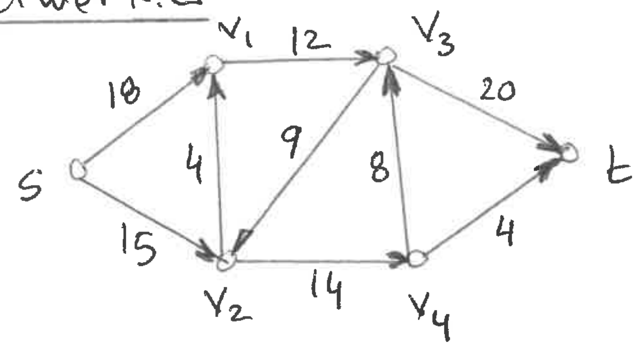
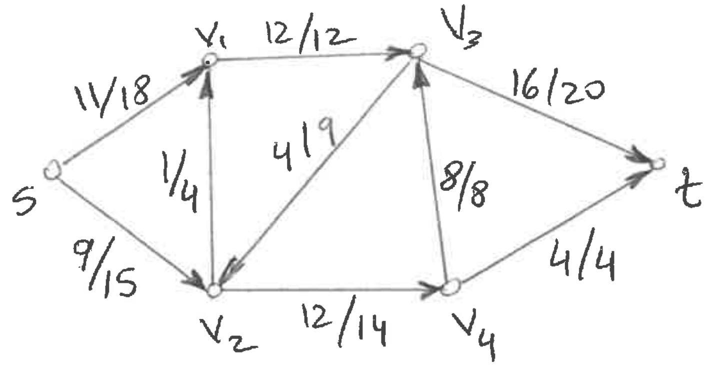
Maximum - flow problem

• flow network: G



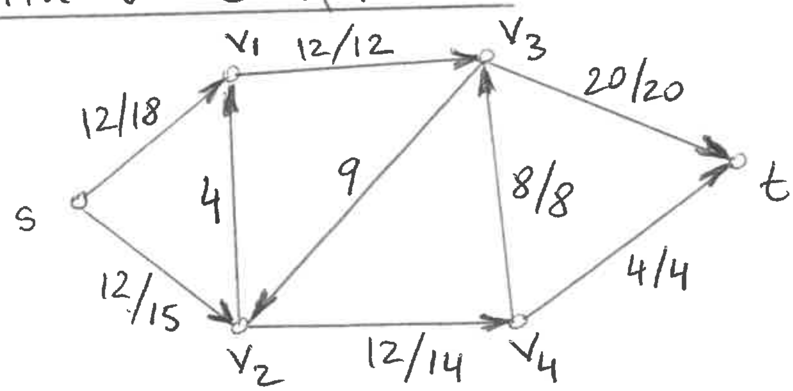
• A flow with value $|f|=20$

flow/capacity
11/18



- a flow must satisfy
  - capacity constraint
  - flow conservation

• A flow with value $|f|=24$



Maximum - flow problem: given a flow network G with source s and sink t, find a flow of maximum value.

Maximum-flow (standard form)

- directed graph $G(V,E)$
- source s, sink t
- capacity c of each edge

→ **Max-flow solver** (Ford-Fulkerson) → maximum-flow f

Network with antiparallel edges

Network:

- directed graph $G(V,E)$
- source s, sink t
- G has antiparallel edge(s)
- capacity c of each edge

$\xrightarrow{\text{(see slides)}}$

flow-network (standard form)

- directed graph $G'(V',E')$
- source s, sink t
- capacity c of each edge

→ **Max-flow solver** (Ford-Fulkerson) → max-flow f' of G' → max-flow f of G

Network with multiple sources, multiple sinks

Network

- directed graph $G(V,E)$
- multiple sources, multiple sinks
- capacity c of each edge

$\xrightarrow{\text{(see slides)}}$

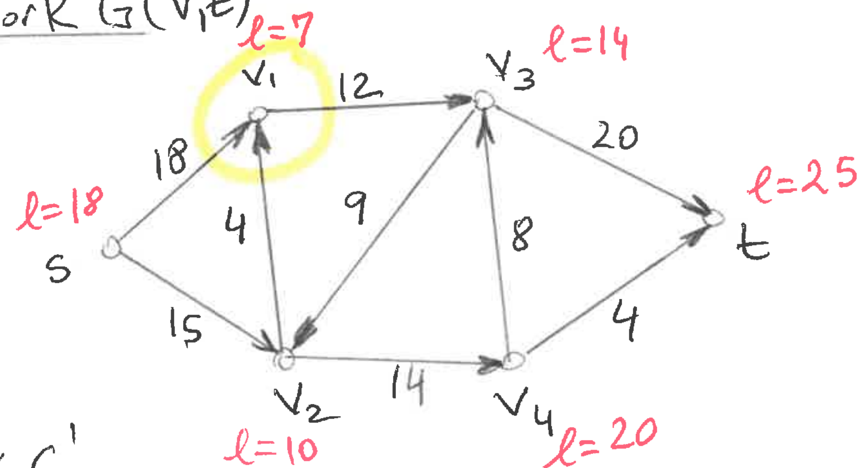
flow-network (standard form)

- directed graph $G'(V',E')$
- source s, sink t
- capacity c of each edge

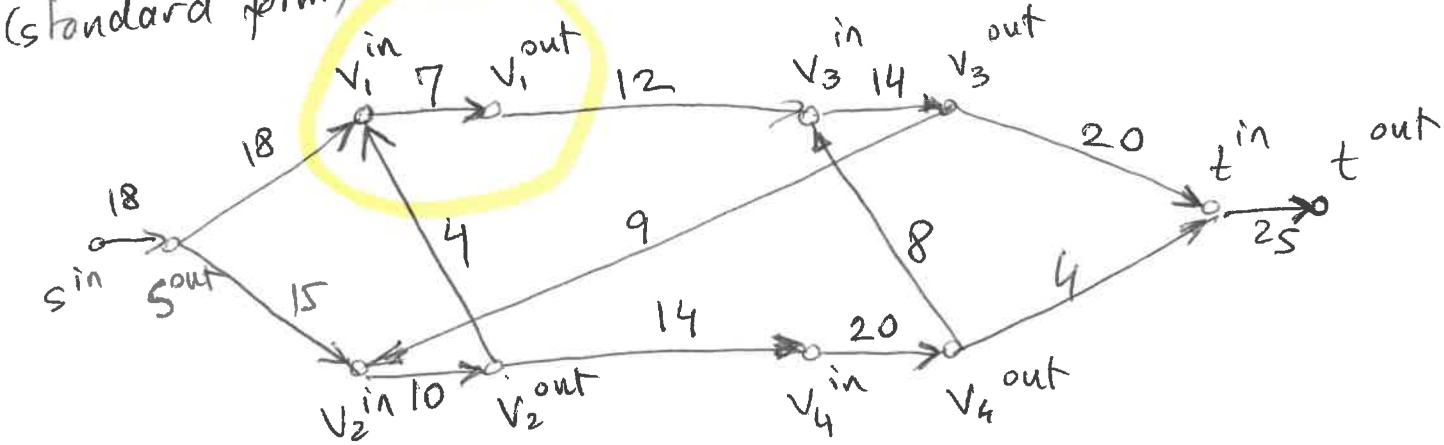
→ **Max-flow solver** (Ford-Fulkerson) → max-flow f' of G' → max-flow f of G

Suppose that in addition to edge capacities, a flow network has ***vertex capacities***. That is each vertex *v* has a limit *l*(*v*) on how much flow can pass through *v*. Show how to transform a flow network G = (V, E) with vertex capacities into an equivalent flow network G' = (V', E') without vertex capacities, such that a maximum flow in G' has the same value as a maximum flow in G. How many vertices and edges does G' have?

flow network G(V,E)



flow network G'
(standard form)



$G'(V', E')$

$|V'| = 2 \cdot |V|$

$|E'| = |E| + |V|$

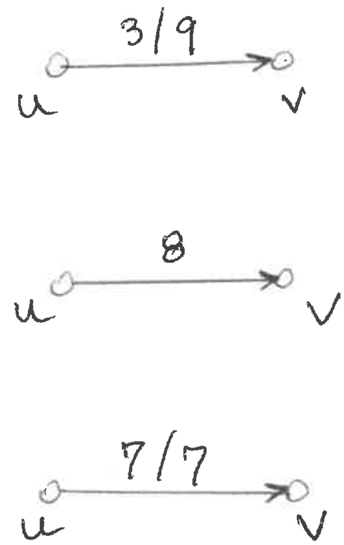
Residual network

residual capacity

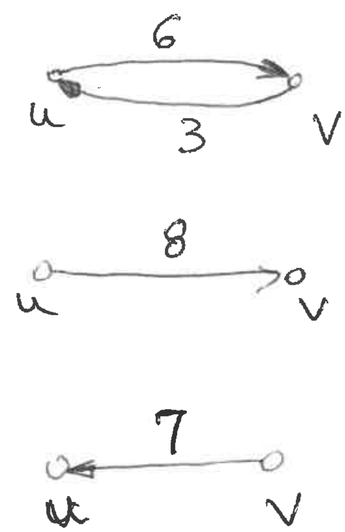
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise} \end{cases}$$

examples

flow network G



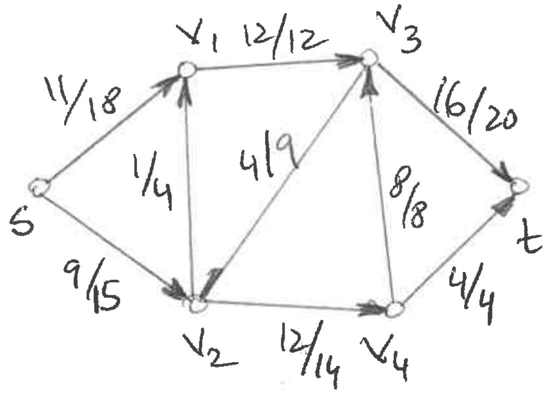
residual network G_f



Note: residual network $G_f(V, E_f)$

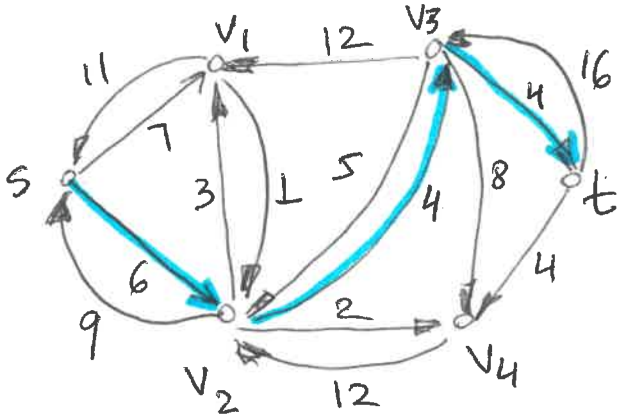
where $|E_f| \leq 2 \cdot |E|$

$\underline{G}$
(flow network)



$|f| = 20$

$\underline{G_f}$
(residual network)

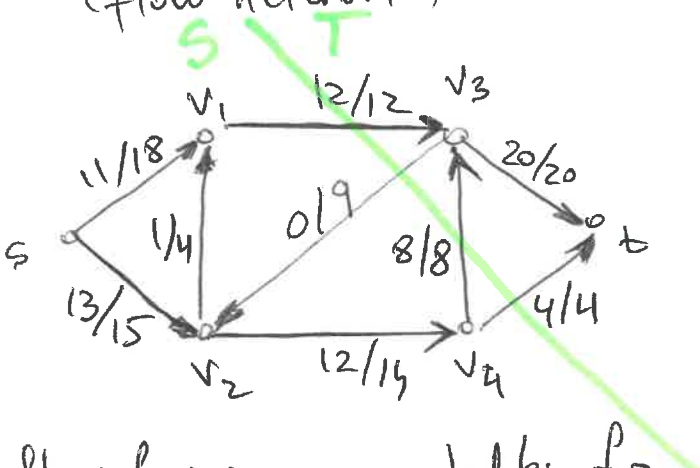


augmenting path $p = \langle s, v_2, v_3, t \rangle$

residual capacity $c_f(p) = \min\{c_f(s, v_2), c_f(v_2, v_3), c_f(v_3, t)\} = \min\{6, 4, 4\} = 4$

$|f_p| = 4$

$\underline{G}$
(flow network)



- flow f was augmented by f_p

$|f \uparrow f_p| = |f| + |f_p| = 20 + 4 = 24$

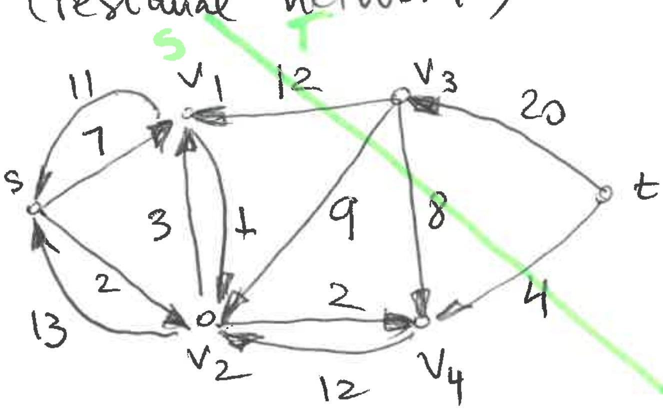
- the value of the flow is now $\underline{24}$

$c(S, T) = \underline{24}$

since $|f| = c(S, T) \Rightarrow$

$\Rightarrow f$ is $\underline{\text{a maximum-flow}}$

$\underline{G_f}$
(residual network)



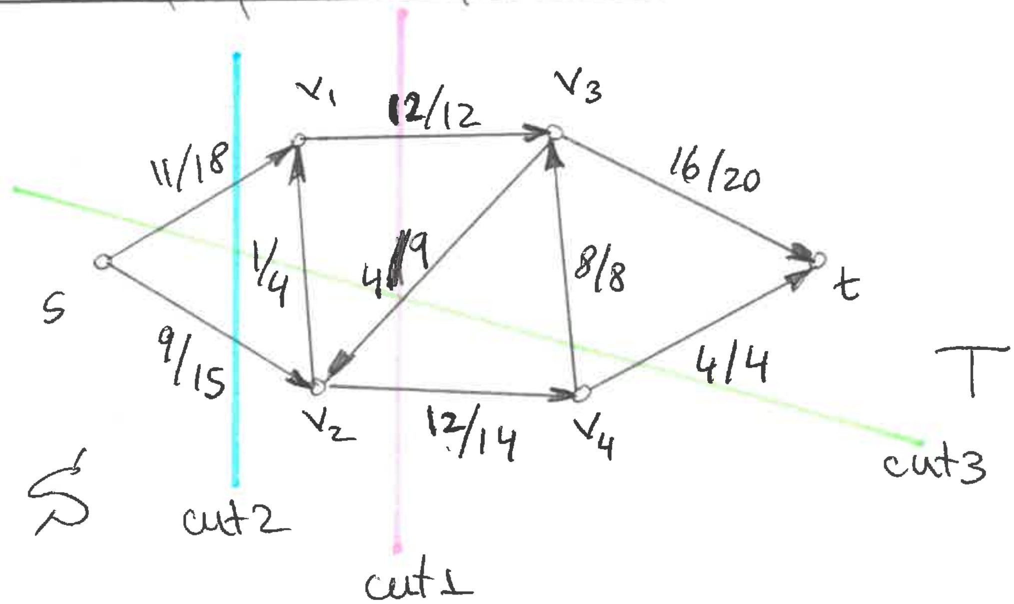
- cannot find an augmenting path from s to t $\Rightarrow$ we have reached the $\underline{\text{maximum-flow}}$!

$\underline{\text{cut } (S, T)}$

$S = \{s, v_1, v_2, v_4\}$

$T = \{v_3, t\}$

Cuts of flow networks



$|f| = 20$

cut1

$S = \{s, v_1, v_2\}$

$T = \{t, v_3, v_4\}$

$f(S,T) = 20$

$c(S,T) = 26$

cut2

$S = \{s\}$

$T = \{t, v_1, v_2, v_3, v_4\}$

$f(S,T) = 20$

$c(S,T) = 33$

cut3

$S = \{s, v_2, v_4\}$

$T = \{t, v_1, v_3\}$

$f(S,T) = 20$

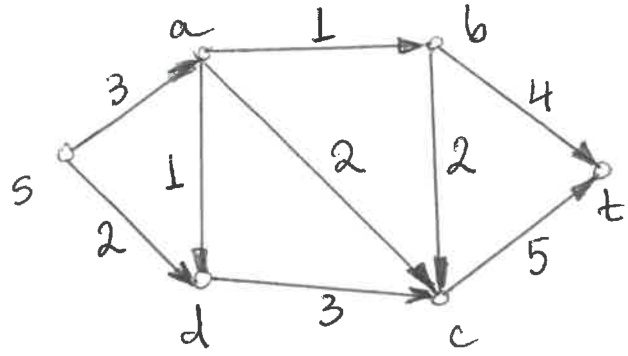
$c(S,T) = 34$

- For a given flow f, the net flow across any cut is the same, and it equals the flow value $|f|$
- For any cut (S,T), the net flow $f(S,T)$ is upper-bounded by the capacity of the cut $c(S,T)$.

$$f(S,T) \le c(S,T)$$

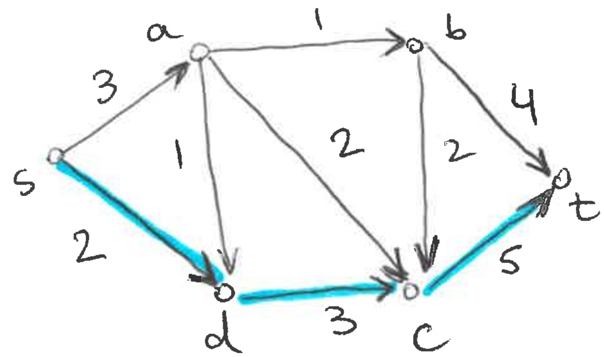
Use Ford-Fulkerson to compute the maximum-flow for the network G below.

flow network G



$|f| = 0$

G_f
(residual network)

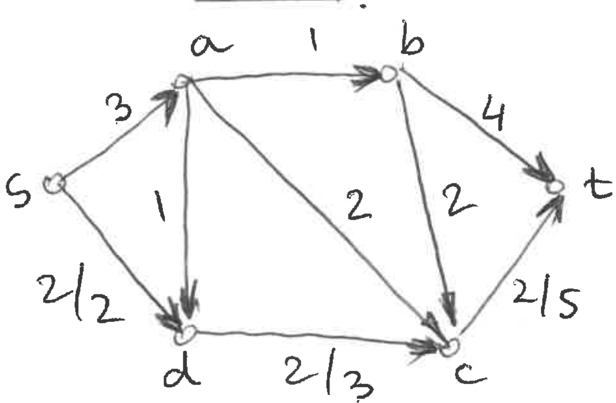


$p = \langle s, d, c, t \rangle$

$c_f(p) = 2$

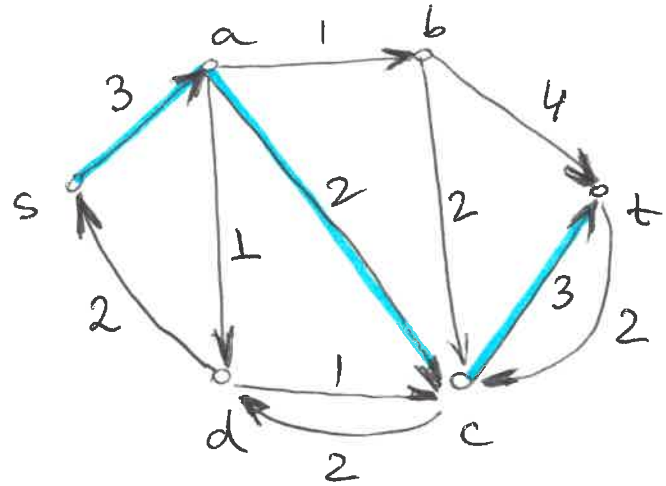
$|f_p| = 2$

G



$|f| = 2$

G_f

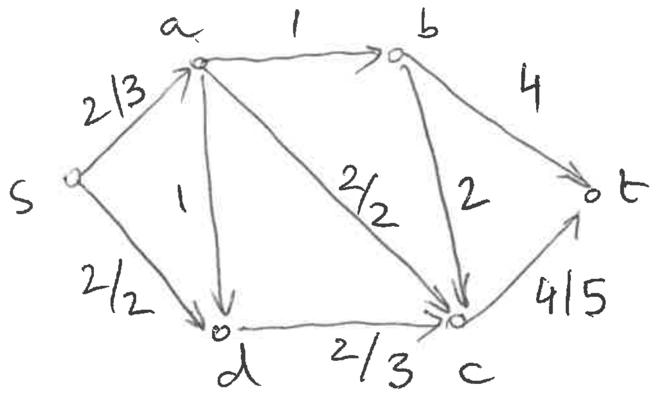


$p = \langle s, a, c, t \rangle$

$c_f(p) = 2$

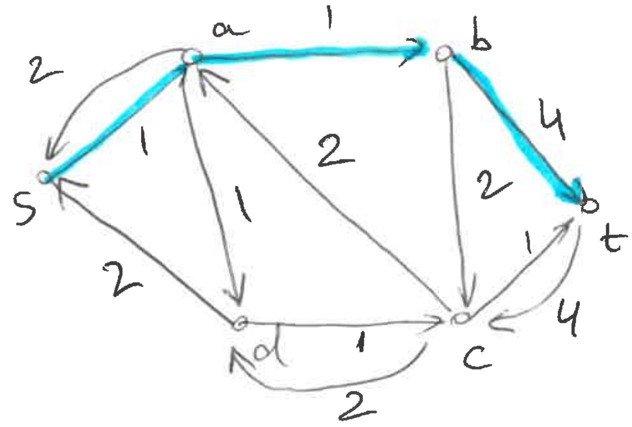
$|f_p| = 2$

G



$|f| = 4$

G_f



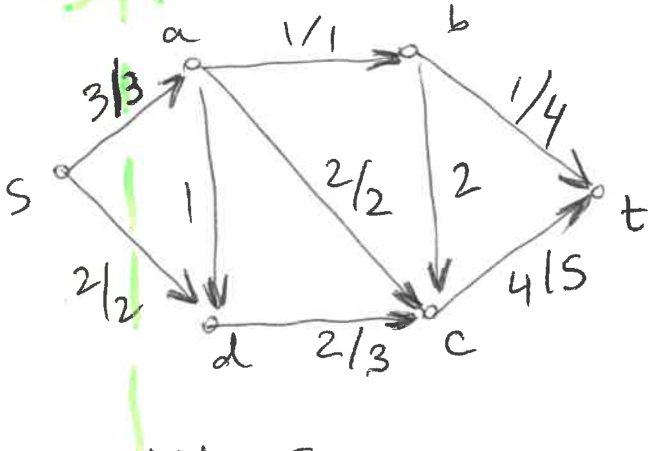
$p = \langle s, a, b, t \rangle$

$c_f(p) = 1$

$|f_p| = 1$

S|T

G



$|f| = 5$

this is a max-flow! ⟸

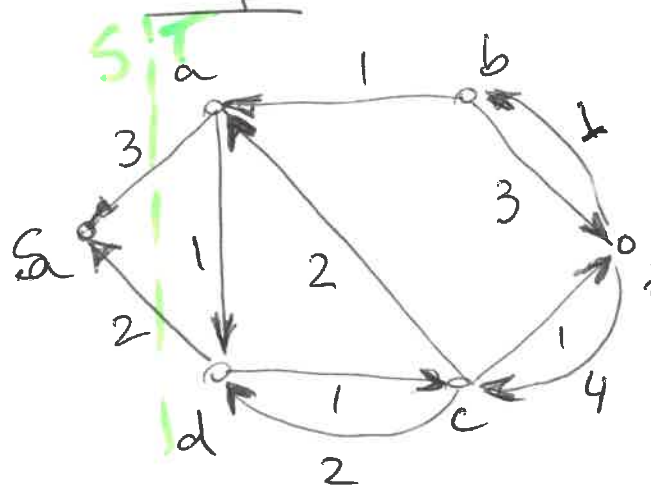
cut (S, T)

$c(S,T) = 5$

Since $|f| = c(S,T) \Rightarrow$

$\Rightarrow$ flow f is a max-flow!

G_f



no augmenting path $\Rightarrow$
we have reached the max-flow!

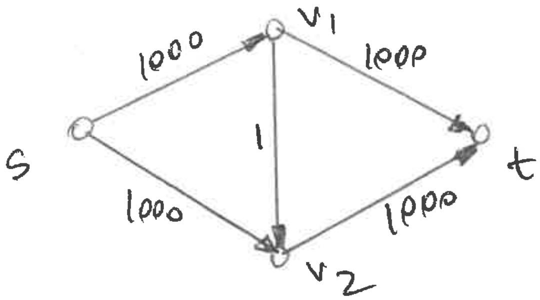
cut (S, T)

$S = \{s\}$

$T = \{t, a, b, c, d\}$

Example where max-flow converges slow:

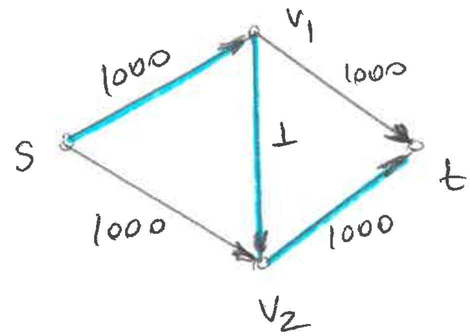
flow network G



$|f| = 0$

G_f

(residual network)



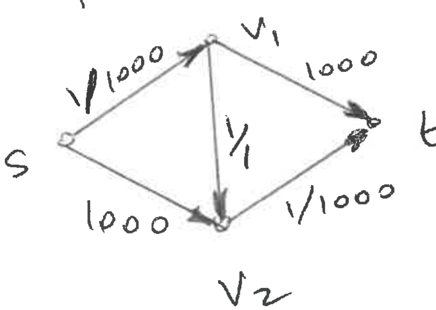
$p = \langle s, v_1, v_2, t \rangle$

$c_f(p) = 1$

$|f_p| = 1$

G

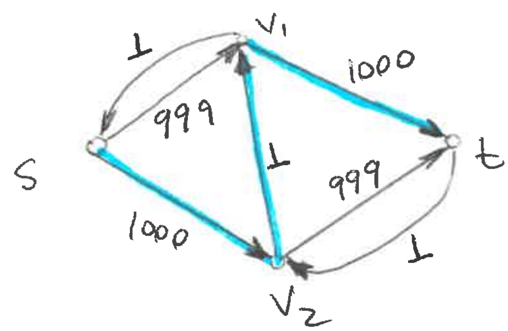
(flow network)



$|f| = 1$

G_f

(residual network)



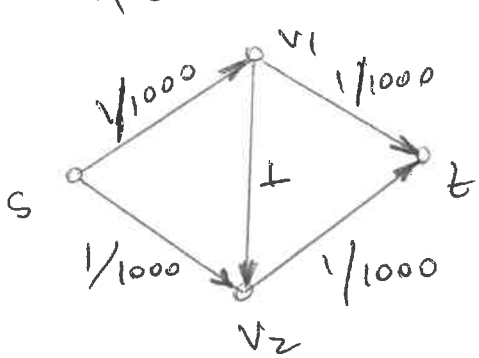
$p = \langle s, v_2, v_1, t \rangle$

$c_f(p) = 1$

$|f_p| = 1$

G

(flow network)



$|f| = 2$

. . . so on

Observations

- the flow increases 1 unit per iteration
- uses 2000 iterations to reach the maximum-flow $|f^*| = 2000$

Flow network

(standard form)

- directed graph $G(V, E)$
- source s, sink t
- capacity c of each edge

$\longrightarrow$ $\boxed{\text{Max-flow solver}}$ $\longrightarrow$ max-flow f

Ford-Fulkerson: $RT = O(E \cdot |f^*|)$

Edmonds-Karp: $RT = O(V \cdot E^2)$