A Simple Form of the Bianchi I Constraint

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I. INTRODUCTION

Loop Quantum Cosmology has quickly developed into an active field of research. The FLRW model has been intensely analyzed. The k=0 model is exactly soluble - allowing one to see the bounce analytically as well as put an upper bound on the energy density. Extensive numerical evolutions have been run for the homogeneous models. For the anisotropic models though the constraint operator has been to complex to allow for analysis beyond the effective classical equations. Where the FLRW constraint operator preserved a regular lattice in the volume, the Bianchi one constraint preserves a regular lattice in the volume but takes values on a dense subset of the reals in the anisotropic parameters. This poses complications with both the numerical evolution looking for eigenstates of the constraint and with numerical evolution of semiclassical states. Further it poses great complications for the analytical analysis of the theory.

In this paper we present a dramatic simplification of the Bianchi one constraint allowing for both numerical evolution and possibly analytical study. I will overview the classical framework then the loop quantum quantization and finally the simplification. Given the simplification I will probe the spectrum of the constraint operator.

II. CLASSICAL FRAMEWORK

We introduce a fiducial metric ${}^oq_{ab}$, ${}^ods^2 = dx^2 + dy^2 + dz^2$, and a fiducial cell V whose edges lie along the coordinates directions x, y, z. The fiducial cell has edge lengths L_i and volume ${}^oV = L_1L_2L_3$ as measured by the fiducial metric.

Due the symmetries of the model the connection and triad can be reduced to the following form

$$A^i_a =: c^i(L^i)^{-1\,o}\omega^i_a \quad E^a_i =: p_iL_iV_o^{-1}\sqrt{{}^oq}\,{}^oe^a_i \tag{2.1}$$

Where the connection is determined entirely by the three variables c_i and the triad by the three variables p_i. They have the following simple Poisson bracket.

$$\{c^i, p_j\} = 8\pi G\delta^i_j \tag{2.2}$$

In this form the connection and triad automatically solve the Gauss and diffeomorphism

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constraints leaving only the scalar constraint. The choice of harmonic time, catered to the use of the scalar field as a clock, $N = \sqrt{|p_1 p_2 p_3|}$ leads to the following form of the constraint

$$\mathcal{C}_H = -\frac{1}{8\pi G\gamma^2}(p_1 p_2 c_1 c_2 + p_1 p_3 c_1 c_3 + p_2 p_3 c_2 c_3) + \frac{p_T^2}{2} \tag{2.3}$$

Here the constraint is for Bianchi I with a massless the scalar field, but in this paper we will focus mostly on the gravitational part of the constraint. From the constraint we find that we there are three constants of motion $c_i p_i$ (no summation), which commute trivially with the constraint. **signifigance to simplification**

The relation of c, p to the scale factor is given by

$$p_1 = \mathrm{sgn}(a_1)|a_2 a_3| L_2 L_3 \quad p_2 = \mathrm{sgn}(a_2)|a_1 a_3| L_1 L_3 \quad p_3 = \mathrm{sgn}(a_3)|a_1 a_2| L_1 L_2 \tag{2.4}$$

and

$$c_i = \gamma L_i V_o^{-1} (a_1 a_2 a_3)^{-1} \frac{da_i}{d\tau} \tag{2.5}$$

where the equations of motion are used to show that the c's are related to the time derivative of the scale factor. The constants of motion are then

$$p_i c_i = \gamma \frac{1}{a}\frac{da_i}{d\tau}, \tag{2.6}$$

which are related to the expansion rate along each direction. These constants will be relevant to the interpretation of the simplification of the quantum constraint.

III. QUANTUM KINEMATICS

As in Loop Quantum Gravity we take the elementary variables to be the fluxes and holonomies. The fluxes are determined by the triad variable p_i. The holonomy along one of the coordinate directions is given by

$$h_1^{(\ell)}(c_1, c_2, c_3) = \cos\frac{c_1 \ell}{2}\mathbb{I} + 2\sin\frac{c_1 \ell}{2}\tau_1 \tag{3.1}$$

which is an almost period function in c. Hence as our basic phase space variables we take $\exp(i\ell c_j)$ and p_i.

$$\{e^{i\ell c_j}, p_i\} = i\ell e^{i\ell c_j}\delta_{ij} \tag{3.2}$$

In LQC we have the luxury of working in the position, holonomy representation, or the momentum, flux, representation. Whereas in the full theory we have only the holonomy representation, although recently there has been progress in constructing a flux representation of LQG.

In the holonomy representation the Hilbert space consists of wave functions $\Psi(c)$ with finite norm

$$||\Psi||^2 = \lim_{\alpha\to\infty}\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} \mathrm{d}c|\Psi(c)|^2 \tag{3.3}$$

Where holonomies act by multiplication and p_i acts by differentiation.

$$\hat{e^{i\ell c}}\Psi(c) = e^{i\ell c}\Psi(c) \quad \hat{p}_i\Psi(c) = -i\partial_c\Psi(c) \tag{3.4}$$

The space is the space of square integrable function on the Bohr compactification of R. The non-existence of the operator c can be seen clearly since normalizable states are those that are asymptotically constant. Any state has non-zero finite norm will be mapped to a state with infinite norm by the action of c.

On the other hand we can work in the conjugate representation where the Hilbert space consists of wave function $\Psi(p)$ with finite norm

$$||\Psi||^2 = \sum_p |\Psi(p)|^2 \tag{3.5}$$

where the sum here is over all real numbers. In the representation the p's act by multiplication and the holonomies by translations.

$$\hat{p}\Psi(p) = p\Psi(p) \quad \hat{e^{i\ell c}}\Psi(p) = \Psi(c+\ell) \tag{3.6}$$

The 'fourier transform' between these two representations is given by

$$\tilde{\Psi}(p) = \lim_{\alpha\to\infty} \frac{1}{2\alpha}\int_{-\alpha}^{\alpha} \mathrm{d}c\, e^{icp}\Psi(c) \tag{3.7}$$

with the inverse given by

$$\Psi(c) = \sum_p e^{-icp}\tilde{\Psi}(p) \tag{3.8}$$

For simplicity the dynamics are presented in the triad representation, but the simplification of the constraint is present in the holonomy representation.

IV. CONSTRAINT

To obtain the dynamics of the theory we need to represent the constraint as an operator on our kinematical hilbert space. This requires writing our constraint

$$\mathcal{H} = \frac{E^a_i E^b_j}{16\pi G\sqrt{|q|}}\left(\epsilon^{ij}{}_k F_{ab}{}^k - 2(1+\gamma^2)K^i_{[a}K^j_{b]}\right) \tag{4.1}$$

in terms of the basic variables, which requires writing the curvature in terms of the holonomies around square loops along the coordinate directions.

$$F_{ab}{}^k = 2\lim_{Ar_\square\to 0} \mathrm{Tr}\left(\frac{h_{\square_{ij}} - \mathbb{I}}{Ar_\square}\tau^k\right) {}^o\omega^i_a\, {}^o\omega^j_b, \tag{4.2}$$

From the non-existence of an operator corresponding to the connection the limit $Ar_\square \to 0$ does not exist. If we chose the physical area of the loop to be the smallest area eigenvalue of loop quantum gravity, the lengths of the holonomies are fixed to be

$$\bar{\mu}_1 = \sqrt{\frac{|p_1|\Delta\,\ell_{\rm Pl}^2}{|p_2p_3|}}, \quad \bar{\mu}_2 = \sqrt{\frac{|p_2|\Delta\,\ell_{\rm Pl}^2}{|p_1p_3|}}, \quad \bar{\mu}_3 = \sqrt{\frac{|p_3|\Delta\,\ell_{\rm Pl}^2}{|p_1p_2|}}\,. \tag{4.3}$$

and the constraint is written in terms of operators

$$\exp^{ic_i\bar{\mu}_i} \tag{4.4}$$

which have very complex action on the states $\Psi(p_1, p_2, p_3)$.

Much like in the FLRW models the action of these holonomies is simplified if we change variables to λ_i defined by,

$$\lambda_i \;=\; \frac{\mathrm{sgn}(p_i)\,\sqrt{|p_i|}}{(4\pi|\gamma|\,\sqrt{\Delta}\,\ell_{\rm Pl}^3)^{1/3}}\,, \tag{4.5}$$

and the action of the constraint is further simplified is we introduce the volume

$$v = 2\lambda_1\lambda_2\lambda_3 \tag{4.6}$$

under these change of variables our Hilbert space is now functions of λ_1, λ_2, and v, all restricted to be positive, with finite norm.

$$||\Psi||^2 = \sum_{\lambda_1,\lambda_2,v} |\Psi(\lambda_1, \lambda_2, v)|^2 < \infty \tag{4.7}$$

The action of the constraint on wave functions of λ_1, λ_2, and v takes the following simplified form.

$$\Theta\Psi(\lambda_1, \lambda_2, v) = \frac{\pi G}{2}\sqrt{v}\,\Big[-(v+2)\sqrt{v+4}\Psi_4^+(\lambda_1, \lambda_2, v) + (v+2)\sqrt{v} \tag{4.8}$$

$$\Psi_0^+(\lambda_1, \lambda_2, v) + (v-2)\sqrt{v}\Psi_0^-(\lambda_1, \lambda_2, v) - (v-2)\sqrt{v-4}\Psi_4^-(\lambda_1, \lambda_2, v)\Big] \tag{4.9}$$

Which surprisingly acts as a difference operator in the volume v much like the FLRW models. There is one term that shifts the volume forward by 4, two that leave the volume the same, and one that shifts the volume back by 4. The complexity of the action of the constraint arises in how it changes the anistropies labelled by λ_i. The action on the anisotropies is given by multiplicative shifts depending on the volume.

$$\begin{aligned}\Psi_4^\pm(\lambda_1, \lambda_2, v) = {} & \Psi\left(\frac{v\pm4}{v\pm2}\cdot\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\pm4;\right) + \Psi\left(\frac{v\pm4}{v\pm2}\cdot\lambda_1, \lambda_2, v\pm4;\right) \\ & + \Psi\left(\frac{v\pm2}{v}\cdot\lambda_1, \frac{v\pm4}{v\pm2}\cdot\lambda_2, v\pm4;\right) + \Psi\left(\frac{v\pm2}{v}\cdot\lambda_1, \lambda_2, v\pm4;\right) \\ & + \Psi\left(\lambda_1, \frac{v\pm2}{v}\cdot\lambda_2, v\pm4;\right) + \Psi\left(\lambda_1, \frac{v\pm4}{v\pm2}\cdot\lambda_2, v\pm4;\right),\end{aligned} \tag{4.10}$$

$$
\begin{aligned}
\Psi_0^{\pm}(\lambda_1,\lambda_2,v;) = {} & \Psi\left(\frac{v\pm 2}{v}\cdot\lambda_1, \frac{v}{v\pm 2}\cdot\lambda_2, v;\right) + \Psi\left(\frac{v\pm 2}{v}\cdot\lambda_1, \lambda_2, v;\right) \\
& + \Psi\left(\frac{v}{v\pm 2}\cdot\lambda_1, \frac{v\pm 2}{v}\cdot\lambda_2, v;\right) + \Psi\left(\frac{v}{v\pm 2}\cdot\lambda_1, \lambda_2, v;\right) \\
& + \Psi\left(\lambda_1, \frac{v}{v\pm 2}\cdot\lambda_2, v;\right) + \Psi\left(\lambda_1, \frac{v\pm 2}{v}\cdot\lambda_2, v;\right), \qquad (4.11)
\end{aligned}
$$

Where the action on the volume picks out nice invariant lattices. It can be seen that the invariant lattices in the anisotropies are isomorphic to the rationals and are thus dense in the space of all reals. It is here that the difficulty in analytical and numerical evaluation arise.

V. SIMPLIFICATION OF CONSTRAINT

Now we introduce a further simplification of the constraint. The first step is to introduce the simple change of variables $x_i = \ln(\lambda_i$. Under this change multiplicative shifts in λ_i become translations in x_i. The range of variables goes from λ ranging from $(0,\infty)$ to x ranging from $(-\infty,\infty)$. And the measure changes simply since the change of variables in one to one $\sum_\lambda = \sum_x$. Further it is simple to see that the constraint commutes with all translations in x. We can then simultaneously diagonalize the constraint operator and the operators' generating translations.

$$
U_1(a)\Psi(\lambda_1,\lambda_2,v) = \Psi(\lambda_1 + a, \lambda_2, v) \qquad (5.1)
$$

$$
[\Theta, U_i(a)] = 0 \qquad (5.2)
$$

The eigenstates of translations are exponentials as we expect, $e^{ix_i m_i}$. We can then go to the dual representation, $\Psi(m_i, m_2, v)$.

$$
\Psi(m_1,m_2,v) = \sum_{x_1,x_2} e^{-im_1x_1}e^{-im_2x_2}\Psi(x_1,x_2,v) \qquad (5.3)
$$

Surprisingly in this representation the constraint takes a very simple form, namely it reduces to a difference equation much like in the FRW models. The action of the constraint is given by

$$
\Theta\Psi(m_1,m_2,v) = -f_+^4(m_1,m_2,v)\Psi(m_1,m_2,v+4) + (f_+^0 + f_-^0)\Psi(m_1,m_2,v) \qquad (5.4)
$$

$$
-f_-^4(m_1,m_2,v)\Psi(m_1,m_2,v-4) \qquad (5.5)
$$

The m labels are unchanged by the action of Θ as we expect. The operator simply shifts the volume in steps of 4 as with the FLRW Θ. The difference lies simply in the functions that multiply the shifts in volume. These functions are given by,

$$
f_\pm^4 = \frac{\pi G}{2}\sqrt{v}(v\pm 2)\sqrt{v\pm 4}\left(e^{im_1\ln(\frac{v\pm 4}{v\pm 2})}e^{im_2\ln(\frac{v\pm 2}{v})} + e^{im_1\ln(\frac{v\pm 4}{v\pm 2})} + e^{im_1\ln(\frac{v\pm 2}{v})} + m_1 \leftrightarrow m_2\right) \qquad (5.6)
$$

and

$$f^0_{\pm} = \frac{\pi G}{2}\sqrt{v}(v \pm 2)\sqrt{v}\left(e^{im_1 \ln(\frac{v}{v\pm 2})}e^{im_2 \ln(\frac{v\pm 2}{v})} + e^{im_1 \ln(\frac{v}{v\pm 2})} + e^{im_1 \ln(\frac{v\pm 2}{v})} + m_1 \leftrightarrow m_2\right) \quad (5.7)$$

While these functions are complex for fixed values of m_i it is a standard difference equation, which can be analyzed using standard tools.

VI. INTERPRETATION

We have seen that by carrying out a fourier transform to the variables conjugate to x_i the constraint simplifies dramatically, but what has happened physically? What is the classical variable associated to the m_i? Classically we find that the canonically pair is given by

$$x_i = \ln\left(\frac{\sqrt{|p_i|}}{(4\pi|\gamma|\sqrt{\Delta}\,\ell_{\rm Pl}^3)^{1/3}}\right) \quad (6.1)$$

$$m_1 = 2(p_1c_1 - p_3c_3) \quad m_2 = 2(p_2c_2 - p_3c_3) \quad (6.2)$$

We see that these commute with the volume

$$\{m_i, v\} = 0 \quad (6.3)$$

The conjugate to the volume is given by

$$b = \frac{2}{3v}(p_1c_1 + p_2c_2 + p_3c_3) \quad (6.4)$$

So we can similarly that the m_i commute with the conjugate to the volume. We then have a canonical transformation from (c_i, p_i) to (x_i, m_i, v, b). Recall that classically p_ic_i was related to the hubble rate in the i-th direction. The m_i then define the difference in the Hubble rates between the 3 direction - or describe the anisotropy of the system.

We then see that taking $m_i = 0$ is equivalent to enforcing isotropy, and indeed if $m_i = 0$ the constraint reduces to that of the $k = 0$ FRLW model. How does this projection compare with that done in [**?**]. The original projection introduced was,

$$\Psi_{FRW} = \sum_{\lambda_1,\lambda_2} \Psi(\lambda_1, \lambda_2, v) \quad (6.5)$$

Translating this to the variables x_i and then taking the fourier transform we find that

$$\Psi_{FRW} = \sum_{x_1,x_2} \lim_{\alpha\to\infty}\frac{1}{2\alpha_1}\int_{-\alpha_1}^{\alpha_1} \mathrm{d}m_1 \lim_{\alpha\to\infty}\frac{1}{2\alpha_2}\int_{-\alpha_2}^{\alpha_2} \mathrm{d}m_2 e^{im_1x_1}e^{im_2x_2}\Psi(m_1, m_2, v) \quad (6.6)$$

which reduces to

$$\Psi(v)_{FRW} = \lim_{\alpha\to\infty}\frac{1}{2\alpha_1}\int_{-\alpha_1}^{\alpha_1} \mathrm{d}m_1 \lim_{\alpha\to\infty}\frac{1}{2\alpha_2}\int_{-\alpha_2}^{\alpha_2} \mathrm{d}m_2 \bar{\delta}(m_1)\bar{\delta}(m_2)\Psi(m_1, m_2, v) = \Psi(0, 0, v) \quad (6.7)$$

Where $\bar{\delta}(m)$ is the generalization of the Dirac delta distribution for the Bohr compactification, defined here in terms of its Fourier transform.

$$\bar{\delta}(m) = \sum_x e^{ixm} \tag{6.8}$$

VII. SPECTRUM

We can make a quick analysis of the spectrum of the operator to gain insight into the model. We know that in the FLRW k=0 model the Theta operator has a continuous spectrum $(0, \infty)$ and has 0 as a discrete eigenvalue. The discrete eigenvalue given by the state with support on only the zero volume. This is reasonable since there are non-trivial solutions to the classical equations of motion without matter. On the other hand there are non-trivial solutions for vacuum Bianchi I, so how does the spectrum change. We would expect to see that the spectrum changes to include a non-trivial zero eigenvalue.

We can carry out an asymptotic expansion of the difference equation for large v, since the nature of the spectrum depends only on the asymptotic behavior - i.e. the normalizability of the eigenstates.

For large v the eigenvalue equation can be expanded as,

$$-f_+^4(m_1, m_2, v)\Psi(m_1, m_2, v+4) + (f_+^0 + f_-^0)\Psi(m_1, m_2, v) \tag{7.1}$$

$$-f_-^4(m_1, m_2, v)\Psi(m_1, m_2, v-4) = \omega\Psi(m_1, m_2, v) \tag{7.2}$$

where now the functions have simplified to

$$f_\pm^4 \to= 6v^2 \pm 8v\,(3 + i(m_1 + m_2)) + 4\left(3 + 4i(m_1 + m_2) - (m_1^2 + m_2^2 + (m_1 + m_2)^2)\right) \tag{7.3}$$

$$(f_+^0 + f_-^0) = 12v^2 - 8(m_1^2 + m_2^2 + (m_1 - m_2)^2) \tag{7.4}$$

Should we work with this equation - or change the normalization by $\sqrt{v}$ Might be easier to study that equation.

We can rearrange this equation by writing

$$-f_+^4(m_1, m_2, v)\Psi(m_1, m_2, v+4) + 12v^2\Psi(m_1, m_2, v) \tag{7.5}$$

$$-f_-^4(m_1, m_2, v)\Psi(m_1, m_2, v-4) = (\omega + 8(m_1^2 + m_2^2 + (m_1 - m_2)^2))\Psi(m_1, m_2, v) = \omega'\Psi(m_1, m_2, v) \tag{7.6}$$

If I can prove that the resulting left side is positive definite then we're done, but I don't think that it true.

Now we have an operator whose diagonal term is postiive, so it is more natural to assume that the constraint is positive definite. If it were we should see that the spectrum is continuous and runs from $(-m_1^2 + m_2^2 + (m_1 - m_2)^2)$ to ∞.

In this case if we insert

Check this expansion - it's good

$$f_{\pm 4} = \frac{\pi G}{2}\sqrt{v}(v \pm 2)\sqrt{v \pm 4}\left[6 \pm \frac{8i(m_1 + m_2)}{v} - \frac{16i(p_1 + p_2)}{v^2} - \frac{8}{v^2}(m_1^2 + m_1 m_2 + m_2^2)\right] \tag{7.7}$$

$$f_{\pm 0} = \frac{\pi G}{2} v(v \pm 2)\left[6 - \frac{8}{v^2}(m_1^2 - m_1 m_2 + m_2^2)\right] \tag{7.8}$$

We can numerically evolve the eigenvalue equation and look for the asymptotic behavior. When the state is normalizable with the kinematic inner product we have a state with a discrete spectrum. When the state is oscillatory and not divergent the value is part of the continuous spectrum, and when the state diverges there is no solution corresponding to that eigenvalue.

For the data obtained the eigenvalues e are related to the original eigenvalues by

$$\omega = 8\pi G\, e \tag{7.9}$$

As expected the Θ operator has a continuous spectrum that extends into the negative numbers. Further as the parameters labelling the anisotropy are increased the minimum eigenvalue becomes more and more negative and approaches 0 as the parameters go to zero. From the numerical analysis we determine an approximate form for the minimum value to be $-\frac{16\pi G}{3}(m_1^2 + m_2^2 + (m_1 - m_2)^2)$.