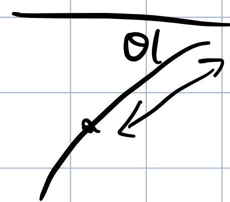
Step 2,

Continuity equation in cylindrical coordinates.

$$0 = \frac{\partial (r \cdot V_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta}$$



relate viscous stress to velocity:

$$\tau_{r\theta} = \mu \left[\frac{1}{r} \frac{\partial V_r}{\partial \theta} + r \frac{\partial}{\partial r} \left(\frac{V_\theta}{r} \right) \right] \qquad \text{B1}$$

$$\tau_{rr} = -\tau_{\theta\theta} = 2\mu \cdot \frac{\partial V_r}{\partial r} \qquad \text{B2}$$

pressure gradient along θ and r

$$\frac{\partial P}{\partial r} = \frac{1}{r} \left[\frac{\partial (r \tau_{rr})}{\partial r} - \tau_{\theta\theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} \right] \qquad \text{B3}$$

$$\frac{\partial P}{\partial \theta} = \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \tau_{r\theta} + \frac{\partial (r \cdot \tau_{r\theta})}{\partial r} \qquad \text{B4}$$

Step 2. do approximation:

To derive an approximate flow profile use $V_r(\theta)$ without require $V_r(r)$ or $V_\theta(r)$

Assume V_r and V_θ vary more rapid with $r\theta$ than r

↓

ignore derivatives with respect to r

⇓

Then

$$\text{B1}\quad \tau_{r\theta} = \mu\left[\frac{1}{r}\cdot\frac{\partial V_r}{\partial\theta} + \cancel{r\cdot\frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right)}\right] = \frac{\mu}{r}\cdot\frac{\partial V_r}{\partial\theta} \qquad \text{B5}$$

$$\text{B2}\quad \tau_{rr} = -\tau_{\theta\theta} = 2\mu\cancel{\frac{\partial V_r}{\partial r}} = 0 \qquad \text{B6}$$

$$\text{B3}\quad \frac{\partial P}{\partial r} = \frac{1}{r}\left[\cancel{\frac{\partial(r\cdot\tau_{rr})}{\partial r}} - \underbrace{\cancel{\tau_{\theta\theta}}}_{=0} + \frac{\partial\tau_{r\theta}}{\partial\theta}\right] = \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} = \frac{\partial S_{\theta\theta}}{\partial r} \qquad \text{B7}$$

$$\text{B4}\quad \frac{\partial P}{\partial\theta} = \cancel{\frac{\partial\tau_{\theta\theta}}{\partial\theta}}^{=0} + \tau_{r\theta} + \frac{\partial(r\tau_{r\theta})}{\partial r} = \tau_{r\theta} + \frac{\partial(r\tau_{r\theta})}{\partial r} = \frac{\partial S_{\theta\theta}}{\partial\theta} \qquad \text{B8}$$

($S_{\theta\theta}$: total normal stress in θ direction)

Step 2.

Get derivatives and get expression for $\tau_{r\theta}(r)$

$$\frac{\partial\,\text{B7}}{\partial\theta} = \frac{\partial^2 S_{\theta\theta}}{\partial r\,\partial\theta} = \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial^2\theta}$$

$$\| \qquad \rightarrow \qquad \|$$

$$\frac{\partial\,\text{B8}}{\partial r} = \frac{\partial^2 S_{\theta\theta}}{\partial r\,\partial\theta} = \frac{\partial}{\partial r}\left[\tau_{r\theta} + \frac{\partial(r\cdot\tau_{r\theta})}{\partial r}\right]$$

$\Downarrow$

$$\frac{1}{r}\frac{\partial^2 \tau_{r\theta}}{\partial^2\theta} = \frac{\partial}{\partial r}\left[\tau_{r\theta} + \frac{\partial(r\cdot\tau_{r\theta})}{\partial r}\right] \qquad (B9)$$

Drop the derivatives with respect to r

$\Downarrow$

right part of (B9) $= 0$

$$\begin{cases} (B9) = 0 \Rightarrow \dfrac{1}{r}\dfrac{\partial^2 \tau_{r\theta}}{\partial^2\theta} = 0 \\ (B7) : \dfrac{\partial S_{\theta\theta}}{\partial r} = \dfrac{\partial P}{\partial r} = \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} \end{cases}$$

$\because$ (B9) : second derivative $= 0$

$\therefore \quad \tau_{r\theta}(r) = A(r)\cdot\theta + B(r)$

$\because$ (B7) .

$$\therefore \quad \tau_{r\theta}(r) = \frac{\partial S_{\theta\theta}}{\partial r}\cdot r\cdot\theta + B(r)$$

When $\theta = 0$. $\tau_{r\theta}(r) = B(r) \Rightarrow$ shear stress at $\theta = 0$

name it $\tau_t(r)$

$$\therefore \quad \tau_{r\theta}(r) = \frac{\partial S_{\theta\theta}}{\partial r}\cdot r\cdot\theta + \tau_t(r) \qquad (B10)$$

Step 4. Get solution for V_r and V_θ

For uniform viscosity situation:

$$g_1(\theta) = \int_0^\theta \frac{d\theta'}{\mu(\theta')} = \int_0^\theta \frac{d\theta'}{\mu} = \frac{\theta}{\mu}$$

$$g_2(\theta) = \int_0^\theta \frac{\theta' d\theta'}{\mu(\theta')} = \int_0^\theta \frac{\theta' d\theta}{\mu} = \frac{1}{\mu} \cdot \frac{\theta'^2}{2}\Big|_0^\theta = \frac{\theta^2}{2\mu}$$

$$g_3(\theta) = \int_0^\theta \frac{(\theta-\theta')}{\mu(\theta')} d\theta' = \int_0^\theta \frac{(\theta-\theta')}{\mu} d\theta'$$

$$= \frac{1}{\mu}\left(\int_0^\theta \theta \, d\theta' - \int_0^\theta \theta' \, d\theta'\right)$$

$$= \frac{1}{\mu}\left(\theta\theta'\Big|_0^\theta - \frac{\theta'^2}{2}\Big|_0^\theta\right)$$

$$= \frac{1}{\mu}\left(\theta^2 - \frac{\theta^2}{2}\right) = \frac{\theta^2}{2\mu}$$

$$g_4(\theta) = \int_0^\theta \frac{\theta'(\theta-\theta')}{\mu(\theta')} d\theta' = \int_0^\theta \frac{\theta'(\theta-\theta')}{\mu} d\theta'$$

$$= \frac{1}{\mu}\left(\int_0^\theta \theta' \cdot \theta \, d\theta' - \int_0^\theta \cdot \theta'^2 d\theta\right)$$

$$= \frac{1}{\mu} \cdot \left(\theta \cdot \frac{\theta'^2}{2}\Big|_0^\theta - \frac{\theta'^3}{3}\Big|_0^\theta\right)$$

$$= \frac{1}{\mu}\left(\frac{\theta^3}{2} - \frac{\theta^3}{3}\right) = \frac{\theta^3}{6\mu}$$

From (B5) $T_{r\theta} = \frac{\mu}{r} \cdot \frac{\partial V_r}{\partial \theta} \rightarrow V_r = T_{r\theta} \cdot \theta \cdot \frac{r}{\mu} + C(r)$

| B10 $\tau_{r\theta}(r) = r\cdot\theta\cdot\frac{\partial S_{\theta\theta}}{\partial r} + \tau_t(r)$

We have

$$\frac{\partial V_r}{\partial \theta} = \tau_{r\theta}\cdot\frac{r}{u}$$

$$V_r = \int_0^\theta \left(\tau_{r\theta}\cdot\frac{r}{u}\right)d\theta' + C(r)$$

$$= \int_0^\theta \cdot\left(r\cdot\theta'\cdot\frac{\partial S_{\theta\theta}}{\partial r} + \tau_t(r)\right)\cdot\frac{r}{u}\cdot d\theta' + C(r)$$

$$= r^2\cdot\frac{\partial S_{\theta\theta}}{\partial r}\cdot\int_0^\theta \frac{\theta'}{u}d\theta' + r\int_0^\theta \frac{\tau_t(r)}{u}\,d\theta' + C(r)$$

$$V_r = r^2\frac{\partial S_{\theta\theta}}{\partial r}\cdot g_2(\theta) + \tau_t(r)\cdot r\cdot g_1(\theta) + C(r) \quad \text{B11}$$

$$0 = \frac{\partial(r\cdot V_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} \quad \text{B1}$$

integrating over r and θ, and substituting for V_r.

$$\frac{\partial V_\theta}{\partial \theta} = -\frac{\partial(r\cdot V_r)}{\partial r}$$

$$\int^r \frac{\partial V_\theta}{\partial \theta}dr = \int^r -\frac{\partial(r\cdot V_r)}{\partial r}dr$$

$$= -r\cdot V_r$$

$$\int^\theta\int^r \frac{\partial V_\theta}{\partial \theta}dr\,d\theta = -\int^\theta r\cdot V_r\,d\theta + D(r)$$

$$\int^{r} V_\theta \, dr = -\int^{\theta} r \cdot V_r \cdot d\theta + D(r)$$

$$\int^{r} V_\theta \, dr = -r \int^{\theta} V_r \, d\theta + D(r)$$

$$= -r \int^{\theta} \left[r^2 \frac{\partial \delta_{\theta\theta}}{\partial r} g_2(\theta) + \tau_t \, r g_1(\theta) + C(r) \right] d\theta + D(r)$$

$$= -r \left[\int^{\theta} r^2 \frac{\partial \delta_{\theta\theta}}{\partial r} g_2(\theta) \, d\theta + \int^{\theta} \tau_t(r) \cdot r \cdot g_1(\theta) \, d\theta + \int^{\theta} C(r) \, d\theta \right] + D(r)$$

$$= -r^3 \frac{\partial \delta_{\theta\theta}}{\partial r} \cdot \int^{\theta} g_2(\theta) d\theta - r^2 \tau_t(r) \int^{\theta} g_1(\theta) d\theta - r \cdot \int^{\theta} C(r) \, d\theta + D(r)$$

$$= -r^3 \frac{\partial \delta_{\theta\theta}}{\partial r} \cdot \int^{\theta} \frac{\theta^2}{2u} d\theta - r^2 \tau_t(r) \cdot \int^{\theta} \frac{\theta}{u} d\theta - r \cdot C(r) \cdot \theta + D(r)$$

$$= \frac{-r^3}{2u} \frac{\partial \delta_{\theta\theta}}{\partial r} \cdot \int^{\theta} \theta^2 d\theta - \frac{r^2}{u} \tau_t(r) \cdot \int^{\theta} \theta \, d\theta - r \cdot C(r) \cdot \theta + D(r)$$

$$= \frac{-r^3}{2u} \cdot \frac{\partial \delta_{\theta\theta}}{\partial r} \cdot \frac{\theta^3}{3} - r^2 \cdot \tau_t(r) \cdot \frac{\theta^2}{2u} - r \cdot \theta \cdot C(r) + D(r)$$

$$= -r^3 \cdot \frac{\partial S_{\theta\theta}}{\partial r} \cdot \frac{\theta^3}{6\mu} - r^2 \cdot T_t(r) \cdot \frac{\theta^2}{2\mu} - r \cdot \theta \cdot C(r) + D(r)$$

$$= -r^3 \cdot \frac{\partial S_{\theta\theta}}{\partial r} \cdot g_4(\theta) - r^2 \cdot T_t(r) \cdot g_3(\theta) - r \cdot \theta \cdot C(r) + D(r)$$

B12

Step 5.

Find approximate expressions for viscous stress

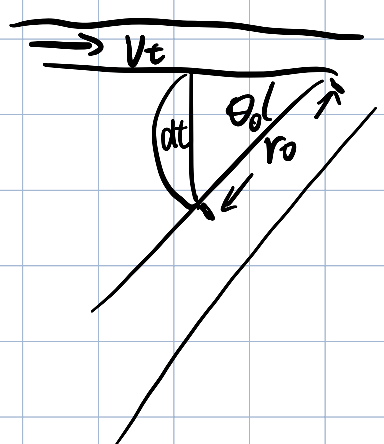
use

B10 $T_{r\theta}(r) = A(r)\cdot\theta + B(r) = r\cdot\theta\cdot\frac{\partial S_{\theta\theta}}{\partial r} + T_t(r)$

B11 $V_r = (r^2 \cdot \frac{\partial S_{\theta\theta}}{\partial r}) \cdot \underbrace{g_2(\theta)}_{\frac{\theta^2}{2\mu}} + T_t \cdot r \cdot \underbrace{g_1(\theta)}_{\frac{\theta}{\mu}} + C(r)$

B12 $\int^r V_\theta \, dr = -r^3 \frac{\partial S_{\theta\theta}}{\partial r} \underbrace{g_4(\theta)}_{\frac{\theta^3}{6\mu}} - r^2 \cdot T_t(r) \cdot \underbrace{g_3(\theta)}_{\frac{\theta^2}{2\mu}} - r\cdot\theta\cdot C(r) + D(r)$

Apply boundary condition, 1

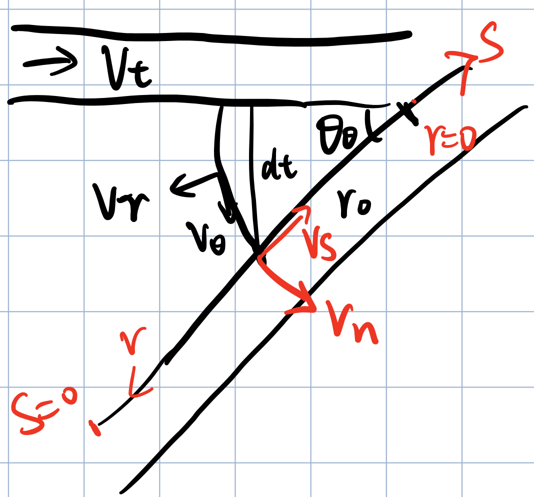


At $\theta = 0$,

$V_\theta = 0 \xRightarrow{+B12} D(r) = 0$

$V_r = -V_t \xRightarrow{+B11} C(r) = -V_t$

Apply boundary condition ②



At $\theta=\theta_0$

$V_r=-V_s$

$V_\theta=V_n$

$\because\ V_r=-V_s$ ⊕ (B11)

$\Downarrow$

$$-V_s = r_0^2 g_2(\theta_0)\frac{\partial \delta_{\theta\theta}}{\partial r} + r_0 T_t(r_0) g_1(\theta_0) - V_t$$

$\because$ locally $dr=-ds$

$$\therefore\ -V_s = -r_0^2\, g_2(\theta_0)\cdot\frac{\partial \delta_n}{\partial s} + r_0 T_t(r_0) g_1(\theta_0) - V_t$$ (B13)

$\because$ Similarly, $V_\theta=V_n$ ⊕ (B12)

$$\therefore\ \int_0^s V_n\, ds = r_0^3 g_4(\theta_0)\frac{\partial \delta_n}{\partial s} - r_0^2 T_t(r_0)\, g_3(\theta_0) + V_t\cdot r_0\theta_0,$$ (B14)

$\Rightarrow$ From B13,

$$r_0 \tau_t(r_0) \cdot g_1(\theta_0) = V_t - V_s + r_0^2 \cdot g_2(\theta_0) \cdot \frac{\partial S_n}{\partial s}$$

$$\tau_t(r_0) = \frac{V_t - V_s}{r_0 \cdot g_1(\theta_0)} + \frac{r_0^{\cancel{2}}\, g_2(\theta_0)}{\cancel{r_0} \cdot g_1(\theta_0)} \cdot \frac{\partial S_n}{\partial s}$$

$$= \frac{V_t - V_s}{r_0 \cdot g_1(\theta_0)} + \frac{r_0 \cdot g_2(\theta_0)}{g_1(\theta_0)} \cdot \frac{\partial S_n}{\partial s} \qquad \text{B15}$$

$\Rightarrow$ From B14 + B15

B14 $\Rightarrow$ $r_0^3 \cdot g_4(\theta_0) \cdot \frac{\partial \delta_n}{\partial s} = \int_0^s v_n \, ds + r_0^2 \tau_t(r_0) g_3(\theta_0) - V_t \cdot r_0 \cdot \theta_0$

$$\frac{\partial \delta_n}{\partial s} = \frac{\int_0^s v_n \, ds + r_0^2 \cdot \tau_t(r_0) \cdot g_3(\theta_0) - V_t \cdot r_0 \cdot \theta_0}{r_0^3 \cdot g_4(\theta_0)}$$

replace $\tau_t(r_0)$ with B15

$$\frac{\partial \delta_n}{\partial s} = \frac{\int_0^s v_n \, ds + r_0^2 \left[\frac{V_t - V_s}{r_0 \cdot g_1(\theta_0)} + \frac{\partial \delta_n}{\partial s} \cdot \frac{r_0 \cdot g_2(\theta_0)}{g_1(\theta_0)} \right] \cdot g_3(\theta_0) - V_t \cdot r_0 \cdot \theta_0}{r_0^3 \cdot g_4(\theta_0)}$$

$$
= \frac{\int_0^s v_n \, ds}{r_0^3 \cdot g_4(\theta_0)} + \frac{\frac{V_t - V_s}{r_0 \cdot g_1(\theta_0)} \cdot r_0^2 \cdot g_3(\theta_0)}{r_0^3 \cdot g_4(\theta_0)} + \frac{\frac{\partial S_n}{\partial s} \cdot \frac{r_0 \cdot g_2(\theta_0)}{g_1(\theta_0)} \cdot g_3(\theta_0) \cdot r_0^2}{r_0^3 \cdot g_4(\theta_0)}
$$

$$
- \frac{V_t \cdot r_0 \cdot \theta_0}{r_0^3 \cdot g_4(\theta_0)}
$$

$$
= \frac{\int_0^s v_n \, ds}{r_0^3 \cdot g_4(\theta_0)} + \frac{(V_t - V_s) \cdot g_3(\theta_0) \cdot \cancel{r_0^2}}{r_0^{\cancel{4}\,2} \cdot g_1(\theta_0) g_4(\theta_0)} + \frac{\partial S_n}{\partial s} \cdot \frac{\cancel{r_0} \cdot g_2(\theta_0) \cdot g_3(\theta_0) \cdot \cancel{r_0^2}}{g_1(\theta_0) \cdot g_4(\theta_0) \cdot \cancel{r_0^3}}
$$

$$
- \frac{V_t \cdot \cancel{r_0} \cdot \theta_0}{r_0^{\cancel{3}\,2} \cdot g_4(\theta_0)}
$$

$$
= \frac{\int_0^s v_n \, ds}{r_0^3 \cdot g_4(\theta_0)} + \frac{(V_t - V_s) \cdot g_3(\theta_0)}{r_0^2 \cdot g_1(\theta_0) \cdot g_4(\theta_0)} + \frac{\partial S_n}{\partial s} \cdot \frac{g_2(\theta_0) \cdot g_3(\theta_0)}{g_1(\theta_0) \cdot g_4(\theta_0)}
$$

$$
- \frac{V_t \cdot \theta_0}{r_0^2 \cdot g_4(\theta_0)}
$$

$$
\left[1 - \frac{g_2(\theta_0) g_3(\theta_0)}{g_1(\theta_0) \cdot g_4(\theta_0)} \right] \frac{\partial S_n}{\partial s} = \frac{\int_0^s v_n \, ds}{r_0^3 \cdot g_4(\theta_0)} + \frac{(V_t - V_s) g_3(\theta_0)}{r_0^2 \cdot g_1(\theta_0) g_4(\theta_0)} - \frac{V_t \cdot \theta_0}{r_0^2 \cdot g_4(\theta_0)}
$$

$$\frac{g_1(\theta_0)g_4(\theta_0)-g_2(\theta_0)g_3(\theta_0)}{g_1(\theta_0)g_4(\theta_0)}\cdot\frac{\partial S_n}{\partial s}=\frac{g_1(\theta_0)\cdot\int_0^s V_n ds+r_0 g_3(\theta_0)(V_t-V_s)-r_0 g_1(\theta_0)\cdot V_t\cdot\theta_0}{r_0^3\, g_1(\theta_0)\, g_4(\theta_0)}$$

$$\Downarrow$$

$$\frac{\partial S_n}{\partial s}=-\frac{\partial S_n}{\partial r}=\frac{-g_1(\theta_0)\cdot\int_0^s V_n ds+r_0\cdot g_3(\theta_0)(V_s-V_t)+V_t\cdot r_0\cdot\theta_0\cdot g_1(\theta_0)}{r_0^3[g_2(\theta_0)g_3(\theta_0)-g_1(\theta_0)g_4(\theta_0)]} \quad \text{B16}$$

For uniform viscosity situation.

$$g_1(\theta)=\frac{\theta}{u},\quad g_2(\theta)=\frac{\theta^2}{2u},\quad g_3(\theta)=\frac{\theta^2}{2u},\quad g_4(\theta)=\frac{\theta^3}{6u}$$

B11 : $$V_r=r^2\cdot\frac{\partial S_{\theta\theta}}{\partial r}\cdot g_2(\theta)+T_t\cdot r g_1(\theta)-V_t$$

$$=-\frac{r_0^2\theta^2}{2u}\cdot\frac{\partial S_n}{\partial s}+T_t\cdot r_0\cdot\frac{\theta}{u}-V_t \quad \text{B17}$$

⇐ compare

B12 :
$$\int^r V_\theta dr=-(r^3\frac{\delta_{\theta\theta}}{\partial r})g_4(\theta)-r^2\cdot T_t(r)\cdot g_3(\theta)+r\cdot\theta\cdot V_t$$

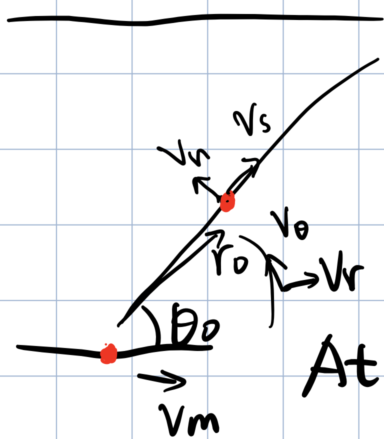
$$\frac{\partial(\int^r V_\theta dr)}{\partial r}=\frac{\partial(\int^r V_\theta dr)}{-\partial s}=V_\theta$$

$$=-\frac{\partial}{\partial s}\Big[-r_0^3\cdot(-\frac{\delta n}{\delta s})\frac{\theta^3}{6u}-r_0^2\cdot T_t\cdot\frac{\theta^2}{2u}+r_0\theta V_t\Big)$$

$$V_\theta = \frac{\partial}{\partial s}\left[-\frac{r_0^3\theta^3}{6\mu}\cdot\frac{\partial S_n}{\partial s} + T_t\cdot\frac{r^2\theta^2}{2\mu} - r_0\cdot\theta\cdot V_t\right] \quad \text{(B18)}$$

Compare

For lower mantle



radius $r_0 = d_b/\sin\theta_0$

arc length $\doteq \theta_0 \cdot d_b/\sin\theta_0$

At $\theta = 0$. $V_\theta = 0,\ V_r = V_m$ (opposite relation with upper mantle)

$\theta = \theta_0,\ V_\theta = -V_n,\ V_r = V_s$

From (B15): $$T_t = \frac{V_t - V_s}{r_0 g_1(\theta_0)} + \frac{\partial S_n}{\partial s}\left[\frac{r_0\cdot g_2(\theta_0)}{g_1(\theta_0)}\right]$$

⇓ Apply for lower mantle

$$T_m = \frac{V_s - V_m}{r_0 g_1(\theta_0)} - \frac{\partial S_n}{\partial s}\cdot\frac{r_0\cdot g_2(\theta_0)}{g_1(\theta_0)} \quad \text{(B19)}$$

From (B16).

$$\frac{\partial S_n}{\partial s} = -\frac{\partial S_n}{\partial r} = \frac{-g_1(\theta_0)\cdot\int_0^s V_n\,ds + r_0 g_3(\theta_0)(V_s - V_t) + V_t\cdot r_0\cdot\theta_0\, g_1(\theta_0)}{r_0^3\left[g_2(\theta_0)\cdot g_3(\theta_0) - g_1(\theta_0)\cdot g_4(\theta_0)\right]}$$

⇓ Apply for lower mantle

$$\frac{\partial S_n}{\partial s} = -\frac{\partial S_n}{\partial r} = \frac{-g_1(\theta_0)\cdot\int_0^s V_n\,ds + r_0 g_3(\theta_0)(V_s - V_m) + V_m\cdot r_0\cdot\theta_0\cdot g_1(\theta_0)}{r_0^3\cdot[g_2(\theta_0)\cdot g_3(\theta_0) - g_1(\theta_0)\,g_4(\theta_0)]}$$

B20

τ_m is shear stress on the base of upper mantle

For uniform viscosity, we can replace $g_i(\theta)$

B17 : $V_r = -\frac{r_0^2\theta^2}{2}\frac{\partial S_n}{\partial s} + \tau_t\cdot r_0\theta - V_t$ compare

$$\Downarrow$$

$$= -\frac{r_0^2\theta^2}{2\mu}\cdot\frac{\partial S_n}{\partial s} + \tau_m\cdot\frac{r_0\cdot\theta}{\mu} - V_m$$

B23

B18 . $V_\theta = \frac{\partial}{\partial s}\left[-\frac{\partial S_n}{\partial s}\left(\frac{r_0^3\theta^3}{6\mu}\right) + \tau_t\left(\frac{r_0^2\theta^2}{2\mu}\right) - V_t\cdot r_0\cdot\theta\right]$

$$\Downarrow$$

$$= \frac{\partial}{\partial s}\left[-\frac{\partial S_n}{\partial s}\left(\frac{r_0^3\theta^3}{6}\right) - \tau_m\cdot\frac{r_0^2\theta^2}{2} - V_m r_0\cdot\theta\right]$$

B24

B20

$$\frac{\partial S_n}{\partial s} = \frac{-g_1(\theta_0)\cdot\int_0^s V_n\,ds + r_0\cdot g_3(\theta_0)(V_s - V_m) + V_m\cdot r_0\cdot\theta_0\cdot g_1(\theta_0)}{r_0^3[g_2(\theta_0)g_3(\theta_0) - g_1(\theta_0)\cdot g_4(\theta_0)]}$$

$\left(g_1(\theta) = \frac{\theta}{\mu},\ g_2(\theta) = \frac{\theta^2}{2\mu},\ g_3(\theta) = \frac{\theta^2}{2\mu},\ g_4(\theta) = \frac{\theta^3}{6\mu}\right)$

$$= \frac{-\frac{\theta_0}{u}\cdot\int_0^s V_n\,ds + r_0\cdot\frac{\theta_0^2}{2u}\cdot(V_s-V_m) + V_m\cdot r_0\cdot\theta_0\cdot\frac{\theta_0}{u}}{\underbrace{r_0^3\left[\frac{\theta_0^2}{2u}\cdot\frac{\theta_0^2}{2u} - \frac{\theta_0}{u}\cdot\frac{\theta_0^3}{6u}\right]}_{\parallel\ \frac{\theta_0^4}{12u^2}}}$$

$$= \frac{-\cancel{\theta_0}}{\cancel{u}}\cdot\frac{12u^{\cancel{2}}}{r_0^3\cdot\theta_0^{\cancel{4}\,3}}\cdot\int_0^s V_n\,ds$$

$$+ \frac{\cancel{r_0}\cdot\cancel{\theta_0^2}(V_s-V_m)\cdot\overset{6}{\cancel{12}}u^{\cancel{2}}}{\cancel{2}\cancel{u}\cdot r_0^{\cancel{3}\,2}\cdot\theta_0^{\cancel{4}\,2}} + \frac{V_m\cdot\cancel{r_0}\cdot\cancel{\theta_0^2}}{r_0^{\cancel{3}\,2}\cdot\theta_0^{\cancel{4}\,2}\cdot\cancel{u}}\cdot 12u^{\cancel{2}}$$

$$= \frac{-12u}{r_0^3\cdot\theta_0^3}\cdot\int_0^s V_n\,ds + \frac{6u(V_s+V_m)}{r_0^2\cdot\theta_0^2}$$

B21

B19

$$\tau_m = \frac{V_s-V_m}{r_0 g_1(\theta_0)} - \frac{\partial S_n}{\partial s}\cdot\left[\frac{r_0 g_2(\theta_0)}{g_1(\theta_0)}\right]$$

$$= \frac{V_s-V_m}{r_0\cdot\frac{\theta}{u}} - \frac{\partial S_n}{\partial s}\cdot\left[\frac{r_0\cdot\frac{\theta_0^2}{2u}}{\frac{\theta}{u}}\right]$$

$$= \frac{(V_s-V_m)u}{r_0\cdot\theta} - \frac{\partial S_n}{\partial s}\cdot\frac{r_0\cdot\theta_0}{2}$$

B22

compare