FSI

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1 Fluid

Let $\Omega \subset \mathbb{R}^d$ be the reference domain (the fixed domain where we solve the problem numerically, it has the same shape as the mesh) with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. Let $\chi : \Omega \times I \to \mathbb{R}^d$ be such that $\chi(\Omega, t) = \Omega(t) \subset \mathbb{R}^d$ and $\partial\Omega(t) = \Gamma_D(t) \cup \Gamma_N(t)$ where $t \in I = (0, T)$ is the time interval.

We seek the velocity field $\mathbf{v} : \Omega(t) \times I \to \mathbb{R}^d$ and pressure $p : \Omega(t) \times I \to \mathbb{R}$ such that:

$$
\begin{aligned}
\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) &= \nabla\cdot\mathbb{T} && \text{in } \Omega(t)\times I, && (1)\\
\nabla\cdot\mathbf{v} &= 0 && \text{in } \Omega(t)\times I, && (2)\\
\mathbb{T} &= -p\mathbb{I} + \mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\right) && \text{in } \Omega(t)\times I, && (3)\\
\mathbf{v}(\mathbf{x}, 0) &= \mathbf{v}_0(\mathbf{x}) && \text{in } \Omega(t), && (4)\\
\mathbf{v} &= \mathbf{g} && \text{on } \Gamma_D(t)\times I, && (5)\\
\mathbb{T}\cdot\mathbf{n} &= \mathbf{h} && \text{on } \Gamma_N(t)\times I, && (6)
\end{aligned}
$$

where $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega(t)$. The weak formulation is to find $\mathbf{v} : \Omega(t) \times I \to \mathbb{R}^d$ and pressure $p : \Omega(t) \times I \to \mathbb{R}$ such that for all φ, q:

$$\int_{\Omega(t)} \rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right)\cdot\varphi\,dx = -\int_{\Omega(t)} \mathbb{T} : \nabla\varphi\,dx + \int_{\Gamma_N(t)} \mathbf{h}\cdot\varphi\,ds, \tag{7}$$

$$\int_{\Omega(t)} q\,\nabla\cdot\mathbf{v}\,dx = 0. \tag{8}$$

We transform this problem into Ω. The left-hand side is

$$
\begin{aligned}
&\int_{\chi(\Omega,t)} \rho(x,t)\left(\frac{\partial \mathbf{v}(x,t)}{\partial t} + (\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t)\right)\cdot\varphi(x,t)\,dx = \\
&\int_\Omega \hat\rho(X,t)\left(\frac{\partial \mathbf{v}(x,t)}{\partial t} + (\mathbf{v}(x,t)\cdot\nabla)\mathbf{v}(x,t)\right)\Bigg|_{x=\chi(X,t)}\cdot\hat\varphi(X,t)\,\det(\mathbb{F}(X,t))\,dX = \\
&\int_\Omega \hat\rho(X,t)\left(\frac{\partial \hat{\mathbf{v}}(X,t)}{\partial t} + (\hat\nabla\hat{\mathbf{v}}(X,t))\mathbb{F}^{-1}(X,t)(\hat{\mathbf{v}}(X,t) - \hat{\mathbf{v}}_{ALE}(X,t))\right)\cdot\hat\varphi(X,t)\,\det(\mathbb{F}(X,t))\,dX = \\
&\int_\Omega \hat\rho(X,t)\left(\frac{\partial \hat{\mathbf{v}}(X,t)}{\partial t} + \mathrm{Grad}(\hat{\mathbf{v}}(X,t))(\hat{\mathbf{v}}(X,t) - \hat{\mathbf{v}}_{ALE}(X,t))\right)\cdot\hat\varphi(X,t)\,\det(\mathbb{F}(X,t))\,dX,
\end{aligned}
$$

where we denote $f(x,t) = \hat{f}(\chi^{-1}(x,t),t)$,

$$\hat{\mathbf{v}}_{ALE}(X,t) = \frac{\partial \chi(X,t)}{\partial t}, \quad (\hat{\nabla}\hat{f}(X,t))_{ij} = \frac{\partial \hat{f}_i(X,t)}{\partial X_j}, \tag{9}$$

$$\mathbb{F}_{ij}(X,t) = (\hat{\nabla}\chi(X,t))_{ij} \quad \text{and} \quad \mathrm{Grad}(\hat{f}(X,t))_{ij} = (\hat{\nabla}\hat{f}(X,t))_{ik}\mathbb{F}^{-1}_{kj}(X,t). \tag{10}$$

The right-hand side is

$$-\int_{\chi(\Omega,t)} \mathbb{T}(x,t) : \nabla\varphi(x,t)\,dx + \int_{\Gamma_N(t)} \mathbf{h}(x,t)\cdot\varphi(x,t)\,ds = \tag{11}$$

$$-\int_{\Omega} \mathbb{T}(x,t)|_{x=\chi(X,t)} : \mathrm{Grad}(\hat{\varphi}(X,t))\det(\mathbb{F}(X,t))\,dX \tag{12}$$

$$+\int_{\Gamma_N} \hat{\mathbf{h}}(X,t)\cdot\hat{\varphi}(X,t)\left\|\mathbb{F}^{-T}(X,t)\hat{\mathbf{n}}(X)\right\|\,dS, \tag{13}$$

where

$$\mathbb{T}(x,t)|_{x=\chi(X,t)} = -\hat{p}(X,t)\mathbb{I} + \mu\left(\mathrm{Grad}(\hat{\mathbf{v}}(X,t)) + (\mathrm{Grad}(\hat{\mathbf{v}}(X,t)))^\top\right) \tag{14}$$

Together we seek for $\hat{\mathbf{v}} : \Omega\times I \to \mathbb{R}^d$ and pressure $\hat{p} : \Omega\times I \to \mathbb{R}$ such that for all $\hat{\varphi}$, $\hat{q}$:

$$\begin{aligned}\int_\Omega \hat{\rho}\left(\frac{\partial\hat{\mathbf{v}}}{\partial t} + \mathrm{Grad}(\hat{\mathbf{v}})(\hat{\mathbf{v}} - \hat{\mathbf{v}}_{ALE})\right)\cdot\hat{\varphi}\,\det(\mathbb{F})\,dX = &-\int_\Omega \hat{\mathbb{T}} : \mathrm{Grad}(\hat{\varphi})\det(\mathbb{F})\,dX \\ &+\int_{\Gamma_N}\hat{\mathbf{h}}\cdot\hat{\varphi}\left\|\mathbb{F}^{-T}\hat{\mathbf{n}}\right\|\,dS, \qquad (15)\end{aligned}$$

$$\int_\Omega \mathrm{Tr}(\mathrm{Grad}(\mathbf{v}))\hat{q}\det(\mathbb{F})\,dX = 0. \tag{16}$$

2 Solid

We use the St. Vennant - Kirchhoff model for elasticity. We seek displacement $\mathbf{u} : \hat{\Omega}_s \times I \to \mathbb{R}^d$, velocity $\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t}$ and pressure multiplier $p : \hat{\Omega}_s\times I \to \mathbb{R}$ such that:

$$\begin{aligned}\rho_s\frac{\partial\mathbf{u}}{\partial t} &= \nabla\cdot\mathbf{S}_e - \nabla p_s && \text{in } \hat{\Omega}_s\times I,\\ p_s &= \frac{1}{\lambda_s}p + (J^2 - 1) && \text{in } \hat{\Omega}_s\times I.\end{aligned}$$

with:

$$\begin{aligned}\mathbf{F} &= \nabla\mathbf{u} + \mathbb{I},\\ \mathbf{C} &= \mathbf{F}^\top\mathbf{F},\\ J &= \det(\mathbf{F}),\\ \mathbf{S}_e &= \mathbf{F}\left(\tfrac{1}{2}\lambda_s\,\mathrm{tr}(\mathbf{C} - \mathbb{I})\,\mathbb{I} + \mu_s(\mathbf{C} - \mathbb{I})\right).\end{aligned}$$

The weak formulation is to find $\mathbf{u}, \mathbf{v}, p$ such that for all test functions ψ, φ, q:

$$\int_{\hat{\Omega}_s} \left(\frac{\partial \mathbf{u}}{\partial t} - \hat{\mathbf{v}} \right) \cdot \hat{\psi} \, dX = 0,$$

$$\int_{\hat{\Omega}_s} \rho_s \frac{\partial \mathbf{v}}{\partial t} \cdot \hat{\varphi} \, dX + \int_{\hat{\Omega}_s} \mathbf{S}_e : \nabla \hat{\varphi} \, dX + \int_{\hat{\Omega}_s} \left(\frac{1}{\lambda_s} p + (J^2 - 1) - p_s \right) \cdot \hat{q} \, dX = 0,$$

3 Mesh displacement

The moving domain $\Omega(t)$ is described using an Arbitrary Lagrangian-Eulerian (ALE) mapping $\chi(\cdot, t)$, and its time derivative $\mathbf{v}_{ALE} = \partial\chi/\partial t$ appears in the transformed ALE equations for the fluid. The displacement $\mathbf{u}$ of the mesh is computed such that it is continuous across the fluid-solid interface and its normal component vanishes at the walls and outlets.

To preserve the quality of the mesh during the deformation caused by the motion of the structure, the fluid domain mesh is treated as a pseudo-elastic medium. A linear elasticity problem is solved in the fluid domain with artificially chosen parameters to determine the mesh displacement. Specifically, we solve the following:

$$-\nabla \cdot (2\mu_m \, \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda_m \, \nabla \cdot \mathbf{u} \, \mathbb{I}) = 0 \quad \text{in } \Omega_f, \tag{17}$$

where μ_m, λ_m are pseudo-material parameters, chosen depending on local mesh size to penalize compression and avoid mesh tangling:

$$\mu_m = \frac{E_m}{2(1 + \nu_m)}, \quad \lambda_m = \frac{\nu_m E_m}{(1 + \nu_m)(1 - 2\nu_m)}, \quad E_m = \frac{1}{\alpha h^\gamma}. \tag{18}$$

Here, h is the local mesh size, α and γ are user-defined constants. A small negative value for ν_m encourages expansion rather than compression in the tangential direction.

This pseudoelastic mesh displacement is added to the coupled variational formulation and solved together with the FSI system at each time step, ensuring consistency between the ALE velocity and the actual motion of the domain.

4 Numerical method

We discretize the time derivative using the second-order backward differentiation formula (BDF2). Let t^n be the time after the nth time step. For a function f, we denote $f^n \approx f(t^n)$. The BDF2 scheme for a quantity f reads:

$$\frac{1}{\Delta t^n} \left(\frac{1 + 2r}{1 + r} f^n - (1 + r) f^{n-1} + \frac{r^2}{1 + r} f^{n-2} \right), \tag{19}$$

where $r = \frac{\Delta t^n}{\Delta t^{n-1}}$. This formulation allows for variable time stepping, which is particularly useful in nonlinear problems such as FSI. The velocity and displacement fields in both solid and fluid are evolved in time using this scheme. To handle the nonlinearities in both fluid and solid problems, we employ Newton's method with an assembled Jacobian. The resulting linear systems are solved using PETSc solvers, typically GMRES with block preconditioners and adaptively chosen tolerances.