Standard form of a Linear Program (LP)

maximize $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

subject to

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq b_2$$
$$\dots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$\dots$$
$$x_n \geq 0$$

Express a LP using summation notation

maximize $\sum_{j=1}^{n} c_j x_j$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m$$
$$x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n$$

Compact form of a LP

maximize $c^T x$

subject to $Ax \leq b$

$x \geq 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Note: we can specify a LP using the tuple (A, b, c)

Slack form of a LP

maximize $x_1 + x_2$

subject to

$$4x_1 - x_2 \le 8$$
$$2x_1 + x_2 \le 10$$
$$-5x_1 + 2x_2 \le 2$$
$$x_1, x_2 \ge 0$$

solution

$$x_3 = 8 - (4x_1 - x_2) = 8 - 4x_1 + x_2$$
$$x_4 = 10 - (2x_1 + x_2) = 10 - 2x_1 - x_2$$
$$x_5 = 2 - (-5x_1 + 2x_2) = 2 + 5x_1 - 2x_2$$
$$x_3, x_4, x_5 \ge 0$$

Slack form of the LP:

$$\cancel{4x_1 - x_2 + x_3 = 8}$$
$$\cancel{2x_1 + x_2 + x_4 = 10}$$
$$\cancel{-5x_1 + 2x_2 + x_5 = 2}$$

$$\boxed{\begin{array}{l} \text{maximize } x_1 + x_2 \\ \text{subject to} \\ \quad \left\{ \begin{array}{l} 4x_1 - x_2 + x_3 = 8 \\ 2x_1 + x_2 + x_4 = 10 \\ -5x_1 + 2x_2 + x_5 = 2 \\ x_1, x_2, x_3, x_4, x_5 \ge 0 \end{array} \right. \end{array}}$$

Formulate the following problem as an LP:

A carpenter makes tables and chairs. Each table can be sold for a profit of \$30 and each chair for a profit of \$10. The carpenter can afford to spend up to 40 hours per week working and takes 6 hours to make a table and 3 hours to make a chair. Customer demand requires that he makes at least three times as many chairs as tables. Tables take up four times as much storage space as chairs and there is room for at most four tables each week. The objective is to maximize the profit per week.

Hint: use two variables, X_T – number of tables made per week, and X_C - number of chairs made per week

Solution

Integer Linear Program

maximize $30 \cdot X_T + 10 \cdot X_C$

subject to

$$6 \cdot X_T + 3 \cdot X_C \leq 40$$

$$3 \cdot X_T \leq X_C$$

$$X_T + \frac{1}{4} \cdot X_C \leq 4$$

$$X_C, X_T \geq 0$$

$$X_C, X_T \in \mathbb{N} \quad (\{0, 1, 2, 3, \ldots\})$$

- NP-hard

The Knapsack Problem

Example: Knapsack capacity W = 16		
item	weight	value
1	2	$20
2	5	$30
3	10	$50
4	5	$10

Given:

- n objects and a knapsack
- object i (i = 1,..,n) has a positive weight w_i and a positive value v_i
- the knapsack can carry a weight ≤ W

Objective: fill the knapsack s.t. to maximize the value of the included objects, while respecting the capacity constraints

Two variations:

- **0-1 knapsack problem**: you can only take the whole object
- **fractional knapsack problem**: you can take fractions of objects

Formulate this problem using Linear Programming.

Solution

0-1 Knapsack problem

n variables : $x_1, x_2 \dots x_n$

$x_i = \begin{cases} 0 & \text{item } i \text{ is NOT selected} \\ 1 & \text{item } i \text{ is selected} \end{cases}$

Boolean/Integer Linear Program

maximize $v_1x_1 + v_2x_2 + \dots + v_nx_n$

subject to $w_1x_1 + w_2x_2 + \dots + w_nx_n \leq W$

$x_1, x_2, \dots, x_n \in \{0, 1\}$

· NP-hard

Fractional Knapsack Problem

n variables : $x_1, x_2, \dots, x_n$, such that $0 \leq x_i \leq 1$

x_i - fraction of item i which is being added to the knapsack

Linear Program

maximize $\sum_{i=1}^{n} v_i x_i$

subject to $\sum_{i=1}^{n} w_i x_i \leq W$

$x_i \leq 1$ for $i = 1$ to n

$x_i \geq 0$ for $i = 1$ to n

RT is polynomial

The Assignment Problem

There are n people who need to be assigned to n jobs, one person per job. The cost of assigning person i to job j is C[i,j]. Find an assignment that minimizes the total cost.

	Job 1	Job 2	Job 3	Job 4
Person 1	0 9	1 2	0 7	0 8
Person 2	0 6	0 4	1 3	0 7
Person 3	1 5	0 8	0 1	0 8
Person 4	0 7	0 6	0 9	1 4

Formulate this problem using Linear Programming.

Solution

n^2 variables x_{ij} $\quad i,j = 1,2,\dots,n$

$$x_{ij} = \begin{cases} 0 & \text{if person } i \text{ is NOT assigned job } j \\ 1 & \text{if person } i \text{ is assigned job } j \end{cases}$$

Boolean/Integer Linear Program

minimize $\displaystyle\sum_{\substack{i=1..n \\ j=1..n}} c_{ij} \cdot x_{ij}$

job j is assigned to one person

subject to

n constraints $\displaystyle\sum_{i=1}^{n} x_{ij} = 1$ for each $j = 1..n$

person i is assigned one job

n constraints $\displaystyle\sum_{j=1}^{n} x_{ij} = 1$ for each $i = 1..n$

n^2 constraints $x_{ij} \in \{0,1\}$ for each $i=1..n, j=1..n$

- no. of variables is n^2
- no. of constraints is $n^2 + 2n$

- NP-hard

Change-making problem

Consider a given amount of money n and the coin system consisting of quarters, dimes, nickels, and pennies. How can the given amount of money n be made with the least number of coins?

Formulate this problem using Linear Programming.

solution

x_1 - no. of quarters
x_2 - no. of dimes
x_3 - no. of nickels
x_4 - no. of pennies

Integer Linear Program

minimize $x_1 + x_2 + x_3 + x_4$

subject to

$$25x_1 + 10 \cdot x_2 + 5 \cdot x_3 + x_4 = n$$
$$x_1 \in \mathbb{N} \quad \{0,1,2,\ldots\}$$
$$x_2 \in \{0,1,2\}$$
$$x_3 \in \{0,1\}$$
$$x_4 \in \{0,1,2,3,4\}$$

- NP-hard.