Theory of Heat Equations for Sigma Functions

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0 Introduction

Classically, the Weierstrass function $\sigma(u)$ is defined through the Weierstrass elliptic function $\wp(u)$,

$$\sigma(u) = u \exp\bigg(\int_0^u \int_0^u \Big(\frac{1}{u^2} - \wp(u)\Big) du du\bigg).$$

The modern approach is to define the sigma function starting from a general elliptic curve. However, in this introduction, we treat only the curve (Weierstrass form) defined by

$$(0.1) y^2 = x^3 + \mu_4 x + \mu_6,$$

since the theory of heat equations is rather incomplete for general elliptic curves. Here we should be conscious of the field of definition of the curve. For this curve, we define the function $\sigma(u)$ by

(0.2)
$$\sigma(u) = \left(\frac{2\pi}{\omega'}\right)^{1/2} \Delta^{-\frac{1}{8}} \exp\left(-\frac{1}{2}{\omega'}^{-1}\eta' u^2\right) \cdot \vartheta\left[\frac{1}{\frac{1}{2}}\right] (\omega^{-1}u, \omega''/\omega')$$

where $\Delta = -16(4\mu_4^3 + 27\mu_6^2)$ is the discriminant of the curve, and ω' , ω'' , η' , and η'' are the periods of the two differential forms (though η'' does not appear explicitly)

$$\frac{dx}{2y}, \quad \frac{xdx}{2y}$$

and with respect to a pair of fixed standard closed paths α_1 and β_1 which represents a symplectic basis of the first homology group. The last part of (0.2) is Jacobi's theta series defined by

(0.3)
$$\vartheta \begin{bmatrix} b \\ a \end{bmatrix} (z, \tau) = \sum_{n \in \mathbb{Z}} \exp 2\pi i \left(\frac{1}{2} \tau (n+b)^2 + (n+b)(z+a) \right) \quad (a, b \in \frac{1}{2} \mathbb{Z}).$$

From now on, we suppose the function $\sigma(u)$ is defined by (0.2). In using this definition, it is not clear that $\sigma(u)$ is independent of the choice of α_1 and β_1 . Indeed, both of the later part (Jacobi's theta series) of (0.2) and former part are not invariant when we choose another pair of α_1 and β_1 . However these changes offset each other and $\sigma(u)$ itself is invariant.

Using the Dedekind eta function $\eta(\tau)$ (not to be confused with the η s above and in Section 1.1), the discriminant Δ of the curve above is given by $\Delta = \left(\frac{2\pi}{\omega'}\right)^{12} \eta(\omega''/\omega')^{24}$, and the first terms in (0.2) can be explicitly written as

(0.4)
$$\left(\frac{2\pi}{\omega'}\right)^{1/2} \Delta^{-\frac{1}{8}} = -\frac{\omega'}{2\pi} \eta (\omega''/\omega')^{-3}.$$

Although Δ is invariant with respect to a change of α_1 and β_1 , both sides of (0.4) and ω' are not invariant. The function $\sigma(u)$ has a power series expansion at the origin as follows:

(0.5)
$$\sigma(u) = u \sum_{n_4, n_6 \ge 0} b(n_4, n_6) \frac{(\mu_4 u^4)^{n_4} (\mu_6 u^6)^{n_6}}{(1 + 4n_4 + 6n_6)!}$$

$$= u + 2\mu_4 \frac{u^5}{5!} + 24\mu_6 \frac{u^7}{7!} - 36\mu_4^2 \frac{u^9}{9!} - 288\mu_4 \mu_6 \frac{u^{11}}{11!} + \cdots,$$

where $b(n_4, n_6) \in \mathbb{Z}$. (This expansion also shows the independence of $\sigma(u)$ with respect to

the choice of α_1 and β_1). One of the motivations of a theory of heat equations for the sigma functions is to know a recurrence relation for $b(n_4, n_6)$. But, there is another motivation as follows. For an arbitrary non-singular curve, especially a plane telescopic curve, there is an intrinsic definition (characterization) of its sigma function (see Proposition 2.1). It would be useful to have a theorem such that the sigma function is expressed in a similar form as (0.2). Indeed it is not so difficult to show that the natural generalization of the right hand side of (0.2), except for the factor (0.4), satisfies such a characterization. However, except for curves of genus one and two, the validity of the expression including the natural generalization of the factor (0.4) was not yet known. We shall discuss this again at the end of this introduction.

It is well known that the sigma function for a general non-singular algebraic curve is expressed by Riemann's theta series with a characteristic coming from the Riemann constant of the curve multiplied by some exponential factor and some constant factor. The last constant factor might be a natural generalization of (0.4). However, the authors believe that there is no proof on the determination of the last constant except in genus one and two (we mention this again later). To fix the last constant is another motivation of the theory.

Now we review the classical theory of the heat equations for $\sigma(u)$. Let z and τ be complex numbers with the imaginary part of τ positive. We define $L = 4\pi i \frac{\partial}{\partial \tau}$ and $H = \frac{\partial^2}{\partial z^2}$. Then Jacobi's theta function (0.3) satisfies the following equation, which is known as the heat equation,

(0.6)
$$(L - H) \vartheta \begin{bmatrix} b \\ a \end{bmatrix} (z, \tau) = 0.$$

Weierstrass' result in [24] is regarded as an interpretation of (0.6) in the form attached to his function $\sigma(u)$, which is displayed as (0.13) below. Strictly speaking, he did not interpret it directly, but by repeated technical integrations from the well-known differential equation satisfied by the $\wp(u)$, to eventually obtain a recurrence relation (extract from Subsection 3.2)

$$(0.7) b(n_4, n_6) = \frac{2}{3}(4n_4 + 6n_6 - 1)(2n_4 + 3n_6 - 1)b(n_4 - 1, n_6) - \frac{8}{3}(n_6 + 1)b(n_4 - 2, n_6 + 1) + 12(n_4 + 1)b(n_4 + 1, n_6 - 1)$$

for the coefficients in the power series expansion of $\sigma(u)$ as in (0.5) (see also [20]).

Frobenius and Stickelberger approached the same result via a different method. In the paper [10] by Frobenius and Stickelberger, which was published in the same year as [24], using

$$\wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \frac{g_2^2}{1200}u^6 + \cdots$$

and a corresponding expansion of the Weierstrass function $\zeta(u)$, with

$$(0.9) g_2 = 60 \sum_{(n',n'')\neq(0,0)} \frac{1}{(n'\omega' + n''\omega'')^4}, g_3 = 140 \sum_{(n',n'')\neq(0,0)} \frac{1}{(n'\omega' + n''\omega'')^6},$$

they obtained the formulae

$$\omega' \frac{\partial g_2}{\partial \omega'} + \omega'' \frac{\partial g_2}{\partial \omega''} = -4g_2, \qquad \omega' \frac{\partial g_3}{\partial \omega'} + \omega'' \frac{\partial g_3}{\partial \omega''} = -6g_3,$$
$$\eta' \frac{\partial g_2}{\partial \omega'} + \eta'' \frac{\partial g_2}{\partial \omega''} = -6g_3, \qquad \eta' \frac{\partial g_3}{\partial \omega'} + \eta'' \frac{\partial g_3}{\partial \omega''} = -\frac{1}{3}g_2^2$$

and hence

(0.10)
$$\omega' \frac{\partial}{\partial \omega'} + \omega'' \frac{\partial}{\partial \omega''} = -4g_2 \frac{\partial}{\partial g_2} - 6g_3 \frac{\partial}{\partial g_3},$$
$$\eta' \frac{\partial}{\partial \omega'} + \eta'' \frac{\partial}{\partial \omega''} = -6g_3 \frac{\partial}{\partial g_2} - \frac{1}{3}g_2^2 \frac{\partial}{\partial g_3}.$$

On p.326 of [10], we find the same system of heat equations as in [24]. Observing the work of Weierstrass from the viewpoint of the paper [6], the left hand sides of (0.10) correspond to $L=4\pi i\frac{\partial}{\partial\tau}$, namely the operations with respect to the period integrals τ or $\{\omega', \omega'', \eta', \eta''\}$ of the curve, which fit the expression (0.2); while, the right hand sides of (0.10) is an interpretation of such operations in order to fit the expansion (0.5) of (0.2) given by [24]. Although we suspect there are fruitful correspondences between Weierstrass, Frobenius, and Stickelberger, the authors have no details of these.

It appears difficult to generalize the Weierstrass method to the higher genus cases. There is some hint in the work of Frobenius-Stickelberger to generalize the result to the higher genus case. In order to do so, it seems necessary to have generalization of relations (0.10). But we do not have naive generalizations of (0.8) and (0.9).

Recently Buchstaber and Leykin succeeded to generalize the above results to the sigma functions of higher genus curves ([6], see also [3, 4, 5]). In [6], Buchstaber and Leykin generalize (0.10) to higher genus curves by using the first cohomology (over the base field) given by the differential forms of the second kind modulo the exact forms (see Section 2.2). The paper [6] is our main reference for our work. Understanding that paper requires some background on the basic theory of heat equations and singularity theory, so we will summarize their arguments in a self-contained way.

We shall explain their method by taking the curve (0.1) as an example. Firstly, we introduce a certain heat equation (primary heat equation) satisfied by a certain function defined in (0.11) below, which is a generalization of the individual terms of the series appearing in the definition (0.2) of $\sigma(u)$. Let

$$L \in \mathbb{Q}[\mu_4, \mu_6] \frac{\partial}{\partial \mu_4} \oplus \mathbb{Q}[\mu_4, \mu_6] \frac{\partial}{\partial \mu_6}$$

be an arbitrary element. We denote by H^1 the space of the first cohomology (over the base field) given by the differential forms of the second kind modulo the exact forms (see Section 2.2, especially (1.11)). Thanks to a lemma due to Chevalley (Lemma 2.5), we see that $\frac{\partial}{\partial \mu_j}$ acts on H^1 and the operator L acts naturally on H^1 . Let

$$\Gamma^L = \left[\begin{array}{cc} -\beta & \alpha \\ -\gamma & \beta \end{array} \right]$$

be the representation matrix of L (see (2.6)), which is called a Gauss-Manin connection in [6]. Then we see that α , β , and γ belong to $\mathbb{Q}[\mu_4, \mu_6]$. Take a symplectic base $(\frac{dx}{2y}, \frac{xdx}{2y})$ of H^1 with respect to a naturally defined inner product in H^1 (See (1.6)). Taking integrals along the set of closed paths α_1 and β_1 which is a homology basis of the curve, we see the action of L gives a linear transform of the period matrix (see (1.7))

$$\Omega = \left[\begin{array}{cc} \omega' & \omega'' \\ \eta' & \eta'' \end{array} \right],$$

which is also represented by Γ^L (see (2.7)). Moreover, we introduce another operator

$$H^{L} = \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial u} & u \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial u} \\ u \end{bmatrix} = \frac{1}{2} \left(\alpha \frac{\partial^{2}}{\partial u^{2}} + 2\beta u \frac{\partial}{\partial u} + \gamma u^{2} + \beta \right).$$

Then the function

$$(0.11) G(b, u, \Omega) = \left(\frac{2\pi}{\omega'}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}\eta'\omega'^{-1}u^{2}\right) \exp\left(2\pi i\left(\frac{1}{2}\omega'^{-1}\omega''b''^{2} + b''(\omega'^{-1}u + b')\right)\right),$$

where $b = [b' \ b'']$ is an arbitrary constant vector, satisfies the heat equation

(0.12)
$$(L - H^{L})G(b, u, \Omega) = 0.$$

We call this and its generalization (Theorem 2.20) the *primary heat equation*. To check the validity of (0.12) is difficult, and no details are given by Buchstaber and Leykin, and the description of this equation in [6] is not entirely consistent.

We denote the expression of the right hand side (0.2) without $\Delta^{-\frac{1}{8}}$ by $\tilde{\sigma}(u)$ (see (2.25)). According to the above equation and the fact that that $\tilde{\sigma}(u)$ is a infinite sum of the $G(b, u, \Omega)$ s for various b' and b'', we see that $(L - H^L) \tilde{\sigma}(u) = 0$.

At the next stage, we shall find suitable Ls which give rise to a system of heat equations which are satisfied by the right hand side of (0.2), including the factor $\Delta^{-\frac{1}{8}}$. It is easy to see that such an operator must belong to the tangent space of the discriminant Δ (see the former part of Subsection 2.4). We import techniques of calculation from singularity theory to get Δ (Lemma 2.39) as well as its tangent space. This stage is carried out in Subsections 2.4 and 2.5 and the result is given in (2.38). For the curve 0.1, the operators finally obtained are

$$L_0 = 4\mu_4 \frac{\partial}{\partial \mu_4} + 6\mu_6 \frac{\partial}{\partial \mu_6}, \quad L_2 = 6\mu_6 \frac{\partial}{\partial \mu_4} - \frac{4}{3}\mu_4^2 \frac{\partial}{\partial \mu_6},$$

which are, of course, exactly same as those in [24] and [10]. The paper [6] uses knowledge of singularity theory and succeeds in generalizing nicely the result of Weierstrass and Frobenius-Stickelberger.

Here, we shall summarize the system of heat equations for the curve (0.1)

$$(0.13) (L_0 - H^{L_0}) \, \sigma(u) = \left(4\mu_4 \frac{\partial}{\partial \mu_4} + 6\mu_6 \frac{\partial}{\partial \mu_6} - u \frac{\partial}{\partial u} + 1\right) \sigma(u) = 0,$$

$$(L_2 - H^{L_2}) \, \sigma(u) = \left(6\mu_6 \frac{\partial}{\partial \mu_4} - \frac{4}{3}\mu_4^2 \frac{\partial}{\partial \mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2\right) \sigma(u) = 0,$$

which is a consequence of (3.6). For the general curves, the corresponding result is given as Theorem 2.49 in the text.

It is very important to determine whether the system of heat equations obtained characterize the sigma function. For the genus one case, it was seen by Weierstrass that the recurrence (0.7) determines all the coefficients if we give an arbitrary value for b(0,0). That is, the solution space of (0.7), as well as (0.13), is of dimension one.

For a general non-singular curve, we consider the multivariate function $\sigma(u)$ defined similarly to (0.2). We can check that the operators obtained (of Theorem 2.49) kill $\sigma(u)$. However, it is not clear whether the solution space is of dimension one over the base field. The authors could not find any reason which suggests that the solution space is one dimensional. We speculate that the system of heat equations for a curve of modality 0 (explained below) might not

characterize the sigma function.

Nevertheless, we shall show that, for any curve of genus less than or equal to three, the solution space is one dimensional by giving an explicit recurrence relation from the system of heat equations obtained, which is described in Section 3. Although our main result is in Subsections 3.5, 3.4, 3.7, and 3.6, we give a number of useful results, which may be known by specialists but are not well known to other researchers, with detailed proofs in Section 2 and Subsection 3.1. Subsection 3.2 reproduces the classical result and it would be helpful to read the following Subsections. Subsection 3.3 is rewritten in a slightly different formulation (Hurwitz-type series expansion of $\sigma(u)$) from [5].

In this work, the family of curves which we will investigate are called (n,s)-curves or plane telescopic curve (see Section 1.1 for its definition). We shall explain a notion called modality, which is introduced by Arnol'd (see 1.3 for details). For any coprime positive integers (e, q) with e < q, we consider the deformation of the singularity at the origin of the curve $y^e = x^q$. Then any deformed curve is called a plane telescopic curve. The number of parameters necessary for such deformation is less than or equal to (e-1)(q-1) that is twice the genus of the generic deformed curve. The difference between this number and (e-1)(q-1) is called the *modality* of this deformation. For instance, the curve (0.1) is regarded as a deformation of $y^2 = x^3$ with two parameters μ_4 and μ_6 , which is equal to twice its genus (i.e. $2 = 1 \times 2$). So that the modality is 0 in this case. It is known that any hyperelliptic curve as well as any elliptic curve is of modality 0. There are only two kinds of non-hyperelliptic plane telescopic curves of modality 0, which are trigonal quartic curve (genus three) ((3,4)-curve) and trigonal quintic curve (genus four) ((3,5)-curve). In this paper we treat those curves of modality 0 except for the (3,5)-curve, which we leave to a description elsewhere. For a general hyperelliptic curve, we give its corresponding system of heat equations in Lemma 3.1. For any plane telescopic curve, we gave a simple formula for its modality in Proposition 1.16.

We shall mention two additional consequences of this theory. Firstly, for hyperelliptic curves of genus less than or equal to three, we again prove partially the result of [19] on Hurwitz integrality of the expansion of the sigma function. For example it is obvious from (0.7) that $b(n_4, n_6) \in \mathbb{Z}[\frac{1}{3}]$. Similar results are shown for the hyperelliptic curves of genus two and three. This idea was suggested to Y.Ô. by Buchstaber. Secondly, this theory of heat equations in turn helps the construction of the sigma function, as explained in Lemma 4.17 of [5] (see also Section 2.6). The formula (0.2) is well-known for the curve (0.1), and its generalization (see (2.3)) is proved for the genus two hyperelliptic curve by Grant [12] by using Thomae's formula. For any curve in the family we have investigated, there is a rough explanation in Lemma 2.3 in p.98 of [4], but without using Thomae's formula. We give a detailed proof of this for the curves of genus less than or equal to three, which is a quote from Lemma 4.17 of [5].

Finally, one of the authors S.Y. wishes to point out to the reader that his contribution on this paper is limited to the proof of the case e = 2 in Lemma 2.41.

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1 Preliminaries

1.1 The curves

We shall use e and q instead of n and s of (n,s)-curves, respectively. Although the name (n,s)-curve (which comes from singularity theory) is used by Buchstaber and Leykin in their papers, we wish to avoid confusion with the many n used as subscripts in sections from Section 3 onward, and the use of s for Schur polynomials in Subsection 2.1. Let e and q be two fixed positive integers such that e < q and $\gcd(e,q) = 1$. We define, for these integers, a polynomial of x and y

(1.1)
$$f(x,y) = y^e - p_1(x)y^{e-1} - \dots - p_{e-1}(x)y - p_e(x),$$

where $p_j(x)$ is a polynomial of x of degree $\lceil \frac{jq}{e} \rceil$ or smaller and its coefficients are denoted by

$$(1.2) p_j(x) = \sum_{k: jq - ek > 0} \mu_{jq - ek} x^k (1 \le j \le e - 1), p_e(x) = x^q + \mu_{e(q - 1)} x^{q - 1} + \dots + \mu_{eq}.$$

Please note that the sign at the front of each $p_j(x)$ with $j \neq e$ in f(x,y) is different from previous papers written by the authors. The base ring over which we work is quite general. However, for simplicity we start by letting it be the field \mathbb{C} of complex numbers and assume the μ_i s to be constants belonging to this field. Let $\mathscr{C} = \mathscr{C}_{\mu}^{e,q}$ be the projective curve defined by

$$(1.3) f(x,y) = 0$$

having a unique point ∞ at infinity. This should be called an (e,q)-curve following Buchstaber, Enolskii, and Leykin [7], or a plane telescopic curve after the paper [16]. If this is non-singular, its genus is (e-1)(q-1)/2, which is denoted by g whether $\mathscr E$ is non-singular or singular:

$$g = \frac{(e-1)(q-1)}{2}.$$

As the general elliptic curve is defined by an equation of the form

$$y^2 - (\mu_1 x + \mu_3)y = x^3 + \mu_2 x^2 + \mu_4 x + \mu_6$$

the curves \mathscr{C} discussed here are a natural generalization of elliptic curves.

We introduce a weight function as follows:

$$\operatorname{wt}(\mu_i) = -j, \quad \operatorname{wt}(x) = -e, \quad \operatorname{wt}(y) = -q.$$

Then all the equations for functions, power series, differential forms, and so on in this paper are of homogeneous weight. For example, $\operatorname{wt}(f(x,y)) = -eq$.

We give the following definitions as in [19]. For the given pair (e, q) $(q > e > 0, \gcd(e, q) = 1)$ with $g = \frac{(e-1)(q-1)}{2}$, we define the Weierstrass non-gap sequence

$$(1.4) w_1, w_2, \cdots, w_g$$

as an increasing sequence of positive integers of the form ae + bq with non-negative integers a, b. We shall write the increasing sequence of non-negative integers of the form ae + bq with non-negative integers a, b by

$$0 = a_0 e + b_0 q$$
, $a_1 e + b_1 q$, \cdots , $a_{n-1} e + b_{n-1} q$.

Then we define for $j = 0, \dots, g - 1$

$$\omega_{w_{g-j}} = \frac{x^{a_j} y^{b_j}}{f_y(x, y)} dx,$$

where $f_y = \frac{\partial}{\partial y} f$. We define g more differential forms of the second kind η_{-w_j} $(j = g, \dots, 1)$ of weight $-w_j$, such that these 2g forms give rise to a symplectic base of $H^1(\mathscr{C}, \mathbb{C})$, as we explain below. We denote $\boldsymbol{\omega} = (\omega_{w_1}, \dots, \omega_{w_g}, \eta_{-w_g}, \dots, \eta_{-w_1})$.

The space $H^1(\mathscr{C},\mathbb{C})$ is regarded as the space of the second kind modulo the exact forms:

$$(1.5) H^1(\mathscr{C}, \mathbb{C}) \simeq \varinjlim_{n} H^0(\mathscr{C}, d\mathscr{O}(n\infty)) / \varinjlim_{n} dH^0(\mathscr{C}, \mathscr{O}(n\infty)).$$

(see [18] or [22]). Indeed, since $m \ge 2g$ is written as m = ae + bq $(a, b \in \mathbb{N})$, any differential form which has a pole of order less than or equal to 2g + 1 at ∞ and is holomorphic elsewhere is expressed as a linear combination of elements in ω and the forms $d(x^a y^b)$. An antisymmetric inner product on this space is defined by

$$\omega\star\eta=\mathop{\mathrm{res}}_{P=\infty}\Big(\int_{\infty}^{P}\omega\Big)\eta(P),$$

for the forms ω and η which are holomorphic outside ∞ . We choose η_{-w_i} to satisfy the symplectic relations

(1.6)
$$\omega_{w_i} \star \omega_{w_i} = 0, \quad \eta_{-w_i} \star \eta_{-w_i} = 0, \quad \omega_{w_i} \star \eta_{-w_i} = -\eta_{-w_i} \star \omega_{w_i} = \delta_{ij}.$$

The choice of η_{-w_j} is not unique. For a more concrete construction of these forms, we refer the reader to [19]. From the relations (1.6), the entries of ω form a symplectic basis of $H^1(\mathscr{C}_{\mu},\mathbb{C})$. The period matrices are defined by:

$$(1.7) \qquad \Omega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}, \quad \omega' = \begin{bmatrix} \int_{\alpha_j} \omega_i \end{bmatrix}, \quad \omega'' = \begin{bmatrix} \int_{\beta_j} \omega_i \end{bmatrix}, \quad \eta' = \begin{bmatrix} \int_{\alpha_j} \eta_i \end{bmatrix}, \quad \eta'' = \begin{bmatrix} \int_{\beta_j} \eta_i \end{bmatrix},$$

where $\{\alpha_j, \beta_j \mid j=1,\dots,g\}$ is a symplectic basis of the homology group $H_1(\mathscr{C},\mathbb{Z})$ as usual. We introduce the g-dimensional space \mathbb{C}^g with the coordinates $u=(u_{w_g},u_{w_{g-1}},\dots,u_{w_1})$ for the domain for which the sigma function is defined. We define a lattice in this space by

(1.8)
$$\Lambda = \omega' \mathbb{Z}^g + \omega'' \mathbb{Z}^g.$$

For any $u \in \mathbb{C}^g$, we define u', $u'' \in \mathbb{R}^g$ by $u = \omega' u' + \omega'' u''$. Likewise, for $\ell \in \Lambda$, we write $\ell = \omega' \ell' + \omega'' \ell''$. In addition we write $\omega'^{-1} t(\omega_{w_g}, \dots, \omega_{w_1}) = t(\hat{\omega}_1, \dots, \hat{\omega}_g), \, \omega'^{-1} \omega'' = [\tau_{ij}]$, and define

$$\delta_j = -\frac{1}{2}\tau_{jj} - \int_{\infty}^{P_j} \hat{\omega}_j + \sum_{i=1}^g \int_{\alpha_i} \left(\int_{\infty}^P \hat{\omega}_j \right) \hat{\omega}_i(P), \quad \delta = \omega'^t(\delta_1, \dots, \delta_g).$$

Here P_j is a fixed initial point of the path α_j . Then we define the Riemann constant by $\delta = [\delta' \ \delta'']$. It is well known that, for our curve, $\delta \in \frac{1}{2}\Lambda$, namely, δ' , $\delta'' \in (\frac{1}{2}\mathbb{Z})^g$. The δ can be taken independent of the values of μ_j s.

Using the above notation, we define a linear form $L(\ ,\)$ on $\mathbb{C}^g \times \mathbb{C}^g$ by

(1.9)
$$L(u,v) = u (v'\eta' + v''\eta'').$$

This is C-linear with respect to the first variable, and R-linear with respect to the second one.

Moreover, we define

(1.10)
$$\chi(\ell) = \exp\left(2\pi i \left(t \delta' \ell'' - t \delta' \ell' + \frac{1}{2} t \ell' \ell''\right)\right) \in \{1, -1\}$$

for any $\ell \in \Lambda$. These are used for defining the sigma function in 2.1.

We denote $\mathbb{Q}[\mu] = \mathbb{Q}[\{\mu_k\}]$. In this paper, we should consider the curve \mathscr{C} and other objects arising from \mathscr{C} to be defined over the ring $\mathbb{Q}[\mu]$, in which the period matrix Ω is exceptional and is defined over only the field \mathbb{C} . It is known that the η_{-w_j} s as well as the ω_{w_j} s are defined over $\mathbb{Q}[\mu]$, namely, they are of the form $\frac{h(x,y)}{f_y(x,y)}dx$ with $h(x,y) \in \mathbb{Q}[\mu][x,y]$. Therefore, from (1.5), it is natural for us to define

$$(1.11) H^{1}(\mathscr{C}_{\mu}, \mathbb{Q}[\mu]) = \{ h(x,y)/f_{y}(x,y) dx \mid h(x,y) \in \mathbb{Q}[\mu][x,y] \} / d\mathbb{Q}[\mu][x,y],$$

which is a $\mathbb{Q}[\mu]$ -module and represented by linear combinations of the ω_{w_j} s and η_{-w_j} s over $\mathbb{Q}[\mu]$. In this paper we frequently switch between regarding the μ_j s as indeterminates or complex numbers.

1.2 Definition of the discriminant

We shall define the discriminant of the curve \mathscr{C} .

Definition 1.12. The discriminant Δ of the form f(x,y) or of the curve \mathscr{C} defined by f(x,y) = 0 is the polynomial of the least degree in the $\mu_j s$ with integer coefficients such that the greatest common divisor of the coefficients is 1, and every zero of Δ corresponds exactly to the case that \mathscr{C} has a singular point.

For (e,q)=(2,2g+1), as in Section 1.3, we rewrite the equation as $y^2=x^{2g+1}+\cdots$, where the right hand side is a polynomial of x only. Then the discriminant of this curve is the discriminant of the right hand side as a polynomial of x only. For (3,4)-curve, we have Sylvester's method explained in [11] pp.118-120, as explained to the authors by C. Ritzenthaler. However, Lemma 2.39 below give a much general method than it, which covers (3,5)-curve, too. The authors do not have general method to compute the discriminants of the other curves. We do have explicit forms of the discriminants of the curves with (e,q)=(2,3), (2,5), (2,7), (3,4) which we treat in this paper. Using the resultant of two forms, we mention here an alternative construction (but is conjectural) as an information for the reader, though it is not used in this paper.

Definition 1.13. Let the coefficients μ_i of (1.1) be indeterminates, and define

$$R_{1} = \operatorname{rslt}_{x} \left(\operatorname{rslt}_{y} \left(f(x, y), \frac{\partial}{\partial x} f(x, y) \right), \operatorname{rslt}_{y} \left(f(x, y), \frac{\partial}{\partial y} f(x, y) \right) \right),$$

$$R_{2} = \operatorname{rslt}_{y} \left(\operatorname{rslt}_{x} \left(f(x, y), \frac{\partial}{\partial x} f(x, y) \right), \operatorname{rslt}_{x} \left(f(x, y), \frac{\partial}{\partial y} f(x, y) \right) \right),$$

$$R = \gcd(R_{1}, R_{2}) \quad in \ \mathbb{Z}[\mu].$$

Here rslt_z is the Sylvester resultant with respect to z.

Now we recall the conjecture from the paper [9].

Conjecture 1.14. (1) R is always a perfect square in $\mathbb{Z}[\mu]$;

(2) Further, on restoring the μ_i to their original values, $R = \Delta^2$.

Remark 1.15. (1) We have checked that for (d,q) = (2,3), (2,5), (2,7), (3,4), (3,5) this

construction gives the correct discriminant in $\mathbb{Z}[\mu]$. However, the authors have no proof of the conjecture.

(2) The discriminant Δ of the (e,q)-curve has weight -2eqg = -eq(e-1)(q-1), as proved in Lemma 2.36.

1.3 The Weierstrass form of the curve and its modality

As explained in [5], the method we use here works only for hyperelliptic curves and for some trigonal curves as explained below.

Starting from the equation f(x,y) = 0 in (1.1) and removing the terms of y^{e-1} and x^{q-1} by replacing y by $y + \frac{1}{e} p_1(x)$, and x by $x + \frac{1}{q} \mu_{(q-1)e}$, respectively, we get a new equation f(x,y) = 0 which is called the *Weierstrass form* of the original one. After making such transformations, we re-label the coefficients by μ_i .

For example, if (e,q) = (2,2g+1), the new equation is

$$f(x,y) = y^2 - (x^{2g+1} + \mu_{4g-2}x^{2g-1} + \mu_{4g-4}x^{2g-2} + \dots + \mu_{4g+2}) = 0;$$

and if (e,q) = (3,4), the new one is

$$f(x,y) = y^3 - (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0.$$

In these cases, the number of remaining μ_j s is 2g. However, in general, we can have cases such that this number is less than 2g. The difference

$$2g$$
 – "the number of μ_i "

is called the $modality^1$ of the (e,q)-curve.

We give here a simple formula giving modalities and, especially, determine all the curves of modality 0.

Proposition 1.16. The modality of an (e,q)-curve is given by $\frac{1}{2}(e-3)(q-3) + \lfloor \frac{q}{e} \rfloor - 1$. The only curves of modality 0 are the (2,2g+1)-, (3,4)-, and (3,5)-curves.

Proof. The number of μ_i 's appearing in the defining equation of Weierstrass form is

$$\sum_{j=1}^{e-2} \left(\left\lfloor \frac{(e-j)q}{e} \right\rfloor + 1 \right) + (q-1) = \frac{1}{2} \sum_{j=1}^{e-1} \left(\left\lfloor \frac{(e-j)q}{e} \right\rfloor + \left\lfloor \frac{jq}{e} \right\rfloor \right) - \left\lfloor \frac{q}{e} \right\rfloor + (e-2) + (q-1)$$

$$= \frac{1}{2} \sum_{j=1}^{e-1} (q-1) - \left\lfloor \frac{q}{e} \right\rfloor + e + q - 3 = \frac{1}{2} (e-1)(q-1) + e + q - 3 - \left\lfloor \frac{q}{e} \right\rfloor.$$

Therefore the modality is

$$2g - \left(\frac{1}{2}(e-1)(q-1) + e + q - 3 - \left\lfloor \frac{q}{e} \right\rfloor\right) = \frac{1}{2}(e-3)(q-3) + \left\lfloor \frac{q}{e} \right\rfloor - 1.$$

The later part follows directly from this. This complete the proof.

If the modality of the curve under consideration is positive, there is a description in [6] to define the operators L_{v_i} of (2.38).

¹A terminology used in singularity theory.

2 Theory of heat equations

2.1 Classical construction of the sigma functions

In this section, we summarise the theory of the sigma function. We use the classical notation of matrices concerning theta series, so our notation is transposed from the notation of [6]. Specifically, we will denote the period matrix of a curve as (1.7) whereas Buchstaber and Leykin's papers use the transpose of this matrix. Other differences between their notation and ours will follow from this.

We introduce generalized notation for the sigma function (0.2) for the general curve \mathscr{C} . This is now a function of g variables $u=(u_{w_g},\cdots,u_{w_1})$ and written as $\sigma(u)=\sigma(u_{w_g},\cdots,u_{w_1})$. It is called, analogously, the sigma function for \mathscr{C} . To define it precisely, we need to introduce the Schur polynomial. Letting $T=\sum_{j=1}^g u_{w_j} T^{w_j}$, we define $\{p_k\}$ by

$$\sum_{k=0}^{\infty} p_k T^k = \sum_{n=0}^{\infty} \frac{T^n}{n!}.$$

Then we define $s(u) = s(u_{w_q}, \dots, u_{w_1})$ by

$$s(u) = \det([p_{w_{g-i}+j-g}]).$$

$$1 \le i \le g, \ 1 \le j \le g$$

This is called the Schur polynomial with respect to the sequence (w_q, \dots, w_1) .

We characterise the sigma function as follows.

Proposition 2.1. Suppose that the $\{\mu_j\}$ are constants in \mathbb{C} and Δ is not zero. There exists a unique function on \mathbb{C}^g having the following properties:

- (1) $\sigma(u)$ is an entire function on \mathbb{C}^g ;
- (2) $\sigma(u+\ell) = \chi(\ell) \, \sigma(u) \exp L(u+\frac{1}{2}\ell,\ell)$ for any $u \in \mathbb{C}^g$ and $\ell \in \Lambda$, where Λ , L, and χ are defined in (1.8), (1.9), and (1.10), respectively;
- (3) $\sigma(u)$ can be expanded as a power series around the origin with coefficients in $\mathbb{Q}[\mu]$ of homogeneous weight $(e^2-1)(q^2-1)/24$;
- (4) $\sigma(u)|_{\mu=0}$ is the Schur polynomial s(u);
- (5) $\sigma(u) = 0 \iff u \in \kappa^{-1}(\Theta).$

Proof. We define

(2.2)
$$\tilde{\sigma}(u) = \tilde{\sigma}(u,\Omega) = \left(\frac{(2\pi)^g}{\det \omega'}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}{}^t u \, \eta' \omega'^{-1} u\right) \\ \cdot \sum_{n \in \mathbb{Z}^g} \exp\left(\frac{1}{2}{}^t (n+\delta'') \, \omega'^{-1} \omega'' (n+\delta'') + {}^t (n+\delta'') (\omega'^{-1} u + \delta')\right),$$

where det denotes the determinant. It is obvious that this function has the property (1), and it is easy to show that this function satisfies (2) using (2.8). Frobenius' method shows that the solutions of the equation (1) form a one dimensional space (see p.93 of [13]). Although this function is constructed by using Ω , it is independent of Ω . Namely, the function $\hat{\sigma}(u,\Omega)$ is invariant under a modular transform, that is the transform of Ω by $\operatorname{Sp}(2g,\mathbb{Z})$. Therefore, it is expressed as a power series of u with coefficients being functions of the μ_j s. On (3), we refer the reader to [17] and [19]. In these papers, $\tilde{\sigma}(u)$ times some constant is expressed as a determinant of infinite size, and, by that expression, (3) is proved. It is well known that any non-zero multiple of $\tilde{\sigma}(u)$ has the property (5). We can show that $\tilde{\sigma}(u)$ times some constant

satisfies (2) by means of the paper [19]. Another proof of this is given by [18].

It is known for g = 1 and 2 that

(2.3)
$$\hat{\sigma}(u) = \Delta^{-\frac{1}{8}} \tilde{\sigma}(u),$$

with the choice of a certain $\frac{1}{8}$ th root of Δ , exactly satisfies all the properties in 2.1. Namely, we show that $\sigma(u) = \hat{\sigma}(u)$. For g = 1, it is shown as in [21] by a transformation formulae for $\eta(\tau)$ and theta series described in pp.176–180 of [23], and for g = 2 the paper [12] by D. Grant, in which the property (4) is shown by using Thomae's formula.

In this paper, we show that $\sigma(u) = \hat{\sigma}(u)$, in other words, that the function (2.3) for the genus 3 curves, i.e. for the (2,7)-curve and the (3,4)-curve, also satisfies (4) up to a numerical multiplicative constant.

Lemma 2.4. The constant $\left(\frac{(2\pi)^g}{\det \omega'}\right)^{1/2} \Delta^{-\frac{1}{8}}$ is of weight $\frac{(e^2-1)(q^2-1)}{24}$, which is equal to the weight of $\sigma(u)$.

Proof. The weight of $\Delta^{\frac{1}{8}}$ is -eq(e-1)(q-1)/8. The weight of π is 0. The weight of $\det(\omega')$ is $\sum_{j=1}^{g} w_j$, which equals

$$\sum_{i=1}^{g} w_i = \frac{eq(e-1)(q-1)}{4} - \frac{(e^2-1)(q^2-1)}{12}$$

by p.97 of [4]. Hence the weight of the constant under consideration is $\frac{(e^2-1)(q^2-1)}{24}$.

2.2 Generalization of the Frobenius-Stickelberger theory [10] to higher genus

This and the following section are devoted to explaining the theory of Buchstaber and Leykin [6], on the differentiation of Abelian functions with respect to their parameters, as clearly as we can. That generalises the work of Frobenius and Stickelberger [10], discussed above, on the elliptic case of this problem.

For higher genus cases, we do not have a naive generalization of (0.8) and (0.9), which are mentioned in the Introduction. However, we can give a natural generalization of the relations (0.10) to the curve \mathcal{C}_{μ} as explained in the next section.

We shall consider operators in $\mathbb{Q}[\mu][\partial_{\mu}]$. Here we denote by ∂_{μ} the set $\{\frac{\partial}{\partial \mu_{j}}\}$. We first explain the symbol $\frac{\partial}{\partial \mu_{j}}$. To do so, we recall the following lemma which is essentially the same as Lemma 1 in [14]. We denote by R the function field $\mathbb{Q}(\mu, x, y)$ of \mathscr{C} . Let ξ be an element in R. If ξ' is another element in R. We shall treat derivations on R as follows. For a derivation D on R such that $D\xi = 0$ and $\omega \in R d\xi$, we define

$$D(\omega) = D\left(\frac{\omega}{d\xi}\right)d\xi.$$

Now we need the following lemma due to Chevalley.

Lemma 2.5. (Chevalley [8]) Any derivation is independent of the choice of the element ξ in the following sense. Let ξ and ξ' be two elements in R transcendental over $\mathbb{Q}(\mu)$. Let D and D' be derivations on R such that Ds = D's for any $s \in \mathbb{Q}(\mu)$, and hence in particular

 $D\xi = D'\xi' = 0$. Then we have for any $w \in R$

$$D(wd\xi) - D'(wd\xi) = d(-wD'\xi).$$

Proof. (From Manin [14]) The operator $D - D' + (D'\xi)\frac{d}{d\xi}$ is a derivative on R which vanish on $\mathbb{Q}(\mu)$ and also kills ξ . Hence this vanishes on R, so that

$$(D - D')w + (D'\xi)\frac{dw}{d\xi} = 0.$$

Moreover $(D-D')(wd\xi) = (D-D')w \cdot d\xi + w \cdot (D-D')d\xi$. Since ξ' is transcendental, we see $\frac{d}{d\xi'}D' = D'\frac{d}{d\xi'}$ and

$$Dd\xi = D\left(\frac{d\xi}{d\xi}\right)d\xi = 0, \quad D'd\xi = D'\left(\frac{d\xi}{d\xi'}\right)d\xi' = \frac{d}{d\xi'}(D'\xi)d\xi' = d(D'\xi).$$

Therefore

$$(D - D')(wd\xi) = -(D'\xi)\frac{dw}{d\xi}d\xi - w \cdot d(D'\xi) = -d(wD'\xi)$$

as desired. \Box

For a sample usage of $\frac{\partial}{\partial \mu_j}$, see Section 3.2. By Lemma 2.5, any element in $\mathbb{Q}[\mu][\partial_{\mu}]$ operates linearly on the space $H^1(\mathcal{C}_{\mu}, \mathbb{Q}[\mu])$. More precisely, we let L operate firstly on the forms with variable μ_j s of representative in $H^1(\mathcal{C}_{\mu}, \mathbb{Q}[\mu])$, then we restore μ_j s to the original values in \mathbb{C} . Let L be a first order operator, namely an element in $\bigoplus_j \mathbb{Q}[\mu] \frac{\partial}{\partial \mu_j}$. We define $\Gamma^L \in \mathrm{Mat}(2g, \mathbb{Q}[\mu])$ as the representation matrix for the following action of L:

(2.6)
$$L^{t}\boldsymbol{\omega} = \Gamma^{Lt}\boldsymbol{\omega} \quad \text{for } \boldsymbol{\omega} \in H^{1}(\mathscr{C}_{\mu}, \mathbb{Q}[\mu]).$$

In [6], the matrix $-^t(\Gamma^L)$ defined by (2.6) is called the *Gauss-Manin connection* for the vector field L. Accordingly, by integrating (2.6) along each element in the chosen symplectic basis of $H_1(\mathcal{C}_{\mu}, \mathbb{Z})$, we get natural action

(2.7)
$$L(\Omega) = \Gamma^L \Omega \quad \text{of } L \text{ on } \Omega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}.$$

This is no other than the generalization (0.10) of Frobenius-Stickelberger. Of course, since Ω is the period matrix of a symplectic basis, these elements must satisfy the constraint

(2.8)
$${}^t\Omega J\Omega = 2\pi i J, \text{ where } J = \begin{bmatrix} 1_g \\ -1_g \end{bmatrix}$$

by (1.6). This is to say that $(2\pi i)^{-\frac{1}{2}}\Omega \in \operatorname{Sp}(2g,\mathbb{C})$. It follows immediately that

(2.9)
$$\Omega^{-1} = \frac{1}{2\pi i} J^{-1} {}^{t}\Omega J = \frac{1}{2\pi i} \begin{bmatrix} {}^{t}\eta'' & {}^{-t}\omega'' \\ {}^{-t}\eta' & {}^{t}\omega' \end{bmatrix}.$$

Operating on both sides of (2.8) with L, using (2.7) and (2.8) that the matrix Γ^L must satisfy

(2.10)
$${}^t\Gamma^L J + J\Gamma^L = 0, \text{ i.e. } {}^t(\Gamma^L J) = \Gamma^L J,$$

which is to say that $\Gamma^L \in \mathfrak{sp}(2g, \mathbb{Q}[\mu])$. Thus we may write

$$\Gamma^L J = \begin{bmatrix} \alpha & \beta \\ {}^t\beta & \gamma \end{bmatrix}, \quad \Gamma^L = \begin{bmatrix} -\beta & \alpha \\ -\gamma & {}^t\beta \end{bmatrix}$$

with ${}^t\alpha = \alpha$ and ${}^t\gamma = \gamma$.

Remark 2.11. We use different notation for $D(x, y, \lambda)$ in p.273 of [6], and for Ω , Γ and β in p.274 of loc. cit. Our ${}^t\omega$ equals to $D(x, y, \lambda)$ by transposing and changing the sign on the latter half entries. The others are naturally modified according to this difference and taking transposes. We will give a detailed comparison of our notation with theirs below, in 2.51, at the end of 2.3.

Using the notation above, from (2.7), it is natural for us to understand the operator L above as being the operator

$$L = \sum_{i=1}^{g} \sum_{j=1}^{g} (\Gamma \Omega)_{i,j} \frac{\partial}{\partial \omega'_{ij}} + \sum_{i=1}^{g} \sum_{j=1}^{g} (\Gamma \Omega)_{i,j+g} \frac{\partial}{\partial \omega''_{ij}} + \sum_{i=1}^{g} \sum_{j=1}^{g} (\Gamma \Omega)_{i+g,j+g} \frac{\partial}{\partial \eta''_{ij}},$$

since

(2.12)
$$\Gamma^{L}\Omega = \begin{bmatrix} -\beta & \alpha \\ -\gamma & {}^{t}\beta \end{bmatrix} \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix} = \begin{bmatrix} -\beta\omega' + \alpha\eta' & -\beta\omega'' + \alpha\eta'' \\ -\gamma\omega' + {}^{t}\beta\eta' & -\gamma\omega'' + {}^{t}\beta\eta'' \end{bmatrix}.$$

Conversely, starting from a matrix

$$\Gamma = \left[\begin{array}{cc} -\beta & \alpha \\ -\gamma & {}^t\beta \end{array} \right] \in \mathfrak{sp}(2g,\mathbb{Q}[\mu]),$$

with ${}^t\alpha=\alpha$ and ${}^t\gamma=\gamma$, we get uniquely an operator $L\in\bigoplus_j\mathbb{Q}[\mu]\frac{\partial}{\partial\mu_j}$ such that $\Gamma^L=\Gamma$. So far, this is a natural generalization of the situation investigated by Frobenius-Stickelberger [10].

2.3 Primary heat equations

In this Section, we review the general heat equations satisfied by the sigma functions. If we want to find second-order linear parabolic partial differential equations (heat equations) satisfied by the sigma function, we should proceed in as general a way as possible. Here, note that the equation (0.6) is satisfied not only by the Jacobi theta function (0.3) but also by each individual term of the sum in (0.3). This corresponds a statement in the proof of Theorem 13 in page 274 of [6]. Here we will review their theory of such equations, correcting a few minor errors, and apply it explicitly to more general curves than considered in [6]. We shall start from this point of view.

In this paper, we start with a first order operator as in 2.6: $L \in \bigoplus_{j} \mathbb{Q}[\mu] \frac{\partial}{\partial \mu_{j}}$. Then we have the symmetric matrix $-\Gamma J = \begin{bmatrix} \alpha & \beta \\ t_{\beta} & \gamma \end{bmatrix}$, and we also associate with it a second-order differential operator H^{L} , given by

(2.13)
$$H^{L} = \frac{1}{2} \begin{bmatrix} {}^{t}\partial_{u} {}^{t}u \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ {}^{t}\beta & \gamma \end{bmatrix} \begin{bmatrix} \partial_{u} \\ u \end{bmatrix}$$
$$= \sum_{i=1}^{g} \sum_{j=1}^{g} \left(\frac{1}{2} \alpha_{ij} \frac{\partial^{2}}{\partial u_{i}\partial u_{j}} + \beta_{ij} u_{i} \frac{\partial}{\partial u_{j}} + \frac{1}{2} \gamma_{ij} u_{i} u_{j} \right) + \frac{1}{2} \operatorname{Tr} \beta.$$

Here ∂_u denotes the column vector with g components $\frac{\partial}{\partial u_i}$, and u the column vector with g components u_i . It is then straightforward to verify the following

Lemma 2.14. If we define a Green's function $G_0(u,\Omega)$ by

$$G_0(u,\Omega) = \left(\frac{(2\pi)^g}{\det \omega'}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2} {}^t u \, \eta' \omega'^{-1} u\right),$$

then, for any Γ satisfying (2.10), the heat equation holds:

$$(2.15) (L - H^L)G_0 = 0.$$

Proof. We have

$$LG_{0} = -\frac{1}{2} \text{Tr}(\omega'^{-1}L(\omega'))G_{0} - \frac{1}{2} ({}^{t}u L(\eta')\omega'^{-1}u)G_{0} + \frac{1}{2} ({}^{t}u \eta'\omega'^{-1}L(\omega')\omega'^{-1}u)G_{0}$$

$$= \frac{1}{2} \left[\text{Tr}(\omega'^{-1}(\beta\omega' - \alpha\eta')) + ({}^{t}u ((\gamma\omega' - {}^{t}\beta\eta')\omega'^{-1} - \eta'\omega'^{-1}(\beta\omega' - \alpha\eta')\omega'^{-1})u) \right] G_{0}$$

$$= \frac{1}{2} \left[\text{Tr}(\beta - \alpha\eta'\omega'^{-1}) + ({}^{t}u (\gamma - {}^{t}\beta\eta'\omega'^{-1} - \eta'\omega'^{-1}\beta + \eta'\omega'^{-1}\alpha\eta'\omega'^{-1})u) \right] G_{0}.$$

This is the same as

$$H^{L_0}G_0 = \frac{1}{2} \begin{bmatrix} t \partial_u & t u \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ t \beta & \gamma \end{bmatrix} \begin{bmatrix} \partial_u \\ u \end{bmatrix} G_0 = \frac{1}{2} \begin{bmatrix} t \partial_u & t u \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ t \beta & \gamma \end{bmatrix} \begin{bmatrix} -\eta'\omega'^{-1}u \\ u \end{bmatrix} G_0$$
$$= \frac{1}{2} \operatorname{Tr}(t^*\beta - \alpha \eta'\omega'^{-1}) G_0 + \frac{1}{2} \begin{bmatrix} -t u \eta'\omega'^{-1} & u \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ t \beta & \gamma \end{bmatrix} \begin{bmatrix} -\eta'\omega'^{-1}u \\ u \end{bmatrix} G_0.$$

Here we have used one of the generalized Legendre relations and the symmetry of $\eta'\omega'^{-1}$. \square

Now we recall that the different terms in the expansion of the theta function are periodic translates of one another. Analogously, to construct the different terms appearing in the expansion of the sigma function, we act on G_0 by iterating an element of the Heisenberg group. We denote, for a scalar z and two g-component column vectors p, q,

$$F(z, p, q) = \exp(z) \exp({}^{t}pu) \exp({}^{t}q\partial_{u}).$$

We write its inverse operator as

$$F^{-1}(z, p, q) = \exp(-tq\partial_u)\exp(-tpu)\exp(-z).$$

Then we have

Lemma 2.16. Defining F(z, p, q), L, and H^L as above, the operator equality

(2.17)
$$F^{-1}(z, p, q)(L - H^{L})F(z, p, q) = L - H^{L}$$

holds, if and only if z, p, q satisfy

$$L\begin{bmatrix} q \\ p \end{bmatrix} = \Gamma \begin{bmatrix} q \\ p \end{bmatrix}, \quad L(z) = \frac{1}{2} \left({}^t p \alpha p - {}^t q \gamma q \right).$$

Proof. We calculate directly that

$$F^{-1}(z, p, q) LF(z, p, q) = F^{-1}(z, p, q) (L(z) + L({}^tp) u + L({}^tq) \partial_u) F(z, p, q) + L$$
$$= L + (L(z) + L({}^tp) (u - q) + L({}^tq) \partial_u)$$

Similarly, we find

$$\begin{split} F^{-1}(z,p,q) \, H^L F(z,p,q) &= \frac{1}{2} {\begin{bmatrix} }^t \partial_u + {}^t p & {}^t u - {}^t q {\end{bmatrix}} {\begin{bmatrix} \alpha & \beta \\ {}^t \beta & \gamma {\end{bmatrix}} {\begin{bmatrix} \partial_u + p \\ u - q {\end{bmatrix}}} \\ &= H^\Gamma + {\begin{bmatrix} }^t p & - {}^t q {\end{bmatrix}} {\begin{bmatrix} \alpha & \beta \\ {}^t \beta & \gamma {\end{bmatrix}} {\begin{bmatrix} \partial_u \\ u {\end{bmatrix}}} + \frac{1}{2} {\begin{bmatrix} }^t p & - {}^t q {\end{bmatrix}} {\begin{bmatrix} \alpha & \beta \\ {}^t \beta & \gamma {\end{bmatrix}} {\begin{bmatrix} p \\ -q {\end{bmatrix}}}. \end{split}$$

Matching coefficients of ∂_u , u, and 1, and transposing and rearranging, we see that, respectively,

$$L(q) = \begin{bmatrix} -\beta & \alpha \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}, \quad L(p) = \begin{bmatrix} -\gamma & t\beta \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}, \quad L(z) = \frac{1}{2} \begin{pmatrix} tp\alpha p - tq\gamma q \end{pmatrix}$$

as desired. \Box

Corollary 2.18. It follows that, if the formula (2.17) holds, then

where b' and b'' are arbitrary constant vectors, and z_0 is an irrelevant constant, which we set to zero below.

Proof. Since
$$L\left(\Omega^{-1}\begin{bmatrix}q\\p\end{bmatrix}\right) = \mathbf{0}$$
, and $L(z - \frac{1}{2}^t qp) = 0$ by (2.19), we see that $\Omega^{-1}\begin{bmatrix}q\\p\end{bmatrix}$ and $z - \frac{1}{2}^t qp$ are constants.

Denoting the constant vector (b', b'') simply by b, we denote p, q, and z with $z_0 = 0$ in (2.19) by p(b), q(b), and z(b), respectively. We define

$$G(b, u, \Omega) = F(z(b), p(b), q(b)) G_0(u, \Omega).$$

Using the Legendre relations in the form (2.9), we note that ${}^tp\omega' - {}^tq\eta' = 2\pi i {}^tb''$, and hence we obtain

$$G(b, u, \Omega) = F(z(b), p(b), q(b)) G_0(u, \Omega)$$

$$= \exp(\frac{1}{2}t^{p}pq) \exp(t^{p}pu) \exp(-t^{q}q\eta'\omega'^{-1}u) \exp(-\frac{1}{2}t^{q}q\eta'\omega'^{-1}q) G_0$$

$$= \exp(-\frac{1}{2}(2\pi i^{t}b''\omega'^{-1}q) \exp(2\pi i^{t}b''\omega'^{-1}u) G_0$$

$$= \exp(-\frac{1}{2}(2\pi i^{t}b''\omega'^{-1}(\omega'b' + \omega''b'') \exp(2\pi i^{t}b''\omega'^{-1}u) G_0$$

$$= \left(\frac{(2\pi)^g}{\det(\omega')}\right)^{\frac{1}{2}} \exp\left(-\frac{1}{2}tu\eta'\omega'^{-1}u\right) \exp\left(2\pi i(\frac{1}{2}tb''\omega'^{-1}\omega''b'' + tb''(\omega'^{-1}u + b'))\right).$$

Now, the following theorem, which is the foundation of our theory, is obvious from (2.15) and (2.17).

Theorem 2.20. (Primary heat equation) For the function $G(b, u, \Omega)$ above, one has

$$(2.21) (L-HL) G(b, u, \Omega) = 0.$$

2.4 The algebraic heat operators

For the coordinates of the space where the sigma function is defined, we do not use (u_1, \dots, u_g) for subscripts of the variable u, but denote instead

$$u = (u_{w_q}, \cdots, u_{w_1}).$$

That is, the components of u are labeled by their weights, which are the Weierstrass gaps. As in Section 2.3, we suppose $L \in \bigoplus_j \mathbb{Q}[\mu] \frac{\partial}{\partial \mu_j}$. Then we have the representation matrix $\Gamma^L \in \operatorname{sp}(2g,\mathbb{Q}[\mu])$. As a corollary to Theorem 2.20, we have

Corollary 2.22. Let

(2.23)
$$\rho(u) = \sum_{b} G(b, u, \Omega),$$

where b runs through the elements of any set $\subset \mathbb{C}^{2g}$ such that the sum converges absolutely. Then we have

$$(L - H^L) \rho(u, \Omega) = 0.$$

Proof. Since both of L and H^L are independent of b, each term of $\rho(u)$ satisfies (2.21).

Remark 2.24. Assume that \mathcal{C}_{μ} is non-singular. Let Ω be the usual period matrix defined by (1.7), and δ be its Riemann constant. The function defined at (2.2) is written as

(2.25)
$$\tilde{\sigma}(u) = \tilde{\sigma}(u, \Omega) = \sum_{n \in \mathbb{Z}^g} G({}^t[\delta' \ n + \delta''], u, \Omega).$$

Since the imaginary part of $\omega'^{-1}\omega''$ is positive definite, this series converges absolutely. This is a special case of $\rho(u)$ of (2.22).

Because both L and H^L are independent of b, there are infinitely many linearly independent entire functions $\rho(u)$ on \mathbb{C}^g satisfying $(L - H^L)\rho(u) = 0$. Moreover, since, for a fixed b, the function $G(b, u, \Omega)$ is independent of L, we see that, by switching the choice of L, there are infinitely many linearly independent operators of the form $(L - H^L)$ which satisfy $(L - H^L) \rho(b, u, \Omega) = 0$ for some fixed $\rho(u)$.

However, because our aim is to find a method to calculate the power series expansion of the sigma function, we need a more detailed discussion. For our purpose,

- (A1) we need to find operators in the ring $\mathbb{Q}[\mu][\partial_{\mu}, \partial_{u}]$ which annihilate the sigma function (2.3), and
- (A2) we require that any function killed by all such operators belongs to $\mathbb{Q}[\mu][[u_{w_q}, \cdots, u_{w_1}]]$.

Since L is a derivation with respect to the μ_j s but H^L is a differential operator with respect to the u_{w_j} s, we see that, for some function Ξ depending only on the μ_j s,

$$L \Xi \tilde{\sigma}(u) = (L\Xi) \tilde{\sigma}(u) + \Xi (L\tilde{\sigma}(u)), \quad H^L \Xi \tilde{\sigma}(u) = \Xi (H^L \tilde{\sigma}(u)).$$

Therefore, $\Xi \tilde{\sigma}(u)$ satisfies

(2.26)
$$(L - H^{L})(\Xi \tilde{\sigma}(u)) = \frac{L\Xi}{\Xi} \Xi \tilde{\sigma}(u) = (L \log \Xi)\Xi \tilde{\sigma}(u).$$

If $\Xi \tilde{\sigma}(u)$ is the correct sigma function, the left hand side of the above is in $\mathbb{Q}[\mu][u]$, So, if we suppose that the correct $\sigma(u)$ is equal to $\hat{\sigma}(u)$ of (2.3), then we should narrow down the choice

2.5 The operators tangent to the discriminant

Throughout this section, we suppose that all the μ_j s are variables or indeterminates. In view of our approach, we require that $L \log \Delta$ belongs to $\mathbb{Q}[\mu]$ because of conditions (A1), (A2) and equation (2.26). Now we explain suitable choices for Γ for which L satisfies the condition. Let M(x,y) be a vector consisting of 2g monomials of x and y which appear in f(x,y) (with non-zero coefficient) displayed in ascending order of weight (i.e. descending order of absolute value of weight) in the first g entries and in the ascending order (i.e. descending order of absolute value of weight) in the remaining g entries.

For $\{w_1, \dots, w_g\}$ in (1.4) with respect to (e,q), we define a sequence $\{v_j\}_{j=1}^{2g}$ by

$$v_j = \begin{cases} 2g - 1 - w_{g-j+1} & \text{if } 1 \le j \le g, \\ 2g - 1 + w_{j-g} & \text{if } g+1 \le j \le 2g. \end{cases}$$

We denote by $M_j(x,y)$ the monomial in x and y of weight $-v_j$. Then we let

$$M(x,y) = {}^{t}[M_{j}(x,y) \ (j=1,2,\cdots,2g)], \ \check{M}(x,y) = {}^{t}[M_{j}(x,y) \ (j=2g,2g-1,\cdots,1)].$$

This is equivalent to saying that M(x,y) is the vector whose entries are displayed as the elements of the set

$$M(x,y) = \{ x^i y^j \mid 0 \le i \le e - 2, \ 0 \le j \le q - 2 \}.$$

in decreasing order of weight. We see that all the terms $\{M_j(x,y)\}$ appear in f(x,y) provided the *modality* of the curve is 0.

Example 2.27. If (e,q) = (2,2g+1), then M_j s are given as follows:

j	1	2	 g	g+1	g+2	 2g	
w_{g-j+1}	2g - 1	2g - 3	 1	1	3	 2g - 1	w_{j-g}
v_{j}			2g - 2				
$M_j(x,y)$	1	x	 x^{g-1}	x^g	x^{g+1}	 x^{2g-1}	

i 1 2 3 4 5

If e = 3 and q = 4 or 5, then M_i s are given as follows:

(3,4)-curv	· ·	j+1		5 2	2 1	1	2	5	w_{j-1}	3	
(3, 4)-curv	С.	v_{j}) ;	3 4	6	7	10		
		$M_j(x)$	(x,y)) [1 :	x y	x^2	xy	x^2y		
										_	
		j	1	2	3	4	5	6	7	8	
(3,5)-curve:	w_{4-}	-j+1	7	4	2	1	1	2	4	7	w_{j-4}
(3, 3)-curve.	ı	j_j	0	3	5	6	8	9	11	14	
	M_j	(x,y)	1	\boldsymbol{x}	y	x^2	xy	x^3	x^2y	x^3y	

All these examples, and only these, have modality zero.

Example 2.28. In contrast, for e = 3 and q = 7, we have, similarly,

	j	1	2	3	4	5	6	7	8	9	10	11	12	
(3,7)-curve:	w_{6-j+1}	11	8	5	4	2	1	1	2	4	5	8	11	w_{j-6}
(3, 1)-curve.	v_j	0	3	6	7	9	10	12	13	15	16	19	22	
	$M_j(x,y)$	1	x	x^2	y	x^3	xy	x^4	x^2y	x^5	x^3y	x^4y	x^5y	

However, the (full) defining equation of a (3,7)-curve is given by

$$y^{3} - (\mu_{1}x^{2} + \mu_{4}x + \mu_{7})y^{2} - (\mu_{2}x^{4} + \mu_{5}x^{3} + \mu_{8}x^{2} + \mu_{11}x + \mu_{14})y$$
$$= x^{7} + \mu_{3}x^{6} + \mu_{6}x^{5} + \mu_{9}x^{4} + \mu_{12}x^{3} + \mu_{15}x^{3} + \mu_{18}x + \mu_{21}$$

and this equation does not include a term in $M_{12}(x,y) = x^5y$. This curve is of modality 1. We will not discuss this example further in this paper.

Now, for the M(x,y), corresponding to the given curve of modality 0, we define a matrix

$$T = [T_{ij}]_{i,j=1,\cdots,2g}$$

with $\operatorname{wt}(T_{ij}) = -(eq + v_i - v_{2g+1-j})$ by the equalities

(2.29)
$$-eq f(x,y) M_i(x,y) \equiv \sum_{j=1}^{2g} T_{ij} M_{2g+1-j}(x,y) \mod (f_x, f_y)$$
(i.e. $-eq f(x,y) M(x,y) \equiv T \check{M}(x,y) \mod (f_x, f_y)$),

where $f_x = \frac{\partial}{\partial x} f(x,y)$, $f_y = \frac{\partial}{\partial y} f(x,y)$. The factor -eq ensures that the coefficients of L_{v_i} will be integral, and that the signs of the first row and column of T are negative. These T_{ij} are uniquely determined because $\mathbb{Q}[x,y]/(f_x,f_y)$ is a $\mathbb{Q}[\mu]$ -module of rank 2g spanned by M(x,y) and we can reduce the order of x in the left hand side by using $f_x(x,y) = \cdots - qx^{q-1} + \cdots$ and that of y by using $f_y(x,y) = ey^{e-1} + \cdots$.

Now we are going to construct the discriminant Δ and a basis of the space of the vector fields tangent to the variety defined by $\Delta = 0$ by following a method known in singularity theory. First of all, we let T be the representation matrix of the map of -eq f(x, y)-multiplication from the free $\mathbb{Q}[\mu]$ -module of rank 2g

$$\mathbb{Q}[x,y] / (f_1(x,y), f_2(x,y)),$$

where $f_1 = \frac{\partial}{\partial x} f$ and $f_2 = \frac{\partial}{\partial y} f$, to itself with respect to the basis $\{M_i(x,y)\}$ given above. Then the discriminant Δ of $\mathscr C$ is given by the determinant of T: $\Delta = \det(T)$. However, following p.112 of [5], we also define a function: H = H((x,y),(z,w)) is defined by²

$$H = \frac{1}{2} \begin{vmatrix} \frac{f_1(x,y) - f_1(z,w)}{x - z} & \frac{f_2(x,y) - f_2(z,w)}{x - z} \\ \frac{f_1(z,y) - f_1(x,w)}{y - w} & \frac{f_2(z,y) - f_2(x,w)}{y - w} \end{vmatrix}.$$

Now let $[H_{ij}]$ be the matrix defined by

$$H((x,y),(z,w)) = {}^{t}M(x,y) [H_{ij}] M(z,w).$$

Below, we will use, instead of T, the symmetric $2g \times 2g$ matrix $V = [v_{ij}]$ with entries in

²Not to be confused with the heat operator H.

 $\mathbb{Q}[\mu]$ defined by the equation

(2.30)
$${}^{t}M(x,y) V M(z,w) = f(x,y) H$$

in the ring

$$\mathbb{Q}[x,y,z,w] / (f_1(x,y), f_2(x,y), f_1(z,w), f_2(z,w)).$$

We define $[H_{ij}]$ as the matrix given by

$$H((x,y),(z,w)) = {}^{t}M(x,y) [H_{ij}] M(z,w).$$

Lemma 2.31. The matrix $[H_{ij}]$ is of the form

(2.32)
$$[H_{ij}] = \begin{bmatrix} * & \cdots & * & -eq \\ * & \cdots & -eq \\ \vdots & \ddots & \\ -eq & \end{bmatrix}.$$

If e = 2, then we have

$$[H_{ij}] = \begin{bmatrix} -4\mu_{2(q-2)} & -6\mu_{2(q-3)} & -8\mu_{2(q-4)} & \cdots & -2(q-2)\mu_4 & 0 & -2q \\ -6\mu_{2(q-3)} & -8\mu_{2(q-4)} & -10\mu_{2(q-5)} & \cdots & 0 & -2q \\ -8\mu_{2(q-4)} & -10\mu_{2(q-5)} & -12\mu_{2(q-6)} & \cdots & -2q \\ \vdots & \vdots & \vdots & \ddots & \\ -2(q-2)\mu_4 & 0 & -2q \\ 0 & -2q \\ -2q \end{bmatrix}.$$

Proof. Any entry H_{ij} belongs to $\mathbb{Q}[\mu]$. Indeed, by expanding the matrix, it is seen that the numerator

$$(f_1(x,y)f_2(z,y) - f_1(x,y)f_2(x,w) - f_1(z,w)f_2(z,y) + f_1(z,w)f_2(x,w))$$

$$- (f_1(z,y)f_2(x,y) - f_1(z,y)f_2(z,w) - f_1(x,w)f_2(x,y) + f_1(x,w)f_2(z,w))$$

$$= (f_1(x,y)f_2(z,y) - f_1(z,y)f_2(x,y)) - (f_1(x,y)f_2(x,w) - f_1(z,y)f_2(z,w))$$

$$- (f_1(z,w)f_2(z,y) - f_1(x,w)f_2(x,y)) + (f_1(z,w)f_2(x,w) - f_1(x,w)f_2(z,w))$$

$$= (f_1(x,y)f_2(z,y) - f_1(x,w)f_2(z,w)) - (f_1(x,y)f_2(x,w) - f_1(x,w)f_2(x,y))$$

$$- (f_1(z,w)f_2(z,y) - f_1(z,y)f_2(z,w)) + (f_1(z,w)f_2(x,w) - f_1(z,y)f_2(x,y))$$

is divisible by (z-x)(w-y), because the second expression is clearly divisible by (z-x), while the third expression is divisible by (w-y). It is not difficult to check that the entries H_{ij} belong to $\mathbb{Z}[\mu]$ by looking at the terms in f(x,y) precisely. Setting all the μ_j to be 0, we have

$$\frac{1}{2} \left| \frac{q(-x^{q-1} + z^{q-1})}{x - z} \quad \frac{e(y^{e-1} - w^{e-1})}{x - z} \right| \\
\frac{q(-x^{q-1} + z^{q-1})}{y - w} \quad \frac{e(y^{e-1} - w^{e-1})}{y - w} \right| \\
= -eq(x^{q-2} + x^{q-2}z + \dots + z^{q-2})(y^{e-2} + y^{e-2}w + \dots + w^{e-2}).$$

It follows that the counter-diagonal entries of $[H_{ij}]$ are -eq. From the definitions, the weight of H is 2(eq - e - q) and wt $(M_{2g}(x, y)) = 2g - 1 + w_g = 2(eq - q - e)$. Therefore the entries

below the counter-diagonal must be 0. For the case e=2, we have

$$\frac{1}{2} \begin{vmatrix}
-\frac{q(x^{q-1}-z^{q-1})+(q-1)\mu_4(x^{q-2}-z^{q-2})+\cdots+\mu_{eq-e}(x-z)}{x-z} & \frac{2y-2w}{x-z} \\
\frac{q(x^{q-1}-z^{q-1})+(q-1)\mu_4(x^{q-2}-z^{q-2})+\cdots+\mu_{eq-e}(x-z)}{y-w} & \frac{2y-2w}{y-w}
\end{vmatrix}$$

$$= -2 \frac{q(x^{q-1}-z^{q-1})+(q-1)\mu_4(x^{q-2}-z^{q-2})+\cdots+\mu_{eq-e}(x-z)}{x-z}$$

$$= -2 \left(q(x^{q-2}+x^{q-3}z+\cdots+z^{q-2})+(q-1)\mu_4(x^{q-3}+x^{q-4}z+\cdots+z^{q-3})+\cdots+\mu_{eq-e}\right),$$
giving the desired form of $[H_{ij}]$.

Lemma 2.34. We have

$$\det(V) = \det(T)$$
.

Proof. Since

$$f(x,y) H((x,y),(z,w)) = f(x,y)^{t} M(z,w) [H_{jk}] M(x,y)$$

= ${}^{t} \check{M}(z,w) [-\frac{1}{eq} T_{ij}] [H_{jk}] M(x,y),$

we see that V equals $-\frac{1}{eq}T[H_{jk}]$ with sorted rows in reverse order. Since $[H_{jk}]$ is a skew-upper-triangular matrix of the form (2.32), we have demonstrated $\det(V) = \det(T)$ as desired. \square

Remark 2.35. It is easy to see

$$\operatorname{wt}(V_{ij}) = \operatorname{wt}(T_{ij}) = eq - v_j.$$

We now compute the weight of T_{ij} and V_{ij} . If $1 \leq i \leq g$ and $1 \leq j \leq g$, then, by the definition,

$$wt(V_{ij}) = wt(T_{ij}) = -(eq + v_i - v_{2g+1-j})$$

$$= -eq - (2g + 1 - w_{g+1-i}) + (2g + 1 + w_{g+1-j})$$

$$= -eq + w_{g+1-i} + w_{g+1-j}.$$

If $g + 1 \le i \le 2g$ and $1 \le j \le g$, then

$$wt(V_{ij}) = wt(T_{ij}) = -(eq + v_i - v_{2g+1-j})$$

$$= -eq - (2g + 1 + w_{i-g}) + (2g + 1 + w_{g+1-j})$$

$$= -eq - w_{i-g} + w_{g+1-j}.$$

For the other i and j, we have similar formulae.

Lemma 2.36. $\operatorname{wt}(\Delta) = -eq(e-1)(q-1)$.

Proof. This follows from $\operatorname{wt}(T_{ii}) + \operatorname{wt}(T_{2q-i,2q-i}) = -2eq$.

Example 2.37. If (e, q) = (2, 2g + 1), we see, for $1 \le i \le g$ and $1 \le j \le g$, that

$$wt(V_{ij}) = wt(T_{ij}) = -eq + w_{g+1-i} + w_{g+1-j}$$
$$= -2(2g+1) + 2(g+1-i) - 1 + 2(g+1-j) - 1$$
$$= -2(i+j).$$

If $g + 1 \le i \le 2g$ and $1 \le j \le g$, then

$$wt(V_{ij}) = wt(T_{ij}) = -eq - w_{i-g} + w_{g+1-j}$$
$$= -2(2g+1) - 2(i-g) + 1 + 2(g+1-j) - 1 = -2(i+j).$$

For the other i and j, we have the same result.

For any v_i , the coefficient μ_{eq-v_i} appears in the defining equation of the Weierstrass form. Let

(2.38)
$$L_{v_i} = \sum_{i=1}^{2g} V_{ij} \frac{\partial}{\partial \mu_{eq-v_j}} \quad (i = 0, \dots, 2g).$$

Note that $\operatorname{wt}\left(\frac{\partial}{\partial \mu_j}\right) = j$ and $\operatorname{wt}(L_{v_i}) = v_i$.

Lemma 2.39. The determinant det(T) = det(V) is a constant multiple of Δ .

Proof. (From Theorem A8 in [2]) We assume $\mu \subset \mathbb{C}$. Take the map given by multiplication by f(x,y)

$$f(x,y): \mathbb{Q}[\mu][x,y]/(f_x,f_y) \longrightarrow \mathbb{Q}[\mu][x,y]/(f_x,f_y)$$

The vector M(x,y) consists of the elements of a basis of $\mathbb{Q}[\mu][x,y]/(f_x,f_y)$ as a $\mathbb{Q}[\mu]$ -module. The matrix T is no other than the representation matrix with respect to this basis. So $\det(T) = 0$ if and only if the rank³ of the co-kernel $\mathbb{Q}[\mu][x,y]/(f,f_x,f_y)$ of the map is positive. This is exactly the case that the ideal (f,f_x,f_y) does not contain $1 \in \mathbb{Q}[\mu][x,y]$. By Theorem 5.4 i) in [15] (Hilbert's Nullstellensatz), we see this is equivalent to saying that there exists a set $(x,y) \in \mathbb{C}^2$ such that

$$f(x,y) = f_x(x,y) = f_y(x,y) = 0.$$

Therefore the discriminant Δ is a factor of $\det(T)$. Moreover, since the matrix T reflects precisely the rank of $\mathbb{Q}[\mu][x,y]/(f,f_x,f_y)$, $\det(T)$ must be a factor of the discriminant Δ .

On the operators (2.38) and the discriminant, we have the following important result

Proposition 2.40. Any L_{v_i} is tangent to the discriminant Δ , and $(L_{v_i}\Delta)/\Delta \in \mathbb{Q}[\mu]$. Moreover, any operator $D \in \mathbb{Q}[\mu][\partial_{\mu}]$ which is tangent to Δ is a linear combination of the $L_{v_i}s$.

Proof. This follows from Kyoji Saito's theorem (see Theorem A4 in [2]). See also Corollary 3 to Theorem (p.2716) of [25] of Zakalyukin, and Corollary 3.4 in [1]. However, the statement for the cases (e, q) = (2, q), (3, 4) is contained in 2.41.

From 2.40 and Frobenius integrability theorem, we see that the set $\{L_{v_j}\}$ of operators form a basis of a Lie algebra. The subvariety defined by $\Delta = 0$ is the maximal integral manifold of that set. The structure constants of this algebra are polynomials in the μ_i , so it is a polynomial Lie algebra, as discussed in [3]. The corresponding fundamental relations of $\{L_{v_j}\}$ for (e,q)=(2,3), (2,5), (2,7), and (3,4) are available on request.

For any $g \in \mathbb{Q}[x,y]$, we define

$$\operatorname{Hess} g = \begin{vmatrix} \frac{\partial^2}{\partial x^2} g & \frac{\partial^2}{\partial x \partial y} g \\ \frac{\partial^2}{\partial y \partial x} g & \frac{\partial^2}{\partial y^2} g \end{vmatrix}.$$

³the Tjurina number at μ .

Since [6] gives no proof of the following proposition, we give a proof here.

Proposition 2.41. (Buchstaber-Leykin) Let

$$L = [L_{v_1} \ L_{v_2} \ \cdots \ L_{v_{2q}}]^t M(x, y).$$

Then we have

$$(2.42) L(\Delta) = \operatorname{Hess} f \cdot \Delta$$

in the ring $\mathbb{Q}[\mu][x,y]/(f_x,f_y)$.

Proof. For the case e=2, we have f(x,y) is of the form $y^2-p_2(x)$. We denote $p_2(x)=p(x)$ for simplicity. Moreover, we denote $p'(x)=\frac{\partial}{\partial x}p(x)$ and $p''(x)=\frac{\partial^2}{\partial x^2}p(x)$. In this case, $\mathbb{Q}[\mu][x,y]/(f_1,f_2)$ is identified with $\mathbb{Q}[\mu][x]/(p'(x))$ since $f_1(x,y)=\frac{\partial}{\partial y}f(x,y)=2y$. Let F be a splitting field of p(x). We write the factorisation of p(x) in F as $p(x)=(x-a_1)\cdots(x-a_q)$. Then μ_{2i} is $(-1)^i$ times the fundamental symmetric function of a_1,\cdots,a_q of degree i. Of course the ring $\mathbb{Q}[\mu]$ is a sub-ring of $\mathbb{Q}[a_1,\cdots,a_q]$. The main idea is to consider $\frac{\mathrm{Hess}\,f}{f}=-\frac{p''(x)}{p(x)}$ in the localized ring

$$(F[x]/(p'(x)))_{p(x)}$$

of F[x]/(p'(x)) with respect to the multiplicative set $\{1, p(x), p(x)^2, \dots\}$ (see [15], Section 4). The following calculation is done in the localized ring above, by which we see $(F[x]/(p'(x)))_{p(x)} = F[x]/(p'(x))$.

$$\frac{p''(x)}{p(x)} = \sum_{(i,j),i < j} \frac{2}{(x - a_i)(x - a_j)} = \sum_{(i,j),i < j} \frac{2}{a_i - a_j} \left(\frac{1}{x - a_i} - \frac{1}{x - a_j}\right)$$

$$= 2\sum_{i=1}^q \left(\sum_{j \neq i} \frac{1}{a_i - a_j}\right) \frac{1}{x - a_i} = -2\sum_{i=1}^q \left(\sum_{j \neq i} \frac{1}{a_i - a_j}\right) \frac{1}{p'(a_i)} \frac{-p'(a_i)}{x - a_i}$$

$$= -2\sum_{i=1}^q \left(\sum_{j \neq i} \frac{1}{a_i - a_j}\right) \frac{1}{p'(a_i)} \frac{p'(x) - p'(a_i)}{x - a_i} = -2\sum_{i=1}^q c_i \frac{p'(x) - p'(a_i)}{x - a_i},$$

where

$$c_i = \left(\sum_{j \neq i} \frac{1}{a_i - a_j}\right) \frac{1}{p'(a_i)}.$$

Since $\operatorname{Hess} f(x,y) = -p''(x)$ and p(x) = -f(x,y) in the localized ring, it is sufficient to show that

$$\frac{\partial}{\partial \mu_{2i}} \log \Delta = \sum_{j=1}^{q} c_j a_j^{q-i}$$

up to some non-zero constant multiple. Indeed, if we have the formula above, we have

$$\frac{L(x)\Delta}{\Delta} = \sum_{i,k} M_i(x,y) v_{ik} \frac{\partial}{\partial \mu_{2i}} \log \Delta = \sum_{i,k} M_i(x,y) v_{ik} \sum_{j=1}^q c_j a_j^{q-k}$$

$$= \sum_{j=1}^q c_j \sum_{i,k} M_i(x,y) v_{ik} a_j^{q-k} = \sum_{j=1}^q c_j f(x,y) H((x,y), (a_j,0))$$

$$= f(x,y) \sum_{j=1}^q c_j 2 \frac{p'(x) - p'(a_j)}{x - a_j} = f(x,y) \frac{p''(x)}{p(x)} = -p''(x) = \text{Hess } f(x,y).$$

To calculate $\frac{\partial \log(\Delta)}{\partial \mu_{2i}}$, we remove the assumption $\mu_2 = 0$. Since Δ is some non-zero constant multiple of

$$\prod_{i < j} (a_i - a_j)^2,$$

we easily get the $q \times q$ -matrix $\left(\frac{\partial \mu_{2i}}{\partial a_j}\right)$, and then we get $\frac{\partial \log(\Delta)}{\partial \mu_{2i}}$ by using its inverse matrix. For $(e,q)=(3,4),\ (3,5)$, we know only a proof by direct calculation with Maple by using explicit Δ and the operators L_{v_i} s.

2.6 Proof of $\sigma(u) = \hat{\sigma}(u)$

Now we start to show the expression (2.3). From the definition of L_0 and 2.35, we see that

(2.43)
$$L_0 = \sum_{j} (eq - v_j) \mu_{eq - v_j} \frac{\partial}{\partial \mu_{eq - v_j}}.$$

and $L_0(F(\mu)) = -\text{wt}(F(\mu)) F(\mu)$ for any homogeneous form $F(\mu) \in \mathbb{Q}[\mu]$.

Lemma 2.44.

$$L_0{}^t\omega = \left[egin{array}{c|cccc} -w_g & & & & & & \\ & \ddots & & & & & \\ & & -w_1 & & & & \\ & & & w_g & & & \\ & & & \ddots & & \\ & & & & w_1 \end{array}
ight]{}^t\omega = \Gamma_0{}^t\omega$$

on $H^1(\mathscr{C}, \mathbb{Q}[\mu])$.

Proof. For any 1-form of homogeneous weight w

$$\omega = \sum_{j} c_j t^{j+w} dt,$$

where t is a local parameter at ∞ of weight 1, $c_j = c_j(\mu) \in \mathbb{Q}[\mu]$, and $\operatorname{wt}(c_j) = -j$, we see

$$d\left(\sum_{j} c_{j} t^{j+w} dt\right) = d\left(\sum_{j} c_{j} t^{j} dt \cdot t^{w}\right)$$

$$= d\left(\sum_{j} c_{j} t^{j} dt\right) \cdot t^{w}\right) + \left(\sum_{j} c_{j} t^{j} dt\right) \cdot d(t^{w})$$

$$= \left(\sum_{j} j c_{j} t^{j-1} dt\right) \cdot t^{w} + \left(\sum_{j} c_{j} t^{j} dt\right) \cdot w t^{w-1}$$

$$= L_{0}\left(\sum_{j} c_{j} t^{j+w-1} dt\right) + w \sum_{j} c_{j} t^{j+w-1} dt.$$

So that

$$L_0(\omega) \equiv -\operatorname{wt}(\omega) \omega \mod d(\mathbb{Q}[\mu][[t]]dt).$$

Since $\operatorname{wt}(\omega_i) = i$ and $\operatorname{wt}(\eta_{-i}) = -i$, the statement is now obvious.

We would like to reproduce the proof of Lemma 4.17 in [6], that is

$$\sigma(u) = \Delta^{-\frac{1}{8}} \tilde{\sigma}(u) \ (= \hat{\sigma}(u)).$$

If we had a direct explanation of why the sigma function is given as $\Delta^{-M}\tilde{\sigma}(u)$ with some constant M, the theory of heat equation of the sigma function would become very clear. Unfortunately, the authors know only of a proof which starts by narrowing the range of possible existence of the sigma function as follows.

First of all, the function $\sigma(u)$ characterized in 2.1, is a power series of homogeneous weight, which must be written as

$$\sigma(u) = u_1^{(e^2 - 1)(q^2 - 1)/24}$$

$$(2.45) \qquad \sum_{\{n_{eq-v_j}, \ell_{w_i}\}} a(\ell_{w_2}, \cdots, \ell_{w_g}, n_{eq-v_1}, \cdots, n_{eq-v_{2g}}) \prod_{j=1}^{2g} (\mu_{eq-v_j} u_1^{eq-v_j})^{n_{eq-v_j}} \prod_{i=2}^{g} \left(\frac{u_{w_i}}{u_1^{w_i}}\right)^{\ell_{w_i}},$$

where the $a(\ell_{w_2}, \dots, \ell_{w_g}, n_{eq-v_1}, \dots, n_{eq-v_{2g}})$'s are absolute constants and the set of 3g-1 variables $\{n_{eq-v_j}, \ell_{w_i}\}$ runs through the non-negative integers such that

$$\frac{(e^2 - 1)(q^2 - 1)}{24} - \sum_{i=2}^{g} \ell_{w_i} + \sum_{i=1}^{2g} n_{eq - v_j} \ge 0.$$

It is obvious that the operator

(2.46)
$$\sum_{j=1}^{g} (eq - v_j) \mu_{eq - v_j} \frac{\partial}{\partial \mu_{eq - v_j}} - \sum_{j=1}^{g} w_j u_{w_j} \frac{\partial}{\partial u_{w_j}} + \operatorname{wt}(\sigma(u))$$

$$= L_0 - \sum_{j=1}^{g} w_j u_{w_j} \frac{\partial}{\partial u_{w_j}} + \operatorname{wt}(\sigma(u)) = L_0 - \sum_{j=1}^{g} w_j u_{w_j} \frac{\partial}{\partial u_{w_j}} + \frac{(e^2 - 1)(q^2 - 1)}{24}.$$

kills the series (2.45).

Now, we assume that there exists a function Ξ of the μ_j s which is independent of the u_j s such that

$$\sigma(u) = \Xi \,\tilde{\sigma}(u).$$

From (2.26) and the condition (A1), (A2) in Section 2.4, the operators L_{v_j} s should be tangent to Ξ . As in the discussion just after 2.40, we see Ξ should be a power of the discriminant Δ times a numerical constant by the latter part of 2.40. So, $\sigma(u)$ would be written as $\sigma(u) = \Delta^{-M} \tilde{\sigma}(u)$ with a numerical constant M. Then, by (2.26), we have

$$(2.47) (L_0 - H^{L_0} + ML_0 \log \Delta) \sigma(u) = 0.$$

By Lemma 2.44 and (2.13) (or by 2.36 and (2.43)), we have

$$H^{L_0} = -\sum_{j=1}^{g} w_j \, u_{w_j} \frac{\partial}{\partial u_{w_j}} - \frac{1}{2} \sum_{j=1}^{g} w_j.$$

On the other hand, (2.42) and (2.32) yield that

$$L_0 \Delta = \operatorname{wt}(\Delta) \Delta = eq(e-1)(q-1) \cdot \Delta.$$

Then

(2.48)
$$L_0 - H^{L_0} + ML_0 \log \Delta = L_0 - H^{L_0} + M \operatorname{wt}(\Delta)$$

$$= L_0 - \sum_{j=1}^g w_j u_{w_j} \frac{\partial}{\partial u_{w_j}} - \frac{1}{2} \left(\frac{eq(e-1)(q-1)}{4} - \frac{(e^2-1)(q^2-1)}{12} \right)$$

$$+ Meq(e-1)(q-1).$$

Therefore the operator (2.48) should equal to the operator (2.46). Hence,

$$-\frac{1}{2}\left(\frac{eq(e-1)(q-1)}{4} - \frac{(e^2-1)(q^2-1)}{12}\right) + Meq(e-1)(q-1) = \frac{(e^2-1)(q^2-1)}{24},$$

and it follows that $M = \frac{1}{8}$ as desired.

In the rest of the paper, we denote

$$H_{v_i} = H^{L_{v_i}} + \frac{1}{8}L_{v_i}\log\Delta.$$

Then, (2.26) implies the following.

Theorem 2.49. One has

$$(L_{v_j} - H_{v_j})\hat{\sigma}(u) = 0 \quad (j = 1, \dots, 2g).$$

In Section 3, for (e,q)=(2,3), (2,5), (2,7),and (3,4),we will solve the system of equations

$$(2.50) (L_{v_i} - H_{v_i}) \varphi(u) = 0 (i = 1, \dots, 2g)$$

for an unknown holomorphic function $\varphi(u)$, and below we will show that the solution space of this system is of dimension 1, and any solution satisfies the properties of $\sigma(u)$ in (2.1). Hence we have proved $\sigma(u) = \hat{\sigma}(u)$.

From now on, though out this paper, we denote

$$\Gamma_{v_j} = \Gamma^{L_{v_j}}$$
.

Especially, $\Gamma_0 = \Gamma^{L_0} = \Gamma^{L_{v_1}}$.

Remark 2.51. As noted above in 2.11, our notation differs from that of Buchstaber and Leykin; we denote the matrix Γ_j in p.274 of [6] by $\Gamma_j^{\rm BL}$ and we define the sub-matrices of $-J\Gamma_j^{\rm BL}$ and $\Gamma_{v_j}J$ by

$$-J\Gamma_{j}^{\mathrm{BL}} = \begin{bmatrix} \alpha_{j}^{\mathrm{BL}} & t(\beta_{j}^{\mathrm{BL}}) \\ \beta_{j}^{\mathrm{BL}} & \gamma_{j}^{\mathrm{BL}} \end{bmatrix}, \quad \Gamma_{v_{j}}J = \begin{bmatrix} \alpha_{v_{j}} & \beta_{v_{j}} \\ t(\beta_{v_{j}}) & \gamma_{v_{j}} \end{bmatrix}$$

by following the notation of [6] and the present paper. Then we have

$$\alpha_j^{\mathrm{BL}} = \alpha_{v_j}, \quad \beta_j^{\mathrm{BL}} = {}^t(\beta_{v_j}), \quad \gamma_j^{\mathrm{BL}} = \gamma_{v_j}, \quad \Gamma_j^{\mathrm{BL}} = {}^t(\Gamma_{v_j}).$$

3 Solving the heat equations

3.1 General results for the (2, q)-curve

In this section, we discuss the hyperelliptic case e = 2. Firstly, we give the explicit expression for the entries of the matrix V of (2.30).

Lemma 3.1. We have

$$V_{ij} = -\frac{2i(q-j)}{q} \mu_{2i}\mu_{2j} + \sum_{m=1}^{m_0} 2(j-i+2m) \mu_{2(i-m)} \mu_{2(j+m)}$$
$$= -\frac{2i(q-j)}{q} \mu_{2i}\mu_{2j} + \sum_{\ell=\ell_0}^{i-1} 2(i+j-2\ell) \mu_{2\ell} \mu_{2(i+j-\ell)},$$

where $\mu_0 = 1$, $\mu_2 = 0$, $m_0 = \min\{i, q - j\}$, and $\ell_0 = \max\{0, i + j - q\}$.

Proof. First of all, assuming the first equality, we show the second equality. To change the first expression to the second with summation to i-1, we use the substitution $\ell=i-m$. It is obvious that the second equality with summation to i-1 is equal to one with summation to j for $i=j,\,j+1$. For the case on i< j, the difference of the two is expressed as

$$\sum_{\ell=i}^{j} 2(i+j-2\ell) \,\mu_{2\ell} \,\mu_{2(i+j-\ell)} = -\sum_{\ell'=i}^{j} 2(i+j-2\ell') \,\mu_{2(i+j-\ell')} \,\mu_{2\ell'}$$

setting $\ell' = i + j - \ell$, it is clear that it vanishes. We see the case j < i in a similar way. The matrix $V = [V_{ij}]$ is symmetric by definition. However, if the Lemma is proved, we see this directly, by subtracting the term for $\ell = j$ from the first term. Now, noting that in the hyperelliptic case, the $M_j(x,y)$ are independent of x, we define $M^{(i)} = M^{(i)}(x) \in \mathbb{Z}[\mu][x]$ by using $[H_{ij}]$ of (2.33):

$$M^{(i)}(x) = \sum_{j=1}^{q-1} H_{q-i,j} M_j(x,y) = \sum_{m=0}^{i-1} 2(q+1+m-i) \,\mu_{2(i-m-1)} \, x^m.$$

While we are treating $f(x,y) = y^2 - p_2(x)$, we denote $p_2(x)$ by p(x) in this proof, for a less cumbersome notation.

Since $\mathbb{Q}[\mu][x,y]/(f_x(x,y),f_y(x,y)) = \mathbb{Q}[\mu][x,y]/(p'(x),2y)$, which is isomorphic to $\mathbb{Q}[\mu][x]/(p'(x))$, it suffice to know explicitly the residue $V^{(i)} = V^{(i)}(x)$ of degree less than q-1 of the division $p(x) M^{(i)}(x)$ by p'(x) for $1 \le i \le q-1$. The key to this proof is that we actually know the quotient $Q^{(i)} = Q^{(i)}(x) \in \mathbb{Q}[\mu][x]$, as well as $V^{(i)}$, of this division! Namely, we will show that, if we define functions

$$Q^{(i)}(x) = \sum_{m=1}^{i} 2\mu_{2(i-m)} x^m + \frac{2i}{q} \mu_{2i},$$

then the expression

(3.2)
$$V^{(i)}(x) = p(x) M^{(i)}(x) - p'(x) Q^{(i)}(x)$$

is of degree less than q-1. Moreover, we can calculate all the terms of $V^{(i)}$ explicitly, which are no other than the V_{ij} s.

Let us start to calculate each term of x^k of the right hand side of (3.2) for any $k \geq 0$. We

divide the calculation into four cases.

(i) The case $k \geq q$.

In this case $M^{(i)}$ has terms only up to x^{i-1} $(i-1 \le q-2 < q \le k)$, and p(x) has terms up to x^q $(q \le k)$. Therefore, we find that the coefficient C_k of x^k in $M^{(i)}(x)$ p(x) is given by

$$C_k = \sum_{m=k-q}^{i-1} 2(q+1-i+m) \,\mu_{2(i-1-m)} \,\mu_{2(q-k+m)}$$
$$= \sum_{m'=q-1-k}^{i} 2(k-m'+1) \,\mu_{2(q-1-k+m')} \,\mu_{2(i-m')},$$

where we have changed the summation index by q - k + m = i - m'. On the other hand, $Q^{(i)}$ has terms up to x^i $(i \le q - 1 < q \le k)$, and p'(x) has terms up to x^{q-1} (q - 1 < k), so we see

"Coeff. of
$$x^k Q^{(i)}(x) p'(x)$$
" = $\sum_{m=k-q+1}^{i} 2 \mu_{2(i-m)} \mu_{2(q-1-k+m)}(k-m+1)$.

So the right hand side of (3.2) has no term in x^k for $k \geq q$.

(ii) The case k = q - 1.

Since $M^{(i)}$ has terms only up to x^{i-1} $(i-1 \le q-2 < q-1 = k)$, we see that

"Coeff. of
$$x^k$$
 $M^{(i)}(x) p(x)$ " = $\sum_{m=0}^{i-1} 2(q+1-i+m) \mu_{2(i-1-m)} \mu_{2(1+m)}$
= $\sum_{m'=0}^{i-1} 2(q-m') \mu_{2m'} \mu_{2(i-m')}$
= $\sum_{m'=1}^{i-1} 2(q-m') \mu_{2m'} \mu_{2(i-m')} + 2q \mu_0 \mu_{2i}$,

where we have changed the index of summation by m+1=i-m'. In this case $Q^{(i)}$ has terms up to x^i $(i \leq q-1=k)$, and p'(x) has terms up to x^{q-1} (q=k), we have the coefficient C_k x^k in $Q^{(i)}(x)$ p'(x) is given by

$$C_k = \sum_{m=1}^{i} 2 \mu_{2(i-m)} \mu_{2m}(q-m) + \frac{2i}{q} \mu_{2i} \mu_0 q$$

$$= \sum_{m=1}^{i-1} 2 \mu_{2(i-m)} \mu_{2m}(q-m) + 2(q-i) \mu_0 \mu_{2i} + 2i \mu_{2i} \mu_0.$$

So the right hand side of (3.2) has no term of x^{q-1} .

(iii) The case i - 1 < k < q - 1.

Since $M^{(i)}$ has terms only up to x^{i-1} , we see that the coefficient D_k of x^k in $M^{(i)}(x) p(x)$ is given by

$$C_k = \sum_{m=0}^{i-1} 2(q+1+m-i) \,\mu_{2(i-m-1)} \,\mu_{2(q-k+m)}$$
$$= \sum_{m=1}^{i} 2(q+m-i) \,\mu_{2(i-m)} \,\mu_{2(q-1-k+m)}$$

by rewriting m to m-1. On the other hand, the coefficient C_k of x^k in $Q^{(i)}(x) p'(x)$

$$C_k = \frac{2i}{q} \mu_{2i} (k+1) \mu_{q-1-k} + \sum_{m=1}^{i} 2 \mu_{2(i-m)} (k-m-1) \mu_{2(q-1-k+m)}$$

So the coefficient of x^k in right hand side of (3.2) is

$$\sum_{m=1}^{i} 2(q-1-k+2m-i) \,\mu_{2(i-m)} \,\mu_{2(q-1-k+m)} + \frac{2i}{q}(k+1)\mu_{2i} \,\mu_{2(q-1-k)}$$

and V_{ij} , which is no other than the value of this at k = q - 1 - j, is given by

$$\sum_{m=1}^{i} 2(j+2m-i) \,\mu_{2(i-m)} \,\mu_{2(j+m)} + \frac{2i}{q} (q-j) \mu_{2i} \,\mu_{2j}$$

as desired since i < k + 1 = q - j.

(iv) The case k < i - 1.

Since $M^{(i)}$ has terms up to x^{i-1} , of higher degree than x^k , we see that the coefficient D_k of x^k in $M^{(i)}(x) p(x)$ is given by

$$D_k = \sum_{m=0}^{k} 2(q+1+m-i) \,\mu_{2(i-m-1)} \,\mu_{2(q-k+m)} = \sum_{m=1}^{k+1} 2(q+m-i) \,\mu_{2(i-m)} \,\mu_{2(q-1-k+m)}$$

on replacing the summation index m + 1. Similarly, $Q^{(i)}$ has terms up to x^i exceeding x^k again, and

"Coeff. of
$$x^k$$
 $p'(x) Q^{(i)}(x)$ " = $\frac{2i}{q} \mu_{2i} \mu_{q-1-k} + \sum_{m=1}^{k+1} 2 \mu_{2(i-m)} (k-m-1) \mu_{2(q-1-k+m)}$

with an extra term for m = k + 1 which is zero. So the coefficient of x^k in right hand side of (3.2) is

$$\sum_{m=1}^{k+1} 2(q-1-k+2m-i)\,\mu_{2(i-m)}\,\mu_{2(q-1-k+m)} + \frac{2i}{q}(k+1)\mu_{2i}\,\mu_{2(q-1-k)}$$

and then V_{ij} , which is no other than the value of this at k = q - 1 - j, is given by

$$V_{ij} = \sum_{m=1}^{q-j} 2(j+2m-i) \,\mu_{2(i-m)} \,\mu_{2(j+m)} + \frac{2i}{q} (q-j) \mu_{2i} \,\mu_{2j}$$

as desired since $q - j = k + 1 \le i$.

Lemma 3.3. We have

$$L_{v_j}(\Delta) = 2(q-j)(q+1-j)\mu_{2j} \Delta$$
 (for $j = q-1, q-2, \dots, 2, 1$).

Proof. Since the Hessian of $f(x,y) = y^2 - p(x)$ is

$$\begin{vmatrix} p_2''(x) & 0 \\ 0 & 2 \end{vmatrix} = 2 p_2''(x) = 2 (q(q-1)x^{q-2} + (q-2)(q-3)\mu_4 x^{q-4} + \dots + 2 \cdot 1 \mu_{q-3}),$$

this lemma follows from (2.42).

3.2 Heat equations for the (2,3)-curve

In this section we recall Weierstrass' result which gives a recursive relation for the coefficients of the power series expansion of his sigma function at the origin. We refer the reader to (12) and (13) in p. 314 of [10] also. Here we derive Weierstrass' result by following the method of [6], namely, following the theory described in previous sections, but without using the general results 3.1 and 3.3, in order to demonstrate the ideas of the theory.

Weierstrass' original method is explained in [24] and some explanation of it is available in [20]. It is easy to get L_0 and L_2 :

(3.4)
$$V = \begin{bmatrix} 4\mu_4 & 6\mu_6 \\ 6\mu_6 & -\frac{4}{3}\mu_4^2 \end{bmatrix}, \text{ and } \begin{cases} L_0 = 4\mu_4 \frac{\partial}{\partial \mu_4} + 6\mu_6 \frac{\partial}{\partial \mu_6} \\ L_2 = 6\mu_6 \frac{\partial}{\partial \mu_4} - \frac{4}{3}\mu_4^2 \frac{\partial}{\partial \mu_6} \end{cases}$$

In this case, we see V = T since

$$H((x,y),(z,w)) = -6x - 6z = \begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} -6 \\ -6 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}.$$

The differential forms

$$\omega_1 = \frac{dx}{2y}, \quad \eta_{-1} = \frac{xdx}{2y}$$

form a symplectic basis of $H^1(\mathscr{C}_{\mu}, \mathbb{Q}[\mu])$. We have $\boldsymbol{\omega} = (\omega_1, \eta_{-1})$. Bearing in mind Lemma 2.5, we proceed by using $x^{-\frac{1}{2}}$ as the local parameter satisfying that $\frac{\partial}{\partial \mu_j} x = 0$ for j = 4, 6, and we compute the matrix Γ as follows. Using f(x,y) = 0, we see $2y \frac{\partial}{\partial \mu_4} y = x$ and $2y \frac{\partial}{\partial \mu_6} y = 1$, so that

$$\frac{\partial}{\partial \mu_4} y = \frac{x}{2y}, \quad \frac{\partial}{\partial \mu_6} y = \frac{1}{2y}.$$

Therefore, we have

$$\frac{\partial}{\partial \mu_6} \omega_1 = -\frac{1}{4y^3} dx, \quad \frac{\partial}{\partial \mu_4} \omega_1 = \frac{\partial}{\partial \mu_6} \eta_{-1} = -\frac{x}{4y^3} dx, \quad \frac{\partial}{\partial \mu_4} \eta_{-1} = -\frac{x^2}{4y^3} dx.$$

In the calculation below, the symbol "≡" stands for an equality modulo the exact forms. We have

$$(3.5) d\left(\frac{1}{y}\right) = -\frac{1}{y^2}dy = -\frac{1}{y^2}\frac{dy}{dx}dx = -\frac{1}{y^2}\frac{3x^2 + \mu_4}{2y}dx = 6\frac{\partial}{\partial\mu_4}\eta_{-1} + 2\mu_4\frac{\partial}{\partial\mu_6}\omega_1,$$

$$d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2} = \frac{y - x\frac{dy}{dx}}{y^2}dx = \frac{y - x\frac{3x^2 + \mu_4}{2y}}{y^2}dx = \frac{y - \frac{3y^2 - 2\mu_4x - 3\mu_6}{2y}}{y^2}dx$$

$$= -\omega_1 - 4\mu_4\frac{\partial}{\partial\mu_4}\omega_1 - 6\mu_6\frac{\partial}{\partial\mu_6}\omega_1 = -\omega_1 - L_0\omega_1,$$

$$d\left(\frac{x^2}{y}\right) = \frac{2xydx - x^2dy}{y^2} = \frac{2xy - x^2\frac{dy}{dx}}{y^2}dx = \frac{2xy - x^2\frac{3x^2 + \mu_4}{2y}}{y^2}dx$$

$$= \frac{2xy - \frac{3x(y^2 - \mu_4x - \mu_6) + \mu_4}{2y}}{y^2}dx = \frac{xdx}{2y} + \frac{\mu_4x^2}{y^3}dx + \frac{3\mu_6x}{2y^3}dx = \eta_{-1} - L_0\eta_{-1}$$

$$\equiv \eta_{-1} + \frac{4}{3}\mu_4^2\frac{\partial}{\partial\mu_6}\omega_1 - 6\mu_6\frac{\partial}{\partial\mu_4}\omega_1 \quad (\text{by (3.5)})$$

$$= \eta_{-1} - L_2\omega_1.$$

Therefore, we see

$$L_{2}\eta_{-1} = 6\mu_{6}\frac{\partial}{\partial\mu_{4}}\eta_{-1} - \frac{4}{3}\mu_{4}^{2}\frac{\partial}{\partial\mu_{6}}\eta_{-1} \equiv -2\mu_{6}\mu_{4}\frac{\partial}{\partial\mu_{6}}\omega_{1} - \frac{4}{3}\mu_{4}^{2}\frac{\partial}{\partial\mu_{6}}\omega_{1} \quad (\text{by } (3.5))$$

$$= -\frac{\mu_{4}}{3}\left(6\mu_{6}\frac{\partial}{\partial\mu_{6}}\omega_{1} + 4\mu_{4}\frac{\partial}{\partial\mu_{4}}\omega_{1}\right) = -\frac{\mu_{4}}{3}L_{0}\omega_{1} = \frac{\mu_{4}}{3}\omega_{1}.$$

Summarising these results, we have on $H^1(\mathscr{C}, \mathbb{Q}[\mu])$ that

$$L_0{}^t\omega = \Gamma_0{}^t\omega, \ L_2{}^t\omega = \Gamma_2{}^t\omega, \ \text{where} \ \Gamma_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \Gamma_2 = \begin{bmatrix} 1 \\ \frac{\mu_4}{3} \end{bmatrix}$$

Note that, by these equation, we have $L_j\Omega = \Gamma_j\Omega$ with $\Omega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$ as in (2.12). Since $L_0 \det(T) = 12 \det(T)$, $L_2 \det(T) = 0$, $\Delta = \det(T)$, and (2.44), we have arrived at

(3.6)
$$(L_0 - H_0)\sigma(u) = \left(4\mu_4 \frac{\partial}{\partial \mu_4} + 6\mu_6 \frac{\partial}{\partial \mu_6} - u \frac{\partial}{\partial u} + 1\right)\sigma(u) = 0,$$

$$(L_2 - H_2)\sigma(u) = \left(6\mu_6 \frac{\partial}{\partial \mu_4} - \frac{4}{3}\mu_4^2 \frac{\partial}{\partial \mu_6} - \frac{1}{2}\frac{\partial^2}{\partial u^2} + \frac{1}{6}\mu_4 u^2\right)\sigma(u) = 0.$$

where $H_j = H^{L_j} + \frac{1}{8}L_j \log \Delta$ for j = 0 and 2. From the first of (3.6) it follows that the sigma function is of the form

$$\sigma(u) = u \sum_{n_4, n_6 \ge 0} b(n_4, n_6) \frac{(\mu_4 u^4)^{n_4} (\mu_6 u^6)^{n_6}}{(1 + 4n_4 + 6n_6)!}.$$

Using the second equation we then have a recurrence relation

(3.7)
$$b(n_4, n_6) = \frac{2}{3}(4n_4 + 6n_6 - 1)(2n_4 + 3n_6 - 1)b(n_4 - 1, n_6) - \frac{8}{3}(n_6 + 1)b(n_4 - 2, n_6 + 1) + 12(n_4 + 1)b(n_4 + 1, n_6 - 1)$$

with $b(n_4, n_6) = 0$ if $n_4 < 0$ or $n_6 < 0$. Since the term $b(n_4, n_6)$ on the left hand side has weight $4n_4 + 6n_6$, and the terms b(i, j) on the right hand side have weight smaller than this, all terms may be found from (3.7). Therefore, any solution of (3.6) is a constant times the function

$$\sigma(u) = u + 2\mu_4 \frac{u^5}{5!} + 24\mu_6 \frac{u^7}{7!} - 36\mu_4^2 \frac{u^9}{9!} - 288\mu_4 \mu_6 \frac{u^{11}}{11!} + \cdots$$

3.3 Heat equations for the (2,5)-curve

In this section, we list the analogous results on the heat equations for the curve

$$\mathscr{C}_{\mu}: y^2 = x^5 + \mu_4 x^3 + \mu_6 x^2 + \mu_8 x + \mu_{10}.$$

We note here that our results correct a sign in [5]; the overall constant $\frac{1}{80}$ at the 4th line from bottom in page 68 of [5] should be $-\frac{1}{80}$. Here we give the Hurwitz series version of the algorithm. Now, we take a usual symplectic basis of differentials

$$\omega_3 = \frac{1}{2y}dx$$
, $\omega_1 = \frac{x}{2y}dx$, $\eta_{-3} = \frac{3x^3 + \mu_4 x}{2y}dx$, $\eta_{-1} = \frac{x^2}{2y}dx$.

of $H^1(\mathscr{C}_{\mu}, \mathbb{Q}[\mu])$. The matrix V for this case is given by

$$V = \begin{bmatrix} 4\mu_4 & 6\mu_6 & 8\mu_8 & 10\mu_{10} \\ 6\mu_6 & -\frac{12}{5}\mu_4^2 + 8\mu_8 & -\frac{8}{5}\mu_4\mu_6 + 10\mu_{10} & -\frac{4}{5}\mu_4\mu_8 \\ 8\mu_8 & -\frac{8}{5}\mu_4\mu_6 + 10\mu_{10} & -\frac{12}{5}\mu_6^2 + 4\mu_4\mu_8 & 6\mu_4\mu_{10} - \frac{6}{5}\mu_6\mu_8 \\ 10\mu_{10} & -\frac{4}{5}\mu_4\mu_8 & 6\mu_4\mu_{10} - \frac{6}{5}\mu_6\mu_8 & 4\mu_{10}\mu_6 - \frac{8}{5}\mu_8^2 \end{bmatrix}.$$

So the operators L_i are given by

$${}^{t}[L_{0}\ L_{2}\ L_{4}\ L_{6}] = V\ {}^{t}\left[\frac{\partial}{\partial\mu_{4}}\ \frac{\partial}{\partial\mu_{6}}\ \frac{\partial}{\partial\mu_{8}}\ \frac{\partial}{\partial\mu_{10}}\right].$$

While the authors have the explicit commutation relations of these L_i , we shall not include these here because their explicit forms are not needed in this paper. However, these commutators are all in the span of the L_i . By (2.41), we see that these L_j 's operate on the discriminant Δ as follows:

$$\begin{bmatrix} L_0 & L_2 & L_4 & L_6 \end{bmatrix} \Delta = \begin{bmatrix} 40 & 0 & 12 \mu_4 & 4 \mu_6 \end{bmatrix} \Delta.$$

The representation matrices Γ_j for the L_j acting on the space $H^1(\mathscr{C}_{\mu}, \mathbb{Q}[\mu])$ are

Therefore, we find the following operators H_i :

$$\begin{split} H_0 &= 3u_3\frac{\partial}{\partial u_3} + u_1\frac{\partial}{\partial u_1} - 3, \\ H_2 &= \frac{1}{2}\frac{\partial^2}{\partial u_1^2} + u_1\frac{\partial}{\partial u_3} - \frac{4}{5}\mu_4u_3\frac{\partial}{\partial u_1} - \frac{3}{10}\mu_4u_1^2 - \left(\frac{3}{2}\mu_8 - \frac{2}{5}\mu_4^2\right)u_3^2, \\ H_4 &= \frac{\partial^2}{\partial u_1\partial u_3} - \frac{6}{5}\mu_6u_3\frac{\partial}{\partial u_1} + \mu_4u_3\frac{\partial}{\partial u_3} - \frac{1}{5}\mu_6u_1^2 + \mu_8u_1u_3 + \left(3\mu_{10} - \frac{3}{5}\mu_4\mu_6\right)u_3^2 - \mu_4, \\ H_6 &= \frac{1}{2}\frac{\partial^2}{\partial u_3^2} - \frac{3}{5}\mu_8u_3\frac{\partial}{\partial u_1} - \frac{1}{10}\mu_8u_1^2 + 2\mu_{10}u_3u_1 - \frac{3}{10}\mu_8\mu_4u_3^2 - \frac{1}{2}\mu_6. \end{split}$$

By the first heat equation $(L_0 - H_0) \sigma(u) = 0$, the sigma function must be of the form

$$\sigma(u_3, u_1) = \sum_{\substack{m, n_4, n_6, n_8, n_{10} \ge 0 \\ 3 = 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10} \ge 0}} b(m, n_4, n_6, n_8, n_{10}) \cdot \frac{u_1^3 \left(\frac{u_3}{u_1^3}\right)^m \left(\mu_4 u_1^4\right)^{n_4} \left(\mu_6 u_1^6\right)^{n_6} \left(\mu_8 u_1^8\right)^{n_8} \left(\mu_{10} u_1^{10}\right)^{n_{10}}}{m! \left(3 - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10}\right)!}.$$

Let

$$k = 3 - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10}$$
.

Then the other heat equations $(L_i - H_i) \sigma(u) = 0$ imply the following recursion scheme:

$$b(m, n_4, n_6, n_8, n_{10}) = \begin{cases} B_2 & \text{(if } k > 1 \text{ and } m \ge 0) \\ B_1 & \text{(if } k = 1 \text{ and } m > 0) \\ B_0 & \text{(if } k = 0 \text{ and } m > 1), \end{cases}$$

where the B_i are given by:

$$B_2 = 20(n_8+1) b(m,n_4,n_6,n_8+1,n_{10}-1) \\ +16(n_6+1) b(m,n_4,n_6+1,n_8-1,n_{10}) \\ +12(n_4+1) b(m,n_4+1,n_6-1,n_8,n_{10}) \\ -\frac{24}{5}(n_6+1) b(m,n_4-2,n_6+1,n_8,n_{10}) \\ +\frac{3}{5}(k-3)(k-2) b(m,n_4-1,n_6,n_8,n_{10}) \\ -\frac{8}{5}(n_{10}+1) b(m,n_4-1,n_6,n_8-1,n_{10}+1) \\ -\frac{16}{5}(n_8+1) b(m,n_4-1,n_6-1,n_8+1,n_{10}) \\ -2(k-2) b(m+1,n_4,n_6,n_8,n_{10}) \\ -3m(m-1) b(m-2,n_4,n_6,n_8-1,n_{10}) \\ +\frac{4}{5}m(m-1) b(m-2,n_4-2,n_6,n_8,n_{10}) \\ +\frac{8}{5}m b(m-1,n_4-1,n_6,n_8,n_{10}), \\ B_1 = +10(n_6+1) b(m-1,n_4,n_6+1,n_8,n_{10}-1) \\ -\frac{12}{5}(n_8+1) b(m-1,n_4,n_6-1,n_8-1,n_{10}) \\ -\frac{6}{5}(n_{10}+1) b(m-1,n_4,n_6-1,n_8-1,n_{10}) \\ -\frac{1}{6}(5m-10+8n_6-20n_8-30n_{10}) b(m-1,n_4+1,n_6,n_8,n_{10}) \\ -3(m-1)(m-2) b(m-3,n_4,n_6,n_8,n_{10}-1) \\ +\frac{3}{5}(m-1) (m-2) b(m-3,n_4,n_6,n_8,n_{10}-1) \\ +\frac{6}{5}(m-1) b(m-2,n_4,n_6-1,n_8,n_{10}), \\ B_0 = -\frac{16}{5}(1+n_{10}) b(m-2,n_4,n_6-1,n_8,n_{10}) \\ +20(n_4+1) b(m-2,n_4,n_6-1,n_8,n_{10}-1) \\ +12(n_8+1) b(m-2,n_4+1,n_6,n_8+1,n_{10}-1) \\ -\frac{8}{5}(n_6+1) b(m-2,n_4-1,n_6,n_8+1,n_{10}-1) \\ -\frac{8}{5}(n-2)(m-3) b(m-4,n_4-1,n_6,n_8-1,n_{10}) \\ +\frac{3}{5}(m-2)(m-3) b(m-4,n_4-1,n_6,n_8-1,n_{10}).$$

From these, we see that the expansion of $\sigma(u)$ is Hurwitz integral over $\mathbb{Z}[\frac{1}{5}]$.

Remark 3.8. Actually the above recurrence scheme is one of several possible recurrence relations. However, we see any such system gives the same solution space by the following

argument. Here, of course, we suppose that $b(m,n_4,\cdots,n_{10})=0$ if k or any of the explicit arguments is negative. For any finite subset $S\subset\{(m,n_4,\cdots,n_{10})\,|\,k,n_4,\cdots,n_{10}\geq 0\}$, we take the set E_S of relations h between $\{b(m,n_4,\cdots,n_{10})\}$ such that any $b(m,n_4,\cdots,n_{10})$ appears as a term in h provided that $(m,n_4,\cdots,n_{10})\in S$. For instance, if we consider

$$S = \{b(1,0,0,0,0), b(0,0,0,0,0), b(0,1,0,0,0), b(1,1,0,0,0), b(2,1,0,0,0)\},\$$

then E_S consists of the following 4 equations:

$$b(0,0,0,0,0) = -2b(1,0,0,0,0),$$

$$b(0,1,0,0,0) = 12b(0,0,0,0,0) - 10b(1,1,0,0,0),$$

$$b(1,1,0,0,0) = \frac{6}{5}b(1,0,0,0,0) - 4b(2,1,0,0,0) - \frac{16}{5}b(0,0,0,0,0),$$

$$b(2,1,0,0,0) = \frac{1}{5} \cdot 0 \cdot b(1,0,0,0,0).$$

The solution space of such a system of linear equations E_S is of dimension 1 or larger because we have at least one iteration system as above whose solution space is of dimension 1. Since E_S is independent of the choice of recursion system, any recursion system must include the same solution space of dimension 1.

The first few terms of the sigma expansion are given as follows (up to a constant multiple):

$$\sigma(u_3, u_1) = u_3 - 2\frac{u_1^3}{3!} - 4\mu_4 \frac{u_1^7}{7!} - 2\mu_4 \frac{u_3 u_1^4}{4!} + 64\mu_6 \frac{u_1^9}{9!} - 8\mu_6 \frac{u_3 u_1^6}{6!} - 2\mu_6 \frac{u_3^2 u_1^3}{2!3!} + \mu_6 \frac{u_3^3}{3!} + (1600\mu_8 - 408\mu_4^2) \frac{u_1^{11}}{11!} - (4\mu_4^2 + 32\mu_8) \frac{u_3 u_1^8}{8!} - 8\mu_8 \frac{u_3^2 u_1^5}{2!5!} - 2\mu_8 \frac{u_3^3 u_1^2}{3!2!} + \cdots$$

3.4 Heat equations for the (2,7)-curve

We take the hyperelliptic genus g=3 curve $\mathscr C$ in "Weierstrass" form

$$y^{2} = f(x) = x^{7} + \mu_{4}x^{5} + \mu_{6}x^{4} + \mu_{8}x^{3} + \mu_{10}x^{2} + \mu_{12}x + \mu_{14}.$$

The discriminant Δ of \mathscr{C} is the resultant of f and f_x . It has 320 terms and is of weight 84. The matrix V is given by

$$V = \begin{bmatrix} 4\,\mu_4 & 6\,\mu_6 & 8\,\mu_8 \\ 6\,\mu_6 & -\frac{4}{7}\left(5\,\mu_4^2 - 14\,\mu_8\right) & -\frac{2}{7}\left(8\,\mu_6\mu_4 - 35\,\mu_{10}\right) \\ 8\,\mu_8 & -\frac{2}{7}\left(8\,\mu_6\mu_4 - 35\,\mu_{10}\right) & \frac{4}{7}\left(21\,\mu_{12} - 6\,\mu_6^2 + 7\,\mu_4\mu_8\right) \\ 10\,\mu_{10} & -\frac{12}{7}\left(\mu_4\mu_8 - 7\,\mu_{12}\right) & \frac{2}{7}\left(49\,\mu_{14} - 9\,\mu_6\mu_8 + 21\,\mu_4\mu_{10}\right) \\ 12\,\mu_{12} & -\frac{2}{7}\left(4\,\mu_4\mu_{10} - 49\,\mu_{14}\right) & \frac{4}{7}\left(14\,\mu_4\mu_{12} - 3\,\mu_6\mu_{10}\right) \\ 14\,\mu_{14} & -\frac{4}{7}\,\mu_4\mu_{12} & \frac{2}{7}\left(35\,\mu_4\mu_{14} - 3\,\mu_6\mu_{12}\right) \\ & & 10\,\mu_{10} & 12\,\mu_{12} & 14\,\mu_{14} \\ -\frac{12}{7}\left(\mu_4\mu_8 - 7\,\mu_{12}\right) & -\frac{2}{7}\left(4\,\mu_4\mu_{10} - 49\,\mu_{14}\right) & -\frac{4}{7}\,\mu_4\mu_{12} \\ \frac{2}{7}\left(21\,\mu_4\mu_{10} - 9\,\mu_6\mu_8 + 49\,\mu_{14}\right) & \frac{4}{7}\left(14\,\mu_4\mu_{12} - 3\,\mu_6\mu_{10}\right) & \frac{2}{7}\left(35\,\mu_4\mu_{14} - 3\,\mu_6\mu_{12}\right) \\ \frac{4}{7}\left(7\,\mu_6\mu_{10} - 6\,\mu_8^2 + 14\,\mu_4\mu_{12}\right) & \frac{2}{7}\left(21\,\mu_6\mu_{12} - 8\,\mu_{10}\mu_8 + 35\,\mu_4\mu_{14}\right) & \frac{8}{7}\left(7\,\mu_6\mu_{14} - \mu_{12}\mu_8\right) \\ \frac{2}{7}\left(21\,\mu_6\mu_{12} - 8\,\mu_{10}\mu_8 + 35\,\mu_4\mu_{14}\right) & \frac{2}{7}\left(21\,\mu_4\mu_8 - 5\,\mu_{12}\mu_{10}\right) & \frac{4}{7}\left(7\,\mu_{14}\mu_{10} - 3\,\mu_{12}^2\right) \end{bmatrix}$$

So we have

$${}^{t}[L_{0} \ L_{2} \ L_{4} \ L_{6} \ L_{8} \ L_{10}] = V \ {}^{t}\left[\frac{\partial}{\partial \mu_{4}} \ \frac{\partial}{\partial \mu_{6}} \ \frac{\partial}{\partial \mu_{8}} \ \frac{\partial}{\partial \mu_{10}} \ \frac{\partial}{\partial \mu_{12}} \ \frac{\partial}{\partial \mu_{14}}\right].$$

Using (2.41), their operation on Δ are given by

$$[L_0 \ L_2 \ L_4 \ L_6 \ L_8 \ L_{10}](\Delta) = [84 \ 0 \ 40\mu_4 \ 24\mu_6 \ 12\mu_8 \ 4\mu_{10}]\Delta.$$

As for the (2,5)-case, we have fundamental relations for these L_i as a set of generators of certain Lie algebra, which we do not include here. The symplectic base of $H^1(\mathscr{C}, \mathbb{Q}[\mu])$ is consist of

$$\omega_5 = \frac{dx}{2y}, \quad \omega_3 = \frac{xdx}{2y}, \quad \omega_1 = \frac{x^2dx}{2y},$$

$$\eta_{-5} = \frac{(5x^5 + 3\mu_4x^3 + 2\mu_6x^3 + \mu_8x^2)dx}{2y}, \quad \eta_{-3} = \frac{(3x^4 + \mu_4x^2)dx}{2y}, \quad \eta_{-1} = \frac{x^3dx}{2y}.$$

With respect to these, the matrices $\Gamma_j = \begin{bmatrix} -\beta_j & \alpha_j \\ -\gamma_j & t\beta_j \end{bmatrix}$ are given as follows ⁴:

These give a set of heat equations $(L_j - H_j) \sigma(u) = 0$ as before.

3.5 Series expansion of the sigma function for the (2,7)-curve

Following [5] but in the Hurwitz series form as [24], for the (2,7) case we write the sigma function as

$$\sigma(u_{5}, u_{3}, u_{1}) = \sum_{\substack{\ell, m, n_{4}, n_{6}, n_{8}, \\ n_{10}, n_{12}, n_{14}}} b(\ell, m, n_{4}, n_{6}, n_{8}, n_{10}, n_{12}, n_{14})$$

$$\cdot (\mu_{4}u_{1}^{4})^{n_{4}} (\mu_{6}u_{1}^{6})^{n_{6}} (\mu_{8}u_{1}^{8})^{n_{8}} (\mu_{10}u_{1}^{10})^{n_{10}} (\mu_{12}u_{1}^{12})^{n_{12}} (\mu_{14}u_{1}^{14})^{n_{14}}$$

$$\cdot \frac{u_{1}^{6} (\frac{u_{5}}{u_{1}^{5}})^{\ell} (\frac{u_{3}}{u_{1}^{3}})^{m}}{(6 - 5\ell - 3m + 4n_{4} + 6n_{6} + 8n_{8} + 10n_{10} + 12n_{12} + 14n_{14})! \ell! m!},$$

⁴These should not be confused with the symplectic basis of cycles α_i and β_i in (1.7)

giving a solution of $(L_0 - H_0) \sigma(u) = 0$. Note that we need the leading term u_1^6 to give an even series, and to match the leading term, the Schur function, in the expansion of sigma. If we define

$$k = 6 - 5\ell - 3m + 4n_4 + 6n_6 + 8n_8 + 10n_{10} + 12n_{12} + 14n_{14}$$

we can rewrite the above expression as

$$(3.9) \quad \sigma = \sum b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \,\mu_4^{n_4} \,\mu_6^{n_6} \,\mu_8^{n_8} \,\mu_{10}^{n_{10}} \,\mu_{12}^{n_{12}} \,\mu_{14}^{n_{14}} \,\frac{u_5^{\ell}}{\ell!} \,\frac{u_3^m}{m!} \,\frac{u_1^k}{k!},$$

where we require all the integer indices k, ℓ , m, n_4 , n_6 , n_8 , n_{10} , n_{12} , n_{14} to be non-negative.

Note that the *u*-weight of this expression is $k_0 = 6 + 4n_4 + 6n_6 + 8n_8 + 10n_{10} + 12n_{12} + 14n_{14}$, which does not depend on ℓ or m. (Note also that $k = k_0 - 5\ell - 3m$). For fixed n_4 , n_6 , n_8 , n_{10} , n_{12} , $n_{14} \ge 0$, $k_0 \ge 0$ is fixed, and for non-negative k, we require $\ell = 0, \ldots, \lfloor (k_0 + 6)/5 \rfloor$, $m = 0, \ldots, \lfloor (6 + k_0 - 5\ell)/3 \rfloor$.

As noted above, if we insert this ansatz into the equation for $(L_0 - H_0)\sigma = 0$, we get an expression which is identically zero, for for any set of $b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$.

If we insert this ansatz into the expression for $(L_2 - H_2)\sigma = 0$, we get (after some algebra, and providing k > 0) the recurrence relation shown below, involving 20 terms (compare the equations on p.68 of [5] for the genus 2 case). We can structure this relation by the weight of each b coefficient of (3.9) (more precisely by the weight of the corresponding term in the sigma expansion). We will call this P_2 :

$$-7b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +14(2-k)b(\ell, m+1, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ -42mb(\ell+1, m-1, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +196(n_{12}+1)b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}+1, n_{14}-1) \\ +168(n_{10}+1)b(\ell, m, n_4, n_6, n_8, n_{10}+1, n_{12}-1, n_{14}) \\ +140(n_8+1)b(\ell, m, n_4, n_6, n_8+1, n_{10}-1, n_{12}, n_{14}) \\ +112(n_6+1)b(\ell, m, n_4, n_6+1, n_8-1, n_{10}, n_{12}, n_{14}) \\ +20(n_6+1)b(\ell, m, n_4-2, n_6+1, n_8, n_{10}, n_{12}, n_{14}) \\ +5(3-k)(2-k)b(\ell, m, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ -8(n_{14}+1)b(\ell, m, n_4-1, n_6, n_8, n_{10}, n_{12}-1, n_{14}+1) \\ -16(n_{12}+1)b(\ell, m, n_4-1, n_6, n_8, n_{10}, n_{12}-1, n_{14}+1) \\ -24(n_{10}+1)b(\ell, m, n_4-1, n_6, n_8-1, n_{10}+1, n_{12}, n_{14}) \\ -32(n_8+1)b(\ell, m, n_4-1, n_6-1, n_8+1, n_{10}, n_{12}, n_{14}) \\ +84(n_4+1)b(\ell, m, n_4+1, n_6-1, n_8, n_{10}, n_{12}, n_{14}) \\ +84(n_4+1)b(\ell, m-2, n_4+1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8m(m-1)b(\ell, m-2, n_4-2, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +16mb(\ell, m-1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +35\ell(\ell-1)b(\ell-2, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +4\ell(\ell-1)b(\ell-2, m, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_8, n_{10}, n_{12}, n_{14}) \\ +8\ell b(\ell-1, m+1, n_4-1, n_6, n_$$

This relation applies only for k > 1, and can be written as

$$P_2: b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) = 2(2-k)b(\ell, 1+m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) - 6mb(1+\ell, m-1, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) + \text{"lower weight terms"},$$

where the lower weight terms have coefficients which are quadratic or linear in ℓ , m, n_4 , n_6 , n_8 , n_{10} , n_{12} , n_{14} , times integer numbers or rational numbers with denominators 7. Here the number $4n_4+6n_6+8n_8+10n_{10}+12n_{12}+14n_{14}$ for $b(\ell,m,n_4,n_6,n_8,n_{10},n_{12},n_{14})$ is the μ -weight of the term. For P_2 , the left hand side and the first two terms on the right hand side all have the μ -weight $W=4n_4+6n_6+8n_8+10n_{10}+12n_{12}+14n_{14}$. The next highest μ -weight terms of the "lower weight terms" are of μ -weight W-2, and the lowest weight terms are of μ -weight W-12.

Putting the same ansatz into $(L_4 - H_4)\sigma = 0$ we get another recurrence P_4 with 20 terms, providing m > 0 and k > 0. We can write this as

$$P_4: b(\ell, 1+m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) = -7 (k-1) b(1+l, m-1, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$
 + "lower weight terms".

Here the lower weight terms have the same property as P_2 .

+ "lower weight terms".

We have another relation from the equation $(L_6 - H_6)\sigma = 0$

$$P_6: b(l, m+2, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) + 2b(1+l, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$

= "lower weight terms".

We can write this in two different ways which will each come in useful

$$P_6^{(a)}: b(l, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) = -2b(l+1, m-2, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$
+ "lower weight terms",
$$P_6^{(b)}: 2b(l, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}) = -b(l-1, m+2, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$

Continuing, we have two further relations, from the equations $(L_8 - H_8)\sigma = 0$ and $(L_{10} - H_{10})\sigma = 0$

```
\begin{split} P_8 \ : \ b(\ell+1,m+1,n_4,n_6,n_8,n_{10},n_{12},n_{14}) = \text{``lower $\mu$-weight terms''$}, \\ P_{10} \ : \ b(\ell+2,m,n_4,n_6,n_8,n_{10},n_{12},n_{14}) = \text{``lower $\mu$-weight terms''$}, \end{split}
```

where the lower μ -weight terms have the same properties as P_2 and P_4 . The relations P_6 , P_8 , P_{10} have a total of 24, 24, 19 terms respectively.

As before, we need to normalize the sigma expansion, so we choose b(1,0,0,0,0,0,0,0,0) = 1. We need to find relations which either express coefficients in terms of ones with lower or equal μ -weight. Clearly we must take care with our recurrence relation to avoid infinite looping. We find that the following choice of recurrence scheme results in a sequence which decreases the μ -weight after no more than one extra step at any point in the recurrence.

$$b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$

$$= \begin{cases}
0 & \text{if } \min\{k, \ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}\} < 0, \\
1 & \text{if } \ell = 1, m = n_4 = n_6 = n_8 = n_{10} = n_{12} = n_{14} = 0, \\
\text{rhs}(P_2) & \text{if } k > 1, \\
\text{rhs}(P_4) & \text{if } k = 1, m > 0, \\
\text{rhs}(P_{6a}) & \text{if } k = 1, m = 0 \text{ (and } \ell > 0), \\
\text{rhs}(P_{6b}) & \text{if } k = 0, m > 1, \\
\text{rhs}(P_8) & \text{if } k = 0, m = 1 \text{ and } \ell > 0, \\
\text{rhs}(P_{10}) & \text{if } k = 0, m = 0 \text{ and } \ell > 1.
\end{cases}$$

Note that the structure of this complicated linear recurrence relation does *not* depend on the moduli μ_i . We have used this to calculate the terms on the Hurwitz series for sigma up to weight 40 in u_i (weight 34 in the μ_i).

As for the (2,5)-curve, there is another possible recursion scheme:

$$b(\ell, m, n_4, n_6, n_8, n_{10}, n_{12}, n_{14})$$

$$= \begin{cases}
0 & \text{if } \min\{k, \ell, m, n_4, n_8, n_{10}, n_{12}, n_{14}\} < 0, \\
1 & \text{if } \ell = 1, m = n_4 = n_8 = n_{10} = n_{12} = n_{14} = 0, \\
\text{rhs}(P_{10}) & \text{if } \ell > 1, \\
\text{rhs}(P_8) & \text{if } \ell = 1, m > 0, \\
\text{rhs}(P_{6a}) & \text{if } \ell = 1, m = 0 \text{ and } k > 0, \\
\text{rhs}(P_4) & \text{if } \ell = 0, m > 0 \text{ and } k > 0, \\
\text{rhs}(P_{6b}) & \text{if } \ell = 0, m > 0 \text{ and } k = 0, \\
\text{rhs}(P_2) & \text{if } \ell = 0, \text{ and } m = 0.
\end{cases}$$

We have used this to calculate the terms in the sigma series up to weight 40 in $\{u_j\}$, or equivalently, weight 35 in the $\{\mu_i\}$. The first few terms of the sigma expansion are given as follows (up to a constant multiple):

$$\sigma(u_5, u_3, u_1) = 16 \frac{u_1^6}{6!} - 2 \frac{u_1^3 u_3}{3!} - 2 \frac{u_3^2}{2!} + u_5 u_1 + 64 \mu_4 \frac{u_1^{10}}{10!} + 36 \mu_4 \frac{u_1^7 u_3}{7!} - 4 \mu_4 \frac{u_1^4 u_3^2}{4!2!}$$

$$- 2 \mu_4 \frac{u_1 u_3^3}{3!} + 2 \mu_4 \frac{u_5 u_1^5}{5!} - 512 \mu_6 \frac{u_1^{12}}{12!} + 64 \mu_6 \frac{u_1^9 u_3}{9!} + 16 \mu_6 \frac{u_1^6 u_3^2}{6!2!}$$

$$- 8 \mu_6 \frac{u_1^3 u_3^3}{3!^2} - 8 \mu_6 \frac{u_3^4}{4!} + 24 \mu_6 \frac{u_5 u_1^7}{7!} + \cdots.$$

Further studies are required to establish whether there are other recursion schemes which can be used to generate the series, and which recursions could be considered the most efficient in some sense.

3.6 Heat equations for the (3,4)-curve

We take the trigonal genus 3 curve $\mathscr{C}=\mathscr{C}_{\mu}^{3,4}$ in the Weierstrass form

$$y^{3} = (\mu_{8} + \mu_{5}x + \mu_{2}x^{2})y + x^{4} + \mu_{6}x^{2} + \mu_{9}x + \mu_{12}.$$

The matrix V is given by

$$V = \begin{bmatrix} V_{1..3,1..3} & V_{1..3,4..6} \\ {}^{t}V_{1..3,4..6} & V_{4..6,4..6} \end{bmatrix},$$

where

$$V_{1..3,1..3} = \begin{bmatrix} 2\,\mu_2 & 5\,\mu_5 & 6\,\mu_6 \\ 5\,\mu_5 & \frac{1}{6}\,\mu_2^4 - 4\,\mu_2\mu_6 + 8\,\mu_8 & -\frac{1}{2}\,\mu_2^2\mu_5 + 9\,\mu_9 \\ 6\,\mu_6 & -\frac{1}{2}\,\mu_2^2\mu_5 + 9\,\mu_9 & \frac{2}{3}\,\mu_2^2\mu_6 + \frac{10}{3}\,\mu_2\mu_8 + \frac{5}{3}\,\mu_5^2 \end{bmatrix}$$

and

$$V_{1...3,4...6} = \begin{bmatrix} 8 \mu_8 & 9 \mu_9 \\ \frac{1}{12} \mu_2^3 \mu_5 - \frac{1}{2} \mu_2 \mu_9 - \frac{3}{2} \mu_5 \mu_6 & \frac{1}{6} \mu_2^3 \mu_6 - \frac{1}{3} \mu_2^2 \mu_8 - \frac{1}{6} \mu_2 \mu_5^2 - 3 \mu_6^2 + 12 \mu_{12} \\ \frac{4}{3} \mu_2^2 \mu_8 - \frac{7}{12} \mu_2 \mu_5^2 + 12 \mu_{12} & \frac{4}{3} \mu_2^2 \mu_9 - \frac{7}{6} \mu_2 \mu_5 \mu_6 + \frac{13}{3} \mu_5 \mu_8 \\ & 12 \mu_{12} \\ \frac{1}{12} \mu_2^3 \mu_9 - \frac{1}{6} \mu_2 \mu_5 \mu_8 - \frac{3}{2} \mu_6 \mu_9 \\ & 2 \mu_2^2 \mu_{12} - \frac{7}{12} \mu_2 \mu_5 \mu_9 + \frac{8}{3} \mu_8^2 \end{bmatrix}$$

and the remaining elements given by

$$\begin{split} V_{4,4} &= \frac{1}{24} \, \mu_2^2 \mu_5^2 + 6 \, \mu_2 \mu_{12} - \frac{7}{2} \, \mu_5 \mu_9 + 4 \, \mu_6 \mu_8, \\ V_{4,5} &= V_{5,4} = \frac{1}{12} \, \mu_2^2 \mu_5 \mu_6 + \frac{7}{6} \, \mu_2 \mu_5 \mu_8 - \frac{5}{12} \, \mu_5^3 - \frac{3}{2} \, \mu_6 \mu_9, \\ V_{4,6} &= V_{6,4} = \frac{1}{24} \, \mu_2^2 \mu_5 \mu_9 + \frac{4}{3} \, \mu_2 \mu_8^2 - \frac{5}{12} \, \mu_5^2 \mu_8 + 6 \, \mu_6 \mu_{12} - \frac{9}{4} \, \mu_9^2, \\ V_{5,5} &= \frac{1}{6} \, \mu_2^2 \mu_6^2 + 2 \, \mu_2^2 \mu_{12} + \frac{5}{3} \, \mu_2 \mu_5 \mu_9 - \frac{8}{3} \, \mu_2 \mu_6 \mu_8 - \frac{4}{3} \, \mu_5^2 \mu_6 + \frac{8}{3} \, \mu_8^2, \\ V_{5,6} &= V_{6,5} &= \frac{1}{12} \, \mu_2^2 \mu_6 \mu_9 + 3 \, \mu_2 \mu_5 \mu_{12} - \frac{5}{6} \, \mu_2 \mu_8 \mu_9 - \frac{5}{12} \, \mu_5^2 \mu_9 - \frac{1}{2} \, \mu_5 \mu_6 \mu_8, \\ V_{6,6} &= \frac{1}{24} \, \mu_2^2 \mu_9^2 + 2 \, \mu_2 \mu_8 \mu_{12} + \mu_5^2 \mu_{12} - \frac{11}{6} \, \mu_5 \mu_8 \mu_9 + \frac{4}{3} \, \mu_6 \mu_8^2. \end{split}$$

The discriminant Δ for $\mathscr C$ is calculated by an algorithm provided by Sylvester, has 670 terms and is of weight 72. Then we have

$$[L_0 \ L_3 \ L_4 \ L_6 \ L_7 \ L_{10}] \varDelta = [72 \ 0 \ -8\mu_2^2 \ 12\mu_6 \ -8\mu_2\mu_5 \ -\mu_5^2 -4\mu_2\mu_8] \varDelta.$$

As for the (2,7)-case, we have fundamental relations for these L_i as a set of generators of certain Lie algebra.

The symplectic base of $H^1(\mathscr{C}, \mathbb{Q}[\mu])$ in this case is given by

$$\omega_5 = \frac{dx}{3v^2}, \quad \omega_2 = \frac{xdx}{3v^2}, \quad \omega_1 = \frac{ydx}{3v^2}, \quad \eta_{-5} = \frac{5x^2ydx}{3v^2}, \quad \eta_{-2} = \frac{2xydx}{3v^2}, \quad \omega_{-1} = \frac{x^2dx}{3v^2}.$$

The matrices $\Gamma_j = \begin{bmatrix} -\beta_j & \alpha_j \\ -\gamma_i & {}^t\beta_j \end{bmatrix}$ are given as follows⁵:

$$\begin{split} &\alpha_0 = O, \; \beta_0 = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; \beta_3 = \begin{bmatrix} 0 & \frac{1}{12} \mu_2^3 - \frac{3}{2} \mu_6 & -\frac{1}{6} \mu_5 \mu_2 \\ 2 & 0 & -\frac{1}{3} \mu_2^3 \\ 0 & -\frac{1}{2} \mu_2 & 0 \end{bmatrix}, \\ &\gamma_3 = \begin{bmatrix} 5 \mu_5 \mu_8 - \frac{1}{3} \mu_2^2 \mu_9 + \frac{1}{12} \mu_2^4 \mu_5 - \frac{5}{8} \mu_2 \mu_5 \mu_6 & -\mu_2 \mu_8 & \mu_9 \\ -\mu_2 \mu_8 & \frac{2}{3} \mu_5 \mu_2 & \frac{5}{6} \mu_3^3 - \mu_0 & \mu_5 \end{bmatrix}, \\ &\alpha_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \mu_2 \end{bmatrix}, \; \beta_4 = \begin{bmatrix} \frac{2}{3} \mu_2^2 - \frac{7}{12} \mu_5 \mu_2 & \frac{5}{3} \mu_8 \\ 0 & 0 & \frac{2}{3} \mu_5 \end{bmatrix}, \\ &\gamma_4 = \begin{bmatrix} -\frac{7}{12} \mu_2^2 \mu_5^2 + 9 \mu_2 \mu_{12} + \frac{5}{3} \mu_5 \mu_9 + \frac{11}{3} \mu_6 \mu_8 & \frac{2}{3} \mu_2 \mu_9 & -\frac{1}{3} \mu_2 \mu_8 \\ \frac{2}{3} \mu_2 \mu_9 & \frac{4}{3} \mu_2 \mu_6 + \frac{4}{3} \mu_8 & -\frac{1}{2} \mu_5 \mu_2 \\ -\frac{1}{3} \mu_2 \mu_8 & \frac{2}{3} \mu_2 \mu_9 & \frac{1}{3} \mu_2 \mu_5 - \frac{5}{12} \mu_2 \end{bmatrix}, \\ &\alpha_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; \beta_6 = \begin{bmatrix} \mu_6 & \frac{1}{12} \mu_2^2 \mu_5 - \frac{5}{5} \mu_9 & \frac{1}{3} \mu_2 \mu_5 - \frac{5}{12} \mu_2 \\ 0 & -\frac{1}{4} \mu_5 & 0 \end{bmatrix}, \\ &\gamma_6 = \begin{bmatrix} \frac{1}{12} m_2^3 \mu_3^2 + 4 \mu_2^2 \mu_1^2 - \frac{20}{12} \mu_2 \mu_5 \mu_5 \\ \frac{1}{2} \mu_5 \mu_8 & \frac{3}{8} \mu_2 \mu_6 - \frac{1}{3} \mu_5 \mu_5 & \frac{3}{12} \mu_2 \\ \frac{1}{2} \mu_5 \mu_8 & \frac{3}{8} \mu_2 \mu_6 - \frac{1}{3} \mu_5 \mu_5 & \frac{1}{12} \mu_2 \mu_5 - \frac{1}{2} \mu_9 \\ \frac{1}{3} \mu_5 \mu_5 & \frac{1}{3} \mu_5 \mu_5 & \frac{1}{2} \mu_5 \mu_5 \\ \frac{1}{3} \mu_5 \mu_5 & \frac{3}{3} \mu_2 & \frac{1}{2} \mu_5 \mu_5 & \frac{1}{2} \mu_5 \mu_5 \\ \frac{1}{3} \mu_5 \mu_5 & \frac{1}{2} \mu_5 \mu_5 & \frac{1}{2} \mu_5 \mu_6 & \frac{1}{3} \mu_5 \mu_5 & \frac{1}{12} \mu_5 \mu_6 \\ \frac{2}{3} \mu_5 \mu_5 & \frac{1}{2} \mu_5 \mu_5 & \frac{3}{8} \mu_2 \mu_6 - \frac{1}{6} \mu_2 \mu_5 & \frac{5}{12} \mu_3 \mu_5 \\ \frac{2}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 & \frac{1}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 \mu_5 & \frac{1}{3} \mu_5 \mu_5 \\ \frac{2}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 & \frac{1}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 \mu_5 & \frac{1}{3} \mu_5 \mu_5 \\ \frac{2}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 & \frac{1}{3} \mu_5 \mu_6 & -\frac{1}{6} \mu_5^2 \mu_5 & \frac{1}{3} \mu_5 \mu_5 \\ \frac{2}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 \mu_5 & \frac{1}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 \mu_5 & \frac{1}{3} \mu_5 \mu_5 \\ \frac{2}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 \mu_5 & \frac{1}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5^2 \mu_5 & \frac{1}{3} \mu_5 \mu_5 \\ 0 & 0 & -\frac{1}{6} \mu_5 \mu_5 & \frac{1}{3} \mu_5 \mu_5 \\ 0 & 0 & -\frac{1}{6} \mu_5 \mu_5 \\ 0 & 0 & -\frac{1}{6} \mu_5 \mu_5 \\ \frac{2}{3} \mu_5 \mu_5 & -\frac{1}{6} \mu_5 \mu_5 & \frac$$

⁵These should not be confused with the symplectic basis of cycles α_i and β_i in (1.7)

3.7 Series expansion of the sigma function for the (3,4)-curve

It is not clear to the authors which part in [6] shows that the space of the solutions $\varphi(\mu, u_5, u_2, u_1) \in \mathbb{Q}[\mu][[u_5, u_2, u_1]]$ of the system

$$(L_j - H_j)\varphi(\mu, u_5, u_2, u_1) = 0$$
 $(j = 0, 3, 4, 6, 7, 10)$

is one dimensional. In this Section we shall give a proof of this for the case of the (3,4)-curves. Following [5], but using the Hurwitz series form, for the (3,4)-curve we define

$$\sigma(u_{5}, u_{2}, u_{1}) = \sum_{\ell, m, n_{2}, n_{5}, n_{6}, n_{8}, n_{9}, n_{12}} b(\ell, m, n_{2}, n_{5}, n_{6}, n_{8}, n_{9}, n_{12}) u_{1}^{5} \left(\frac{u_{5}}{u_{1}^{5}}\right)^{\ell} \left(\frac{u_{2}}{u_{1}^{2}}\right)^{m} \cdot \left(\mu_{2}u_{1}^{2}\right)^{n_{2}} \left(\mu_{5}u_{1}^{5}\right)^{n_{5}} \left(\mu_{6}u_{1}^{6}\right)^{n_{6}} \left(\mu_{8}u_{1}^{8}\right)^{n_{8}} \left(\mu_{9}u_{1}^{9}\right)^{n_{9}} \left(\mu_{12}u_{1}^{12}\right)^{n_{12}} / \left(\ell! \, m! \, (5 - 5\ell - 2m + 2n_{2} + 5n_{5} + 6n_{6} + 8n_{8} + 9n_{9} + 12n_{12})!\right)$$

If we define

$$k = 5 - 5\ell - 2m + 2n_2 + 5n_5 + 6n_6 + 8n_8 + 9n_9 + 12n_{12}$$

we can rewrite the above expression as

$$\sigma(u_5, u_2, u_1) = \sum b(\ell, m, n_2, n_5, n_6, n_8, n_9, n_{12}) \cdot \frac{\mu_2^{n_2} \mu_5^{n_5} \mu_6^{n_6} \mu_8^{n_8} \mu_9^{n_9} \mu_{12}^{n_{12}} u_5^{\ell} u_2^{m} u_1^{k}}{\ell! \, m! \, k!}$$

where we require all the integer indices k,ℓ , $m, n_2, n_5, n_6, n_8, n_9, n_{12}$ to be non-negative. Note that the u-weight of this expression is $k_0 = 5 + 2n_2 + 5n_5 + 6n_6 + 8n_8 + 9n_9 + 10n_{12}$, which does not depend on ℓ or m. (Note also that $k = k_0 - 5\ell - 2m$). For fixed $n_2, n_5, n_8, n_6, n_9, n_{12} \ge 0$, $k_0 \ge 0$ is fixed, and for non-negative k, we require $\ell = 0, \ldots, \lfloor k_0/5 \rfloor, m = 0, \ldots, \lfloor (k_0 - 5\ell)/2 \rfloor$. In addition, we can use the condition that σ is an odd function, $\sigma(-u) = -\sigma(u)$; this tells us that if k_0 is even(odd) then we should restrict ourselves to m even(odd) respectively.

If we insert this ansatz into the equation for $(L_0 - H_0)\sigma = 0$, we get an expression which is identically zero, whatever the values for the $b(\ell, m, n_2, n_5, n_6, n_8, n_9, n_{12})$. If we insert the ansatz into the equation for $(L_3 - H_3)\sigma = 0$, we get (after some algebra) the recurrence relation shown below, involving 34 terms (compare the equations on p.68 of [5] for the genus 2 case). We can structure the relation by the weight of each b coefficient (more precisely by the weight of the corresponding term in the sigma expansion).

Contrarily to the (2,3)-, (2,5)-, (2,7)-curves, we could not find any method for (3,4)-curve to prove Hurwitz integrality of the expansion of $\sigma(u)$.

We will call the recurrence relation, generated from $(L_3 - H_3)\sigma = 0$, R_3 :

$$\begin{aligned} 24b(\ell,m+1,n_2,n_5,n_6,n_8,n_9,n_{12}) &+ 48m \, b(\ell+1,m-1,n_2,n_5,n_6,n_8,n_9,n_{12}) \\ &= -12(4-k)(3-k) \, b(\ell,m,n_2,n_5-1,n_6,n_8,n_9,n_{12}) \\ &- 36(n_8+1) \, b(\ell,m,n_2,n_5-1,n_6-1,n_8+1,n_9,n_{12}) \\ &+ 4(n_9+1) \, b(\ell,m,n_2-3,n_5,n_6-1,n_8,n_9+1,n_{12}) \\ &+ 2(n_{12}+1) \, b(\ell,m,n_2-3,n_5,n_6,n_8,n_9-1,n_{12}+1) \\ &+ 4(n_5+1) \, b(\ell,m,n_2-4,n_5+1,n_6,n_8,n_9,n_{12}) \\ &- 8(n_9+1) \, b(\ell,m,n_2-4,n_5+1,n_6,n_8,n_9,n_{12}) \\ &- 8(n_9+1) \, b(\ell,m,n_2-2,n_5,n_6,n_8-1,n_9+1,n_{12}) \\ &+ 2(n_8+1) \, b(\ell,m,n_2-3,n_5-1,n_6,n_8+1,n_9,n_{12}) \\ &- 96(n_5+1) \, b(\ell,m,n_2-1,n_5+1,n_6-1,n_8,n_9,n_{12}) \\ &- 12(n_8+1) \, b(\ell,m,n_2-1,n_5,n_6,n_8+1,n_9-1,n_{12}) \\ &- 12(n_6+1) \, b(\ell,m,n_2-1,n_5-1,n_6,n_8-1,n_9,n_{12}) \\ &- 12(n_6+1) \, b(\ell,m,n_2-1,n_5-1,n_6,n_8-1,n_9,n_{12}) \\ &- 4(n_{12}+1) \, b(\ell,m,n_2-1,n_5-2,n_6,n_8,n_9+1,n_{12}) \\ &+ 120(n_2+1) \, b(\ell,m,n_2+1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &- 12(3-k) \, b(\ell,m+1,n_2-1,n_5,n_6,n_8,n_9,n_{12}) \\ &- 12(3-k) \, b(\ell,m+1,n_2-1,n_5,n_6,n_8,n_9,n_{12}) \\ &- 24m(3-k) \, b(\ell,m-1,n_2,n_5,n_6-1,n_8,n_9,n_{12}) \\ &- 24m(3-k) \, b(\ell,m-1,n_2,n_5,n_6,n_8,n_9-1,n_{12}) \\ &- 8m(m-1) \, b(\ell,m-2,n_2-1,n_5-1,n_6,n_8,n_9-1,n_{12}) \\ &+ 24\ell(3-k) \, b(\ell-1,m,n_2,n_5,n_6,n_8,n_9-1,n_{12}) \\ &+ 24\ell(3-k) \, b(\ell-1,m,n_2,n_5,n_6,n_8,n_9-1,n_{12}) \\ &+ 24\ell(\ell-1) \, b(\ell-2,m,n_2-4,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+ 24\ell(\ell-1) \, b(\ell-2,m,n_2-2,n_5,n_6,n_8,n_9-1,n_{12}) \\ &+ 24\ell(\ell-1) \, b(\ell-2,m,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+ 24\ell m \, b(\ell-1,m-1,n_2-1,n_5,n_6,n_8,n_9,n_{12}) \\ &+ 24\ell m \, b(\ell-1,m,n_2-1,n_5,n_6,n_8,n_9,n_{12}) \\ &+ 26\ell(\ell-1) \, b(\ell-2,m,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+ 26\ell(\ell-1) \, b(\ell-2,m,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+ 26\ell(\ell-1) \, b(\ell-2,m,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+ 288(n_9+1) \, b(\ell,m,n_2,n_5,n_6-1,n_8,n_9+1,n_{12}-1) \\ &+ 288(n_9+1) \, b(\ell,m,n_2,n_5,n_6,n_8,n_9+1,n_{12}-1) \\ &+ 288(n_9+1) \, b(\ell,m,n_2,n_5,n_6,n_8,n_9+1,n_{12}-1) \\ &+ 216(n_8+1) \, b(\ell,m,n_2,n_5,n_6,n_8,n_9+1,n_{12}-1) \\ &+ 216(n_8+1) \, b(\ell,m,n_2,n_5,n_6,n_8,n_9+1,n_{12}-1) \\ &+ 216(n_8+1) \, b(\ell,m,n_2,n_5,n_6,n_8,n_9+1,n_{12}-1) \\ &+ 216$$

Note the two expressions on the left hand side, which are the highest weight terms, at weight $W = 2n_2 + 5n_5 + 6n_6 + 8n_8 + 9n_9 + 12n_{12}$. The next highest weight term (underlined) is of weight W - 2, and the lowest weight terms are of weight W - 13.

Putting the ansatz into the equation for $(L_4 - H_4)\sigma$ we get another recurrence with 27

terms, which we call R_4 .

$$\begin{aligned} -12\,b(\ell,m+2,n_2,n_5,n_8,n_6,n_9,n_{12}) + 24(4-k)\,b(\ell+1,m,n_2,n_5,n_6,n_8,n_9,n_{12}) \\ &= 4\,\underline{b(\ell,m,n_2-1,n_5,n_6,n_8,n_9,n_{12})} \\ +16\ell m\,b(\ell-1,m-1,n_2-1,n_5,n_6,n_8,n_9-1,n_{12}) \\ &-14\ell\,b(\ell-1,m+1,n_2-1,n_5-1,n_6,n_8,n_9-1,n_{12}) \\ &+40\ell\,b(\ell-1,m,n_2,n_5,n_6,n_8-1,n_9,n_{12}) \\ &+16m\,b(\ell,m-1,n_2,n_5-1,n_6,n_8,n_9,n_{12}) \\ &-7\ell(\ell-1)\,b(\ell-2,m,n_2-2,n_5-2,n_6,n_8,n_9,n_{12}) \\ &+108\ell(\ell-1)\,b(\ell-2,m,n_2-1,n_5,n_6,n_8,n_9,n_{12}-1) \\ &+44\ell(\ell-1)\,b(\ell-2,m,n_2-1,n_5,n_6,n_8,n_9,n_{12}-1) \\ &+42\ell(\ell-1)\,b(\ell-2,m,n_2,n_5,n_6-1,n_8,n_9,n_{12}) \\ &+12(5-k)(4-k)\,b(\ell,m,n_2,n_5,n_6-1,n_8,n_9,n_{12}) \\ &+2\ell(\ell-1)\,b(\ell-2,m,n_2,n_5-1,n_6,n_8-1,n_9,n_{12}) \\ &+2\ell(\ell-1)\,b(\ell-2,m,n_2,n_5-1,n_6,n_8-1,n_9,n_{12}) \\ &-288(n_8+1)\,b(\ell,m,n_2,n_5,n_6,n_8+1,n_9,n_{12}-1) \\ &-64(n_{12}+1)\,b(\ell,m,n_2,n_5,n_6,n_8-2,n_9,n_{12}+1) \\ &-40(n_6+1)\,b(\ell,m,n_2,n_5,n_6,n_8-2,n_9,n_{12}+1) \\ &-40(n_6+1)\,b(\ell,m,n_2,n_5+1,n_6,n_8,n_9-1,n_{12}) \\ &-216(1+n_5)\,b(\ell,m,n_2,n_5+1,n_6,n_8,n_9-1,n_{12}) \\ &-4(\ell-1-n_9-8n_5-2n_6+k+2m-2n_2)\,b(\ell,m,n_2-2,n_5,n_6,n_8+1,n_9,n_{12}) \\ &+14(n_8+1)\,b(\ell,m,n_2-1,n_5-2,n_6,n_8+1,n_9,n_{12}) \\ &+14(n_8+1)\,b(\ell,m,n_2-1,n_5-1,n_6,n_8,n_9-1,n_{12}+1) \\ &+28(n_9+1)\,b(\ell,m,n_2-1,n_5-1,n_6,n_8,n_9-1,n_{12}+1) \\ &+28(n_9+1)\,b(\ell,m,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+16m(m-1)\,b(\ell,m-2,n_2,n_5,n_6,n_8-1,n_9,n_{12}) \\ &+16m(m-1)\,b(\ell,m-2,n_2,n_5,n_6,n_8-1,n_9,n_{12}) \\ &+16m(m-1)\,b(\ell,m-2,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+12m(4-k)\,b(\ell,m-1,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+12m(4-k)\,b(\ell,m-1,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+12m(4-k)\,b(\ell,m-1,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+12m(4-k)\,b(\ell,m-1,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\ &+12m(4-k)\,b(\ell,m-1,n_2-1,n_5-1,n_6,n_8,n_9,n_{12}) \\$$

As for R_3 , the two expressions on the left hand side, are the highest weight terms, at weight $W = 2n_2 + 5n_5 + 6n_6 + 8n_8 + 9n_9 + 12n_{12}$. The next highest weight terms are of weight W - 2, and the lowest weight terms are of weight W - 14.

We see that the two recurrence relations have the same terms in a. Hence we can take linear combinations to get two relations, each with only one leading term at weight W

$$S_{3,4}: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \frac{1}{k - m + 1} \text{ (lower weight terms)} \quad (m \neq 0),$$

 $T_{3,4}: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \frac{1}{k - m} \text{ (lower weight terms)} \quad (l \neq 0).$

These $S_{3,4}$ and $T_{3,4}$ connect the left hand side with terms of relative weight -2 and lower, down to -14. In addition we have other relations from the equations $(L_6 - H_6)\sigma = 0$, $(L_7 - H_7)\sigma = 0$,

and $(L_{10} - H_{10})\sigma = 0$ that $R_6: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \text{``lower weight terms''},$ $R_7: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \text{``lower weight terms''},$ $R_{10}: b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) = \text{``lower weight terms''},$

respectively. Here the right hand sides are linear in the coefficients b with coefficient in at most quadratic in k, ℓ , m, n_2 , n_5 , n_8 , n_6 , n_9 , n_{12} over the rationals but each denominator is a divisor of 24.

 R_6 , R_7 , R_{10} have a total of 37, 47, 42 terms respectively and connect the left hand side with terms of relative weight -5, -5, -8 and lower, down to -16, -17, -20 respectively.

Ideally we would like to proceed as follows. Suppose we have already calculated the b coefficients at weight W-2. Then we would like to use one of the above to calculate each coefficient at weight W. We could proceed in this manner to calculate coefficients at successive weight levels to the required number of terms. However this approach needs some modification. Recall that the weight does not depend on ℓ or m. Clearly if $\ell > 1$ we can use R_6 , and if $\ell = 1$, m > 0, we can use R_7 . Similarly if $\ell = 1$, k > 0, we can use R_6 . A short calculation shows that if $\ell = 1$, one of these two possibilities holds except in the special case $\ell = 1$, $m = n_2 = n_5 = n_8 = n_6 = n_9 = n_{12} = 0$ which is covered later. For the case $\ell = 0$ we cannot use R_6 , R_7 , R_{12} . If $m \neq 0$ and $m \neq (k+1)$ we can use $S_{3,4}$. All the possibilities considered so far will reduce the weight by 2. There remain the cases m = 0 and m = (k+1) to deal with.

The case $\ell = 0$, m = 0 is handled as follows. Take $48(k - m) T_{3,4}$:

$$48(m-k) b(\ell, m, n_2, n_5, n_6, n_8, n_9, n_{12})$$

$$= 8 b(\ell-1, m, n_2-1, n_5, n_6, n_8, n_9, n_{12}) + 12 k b(\ell-1, m+2, n_2, n_5, n_6, n_8, n_9, n_{12}) + \cdots$$

Shifting by $n_2 \to n_2 + 1$, we get

$$48(m-k-2)b(\ell,m,n_2+1,n_5,n_6,n_8,n_9,n_{12})$$

$$=8b(\ell-1,m,n_2,n_5,n_6,n_8,n_9,n_{12})+12(k+2)b(\ell-1,m+2,n_2,n_5,n_6,n_8,n_9,n_{12})+\cdots.$$

and

$$b(\ell-1, m, n_2, n_5, n_6, n_8, n_9, n_{12}) = -6(m-k-2) b(\ell, m, n_2+1, n_5, n_6, n_8, n_9, n_{12})$$
$$-\frac{3}{2}(k+2) b(\ell-1, m+2, n_2, n_5, n_6, n_8, n_9, n_{12}) + \cdots$$

On the right hand side we now have two terms of non-negative relative weight as above of relative weight +2 which comes from the underlined term in R_3 and of relative weight 0 which comes from the underlined term in R_4 . Putting $\ell = 1$, m = 0, we have, say $T_{3,4}^{(0)}$, that

$$b(0, 0, n_2, n_5, n_6, n_8, n_9, n_{12}) = -6(k - 3 + 5l + 2m) b(1, 0, n_2 + 1, n_5, n_6, n_8, n_9, n_{12})$$
$$-\frac{3}{2}(k - 3 + 5l + 2m) b(0, 2, n_2, n_5, n_6, n_8, n_9, n_{12}) + \cdots$$

The first term in the right hand side has $\ell=1, m=0$, and $k=5+2n_2+\cdots>0$. Hence we can apply R_7 to this term to give a term with maximum relative weight +2-5=-3. The second term has $\ell=0, m=2$, and $k=1+2n_2+5n_5+\cdots$ so k+1>2 and hence $k+1\neq m$. For this term we can apply $S_{3,4}$ to produce a term of maximum relative weight 0-2=-2. Hence both terms of weight ≥ 0 can be expressed as terms of relative weight ≤ -2 , so our chain eventually decreases in weight.

The case $\ell = 0$, m = (k + 1) is treated as follows. Take R_3 , shift by $m \to m - 1$, and set $\ell = 0$ to get

$$\begin{split} R_2^{(0)}: \ b(0, m, n_2, n_5, n_6, n_8, n_9, n_{12}) \\ &= -2(m-1) \, b(1, m-2, n_2, n_5, n_6, n_8, n_9, n_{12}) + \text{``lower weight terms''}. \end{split}$$

Now the first term on the right, $b(1, m-2, n_2, n_5, n_6, n_8, n_9, n_{12})$, is of the same weight as the term on the left. Write this as $b(1, m', n_2, n_5, n_6, n_8, n_9, n_{12})$, with corresponding k-value k'. If $\min(k', m') < 0$ then this term is zero as discussed above. If k' = 0, m' = 0, it is easy to show that $m' = n_2 = n_5 = n_8 = n_6 = n_9 = n_{12} = 0$, and this term (b(1, 0, 0, 0, 0, 0, 0, 0, 0)) cannot be reduced further. Otherwise one or both of k' = 0, m' is positive, so we can apply R_7 or R_6 to reduce the term to terms of relative weight ≤ -2 , so our chain terminates or decreases in weight.

These choices, plus the requirement discussed above that $b(\ell, m, n_2, n_5, n_6, n_8, n_9, n_{12}) = 0$ if any of the $\{k, \ell, m, n_2, n_5, n_6, n_8, n_9, n_{12}\}$ are negative, define all the $b(\ell, m, n_2, n_5, n_6, n_8, n_9, n_{12})$ in terms of the so-far undefined b(1, 0, 0, 0, 0, 0, 0, 0, 0). Therefore the solution of the system

$$(L_i - H_j)\sigma(u) = 0$$
 $(j = 0, 3, 4, 6, 7, 10)$

is of dimension one. Choosing b(1, 0, 0, 0, 0, 0, 0, 0, 0) = 1, we summarise with $k = 5 - 5\ell - 2m + 2n_2 + 5n_5 + 8n_8 + 6n_6 + 9n_9 + 12n_{12}$ as defined as above as follows:

$$b(\ell, m, n_2, n_5, n_8, n_6, n_9, n_{12}) =$$

$$= \begin{cases}
0 & \text{if } \min(k, \ell, m, n_2, n_5, n_6, n_8, n_9, n_{12}) < 0, \\
1 & \text{if } \ell = 1, m = n_2 = n_5 = n_8 = n_6 = n_9 = n_{12} = 0, \\
\text{rhs}(R_{10}) & \text{if } \ell > 1, \\
\text{rhs}(R_7) & \text{if } \ell > 0, m > 0, \\
\text{rhs}(R_6) & \text{if } \ell > 0, k > 0, \\
\text{rhs}(S_{3,4}) & \text{if } \ell = 0, m \neq 0 \text{ and } m \neq (k+1), \\
\text{rhs}(T_{3,4}^{(0)}) & \text{if } \ell = 0 \text{ and } m = (k+1).
\end{cases}$$

We have used this to calculate the terms in the sigma series up to weight 40 in $\{u_j\}$, or equivalently, weight 35 in the $\{\mu_i\}$. The first few terms of the sigma expansion are given as follows (up to a constant multiple):

$$\begin{split} \sigma(u_5,u_2,u_1) &= u_5 + \frac{6\,u_1^5}{5!} - 2\frac{u_1u_2^2}{2!} - 2\mu_2\frac{u_1^3u_2^2}{2!\,3!} + 30\mu_2\frac{u_1^7}{7!} - 2\mu_2^2\frac{u_1u_2^4}{4!} - 2\mu_2^2\frac{u_1^5u_2^2}{2!\,5!} \\ &+ 126\,\mu_2^2\frac{u_1^9}{9!} + 24\mu_5\frac{u_1^8u_2}{8!} - \mu_5\frac{u_5u_2u_1^3}{3!} + 8\mu_5\frac{u_2^5}{5!} - 2\mu_6\frac{2\,u_5u_2^2u_1^2}{2!\,2!} + 6\mu_6\frac{u_5u_1^6}{6!} \\ &+ 24\mu_6\frac{u_1^7u_2^2}{2!\,7!} - 2\mu_3^3\frac{u_1^7u_2^2}{2!\,7!} - 2\mu_2^3\frac{u_1^3u_2^4}{4!\,3!} + 432\mu_6\frac{u_1^{11}}{11!} + 510\mu_2^3\frac{u_1^{11}}{11!} \\ &- \mu_2\mu_5u_1^5\frac{u_5u_2}{5!} - \mu_2\mu_5\frac{u_5u_2^3u_1}{3} + 288\mu_2\mu_5\frac{u_1^{10}u_2}{10!} + \dots \end{split}$$

References

- [1] Arnol'd, V.I., Wave front evolution and equivariant Morse lemma, Comm. Pure and Appl. Math. XXIX (1976), 557–582.
- [2] Bruce, J.W., Functions on discriminants, J. London Math. Soc.(2) 30 (1984), 551–567.
- [3] Buchstaber, V.M. and Leykin, D.V., *Polynomial Lie Algebras*, Functional Analysis and Its Applications, **36** (2002), 267–280.
- [4] _____, Heat equations in a nonholonomic frame, Functional Anal. Appl., **38** (2004), 88–101.
- [5] _____, Addition laws on Jacobian varieties of plane algebraic curves, Proceedings of The Steklov Institute of Mathematics, **251** (2005), 1–72.
- [6] _____, Solution of the problem of differentiation of Abelian functions over parameters for families of (n, s)-curves, Functional Analysis and Its Applications, 42 (2008), 268–278.
- [7] Buchstaber, V.M., Enolskii, V.Z. and Leykin, D.V., *Kleinian functions, hyperelliptic Jacobians and applications*, Reviews in Math. and Math. Physics, **10** (1997), 1–125.
- [8] Chevalley, C., Introduction to the theory of algebraic functions of one variable, Math.Surv. VI, A.M.S., 1951.
- [9] Eilbeck, J.C., Enol'skii, V.Z., Matsutani, S., Ônishi, Y. and Previato, E., Abelian functions for trigonal curves of genus three, International Mathematics Research Notices, 2007 (2007), 102–139.
- [10] Frobenius, G.F. and Stickelberger, L., Ueber die Differentiation der elliptischen Functionen nach den Perioden und Invarianten, J. reine angew. Math. 92 (1882), 311–327.
- [11] Gelfand, I. M, Kapranov, M.M, and Zelevinsky, A.V., Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, 1994.
- [12] Grant. D., A generalization of Jacobi's derivative formula to dimension two, J. reine angew. Math. 392(1988), 125136 392 (1988), 125–136.
- [13] Lang, S., An introduction to algebraic and Abelian functions, (2nd ed.) ed., Springer-Verlag, 1982.
- [14] Manin, Ju.I., Algebraic curves over fields with differentiation, A. M. S. Transl., Twenty two papers on algebra, number theory and differential geometry **37** (1964), 59–78.
- [15] Matsumura, H., Commutative Ring Theory, Cambridge studies in adv. math., vol. 8, Cambridge Univ. Press., 1980.
- [16] Miura, S., Linear codes on affine algebraic curves (in Japanese, Affine Dai-su-kyoku-sen-zyo no sen-kei-fu-gou), Journal of the institute of electronics, information and communication engineering A, J81-A (1998), 1398–1421.
- [17] Nakayashiki, A., Sigma function as a tau function, International Mathematical Research Notices, **2010** (2010), 373–394.

- [18] Nayakashiki, A., On algebraic expressions of sigma functions for (n, s) curves, Asian J. Math. 14 (2010), no. 2, 175–212.
- [19] Ônishi, Y., Arithmetical power series expansion of the sigma function for a plane curve, to appear in Proc. Edinburgh Math. Soc.
- [20] _____, On Weierstrass' paper "Zur Theorie der elliptischen Functionen", http://www2.meijo-u.ac.jp/~yonishi/#publications.
- [21] _____, Universal elliptic functions, http://arxiv.org/abs/1003.2927, 2010.
- [22] ______, Theory of Abelian functions (in Japanese, Abel Kan-su-ron), Department of Math., Chuo University, 2013.
- [23] Rademacher, H., *Topics in analytic number theory*, Die Grundlehren der math. Wiss., vol. 169, Springer-Verlag, Berlin, Heidelberg, New York, 1973.
- [24] Weierstrass, K., Zur Theorie der elliptischen Functionen, Königl. Akademie der Wissenschaften 27 (1882), (Werke II, pp.245–255).
- [25] Zakalyukin, V.M., Reconstructions of fronts and caustics depending on a parameter and versality of mappings, Journal of Soviet Mathematics 27 (1984), 2713–2735.

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