Stochastic Models

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Abstract

The text deals with the analysis of stochastic differential equations and ideas for portfolio optimization that trades stocks subject to a log-normal distribution. The text was created without any knowledge of finance. The only assumption is that stock prices are given by the equation $\ln \dot{\mathrm{X}}(t) = \boldsymbol{\gamma} + \boldsymbol{\xi}$. The text is not complete and probably never will be; it is more of a collection of ideas, useful formulas, and unfinished thoughts, nothing serious.

1 Langevin white noise $\xi(t)$

Properties in one dimension:

$$\langle \xi(s) \rangle = 0 \quad \text{and} \quad \langle \xi(s)\xi(s') \rangle = \sigma^2 \delta(s - s') \quad \forall s, s' \in [0, T]. \tag{1}$$

Where σ is intensity of the noise.

In more dimension we have

$$\langle \boldsymbol{\xi}(s) \rangle = 0 \quad \text{and} \quad \left\langle \boldsymbol{\xi}(s)\boldsymbol{\xi}^T(s') \right\rangle = \Sigma \delta(s - s') \quad \forall s, s' \in [0, T]. \tag{2}$$

Where Σ is covariance matrix. Note that there exist matrix, denoted by $\sqrt{\Sigma}$, such that $\boldsymbol{\xi}(s) = \sqrt{\Sigma}\boldsymbol{N}(s)$ and $\boldsymbol{N}(s)$ is not correlated normalized Langevin white noise i.e. $\left\langle \boldsymbol{N}(s)\boldsymbol{N}^T(s') \right\rangle = I\delta(s - s') \quad \forall s, s' \in [0, T]$.

1.1 Discrete analogy in one dimension

Discrete analogy is probably not mathematically correct, but it help with intuition. Set

$$\Delta t = \frac{T}{n} \quad \text{and} \quad \xi_n(s) = \sum_{j=0}^{n-1} \frac{\sigma}{\sqrt{\Delta t}} N_j(0,1) \chi_{[j\Delta t, (j+1)\Delta t]}(s), \tag{3}$$

where $N_j(0, 1)$ are identically independent distributed random variables with normal distribution. And $\chi_{[j\Delta t,(j+1)\Delta t]}(s)$ is characteristic function. Then we have

$$\int_0^T \xi_n(s)\,ds = \sum_{j=0}^{n-1} \sigma\sqrt{\Delta t} N_j(0,1) = N(0,\sigma^2 n\Delta t) = N(0,\sigma^2 T). \tag{4}$$

The result is independent on n. And the limit

$$\xi(s) = \lim_{n\to\infty} \xi_n(s) \tag{5}$$

has properties (1). First equality $\langle \xi(s) \rangle = 0$ is obvious. Let us show the second equality

$$\begin{aligned}\langle \xi_n(s)\xi_n(s')\rangle &= \frac{\sigma^2}{\Delta t}\sum_{j,i=0}^{n} \langle N_j(0,1)N_i(0,1)\rangle\, \chi_{[i\Delta t,(i+1)\Delta t]}(s')\chi_{[j\Delta t,(j+1)\Delta t]}(s) \\ &= \frac{\sigma^2}{\Delta t}\sum_{i=0}^{n} \left\langle N_i^2(0,1)\right\rangle \chi_{[i\Delta t,(i+1)\Delta t]}(s',s) \xrightarrow{*} \sigma^2\delta(s-s') \quad \text{as} \quad n\to\infty.\end{aligned} \tag{6}$$

Where we use $\langle N_j(0,1)N_i(0,1)\rangle = 0$ if $i \neq j$ and $\langle N_j(0,1)N_i(0,1)\rangle = \left\langle N_i^2(0,1)\right\rangle = 1$ if $i = j$.

2 Solution of arbitrary system of stochastic ODE

Let us have a model governed by stochastic differential equation

$$\begin{aligned}\dot{\mathbf{X}}(t) =&\mathbf{A}(t)\mathbf{X}(t) + \boldsymbol{\xi}(t), \\ \mathbf{X}_0 =&\mathbf{X}(0).\end{aligned} \tag{7}$$

Where $\mathbf{A}(t) \in \mathrm{R}^{\mathrm{N}\times\mathrm{N}}$ and $\boldsymbol{\xi}(t) = \sqrt{\Sigma}(t)\boldsymbol{N}(t)$, so we allow time-dependent intensity of the white noise. Suppose we have a fundamental matrix $\Phi(t)$ of the system $\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t)$ satisfying $\dot{\Phi}(t) = \mathbf{A}(t)\Phi(t)$ and $\Phi(0) = I$. Then the solution is given by

$$\mathbf{X}(t) = \Phi(t)\mathbf{X}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{\xi}(s)\,ds. \tag{8}$$

2.1 Mean value and covariance of the system

We can compute mean value of $\mathbf{X}(t)$ as follows

$$\begin{aligned}\langle \mathbf{X}(t)\rangle =&\left\langle \Phi(t)\mathbf{X}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{\xi}(s)\,ds \right\rangle \\ =&\Phi(t)\mathbf{X}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\left\langle \boldsymbol{\xi}(s)\right\rangle\,ds \\ =&\Phi(t)\mathbf{X}_0,\end{aligned} \tag{9}$$

because $\langle \boldsymbol{\xi}(s) \rangle = 0$. The covariance matrix $\mathbf{C}(t)$ can be computed from definition

$$\mathbf{C}(t) = \left\langle \mathbf{X}(t)\mathbf{X}^T(t) \right\rangle - \langle \mathbf{X}(t) \rangle \left\langle \mathbf{X}^T(t) \right\rangle. \tag{10}$$

The first part of this matrix is

$$\begin{aligned}
&\left\langle \mathbf{X}(t)\mathbf{X}^T(t) \right\rangle = \Bigg\langle \Phi(t)\mathbf{X}_0\mathbf{X}_0^T\Phi^T(t) + \Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{\xi}(s)\,ds\,\mathbf{X}_0^T\Phi^T(t) \\
&+ \Phi(t)\mathbf{X}_0\int_0^t \boldsymbol{\xi}^T(s)\Phi^{-T}(s)\,ds\,\Phi^T(t) + \Phi(t)\int_0^t \Phi^{-1}(s)\boldsymbol{\xi}(s)\,ds\int_0^t \boldsymbol{\xi}^T(s)\Phi^{-T}(s)\,ds\,\Phi^T(t)\Bigg\rangle \\
&= \langle \mathbf{X}(t) \rangle \left\langle \mathbf{X}^T(t) \right\rangle + \Bigg\langle \Phi(t)\int_0^t\int_0^t \Phi^{-1}(s)\boldsymbol{\xi}(s)\boldsymbol{\xi}^T(s')\Phi^{-T}(s')\,ds'\,ds\,\Phi^T(t)\Bigg\rangle \\
&= \langle \mathbf{X}(t) \rangle \left\langle \mathbf{X}^T(t) \right\rangle + \Phi(t)\int_0^t\int_0^t \Phi^{-1}(s)\left\langle \boldsymbol{\xi}(s)\boldsymbol{\xi}^T(s') \right\rangle \Phi^{-T}(s')\,ds'\,ds\,\Phi^T(t) \\
&= \langle \mathbf{X}(t) \rangle \left\langle \mathbf{X}^T(t) \right\rangle + \Phi(t)\int_0^t \Phi^{-1}(s)\Sigma(s)\Phi^{-T}(s)\,ds\,\Phi^T(t).
\end{aligned} \tag{11}$$

Than the correlation matrix is given by

$$\mathbf{C}(t) = \Phi(t)\int_0^t \Phi^{-1}(s)\Sigma(s)\Phi^{-T}(s)\,ds\,\Phi^T(t). \tag{12}$$

The solution is multivariate normal distribution, given by

$$\mathbf{X}(t) = N(\Phi(t)\mathbf{X}_0, \mathbf{C}(t)) \tag{13}$$

2.2 Numerical validation

For numerical validation we choose following system

$$\begin{pmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} & -2 \\ 2 & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} + \boldsymbol{\xi}(t). \tag{14}$$

And Langevin white noise satisfies

$$\left\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}^T(t') \right\rangle = \Sigma\delta(t-t') = \begin{pmatrix} 0.4 & 0.1 \\ 0.1 & 0.2 \end{pmatrix} \delta(t-t'). \tag{15}$$

The fundamental matrix of the system is

$$\Phi(t) = e^{-\frac{t}{5}} \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}. \tag{16}$$

We solve this problem numerically and compare the mean value and the covariance matrix with the sample mean and the sample covariance matrix from 50 000 simulations.

```
a=0;
b=10;
n=100000;
x0=1;
y0=2;
t=linspace(a,b,n);
M=(exp(-t/5).*[x0*cos(2*t)-y0*sin(2*t) ; x0*sin(2*t)+y0*cos(2*t)]);

A=[-1/5,-2;2,-1/5];
h=(b-a)/n;
SUMX=zeros(2,n);
Cov=zeros(2,2*n);
Mu=zeros(2,n);
Mu(:,1)=[x0;y0];
for i=1:(n-1)
    Mu(:,i+1)=Mu(:,i)+h*A*Mu(:,i);
end
m=50000;
mu = [0 0];
Sigma = [0.4 0.1; 0.1 0.2];
for j=1:m
    R = mvnrnd(mu,Sigma,n);
    X=zeros(2,n);
    C=zeros(2,2*n);
    X(:,1)=[x0;y0];
    for i=1:(n-1)
        X(:,i+1)=X(:,i)+h*A*X(:,i)+(R(i,:))'*sqrt(h);
        C(:,(2*i-1):(2*i))=X(:,i)*(X(:,i))';
    end
    Cov=Cov+C/(m);
    SUMX=SUMX+X;
end
AVX=SUMX/m;
for i=1:n
    Cov(:,(2*i-1):(2*i))=Cov(:,(2*i-1):(2*i))-AVX(:,i)*(AVX(:,i))';
end
```

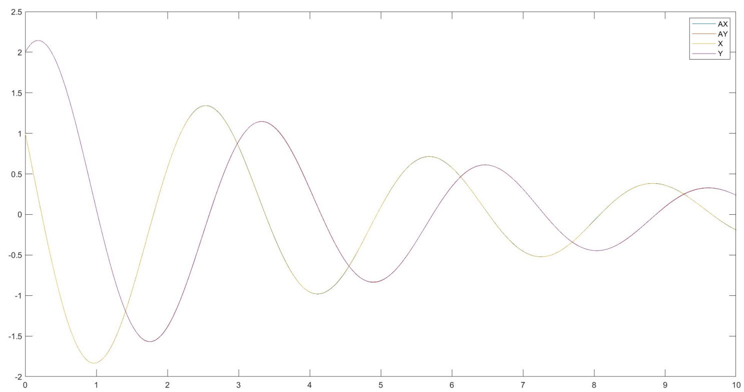

Figure 1: Comparison of sample mean value of X(t) and Y(t) with analytic solution from 50 000 simulations.

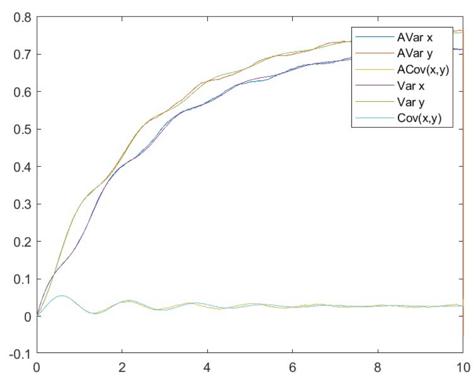

Figure 2: Comparison of component of sample covariance matrix with analytic solution from 50 000 simulations.

3 Optimization of portfolio

3.1 Model description and solution of the evolution of stocks

Let us have a system, which should correspond to the model behavior of a set of stocks

$$\begin{aligned} \frac{d\ln(\mathbf{X}(t))}{dt} =& \boldsymbol{\gamma}(t) + \boldsymbol{\xi}(t), \\ \mathbf{X}_0 =& \mathbf{X}(t_0), \end{aligned} \tag{17}$$

where $\boldsymbol{\gamma}(t)$ is given trend and $\boldsymbol{\xi}(t)$ is given white noise, $\mathbf{X}(t) = (X_1(t), \dots, X_n(t))^T$ denote the price of of stocks. It is easy to solve the system and earn the mean value and the covariance matrix for random vector $\ln(\mathbf{X}(t))$ using the formula from section above

$$\ln(\mathbf{X}(t)) = N(\boldsymbol{\mu}(t), \mathbf{C}(t)). \tag{18}$$

Than the price of stocks has multivariate log-normal distribution (MLND)

$$\mathbf{X}(t) = e^{N(\boldsymbol{\mu}(t), \mathbf{C}(t))}. \tag{19}$$

3.2 Preparatory Work and List of Formulas

In this subsection we introduce a few formulae which we use in following section

3.2.1 Mean and covariance matrix of MLND

Let us have a multivariate normal random variable $N(\boldsymbol{\mu}, \Sigma)$, than the mean of variable $e^{N(\boldsymbol{\mu}, \Sigma)}$ is given by

$$\left\langle e^{N(\boldsymbol{\mu}, \Sigma)} \right\rangle_i = e^{\mu_i + \frac{1}{2}\Sigma_{ii}},$$

and covariance matrix of variable $e^{N(\boldsymbol{\mu}, \Sigma)}$ is given by

$$\left(\operatorname{Var} e^{N(\boldsymbol{\mu}, \Sigma)}\right)_{ij} = e^{\mu_i + \mu_j + \frac{1}{2}(\Sigma_{ii} + \Sigma_{jj})}(e^{\Sigma_{ij}} - 1).$$

3.2.2 Algebra of the Mean and Covariance Matrix for the Sum and Multiplication by a Constant of Random Variables

Let us have a constant diagonal matrix $\mathbf{K} = \operatorname{diag}(K_1, \dots, K_n)$, random vector $\mathbf{X} = (X_1, \dots, X_n)^T$ with covariance matrix $\operatorname{Var}(\mathbf{X})$, than

$$\left\langle \sum_{i=1}^{n} X_i \right\rangle = \sum_{i=1}^{n} \langle X_i \rangle,$$

and

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \mathbf{1}^T \operatorname{Var}(\mathbf{X})\mathbf{1},$$

where $\mathbf{1} = (1, 1, \dots, 1)^T$ and finally

$$\operatorname{Var}(\mathbf{K}\mathbf{X}) = \mathbf{K} \operatorname{Var}(\mathbf{X})\mathbf{K},$$

thus we have

$$\operatorname{Var}(\mathbf{1}^T\mathbf{K}\mathbf{X}) = \mathbf{1}^T\mathbf{K} \operatorname{Var}(\mathbf{X})\mathbf{K}\mathbf{1}.$$

3.2.3 Solution of a Maximization Problem with Constraints Using Lagrange Multipliers

We want to find maximum or minimum of the following function

$$f(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x} + \boldsymbol{b}\boldsymbol{x},$$

where A is given symmetric regular matrix and $\boldsymbol{b} \in \mathrm{R}^n$ with the constraint

$$\mathbf{1}^T\boldsymbol{x} = 1.$$

Lagrange function is

$$\mathcal{L}(\boldsymbol{x}, \lambda) = \boldsymbol{x}^T A\boldsymbol{x} + \boldsymbol{b}\boldsymbol{x} + \lambda(\mathbf{1}^T\boldsymbol{x} - 1),$$

And corresponding Lagrange equations are

$$\begin{aligned} \nabla\mathcal{L}(\boldsymbol{x}, \lambda) =& 2A\boldsymbol{x} + \boldsymbol{b} + \lambda\mathbf{1} = 0, \\ \frac{\partial\mathcal{L}(\boldsymbol{x}, \lambda)}{\partial\lambda} =& \mathbf{1}^T\boldsymbol{x} - 1 = 0. \end{aligned} \tag{20}$$

The solution is

$$\boldsymbol{x} = -\frac{1}{2}A^{-1}\left(\boldsymbol{b} - \frac{2 + \mathbf{1}^T A^{-1}\boldsymbol{b}}{\mathbf{1}^T A^{-1}\mathbf{1}}\mathbf{1}\right). \tag{21}$$

3.3 Derivation and solution of governing equations

Suppose that we know the current value of the price of stocks $\mathbf{X}_0 = \mathbf{X}(t_0) = (X_1(t_0), \ldots, X_n(t_0))^T$, so it is determined vector of numbers. As we show in subsection 3.1, the distribution of $\mathbf{X}(t_0 + \Delta t)$ is given by

$$\mathbf{X}(t_0 + \Delta t) = e^{N(\boldsymbol{\mu}(t_0+\Delta t), \mathbf{C}(t_0+\Delta t))}. \tag{22}$$

The goal of our algorithm is to determine how to allocate the portfolio's capital among individual stocks. In order to do that we introduce an investment strategy $\boldsymbol{\pi}(t) = (\pi_1(t), \ldots, \pi_n(t))^T$, the quantity $\pi_i(t)$ represents the proportion of total wealth invested in the i-th stock at time t. Negative weights correspond to short positions and if holds $\mathbf{1}^T\boldsymbol{\pi}(t) = 1\,\forall\, t$ then $\boldsymbol{\pi}$ is called portfolio, in our work it holds. Z_π represents the value of the portfolio managed by the strategy π at time t. It satisfies

$$\frac{Z_\pi(t_0 + \Delta t)}{Z_\pi(t_0)} = \sum_{i=1}^{n} \pi_i(t_0)\frac{X_i(t_0 + \Delta t)}{X_i(t_0)}, \tag{23}$$

if we do not change $\boldsymbol{\pi}(t)$ in this time interval. The equality is clear, because if some $\pi_i = 1$ then the ratio between $Z_\pi(t_0 + \Delta t)$ and $Z_\pi(t_0)$ should be equal the ration between $X_i(t_0 + \Delta t)$ and $X_i(t_0)$. Hence

$$Z_\pi(t_0 + m\Delta t) = Z_\pi(t_0)\prod_{j=0}^{m-1}\left(\sum_{i=1}^{n}\pi_i(t_0 + j\Delta t)\frac{X_i(t_0 + (j+1)\Delta t)}{X_i(t_0 + j\Delta t)}\right). \tag{24}$$

Our goal is to manage the behaviour of this variable by clever changing of $\boldsymbol{\pi}(t)$. We apply the logarithm on $Z_\pi(t_0 + m\Delta t)$ in order to eliminate the product

$$\ln Z_\pi(t_0 + m\Delta t) = \ln Z_\pi(t_0) + \sum_{j=0}^{m-1} \ln\left(\sum_{i=1}^{n} \pi_i(t_0 + j\Delta t)\frac{X_i(t_0 + (j+1)\Delta t)}{X_i(t_0 + j\Delta t)}\right). \tag{25}$$

By subtracting $\ln Z_\pi(t_0 + (m-1)\Delta t)$ from this equality, it is possible to write it in form of differential equation

$$\ln Z_\pi(t + \Delta t) - \ln Z_\pi(t) = \ln\left(\sum_{i=1}^{n} \pi_i(t)\frac{X_i(t + \Delta t)}{X_i(t)}\right) \tag{26}$$

We set $X_i(t + \Delta t) = X_i(t) + dX_i$ and using $\mathbf{1}^T\boldsymbol{\pi}(t) = 1$ and Taylor polynomials we obtain

$$\begin{aligned}
&\ln\left(\sum_{i=1}^{n} \pi_i(t)\frac{X_i(t + \Delta t)}{X_i(t)}\right) = \ln\left(1 + \sum_{i=1}^{n} \pi_i(t)\frac{dX_i}{X_i(t)}\right) \\
&= \sum_{i=1}^{n} \pi_i(t)\frac{dX_i}{X_i(t)} - \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\frac{dX_i}{X_i(t)}\right)^2 \\
&= \sum_{i=1}^{n} \pi_i(t)\left(\ln\left(1 + \frac{dX_i}{X_i(t)}\right) + \frac{1}{2}\left(\frac{dX_i}{X_i(t)}\right)^2\right) - \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\frac{dX_i}{X_i(t)}\right)^2 \\
&= \sum_{i=1}^{n} \pi_i(t)\ln\left(1 + \frac{dX_i}{X_i(t)}\right) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\left(\frac{dX_i}{X_i(t)}\right)^2 - \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\frac{dX_i}{X_i(t)}\right)^2
\end{aligned} \tag{27}$$

Note that we can not neglect $\left(\frac{dX_i}{X_i(t)}\right)^2$ in stochastic differential equation, indeed

$$\begin{aligned}
&\frac{dX_i}{X_i(t)}\frac{dX_j}{X_j(t)} = \left(\ln\left(1 + \frac{dX_i}{X_i(t)}\right) + \mathcal{O}(dX_i^2)\right)\left(\ln\left(1 + \frac{dX_j}{X_j(t)}\right) + \mathcal{O}(dX_j^2)\right) \\
&= d\,\ln X_i(t) d\,\ln X_j(t) + \mathcal{O}(dX_j^3)
\end{aligned} \tag{28}$$

where we use

$$\ln\left(1 + \frac{dX_i}{X_i(t)}\right) = \ln\frac{X_i(t + \Delta t)}{X_i(t)} = \ln X_i(t + \Delta t) - \ln X_i(t) = d\,\ln X_i(t) = \mathcal{O}(dX_i)$$

from equation (17) we have $d\,\ln X_i(t) = (\gamma_i(t) + \xi_i(t))\Delta t$ and

$$d\,\ln X_i(t) d\,\ln X_j(t) = (\gamma_i(t) + \xi_i(t))(\gamma_j(t) + \xi_j(t))\Delta t^2 = \xi_i(t)\xi_j(t)\Delta t^2 + \mathcal{O}(\Delta t^2)$$

Note, from subsection discrete analogy we have

$$\xi_i(t)\xi_j(t) = \left(\sum_{l=1}^{n} \sqrt{\Sigma}_{il}\boldsymbol{N}_l(t)\right)\left(\sum_{k=1}^{n} \sqrt{\Sigma}_{jk}\boldsymbol{N}_k(t)\right) = \sum_{r=1}^{n} \sqrt{\Sigma}_{ir}\sqrt{\Sigma}_{jr}\boldsymbol{N}_r^2(t) = \frac{\Sigma_{ij}}{\Delta t}N^2(0,1)$$

thus we have equality

$$\frac{dX_i}{X_i(t)}\frac{dX_j}{X_j(t)} = \Sigma_{ij} N_t^2(0,1)\Delta t$$

$N_t^2(0,1)$ denotes square of normal variable, it is different in each time t. Note that the following integral is not random variable, it converge from strong law of large numbers

$$\int_0^t \Sigma_{ij}(t) N_t^2(0,1)\, dt = \int_0^t \Sigma_{ij}(t)\, dt.$$

Hence, the expression in equation (27)=(26) is equal

$$\sum_{i=1}^n \pi_i(t) d \ln X_i(t) + \left(\frac{1}{2}\sum_{i=1}^n \pi_i(t)\Sigma_{ii} - \frac{1}{2}\sum_{i,j=1}^n \pi_i(t)\Sigma_{ij} \right) dt = d \ln Z_\pi. \quad (29)$$

This is our desired equation, it coincide with equation on wikipedia–Stochastic portfolio theory, it was not our aim but it happens. Substituting from (17) we obtain

$$\frac{d \ln Z_\pi}{dt} = \boldsymbol{\pi}^T(t)\boldsymbol{\gamma}(t) + \frac{1}{2}\boldsymbol{\pi}^T(t)\boldsymbol{\sigma}(t) - \frac{1}{2}\boldsymbol{\pi}^T(t)\Sigma(t)\boldsymbol{\pi}(t) + \boldsymbol{\pi}^T(t)\boldsymbol{\xi}(t), \quad (30)$$

where $\boldsymbol{\sigma}(t) = (\Sigma_{11}, \Sigma_{22}, \ldots, \Sigma_{nn})$. Only this expression $\boldsymbol{\pi}^T(t)\boldsymbol{\xi}(t)$ is random variable. It is Langevin white noise

$$\left\langle \boldsymbol{\pi}^T(t)\boldsymbol{\xi}(t)\boldsymbol{\xi}^T(t')\boldsymbol{\pi}(t') \right\rangle = \boldsymbol{\pi}^T(t) \left\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}^T(t') \right\rangle \boldsymbol{\pi}(t') = \boldsymbol{\pi}^T(t)\Sigma\boldsymbol{\pi}(t')\delta(t-t') \quad (31)$$

with intensity $\sigma^2(t) = \boldsymbol{\pi}^T(t)\Sigma(t)\boldsymbol{\pi}(t)$. By integration with respect to the time we obtain the solution

$$\ln Z_\pi(t) = N\left(\int_{t_o}^t \boldsymbol{\pi}^T(s)\boldsymbol{\gamma}(s) + \frac{1}{2}\boldsymbol{\pi}^T(s)\boldsymbol{\sigma}(s) - \frac{1}{2}\boldsymbol{\pi}^T(s)\Sigma(s)\boldsymbol{\pi}(s)\, ds, \int_{t_0}^t \sigma^2(s)\, ds \right), \quad (32)$$

Logarithm of $\ln Z_\pi$ is normal random variable with corresponding mean and variance.

3.4 First idea of optimization of portfolio

The idea is to control mean and variance of equation (23)

$$\frac{Z_\pi(t_0+\Delta t)}{Z_\pi(t_0)} = \sum_{i=1}^n \pi_i(t_0)\frac{X_i(t_0+\Delta t)}{X_i(t_0)}, \quad (33)$$

in each time step using the solution (22). We denote

$$Z_i = \frac{X_i(t_0+\Delta t)}{X_i(t_0)} \quad (34)$$

and $Z = (Z_1, \ldots, Z_n)$. It is possible to compute mean value of (23) as follows (we reduce the notation $\boldsymbol{\mu}_i := \boldsymbol{\mu}_i(t + \Delta t)$ and $\mathbf{C}_{ii} := \mathbf{C}_{ii}(t_0 + \Delta t)$)

$$\left\langle \sum_{i=1}^{n} \pi_i(t_0) \frac{X_i(t_0 + \Delta t)}{X_i(t_0)} \right\rangle = \sum_{i=1}^{n} \frac{\pi_i}{X_i(t_0)} e^{\boldsymbol{\mu}_i + \frac{1}{2}\mathbf{C}_{ii}} = \boldsymbol{\pi}^T \langle Z \rangle \tag{35}$$

and covariance as follows

$$\operatorname{Var}\left(\boldsymbol{\pi}^T Z\right) = \boldsymbol{\pi}^T \operatorname{Var}(Z)\, \boldsymbol{\pi} \tag{36}$$

where

$$\operatorname{Var}(Z)_{ij} = \frac{\operatorname{Var}(\mathbf{X}(t_0 + \Delta t))_{ij}}{X_i(t_0) X_j(t_0)} = \frac{e^{\boldsymbol{\mu}_i + \boldsymbol{\mu}_j + \frac{1}{2}(\mathbf{C}_{ii} + \mathbf{C}_{jj})} \left(e^{\mathbf{C}_{ij}} - 1\right)}{X_i(t_0) X_j(t_0)}. \tag{37}$$

We introduce the parameter $\alpha \in [0, 1]$ and $q = \frac{1-\alpha}{\alpha}$, $\alpha = 1$ maximize the profit without care on variance and $\alpha = 0$ means minimizing the risk without care on profit. We maximize the function

$$\begin{aligned} \mathcal{L}(\boldsymbol{\pi}) &= \alpha \boldsymbol{\pi}^T \langle Z \rangle - (1 - \alpha) \boldsymbol{\pi}^T \operatorname{Var}(Z)\, \boldsymbol{\pi} \\ \mathbf{1}^T \boldsymbol{\pi} &= 1 \end{aligned} \tag{38}$$

Solution by Lagrange multipliers is

$$\boldsymbol{\pi} = \frac{1}{2q} \left(\operatorname{Var}(Z)\right)^{-1} \left(\langle Z \rangle + \frac{2q - \mathbf{1}^T \left(\operatorname{Var}(Z)\right)^{-1} \langle Z \rangle}{\mathbf{1}^T \left(\operatorname{Var}(Z)\right)^{-1} \mathbf{1}} \mathbf{1} \right) \tag{39}$$