Lagrangian Mechanics and Equations of Motion for a Chaotic Pendulum

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April 10, 2025

We started by drawing a diagram to define our system:

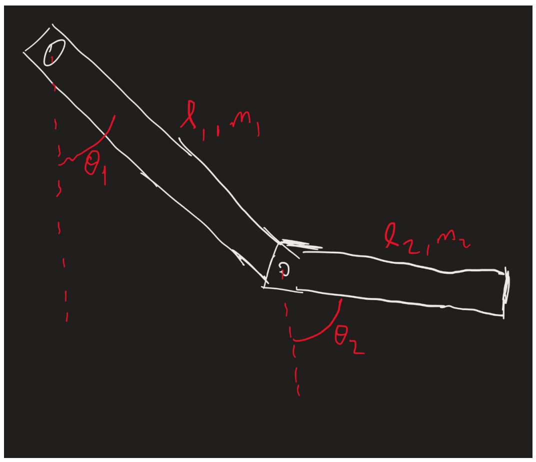

Figure 1. Defining coordinates of the system.

Now we make the expression for position of the center of mass of each pendulum arm:

$$(x_1, y_1) = \left(\frac{l_1}{2}\sin\theta_1, -\frac{l_1}{2}\cos\theta_1\right)$$

$$(x_2, y_2) = \left(l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2, -l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2\right)$$

Next, thinking ahead to kinetic energy, let's take the time derivative of these expression:

$$(\dot{x}_1, \dot{y}_1) = \left(\frac{l_1}{2}\dot{\theta}_1\cos\theta_1, \frac{l_1}{2}\dot{\theta}_1\sin\theta_1\right)$$

$$(\dot{x}_2, \dot{y}_2) = \left(l_1\dot{\theta}_1\cos\theta_1 + \frac{l_2}{2}\dot{\theta}_2\cos\theta_2, l_1\dot{\theta}_1\sin\theta_1 + \frac{l_2}{2}\dot{\theta}_2\sin\theta_2\right)$$

Then the kinetic and potential energy is given by:

$$T = \frac{1}{6}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{6}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

Plugging $\dot{x}$ and $\dot{y}$ we get:

$$T = \frac{1}{6}m_1\left(\left(\frac{l_1}{2}\dot{\theta}_1\cos\theta_1\right)^2 + \left(\frac{l_1}{2}\dot{\theta}_1\sin\theta_1\right)^2\right)$$

$$+\frac{1}{6}m_2\left(\left(l_1\dot{\theta}_1\cos\theta_1 + \frac{l_2}{2}\dot{\theta}_2\cos\theta_2\right)^2 + \left(l_1\dot{\theta}_1\sin\theta_1 + \frac{l_2}{2}\dot{\theta}_2\sin\theta_2\right)^2\right)$$

Now expanding the squared sums gives:

$$T = \frac{1}{24}m_1l_1^2\dot{\theta}_1^2 + \boxed{\frac{1}{6}m_2l_1^2\dot{\theta}_1^2\cos^2\theta_1}_A + \frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + \boxed{\frac{1}{24}m_2l_2^2\dot{\theta}_2^2\cos^2\theta_2}_B$$

$$+\boxed{\frac{1}{6}m_2l_1^2\dot{\theta}_1^2\sin^2\theta_1}_A + \frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2 + \boxed{\frac{1}{24}m_2l_2^2\dot{\theta}_2^2\sin^2\theta_2}_B$$

The boxed terms with the same subscript can have their coefficients factored out, and the trig identity $\cos^2 + \sin^2 = 1$ applied to get:

$$T = \frac{1}{24}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{6}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{24}m_2l_2^2\dot{\theta}_2^2 + \frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)$$

Now lets use the following trig identities:

$$\cos\theta_1\cos\theta_2 = \frac{1}{2}\left(\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2)\right)$$

$$\sin\theta_1\sin\theta_2 = \frac{1}{2}\left(\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2)\right)$$

To simplify further, giving:

$$T = \frac{1}{24}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{6}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{24}m_2l_2^2\dot{\theta}_2^2 + \frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$

We can also combine the $\dot{\theta}_1^2$ terms to get:

$$T = \frac{1}{24}(m_1+4m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{24}m_2l_2^2\dot{\theta}_2^2 + \frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$

Now for potential energy:

$$V = m_1gy_1 + m_2gy_2$$

$$V = m_1g\left(-\frac{l_1}{2}\cos\theta_1\right) + m_2g\left(-l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2\right)$$

Multiplying out the parentheses and combine the $\cos\theta_1$ terms gives:

$$V = -\frac{1}{2}(m_1 + 2m_2)gl_1\cos\theta_1 - \frac{1}{2}m_2gl_2\cos\theta_2$$

Now making our Lagrangian ($L = T - V$), we get the following expression:

$$L = \frac{1}{24}(m_1 + 4m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{24}m_2l_2^2\dot{\theta}_2^2 + \frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

$$+\frac{1}{2}(m_1 + 2m_2)gl_1\cos\theta_1 + \frac{1}{2}m_2gl_2\cos\theta_2$$

The Euler-Lagrange equations are as follows:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_1}\right) = \frac{dL}{d\theta_1}$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right) = \frac{dL}{d\theta_2}$$

First let me do $\frac{dL}{d\dot{\theta}_1}$:

$$\frac{dL}{d\dot{\theta}_1} = \frac{1}{12}(m_1 + 4m_2)l_1^2\dot{\theta}_1 + \frac{1}{6}m_2l_1l_2\dot{\theta}_2\cos(\theta_1 - \theta_2)$$

Now let me do $\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_1}\right)$:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_1}\right) = \frac{1}{12}(m_1 + 4m_2)l_1^2\ddot{\theta}_1 + \frac{1}{6}m_2l_1l_2\left(\ddot{\theta}_2\cos(\theta_1 - \theta_2) - \dot{\theta}_2\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\right)$$

Multiplying out the parentheses gives:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_1}\right) = \frac{1}{12}(m_1+4m_2)l_1^2\ddot{\theta}_1+\frac{1}{6}m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)-\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)+\frac{1}{6}m_2l_1l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)$$

And now let me do $\frac{dL}{d\theta_1}$:

$$\frac{dL}{d\theta_1} = -\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - \frac{1}{2}(m_1 + 2m_2)gl_1\sin\theta_1$$

Now setting them equal:

$$\frac{1}{12}(m_1+4m_2)l_1^2\ddot{\theta}_1+\frac{1}{6}m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)\boxed{-\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)}+\frac{1}{6}m_2l_1l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)$$

$$= \boxed{-\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2)} - \frac{1}{2}(m_1 + 2m_2)gl_1\sin\theta_1$$

The boxed terms cancel each other giving:

$$(Eq.\ 1) \qquad \frac{1}{12}(m_1+4m_2)l_1^2\ddot{\theta}_1+\frac{1}{6}m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+\frac{1}{6}m_2l_1l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)$$

$$=-\frac{1}{2}(m_1+2m_2)gl_1\sin\theta_1$$

Now for the second Euler-Lagrange equation:
First let me do $\frac{dL}{d\dot{\theta}_2}$:

$$\frac{dL}{d\dot{\theta}_2}=\frac{1}{12}m_2l_2^2\dot{\theta}_2+\frac{1}{6}m_2l_1l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$$

Now let me do $\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right)$:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right)=\frac{1}{12}m_2l_2^2\ddot{\theta}_2+\frac{1}{6}m_2l_1l_2\left(\ddot{\theta}_1\cos(\theta_1-\theta_2)-\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2)\right)$$

Multiplying out the parentheses gives:

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}_2}\right)=\frac{1}{12}m_2l_2^2\ddot{\theta}_2+\frac{1}{6}m_2l_1l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-\frac{1}{6}m_2l_1l_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)+\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)$$

And now let me do $\frac{dL}{d\theta_2}$:

$$\frac{dL}{d\theta_2}=\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)-\frac{1}{2}m_2gl_2\sin\theta_2$$

Now setting them equal:

$$\frac{1}{12}m_2l_2^2\ddot{\theta}_2+\frac{1}{6}m_2l_1l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-\frac{1}{6}m_2l_1l_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)+\boxed{\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)}$$

$$=\boxed{\frac{1}{6}m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)}-\frac{1}{2}m_2gl_2\sin\theta_2$$

The boxed terms cancel each other giving:

$$(Eq.\ 2) \qquad \frac{1}{12}m_2l_2^2\ddot{\theta}_2+\frac{1}{6}m_2l_1l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-\frac{1}{6}m_2l_1l_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)=-\frac{1}{2}m_2gl_2\sin\theta_2$$

Now we have two equations as follows:

$$(Eq.\ 1) \qquad \frac{1}{12}(m_1+4m_2)l_1^2\ddot{\theta}_1+\frac{1}{6}m_2l_1l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+\frac{1}{6}m_2l_1l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)$$

$$=-\frac{1}{2}(m_1+2m_2)gl_1\sin\theta_1$$

$$(Eq.\ 2) \qquad \frac{1}{12}m_2l_2^2\ddot{\theta}_2+\frac{1}{6}m_2l_1l_2\ddot{\theta}_1\cos(\theta_1-\theta_2)-\frac{1}{6}m_2l_1l_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)=-\frac{1}{2}m_2gl_2\sin\theta_2$$

In order to get $\ddot{\theta}_1$ isolated, let me solve for $\ddot{\theta}_2$ in equation 2:

$$\ddot{\theta}_2 = \frac{-\frac{1}{6}m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + \frac{1}{6}m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - \frac{1}{2}m_2 g l_2 \sin\theta_2}{\frac{1}{12}m_2 l_2^2}$$

Simplifying gives:

$$\ddot{\theta}_2 = -2\frac{l_1}{l_2}\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + 2\frac{l_1}{l_2}\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - 6\frac{g}{l_2}\sin\theta_2$$

Now we can plug this expression into equation 1:

$$\frac{1}{12}(m_1+4m_2)l_1^2\ddot{\theta}_1+\frac{1}{6}m_2 l_1 l_2 \left(-2\frac{l_1}{l_2}\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + 2\frac{l_1}{l_2}\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - 6\frac{g}{l_2}\sin\theta_2\right)\cos(\theta_1-\theta_2)$$
$$+\frac{1}{6}m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = -\frac{1}{2}(m_1 + 2m_2) g l_1 \sin\theta_1$$

We want to solve for $\ddot{\theta}_1$ now so multiplying out the parentheses:

$$\frac{1}{12}(m_1+4m_2)l_1^2\ddot{\theta}_1-\frac{1}{3}m_2 l_1^2\ddot{\theta}_1\cos^2(\theta_1-\theta_2)+\frac{1}{3}m_2 l_1^2\dot{\theta}_1^2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)-m_2 l_1 g\sin\theta_2\cos(\theta_1-\theta_2)$$
$$+\frac{1}{6}m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = -\frac{1}{2}(m_1 + 2m_2) g l_1 \sin\theta_1$$

Applying the trig identity:

$$\sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2) = \frac{1}{2}(sin(2\theta_1 - 2\theta_2))$$

Gives the simplification:

$$\frac{1}{12}(m_1+4m_2)l_1^2\ddot{\theta}_1-\frac{1}{3}m_2 l_1^2\ddot{\theta}_1\cos^2(\theta_1-\theta_2)+\frac{1}{6}m_2 l_1^2\dot{\theta}_1^2\sin(2\theta_1-2\theta_2)-m_2 l_1 g\sin\theta_2\cos(\theta_1-\theta_2)$$
$$+\frac{1}{6}m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = -\frac{1}{2}(m_1 + 2m_2) g l_1 \sin\theta_1$$

Isolating $\ddot{\theta}_1$ gives:

$$\ddot{\theta}_1 = \frac{-\frac{1}{2}(m_1 + 2m_2) g l_1 \sin\theta_1 - \frac{1}{6}m_2 l_1^2 \dot{\theta}_1^2 \sin(2\theta_1 - 2\theta_2) + m_2 l_1 g \sin\theta_2 \cos(\theta_1 - \theta_2) - \frac{1}{6}m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)}{\frac{1}{12}(m_1 + 4m_2)l_1^2 - \frac{1}{3}m_2 l_1^2 \cos^2(\theta_1 - \theta_2)}$$

We can cancel a l_1 from every term and fix the constants to get the final simplified expression:

$$\ddot{\theta}_1 = \frac{-6(m_1 + 2m_2) g \sin\theta_1 - 2m_2 l_1 \dot{\theta}_1^2 \sin(2\theta_1 - 2\theta_2) + 12 m_2 g \sin\theta_2 \cos(\theta_1 - \theta_2) - 2m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)}{l_1(m_1 + 4m_2) - 4m_2 l_1 \cos^2(\theta_1 - \theta_2)}.$$

Now that we have $\ddot{\theta}_1$ in terms of just the angles and angular velocities, we now can get $\ddot{\theta}_2$ with the equation we made earlier:

$$\ddot{\theta}_2 = -2\frac{l_1}{l_2}\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + 2\frac{l_1}{l_2}\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - 6\frac{g}{l_2}\sin\theta_2$$

I'm not too keen on plugging in and simplifying this, but now we have equations for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in terms of just $\theta_1, \theta_2, \dot{\theta}_1,$ and $\dot{\theta}_2$. The final equations are here:

$$\ddot{\theta}_1 = \frac{-6(m_1 + 2m_2)g\sin\theta_1 - 2m_2 l_1 \dot{\theta}_1^2 \sin(2\theta_1 - 2\theta_2) + 12 m_2 g \sin\theta_2 \cos(\theta_1 - \theta_2) - 2m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)}{l_1(m_1 + 4m_2) - 4m_2 l_1 \cos^2(\theta_1 - \theta_2)}.$$

$$\ddot{\theta}_2 = -2\frac{l_1}{l_2}\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + 2\frac{l_1}{l_2}\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - 6\frac{g}{l_2}\sin\theta_2$$