



PAPER

OPEN ACCESS

RECEIVED
13 June 2022REVISED
25 July 2022ACCEPTED FOR PUBLICATION
28 July 2022PUBLISHED
5 August 2022

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The half-period addition formulae for genus two hyperelliptic \wp functions and the $\mathrm{Sp}(4, \mathbb{R})$ Lie group structure

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E-mail: masahito.hayashi@oit.ac.jp, shigemoto@tezukayama-u.ac.jp and tsukioka@bukkyo-u.ac.jp**Keywords:** genus two, half-period addition formula, hyperelliptic \wp functions, Lie group structure

Abstract

In the previous study, by using the two-flows Kowalevski top, we have demonstrated that the genus two hyperelliptic functions provide the $\mathrm{Sp}(4, \mathbb{R})/\mathbb{Z}_2 \cong \mathrm{SO}(3, 2)$ Lie algebra structure. In this study, by directly using the differential equations of the genus two hyperelliptic \wp functions instead of using integrable models, we demonstrate that the half-period addition formula for the genus two hyperelliptic functions provides the order two $\mathrm{Sp}(4, \mathbb{R})$ Lie group structure.

1. Introduction

We are interested in the mechanism why there are exact solutions, and further a series of infinitely many solutions in some cases, for some special non-linear differential equations. Soliton equations are the examples of such equations, hence the various methods for studying the soliton systems are beneficial for our objective. Starting from the inverse scattering method [1–3], the soliton theory has many interesting developments, such as the AKNS formulation [4], geometrical approach [5–8], Bäcklund transformation [9–11], Hirota equation [12, 13], Sato theory [14], vertex construction of the soliton solution [15–17], and Schwarzian type mKdV/KdV equation [18].

We expect there is a Lie group structure behind some non-linear differential equation, which is the reason why such non-linear differential equation has a of infinite solutions. Owing to the addition formula of the Lie group structure, there is a series of infinitely many solutions. As the representation of the addition formula of the Lie group, the algebraic functions such as trigonometric/elliptic/hyperelliptic functions will emerge for solutions of special partial differential equations.

The AKNS formalism for the Lax pair is a powerful tool to examine the Lie algebra structure of the soliton equations of the non-linear integrable models. In our previous researches, we deduced the $\mathrm{SL}(2, \mathbb{R})/\mathbb{Z}_2 \cong \mathrm{SO}(2, 1) \cong \mathrm{SU}(1, 1)/\mathbb{Z}_2 \cong \mathrm{Sp}(2, \mathbb{R})/\mathbb{Z}_2$ Lie algebra structure for the two-dimensional KdV/mKdV/sinh-Gordon models; in addition, the $\mathrm{SO}(3, 2) \cong \mathrm{Sp}(4, \mathbb{R})/\mathbb{Z}_2$ Lie algebra structure for the two-flows (two-dimensional) Kowalevski model [19–25].

In this study, in order to examine the Lie group structure, instead of the Lie algebra structure, we use algebraic functions such as the elliptic functions and the genus two hyperelliptic functions directly rather than integrable models indirectly. Especially, by using the half-period addition formula, we have deduced that there is the order two $\mathrm{Sp}(4, \mathbb{R})$ Lie group structure for the genus two hyperelliptic \wp functions.

The paper is organized as follows: In section 2, we demonstrate that the elliptic functions have $\mathrm{SO}(3)$ Lie group structure via the algebraic addition formula. In section 3, we briefly review the genus two Jacobi's inversion problem to explicitly present the genus two hyperelliptic \wp function. Then we review the addition formula of the genus two sigma function, which is used in the next section. In section 4, we first review that the half-period addition formula of the \wp function gives the order two $\mathrm{Sp}(2, \mathbb{R})$ Lie group structure. Next, we

demonstrate that the half-period addition formulae of the genus two hyperelliptic \wp functions give the order two $\mathrm{Sp}(4, \mathbb{R})$ Lie group structure. We devote the final section to the summary and the discussions.

2. The various addition formulae for the elliptic functions

We investigate various types of addition formulae for the elliptic functions, classified into analytic, algebro-geometric, and algebraic ones. For the addition formula which includes derivative terms, we define the analytic addition formula.

2.1. The various addition formulae for the Weierstrass type and Jacobi type elliptic functions

We first examine the $\mathrm{SO}(3)/\mathrm{SO}(2,1)$ Lie group structure for the elliptic functions.

The Weierstrass' \wp function satisfies the differential equation [26]

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3 = 4(\wp(u) - e_1)(\wp(u) - e_2)(\wp(u) - e_3).$$

An analytic addition formula of the Weierstrass \wp function is given by

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \frac{1}{4} \left(\frac{\wp'(u_2) - \wp'(u_1)}{\wp(u_2) - \wp(u_1)} \right)^2. \quad (2.1)$$

While, an algebro-geometric addition formula is given by

$$x_3 = -x_1 - x_2 + \frac{1}{4} \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2, \quad (2.2)$$

$$y_3 = - \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x_3 - x_1) - y_1, \quad (2.3)$$

$$y_i^2 = 4x_i^3 - g_2x_i - g_3, \quad (i = 1, 2, 3), \quad (2.4)$$

which constitutes the Mordell-Weil group in number theory. In addition, there is an algebraic addition formula, which will be discussed in the subsequent section.

The Jacobi's sn function satisfies the differential equation [26]

$$(\mathrm{sn}'(u))^2 = (1 - \mathrm{sn}^2(u))(1 - k^2 \mathrm{sn}^2(u)),$$

with $\mathrm{sn}(0) = 0$. Using the sn function, we define cn and dn functions via

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1 \quad \text{and} \quad k^2 \mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1,$$

with $\mathrm{cn}(0) = \mathrm{dn}(0) = 1$. An analytic addition formula of the Jacobi's elliptic function is given by

$$\mathrm{sn}(u_1 + u_2) = \frac{\mathrm{sn}(u_1)\mathrm{sn}'(u_2) + \mathrm{sn}'(u_1)\mathrm{sn}(u_2)}{1 - k^2 \mathrm{sn}^2(u_1)\mathrm{sn}^2(u_2)}. \quad (2.5)$$

While, by using the Abel's addition theorem, the algebro-geometric addition formula is given by⁴

$$x_3 = - \frac{x_1 y_2 + x_2 y_1}{1 - k^2 x_1^2 x_2^2}, \quad (2.6)$$

$$y_3 = \frac{y_1 y_2 (1 + k^2 x_1^2 x_2^2) - (1 - k^2) x_1 x_2 (1 - k^2 x_1^2 x_2^2) - 2k^2 x_1 x_2 (1 - x_1^2)(1 - x_2^2)}{(1 - k^2 x_1^2 x_2^2)^2}, \quad (2.7)$$

$$y_i^2 = (1 - x_i^2)(1 - k^2 x_i^2), \quad (i = 1, 2, 3). \quad (2.8)$$

In addition, algebraic addition formulae are given by

$$\mathrm{sn}(u_1 + u_2) = \frac{\mathrm{sn}(u_1)\mathrm{cn}(u_2)\mathrm{dn}(u_2) + \mathrm{sn}(u_2)\mathrm{cn}(u_1)\mathrm{dn}(u_1)}{1 - k^2 \mathrm{sn}^2(u_1)\mathrm{sn}^2(u_2)}, \quad (2.9)$$

$$\mathrm{cn}(u_1 + u_2) = \frac{\mathrm{cn}(u_1)\mathrm{cn}(u_2) - \mathrm{sn}(u_1)\mathrm{dn}(u_1)\mathrm{sn}(u_2)\mathrm{dn}(u_2)}{1 - k^2 \mathrm{sn}^2(u_1)\mathrm{sn}^2(u_2)}, \quad (2.10)$$

$$\mathrm{dn}(u_1 + u_2) = \frac{\mathrm{dn}(u_1)\mathrm{dn}(u_2) - k^2 \mathrm{sn}(u_1)\mathrm{cn}(u_1)\mathrm{sn}(u_2)\mathrm{cn}(u_2)}{1 - k^2 \mathrm{sn}^2(u_1)\mathrm{sn}^2(u_2)}. \quad (2.11)$$

These algebraic addition formulae can be rearranged in the relation of the $\mathrm{SO}(3)$ Lie group elements of the form [28, 29]

⁴ We must notice that the last term of the numerator in the right-hand side of equation (2.7) is missing in the Baker's textbook [27], p.208.

$$U(u_1)V(u_3)U(u_2) = V(u_2)U(u_3)V(u_1), \quad (u_3 = u_1 + u_2), \quad (2.12)$$

with

$$U(u) = \exp[i \operatorname{am}(u, k) J_3], \quad V(u) = \exp[i \operatorname{am}(ku, 1/k) J_1], \quad (2.13)$$

where $\operatorname{am}(u, k)$ is the Weierstrass' am function defined by $\operatorname{sn}(u, k) = \sin(\operatorname{am}(u, k))$ and $\operatorname{cn}(u, k) = \cos(\operatorname{am}(u, k))$, and J_1 and J_3 imply spin representations of $\mathrm{SO}(3)$. More explicitly, by using the spin 1 representation of J_1 and J_3 ,

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

we can express the algebraic addition formula in the relation of the Lie group elements $U(u)$ and $V(u)$ of the form

$$\begin{pmatrix} \operatorname{cn}(u_1) & \operatorname{sn}(u_1) & 0 \\ -\operatorname{sn}(u_1) & \operatorname{cn}(u_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{dn}(u_3) & k \operatorname{sn}(u_3) \\ 0 & -k \operatorname{sn}(u_3) & \operatorname{dn}(u_3) \end{pmatrix} \begin{pmatrix} \operatorname{cn}(u_2) & \operatorname{sn}(u_2) & 0 \\ -\operatorname{sn}(u_2) & \operatorname{cn}(u_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{dn}(u_2) & k \operatorname{sn}(u_2) \\ 0 & -k \operatorname{sn}(u_2) & \operatorname{dn}(u_2) \end{pmatrix} \begin{pmatrix} \operatorname{cn}(u_3) & \operatorname{sn}(u_3) & 0 \\ -\operatorname{sn}(u_3) & \operatorname{cn}(u_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{dn}(u_1) & k \operatorname{sn}(u_1) \\ 0 & -k \operatorname{sn}(u_1) & \operatorname{dn}(u_1) \end{pmatrix}, \quad (2.14)$$

with $u_3 = u_1 + u_2$. We have used the relations $\operatorname{sn}(ku, 1/k) = k \operatorname{sn}(u, k)$, $\operatorname{cn}(ku, 1/k) = \operatorname{dn}(u, k)$, and $\operatorname{dn}(ku, 1/k) = \operatorname{cn}(u, k)$. This is the integrability condition, called the Yang-Baxter relation, in the two-dimensional integrable statistical model.

The elliptic function is formulated with complex numbers, hence we cannot distinguish between the $\mathrm{SO}(3)$ Lie group structure and the $\mathrm{SO}(2,1)$ Lie group structure, because we can 'analytically continue' from one to another Lie group structure. For the soliton model, the soliton solution is assumed to be the real number, hence the Lie group structure is fixed to be $\mathrm{SO}(2,1)$.

2.2. Algebraic addition formulae for the Weierstrass' \wp function

In order to obtain an algebraic addition formula for the \wp function, we use relations between the \wp function and the Jacobi's elliptic functions in the form [26]

$$\frac{1}{\operatorname{sn}^2(u)} = \frac{\wp(z) - e_3}{e_1 - e_3}, \quad \frac{\operatorname{dn}^2(u)}{\operatorname{sn}^2(u)} = \frac{\wp(z) - e_2}{e_1 - e_3}, \quad \frac{\operatorname{cn}^2(u)}{\operatorname{sn}^2(u)} = \frac{\wp(z) - e_1}{e_1 - e_3}, \quad (2.15)$$

with $z = u/\sqrt{e_1 - e_3}$. Noticing that

$$\operatorname{sn}(u + 3iK') = \frac{1}{k \operatorname{sn}(u)}, \quad \text{with} \quad K' = i \int_{1/k}^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad (2.16)$$

$\operatorname{sn}(u)$ and $1/(k \operatorname{sn}(u))$ satisfies the same differential equation, thus they have similar properties. Similarly, we have

$$\operatorname{cn}(u + 3iK') = \frac{i \operatorname{dn}(u)}{k \operatorname{sn}(u)}, \quad \operatorname{dn}(u + 3iK') = \frac{i \operatorname{cn}(u)}{\operatorname{sn}(u)}. \quad (2.17)$$

Accordingly, we define $\widehat{\operatorname{sn}}(u)$, $\widehat{\operatorname{cn}}(u)$ and $\widehat{\operatorname{dn}}(u)$ functions as

$$\widehat{\operatorname{sn}}(u) = \frac{1}{k \operatorname{sn}(u)} = \frac{\sqrt{\wp(z) - e_3}}{k\sqrt{e_1 - e_3}}, \quad (2.18)$$

$$\widehat{\operatorname{cn}}(u) = \frac{i \operatorname{dn}(u)}{k \operatorname{sn}(u)} = \frac{i\sqrt{\wp(z) - e_2}}{k\sqrt{e_1 - e_3}}, \quad (2.19)$$

$$\widehat{\operatorname{dn}}(u) = \frac{i \operatorname{cn}(u)}{\operatorname{sn}(u)} = \frac{i\sqrt{\wp(z) - e_1}}{\sqrt{e_1 - e_3}}. \quad (2.20)$$

They satisfy the relations $\widehat{\operatorname{sn}}^2(u) + \widehat{\operatorname{cn}}^2(u) = 1$ and $k^2 \widehat{\operatorname{sn}}^2(u) + \widehat{\operatorname{dn}}^2(u) = 1$.

By using the addition formulae of the Jacobi's elliptic functions, we obtain those of the $\widehat{\operatorname{sn}}(u)$, $\widehat{\operatorname{cn}}(u)$ and $\widehat{\operatorname{dn}}(u)$ functions as follows:

$$\frac{1}{k \widehat{\operatorname{sn}}(u_1 + u_2)} = \frac{\widehat{\operatorname{sn}}(u_1)\widehat{\operatorname{cn}}(u_2)\widehat{\operatorname{dn}}(u_2) + \widehat{\operatorname{sn}}(u_2)\widehat{\operatorname{cn}}(u_1)\widehat{\operatorname{dn}}(u_1)}{1 - k^2 \widehat{\operatorname{sn}}^2(u_1)\widehat{\operatorname{sn}}^2(u_2)}, \quad (2.21)$$

$$\frac{i \widehat{\operatorname{dn}}(u_1 + u_2)}{k \widehat{\operatorname{sn}}(u_1 + u_2)} = \frac{\widehat{\operatorname{cn}}(u_1)\widehat{\operatorname{cn}}(u_2) - \widehat{\operatorname{sn}}(u_1)\widehat{\operatorname{dn}}(u_1)\widehat{\operatorname{sn}}(u_2)\widehat{\operatorname{dn}}(u_2)}{1 - k^2 \widehat{\operatorname{sn}}^2(u_1)\widehat{\operatorname{sn}}^2(u_2)}, \quad (2.22)$$

$$\frac{i \widehat{\text{cn}}(u_1 + u_2)}{\widehat{\text{sn}}(u_1 + u_2)} = \frac{\widehat{\text{dn}}(u_1)\widehat{\text{dn}}(u_2) - k^2 \widehat{\text{sn}}(u_1)\widehat{\text{cn}}(u_1)\widehat{\text{sn}}(u_2)\widehat{\text{cn}}(u_2)}{1 - k^2 \widehat{\text{sn}}^2(u_1)\widehat{\text{sn}}^2(u_2)}. \quad (2.23)$$

Equations (2.21)–(2.23) imply the addition formula of the $\wp(u)$ function via equations (2.18)–(2.20). By using equations (2.16)–(2.17) with the same K' in (2.16), we can prove that $\widehat{\text{sn}}$, $\widehat{\text{cn}}$ and $\widehat{\text{dn}}$ have the same property, hence equations (2.21)–(2.23) are expressed in the convenient forms

$$\widehat{\text{sn}}(u_1 + u_2 + 3iK') = \frac{\widehat{\text{sn}}(u_1)\widehat{\text{cn}}(u_2)\widehat{\text{dn}}(u_2) + \widehat{\text{sn}}(u_2)\widehat{\text{cn}}(u_1)\widehat{\text{dn}}(u_1)}{1 - k^2 \widehat{\text{sn}}^2(u_1)\widehat{\text{sn}}^2(u_2)}, \quad (2.24)$$

$$\widehat{\text{cn}}(u_1 + u_2 + 3iK') = \frac{\widehat{\text{cn}}(u_1)\widehat{\text{cn}}(u_2) - \widehat{\text{sn}}(u_1)\widehat{\text{dn}}(u_1)\widehat{\text{sn}}(u_2)\widehat{\text{dn}}(u_2)}{1 - k^2 \widehat{\text{sn}}^2(u_1)\widehat{\text{sn}}^2(u_2)}, \quad (2.25)$$

$$\widehat{\text{dn}}(u_1 + u_2 + 3iK') = \frac{\widehat{\text{dn}}(u_1)\widehat{\text{dn}}(u_2) - k^2 \widehat{\text{sn}}(u_1)\widehat{\text{cn}}(u_1)\widehat{\text{sn}}(u_2)\widehat{\text{cn}}(u_2)}{1 - k^2 \widehat{\text{sn}}^2(u_1)\widehat{\text{sn}}^2(u_2)}. \quad (2.26)$$

We can express equations (2.24)–(2.26) in the relation of the Lie group elements of the form

$$\begin{aligned} & \begin{pmatrix} \widehat{\text{cn}}(u_1) & \widehat{\text{sn}}(u_1) & 0 \\ -\widehat{\text{sn}}(u_1) & \widehat{\text{cn}}(u_1) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{\text{dn}}(u_3) & k \widehat{\text{sn}}(u_3) \\ 0 & -k \widehat{\text{sn}}(u_3) & \widehat{\text{dn}}(u_3) \end{pmatrix} \begin{pmatrix} \widehat{\text{cn}}(u_2) & \widehat{\text{sn}}(u_2) & 0 \\ -\widehat{\text{sn}}(u_2) & \widehat{\text{cn}}(u_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{\text{dn}}(u_2) & k \widehat{\text{sn}}(u_2) \\ 0 & -k \widehat{\text{sn}}(u_2) & \widehat{\text{dn}}(u_2) \end{pmatrix} \begin{pmatrix} \widehat{\text{cn}}(u_3) & \widehat{\text{sn}}(u_3) & 0 \\ -\widehat{\text{sn}}(u_3) & \widehat{\text{cn}}(u_3) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widehat{\text{dn}}(u_1) & k \widehat{\text{sn}}(u_1) \\ 0 & -k \widehat{\text{sn}}(u_1) & \widehat{\text{dn}}(u_1) \end{pmatrix}, \end{aligned} \quad (2.27)$$

with $u_3 = u_1 + u_2 + 3iK'$.

3. The Rosenhain's solution for the genus two Jacobi's inversion problem

The Weierstrass-Klein type approach to the Jacobi's inversion problem is quite useful to observe the whole structure of the Jacobi's inversion problem. However, it is difficult to obtain explicit expressions of the sigma function for higher genus hyperelliptic \wp functions.

3.1. The Jacobi's inversion problem for the elliptic function

It is instructive to examine the Jacobi's inversion problem for the elliptic function in order to observe the genus two Jacobi's inversion problem.

We adopt the elliptic curve

$$y^2 = 4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3)$$

and consider the problem of finding the function $u = u(x)$

$$u = \int_{\infty}^x \frac{dx}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}. \quad (3.1)$$

Then, Jacobi's inversion problem of obtaining the function $x = x(u)$ is solved by introducing the theta function in such a way as expressing x as a function of the ratio of the theta functions [26], i.e.,

$$x = x(u) = \wp(u) = e_3 + (e_1 - e_3) \left(\frac{\vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix}(0) \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}(u/2\omega_1)}{\vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}(0) \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix}(u/2\omega_1)} \right)^2, \quad (3.2)$$

with

$$\omega_1 = \int_{\infty}^{e_1} \frac{dx}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}. \quad (3.3)$$

We must notice that the $\wp(u)$ is the quadratic function of the ratio of the theta functions instead of the linear function. Furthermore, the argument of the theta function becomes $u/2\omega_1$ instead of the simple u . By introducing the sigma function as the potential of the \wp function in the form $\sigma(u) = e^{\eta_1 u^2/2\omega_1} \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix}(u/2\omega_1)$, we can simply express \wp function in the form

$$\wp(u) = -\frac{d^2}{du^2} \log \sigma(u) = -\frac{\eta_1}{\omega_1} - \frac{d^2}{du^2} \log \vartheta \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right] (u/2\omega_1), \quad (3.4)$$

where $\eta_1 = \zeta(\omega_1)$. The role of the factor $e^{\eta_1 u^2/2\omega_1}$ is to shift the constant value of the \wp function in such a way as $\wp(u)$ has no constant term in the Laurent expansion around $u \approx 0$ in the form

$\wp(u) = 1/u^2 + g_2/20 u^2 + g_3/28 u^4 + \dots$, which is equivalent to set $\lambda_2 = 0$ in the elliptic curve of the form $y^2 = 4x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$.

3.2. The genus two Jacobi's inversion problem

The genus two hyperelliptic \wp functions were given by Göpel [30, 31] and independently by Rosenhain [32, 33] via the solution of the Jacobi's inversion problem. However, they are too complicated to derive the addition formula of the sigma function; which is used in the next section. Nowadays, Göpel and Rosenhain's results are little known. Hence, we sketch the Rosenhain's solution for the genus two Jacobi's inversion problem [34], which provides the explicit expressions of $\wp_{22}(u_1, u_2)$ and $\wp_{12}(u_1, u_2)$ by the theta functions. For the genus two case, we adopt Jacobi's standard form of the hyperelliptic curve in the form

$y^2 = x(1-x)(1-k_0^2 x)(1-k_1^2 x)(1-k_2^2 x) = f_5(x)$. By using three theta function identities, we can consistently parametrize as

$$\left(\frac{\vartheta \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix} \right] (u_1, u_2)}{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right] (u_1, u_2)} \right)^2 = k_0 k_1 k_2 x_1 x_2, \quad (3.5)$$

$$\left(\frac{\vartheta \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right] (u_1, u_2)}{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right] (u_1, u_2)} \right)^2 = -\frac{k_0 k_1 k_2}{k'_0 k'_1 k'_2} (1-x_1)(1-x_2), \quad (3.6)$$

$$\left(\frac{\vartheta \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix} \right] (u_1, u_2)}{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right] (u_1, u_2)} \right)^2 = -\frac{k_1 k_2}{k'_0 k_{01} k_{02}} (1-k_0^2 x_1)(1-k_0^2 x_2), \quad (3.7)$$

$$\left(\frac{\vartheta \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right] (u_1, u_2)}{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right] (u_1, u_2)} \right)^2 = \frac{k_0 k_2}{k'_1 k_{01} k_{12}} (1-k_1^2 x_1)(1-k_1^2 x_2), \quad (3.8)$$

$$\left(\frac{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right] (u_1, u_2)}{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right] (u_1, u_2)} \right)^2 = \frac{k_0 k_1}{k'_2 k_{01} k_{12}} (1-k_2^2 x_1)(1-k_2^2 x_2), \quad (3.9)$$

with $k'_0 = \sqrt{1-k_0^2}$, $k'_1 = \sqrt{1-k_1^2}$, $k'_2 = \sqrt{1-k_2^2}$, $k_{01} = \sqrt{k_0^2 - k_1^2}$, $k_{02} = \sqrt{k_0^2 - k_2^2}$, and $k_{12} = \sqrt{k_1^2 - k_2^2}$. Combining any two of these five relations, we obtain ten different expressions for $x_1 + x_2$ and $-x_1 x_2$. The other ten independent ratios of the theta functions are expressed by the symmetric function of x_1, x_2 in such the form as

$$\left(\frac{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right] (u_1, u_2)}{\vartheta \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right] (u_1, u_2)} \right)^2 = -\frac{F_{01}(x_1)F_{01}(x_2)}{k'_0 k'_1 k'_2 (x_1 - x_2)^2} \left(\frac{\sqrt{f_5(x_1)}}{F_{01}(x_1)} - \frac{\sqrt{f_5(x_2)}}{F_{01}(x_2)} \right)^2, \quad (3.10)$$

with $F_{01}(x) = x(1-x)$.

Next, we differentiate equations (3.5) and (3.6) and express the result with the theta functions by using the addition formulae of the theta functions. In the expression of that addition formulae, other ratios of the theta functions than those of equations (3.5)–(3.9), i.e., equation (3.10) etc come out. Hence, the function $f_5(x)$ naturally emerges in the Jacobi's inversion problem. In order to obtain the standard Jacobi's inversion problem, we can deduce the following equations from equations (3.5) and (3.6) by denoting $U_1 = \xi_1 u_1 + \xi_2 u_2$, $U_2 = \xi_3 u_1 + \xi_4 u_2$,

$$dU_1 = \xi_1 du_1 + \xi_2 du_2 = \frac{dx_1}{\sqrt{f_5(x_1)}} + \frac{dx_2}{\sqrt{f_5(x_2)}}, \quad (3.11)$$

$$dU_2 = \xi_3 du_1 + \xi_4 du_2 = \frac{x_1 dx_1}{\sqrt{f_5(x_1)}} + \frac{x_2 dx_2}{\sqrt{f_5(x_2)}}, \quad (3.12)$$

where ξ_i ($i = 1, 2, 3, 4$) are given by values of the various theta functions and their derivatives at $u_1 = u_2 = 0$, which take the rather complicated expressions. Then u_1 and u_2 are expressed as

$$u_1 = \eta_1 U_1 + \eta_2 U_2, \quad u_2 = \eta_3 U_1 + \eta_4 U_2, \quad (3.13)$$

with

$$\begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix}^{-1} = \frac{1}{\xi_1 \xi_4 - \xi_2 \xi_3} \begin{pmatrix} \xi_4 & -\xi_2 \\ -\xi_3 & \xi_1 \end{pmatrix}$$

By using equations (3.5) and (3.6), we obtain

$$\wp_{22}(U_1, U_2) = x_1 + x_2 = 1 + \frac{1}{k_0 k_1 k_2} \left(\frac{\vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2)} \right)^2 + \frac{k'_0 k'_1 k'_2}{k_0 k_1 k_2} \left(\frac{\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (u_1, u_2)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2)} \right)^2, \quad (3.14)$$

$$\wp_{12}(U_1, U_2) = -x_1 x_2 = -\frac{1}{k_0 k_1 k_2} \left(\frac{\vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2)}{\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (u_1, u_2)} \right)^2. \quad (3.15)$$

Substituting the expressions of u_1 and u_2 in (3.13) into the right-hand side of equations (3.14) and (3.15), we obtain the functional expression of $\wp_{22}(U_1, U_2)$ and $\wp_{12}(U_1, U_2)$. The sigma function $\sigma(u_1, u_2)$ is guaranteed to exist as the potential of the \wp function from the integrability conditions. However, it seems difficult to obtain an explicit form of the sigma function which is expressed by the theta functions.

For the practical use of the sigma function, it is useful to define the sigma function in the Taylor expansion form in such a way as the hyperelliptic \wp functions satisfy the differential equations. Here, we adopt the genus two hyperelliptic curve in the Jacobi's standard form $y^2 = \lambda_5 x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ with $\lambda_5 = 4$ and $\lambda_0 = 0$, because we can easily notice a dual symmetry in this case. The differential equations are given by [35]

$$(1) \wp_{2222} = 6\wp_{22}^2 + \lambda_4 \wp_{22} + \lambda_5 \wp_{21} + \frac{1}{8} \lambda_3 \lambda_5, \quad (3.16)$$

$$(2) \wp_{2221} = 6\wp_{22} \wp_{21} + \lambda_4 \wp_{21} - \frac{1}{2} \lambda_5 \wp_{11}, \quad (3.17)$$

$$(3) \wp_{2211} = 2\wp_{22} \wp_{11} + 4\wp_{21}^2 + \frac{1}{2} \lambda_3 \wp_{21}, \quad (3.18)$$

$$(4) \wp_{2111} = 6\wp_{21} \wp_{11} - \frac{1}{2} \lambda_1 \wp_{22} + \lambda_2 \wp_{21} - \lambda_0, \quad (3.19)$$

$$(5) \wp_{1111} = 6\wp_{11}^2 + \lambda_1 \wp_{21} + \lambda_2 \wp_{11} + \frac{1}{8} \lambda_1 \lambda_3 - \lambda_0 \left(3\wp_{22} + \frac{1}{2} \lambda_4 \right), \quad (3.20)$$

where we set $\lambda_0 = 0$. There is the dual symmetry in the form equations (3.16) \leftrightarrow (3.20), (3.17) \leftrightarrow (3.19), (3.18) \leftrightarrow (3.18) under $u_1 \leftrightarrow u_2$, $\lambda_5 \leftrightarrow \lambda_1$, $\lambda_4 \leftrightarrow \lambda_2$, $\lambda_3 \leftrightarrow \lambda_3$.

One odd sigma function, which satisfies five differential equations, is given in the form

$$\sigma_1(u_1, u_2) = u_1 + \frac{\lambda_2}{24} u_1^3 - \frac{\lambda_5}{12} u_2^3 + \mathcal{O}(\{u_1, u_2\}^5). \quad (3.21)$$

By using $\wp_{ij}(u_1, u_2) = -\partial_i \partial_j \log \sigma_1(u_1, u_2)$, Baker obtained the addition formula for one sigma function [36]

$$\begin{aligned} & \frac{\sigma_1(u_1 + v_1, u_2 + v_2) \sigma_1(u_1 - v_1, u_2 - v_2)}{\sigma_1(u_1, u_1)^2 \sigma_1(v_1, v_2)^2} \\ &= \wp_{22}(u_1, u_2) \wp_{12}(v_1, v_2) - \wp_{12}(u_1, u_2) \wp_{22}(v_1, v_2) - \wp_{11}(u_1, u_2) + \wp_{11}(v_1, v_2). \end{aligned} \quad (3.22)$$

By using the dual symmetry $u_1 \leftrightarrow u_2$, $\lambda_5 \leftrightarrow \lambda_1$, $\lambda_4 \leftrightarrow \lambda_2$, $\lambda_3 \leftrightarrow \lambda_3$, we obtain another odd sigma function, which satisfies five differential equations. This another odd sigma function is given in the form

$$\sigma_2(u_1, u_2) = u_2 + \frac{\lambda_4}{24} u_2^3 - \frac{\lambda_1}{12} u_1^3 + \mathcal{O}(\{u_1, u_2\}^5). \quad (3.23)$$

By using $\widehat{\wp}_{ij}(u_1, u_2) = -\partial_i \partial_j \log \sigma_2(u_1, u_2)$, we obtain the addition formula for another sigma function

$$\begin{aligned} & \frac{\sigma_2(u_1 + v_1, u_2 + v_2) \sigma_2(u_1 - v_1, u_2 - v_2)}{\sigma_2(u_1, u_2)^2 \sigma_2(v_1, v_2)^2} \\ &= \widehat{\wp}_{11}(u_1, u_2) \widehat{\wp}_{12}(v_1, v_2) - \widehat{\wp}_{12}(u_1, u_2) \widehat{\wp}_{11}(v_1, v_2) - \widehat{\wp}_{22}(u_1, u_2) + \widehat{\wp}_{22}(v_1, v_2). \end{aligned} \quad (3.24)$$

Therefore, the addition formula of the sigma function changes depending on what kind of sigma function we adopt.

4. The half-period addition formulae

The half-period addition formula for the elliptic/hyperelliptic functions forms the order two group. We first examine the half-period addition formula for the elliptic function, which will be instructive to observe the half-period addition formula for the genus two hyperelliptic functions.

4.1. The half-period addition formula for the Weierstrass' \wp function

For the genus one case, we adopt the Weierstrass elliptic curve of the form

$$y^2 = 4x^3 - g_2x - g_3 = 4(x - e_1)(x - e_2)(x - e_3).$$

The Jacobi's inversion problem is to obtain $x = \wp(u)$ from

$$u = \int_{\infty}^x \frac{dx}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}. \quad (4.1)$$

Considering on the Riemann surface, if x reaches one of the branch points e_i ($i = 1, 2, 3$), u reaches the corresponding half-period ω_i ($i = 1, 2, 3$),

$$\omega_i = \int_{\infty}^{e_i} \frac{dx}{\sqrt{4(x - e_1)(x - e_2)(x - e_3)}}, \quad (i = 1, 2, 3). \quad (4.2)$$

The half-period addition formula of the \wp function is given by

$$\wp(u + \omega_1) = \frac{e_1 \wp(u) + e_1^2 + e_2 e_3}{\wp(u) - e_1} = \frac{a \wp(u) + b}{c \wp(u) + d}, \quad (4.3)$$

and that of the cyclic permutation of $\{\omega_1, \omega_2, \omega_3\}$ and $\{e_1, e_2, e_3\}$. We have expressed $a = e_1$, $b = e_1^2 + e_2 e_3$, $c = 1$, $d = -e_1$ in (4.3) and observe a matrix defined by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

has $\text{SL}(2, \mathbb{R}) \cong \text{Sp}(2, \mathbb{R})$ Lie algebra structure. Furthermore, we obtain

$$M^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = (2e_1^2 + e_2 e_3) \mathbf{1},$$

which is equivalent to

$$\wp(u + 2\omega_1) = \frac{(a^2 + bc) \wp(u) + (ab + bd)}{(ac + cd) \wp(u) + (bc + d^2)} = \frac{(2e_1^2 + e_2 e_3) \wp(u)}{(2e_1^2 + e_2 e_3)} = \wp(u). \quad (4.4)$$

Hence, the half-period addition formula (4.3) provides the order two $\text{SL}(2, \mathbb{R}) \cong \text{Sp}(2, \mathbb{R})$ Lie group structure in addition to the $\text{SL}(2, \mathbb{R}) \cong \text{Sp}(2, \mathbb{R})$ Lie algebra structure, which suggests that genus one Weierstrass' \wp function has $\text{SL}(2, \mathbb{R}) \cong \text{Sp}(2, \mathbb{R})$ Lie group structure in the general case.

By applying the half-period transformation twice, we obtain the identity transformation. Therefore, the half-period transformation forms the order two Lie group transformation. Thus, we first demonstrate the relation between the Lie algebra element and the order two Lie group element for the general $\text{Sp}(2g, \mathbb{R})$ ($g=1, 2, \dots$) Lie group. By using the almost complex structure J , which is skew symmetric real matrix with $J^2 = -1$, the Lie algebra element A and the order two Lie group element G satisfy

$$JA + A^T J = 0, \quad G^T J G = J, \quad G^2 = \mathbf{1}. \quad (4.5)$$

By using $G^2 = 1$, we obtain $G^T J = JG$ from $G^T J G = J$.

For the projective representation of any matrix M , $(\text{const.}) \times M$ is equivalent to M . Thus, in the right-hand side of $G^T J = JG$, JG is equivalent to $-JG$, hence we obtain the Lie algebra relation $G^T J = -JG$, which implies that the Lie algebra element A becomes also the order two Lie group element G . For the $\text{Sp}(2, \mathbb{R})$ case, we adopt $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the Lie group transformation is given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad (4.6)$$

with $G = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. The projective representation is given by $x'/y' = \frac{ax/y+b}{cx/y+d}$. For the constant multiplied group element $\lambda G = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$, the projective representation of the transformation is given by $x'/y' = \frac{\lambda ax/y + \lambda b}{\lambda cx/y + \lambda d} = \frac{ax/y+b}{cx/y+d}$, i.e., λG is equivalent to G for the projective representation. The above M satisfies $M^T J + J M^T = 0$, $M^2 = (\text{const.})\mathbf{1}$. This implies that M is not only the $\text{Sp}(2, \mathbb{R})$ Lie algebra element but also the order two $\text{Sp}(2, \mathbb{R})$ Lie group element.

4.2. The half-period addition formula for the genus two hyperelliptic \wp functions

We adopt the genus two hyperelliptic curve in the form

$$y^2 = f_5(x) = 4x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 = 4(x - e_1)(x - e_2)(x - e_3)(x - e_4)(x - e_5), \quad (4.7)$$

which gives $\lambda_4 = -4(e_1 + e_2 + e_3 + e_4 + e_5)$, \dots , $\lambda_0 = -4e_1 e_2 e_3 e_4 e_5$. In the Riemann surface, there are six branching points e_1, e_2, e_3, e_4, e_5 and $e_6 = \infty$. The cuts are drawn from e_{2i-1} to e_{2i} ($i = 1, 2, 3$). The Jacobi's inversion problem is given by

$$u_1 = \int_{\infty}^{x_1} \frac{dt}{\sqrt{f_5(t)}} + \int_{\infty}^{x_2} \frac{dt}{\sqrt{f_5(t)}}, \quad u_2 = \int_{\infty}^{x_1} \frac{t dt}{\sqrt{f_5(t)}} + \int_{\infty}^{x_2} \frac{t dt}{\sqrt{f_5(t)}}, \quad (4.8)$$

and the genus two hyperelliptic \wp functions are given by

$$\wp_{22}(u) = x_1 + x_2, \quad \wp_{21}(u) = -x_1 x_2, \quad \wp_{11}(u) = \frac{F(x_1, x_2) - 2y_1 y_2}{4(x_1 - x_2)^2}, \quad (4.9)$$

with

$$F(x_1, x_2) = 2\lambda_0 + \lambda_1(x_1 + x_2) + 2\lambda_2 x_1 x_2 + \lambda_3 x_1 x_2 (x_1 + x_2) + 2\lambda_4 (x_1 x_2)^2 + 4(x_1 x_2)^2 (x_1 + x_2).$$

By setting $x_1 = e_i, x_2 = e_j$, the half-period is given by $\Omega = (\omega_1, \omega_2)$ for $u = (u_1, u_2)$ in the form

$$\omega_1 = \int_{\infty}^{e_i} \frac{dt}{\sqrt{f_5(t)}} + \int_{\infty}^{e_j} \frac{dt}{\sqrt{f_5(t)}}, \quad \omega_2 = \int_{\infty}^{e_i} \frac{t dt}{\sqrt{f_5(t)}} + \int_{\infty}^{e_j} \frac{t dt}{\sqrt{f_5(t)}}, \quad (4.10)$$

which provides $\wp_{22}(\Omega) = e_i + e_j$, $\wp_{22}(\Omega) = -e_i e_j$, $\wp_{11}(\Omega) = F(e_i, e_j)/4(e_i - e_j)^2$, where we use $y_1(\Omega) = 0$ and $y_2(\Omega) = 0$ because $x_1(\Omega) = e_i$ and $x_2(\Omega) = e_j$.

In order to obtain the half-period addition formula for the hyperelliptic \wp functions, we use the addition formula of the sigma function equation (3.22) of the form

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{22}(u)\wp_{21}(v) - \wp_{21}(u)\wp_{22}(v) - \wp_{11}(u) + \wp_{11}(v). \quad (4.11)$$

Next, we set $v = \Omega$ (half period), hence we have

$$\begin{aligned} \frac{\sigma(u+\Omega)^2}{\sigma(\Omega)^2\sigma(u)^2} &= \wp_{21}(\Omega)\wp_{22}(u) - \wp_{22}(\Omega)\wp_{21}(u) - \wp_{11}(u) + \wp_{11}(\Omega) \\ &= d_1 \wp_{22}(u) + d_2 \wp_{21}(u) + d_3 \wp_{11}(u) + d_4, \end{aligned} \quad (4.12)$$

where $d_1 = \wp_{21}(\Omega)$, $d_2 = -\wp_{22}(\Omega)$, $d_3 = -1$, $d_4 = \wp_{11}(\Omega)$. Considering the logarithm of equation (4.12) and differentiating twice, we obtain

$$\begin{aligned} \wp_{ij}(u+\Omega) &= \wp_{ij}(u) - \frac{1}{2} \frac{Q(u, \Omega) \partial_i \partial_j Q(u, \Omega) - (\partial_i Q(u, \Omega))(\partial_j Q(u, \Omega))}{Q(u, \Omega)^2} \\ &= \frac{2Q(u, \Omega)^2 \wp_{ij}(u) - Q(u, \Omega) \partial_i \partial_j Q(u, \Omega) + (\partial_i Q(u, \Omega))(\partial_j Q(u, \Omega))}{2Q(u, \Omega)^2}, \end{aligned} \quad (4.13)$$

with

$$Q(u, \Omega) = d_1 \wp_{22}(u) + d_2 \wp_{21}(u) + d_3 \wp_{11}(u) + d_4,$$

by using $\wp_{ij}(u_1, u_2) = -\partial_i \partial_j \sigma(u_1, u_2)$. Using equations (3.16)–(3.20) and equations (A.1)–(A.10) in the appendix, the numerator and the denominator are expressed by the polynomial of various \wp_{ij} 's. In the numerator of the right-hand side of equation (4.13), we have the third-degree terms of \wp_{ij} 's in general, yet the third-degree terms automatically cancel. Therefore, the numerator starts from the second-degree terms of \wp_{ij} 's. Furthermore, as it is surprisingly enough, the numerator has the factor $Q(u, \Omega)$. Thus, both the numerator and the denominator starts from the first degree terms of \wp_{ij} 's.

Hence, the addition formulae for half-period are given by Baker [36] and Buchstaber *et al* [37] in the form

$$\wp_{22}(u + \Omega) = \frac{a_1 \wp_{22}(u) + a_2 \wp_{21}(u) + a_3 \wp_{11}(u) + a_4}{d_1 \wp_{22}(u) + d_2 \wp_{21}(u) + d_3 \wp_{11}(u) + d_4}, \quad (4.14)$$

$$\wp_{21}(u + \Omega) = \frac{b_1 \wp_{22}(u) + b_2 \wp_{21}(u) + b_3 \wp_{11}(u) + b_4}{d_1 \wp_{22}(u) + d_2 \wp_{21}(u) + d_3 \wp_{11}(u) + d_4}, \quad (4.15)$$

$$\wp_{11}(u + \Omega) = \frac{c_1 \wp_{22}(u) + c_2 \wp_{21}(u) + c_3 \wp_{11}(u) + c_4}{d_1 \wp_{22}(u) + d_2 \wp_{21}(u) + d_3 \wp_{11}(u) + d_4}. \quad (4.16)$$

There are two types of the half-periods. Type I is given by setting $x_1(\Omega) = e_p, x_2(\Omega) = e_j$ ($i \neq j, 1 \leq i, j \leq 5$). Type II is given by setting $x_1(\Omega) = e_p, x_2(\Omega) = e_6 = \infty$, ($1 \leq i \leq 5$).

For the type I half-period addition formula, we consider the following example of $e_i = e_1, e_j = e_2$, and use the expression of Buchstaber *et al*'s paper. In this case, we have the expression $\wp_{22}(\Omega_1) = e_1 + e_2, \wp_{22}(\Omega_1) = -e_1 e_2, \wp_{11}(\Omega_1) = F(e_1, e_2)/4(e_1 - e_2)^2 = e_1 e_2 (e_3 + e_4 + e_5) + e_3 e_4 e_5$, which provides

$$G_I = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & -a_1 & b_3 & b_4 \\ b_4 & -a_4 & c_3 & c_4 \\ -b_3 & a_3 & -1 & -c_3 \end{pmatrix}. \quad (4.17)$$

One of the examples is given by

$$\begin{aligned} a_1 &= S_3 - S_1 s_2, & a_2 &= -s_2 - S_1 s_1 + S_2, & a_3 &= -s_1, & a_4 &= -s_2^2 + S_2 s_2 + S_1 s_2 s_1, \\ b_1 &= -S_3 s_1 + S_2 s_2, & b_2 &= -a_1, & b_3 &= s_2, & b_4 &= -2S_3 s_2 + S_3 s_1^2 - S_2 s_2 s_1, \\ c_1 &= b_4, & c_2 &= -a_4, & c_3 &= -S_3 - S_1 s_2, \\ c_4 &= -S_2^2 s_2 + 4S_3 S_1 s_2 + S_2 S_1 s_2 s_1 + S_3 S_2 s_1 + S_2 s_2^2 - S_3 S_1 s_1^2 - S_3 s_2 s_1, \\ d_1 &= -b_3, & d_2 &= a_3, & d_3 &= -1, & d_4 &= -c_3, \end{aligned}$$

with $s_1 = e_1 + e_2, s_2 = e_1 e_2, S_1 = e_3 + e_4 + e_5, S_2 = e_3 e_4 + e_4 e_5 + e_5 e_3, S_3 = e_3 e_4 e_5$. All type I half-periods are given by arranging $\{e_1, e_2, e_3, e_4, e_5\}$ into two sets $\{e_p, e_j\} \cup \{e_q, e_r, e_s\}$. We can verify $G_I^2 = (\text{const.})\mathbb{I}$ for all type I half-periods.

For the type II half-period addition formula, we consider the following example of $e_i = e_1, e_j = e_6 = \infty$ and we use the expression of Buchstaber *et al*'s paper. We take the most singular term and the limit $e_6 \rightarrow \infty$ at the end. Thus, we have the expression $\wp_{22}(\Omega_{II}) = e_6, \wp_{22}(\Omega_{II}) = -e_1 e_6, \wp_{11}(\Omega_{II}) = e_1^2 e_6$, which provides

$$G_{II} = \begin{pmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 & \hat{a}_4 \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 & \hat{b}_4 \\ \hat{c}_1 & \hat{c}_2 & \hat{c}_3 & \hat{c}_4 \\ \hat{d}_1 & \hat{d}_2 & \hat{d}_3 & \hat{d}_4 \end{pmatrix} = \begin{pmatrix} \hat{a}_1 & 0 & 1 & \hat{a}_4 \\ 0 & \hat{a}_1 & \hat{b}_3 & \hat{b}_4 \\ -\hat{b}_4 & \hat{a}_4 & -\hat{a}_1 & 0 \\ \hat{b}_3 & -1 & 0 & -\hat{a}_1 \end{pmatrix}. \quad (4.18)$$

One of the examples is given by

$$\begin{aligned} \hat{a}_1 &= -e_1^2, & \hat{a}_2 &= 0, & \hat{a}_3 &= 1, & \hat{a}_4 &= e_1^2 T_1 - e_1 T_2, \\ \hat{b}_1 &= 0, & \hat{b}_2 &= \hat{a}_1, & \hat{b}_3 &= -e_1, & \hat{b}_4 &= e_1 T_3 - T_4, \\ \hat{c}_1 &= -\hat{b}_4, & \hat{c}_2 &= \hat{a}_4, & \hat{c}_3 &= -\hat{a}_1, & \hat{c}_4 &= 0, \\ \hat{d}_1 &= \hat{b}_3, & \hat{d}_2 &= -1, & \hat{d}_3 &= 0, & \hat{d}_4 &= -\hat{a}_1, \end{aligned}$$

with $T_1 = e_2 + e_3 + e_4 + e_5, T_2 = e_2 e_3 + e_2 e_4 + e_2 e_5 + e_3 e_4 + e_3 e_5 + e_4 e_5, T_3 = e_2 e_3 e_4 + e_2 e_3 e_5 + e_2 e_4 e_5 + e_3 e_4 e_5, T_4 = e_2 e_3 e_4 e_5$. All type II half-periods are given by arranging $\{e_1, e_2, e_3, e_4, e_5\}$ into two sets $\{e_i\} \cup \{e_k, e_p, e_q, e_r\}$. We can verify $G_{II}^2 = (\text{const.})\mathbb{I}$ for all type II half-periods.

For $\text{Sp}(4, \mathbb{R})$ case, we adopt the representation of almost complex structure J with $J^2 = -1$ in the form⁵

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (4.19)$$

⁵ Depending on the ordering of the elements of the vector $(\wp_{22}, \wp_{21}, \wp_{11}, 1)$, the representation of G_I, G_{II} and J changes. We adopt Baker's ordering [36]. The existence of the $\text{Sp}(4, \mathbb{R})$ Lie group structure is independent of such ordering.

The bases of the $\text{Sp}(4, \mathbb{R})$ Lie algebra, which satisfies $JA + A^T J = 0$, is given by

$$\begin{aligned} I_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ I_5 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_6 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, & I_8 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ I_9 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & I_{10} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

We have verified that G_I and G_{II} satisfies $JG_I + G_I^T J = 0$, and $JG_{II} - G_{II}^T J = 0$. We observe that equation (4.17) is the element of the ordinary $\text{Sp}(4, \mathbb{R})$ Lie algebra; yet equation (4.18) is not that of the ordinary $\text{Sp}(4, \mathbb{R})$ Lie algebra. However, the transformation of the half-period addition formula for the hyperelliptic \wp functions provides the projective representation. Hence, in the projective representation, G_I and G_{II} are not only the elements of the $\text{Sp}(4, \mathbb{R})$ Lie algebra but also the elements of the order two $\text{Sp}(4, \mathbb{R})$ Lie group. This suggests that the genus two hyperelliptic functions have the general continuous $\text{Sp}(4, \mathbb{R})$ Lie group structure.

5. Summary and discussions

First, we have examined various types of addition formulae for the elliptic functions. The algebraic addition formula can be rearranged into the relation of the Lie group elements, which is called the Yang-Baxter's integrable condition. Second, we have reviewed the Rosehnain's approach to the genus two Jacobi's inversion problem. It is difficult to express the explicit form of the sigma function for the genus two case, thus we use the Taylor expansion form for the sigma function. We pointed out that addition formula of the sigma function depends on what kind of sigma function we adopt. Finally, we have obtained the order two addition formula of the genus two hyperelliptic \wp functions by using the addition formula of a sigma function.

In the previous study, via the two flows Kowalevski top, we had demonstrated that the genus two hyperelliptic functions provide the $\text{Sp}(4, \mathbb{R})/\mathbb{Z} \cong \text{SO}(3,2)$ Lie algebra structure. In this study, by directly using the differential equations of the genus two hyperelliptic \wp functions, we have demonstrated that the half-period addition formula for the genus two hyperelliptic \wp functions provides the order two $\text{Sp}(4, \mathbb{R})$ Lie group structure. This suggests that the genus two hyperelliptic \wp functions have the general continuous $\text{Sp}(4, \mathbb{R})$ Lie group structure.

Data availability statement

No new data were created or analysed in this study.

Appendix. The $\wp_{ijk}\wp_{lmn}$ type differential equations for the genus two hyperelliptic equations

The differential equations for the genus two hyperelliptic \wp functions, which corresponds to $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ in the genus one elliptic \wp function, are given by [37]⁶

$$(1) \quad \wp_{222}^2 = 4\wp_{22}^3 + \lambda_4 \wp_{22}^2 + 4\wp_{22}\wp_{21} + \lambda_3 \wp_{22} + 4\wp_{11} + \lambda_2, \quad (\text{A.1})$$

$$(2) \quad \wp_{221}^2 = 4\wp_{22}\wp_{21}^2 + \lambda_4 \wp_{21}^2 - 4\wp_{21}\wp_{11} + \lambda_0, \quad (\text{A.2})$$

$$(3) \quad \wp_{211}^2 = 4\wp_{21}^2\wp_{11} + \lambda_0 \wp_{22}^2 - \lambda_1 \wp_{22}\wp_{21} + \lambda_2 \wp_{21}^2, \quad (\text{A.3})$$

$$\begin{aligned} (4) \quad \wp_{111}^2 &= 4\wp_{11}^3 + \lambda_0 \wp_{21}^2 - 4\lambda_0 \wp_{22}\wp_{11} + \lambda_1 \wp_{21}\wp_{11} + \lambda_2 \wp_{11}^2 + \left(\frac{1}{4}\lambda_1^2 - \lambda_0\lambda_2\right)\wp_{22} \\ &+ \frac{1}{2}\lambda_0\lambda_3\wp_{21} + \left(\frac{1}{4}\lambda_1\lambda_3 - \lambda_0\lambda_4\right)\wp_{11} + \frac{1}{16}(\lambda_1^2\lambda_4 + \lambda_0\lambda_3^2 - 4\lambda_0\lambda_2\lambda_4), \end{aligned} \quad (\text{A.4})$$

⁶ In the Buchstaber *et al*'s paper, the last term of equation (A.4) is given by $-\lambda_0\lambda_3\lambda_4/4$, but this contains a typographical error and is correctly given by $-\lambda_0\lambda_2\lambda_4/4$.

$$(5) \quad \wp_{222}\wp_{221} = 4\wp_{22}^2\wp_{21} + \lambda_4\wp_{22}\wp_{21} - 2\wp_{22}\wp_{11} + 2\wp_{21}^2 + \frac{1}{2}\lambda_3\wp_{21} + \frac{1}{2}\lambda_1, \quad (\text{A.5})$$

$$(6) \quad \wp_{222}\wp_{211} = 2\wp_{22}^2\wp_{11} + 2\wp_{22}\wp_{21}^2 + \frac{1}{2}\lambda_3\wp_{22}\wp_{21} + 4\wp_{21}\wp_{11} - \frac{1}{2}\lambda_1\wp_{22} + \lambda_2\wp_{21}, \quad (\text{A.6})$$

$$(7) \quad \wp_{222}\wp_{111} = 6\wp_{22}\wp_{21}\wp_{11} - 2\wp_{21}^3 - \lambda_1\wp_{22}^2 + 2\lambda_2\wp_{22}\wp_{21} - \frac{1}{2}\lambda_3\wp_{22}\wp_{11} - \lambda_3\wp_{21}^2 \\ + 2\lambda_4\wp_{21}\wp_{11} - 4\wp_{11}^2 - \frac{1}{4}\lambda_1\lambda_4\wp_{22} + \frac{1}{8}(-4\lambda_1 + 4\lambda_2\lambda_4 - \lambda_3^2)\wp_{21} \\ - \lambda_2\wp_{11} - \frac{1}{8}\lambda_1\lambda_3, \quad (\text{A.7})$$

$$(8) \quad \wp_{221}\wp_{211} = 2\wp_{22}\wp_{21}\wp_{11} + 2\wp_{21}^3 + \frac{1}{2}\lambda_3\wp_{21}^2 - \lambda_0\wp_{22} + \frac{1}{2}\lambda_1\wp_{21}, \quad (\text{A.8})$$

$$(9) \quad \wp_{221}\wp_{111} = 2\wp_{22}\wp_{11}^2 + 2\wp_{21}^2\wp_{11} - 2\lambda_0\wp_{22}^2 + \lambda_1\wp_{22}\wp_{21} + \frac{1}{2}\lambda_3\wp_{21}\wp_{11} \\ - \frac{1}{2}\lambda_0\lambda_4\wp_{22} + \frac{1}{4}(\lambda_1\lambda_4 - 4\lambda_0)\wp_{21} - \frac{1}{2}\lambda_1\wp_{11} - \frac{1}{4}\lambda_0\lambda_3, \quad (\text{A.9})$$

$$(10) \quad \wp_{211}\wp_{111} = 4\wp_{21}\wp_{11}^2 - \lambda_0\wp_{22}\wp_{21} - \frac{1}{2}\lambda_1\wp_{22}\wp_{11} + \frac{1}{2}\lambda_1\wp_{21}^2 + \lambda_2\wp_{21}\wp_{11} \\ - \frac{1}{4}\lambda_0\lambda_3\wp_{22} + \frac{1}{8}\lambda_1\lambda_3\wp_{21} - 2\lambda_0\wp_{11} + \frac{1}{8}(-4\lambda_0\lambda_2 + \lambda_1^2). \quad (\text{A.10})$$

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