Toroidal flow

Hele-Shaw $\rightarrow$ $$\frac{dP}{dx} = \frac{\partial}{\partial z}\left(\mu \frac{\partial v_x}{\partial z}\right) \qquad x$$

$$\frac{dP}{dy} = \frac{\partial}{\partial z}\left(\mu \frac{\partial v_y}{\partial z}\right) \qquad y$$

Integrating x:

$$\frac{\partial P}{\partial x} z = \mu \frac{\partial v_x}{\partial z} + C_1$$

$$\frac{z}{\mu}\frac{\partial P}{\partial x} = \frac{\partial v_x}{\partial z} + \frac{C_1}{\mu}$$

$$\frac{z^2}{2\mu}\frac{\partial P}{\partial x} = v_x + \frac{C_1 z}{\mu} + C_2$$

BCs:

$$v_x(z=0) = V_T \Rightarrow C_2 = -V_T$$

$$v_x(z=\lambda) = V_m \Rightarrow \frac{\lambda^2}{2\mu}\frac{\partial P}{\partial x} = V_m + \frac{C_1}{\mu}\lambda - V_T$$

$$\frac{C_1}{\mu}\lambda = \frac{\lambda^2}{2\mu}\frac{\partial P}{\partial x} - V_m + V_T$$

$$C_1 = \frac{\lambda}{2}\frac{\partial P}{\partial x} - \frac{V_m \mu}{\lambda} + \frac{V_T \mu}{\lambda}$$

$$v_x = \frac{z^2}{2\mu}\frac{\partial P}{\partial x} - \frac{C_1}{\mu} z - C_2$$

$$= \frac{z^2}{2\mu}\frac{\partial P}{\partial x} - \frac{z}{\mu}\left(\frac{\lambda}{2}\frac{\partial P}{\partial x} - \frac{V_m \mu}{\lambda} + \frac{V_T \mu}{\lambda}\right) + V_T$$

$$= \frac{\partial P}{\partial x}\left(\frac{z^2}{2\mu} - \frac{\lambda z}{2\mu}\right) + \frac{z V_m}{\lambda} - \frac{z V_T}{\lambda} + V_T$$

$$V_x = \frac{\partial p}{\partial x}\left(\frac{z(z-\lambda)}{2\mu}\right) + \frac{zV_m - zV_T + \lambda V_T}{\lambda}$$

$$= \frac{\partial p}{\partial x}\left(\frac{z(z-\lambda)}{2\mu}\right) + \frac{V_T(\lambda - z) + V_m z}{\lambda}$$

$$\bar{V}_x = \int_0^\lambda V_x \, dz = \frac{1}{2\mu}\frac{\partial p}{\partial x}\int_0^\lambda (z^2 - z\lambda)\,dz + \frac{1}{\lambda}\int_0^\lambda (V_T\lambda - V_T z + V_m z)\,dz$$

$$= \frac{1}{2\mu}\frac{\partial p}{\partial x}\left[\frac{z^3}{3} - \frac{z^2\lambda}{2}\right]_0^\lambda + \frac{1}{\lambda}\left[V_T\lambda z - V_T\frac{z^2}{2} + V_m\frac{z^2}{2}\right]_0^\lambda$$

$$= \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(\frac{\lambda^3}{3} - \frac{\lambda^3}{2}\right) + \frac{1}{\lambda}\left(V_T\lambda^2 - V_T\frac{\lambda^2}{2} + V_m\frac{\lambda^2}{2}\right)$$

$$= \frac{\partial p}{\partial x}\left(\frac{2\lambda^3}{12\mu} - \frac{3\lambda^3}{12\mu}\right) + V_T\left(\lambda - \frac{\lambda}{2}\right) + V_m\frac{\lambda}{2}$$

$$\therefore \quad \bar{V}_x = \lambda\left(\frac{V_T + V_m}{2}\right) - \frac{\partial p}{\partial x}\left(\frac{\lambda^3}{12\mu}\right)$$

$$\bar{V}_y = -\frac{\partial p}{\partial y}\left(\frac{\lambda^3}{12\mu}\right) \quad \text{assuming } V_{T,y} = V_{M,y} = 0$$

Cons. mass $\nabla \cdot \underline{v} = 0$ (Laplace's Eqn)

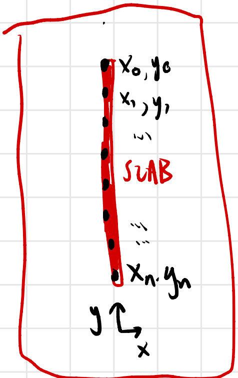
$$\Rightarrow \quad -\frac{\lambda^3}{12\mu}\frac{\partial^2 p}{\partial x^2} - \frac{\lambda^3}{12\mu}\frac{\partial^2 p}{\partial y^2} = 0 \Rightarrow \nabla^2 p = 0$$

$$\nabla^2 p = 0$$

A solution is (didn't check)

$$P_i = \frac{A_i (x - x_i)}{(x-x_i)^2 + (y-y_i)^2}$$

[except at x_i & y_i
$-a < y_i < a$
$x_i = 0$]



$$\frac{\partial P_i}{\partial x} = \frac{A_i\left[(y-y_i)^2 - (x-x_i)^2\right]}{\left[(x-x_i)^2 + (y-y_i)^2\right]^2}$$

putting into the flux expression:

$$\bar{V}_i = \lambda\left(\frac{V_t + V_m}{2}\right) - \frac{\partial P_i}{\partial x}\frac{\lambda^3}{12\mu}$$

$$\bar{V}_i = \lambda\left(\frac{V_t + V_m}{2}\right) + \left[\frac{(x-x_i)^2 - (y-y_i)^2}{(x-x_i)^2 + (y-y_i)^2}\right]\frac{A_i \lambda^3}{12\mu}$$

matrix-solve

$$\underline{A}\,\underline{B} = \underline{C}$$

$$\begin{bmatrix} \frac{(x_0-x_i)^2 - (y_0-y_i)^2}{(x_0-x_i)^2 + (y_0-y_i)^2} \\ \vdots \\ \frac{(x_n-x_i)^2 - (y_n-y_i)^2}{(x_n-x_i)^2 + (y_n-y_i)^2} \end{bmatrix} \begin{bmatrix} A_0 \\ \vdots \\ A_i \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} \uparrow \\ \left[V_R - \lambda\left(\frac{V_t + V_m}{2}\right)\right]\frac{12\mu}{\lambda^3} \\ \downarrow \end{bmatrix}$$

$$\underline{B} = \underline{A}^{-1}\underline{C}$$