6

σ-Functions: Old and New Results

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To Professor Emma Previato on the occasion of her 65th birthday

Abstract. We are considering multi-variable sigma functions of genus g hyperelliptic curve as a function of two groups of variables – Jacobian variables and parameters of the curve. In the theta-functional representation of sigma-function the second group arises as periods of first and second kind differentials of the curve. We develop representation of periods in terms of theta-constants; for the first kind of period, generalization of Rosenhain type formulae are obtained whilst for the second kind of period, theta-constant expressions are presented which are explicitly related to the fixed co-homology basis.

We describe a method of constructing differentiation operators for hyperelliptic analogues of ζ- and $\wp$-functions on the parameters of the hyperelliptic curve. To demonstrate this method, we give the detailed construction of these operators in the case genus 1 and 2.

1 Introduction

Our note belongs to the area in whose development Emma Previato took active part. Ever since the first publication of the present authors [14] she inspired them and gave a lot of suggestions and advice.

The aforementioned area is a construction of Abelian functions in terms of multi-variable σ-functions. Similar to the Weierstrass elliptic function, multi-variable sigma keeps its main property – it remains form-invariant under the action of symplectic groups. Abelian functions appeared to be logarithmic derivatives like $\zeta, \wp$-functions of the Weierstrass theory and similar to the standard theta-functional approach which lead to the Krichever formula for KP solutions, [51]. But the fundamental difference between sigma and theta-functional theories is the following. They are both constructed by the curve and given as series in Jacobian variables, but in the first case the expansion is purely algebraic with respect to the model of the curve because its coefficients are polynomials in parameters of the defining curve equation, whilst in the second case the coefficients are transcendental being built in terms of Riemann matrix periods which are complete Abelian integrals.

In many publications, in particular see [59, 35, 17, 55, 32, 1, 11, 53, 56, 57] and references therein, it was demonstrated that the multi-variable $\wp$-function represents a language which is very suitable for discussion of completely integrable systems of KP type. In particular, very recently, using the fact that sigma is an entire function in parameters of the curve (in contrast with theta-functions) families of degenerate solutions for solitonic equations were obtained [8, 9]. In recent papers [24] and [25], an algebraic construction of a wide class of polynomial Hamiltonian integrable systems was given, and those whose solutions are given by hyperelliptic $\wp$-functions were indicated.

The revival of interest in multi-variable σ-functions is in many respects due to H. Baker's exposition of the theory of Abelian functions which goes back to K. Weierstrass and F. Klein and is well documented and developed in his remarkable monographs [4, 5]. The heart of his exposition is the representation of fundamental bi-differentials of hyperelliptic curves in algebraic form in contrast with the most well-known representation as double differentials of Riemann theta-functions developed in Fay's monograph, [41]. Recent investigations demonstrated that Baker's approach can be extended beyond hyperelliptic curves to wide classes of algebraic curves.

The multi-variable σ-function of an algebraic curve $\mathscr{C}$ is known to be represented in terms of θ-functions of the curve as a function of two groups of variables – Jacobian of the curve, $\mathrm{Jac}(\mathscr{C})$ and Riemann matrix τ. In a vast amount of recent publications properties of σ-functions as a function of the first group of variables are discussed whilst the modular part of variables and relevant objects like θ-constant representations of complete Abelian integrals are considered separately. In this paper we deal with σ as a function over both groups of variables.

Due to the pure modular part of σ-variables we consider problem of expression of complete integrals of the first and second kind in terms of theta-constants. Revival of interest in this classically known matter accords to many recent publications reconsidering such problems like Schottky problem [40], Thomae [37, 36] and Weber [58] formulae, theory of invariants and its applications [52, 31].

The paper is organized as follows. In Section 2 we consider hyperelliptic genus g curves and a complete co-homology basis of $2g$ meromorphic differentials, with g of them chosen as holomorphic ones. We discuss expressions for periods of these differentials in terms of θ-constant with half integer characteristics. Theta-constant representation of periods of holomorphic integrals is known from Rosenhain's memoir [60], where the case of genus two was elaborated. We discuss this case and generalize Rosenhain's expressions to higher genera hyperelliptic curves. Theta-constant representation of periods of the second kind is known after F. Klein [49] who presented closed formula in terms of derived even theta-constants for non-hyperelliptic genus three curves. We re-derive this formula for higher genera hyperelliptic curves.

Section 3 is devoted to the classically known problem, which was resolved in the case of elliptic curves by Frobenius and Stickelberger [44]. The general method of the solution of this problem for a wide class of so-called (n, s)- curves has been developed in [22] and represents an extension of Weierstrass's method for the derivation of a system of differential equations defining a sigma-function. All stages of this derivation are given in detail and the main result is that the sigma-function is completely defined as the solution of a system of heat conductivity equations in a nonholonomic frame. We also consider another widely known problem – description of dependence of the solutions on initial data. This problem is formulated as a description of the dependence of integrals of motion, whose levels are given as half of the curve parameters, from the remaining half of the parameters. The differential formulae obtained permit the presentation of an effective solution of this problem for Abelian functions of hyperelliptic curves. It is worth noting that because integrals of motion can be expressed in terms of periods of the second kind in this place results of Section 2 are demanded. The results obtained in Section 3 are exemplified in details by curves of genera one and two. All consideration is based on explicit uniformization of space universal bundles of the hyperelliptic Jacobian.

2 Modular Representation of Periods of Hyperelliptic Co-Homologies

Modular invariance of the Weierstrass elliptic σ-function, $\sigma = \sigma(u; g_2, g_3)$ follows from its defining in [65] in terms of recursive series in terms of variables (u, g_2, g_3). Alternatively a σ-function can be represented in terms of a Jacobi θ-function and its modular invariance follows from transformation properties of θ-functions. The last representation involves complete elliptic integrals of the first and second kind and their representations in terms of θ-constants are classically known. In this section we are studying generalizations of these representations to hyperelliptic curves of higher genera realized in the form

$$y^2 = P_{2g+1}(x) = (x - e_1)\cdots(x - e_{2g+1}) \tag{1}$$

Here $P_{2g+1}(x)$ is a monic polynomial of degree $2g+1$, $e_i \in \mathbb{C}$ - branch points and the curve is assumed to be non-degenerate, i.e. $e_i \neq e_j$.

2.1 Problems and Methods

Representations of complete elliptic integrals of the first and second kind in terms of Jacobi θ-constants are classically known. In particular, if an elliptic curve is given in Legendre form[1]

$$y^2 = (1 - x^2)(1 - k^2x^2) \tag{2}$$

where k is Jacobian modulus, then complete elliptic integrals of the first kind $K = K(k)$ are represented as

$$K = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{\pi}{2}\vartheta_3^2(0;\tau), \tag{3}$$

and $\vartheta_3 = \vartheta_3(0|\tau)$ and $\tau = \imath\frac{K'}{K}$, $K' = K(k')$, $k^2 + k'^2 = 1$.

Further, for an elliptic curve realized as Weierstrass cubic

$$y^2 = 4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3) \tag{4}$$

recall standard notation for periods of the first and second kind elliptic integrals

$$\begin{aligned} 2\omega &= \oint_{\mathfrak{a}} \frac{\mathrm{d}x}{y}, \quad 2\eta = -\oint_{\mathfrak{a}} \frac{x\mathrm{d}x}{y} \\ 2\omega' &= \oint_{\mathfrak{b}} \frac{\mathrm{d}x}{y}, \quad 2\eta' = -\oint_{\mathfrak{b}} \frac{x\mathrm{d}x}{y} \end{aligned} \qquad \tau = \frac{\omega'}{\omega} \tag{5}$$

and the Legendre relation for them

$$\omega\eta' - \eta\omega' = -\frac{\imath\pi}{2} \tag{6}$$

Then the following Weierstrass relation is valid

$$\eta = -\frac{1}{12\omega}\left(\frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)}\right) \tag{7}$$

In this section we are discussing generalizations of these relations to higher genera hyperelliptic curves realized as in (1).

2.2 Definitions and Main Theorems

In this subsection we reproduce notation from H. Baker [6]. Let $\mathscr{C}$ be a genus g non-degenerate hyperelliptic curve realized as a double cover of a Riemann sphere,

$$y^2 = 4\prod_{j=1}^{2g+1}(x - e_j) \equiv 4x^{2g+1} + \sum_{i=0}\lambda_i x^i, \quad e_i \neq e_j,\ \lambda_i \in \mathbb{C} \tag{8}$$

[1] Here and below we punctiliously follow notations of elliptic functions theory fixed in [7]

Let $(\mathfrak{a};\mathfrak{b}) = (\mathfrak{a}_1,\ldots,\mathfrak{a}_g;\mathfrak{b}_1,\ldots,\mathfrak{b}_g)$ be a canonic homology basis. Introduce a co-homology basis (*Baker co-homology basis*)

$$\begin{aligned}\mathrm{d}u(x,y) &= (\mathrm{d}u_1(x,y),\ldots,\mathrm{d}u_g(x,y))^T, \mathrm{d}r(x,y)\\ &= (\mathrm{d}r_1(x,y),\ldots,\mathrm{d}r_g(x,y))^T\end{aligned}$$

$$\mathrm{d}u_i(x,y) = \frac{x^{i-1}}{y}\mathrm{d}x,\quad \mathrm{d}r_j(x,y) = \sum_{k=j}^{2g+1-j}(k+1-j)\frac{x^k}{4y}\mathrm{d}x,\quad i,j=1,\ldots,g \tag{9}$$

satisfying the generalized Legendre relation,

$$\mathfrak{M}^T J\mathfrak{M} = -\frac{\imath\pi}{2}J,\qquad \mathfrak{M} = \begin{pmatrix}\omega & \omega'\\ \eta & \eta'\end{pmatrix},\quad J = \begin{pmatrix}0_g & 1_g\\ -1_g & 0_g\end{pmatrix} \tag{10}$$

where $g\times g$ period matrices $\omega,\omega',\eta,\eta'$ are defined as

$$\begin{aligned}2\omega &= \left(\oint_{\mathfrak{a}_j}\mathrm{d}u_i\right),\quad 2\omega' = \left(\oint_{\mathfrak{b}_j}\mathrm{d}u_i\right),\\ 2\eta &= -\left(\oint_{\mathfrak{a}_j}\mathrm{d}r_i\right),\quad 2\eta' = -\left(\oint_{\mathfrak{b}_j}\mathrm{d}r_i\right)\end{aligned} \tag{11}$$

We also denote $\mathrm{d}v = (\mathrm{d}v_1,\ldots,\mathrm{d}v_g)^T = (2\omega)^{-1}\mathrm{d}u$ the vector of normalized holomorphic differentials.

Define the Riemann matrix $\tau = \omega^{-1}\omega'$ belonging to the Siegel half-space $\mathscr{S}_g = \{\tau^T = \tau,\ \mathrm{Im}\tau > 0\}$. Define the Jacobi variety of the curve $\mathrm{Jac}(\mathscr{C}) = \mathbb{C}^g/1_g\oplus\tau$. The canonical Riemann θ-function is defined on $\mathrm{Jac}(\mathscr{C})\times\mathscr{S}_g$ by Fourier series

$$\theta(z;\tau) = \sum_{\ltimes\in\mathbb{Z}^n}\mathrm{e}^{\imath\pi n^T\tau n+2\imath\pi z^T n} \tag{12}$$

We will also use θ-functions with half-integer characteristics $[\varepsilon] = \begin{bmatrix}\varepsilon'^T\\ \varepsilon''\end{bmatrix}$, $\varepsilon'_i,\varepsilon''_j = 0$ or 1 defined as

$$\theta[\varepsilon](z;\tau) = \sum_{\ltimes\in\mathbb{Z}^n}\mathrm{e}^{\imath\pi(n+\varepsilon'/2)^T\tau(n+\varepsilon'/2)+2\imath\pi(z+\varepsilon''/2)^T(n+\varepsilon'/2)} \tag{13}$$

The characteristic is even or odd whenever $\varepsilon'^T\varepsilon'' = 0 \pmod 2$ or $1 \pmod 2$ and $\theta[\varepsilon](z;\tau)$ as function of z inherits parity of the characteristic.

Derivatives of θ-functions by arguments z_i will be denoted as

$$\theta_i[\varepsilon](z;\tau) = \frac{\partial}{\partial z_i}\theta[\varepsilon](z;\tau),\quad \theta_{i,j}[\varepsilon](z;\tau) = \frac{\partial^2}{\partial z_i\partial z_j}\theta[\varepsilon](z;\tau),\quad \text{etc.}$$

The fundamental bi-differential $\Omega(P,Q)$ is uniquely defined on the product $(P,Q)\in\mathscr{C}\times\mathscr{C}$ by the following conditions:

i Ω is symmetric, $\Omega(P,Q)=\Omega(Q,P)$
ii Ω is normalized by the condition

$$\oint_{\mathfrak{a}_i}\Omega(P,Q)=0,\quad i=1,\ldots,g \tag{14}$$

iii Let $P=(x,y)$ and $Q=(z,w)$ have local coordinates $\xi_1=\xi(P)$, $\xi_2=\xi(Q)$ in the vicinity of point R, $\xi(R)=0$, then $\Omega(P,Q)$ expands to a power series as

$$\Omega(P,Q)=\frac{\mathrm{d}\xi_1\mathrm{d}\xi_2}{(\xi_1-\xi_2)^2}+\text{ homorphic 2-form} \tag{15}$$

The fundamental bi-differential can be expressed in terms of θ-functions [41]

$$\Omega(P,Q)=\mathrm{d}_x\mathrm{d}_z\theta\left(\int_Q^P\mathrm{d}v+e\right),\quad P=(x,y),Q=(z,w) \tag{16}$$

where $\mathrm{d}v$ is a normalized holomorphic differential and e any non-singular point of the θ-divisor (θ), i.e. $\theta(e)=0$, but not all θ-derivatives, $\partial_{z_i}\theta(z)|_{z=e}$, $i=1,\ldots,g$ vanish.

In the case of a hyperelliptic curve $\Omega(P,Q)$ can be alternatively constructed as

$$\Omega(P,Q)=\frac{1}{2}\frac{\partial}{\partial z}\frac{y+w}{y(x-z)}\mathrm{d}x\mathrm{d}z+\mathrm{d}r(P)^T\mathrm{d}u(Q)+2\mathrm{d}u^T(P)\varkappa\mathrm{d}u(Q) \tag{17}$$

where the first two terms are given as rational functions of coordinates P,Q and a necessarily symmetric matrix $\varkappa^T=\varkappa$, $\varkappa=\eta(2\omega)^{-1}$ is introduced to satisfy the normalization condition **ii**. In shorter form (18) can be rewritten as

$$\Omega(P,Q)=\frac{2yw+F(x,z)}{4(x-z)^2yw}\mathrm{d}x\mathrm{d}z+2\mathrm{d}u^T(P)\varkappa\mathrm{d}u(Q) \tag{18}$$

where $F(x,z)$ is so-called Kleinian 2-polar, given as

$$F(x,z)=\sum_{k=0}^{g}x^kz^k\,(2\lambda_{2k}+\lambda_{2k+1}(x+z)) \tag{19}$$

Recently algebraic representations for $\Omega(P,Q)$ similar to (18) were found in [61], [39] for wide class on algebraic curves, including (n,s)-curves [16].

The main relation underlying the theory is the Riemann formula representing Abelian integrals of the third kind as θ-quotient written in terms of the above described realization of the fundamental differential $\Omega(P,Q)$.

Theorem 2.1 *(Riemann) Let* $P'=(x',y')$ *and* $P''=(x'',y'')$ *be two arbitrary distinct points on* $\mathscr{C}$ *and let* $\mathscr{D}'=\{P_1'+\cdots+P_g'\}$ *and*

$\mathscr{D}'' = \{P_1'' + \cdots + P_g''\}$ *be two non-special divisors of degree g. Then the following relation is valid*

$$\int_{P''}^{P'} \sum_{j=1}^{g} \int_{P_j'}^{P_j''} \left\{ \frac{2yy_i + F(x, x_i)}{4(x - x_i)^2} \frac{\mathrm{d}x}{y} \frac{\mathrm{d}x_i}{y_i} + 2\mathrm{d}u(x, y)\varkappa \mathrm{d}u(x_i, y_i) \right\}$$
$$= \ln \left(\frac{\theta(\mathscr{A}(P') - \mathscr{A}(\mathscr{D}') + K_\infty)}{\theta(\mathscr{A}(P') - \mathscr{A}(\mathscr{D}'') + K_\infty)} \right) - \ln \left(\frac{\theta(\mathscr{A}(P'') - \mathscr{A}(\mathscr{D}') + K_\infty)}{\theta(\mathscr{A}(P'') - \mathscr{A}(\mathscr{D}'') + K_\infty)} \right) \tag{20}$$

where $\mathscr{A}(P) = \int_\infty^P \mathrm{d}v$ *is an Abel map with base point* ∞*,* K_∞ *– vector of Riemann constants with base point* ∞ *which is a half-period.*

Introduce a multi-variable fundamental σ-function,

$$\sigma(u) = C\theta[K_\infty]((2\omega)^{-1}u)\mathrm{e}^{u^T \varkappa u}, \tag{21}$$

where $[K_\infty]$ is characteristic of the vector of Riemann constants, $u = \int_\infty^{P_1} \mathrm{d}u + \cdots + \int_\infty^{P_g} \mathrm{d}u$ with non-special divisor $P_1 + \cdots + P_g$. The constant C is chosen so that the expansion $\sigma(\boldsymbol{u})$ near $u \sim 0$ starts with a Schur-Weierstrass polynomial [16]. The whole expression is proved to be invariant under the action of the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. Klein-Weierstrass multi-variable $\wp$-functions are introduced as logarithmic derivatives,

$$\wp_{i,j}(u) = -\frac{\partial^2}{\partial u_i \partial u_j}, \quad \wp_{i,j,k}(u) = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k}, \quad \text{etc.} \quad i, j, k = 1, \dots g \tag{22}$$

Corollary 2.2 *For* $r \neq s \in \{1, \dots, g\}$ *the following formula is valid*

$$\sum_{i,j=1}^{g} \wp_{i,j} \left(\sum_{k=1}^{g} \int_\infty^{(x_k, y_k)} \mathrm{d}u \right) x_s^{i-1} x_r^{j-1} = \frac{F(x_s, x_r) - 2y_s y_r}{4(x_s - x_r)^2} \tag{23}$$

Corollary 2.3 *Jacobi problem of inversion of the Abel map* $\mathscr{D} \to \mathscr{A}(\mathscr{D})$ *with* $\mathscr{D} = (x_1, y_1) + \cdots + (x_g, y_g)$ *is resolved as*

$$\begin{aligned} &x^g - \wp_{g,g}(u)x^{g-1} - \cdots - \wp_{g,1}(u) = 0 \\ &y_k = \wp_{g,g,g}(u)x_k^{g-1} + \cdots + \wp_{g,g,1}(u), \quad k = 1, \dots, g \end{aligned} \tag{24}$$

2.3 $\wp$-Values at Non-Singular Even Half-Periods

In this section we present a generalization of Weierstrass formulae

$$\wp(\omega) = e_1, \ \wp(\omega + \omega') = e_2, \ \wp(\omega') = e_3 \tag{25}$$

to the case of a genus g hyperelliptic curve (8). To do that introduce partitions

$$\begin{gathered}\{1,\ldots,2g+1\}=\mathscr{I}_0\cup\mathscr{J}_0,\quad \mathscr{I}_0\cap\mathscr{J}_0=\emptyset\\ \mathscr{I}_0=\{i_1,\ldots,i_g\},\quad \mathscr{J}_0=\{j_1,\ldots,j_{g+1}\}\end{gathered}\tag{26}$$

Then any non-singular even half-period $\Omega_{\mathscr{I}}$ is given as

$$\begin{gathered}\Omega_{\mathscr{I}_0}=\int_{\infty}^{(e_{i_1},0)}\mathrm{d}u+\cdots+\int_{\infty}^{(e_{i_g},0)}\mathrm{d}u,\\ \mathscr{I}_0=\{i_1,\ldots,i_g\}\subset\{1,\ldots,2g+1\}\end{gathered}\tag{27}$$

Denote elementary symmetric functions $s_n(\mathscr{I}_0)$, $S_n(\mathscr{J}_0)$ of order n built on branch points $\{e_{i_k}\}$, $i_k\in\mathscr{I}_0$, $\{e_{j_k}\}$, $j_k\in\mathscr{J}_0$ correspondingly. In particular,

$$\begin{aligned}s_1(\mathscr{I}_0)&=e_{i_1}+\cdots+e_{i_g}, & S_1(\mathscr{J}_0)&=e_{j_1}+\cdots+e_{j_{g+1}}\\ s_2(\mathscr{I}_0)&=e_{i_1}e_{i_2}+\cdots+e_{i_{g-1}}e_{i_g}, & S_2(\mathscr{J}_0)&=e_{j_1}e_{j_2}+\cdots+e_{j_g}e_{i_{g+1}}\\ &\vdots & &\vdots\\ s_g(\mathscr{I}_0)&=e_{i_1}\cdots e_{i_g} & S_{g+1}(\mathscr{J}_0)&=e_{j_1}\cdots e_{j_{g+1}}\end{aligned}\tag{28}$$

Because of symmetry, $\wp_{p,q}(\Omega_{\mathscr{I}_0})=\wp_{q,p}(\Omega_{\mathscr{I}_0})$ is enough to find these quantities for $p\le q\in\{1,\ldots,g\}$. The following is valid.

Proposition 2.4 *(Conjectural Proposition) Let an even non-singular half-period $\Omega_{\mathscr{I}_0}$ be associated to a partition $\mathscr{I}_0\cup\mathscr{J}_0=\{1,\ldots,2g+1\}$. Then for all k, $j\ge k$, $k,j=1\ldots,g$ the following formula is valid*

$$\begin{aligned}&\wp_{k,j}(\Omega_{\mathscr{I}_0})\\ &=(-1)^{k+j}\sum_{n=1}^{k}n\big(s_{g-k+n}(\mathscr{I}_0)S_{g-j-n+1}(\mathscr{J}_0)+s_{g-j-n}(\mathscr{I}_0)S_{g+n-k+1}(\mathscr{J}_0)\big).\end{aligned}\tag{29}$$

Proof. The Klein formula written for even non-singular half-period $\Omega_{\mathscr{I}_0}$ leads to a linear system of equations with respect to Kleinian two-index symbols $\wp_{i,j}(\Omega_{\mathscr{I}_0})$

$$\sum_{i=1}^{g}\sum_{j=1}^{g}\wp_{i,j}(\Omega_{\mathscr{I}_0})e_{i_r}^{i-1}e_{i_s}^{j-1}=\frac{F(e_{i_r},e_{i_s})}{4(e_{i_r}-e_{i_s})^2}\qquad i_r,i_s\in\mathscr{I}_0\tag{30}$$

To solve these equations we note that

$$\wp_{k,g}(\Omega_{\mathscr{I}_0})=(-1)^{k+1}s_k(\mathscr{I}_0),\quad k=1,\ldots,g\tag{31}$$

Also note that $F(e_{i_r},e_{i_s})$ is divisible by $(e_{i_r}-e_{i_s})^2$ and

$$\frac{F(e_{i_r},e_{i_s})}{4(e_{i_r}-e_{i_s})^2}=e_{i_r}^{g-1}e_{i_s}^{g-1}\mathfrak{S}_1+e_{i_r}^{g-2}e_{i_s}^{g-2}\mathfrak{S}_2+\cdots+\mathfrak{S}_{2g-1}\tag{32}$$

where $\mathfrak{S}_k$ are order k elementary symmetric functions of elements e_i $i \in \{1, \ldots, 2g+1\} - \{i_r, i_s\}$.

Let us analyse equations (30) for small genera, $g \leq 5$. One can see that plugging in the equation (32) to (30) we get non-homogeneous linear equations solvable by Kramer's rule and the solutions can be presented in the form (29).

Now suppose that (29) is valid for higher $g > 5$ where computer power is insufficient to check that by means of computer algebra. But it's possible to check (29) for arbitrary big genera numerically leading to branch points e_i, $i = 1, \ldots, i = 2g+1$ certain numeric values. Much checking confirmed (29). $\square$

2.4 Modular Representation of $\varkappa$ Matrix

Quantities $\wp_{i,j}(\Omega_{\mathscr{I}_0})$ are expressed in terms of even θ-constants as follows

$$\wp_{i,j}(\Omega_{\mathscr{I}_0}) = -2\varkappa_{i,j} - \frac{1}{\theta[\varepsilon_{\mathscr{I}_0}](0)} \partial^2_{U_i,U_j}\theta[\varepsilon_{\mathscr{I}_0}](0), \quad \forall \mathscr{I}_0,\ i,j = 1, \ldots, g. \tag{33}$$

Here $[\varepsilon_{\mathscr{I}_0}]$ is characteristic of the vector $\left[\Omega_{\mathscr{I}_0} + K_\infty\right]$, where K_∞ is a vector of Riemann constants with base point ∞ and ∂_U is the directional derivative along vector $\boldsymbol{U}_i$, that is the ith column vector of an inverse matrix of $\mathfrak{a}$-periods, $\mathscr{A}^{-1} = (U_1, \ldots, U_g)$. The same formula is valid for all possible partitions $\mathscr{I}_0 \cup \mathscr{J}_0$, there are N_g of those, that is the number of non-singular even characteristics,

$$N_g = \begin{pmatrix} 2g+1 \\ g \end{pmatrix} \tag{34}$$

Therefore one can write

$$\varkappa_{i,j} = \frac{1}{8N_g}\Lambda_{i,j} - \frac{1}{2N_g} \sum_{\text{All even non-singular } [\varepsilon]} \frac{\partial^2_{U_i,U_j}\theta[\varepsilon_{\mathscr{I}_0}](0)}{\theta[\varepsilon_{\mathscr{I}_0}](0)} \tag{35}$$

where

$$\Lambda_{i,j} = -4 \sum_{\text{All partitions } \mathscr{I}_0} \wp_{i,j}(\Omega_{\mathscr{I}_0}) \tag{36}$$

Denote by Λ_g the symmetric matrix

$$\Lambda_g = (\Lambda_{i,j})_{i,j=1,\ldots,g} \tag{37}$$

Proposition 2.5 *Entries $\Lambda_{k,j}$ at $k \leq j$ to the symmetric matrix Λ are given by the formula*

$$\Lambda_{k,j} = \lambda_{k+j} \frac{\begin{pmatrix} 2g+1 \\ g \end{pmatrix}}{\begin{pmatrix} 2g+1 \\ 2g+1-k-j \end{pmatrix}} \sum_{n=1}^{k} n \left[\begin{pmatrix} g \\ g-k+n \end{pmatrix} \begin{pmatrix} g+1 \\ g-j-n+1 \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} g \\ g-j-n \end{pmatrix} \begin{pmatrix} g+1 \\ g-k+n+1 \end{pmatrix} \right] \tag{38}$$

Proof. Execute summation in (29) and find that each $\Lambda_{k,j}$ is proportional to λ_{k+j} with integer coefficients. □

Matrix Λ_g exhibits interesting properties regarding the sum of anti-diagonal elements implemented at derivations in [33],

$$\sum_{i,j,\, i+j=k} \Lambda_{g;i,j} = \lambda_k \frac{N_g}{4g+2} \left[\frac{1}{2} k(2g+2-k) + \frac{1}{4}(2g+1)((-1)^k - 1) \right] \tag{39}$$

Lower genera examples of matrix Λ were given in [32], [33], but the method implemented there was unable to get expressions for Λ at big genera.

Example 2.6 *At $g = 6$ we get matrix*

$$\Lambda_6 = \begin{pmatrix} 792\lambda_2 & 330\lambda_3 & 120\lambda_4 & 36\lambda_5 & 8\lambda_6 & \lambda_7 \\ 330\lambda_3 & 1080\lambda_4 & 492\lambda_5 & 184\lambda_6 & 51\lambda_7 & 8\lambda_8 \\ 120\lambda_4 & 492\lambda_5 & 1200\lambda_6 & 542\lambda_7 & 184\lambda_8 & 36\lambda_9 \\ 36\lambda_5 & 184\lambda_6 & 542\lambda_7 & 1200\lambda_8 & 492\lambda_9 & 120\lambda_{10} \\ 8\lambda_6 & 51\lambda_7 & 184\lambda_8 & 492\lambda_9 & 1080\lambda_{10} & 330\lambda_{11} \\ \lambda_7 & 8\lambda_8 & 36\lambda_9 & 120\lambda_{10} & 330\lambda_{11} & 792\lambda_{12} \end{pmatrix} \tag{40}$$

Collecting all these together we get the following.

Proposition 2.7 *An $\varkappa$-matrix defining a multi-variate σ-function admits the following modular form representation*

$$\varkappa = \frac{1}{8N_g} \Lambda_g - \frac{1}{2N_g} (2\omega)^{-1^T}$$
$$\times \left[\sum_{N_g \text{ even } [\varepsilon]} \frac{1}{\theta[\varepsilon]} \begin{pmatrix} \theta_{1,1}[\varepsilon] & \cdots & \theta_{1,g}[\varepsilon] \\ \vdots & \ldots & \vdots \\ \theta_{1,g}[\varepsilon] & \cdots & \theta_{g,g}[\varepsilon] \end{pmatrix} \right] .(2\omega)^{-1} \tag{41}$$

where 2ω is a matrix of $\mathfrak{a}$-periods of holomorhphic differentials and $\theta_{i,j}[\varepsilon] = \partial^2_{z_i,z_j} \theta[\varepsilon](z)_{z=0}$.

Note that the modular form representation of period matrices η, η' follows from the above formula,

$$\eta = 2\varkappa\omega, \qquad \eta' = 2\varkappa\omega' - \imath\pi(2\omega)^{T^{-1}} \tag{42}$$

Example 2.8 *At $g = 2$ for the curve $y^2 = 4x^5 + \lambda_4 x^4 + \cdots + \lambda_0$ the following representation of the $\varkappa$-matrix is valid*

$$\varkappa = \frac{1}{80}\begin{pmatrix} 4\lambda_2 & \lambda_3 \\ \lambda_3 & 4\lambda_4 \end{pmatrix} - \frac{1}{20}\sum_{10\ even\ [\varepsilon]} \frac{1}{\theta[\varepsilon]}\begin{pmatrix} \partial^2_{U_1^2}\theta[\varepsilon] & \partial^2_{U_1,U_2}\theta[\varepsilon] \\ \partial^2_{U_1,U_2}\theta[\varepsilon] & \partial^2_{U_2^2}\theta[\varepsilon] \end{pmatrix} \tag{43}$$

with $\varkappa = \eta(2\omega)^{-1}$, $\mathscr{A}^{-1} = (2\omega)^{-1} = (U_1, U_2)$ and directional derivatives ∂_{U_i}, $i = 1, 2$.

A representation of the $\varkappa$ matrix of genus 2 and 3 hyperelliptic curves in terms of directional derivatives of non-singular odd constant was found in [30].

2.5 Co-Homologies of Baker and Klein

Calculation of the $\varkappa$-matrix for the hyperelliptic curve (8) were done using the co-homology basis introduced by H. Baker (9). When holomorphic differentials, $\mathrm{d}u(x, y)$, are chosen meromorphic differentials, $\mathrm{d}r(x, y)$, can be found from the symmetry condition **I**. One can check that the symmetry condition is also fulfilled if meromorphic differentials are changed as

$$\mathrm{d}r(x, y) \to \mathrm{d}r(x, y) + M\mathrm{d}u(x, y), \tag{44}$$

where M is an arbitrary constant symmetric matrix $M^T = M$. One can then choose

$$M = -\frac{1}{8N_g}\Lambda_g \tag{45}$$

Then $\varkappa$ will change to

$$\varkappa = -\frac{1}{2}\frac{1}{N_g}\sum_{N_g\ \text{even}\ \left[\varepsilon_{\mathscr{I}_0}\right]} \frac{1}{\theta[\varepsilon_{\mathscr{I}_0}](0)}\left(\partial_{U_i}\partial_{U_j}\theta[\varepsilon_{\mathscr{I}_0}](0)\right)_{i,j=1,\ldots,g}. \tag{46}$$

Following [33] we introduce the co-homology basis of Klein

$$\mathrm{d}u(x, y), \quad \mathrm{d}r(x, y) - \frac{1}{8N_g}\Lambda_g \mathrm{d}u(x, y) \tag{47}$$

with constant matrix, $\Lambda_g = \Lambda_g(\lambda)$ given by (37,38). Therefore we proved

Proposition 2.9 *The $\varkappa$-matrix is represented in the modular form (46) in the co-homology basis (47).*

Formula (46) first appears in F. Klein ([48], [49]). It was recently revisited in a more general context by Korotkin and Shramchenko ([50]) who extended the representation for $\varkappa$ to non-hyperelliptic curves. Correspondence of this representation to the co-homology basis to the best knowledge of the authors was not earlier discussed.

Rewrite formula (46) in the equivalent form,

$$\omega^T \eta = -\frac{1}{4N_g} \sum_{N_g \text{ even } [\varepsilon]} \frac{1}{\theta[\varepsilon]} \begin{pmatrix} \theta_{1,1}[\varepsilon] & \cdots & \theta_{1,g}[\varepsilon] \\ \vdots & \cdots & \vdots \\ \theta_{1,g}[\varepsilon] & \cdots & \theta_{g,g}[\varepsilon] \end{pmatrix} \tag{48}$$

where ω, η are half-periods of holomorphic and meromorphic differentials in the Kleinian basis.

Example 2.10 *For the Weierstrass cubic $y^2 = 4x^3 - g_2 x - g_3$ (48) represents the Weierstrass relation (7).*

Example 2.11 *At $g = 2$ (48) can be written in the form*

$$\omega^T \eta = -\frac{\imath\pi}{10} \begin{pmatrix} \partial_{\tau_{1,1}} & \partial_{\tau_{1,2}} \\ \partial_{\tau_{1,2}} & \partial_{\tau_{2,2}} \end{pmatrix} \ln \chi_5 \tag{49}$$

where χ_5 is a relative invariant of weight 5,

$$\chi_5 = \prod_{10 \textit{ even } [\varepsilon]} \theta[\varepsilon] \tag{50}$$

Example 2.12 *It's worth mentioning how equations of KdV flows look in both bases. For example at $g = 2$ and curve $y^2 = 4x^5 + \lambda_4 x^4 + \cdots + \lambda_0$ in the Baker basis we got* [15]

$$\begin{aligned} \wp_{2222} &= 6\wp_{2,2}^2 + 4\wp_{1,2} + \lambda_4 \wp_{2,2} + \frac{1}{2}\lambda_3 \\ \wp_{1222} &= 6\wp_{2,2}\wp_{1,2} - 2\wp_{1,1} + \lambda_4 \wp_{1,2} \end{aligned} \tag{51}$$

In the Kleinian basis the same equations change only linearly in $\wp_{i,j}$-terms

$$\begin{aligned} \wp_{2222} &= 6\wp_{2,2}^2 + 4\wp_{1,2} - 47\lambda_4 \wp_{2,2} + 92\lambda_4^2 - \frac{7}{2}\lambda_3 \\ \wp_{1222} &= 6\wp_{2,2}\wp_{1,2} - 2\wp_{1,1} - 23\lambda_4\wp_{1,2} - 6\lambda_3\wp_{2,2} + 23\lambda_3\lambda_4 + 8\lambda_2 \end{aligned} \tag{52}$$

2.6 Rosenhain Modular Form Representation of First Kind Periods

Rosenhain [60] was the first to introduce θ-functions with characteristics at $g = 2$. There are 10 even and 6 odd characteristics in that case. Let us denote each of these characteristics as

$$\varepsilon_j = \begin{bmatrix} {\varepsilon'_j}^T \\ {\varepsilon''_j}^T \end{bmatrix}, \quad j = 1, \ldots 10$$

where ε'_j and ε''_j are column 2-vectors with entries equal to 0 or 1.

Rosenhain fixed the hyperelliptic genus two curve in the form

$$y^2 = x(x-1)(x-a_1)(x-a_2)(x-a_3)$$

and presented without proof the expression

$$\mathscr{A}^{-1} = \frac{1}{2\pi^2 Q^2} \begin{pmatrix} -P\theta_2[\delta_2] & Q\theta_2[\delta_1] \\ P\theta_1[\delta_2] & -Q\theta_1[\delta_1] \end{pmatrix} \tag{53}$$

with

$$P = \theta[\alpha_1]\theta[\alpha_2]\theta[\alpha_3], \quad Q = \theta[\beta_1]\theta[\beta_2]\theta[\beta_3]$$

and 6 even characteristics $[\alpha_{1,2,3}], [\beta_{1,2,3}]$ and 2 odd $[\delta_{1,2}]$ which looks chaotic. One of the first proofs can be found in H. Weber [64]; these formulae are implemented in Bolza's dissertation [12] and [13]. Our derivations of these formulae are based on the *Second Thomae relation* [63], see [38] and [34]. To proceed we give the following definitions.

Definition 2.13 *A triplet of characteristics* $[\varepsilon_1]$, $[\varepsilon_2,]$, $[\varepsilon_3]$ *is called azygetic if*

$$\exp \imath\pi \left\{ \sum_{j=1}^{3} {\varepsilon'_j}^T \varepsilon''_j + \sum_{i=1}^{3} {\varepsilon'_i}^T \sum_{i=1}^{3} \varepsilon''_i \right\} = -1$$

Definition 2.14 *A sequence of* $2g+2$ *characteristics* $[\varepsilon_1], \ldots, [\varepsilon_{2g+2}]$ *is called a* special fundamental system *if the first g characteristics are odd, the remaining are even and any triple of characteristics in it is azygetic.*

Theorem 2.15 *(Conjectural Riemann-Jacobi derivative formula) Let g odd* $[\varepsilon_1], \ldots, [\varepsilon_g]$ *and* $g+2$ *even* $[\varepsilon_{g+1}], \ldots, [\varepsilon_{2g+2}]$ *characteristics create a special fundamental system. Then the following equality is valid*

$$\mathrm{Det} \left. \frac{\partial(\theta[\varepsilon_1](v), \ldots, \theta[\varepsilon_g](v))}{\partial(v_1, \ldots, v_g)} \right|_{v=0} = \pm \prod_{k=1\ldots g+2} \theta[\varepsilon_{g+k}](0) \tag{54}$$

Proof. (54) proved up to $g = 5$ in [43], [45], [42] □

Example 2.16 *Jacobi derivative formula for elliptic curve*

$$\vartheta'_1(0) = \pi\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)$$

Example 2.17 *Rosenhain derivative formula for genus two curve is given without proof in the memoir* [60], *namely, let* $[\delta_1]$ *and* $[\delta_2]$ *be any two odd characteristics from all 6 odd, then*

$$\theta_1[\delta_1]\theta_2[\delta_2] - \theta_2[\delta_1]\theta_1[\delta_2] = \pi^2\theta[\gamma_1]\theta[\gamma_2]\theta[\gamma_3]\theta[\gamma_4] \tag{55}$$

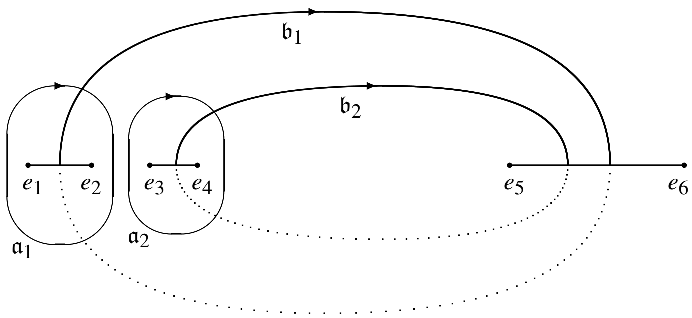

Figure 6.1 Homology basis on the Riemann surface of the curve $\mathscr{C}$ with real branching points $e_1 < e_2 < \cdots < e_6$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i}, $i = 1, 2, 3$. The $\mathfrak{b}$-cycles are completed on the lower sheet (dotted lines).

where 4 even characteristics $[\gamma_1], \ldots, [\gamma_4]$ *are given as* $[\gamma_i] = [\delta_1] + [\delta_2] + [\delta_i]$, $3 \le i \le 6$. *There are 15 Rosenhain derivative formulae.*

The following geometric interpretation of the special fundamental system can be given in the case of hyperelliptic curves. Consider the genus two curve,

$$\mathscr{C}: \quad y^2 = (x - e_1) \cdots (x - e_6)$$

Denote the associated homology basis by $(\mathfrak{a}_1, \mathfrak{a}_2; \mathfrak{b}_1, \mathfrak{b}_2)$. Denote characteristics of Abelian images of branch points with base point e_6 by $\mathfrak{A}_k$, $k = 1, \ldots, 6$. These are half-periods given by their characteristics, $[\mathfrak{A}_k]$ with

$$\mathfrak{A}_k = \int_{(e_6,0)}^{(e_k,0)} u = \frac{1}{2}\tau\varepsilon'_k + \frac{1}{2}\varepsilon''_k, \quad k = 1, \ldots, 6 \tag{56}$$

For the homology basis drawn on the Figure we have

$$[\mathfrak{A}_1] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [\mathfrak{A}_2] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad [\mathfrak{A}_3] = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$[\mathfrak{A}_4] = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_5] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_6] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathfrak{A}_k = \int_{\infty}^{e_k} u = \frac{1}{2}\tau.\varepsilon_k + \frac{1}{2}\varepsilon'_k, \quad [\mathfrak{A}_k] = \begin{bmatrix} \varepsilon_k^T \\ \varepsilon'^T_k \end{bmatrix}$$

One can see that the set of characteristics $[\varepsilon_k] = [\mathfrak{A}_k]$ of Abelian images of branch point contains two odd $[\varepsilon_2]$ and $[\varepsilon_4]$ and the remaining four characteristics are even. One can check that the whole set of these $6 = 2g + 2$ characteristics is azygetic and therefore the set constitutes a special fundamental system. Hence one can write the Rosenhain derivative formula

$$\theta_1[\varepsilon_2]\theta_2[\varepsilon_4] - \theta_2[\varepsilon_2]\theta_1[\varepsilon_4] = \pm\pi^2\theta[\varepsilon_1]\theta[\varepsilon_3]\theta[\varepsilon_5]\theta[\varepsilon_6] \tag{57}$$

In this way we gave a geometric interpretation of the Rosenhain derivative formula and associated set of characteristics in the case when one from even characteristic is zero. The same structure is observed for higher genera hyperelliptic curves even for $g > 5$.

Proposition 2.18 *The characteristics entering to the Rosenhain formula are described as follows. Take any of 15 Rosenhain derivative formulae say,*

$$\theta_1[p]\theta_2[q] - \theta_2[p]\theta_1[q] = \pi^2\theta[\gamma_1]\theta[\gamma_2]\theta[\gamma_3]\theta[\gamma_4]$$

Then 10 even characteristics can be grouped as

$$\underbrace{[\gamma_1], \ldots, [\gamma_4]}_{4}, \quad \underbrace{[\alpha_1], [\alpha_2], [\alpha_3]}_{[\alpha_1]+[\alpha_2]+[\alpha_3]=[p]}, \quad \underbrace{[\beta_1], [\beta_2], [\beta_3]}_{[\beta_1]+[\beta_2]+[\beta_3]=[q]},$$

Then matrix of $\mathfrak{a}$*-periods*

$$\mathscr{A} = \frac{2Q}{PR}\begin{pmatrix} Q\theta_1[q] & Q\theta_2[q] \\ P\theta_1[p] & P\theta_2[p] \end{pmatrix}$$

with

$$P = \prod_{j=1}^{3}\theta[\alpha_j], \quad Q = \prod_{j=1}^{3}\theta[\beta_j], \quad R = \prod_{j=1}^{4}\theta[\gamma_j] \tag{58}$$

Note, that the 15 curves are given as

$$\mathscr{C}_{p,q}: \qquad y^2 = x(x-1)(x-a_1)(x-a_2)(x-a_3) \tag{59}$$

where branch points are computed by *Bolza's formulae* [13],

$$e[\delta_j] = -\frac{\partial_{U_1}\theta[\delta_j]}{\partial_{\boldsymbol{U}_2}\theta[\delta_j]}, \qquad j = 1, \ldots, 6 \tag{60}$$

where ∂_{U_i} is the directional derivative along the vector U_i, $i = 1, 2$, $\mathscr{A}^{-1} = (U_1, U_2)$ and

$$e[p] = 0, \; e[q] = \infty.$$

All 15 curves $\mathscr{C}_{p,q}$ are Möbius equivalent.

2.7 Generalization of the Rosenhain Formula to Higher Genera Hyperelliptic Curve

Generalization of the Rosenhain formula to higher genera hyperelliptic curves was found in [38] and developed further in [34].

$$
\begin{aligned}
&y^2 = \phi(x)\psi(x) \\
&\phi(x) = \prod_{k=1}^{g}(x - e_{2k}), \;\; \psi(x) = \prod_{k=1}^{g+1}(x - e_{2k-1})
\end{aligned}
\tag{61}
$$

Denote by $R = \prod_{k=1}^{g+2}\theta[\gamma_k]$ monomial on the left-hand side of the Riemann-Jacobi formula (54).

Proposition 2.19 *Let a genus g hyperelliptic curve be given as in (61). Then winding vectors* $(U_1, \ldots, U_g) = \mathscr{A}^{-1}$ *are given by the formula*

$$
U_m = \frac{\epsilon}{2\pi^g R}\mathrm{Cofactor}\left(\frac{\partial(\theta[\varepsilon_1](v), \ldots, \theta[\varepsilon_g](v))}{\partial(v_1, \ldots, v_g)}\bigg|_{v=0}\right)\begin{pmatrix} s_{m-1}^{2}\sqrt[4]{\chi_1} \\ \vdots \\ s_{m-1}^{2g}\sqrt[4]{\chi_g}\end{pmatrix}
\tag{62}
$$

Here s_k^i *is an order k symmetric function of elements* $\{e_2, \ldots e_{2g}\}/\{e_{2i}\}$ *and*

$$
\chi_i = \frac{\psi(e_{2i})}{\phi'(e_{2i})}, \quad i = 1, \ldots, g
$$

s_k^i, χ_i *are expressible in* θ*-constants via Thomae formulae* [63].

2.8 Applications of the Above Results

Typical solution of multi-gap integration includes θ-functions $\theta(Ux + Vt + W; \tau)$ where winding vectors U, V are expressed in terms of complete holomorphic integrals and the constant W is defined by initial data. Rosenhain's formulae and their generalizations, express U, V in terms of θ-constants and gives parameters for the equation defining $\mathscr{C}$. In this way the problem of *effectivization of finite gap solutions* [28] can be solved at least for hyperelliptic curves.

Other applications of the Rosenhain formula (53) were presented in [10] where two-gap Lamé and Treibich-Verdier potentials were obtained by the reduction to elliptic functions of general Its-Matveev representations [46] of finite-gap potentials to the Schrödinger equation in terms of multi-variable θ-functions.

Another application is relevant to a computer algebra problem. In the case when, say in Maple, periods of holomorphic differentials are computed then periods of second kind differentials can be obtained by Rosenhain formula (53) and its generalization.

3 Sigma-Functions and the Problem of Differentiation of Abelian Functions

3.1 Problems and Methods

Consider the curve

$$V_\lambda = \left\{ (x, y) \in \mathbb{C}^2 \;:\; y^2 = \mathscr{C}(x; \lambda) = x^{2g+1} + \sum_{k=2}^{2g+1} \lambda_{2k} x^{2g-k+1} \right\} \tag{63}$$

where $g \geqslant 1$ and $\lambda = (\lambda_4, \dots, \lambda_{4g+2}) \in \mathbb{C}^{2g}$ are the parameters. Set $\mathscr{D} = \{\lambda \in \mathbb{C}^{2g} \;:\; \mathscr{C}(x; \lambda)$ has multiple roots$\}$ and $\mathscr{B} = \mathbb{C}^{2g} \setminus \mathscr{D}$. For any $\lambda \in \mathscr{B}$ we obtain the affine part of a smooth projective hyperelliptic curve $\overline{V}_\lambda$ of genus g and the Jacobian variety $Jac(\overline{V}_\lambda) = \mathbb{C}^g / \Gamma_g$, where $\Gamma_g \subset \mathbb{C}^g$ is a lattice of rank $2g$ generated by the periods of the holomorphic differential on cycles of the curve V_λ.

In the general case, an *Abelian function* is a meromorphic function on a complex Abelian torus $T^g = \mathbb{C}^g / \Gamma$, where $\Gamma \subset \mathbb{C}^g$ is a lattice of rank $2g$. In other words, a meromorphic function f on $\mathbb{C}^g$ is Abelian iff $f(u) = f(u + \omega)$ for all $u = (u_1, \dots, u_g) \in \mathbb{C}^g$ and $\omega \in \Gamma$. Abelian functions on T^g form a field $\mathscr{F} = \mathscr{F}_g$ such that:

(1) let $f \in \mathscr{F}$, then $\partial_{u_i} f \in \mathscr{F}$, $i = 1, \dots, g$;
(2) let $f_1, \dots, f_{g+1}$ be any nonconstant functions from $\mathscr{F}$, then there exists a polynomial P such that $P(f_1, \dots, f_{g+1})(u) = 0$ for all $u \in T^g$;
(3) let $f \in \mathscr{F}$ be a nonconstant function, then any $h \in \mathscr{F}$ can be expressed rationally in terms of $(f, \partial_{u_1} f, \dots, \partial_{u_g} f)$;
(4) there exists an entire function $\vartheta : \mathbb{C}^g \to \mathbb{C}$ such that $\partial_{u_i, u_j} \log \vartheta \in \mathscr{F}$, $i, j = 1, \dots, g$.

For example, any elliptic function $f \in \mathscr{F}_1$ is a rational function in the Weierstrass functions $\wp(u; g_2, g_3)$ and $\partial_u \wp(u; g_2, g_3)$, where g_2 and g_3 are parameters of elliptic curve

$$V = \{(x, y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3\}.$$

It is easy to see that the function $\frac{\partial}{\partial g_2} \wp(u; g_2, g_3)$ will no longer be elliptic. This is due to the fact that the period lattice Γ is a function of the parameters g_2 and g_3. In [44] Frobenius and Stickelberger described all the differential operators L in the variables u, g_2 and g_3, such that $Lf \in \mathscr{F}_1$ for any function $f \in \mathscr{F}_1$ (see below Section 7.3).

In [21, 22] the classical problem of differentiation of Abelian functions over parameters for families of (n, s)-curves was solved. In the case of hyperelliptic curves this problem was solved more explicitly.

All genus 2 curves are hyperelliptic. We denote by $\pi : \mathscr{U}_g \to \mathscr{B}_g$ the universal bundle of Jacobian varieties $Jac(\overline{V}_\lambda)$ of hyperelliptic curves. Let us consider the mapping $\varphi\colon \mathscr{B}_g \times \mathbb{C}^g \to \mathscr{U}_g$, which defines the projection $\lambda \times \mathbb{C}^g \to \mathbb{C}^g/\Gamma_g(\lambda)$ for any $\lambda \in \mathscr{B}_g$. Let us fix the coordinates $(\lambda; u)$ in $\mathscr{B}_g \times \mathbb{C}^g \subset \mathbb{C}^{2g} \times \mathbb{C}^g$ where $u = (u_1, \dots, u_{2g-1})$. Thus, using the mapping φ, we fixed in $\mathscr{U}_g$ the structure of the space of the bundle whose fibers J_λ are principally polarized Abelian varieties.

We denote by $F = F_g$ the field of functions on $\mathscr{U}_g$ such that for any $f \in F$ the function $\varphi^*(f)$ is meromorphic, and its restriction to the fiber J_λ is an Abelian function for any point $\lambda \in \mathscr{B}_g$.

Below, we will identify the field F with its image in the field of meromorphic functions on $\mathscr{B} \times \mathbb{C}^g$.

The following **Problem I**:

Describe the Lie algebra of differentiations of the field of meromorphic functions on $\mathscr{B}_g \times \mathbb{C}^g$, generated by the operators L, such that $Lf \in F$ for any function $f \in F$

was solved in [21, 22].

From the differential geometric point of view, Problem I is closely related to **Problem II**:

Describe the connection of the bundle $\pi : \mathscr{U}_g \to \mathscr{B}_g$.

The solution of Problem II leads to an important class of solutions of well-known equations of mathematical physics. In the case $g = 1$, the solution is called the Frobenius-Stikelberger connection (see [29]) and leads to solutions of the Chazy equation.

The space $\mathscr{U}_g$ is a rational variety, more precisely, there is a birational isomorphism $\varphi\colon \mathbb{C}^{3g} \to \mathscr{U}_g$. This fact was discovered by B. A. Dubrovin and S. P. Novikov in [27]. In [27], a fiber of the universal bundle is considered as a level surface of the integrals of motion of gth stationary flow of KdV system, that is, it is defined in $\mathbb{C}^{3g}$ by a system of $2g$ algebraic equations. The degree of the system grows with the growth of genus. In [14], [15] coordinates in $\mathbb{C}^{3g}$ were introduced such that a fiber is defined by $2g$ equations of degree not greater than 3.

The Dubrovin-Novikov coordinates and the coordinates from [14], [15] are the same for the universal space of genus 1 curves. But already in the case of genus 2, these coordinates differ (see [22]).

The integrals of motion of KdV systems are exactly the coefficients $\lambda_{2g+4}, \dots, \lambda_{4g+2}$ of hyperelliptic curve V_λ, in which the coefficients $\lambda_4, \dots, \lambda_{2g+2}$ are free parameters (see (63)). Choosing a point $z \in \mathbb{C}^{3g}$ such that the point $\varphi(z) \in \mathscr{B}_g$ is defined, one can calculate the values of the coefficients $(\lambda_4, \dots, \lambda_{4g+2}) = \pi(z) \in \mathscr{B}_g$ substituting this point in these integrals. Thus, the solution of the Problem I of differentiation of hyperelliptic functions led to the solution of another well-known **Problem III**:

Describe the dependence of the solutions of g-th stationary flow of KdV system on the variation of the coefficients $\lambda_4, \dots, \lambda_{4g+2}$ of hyperelliptic curve, that is, from variation of values of the integrals of motion and parameters.

In [47] results were obtained on Problem III, which use the fact that for a hierarchy KdV the action of polynomial vector fields on the spectral plane is given by the shift of the branch points of the hyperelliptic curve along these fields (see [62]). The deformations of the potential corresponding to this action are exactly the action of the nonisospectral symmetries of the hierarchy KdV.

Let us describe a different approach to Problem III, developed in our works. In [18] we introduced the concept of a polynomial Lie algebra over a ring of polynomials A. For brevity, we shall call them Lie A-algebras. In [21, 22] the ring of polynomials $\mathscr{P}$ in the field F was considered. This ring is generated by all logarithmic derivatives of order $k \geqslant 2$ from the hyperelliptic sigma function $\sigma(u;\lambda)$. The Lie $\mathscr{P}$-algebra $\mathscr{L} = \mathscr{L}_g$ with generators $L_{2k-1},\ k = 1, \dots, g$ and $L_{2l},\ l = 0, \dots, 2g-1$ was constructed. The fields L_{2k-1} define isospectral symmetries, and the fields L_{2l} define nonisospectral symmetries of the hierarchy KdV. The Lie algebra $\mathscr{L}$ is isomorphic to the Lie algebra of differentiation of the ring $\mathscr{P}$ and, consequently, allows us to solve the Problem I (see property (3) of Abelian functions). The generators $L_{2k-1},\ k = 1, \dots, g$, coincide with the operators $\partial_{u_{2k-1}}$, and, consequently, commute. Thus, in the Lie $\mathscr{P}$-algebra $\mathscr{L}$ the Lie $\mathscr{P}$-subalgebra $\mathscr{L}^*$ generated by the operators $L_{2k-1},\ k = 1, \dots, g$ is defined. The generators $L_{2l},\ l = 0, \dots, 2g-1$, are such that the Lie $\mathscr{P}$-algebra $\mathscr{L}^*$ is an ideal in the Lie $\mathscr{P}$-algebra $\mathscr{L}$.

The construction of $L_{2k},\ k = 0, \dots, 2g-1$, is based on the following fundamental fact (see [19]):

The entire function $\psi(u;\lambda)$, satisfying the system of heat equations in a nonholonomic frame

$$\ell_{2i}\psi = H_{2i}\psi,\ i = 0, \dots, 2g-1,$$

under certain initial conditions (see [19]) coincides with the hyperelliptic sigma-function $\sigma(u;\lambda)$. Here ℓ_{2i} are polynomial linear first-order differential operators in the variables $\lambda = (\lambda_4, \dots, \lambda_{4g+2})$ and H_{2i} are linear second-order differential operators in the variables $u = (u_1, \dots, u_{2g-1})$. The methods for constructing these operators are described in [19].

The following fact was used essentially in constructing the operators ℓ_{2i}:

The Lie $\mathbb{C}[\lambda]$-algebra $\mathscr{L}_\lambda$ with generators $\ell_{2i},\ i = 0, \dots, 2g-1$ is isomorphic to an infinite-dimensional Lie algebra $Vect_{\mathscr{B}}$ of vector fields on $\mathbb{C}^{2g}$, that are tangent to the discriminant variety Δ. We recall that the Lie algebra $Vect_{\mathscr{B}}$ is essentially used in singularity theory and its applications (see [2]).

In the Lie $\mathscr{P}$-algebra $\mathscr{L}$ we can choose the generators L_{2k}, $k = 0, \ldots, 2g-1$ such that for any polynomial $P(\lambda) \in \mathbb{C}[\lambda]$ and any k the formula $L_{2k}\pi^* P(\lambda) = \pi^*(\ell_{2k} P(\lambda))$ holds, where π^* is the ring homomorphism induced by the projection $\pi : \mathscr{U}_g \to \mathscr{B}_g$.

Section 3 describes the development of an approach to solving the Problem I. This approach uses:

(a) the graded set of multiplicative generators of the polynomial ring $\mathscr{P} = \mathscr{P}_g$;
(b) the description of all algebraic relations between these generators;
(c) the description of the birational isomorphism $J : \mathscr{U}_g \to \mathbb{C}^{3g}$ in terms of graded polynomial rings;
(d) the description of the polynomial projection $\pi : \mathbb{C}^{3g} \to \mathbb{C}^{2g}$, where $\mathbb{C}^{2g}$ is a space in coordinates $\lambda = (\lambda_4, \ldots, \lambda_{4g+2})$, such that for any $\lambda \in \mathscr{B}$ the space $J^{-1}\pi^{-1}(\lambda)$ is the Jacobian variety $Jac(V_\lambda)$;
(e) the construction of linear differential operators of first order

$$\widehat{H}_{2k} = \sum_{i=1}^{q} h_{2(k-i)+1}(\lambda; u)\partial_{u_{2i-1}}, \; q = \min(k, g),$$

such that $L_{2k} = \ell_{2k} - \widehat{H}_{2k}, \; k = 0, \ldots, 2g-1$.

Here:

- $h_{2(k-i)+1}(\lambda; u)$ are meromorphic functions on $\mathscr{B}_g \times \mathbb{C}^g$;
- $h_{2(k-i)+1}(\lambda; u)$ are homogeneous functions of degree $2(k-i)+1$ in $\lambda = (\lambda_4, \ldots, \lambda_{4g+2}), \; \deg \lambda_{2k} = 2k$, and $u = (u_1, \ldots, u_{2g-1}), \; \deg u_{2k-1} = 1 - 2k$;
- $\partial_{u_{2l-1}} h_{2(k-i)+1}$ are homogeneous polynomials of degree $2(k+l-i)$ in the ring $\mathscr{P}_g$.

This approach was proposed in [23] and found the application in [26]. A detailed construction of the Lie algebra $\mathscr{L}_2$ is given in [23], and the Lie algebra $\mathscr{L}_3$ in [26].

General methods and results (see Section 3.2) will be demonstrated in cases $g = 1$ (see Section 3.3) and $g = 2$ (see Section 3.4).

3.2 Hyperelliptic Functions of Genus $g \geqslant 1$

For brevity, Abelian functions on the Jacobian varieties of (63) will be called *hyperelliptic functions* of genus g. In the theory and applications of these functions, that are based on the sigma-function $\sigma(u; \lambda)$ (see [4, 14, 15, 17]), the grading plays an important role. Below, the variables $u = (u_1, u_3, \ldots, u_{2g-1})$, parameters $\lambda = (\lambda_4, \ldots, \lambda_{4g+2})$ and functions are indexed in a way that clearly indicates their grading. Note that our new notations for the variables differ from the ones in [14, 15, 17] as follows

$$u_i \longleftrightarrow u_{2(g-i)+1},\ i = 1, \dots, g.$$

Let

$$\omega = ((2k_1 - 1) \cdot j_1, \dots, (2k_s - 1) \cdot j_s)$$

where $1 \leqslant s \leqslant g$, $j_q > 0$, $q = 1, \dots, s$ and $j_1 + \cdots + j_s \geqslant 2$. We draw attention to the fact that the symbol "$\cdot$" in the two-component expression $(2k_q - 1) \cdot j_q$ is not a multiplication symbol. Set

$$\wp_\omega(u; \lambda) = -\partial_{u_{2k_1-1}}^{j_1} \cdots \partial_{u_{2k_s-1}}^{j_s} \ln \sigma(u; \lambda). \tag{64}$$

Thus

$$\deg \wp_\omega = (2k_1 - 1) j_1 + \cdots + (2k_s - 1) j_s.$$

Note that our ω differ from the ones in [23, 26].

Say that a multi-index ω is given in normal form if $1 \leqslant k_1 < \cdots < k_s$. According to formula (64), we can always bring the multi-index ω to a normal form using the identifications:

$$\big((2k_p - 1) \cdot j_p, (2k_q - 1) \cdot j_q\big) = \big((2k_q - 1) \cdot j_q, (2k_p - 1) \cdot j_p\big),$$
$$\big((2k_p - 1) \cdot j_p, (2k_q - 1) \cdot j_q\big) = (2k_p - 1) \cdot (j_p + j_q), \quad \text{if } k_p = k_q.$$

In [17] (see also [14, 15]) it was proved that for $1 \leqslant i \leqslant k \leqslant g$ all algebraic relations between hyperelliptic functions of genus g follow from the relations, which in our graded notations have the form

$$\begin{aligned}\wp_{1\cdot 3, (2i-1)\cdot 1} = 6 \big(&\wp_{1\cdot 2}\wp_{1\cdot 1, (2i-1)\cdot 1} + \wp_{1\cdot 1, (2i+1)\cdot 1}\big) \\ &- 2 \big(\wp_{3\cdot 1, (2i-1)\cdot 1} - \lambda_{2i+2}\delta_{i,1}\big).\end{aligned} \tag{65}$$

Here and below, $\delta_{i,k}$ is the Kronecker symbol, $\deg \delta_{i,k} = 0$.

$$\begin{aligned}\wp_{1\cdot 2, (2i-1)\cdot 1}\wp_{1\cdot 2, (2k-1)\cdot 1} = 4 \big(&\wp_{1\cdot 2}\wp_{1\cdot 1, (2i-1)\cdot 1}\wp_{1\cdot 1, (2k-1)\cdot 1} \\ &+ \wp_{1\cdot 1, (2k-1)\cdot 1}\wp_{1\cdot 1, (2i+1)\cdot 1} \\ &+\wp_{1\cdot 1, (2i-1)\cdot 1}\wp_{1\cdot 1, (2k+1)\cdot 1} + \wp_{(2k+1)\cdot 1, (2i+1)\cdot 1}\big) \\ &- 2 \big(\wp_{1\cdot 1, (2i-1)\cdot 1}\wp_{3\cdot 1, (2k-1)\cdot 1} \\ &+ \wp_{1\cdot 1, (2k-1)\cdot 1}\wp_{3\cdot 1, (2i-1)\cdot 1} + \wp_{(2k-1)\cdot 1, (2i+3)\cdot 1} \\ &+\wp_{(2i-1)\cdot 1, (2k+3)\cdot 1}\big) + 2 \big(\lambda_{2i+2}\wp_{1\cdot 1, (2k-1)\cdot 1}\delta_{i,1} \\ &+\lambda_{2k+2}\wp_{1\cdot 1, (2i-1)\cdot 1}\delta_{k,1}\big) \\ &+ 2\lambda_{2(i+j+1)}(2\delta_{i,k} + \delta_{k,i-1} + \delta_{i,k-1}).\end{aligned} \tag{66}$$

Corollary 3.1 *For all $g \geqslant 1$, we have the formulae:*

1. *Setting $i = 1$ in (65), we obtain*

$$\wp_{1\cdot 4} = 6\wp_{1\cdot 2}^2 + 4\wp_{1\cdot 1, 3\cdot 1} + 2\lambda_4. \tag{67}$$

2. *Setting $i = 2$ in (65), we obtain*

$$\wp_{1\cdot 3,3\cdot 1} = 6(\wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} + \wp_{1\cdot 1,5\cdot 1}) - 2\wp_{3\cdot 2}. \tag{68}$$

3. *Setting $i = k = 1$ in (66), we obtain*

$$\wp_{1\cdot 3}^2 = 4\left[\wp_{1\cdot 2}^3 + (\wp_{1\cdot 1,3\cdot 1} + \lambda_4)\wp_{1\cdot 2} + (\wp_{3\cdot 2} - \wp_{1\cdot 1,5\cdot 1} + \lambda_6)\right]. \tag{69}$$

Theorem 3.2 1. *For any $\omega = ((2k_1 - 1)\cdot j_1, \ldots, (2k_s - 1)\cdot j_s)$ the hyperelliptic function $\wp_\omega(u;\lambda)$ is a polynomial from $3g$ functions $\wp_{1\cdot j,(2k-1)\cdot 1},\ 1 \leqslant j \leqslant 3,\ 1 \leqslant k \leqslant g$.*
Note that if $k = 1$, we have $\wp_{1\cdot j,1\cdot 1} = \wp_{1\cdot(j+1)}$.

2. *Set $W_\wp = \{\wp_{1\cdot j,(2k-1)\cdot 1},\ 1 \leqslant j \leqslant 3,\ 1 \leqslant k \leqslant g\}$. The projection of the universal bundle $\pi_g\colon \mathscr{U}_g \to \mathscr{B}_g \subset \mathbb{C}^{2g}$ is given by the polynomials $\lambda_{2k}(W_\wp),\ k = 2, \ldots, 2g+1$ of degree at most 3 from the functions $\wp_{1\cdot j,(2k-1)\cdot 1}$.*

The proof method of Theorem 3.2 will be demonstrated on the following examples:

Example 3.3 1. *Differentiating the relation (67) with respect to u_1, we obtain*

$$\wp_{1\cdot 5} = 12\wp_{1\cdot 2}\wp_{1\cdot 3} + 4\wp_{1\cdot 2,3\cdot 1}. \tag{70}$$

2. *According to formula (67), we obtain*

$$2\lambda_4 = \wp_{1\cdot 4} - 6\wp_{1\cdot 2}^2 - 4\wp_{1\cdot 1,3\cdot 1}. \tag{71}$$

3. *According to formula (68), we obtain*

$$2\wp_{3\cdot 2} = 6(\wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} + \wp_{1\cdot 1,5\cdot 1}) - \wp_{1\cdot 3,3\cdot 1}. \tag{72}$$

4. *Substituting expressions for λ_4 (see (71)) and $\wp_{3\cdot 2}$ (see (72)) into formula (69), we obtain an expression for the polynomial λ_6.*

The derivation of formulae (70)–(72) and the method of obtaining the polynomial λ_6 demonstrate the method of proving Theorem 3.2. Below, this method will be set out in detail in cases $g = 1$ (see Section 3.3) and $g = 2$ (see Section 3.4).

Corollary 3.4 *The operator L of differentiation with respect to $u = (u_1, \ldots, u_{2g-1})$ and $\lambda = (\lambda_1, \ldots, \lambda_{4g+2})$ is a derivation of the ring $\mathscr{P}$ if and only if $L\wp_{1\cdot 1,(2k-1)\cdot 1} \in \mathscr{P}$ for $k = 1, \ldots, g$.*

Proof. According to part 1 of Theorem 3.2, it suffices to prove that $L\wp_{1\cdot j,(2k-1)\cdot 1} \in \mathscr{P}$ for $j = 2$ and 3, $k = 1, \ldots, g$. We have $L\wp_{1\cdot j,(2k-1)\cdot 1} = LL_1\wp_{1\cdot(j-1),(2k-1)\cdot 1} = (L_1L + [L, L_1])\wp_{1\cdot(j-1),(2k-1)\cdot 1}$. Using now that

$[L, L_1] \in \mathscr{L}^*$ and the assumption of Theorem 3.2, we complete the proof by induction. □

Set $\mathscr{A} = \mathbb{C}[X]$, where $X = \{x_{i,2j-1},\ 1 \leqslant i \leqslant 3,\ 1 \leqslant j \leqslant g\}$, $\deg x_{i,2j-1} = i + 2j - 1$.

Corollary 3.5 1. *The birational isomorphism $J\colon \mathscr{U}_g \to \mathbb{C}^{3g}$ is given by the polynomial isomorphism*

$$J^*\colon \mathscr{A} \longrightarrow \mathscr{P} \ :\ J^*X = W_{\wp}.$$

2. *There is a polynomial map*

$$p\colon \mathbb{C}^{3g} \longrightarrow \mathbb{C}^{2g}, \quad p(X) = \lambda,$$

such that

$$p^*\lambda_{2k} = \lambda_{2k}(X),\ k = 2, \dots, 2g+1,$$

where $\lambda_{2k}(X)$ are the polynomials from Theorem 3.2, item 2, obtained by substituting $W_{\wp} \longmapsto X$.

The isomorphism J^* defines the Lie $\mathscr{A}$-algebra $\mathscr{L} = \mathscr{L}_g$ with $3g$ generators $L_{2k-1},\ k = 1, \dots, g$ and $L_{2l},\ l = 0, \dots, 2g-1$. In terms of the coordinates $x_{i,2j-1}$, we obtain the following description of the g-th stationary flow of KdV system.

Theorem 3.6 1. *The commuting operators $L_{2k-1},\ k = 1, \dots, g$, define on $\mathbb{C}^{3g}$ a polynomial dynamical system*

$$L_{2k-1}X = G_{2k-1}(X),\ k = 1, \dots, g, \tag{73}$$

where $G_{2k-1}(X) = \{G_{2k-1,i,2j-1}(X)\}$ and $G_{2k-1,i,2j-1}(X)$ is a polynomial that uniquely defines the expression for the function $\wp_{1\cdot i,(2j-1)\cdot 1,(2k-1)\cdot 1}$ in the form of a polynomial from the functions $\wp_{1\cdot i,(2q-1)\cdot 1}$.

2. *System (73) has $2g$ polynomial integrals $\lambda_{2k} = \lambda_{2k}(X),\ k = 2, \dots, 2g+1$.*

3.3 Elliptic Functions

Consider the curve

$$V_\lambda = \{(x, y) \in \mathbb{C}^2 \ :\ y^2 = x^3 + \lambda_4 x + \lambda_6\}.$$

The discriminant of the family of curves V_λ is

$$\Delta = \{\lambda = (\lambda_4, \lambda_6) \in \mathbb{C}^2 \ :\ 4\lambda_4^3 + 27\lambda_6^2 = 0\}.$$

We have the universal bundle $\pi : \mathscr{U}_1 \to \mathscr{B}_1 = \mathbb{C}^2 \setminus \Delta$ and the mapping

$$\varphi \colon \mathscr{B}_1 \times \mathbb{C} \to \mathscr{U}_1 \ : \ \lambda \times \mathbb{C} \to \mathbb{C}/\,\Gamma_1(\lambda).$$

Consider the field $F = F_1$ of functions on $\mathscr{U}_1$ such that the function $\varphi^*(f)$ is meromorphic, and its restriction to the fiber $\mathbb{C}/\,\Gamma_1(\lambda)$ is an elliptic function for any point $\lambda \in \mathscr{B}_1$. Using the Weierstrass sigma function $\sigma(u;\lambda)$ for $\partial = \frac{\partial}{\partial u}$, we obtain

$$\zeta(u) = \partial \ln \sigma(u;\lambda) \quad \text{and} \quad \wp(u;\lambda) = -\partial\zeta(u;\lambda).$$

The ring of polynomials $\mathscr{P} = \mathscr{P}_1$ in F is generated by the elliptic functions $\wp_{1\cdot i},\ i \geqslant 2$. Set $\wp_{1\cdot i} = \wp_i$. We have $\wp_2 = \wp$ and $\wp_{i+1} = \partial\wp_i = \wp_i'$. All the algebraic relations between the functions $\wp_i$ follow from the relations

$$\wp_4 = 6\wp_2^2 + 2\lambda_4 \quad \text{(see (65))}, \tag{74}$$

$$\wp_3^2 = 4[\wp_2^3 + \lambda_4\wp_2 + \lambda_6] \quad \text{(see (66))}. \tag{75}$$

Thus, we obtain a classical result:

Theorem 3.7 1. *There is the isomorphism* $\mathscr{P} \simeq \mathbb{C}[\wp, \wp', \wp'']$.
2. *The projection* $\pi : \mathscr{U}_1 \to \mathbb{C}^2$ *is given by the polynomials*

$$\frac{1}{2}\wp'' - 3\wp^2 = \lambda_4, \tag{76}$$

$$\left(\frac{\wp'}{2}\right)^2 + 2\wp^3 - \frac{1}{2}\wp''\wp = \lambda_6. \tag{77}$$

Consider the linear space $\mathbb{C}^3$ with the graded coordinates x_2, x_3, x_4, $\deg x_k = k$. Set $\mathscr{A}_1 = \mathbb{C}[x_2, x_3, x_4]$.

Corollary 3.8 *The birational isomorphism* $J : \mathscr{U}_1 \to \mathbb{C}^3$ *is given by the ring isomorphism*

$$J^* \colon \mathscr{A}_1 \to \mathscr{P}_1 \ : \ J^*(x_2, x_3, x_4) = (\wp, \wp', \wp'').$$

Proof. The ring $\mathscr{P}_1$ is generated by elliptic functions $\wp_i$, $i \geqslant 2$, where $\wp_{i+1} = \wp_i'$. It follows from formula (76) that $\wp_5 = 12\wp_2\wp_3$. Hence, each function $\wp_i$ is a polynomial in $\wp_2$, $\wp_3$ and $\wp_4$ for all $i \geqslant 5$. □

Corollary 3.9 1. *The operator* $L_1 = \partial$ *defines a polynomial dynamical system on* $\mathbb{C}^3$

$$x_2' = x_3, \quad x_3' = x_4, \quad x_4' = 12x_2x_3. \tag{78}$$

2. *The system (78) has 2 polynomial integrals*

$$\lambda_4 = \frac{1}{2}x_4 - 3x_2^2 \quad \textit{and} \quad \lambda_6 = \frac{1}{4}x_3^2 + 2x_2^3 - \frac{1}{2}x_4x_2.$$

Let us consider the standard Weierstrass model of an elliptic curve

$$V_g = \{(x, y) \in \mathbb{C}^2 \ : \ y^2 = 4x^3 - g_2 x - g_3\}.$$

The discriminant of this curve has the form $\Delta(g_2, g_3) = g_2^3 - 27g_3^2$. We have $V_\lambda = V_g$ where $g_2 = -4\lambda_4$ and $g_3 = -4\lambda_6$.

The elliptic sigma function $\sigma(u; \lambda)$ satisfies the system of equations

$$\ell_{2i}\,\sigma = H_{2i}\,\sigma, \ i = 0, 1, \tag{79}$$

where

$$\ell_0 = 4\lambda_4\partial_{\lambda_4} + 6\lambda_6\partial_{\lambda_6}; \quad H_0 = u\partial - 1;$$
$$\ell_2 = 6\lambda_6\partial_{\lambda_4} - \frac{4}{3}\lambda_4^2\partial_{\lambda_6}; \quad H_2 = \frac{1}{2}\partial^2 + \frac{1}{6}\lambda_4 u^2.$$

The operators ℓ_0, ℓ_2 and H_0, H_2, characterizing the sigma function $\sigma(u; g_2, g_3)$ of the curve V_g, were constructed in the work of Weierstrass [66]. The operators $L_i \in Der(F_1), \ i = 0, 1$ and 2, were first found by Frobenius and Stickelberger (see [44]). Below, following work [22], we present the construction of the operators L_0 and L_2 on the basis of equations (79).

Let us construct the linear differential operators $\widehat{H}_{2i}, \ i = 0, 1$, of first order, such that $L_{2i} = \ell_{2i} - \widehat{H}_{2i}, \ i = 0, 1$, are the differentiations of the ring $\mathscr{P}_1 = \mathbb{C}[\wp, \wp', \wp'']$.

1. The formula for L_0.

We have $\ell_0\sigma = (u\partial - 1)\sigma$. Therefore, $\ell_0 \ln\sigma = u\zeta(u) - 1$. Applying the operators ∂, ∂^2 and using the fact that operators ∂, ℓ_0 commute, we obtain:

$$\ell_0\zeta = \zeta - u\wp, \quad \ell_0\wp = 2\wp + u\partial\wp.$$

Setting $\widehat{H}_0 = u\partial$, we obtain $L_0 = \ell_0 - u\partial$. Consequently

$$L_0\zeta = \zeta, \quad L_0\wp = 2\wp.$$

2. The formula for L_2.

We have $\ell_2\sigma = \frac{1}{2}\partial^2\sigma - \frac{1}{6}\lambda_4 u^2\sigma$. Therefore $\ell_2 \ln\sigma = \frac{1}{2}\frac{\partial^2\sigma}{\sigma} - \frac{1}{6}\lambda_4 u^2$. We have $\frac{\partial^2\sigma}{\sigma} = -\wp_2 + \zeta^2$. Thus

$$\ell_2 \ln\sigma = -\frac{1}{2}\wp_2 + \frac{1}{2}\zeta^2 - \frac{1}{6}\lambda_4 u^2. \tag{80}$$

Applying the operators ∂ and ∂^2 to (80), we obtain

$$\ell_2\zeta = -\frac{1}{2}\wp_3 + \zeta\,\partial\zeta - \frac{1}{3}\lambda_4 u, \qquad -\ell_2\wp_2 = -\frac{1}{2}\wp_4 + \wp_2^2 - \zeta\,\partial\wp_2 - \frac{1}{3}\lambda_4.$$

Setting $\widehat{H}_2 = \zeta\,\partial$, we obtain $L_2 = \ell_2 - \zeta\,\partial$. Consequently,

$$L_2\zeta = -\frac{1}{2}\wp_3 - \frac{1}{3}\lambda_4 u, \qquad L_2\wp_2 = \frac{1}{2}\wp_4 - \wp_2^2 + \frac{1}{3}\lambda_4 = \frac{2}{3}\wp_4 - 2\wp_2^2.$$

Thus, we get the following result:

Theorem 3.10 *The Lie $\mathscr{P}_1$-algebra $\mathscr{L}_1$ is generated by operators $L_0,\ L_1$ and L_2 such that*

$$[L_0,\ L_k] = kL_k,\ k = 1, 2, \qquad [L_1,\ L_2] = \wp_2 L_1, \tag{81}$$

$$L_0\wp_2 = 2\wp_2; \quad L_1\wp_2 = \wp_3; \quad L_2\wp_2 = \frac{2}{3}\wp_4 - 2\wp_2^2. \tag{82}$$

Proof. Formulas (81)–(82) completely determine the actions of the operators $L_k,\ k = 0, 1, 2$, on the ring $\mathscr{P}_1$ by the following inductive formula:

$$L_k\wp_{i+1} = [L_k,\ L_1]\wp_i + L_1 L_k\wp_i. \tag{83}$$

□

Example 3.11 *Substituting $k = 2$ and $i = 2$ in (83), we obtain*

$$L_2\wp_3 = [L_2,\ L_1]\wp_2 + L_1 L_2\wp_2 = -5\wp_2\wp_3 + \frac{4}{3}\wp_5.$$

3.4 Hyperelliptic Functions of Genus $g = 2$

For each curve with affine part of the form

$$V_\lambda = \left\{(x, y) \in \mathbb{C}^2 \mid y^2 = x^5 + \lambda_4 x^3 + \lambda_6 x^2 + \lambda_8 x + \lambda_{10}\right\},$$

one can construct a sigma-function $\sigma(u; \lambda)$ (see [14]). This function is an entire function in $u = (u_1, u_3) \in \mathbb{C}^2$ with parameters $\lambda = (\lambda_4, \lambda_6, \lambda_8, \lambda_{10}) \in \mathbb{C}^4$. It has a series expansion in u over the polynomial ring $\mathbb{Q}[\lambda_4, \lambda_6, \lambda_8, \lambda_{10}]$ in the vicinity of 0. The initial segment of the expansion has the form

$$\sigma(u; \lambda) = u_3 - \frac{1}{3}u_1^3 + \frac{1}{6}\lambda_6 u_3^3 - \frac{1}{12}\lambda_4 u_1^4 u_3 - \frac{1}{6}\lambda_6 u_1^3 u_3^2 - \\ - \frac{1}{6}\lambda_8 u_1^2 u_3^3 - \frac{1}{3}\lambda_{10} u_1 u_3^4 + \left(\frac{1}{60}\lambda_4\lambda_8 + \frac{1}{120}\lambda_6^2\right) u_3^5 + (u^7). \tag{84}$$

Here (u^k) denotes the ideal generated by monomials $u_1^i u_3^j,\ i + j = k$.

The sigma-function is an odd function in u, i.e. $\sigma(-u; \lambda) = -\sigma(u; \lambda)$.

Set

$$\nabla_\lambda = \left(\frac{\partial}{\partial\lambda_4}, \frac{\partial}{\partial\lambda_6}, \frac{\partial}{\partial\lambda_8}, \frac{\partial}{\partial\lambda_{10}}\right) \quad \text{and} \quad \partial_{u_1} = \frac{\partial}{\partial u_1},\ \partial_{u_3} = \frac{\partial}{\partial u_3}.$$

We need the following properties of the two-dimensional sigma-function (see [15, 20] for details):

1. The following system of equations holds:

$$\ell_i\sigma = H_i\sigma, \quad i = 0, 2, 4, 6, \tag{85}$$

where $(\ell_0\ \ell_2\ \ell_4\ \ell_6)^\top = T\,\nabla_\lambda$,

$$T = \begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ 6\lambda_6 & 8\lambda_8 - \frac{12}{5}\lambda_4^2 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & -\frac{4}{5}\lambda_4\lambda_8 \\ 8\lambda_8 & 10\lambda_{10} - \frac{8}{5}\lambda_4\lambda_6 & 4\lambda_4\lambda_8 - \frac{12}{5}\lambda_6^2 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 \\ 10\lambda_{10} & -\frac{4}{5}\lambda_4\lambda_8 & 6\lambda_4\lambda_{10} - \frac{6}{5}\lambda_6\lambda_8 & 4\lambda_6\lambda_{10} - \frac{8}{5}\lambda_8^2 \end{pmatrix}$$

and

$$\begin{aligned} H_0 &= u_1\partial_{u_1} + 3u_3\partial_{u_3} - 3, \\ H_2 &= \frac{1}{2}\,\partial_{u_1}^2 - \frac{4}{5}\lambda_4 u_3\partial_{u_1} + u_1\partial_{u_3} - \frac{3}{10}\lambda_4 u_1^2 + \frac{1}{10}(15\lambda_8 - 4\lambda_4^2)u_3^2, \\ H_4 &= \partial_{u_1}\partial_{u_3} - \frac{6}{5}\,\lambda_6 u_3\partial_{u_1} + \lambda_4 u_3\partial_{u_3} - \frac{1}{5}\,\lambda_6 u_1^2 + \lambda_8 u_1 u_3 \\ &\quad + \frac{1}{10}(30\lambda_{10} - 6\lambda_6\lambda_4)u_3^2 - \lambda_4, \\ H_6 &= \frac{1}{2}\,\partial_{u_3}^2 - \frac{3}{5}\lambda_8 u_3\partial_{u_1} - \frac{1}{10}\,\lambda_8 u_1^2 + 2\lambda_{10}u_1u_3 - \frac{3}{10}\,\lambda_8\lambda_4 u_3^2 - \frac{1}{2}\,\lambda_6. \end{aligned}$$

2. The equation $\ell_0\,\sigma = H_0\sigma$ implies that σ is a homogeneous function of degree -3 in u_1, u_3, λ_j.

3. The discriminant of the hyperelliptic curve V_λ of genus 2 is equal to $\Delta = \frac{16}{5}\det T$. It is a homogeneous polynomial in λ of degree 40. Set $\mathscr{B} = \{\lambda \in \mathbb{C}^4 \,:\, \Delta(\lambda) \neq 0\}$; then the curve V_λ is smooth for $\lambda \in \mathscr{B}$.

We have

$$\ell_0\,\Delta = 40\Delta, \quad \ell_2\,\Delta = 0, \quad \ell_4\,\Delta = 12\lambda_4\Delta, \quad \ell_6\,\Delta = 4\lambda_6\Delta.$$

Thus, the fields ℓ_0, ℓ_2, ℓ_4 and ℓ_6 are tangent to the variety $\{\lambda \in \mathbb{C}^4 \,:\, \Delta(\lambda) = 0\}$.

The present study is based on the following results.

Theorem 3.12 (uniqueness conditions for the two-dimensional sigma-function) *The entire function $\sigma(u;\lambda)$ is uniquely determined by the system of equations (85) and initial condition $\sigma(u;0) = u_3 - \frac{1}{3}u_1^3$.*

We have the universal bundle $\pi : \mathscr{U}_2 \to \mathscr{B}_2 = \mathbb{C}^4 \setminus \mathscr{D}$ and the mapping

$$\varphi : \mathscr{B}_2 \times \mathbb{C}^2 \to \mathscr{U}_2 \,:\, \lambda \times \mathbb{C}^2 \to \mathbb{C}^2/\,\Gamma_2(\lambda).$$

Consider the field $F = F_2$ of functions on $\mathscr{U}_2$ such that the function $\varphi^*(f)$ is meromorphic, and its restriction to the fiber $\mathbb{C}^2/\,\Gamma_2(\lambda)$ is an hyperelliptic function for any point $\lambda \in \mathscr{B}_2$.

All the algebraic relations between the hyperelliptic functions of genus 2 follow from the relations, which in our notations have the form:

$$\wp_{1\cdot 4} = 6\wp_{1\cdot 2}^2 + 4\wp_{1\cdot 1,3\cdot 1} + 2\lambda_4, \tag{86}$$

$$\wp_{1\cdot 3,3\cdot 1} = 6\wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} - 2\wp_{3\cdot 2}, \tag{87}$$

(see (65) for $i = 1$ and $i = 2$) and

$$\wp_{1\cdot 3}^2 = 4\left[\wp_{1\cdot 2}^3 + (\wp_{1\cdot 1,3\cdot 1} + \lambda_4)\wp_{1\cdot 2} + \wp_{3\cdot 2} + \lambda_6\right], \tag{88}$$

$$\wp_{1\cdot 3}\wp_{1\cdot 2,3\cdot 1} = 4\wp_{1\cdot 2}^2\wp_{1\cdot 1,3\cdot 1} + 2\wp_{1\cdot 1,3\cdot 1}^2 - 2\wp_{1\cdot 2}^2\wp_{3\cdot 2} + 2\lambda_4\wp_{1\cdot 1,3\cdot 1} + 2\lambda_8, \tag{89}$$

$$\wp_{1\cdot 2,3\cdot 1}^2 = 4(\wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1}^2 - \wp_{1\cdot 1,3\cdot 1}\wp_{3\cdot 2} + \lambda_{10}) \tag{90}$$

(see (66) for $(i, k) = (1, 1),\ (1, 2)$ and $(2, 2)$).

Consider the linear space $\mathbb{C}^6$ with the graded coordinates $X = (x_2, x_3, x_4),\ Y = (y_4, y_5, y_6)$, $\deg x_k = k,\ \deg y_k = k$. Set $\mathscr{A}_2 = \mathbb{C}[X, Y]$.

Theorem 3.13 1. *The birational isomorphism* $J_2 \colon \mathscr{U}_2 \to \mathbb{C}^6$ *is given by the isomorphism of polynomial rings*

$$J_2^* \colon \mathscr{A}_2 \longrightarrow \mathscr{P}_2 \ :\ J_2^* X = (\wp_{1\cdot 2}, \wp_{1\cdot 3}, \wp_{1\cdot 4}),$$
$$J_2^* Y = (\wp_{1\cdot 1,3\cdot 1}, \wp_{1\cdot 2,3\cdot 1}, \wp_{1\cdot 3,3\cdot 1}).$$

2. *The projection* $\pi_2 \colon \mathbb{C}^6 \to \mathbb{C}^4$ *is given by the polynomials*

$$\lambda_4 = -3x_2^2 + \frac{1}{2}x_4 - 2y_4, \tag{91}$$

$$\lambda_6 = 2x_2^3 + \frac{1}{4}x_3^2 - \frac{1}{2}x_2x_4 - 2x_2y_4 + \frac{1}{2}y_6, \tag{92}$$

$$\lambda_8 = (4x_2^2 + y_4)y_4 - \frac{1}{2}(x_4y_4 - x_3y_5 + x_2y_6), \tag{93}$$

$$\lambda_{10} = 2x_2y_4^2 + \frac{1}{4}y_5^2 - \frac{1}{2}y_4y_6. \tag{94}$$

Proof. Using the isomorphism J_2^*, we rewrite the relations (86)–(90) in the form

$$x_4 = 6x_2^2 + 4y_4 + 2\lambda_4, \tag{95}$$

$$y_6 = 6x_2y_4 - 2\wp_{3\cdot 2}, \tag{96}$$

$$x_3^2 = 4\left[x_2^3 + (y_4 + \lambda_4)x_2 + \wp_{3\cdot 2} + \lambda_6\right], \tag{97}$$

$$x_3y_5 = 2\left[2x_2^2y_4 + y_4^2 - x_2^2\wp_{3\cdot 2} + \lambda_4y_4 + \lambda_8\right], \tag{98}$$

$$y_5^2 = 4\left[x_2y_4^2 - y_4\wp_{3\cdot 2} + \lambda_{10}\right]. \tag{99}$$

Directly from relations (95)–(99), we obtain the formula for the polynomial mapping π_2, that is, the proof of assertion 2 of the theorem.

Set $x_{i+1} = \wp_{1\cdot(i+1)},\ y_{i+3} = \wp_{1\cdot i,3\cdot 1},\ i \geqslant 1$. Applying the operator ∂_{u_1} to formula (95), we obtain

$$x_5 = 12x_2x_3 + 4y_5 = x_5(X, Y). \tag{100}$$

Substituting the expression for λ_4 from (95) and the expression for $\wp_{3\cdot 2}$ from (96) into the formula (97) and then applying the operator ∂_{u_1}, we obtain

$$y_7 = 4x_3y_4 + x_2(x_5 + 4y_5 - 12x_2x_3) = y_7(X, Y). \tag{101}$$

By induction from formulas (100) and (101), we obtain the polynomial formulas

$$x_{i+1} = x_{i+1}(X, Y), \qquad y_{i+3} = y_{i+3}(X, Y). \tag{102}$$

From formula (96) we obtain

$$\wp_{3\cdot 2} = 3x_2y_4 - \frac{1}{2}y_6 = z_6(X, Y), \qquad \wp_{3\cdot(i+2)} = \partial_{u_3}^i z_6(X, Y) = z_{3i+6}. \tag{103}$$

The following formulae complete the proof of assertion 1 of the theorem

$$\partial_{u_3}x_{i+1} = \partial_{u_1}\wp_{1\cdot i,3\cdot 1} = \partial_{u_1}y_{i+3}(X, Y), \tag{104}$$

$$\partial_{u_3}y_{i+3} = \partial_{u_3}\wp_{1\cdot i,3\cdot 1} = \wp_{1\cdot i,3\cdot 2} = \partial_{u_1}^i z_6(X, Y). \tag{105}$$

□

In the course of the proof of Theorem 3.13, we obtained a detailed proof of Theorem 3.2 in the case $g = 2$.

Set $L_1 = \partial_{u_1}$ and $L_3 = \partial_{u_3}$. We introduce the operators $L_i \in \mathrm{Der}(F_2),\ i = 0, 2, 4, 6$, based on the operators $\ell_i - H_i$.

Theorem 3.14 *The generators of the F_2-module* $\mathrm{Der}(F_2)$ *are given by the formulae*

$$L_{2k-1} = \partial_{u_{2k-1}},\ k = 1, 2, \qquad L_{2k} = \ell_{2k} - \widehat{H}_{2k},\ k = 0, 1, 2, 3,$$

where

$$\widehat{H}_0 = u_1\partial_{u_1} + 3u_3\partial_{u_3}, \qquad \widehat{H}_2 = \left(\zeta_1 - \frac{4}{5}\lambda_4u_3\right)\partial_{u_1} + u_1\partial_{u_3},$$

$$\widehat{H}_4 = \left(\zeta_3 - \frac{6}{5}\lambda_6u_3\right)\partial_{u_1} + (\zeta_1 + \lambda_4u_3)\partial_{u_3}, \qquad \widehat{H}_6 = -\frac{3}{5}\lambda_8u_3\partial_{u_1} + \zeta_3\partial_{u_3}.$$

Proof. We will use the methods of [22] to obtain the explicit form of operators L_i and to describe their action on the ring $\mathscr{P}_2$. Note here that this theorem corrects misprints made in [22, 23].

We have $L_1 = \partial_{u_1} \in \mathrm{Der}(F_2)$ and $L_3 = \partial_{u_3} \in \mathrm{Der}(F_2)$.

Below we use the fact that $[\partial_{u_k}, \ell_q] = 0$ for $k = 1, 3$ and $q = 0, 2, 4, 6$.

1). Derivation of the formula for L_0.

Using (85), we have $\ell_0\sigma = H_0\sigma = (u_1\partial_{u_1} + 3u_3\partial_{u_3} - 3)\sigma$. Therefore

$$\ell_0 \ln \sigma = u_1\partial_{u_1} \ln \sigma + 3u_3\partial_{u_3} \ln \sigma - 3. \tag{106}$$

Applying the operators ∂_{u_1} and ∂_{u_3} to (106), we obtain

$$\ell_0\zeta_1 = \zeta_1 - u_1\wp_{1\cdot2} - 3u_3\wp_{1\cdot1,3\cdot1}, \tag{107}$$

$$\ell_0\zeta_3 = 3\zeta_3 - u_1\wp_{1\cdot1,3\cdot1} - 3u_3\wp_{3\cdot2}. \tag{108}$$

We apply the operator ∂_{u_1} to (107) to obtain

$$-\ell_0\wp_{1\cdot2} = -2\wp_{1\cdot2} - u_1\wp_{1\cdot3} - 3u_3\wp_{1\cdot2,3\cdot1}.$$

Therefore

$$(\ell_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3})\wp_{1\cdot2} = 2\wp_{1\cdot2}.$$

Applying the operator ∂_{u_1} to (108), we obtain

$$-\ell_0\wp_{1\cdot1,3\cdot1} = -\wp_{1\cdot1,3\cdot1} - u_1\wp_{1\cdot2,3\cdot1} - 3\wp_{1\cdot1,3\cdot1} - 3u_3\wp_{1\cdot1,3\cdot2}.$$

Therefore,

$$(\ell_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3})\wp_{1\cdot1,3\cdot1} = 4\wp_{1\cdot1,3\cdot1}.$$

Thus, we have proved that

$$L_0 = \ell_0 - u_1\partial_{u_1} - 3u_3\partial_{u_3} \in \mathrm{Der}(F_2).$$

2). Derivation of the formula for L_2.

Using (85), we have

$$\ell_2\sigma = H_2\sigma = \left(\frac{1}{2}\,\partial_{u_1}^2 - \frac{4}{5}\,\lambda_4u_3\partial_{u_1} + u_1\partial_{u_3} + w_2\right)\sigma$$

where

$$w_2 = w_2(u_1, u_3) = -\frac{3}{10}\,\lambda_4u_1^2 + \frac{1}{10}\,(15\lambda_8 - 4\lambda_4^2)u_3^2.$$

Therefore

$$\ell_2 \ln \sigma = \frac{1}{2}\,\frac{\partial_{u_1}^2\sigma}{\sigma} - \frac{4}{5}\,\lambda_4u_3\partial_{u_1} \ln \sigma + u_1\partial_{u_3} \ln \sigma + w_2.$$

It holds that

$$\frac{\partial_{u_1}^2\sigma}{\sigma} = -\wp_{1\cdot2,0} + \zeta_1^2.$$

We get

$$\ell_2 \ln \sigma = -\frac{1}{2}\wp_{1\cdot 2} + \frac{1}{2}\zeta_1^2 - \frac{4}{5}\lambda_4 u_3 \zeta_1 + u_1\zeta_3 + w_2. \tag{109}$$

Applying the operators ∂_{u_1} and ∂_{u_3} to (109), we obtain

$$\ell_2\zeta_1 = -\frac{1}{2}\wp_{1\cdot 3} - \zeta_1\wp_{1\cdot 2} + \frac{4}{5}\lambda_4 u_3\wp_{1\cdot 2} + \zeta_3 - u_1\wp_{1\cdot 1,3\cdot 1} + \partial_{u_1} w_2,$$
$$\ell_2\zeta_3 = -\frac{1}{2}\wp_{1\cdot 2,3\cdot 1} - \zeta_1\wp_{1\cdot 1,3\cdot 1} - \frac{4}{5}\lambda_4\zeta_1 + \frac{4}{5}\lambda_4 u_3\wp_{1\cdot 1,3\cdot 1} - u_1\wp_{3\cdot 2} + \partial_{u_3} w_2.$$

Applying the operator ∂_{u_1} again, we obtain

$$\begin{aligned}
-\ell_2\wp_{1\cdot 2} &= -\frac{1}{2}\wp_{1\cdot 4} + \wp_{1\cdot 2}^2 - \zeta_1\wp_{1\cdot 3} + \frac{4}{5}\lambda_4 u_3\wp_{1\cdot 3} - 2\wp_{1\cdot 1,3\cdot 1} \\
&\quad - u_1\wp_{1\cdot 2,3\cdot 1} + \partial_{u_1}^2 w_2, \\
-\ell_2\wp_{1\cdot 1,3\cdot 1} &= -\frac{1}{2}\wp_{1\cdot 3,3\cdot 1} + \wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} - \zeta_1\wp_{1\cdot 2,3\cdot 1} + \frac{4}{5}\lambda_4\wp_{1\cdot 2} \\
&\quad + \frac{4}{5}\lambda_4 u_3\wp_{1\cdot 2,3\cdot 1} - \wp_{3\cdot 2} - u_1\wp_{1\cdot 1,3\cdot 2} + \partial_{u_1}\partial_{u_3} w_2.
\end{aligned}$$

Thus, we have proved that

$$L_2 = \left(\ell_2 - \zeta_1\partial_{u_1} - u_1\partial_{u_3} + \frac{4}{5}\lambda_4 u_3\partial_{u_1}\right) \in \mathrm{Der}(F_2).$$

We have $\partial_{u_1}^2 w_2 = -\frac{3}{5}\lambda_4$ and $\partial_{u_1}\partial_{u_3} w_2 = 0$.

3). Derivation of the formula for L_4.
Using (85), we have

$$\ell_4\sigma = H_4\sigma = \left(\partial_{u_1}\partial_{u_3} - \frac{6}{5}\lambda_6 u_3\partial_{u_1} + \lambda_4 u_3\partial_{u_3} + w_4\right)\sigma$$

where

$$w_4 = -\frac{1}{5}\lambda_6 u_1^2 + \lambda_8 u_1 u_3 + \frac{1}{10}(30\lambda_{10} - 6\lambda_6\lambda_4)u_3^2 - \lambda_4.$$

Therefore,

$$\ell_4 \ln\sigma = \frac{\partial_{u_1}\partial_{u_3}\sigma}{\sigma} - \frac{6}{5}\lambda_6 u_3\partial_{u_1}\ln\sigma + \lambda_4 u_3\partial_{u_3}\ln\sigma + w_4.$$

It holds that

$$\frac{\partial_{u_1}\partial_{u_3}\sigma}{\sigma} = -\wp_{1\cdot 1,3\cdot 1} + \zeta_1\zeta_3.$$

We obtain

$$\ell_4 \ln\sigma = -\wp_{1\cdot 1,3\cdot 1} + \zeta_1\zeta_3 - \frac{6}{5}\lambda_6 u_3\zeta_1 + \lambda_4 u_3\zeta_3 + w_4. \tag{110}$$

Applying the operators ∂_{u_1} and ∂_{u_3} to (110), we obtain

$$\ell_4\zeta_1 = -\wp_{1\cdot 2,3\cdot 1} - \wp_{1\cdot 2}\zeta_3 - \zeta_1\wp_{1\cdot 1,3\cdot 1} + \frac{6}{5}\lambda_6 u_3\wp_{1\cdot 2} - \lambda_4 u_3\wp_{1\cdot 1,3\cdot 1} + \partial_{u_1} w_4,$$

$$\begin{aligned}\ell_4\zeta_3 = {} & -\wp_{1\cdot 1,3\cdot 2} - \wp_{1\cdot 1,3\cdot 1}\zeta_3 - \zeta_1\wp_{3\cdot 2} - \frac{6}{5}\lambda_6\zeta_1 + \frac{6}{5}\lambda_6 u_3\wp_{1\cdot 1,3\cdot 1} \\ & + \lambda_4\zeta_3 - \lambda_4 u_3\wp_{3\cdot 2} + \partial_{u_3} w_4.\end{aligned}$$

Applying the operator ∂_{u_1} again, we obtain

$$\begin{aligned}-\ell_4\wp_{1\cdot 2} = {} & -\wp_{1\cdot 3,3\cdot 1} - \wp_{1\cdot 3}\zeta_3 + \wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} + \wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} - \zeta_1\wp_{1\cdot 2,3\cdot 1} \\ & + \frac{6}{5}\lambda_6 u_3\wp_{1\cdot 3} - \lambda_4 u_3\wp_{1\cdot 2,3\cdot 1} + \partial_{u_1}^2 w_4,\\ -\ell_4\wp_{1\cdot 1,3\cdot 1} = {} & -\wp_{1\cdot 2,3\cdot 2} - \wp_{1\cdot 2,3\cdot 1}\zeta_3 + \wp_{1\cdot 1,3\cdot 1}^2 + \wp_{1\cdot 2}\wp_{3\cdot 2} - \zeta_1\wp_{1\cdot 1,3\cdot 2} \\ & + \frac{6}{5}\lambda_6\wp_{1\cdot 2} + \frac{6}{5}\lambda_6 u_3\wp_{1\cdot 2,3\cdot 1} - \lambda_4\wp_{1\cdot 1,3\cdot 1} - \lambda_4 u_3\wp_{1\cdot 1,3\cdot 2} \\ & + \partial_{u_1}\partial_{u_3} w_4.\end{aligned}$$

Therefore, we have proved that

$$L_4 = \left(\ell_4 - \zeta_3\partial_{u_1} - \zeta_1\partial_{u_3} + \frac{6}{5}\lambda_6 u_3\partial_{u_1} - \lambda_4 u_3\partial_{u_3}\right) \in \operatorname{Der}(F_2).$$

We have $\partial_{u_1}^2 w_4 = -\frac{2}{5}\lambda_6$ and $\partial_{u_1}\partial_{u_3} w_4 = \lambda_8$.

4). <u>Derivation of the formula for L_6.</u>
Using (85), we have

$$\ell_6\sigma = H_6\sigma = \left(\frac{1}{2}\partial_{u_3}^2 - \frac{3}{5}\lambda_8 u_3\partial_{u_1} + w_6\right)\sigma$$

where

$$w_6 = -\frac{1}{10}\lambda_8 u_1^2 + 2\lambda_{10} u_1 u_3 - \frac{3}{10}\lambda_8\lambda_4 u_3^2 - \frac{1}{2}\lambda_6.$$

Therefore,

$$\ell_6 \ln\sigma = \frac{1}{2}\frac{\partial_{u_3}^2\sigma}{\sigma} - \frac{3}{5}\lambda_8 u_3\partial_{u_1}\ln\sigma + w_6.$$

We obtain

$$\ell_6 \ln\sigma = -\frac{1}{2}\wp_{3\cdot 2} + \frac{1}{2}\zeta_3^2 - \frac{3}{5}\lambda_8 u_3\zeta_1 + w_6. \tag{111}$$

Applying the operators ∂_{u_1} and ∂_{u_3} to (111), we obtain

$$\ell_6\zeta_1 = -\frac{1}{2}\wp_{1\cdot 1,3\cdot 2} - \zeta_3\wp_{1\cdot 1,3\cdot 1} + \frac{3}{5}\lambda_8 u_3\wp_{1\cdot 2} + \partial_{u_1} w_6,$$

$$\ell_6\zeta_3 = -\frac{1}{2}\wp_{3\cdot 3} - \zeta_3\wp_{3\cdot 2} - \frac{3}{5}\lambda_8\zeta_1 + \frac{3}{5}\lambda_8 u_3\wp_{1\cdot 1,3\cdot 1} + \partial_{u_3} w_6.$$

Applying the operator ∂_{u_1} again, we obtain

$$-\ell_6\wp_{1\cdot 2} = -\frac{1}{2}\wp_{1\cdot 2,3\cdot 2} + \wp_{1\cdot 1,3\cdot 1}^2 - \zeta_3\wp_{1\cdot 2,3\cdot 1} + \frac{3}{5}\lambda_8 u_3\wp_{1\cdot 3} + \partial_{u_1}^2 w_6,$$

$$-\ell_6\wp_{1\cdot 1,3\cdot 1} = -\frac{1}{2}\wp_{1\cdot 1,3\cdot 3} + \wp_{1\cdot 1,3\cdot 1}\wp_{3\cdot 2} - \zeta_3\wp_{1\cdot 1,3\cdot 2} + \frac{3}{5}\lambda_8\wp_{1\cdot 2} + \frac{3}{5}\lambda_8 u_3\wp_{1\cdot 2,3\cdot 1} + \partial_{u_1}\partial_{u_3} w_6.$$

Therefore, we have proved that

$$L_6 = \left(\ell_6 - \zeta_3\partial_{u_3} + \frac{3}{5}\lambda_8 u_3\partial_{u_1}\right) \in \mathrm{Der}(F_2).$$

We have $\partial_{u_1}^2 w_6 = -\frac{1}{5}\lambda_8$ and $\partial_{u_1}\partial_{u_3} w_6 = 2\lambda_{10}$. This completes the proof. □

The description of commutation relations in the differential algebra of Abelian functions of genus 2 was given in [22, 23], see also [17]. We obtain this result directly from Theorem 3.14 and correct some misprints made in [22, 23]. To simplify the calculations, we use the following results:

Lemma 3.15 *The following commutation relations hold for ℓ_k:*

$$[\partial_{u_1}, \ell_k] = 0, \quad k = 0, 2, 4, 6, \quad [\partial_{u_3}, \ell_k] = 0, \quad k = 0, 2, 4, 6,$$

$$[\ell_0, \ell_k] = k\ell_k, \quad k = 2, 4, 6, \quad [\ell_2, \ell_4] = \frac{8}{5}\lambda_6\ell_0 - \frac{8}{5}\lambda_4\ell_2 + 2\ell_6,$$

$$[\ell_2, \ell_6] = \frac{4}{5}\lambda_8\ell_0 - \frac{4}{5}\lambda_4\ell_4, \quad [\ell_4, \ell_6] = -2\lambda_{10}\ell_0 + \frac{6}{5}\lambda_8\ell_2 - \frac{6}{5}\lambda_6\ell_4 + 2\lambda_4\ell_6.$$

Proof. These relations follow directly from (85). □

Lemma 3.16 *The operators L_i, $i = 0, 1, 2, 3, 4, 6$, act on $-\zeta_1$ and $-\zeta_3$ according to the formulae*

$$L_0(-\zeta_1) = -\zeta_1, \qquad L_0(-\zeta_3) = -3\zeta_3,$$

$$L_1(-\zeta_1) = \wp_{1\cdot 2}, \qquad L_1(-\zeta_3) = \wp_{1\cdot 1,3\cdot 1},$$

$$L_2(-\zeta_1) = \frac{1}{2}\wp_{1\cdot 3} - \zeta_3 + \frac{3}{5}\lambda_4 u_1, \qquad L_2(-\zeta_3) = \frac{1}{2}\wp_{1\cdot 2,3\cdot 1} + \frac{4}{5}\lambda_4\zeta_1 + \left(\frac{4}{5}\lambda_4^2 - 3\lambda_8\right)u_3,$$

$$L_3(-\zeta_1) = \wp_{1\cdot 1,3\cdot 1}, \qquad L_3(-\zeta_3) = \wp_{3\cdot 2},$$

$$L_4(-\zeta_1) = \wp_{1\cdot 2,3\cdot 1} + \frac{2}{5}\lambda_6 u_1 - \lambda_8 u_3, \qquad L_4(-\zeta_3) = \wp_{1\cdot 1,3\cdot 2} + \frac{6}{5}\lambda_6\zeta_1 - \lambda_4\zeta_3 - \lambda_8 u_1 + {} + 6\left(\frac{1}{5}\lambda_4\lambda_6 - \lambda_{10}\right) u_3,$$

$$L_6(-\zeta_1) = \frac{1}{2}\wp_{1\cdot 1,3\cdot 2} + \frac{1}{5}\lambda_8 u_1 - 2\lambda_{10} u_3, \; L_6(-\zeta_3) = \frac{1}{2}\wp_{3\cdot 3} + \frac{3}{5}\lambda_8\zeta_1 - 2\lambda_{10} u_1 + \frac{3}{5}\lambda_4\lambda_8 u_3.$$

Proof. For the operators L_1, L_3 this result follows from definitions. For the operators L_0, L_2, L_4 and L_6 this result follows from the proof of Theorem 3.14. □

The following theorem completes the description of the action of generators of the Lie $\mathscr{P}_2$-algebra $\mathscr{L}_2$ on the ring of polynomials $\mathscr{P}_2$.

Theorem 3.17 *The operators* $L_i, \; i = 0, 1, 2, 3, 4, 6$ *act on* $\wp_{1\cdot 2}$ *and* $\wp_{1\cdot 1,3\cdot 1}$ *according to the formulae*

$$L_0\wp_{1\cdot 2} = 2\wp_{1\cdot 2}, \qquad L_0\wp_{1\cdot 1,3\cdot 1} = 4\wp_{1\cdot 1,3\cdot 1},$$

$$L_1\wp_{1\cdot 2} = \wp_{1\cdot 3}, \qquad L_1\wp_{1\cdot 1,3\cdot 1} = \wp_{1\cdot 2,3\cdot 1},$$

$$L_2\wp_{1\cdot 2} = \frac{1}{2}\wp_{1\cdot 4} - \wp_{1\cdot 2}^2 + 2\wp_{1\cdot 1,3\cdot 1} + \frac{3}{5}\lambda_4,$$

$$L_2\wp_{1\cdot 1,3\cdot 1} = \frac{1}{2}\wp_{1\cdot 3,3\cdot 1} - \wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} - \frac{4}{5}\lambda_4\wp_{1\cdot 2} + \wp_{3\cdot 2},$$

$$L_4\wp_{1\cdot 2} = \wp_{1\cdot 3,3\cdot 1} - 2\wp_{1\cdot 2}\wp_{1\cdot 1,3\cdot 1} + \frac{2}{5}\lambda_6,$$

$$L_4\wp_{1\cdot 1,3\cdot 1} = \wp_{1\cdot 2,3\cdot 2} - \wp_{1\cdot 1,3\cdot 1}^2 - \wp_{1\cdot 2}\wp_{3\cdot 2} - \frac{6}{5}\lambda_6\wp_{1\cdot 2} + \lambda_4\wp_{1\cdot 1,3\cdot 1} - \lambda_8,$$

$$L_6\wp_{1\cdot 2} = \frac{1}{2}\wp_{1\cdot 2,3\cdot 2} - \wp_{1\cdot 1,3\cdot 1}^2 + \frac{1}{5}\lambda_8,$$

$$L_6\wp_{1\cdot 1,3\cdot 1} = \frac{1}{2}\wp_{1\cdot 1,3\cdot 3} - \wp_{1\cdot 1,3\cdot 1}\wp_{3\cdot 2} - \frac{3}{5}\lambda_8\wp_{1\cdot 2} - 2\lambda_{10}.$$

Note, that in these formulae the parameters $\lambda_{2k}, \; k = 4, \ldots, 10$, are considered as polynomials $\lambda_{2k}(W_\wp)$ (see 91–94).

Proof. See the derivation of the formulae for the operators $L_{2k}, \; k = 0, 2, 3, 4$, in the proof of Theorem 3.14. □

The following result is based on the formulae of Theorem 3.14.

Theorem 3.18 *The commutation relations in the Lie F_2-algebra* $\mathrm{Der}(F_2)$ *of derivations of the field F_2 have the form*

$$[L_0, L_k] = kL_k, \quad k = 1, 2, 3, 4, 6; \qquad [L_1, L_2] = \wp_{1\cdot 2}L_1 - L_3;$$

$$[L_1, L_3] = 0; \qquad [L_1, L_4] = \wp_{1\cdot 1,3\cdot 1}L_1 + \wp_{1\cdot 2}L_3;$$

$$[L_1, L_6] = \wp_{1\cdot 1,3\cdot 1}L_3; \qquad [L_3, L_2] = \left(\wp_{1\cdot 1,3\cdot 1} + \frac{4}{5}\lambda_4\right) L_1;$$

$$[L_3, L_4] = \left(\wp_{3\cdot 2} + \frac{6}{5}\lambda_6\right) L_1 + \left(\wp_{1\cdot 1,3\cdot 1} - \lambda_4\right) L_3; \qquad [L_3, L_6] = \frac{3}{5}\lambda_8 L_1 + \wp_{3\cdot 2}L_3;$$

$$[L_2, L_4] = \frac{8}{5}\lambda_6 L_0 - \frac{1}{2}\wp_{1\cdot 2,3\cdot 1}L_1 - \frac{8}{5}\lambda_4 L_2 + \frac{1}{2}\wp_{1\cdot 3}L_3 + 2L_6;$$

$$[L_2, L_6] = \frac{4}{5}\lambda_8 L_0 - \frac{1}{2}\wp_{1\cdot 1,3\cdot 2}L_1 + \frac{1}{2}\wp_{1\cdot 2,3\cdot 1}L_3 - \frac{4}{5}\lambda_4 L_4;$$

$$[L_4, L_6] = -2\lambda_{10}L_0 - \frac{1}{2}\wp_{3\cdot 3}L_1 + \frac{6}{5}\lambda_8 L_2 + \frac{1}{2}\wp_{1\cdot 1,3\cdot 2}L_3 - \frac{6}{5}\lambda_6 L_4 + 2\lambda_4 L_6.$$

Proof. Due to linearity, the relation $[L_0, L_k] = kL_k, \quad k = 1, 2, 3, 4, 6$, can be checked independently for every summand in the expression for L_k.

The expressions for $[L_m, L_n]$, where m or n is equal to 1 or 3, can be obtained by simple calculations using Theorem 3.14.

It remains to prove the commutation relations among L_2, L_4 and L_6. We express $[L_m, L_n]$, where $m < n$ and $m, n = 2, 4, 6$, in the form

$$[L_m, L_n] = a_{m,n,0}L_0 + a_{m,n,-1}L_1 + a_{m,n,-2}L_2 + a_{m,n,-3}L_3 + a_{m,n,-4}L_4 + a_{m,n,-6}L_6.$$

We have $\deg a_{i,j,-k} = i + j - k$. Applying both sides of this equation to λ_k and using the explicit expressions for L_k, we get

$$[\ell_m, \ell_n]\lambda_k = (a_{m,n,0}\ell_0 + a_{m,n,-2}\ell_2 + a_{m,n,-4}\ell_4 + a_{m,n,-6}\ell_6)\lambda_k.$$

This formula and Lemma 3.15 yield the values of the coefficients $a_{m,n,-k}$, $k = 0, 2, 4, 6$:

$$[L_2, L_4] = \frac{8}{5}\lambda_6 L_0 + a_{2,4,-1}L_1 - \frac{8}{5}\lambda_4 L_2 + a_{2,4,-3}L_3 + 2L_6; \tag{112}$$

$$[L_2, L_6] = \frac{4}{5}\lambda_8 L_0 + a_{2,6,-1}L_1 + a_{2,6,-3}L_3 - \frac{4}{5}\lambda_4 L_4; \tag{113}$$

$$[L_4, L_6] = -2\lambda_{10}L_0 + a_{4,6,-1}L_1 + \frac{6}{5}\lambda_8 L_2 + a_{4,6,-3}L_3 - \frac{6}{5}\lambda_6 L_4 + 2\lambda_4 L_6. \tag{114}$$

In subsequent calculations we compare the actions of the left- and right-hand sides of the expressions (112)–(114) on the coordinates u_1 and u_3. To this end we use the expressions (85), Theorem 3.14 and Lemma 3.16.

We present the calculation of the coefficient $a_{2,4,-1}$. The left-hand side of (112) gives

$$\begin{aligned}[L_2, L_4]u_1 &= L_2(-\zeta_3 + \frac{6}{5}\lambda_6 u_3) - L_4(-\zeta_1 + \frac{4}{5}\lambda_4 u_3) \\ &= L_2(-\zeta_3) + \frac{6}{5}\ell_2(\lambda_6)u_3 - \frac{6}{5}\lambda_6 u_1 - L_4(-\zeta_1) - \frac{4}{5}\ell_4(\lambda_4)u_3 \\ &\quad - \frac{4}{5}\lambda_4(-\zeta_1 - \lambda_4 u_3) \\ &= -\frac{1}{2}\wp_{1\cdot 2,3\cdot 1} + \frac{8}{5}\lambda_4\zeta_1 - \frac{8}{5}\lambda_6 u_1 + \frac{2}{5}\left(3\lambda_8 - \frac{16}{5}\lambda_4^2\right)u_3.\end{aligned}$$

The right-hand side of (112) gives

$$[L_2, L_4]u_1 = a_{2,4,-1} + \frac{8}{5}\lambda_4\zeta_1 - \frac{8}{5}\lambda_6 u_1 + \frac{2}{5}\left(3\lambda_8 - \frac{16}{5}\lambda_4^2\right)u_3.$$

By equating them, we obtain $a_{2,4,-1} = -\frac{1}{2}\wp_{1\cdot 2,3\cdot 1}$.

The coefficients $a_{2,4,-3}$, $a_{2,6,-1}$, $a_{2,6,-3}$, $a_{4,6,-1}$ and $a_{4,6,-3}$ are calculated in a similar way. $\square$

References

[1] T. Ayano and A. Nakayashiki, *On Addition Formulae for Sigma Functions of Telescopic Curves*, Symmetry, Integrability and Geometry: Methods and Applications (SIGMA) **9** (2013), 046, 14 pages.

[2] V. I. Arnold, *Singularities of Caustics and Wave Fronts*, Kluwer Academic Publishers Group, Dordrecht, 1990.

[3] H. F. Baker, *Abel's theorem and the allied theory of theta functions*, Cambridge University Press, Cambridge, (1897), Reprinted in 1995.

[4] ———, *On the hyperelliptic sigma functions*, Amer. Journ. Math. **20** (1898), 301–384.

[5] ———, *On a system of differential equations leading to periodic functions*, Acta Math. **27** (1903), 135–156.

[6] ———, *Multiply Periodic Functions*, Cambridge University Press, Cambridge, 1907.

[7] H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. 3, New York, Toronto, London McGraw-Hill Book Company, 1955.

[8] J. Bernatska and D. Leykin, *On degenerate sigma-functions in genus two*, Glasgow Math. J. **60** (2018), arXiv:1509.01490.

[9] J. Bernatska, V. Enolski and A. Nakayashiki, *Sato Grassmannian and Degenerate Sigma Function*, (2018), arXiv: 1810.01224 [nlin.SI].

[10] E. D. Belokolos and V. Z. Enol'skii, *Reduction of Theta Functions and Elliptic Finite-Gap Potentials*, Acta Appl. Math. **36** (1994), 87–117.

[11] H. W. Braden, V. Z. Enolski and Yu N. Fedorov, *Dynamics on strata of trigonal Jacobians and some integrable problems of rigid body motion*, Nonlinearity **26** (2013), 1865–1889.

[12] O. Bolza, *Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische insbesondere über die Reduction durch eine Transformation vierten Grades*, Inaugural-Dissertation; Georg-August Universit at zu Göttingen, 1885.

[13] O. Bolza, *Ueber die Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische durch eine Transformation vierten Grades*, Math. Ann. **XXVIII** (1886), 447.

[14] V. M. Buchstaber, V. Z. Enolskii and D. V. Leikin, *Hyperelliptic Kleinian functions and applications*, "Solitons, Geometry and Topology: On the Crossroad", Adv. Math. Sci., AMS Transl., 179:2, Providence, RI, 1997, 1–33.

[15] V. M. Buchstaber, V. Z. Enolskii and D. V. Leykin, *Kleinian functions, hyperelliptic Jacobians and applications*, Rev. Math. and Math. Phys., vol. 10:2, Gordon and Breach, 1997, 3–120.

[16] V. M. Bukhshtaber, D. V. Leikin and V. Z. Enol'skii, *Rational analogues of Abelian functions*, Funkts. Anal. Prilozhen. **33**, No. 2, 1–15 (1999).

[17] V. M. Buchstaber, V. Z. Enolskii and D. V. Leikin, *Multi-Dimensional Sigma-Functions*, arXiv: 1208.0990, 2012, 267 pp.

[18] V. M. Buchstaber and D. V. Leykin, *Polynomial Lie algebras*, Functional Anal. Appl. **36**:4 (2002), 267–280.

[19] V. M. Buchstaber and D. V. Leykin, *Heat Equations in a Nonholonomic Frame*, Funct. Anal. Appl. **38**:2 (2004), 88–101.

[20] V. M. Buchstaber and D. V. Leikin, *Addition Laws on Jacobian Varieties of Plane Algebraic Curves*, Proc. Steklov Inst. Math. **251** (2005), 49–120.

[21] V. M. Buchstaber and D. V. Leikin, *Differentiation of Abelian functions with respect to parameters*, Russian Math. Surveys **62**:4 (2007), 787–789.

[22] V. M. Buchstaber and D. V. Leikin, *Solution of the Problem of Differentiation of Abelian Functions over Parameters for Families of (n, s)-Curves*, Funct. Anal. Appl. **42**:4 (2008), 268–278.

[23] V. M. Buchstaber, *Polynomial dynamical systems and the Kortewegde Vries equation*, Proc. Steklov Inst. Math. **294** (2016), 176–200.

[24] V. M. Buchstaber and A. V. Mikhailov, *The space of symmetric squares of hyperelliptic curves and integrable Hamiltonian polynomial systems on R^4*, arXiv: 1710.00866 v1 [nlin.SI] 2 Oct 2017, 24 pages.

[25] V. M. Buchstaber and A. V. Mikhailov, *Polynomial Hamiltonian integrable systems on symmetric powers of plane curves*, Russian Math. Surveys **73**:6(439) (2018), 1122–1124.

[26] E. Yu. Bunkova, *Differentiation of genus 3 hyperelliptic functions.*, European Journal of Mathematics, **4**:1 (2018), 93-112; arXiv: 1703.03947.

[27] B. A. Dubrovin and S. P. Novikov, *A periodic problem for the Korteweg-de Vries and Sturm-Liouville equations. Their connection with algebraic geometry.*, Dokl. Akad. Nauk SSSR **219**:3 (1974), 531–534.

[28] B. A. Dubrovin, *Theta functions and nonlinear equations*, Russ. Math. Surveys **36** (1981), 11–80.

[29] B. A. Dubrovin, *Geometry of 2D Topological Field Theories*, in Integrable Systems and Quantum Groups (Springer, Berlin, 1996), Lect. Notes in Math. v. 1620, 120–348; `arXiv:hep-th/9407018`.

[30] J. C. Eilbeck, K. Eilers and V. Z. Enolski, *Periods of second kind differentials of (n, s)-curves*, Trans. Moscow Math. Soc. **74**:2 (2013), 245–260.

[31] V. Z. Enolski and Yu. N. Fedorov, *Algebraic description of Jacobians isogeneous to certain Prym varieties with polarization (1,2)*, Experimental Mathematics **25**, 1–32, 2016, Preprint : arXiv: 1120.744 [nlin.SI], [math.AG] 22 Nov. 2014. 52 pp.
[32] V. Enolski, B. Hartmann Betti, V. Kagramanova, J.Kunz Jutta, C. Lämmerzahl and P. Sirimachan, *Inversion of a general hyperelliptic integral and particle motion in Hořava-Lifshitz black hole space-times*, J.Math.Phys. **53** (2012), 012504; arXiv:1106.2408v1 [gr-qc].
[33] K. Eilers, *Modular form representation for periods of hyperelliptic integrals*, SIGMA **12** (2016), 060, 13 pages.
[34] K. Eilers, *Rosenhain-Thomae formulae for higher genera hyperelliptic curves*, Journ. Nonlin. Math. Phys. **25** (2018), 85–105.
[35] J. C. Eilbeck, V. Z. Enolski, S. Matsutani, Y. Ônishi and E. Previato, *Abelian functions for trigonal curves of genus of genus three*, Int. Math. Res. Not. (IMRN) **2007** (2007): rnm 140–168.
[36] A. Eisenmann and H. M. Farkas, *An elementary proof of Thomaes formulae*, Online J. Anal. Comb., (3) Issue 3, (2008) 14p.
[37] V. Z. Enolski and T. Grava, *Thomae type formulae for singular Z_N curves*, Letters in Mathematical Physics (LMP) **76**:2 (2006), 187–214.
[38] V. Enolskii and P. Richter, *Periods of hyperelliptic integrals expressed in terms of θ-constants by means of Thomae formulae*, Phil. Trans. R. Soc. A **366** (2008), 1005–1024.
[39] B. Eynard, *Notes about a combinatorial expression of the fundamental second kind differential on an algebraic curve*, arXiv: 1805.07247 [math-ph], 2018.
[40] H. Farkas, S. Grushevsky and R. Salvati Manni. *An explicit solution to the weak Schottky problem*, https://arxiv.org/abs/1710.02938, 2017.
[41] J. D. Fay, *Theta functions on Riemann surfaces*, Lectures Notes in Mathematics (Berlin), vol. 352, Springer, 1973.
[42] J. Fay, *On the Riemann-Jacobi formula*, Nachr. Akad. Wiss. Gottingen Math.-Phys. Kl. II 1979, no. 5, 61–73.
[43] G. Frobenius, *Uber die constanten Factoren der Thetareihen*, J. Reine Angew. Math.**98** (1885), 244–265.
[44] F. G. Frobenius and L. Stickelberger, *Über die Differentiation der elliptischen Functionen nach den Perioden und Invarianten*, J. Reine Angew. Math. **92** (1882), 311–337.
[45] J.-I. Igusa, *On Jacobis derivative formula and its generalizations*, Amer. J. Math. **102** (1980), 409–446.
[46] A. R. Its and V. B. Matveev, *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg–de Vries equation*, TMF **23**(1) (1975), 51-68; Theoret. and Math. Phys. **23**:1 (1975), 343–355.
[47] P. G. Grinevich and A. Yu. Orlov, *Virasoro action on Riemann surfaces, Grassmanians,* det $\bar\partial$ *and Segal–Wilson τ-function*, in Problem of modern quantum field theory, Springer-Verlag, 1989, 86–106.
[48] F. Klein, *Ueber hyperelliptische Sigmafunctionen*, Math. Ann. **27** (1886), 431–464.
[49] F. Klein, *Ueber hyperelliptische Sigmafunctionen - Zweite Abhandlung*, Math. Ann. **32** (1888), 351–380.

[50] D. Korotkin and V. Shramchenko, *On higher genus Weierstrass sigma-functions*, Physica D (2012).

[51] I. M. Krichever, *Methods of algebraic geometry in the theory of nonlinear equations*, Russ. Math. Surv. **32** (1977), 185–213.

[52] V. Krishnamoorthy, T. Shaska and H. Vlklein, *Invariants of Binary Forms*, In Progress in Galois Theory, pp. 101–122, Dev. Math., 12, New York: Springer, 2005.

[53] S. Matsutani and E. Previato, *Jacobi inversion on strata of the Jacobian of the C_{rs} curve $y^r = f(x)$, I,II*, J. Math. Soc. Japan **60**, No. 4 (2008), 1009–1044. **66**, No. 2 (2014), 647–692.

[54] A. Nakayashiki, *Algebraic Expression of Sigma Functions of (n, s) Curves*, Asian J.Math. **14**:2 (2010), 174–211; arXiv:0803.2083, 2008.

[55] A. Nakayashiki and K. Yori, *Derivatives of Schur, tau and sigma functions, on Abel-Jacobi images*, In Symmetries, Integrable Systems and Representations, K.Iohara, S. Morier-Genoud, B. Remy (eds.), Springer, 2012, 429–462.

[56] A. Nakayashiki, *Tau function approach to theta functions*, Int. Math. Res. Not. (IMRN) **2016-17** (2016), 5202–5248.

[57] A. Nakayashiki, *Degeneration of trigonal curves and solutions of the KP-hierarchy*, Nonlinearity **31** (2018), 3567–3590.

[58] E. Nart and C. Ritzenthaler, *A new proof of a Thomae-like formula for non hyperelliptic genus 3 curves*, In Arithmetic, geometry, cryptography and coding theory, volume 686 of Contemporary Mathematics, pages 137–155. American Mathematical Society, 2017

[59] Y. Ônishi, *Determinant expressions for hyperelliptic functions, with an Appendix by Shigeki Matsutani: Connection of the formula of Cantor and Brioschi-Kiepert type*, Proc. Edinburgh Math. Soc. **48** (2005), 705–742.

[60] G. Rosenhain, *Abhandlung über die Functionen zweier Variablen mit fier Perioden welche die Inversion sind der ultra-elliptische Integrale erster Klasse*, Translation to German from Latin manuscript published in 1851. Ostwald Klassiker der Exacten Wissenschaften, Nr. **65**, pp 1–96, Leipzig, 1895, Verlag von Wilhelm Engelmann.

[61] J. Suzuki, *Kleins Fundamental 2-Form of Second Kind for the C_{ab} Curves*, SIGMA **13** (2017), 017, 13 pages.

[62] M. Schiffer and D. C Spencer, *Functionals of Finite Riemann Surfaces*, Paperback May 21, 2014; originally published as v.16 of the Princeton Mathematical Series, by Princeton University Press, Princeton, New Jersey, 1954.

[63] J. Thomae, *Beitrag zur Bestimmung (0, …, 0) durch die Klassenmoduln algebraischer Funktionen*, Journ. reine angew. Math. **71** (1870), 201–222.

[64] H. Weber, *Anwendung der Thetafunctionen sweier Veräderlicher auf die Theorie der Bewegung eines festen Körpers in einer Flüssigkeit*, Math. Ann. **14** (1879), 173–206.

[65] K. Weierstrass, *Formeln und Lehrsätze der elliptiechen Functionen bearbeited und herausgegeben von H.A.Schwarz*, Göttingen, 1885.

[66] K. Weierstraß, *Zur Theorie der elliptischen Funktionen*, Mathematische Werke, Vol. 2, Teubner, Berlin, 1894, 245–255.

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