Proofs

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June 2022

We prove two lemmas in this document. The first is a lower bound on the accuracy of the single embedding. The second is that the adjacency matrix of a toy network can be recovered up to arbitrary accuracy in three dimensions in the double embedding.

Lemma 1

Lemma 1: Given network $G = (V, E)$ which has a max degree of d_{max} and single embedding X into n dimensions, the number of entries of $l(XX^T)$ which are greater than $1/2$ which should be 0 is bounded below by

$$\frac{|V|^2}{2^n} - d_{max}|V|$$

Proof: We partition the embedding space R^n into regions $R\underline{b}$ where $\underline{b}$ can be any element of $\{-1, 1\}^n$ with $R\underline{b} := \{(x_1, \ldots, x_n) : x_i b_i \geq 0\}$. We note than in the final embedding (initialised randomly) each node is embedded into precisely one region. Suppose in the final embedding X we have nodes (given by rows) X_i and X_j are both in $R\underline{b}$ then every entry has the same sign and so $X_i X_j^T \geq 0$ and thus $l(X_i X_j^T) \geq l(0) = 1/2$.

Let $N\underline{b}$ denote the number of nodes embedded into region $R\underline{b}$. Then for rows of X which are embedded to $R\underline{b}$ there are at least $N\underline{b}$ entries of $l(XX^T)$ which are greater than or equal to $1/2$. So at least $N\underline{b} - d_{max}$ of these entries should be zero. Thus summing over all rows and $\underline{b}$ vectors yields that there are at least

$$\sum_{b \in \{-1,1\}^n} (N\underline{b} - d_{max})N\underline{b}$$

entries which should be zero.

We note that $\sum N\underline{b} = |V|$ as each node is embedded to precisely one region. We conclude there are at least

$$(\sum_{b \in \{-1,1\}^n} N^2\underline{b}) - d_{max}|V|$$

entries which should be zero. This expression under the constraint $\sum N\underline{b} = |V|$ is minimised (using a Lagrange multiplier or symmetry argument) when $N\underline{b} = |V|/2^n$ and the desired bound is yielded.

Corollary: Any matrix $l(XX^T)$ where X is a single three dimensional embedding of `toy` has at least 10350 entries greater than $1/2$ which should be zero.

Lemma 2

We now show that the double embedding doesn't suffer from the same problems.

Lemma 2: We can find three dimensional embeddings X and Y such that $l(XY^T)$ approximates the adjacency matrix of `toy` up to arbitrary accuracy.

Note: The idea of this embedding we inspired by plotting some of the empirical embeddings of this network. Additionally, automorphisms of the network ideally correspond to automorphims of an embedding, as the `toy` network is very symmetrical this idea gave me a lot of insight into how a 'good' embedding may look.

Proof: To begin with we restrict our attention to embeddings for which every node is embedded to a pair of unit vectors. Prompted by symmetry, the rows of X lie on a circle parallel to the xz plane which intersects the y axis at a positive value, and the rows of Y lie on the circle which is the mirror image of the first circle in the xz plane. The X embedding of the one hundred nodes which are only connected to two other nodes is such that they split the upper circle into one hundred arcs of equal length. The Y embedding of these nodes is then the mirror image in the xz plane. We set the distance of the circles to be such that the angle between each of these node's X and Y embeddings is identically $\theta < \pi/2$ and so $X_i Y_i^T > 0$.

We now consider the node i which has $X_i = (cos(\theta/2), sin(\theta/2), 0)$ and $Y_i = (cos(\theta/2), -sin(\theta/2), 0)$, we investigate by what angle we have to traverse along the lower circle for the inner product of X_i and the point we've traversed to remain positive. Let ϕ be the angle traversed along the second circle, the new point P_ϕ is then given by coordinates $(cos(\theta/2)cos(\phi), -sin(\theta/2), sin(\phi))$ and so we're interested in where

$$X_i \cdot P_\phi = cos^2(\theta/2)cos(\phi) - sin^2(\theta/2) > 0$$

By making θ close to $\pi/2$ we can can attain from this inequality that ϕ belongs to an interval $(-k, k)$ where k can be as small a value as our choosing. We therefore choose θ such that $\pi/100 < k < 3\pi/200$. We now embed the remaining two hundred nodes of degree three to be traversal of $\pi/100$ either direction of the degree two node they are adjacent to, and again let the Y embedding be the mirror image. Note that the triangles which are joined to each other in `toy` also have positive dot product, and this determines the ordering of the embeddings of the degree two nodes around the circle. Now to make the approximation arbitrarily close we simply increase the magnitudes of the embedding vectors.

Thoughts

An absolutely SWEET idea has been yielded from this analysis: mirror embeddings. That is of the type described in Lemma 2- each Y_i was a mirror image of X_i. I think these will preserve transitivity?