was discovered by Rosenhain [**Ros51**] :

THEOREM 5.1 (**Rosenhain**). *Let* $[\delta_1], \dots, [\delta_6]$ *be the six odd characteristics and* $[\delta_1], [\delta_2]$ *any two of them. With the remaining four* $[\delta_{i+2}], i = 1, \dots, 4,$ *and* $[\varepsilon_i] = [\delta_1] + [\delta_2] + [\delta_{i+2}]$ (mod) 1, *there exist 15 relations*

$$(5.8) \qquad D[\delta_1, \delta_2] = \theta_1[\delta_1]\theta_2[\delta_2] - \theta_2[\delta_1]\theta_1[\delta_2] = \pm\pi^2\theta[\varepsilon_1]\theta[\varepsilon_2]\theta[\varepsilon_3]\theta[\varepsilon_4].$$

5.3. Example $g = 3$. Consider briefly the hyperelliptic curve V of genus three

$$(5.9) \qquad y^2 = 4(x - e_1)(x - e_2)(x - e_3)(x - e_4)(x - e_5)(x - e_6)(x - e_7).$$

The homology basis of the curve is again fixed by defining the set of half-periods corresponding to the branching points:

$$[\mathfrak{A}_1] = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathfrak{A}_2] = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad [\mathfrak{A}_3] = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$[\mathfrak{A}_4] = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad [\mathfrak{A}_5] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad [\mathfrak{A}_6] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$[\mathfrak{A}_7] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_8] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For any choice of three different numbers k, l, n from the set $\{1, \dots, 7\}$ (there are 35 different possibilities) the following representation is valid (see theorem 3.5):

$$(5.10) \qquad S = \begin{pmatrix} e_l e_n & e_k e_n & e_k e_l \\ e_l + e_n & e_k + e_n & e_k + e_l \\ 1 & 1 & 1 \end{pmatrix},$$

$$N = \theta\{k,l,n\} \times \mathrm{diag}\left(\sqrt{e_k - e_n}\,\frac{\theta\{p,l,n\}\theta\{q,l,n\}\theta\{l,r,s\}}{\theta\{p,q,n\}\theta\{p,r,s\}\theta\{q,r,s\}},\right.$$

(5.11)
$$\sqrt{e_l - e_n}\,\frac{\theta\{p,k,n\}\theta\{q,k,n\}\theta\{n,r,s\}}{\theta\{p,q,k\}\theta\{p,r,s\}\theta\{q,r,s\}},$$

$$\left.\sqrt{e_n - e_k}\,\frac{\theta\{p,k,l\}\theta\{q,k,l\}\theta\{k,r,s\}}{\theta\{p,q,l\}\theta\{p,r,s\}\theta\{q,r,s\}}\right).$$

It is always possible to normalize the curve with $e_k = 0$ and $e_l = 1$; then according to the Corollary 3.2

$$e_n = \frac{\theta^2\{l,p,q\}\theta^2\{l,r,s\}}{\theta^2\{n,p,q\}\theta^2\{n,r,s\}},$$

and therefore (5.11) is expressible entirely in terms of θ-constants. Using this in the formula (3.11)and (4.18), we arrive at the final expression for periods in terms of θ-constants.

6. Conclusion

For the class of hyperelliptic curves we have recalled the lines of thought that allow to express the periods of integrals in terms of θ-constants. The main ingredients were Thomae's two formulae and Picard-Fuchs equations. It should be clear how to derive explicit results for curves of any genus.

These results can be generalized in various directions. First, we believe that the Rosenhain formula (5.7) for periods of the first kind can be given in the same form for higher genera where the entries to the matrix are proportional to minors of the Jacobian $D[\delta_1, \ldots, \delta_g]$, see (3.5). Second, it should be possible to simplify the expression (4.18) to make it more similar to the Weierstrass formula (1.16).

Third, the formulae (5.6) obtained for the winding vectors could be useful to prove the equivalence of the classical Schottky conditions, in the spirit of the derivation given by Farkas [**Far71**], with Dubrovin's relations involving winding vectors and θ-derivatives which he derived in the framework of KdV theory [**Dub81**].

Moreover, since the first Thomae formula has shown to hold also for certain non-hyperelliptic curves [**BR87, Nak97**], there is hope that some of the material developed here can be generalized to those cases.

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