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Design and Analysis of Algorithms

Assignment: Formulating Proofs
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First Proof

Theorem: If x is an odd integer, then x + 1 is even.

Proof:

- Given an odd integer x.
- By definition of an odd integer, x = 2k+1, where k is an integer. Thus,

  x +1 = (2k+1) +1
- By addition, this equals:

  2k+2
- Take out the common value of 2 from both terms and get:

  2(k+1)
- Since k is an integer, k + 1 is also an integer. Accordingly, we can replace k +1 with an integer m,

  2(k+1) = 2m
- By detention of an even integer, 2m is even.
- Therefore, if x is an odd integer, then x + 1 is even.

  QED

<u>Theorem</u>: $\forall n \in N, 3 \mid (n^3 - n)$

<u>Preliminary note</u>:

- According to the definition of divisibility, to prove $3 \mid (n^3 - n)$, we must prove there is an integer q that satisfies the following equation:

  $n^3 - n = 3q$

<u>Base Case</u>:

- $n^3 - n = 3q$
- Replace n with 0 (the base case):

  $0^3 - 0 = 3q$
- Solve the left-hand side of the equation:

  $0 = 3q$
- Divide both sides by three:

  $0 = q$
- 0 is an integer, accordingly, q is an integer as well. Hence, by the definition of divisibility, $3 \mid n^3 - n$, when $n = 0$.

<u>Inductive Hypothesis</u>:

- Assume that for an integer k >= 1, there is an integer r to solve $k^3 - k = 3r$.

<u>Inductive Step</u>:

- Replacing k with k + 1 in the inductive hypothesis yields,

  $(k + 1)^3 - (k + 1) = 3r$

  Now we will prove that there is an integer r that makes this equation true.
- Expand the exponent:

  $(k + 1)(k + 1)(k + 1) - (k + 1) =3r$
- Compute the multiplication on the left-hand side:

  $k^3 + 3k^2 + 3k + 1 - k - 1 = 3r$
- Rearrange using the communitive property, and subtract 1 – 1:

  $k^3 - k + 3k^2 + 3k = 3r$
- Using the inductive hypothesis, replace $k^3 - k$ (highlighted to easily spot) by 3t, where t is an integer,

  $3t + 3k^2 + 3k = 3r$

- Factor out the common 3:

  $3(t + k^2 + k) = 3r$

- Since t and k are both integers, adding and squaring them will yield an integer as well. Thus, $t + k^2 + k$ can be written as an integer x,

  $3(x) = 3r$,

- Since r can be any integer and x is an integer, these two statements are equivalent when $r = x$.
- Therefore, there is an integer r that satisfies the inductive step, $(k + 1)^3 - (k + 1) = 3r$.
- Thus, we have proven the inductive step to be true, thereby proving by induction $\forall n \in N, 3 \mid (n^3 - n)$.

  QED

Theorem: $\forall n \in N$, for $n > 1$ we have $n! < n^n$

Base Case:

- $n! < n^n$
- Replace n with 2 (the base case):

  $2! < 2^2$
- $2 < 4$ is true.

Inductive Hypothesis:

- For an integer $k > 1$, it is true that $k! < k^k$.

Inductive Step:

- Replace k in the inductive hypothesis by $k + 1$:

  $(k + 1)! < (k + 1)^{(k + 1)}$ (1)
- Expand the factorial:

  $(k + 1)(k)(k - 1)\ldots(1) < (k + 1)^{(k + 1)}$ (2)
- $(k)(k - 1)\ldots(1)$ is $k!$ therefore,

  $(k+1)k! < (k + 1)^{(k + 1)}$ (3)
- The expanded exponent on the right hand side is $(k + 1)(k + 1)\ldots$ done $k + 1$ times. Hence, aside from the first $(k + 1)$, the rest of the equation can be written as $(k + 1)^k$. Thus, $(k + 1)(k + 1)^k = (k + 1)^{(k + 1)}$

  Plugging this into equation (3) yields,

  $(k+1)k! < (k+1)(k + 1)^k$ (4)
- Divide each side by $(k+1)$:

  $k! < (k + 1)^k$ (5)
- Adding one to a positive number that is the base of a positive exponent will make the final result have a greater value. Therefore, since k is positive, $k^k < (k+1)^k$ is a true statement.

  Thus, by our inductive hypothesis, $k! < k^k < (k + 1)^k$, is true as well.

  Accordingly, $k! < (k + 1)^k$. Hence, Equation (5), $k! < (k + 1)^k$, is true.
- Thus, the inductive step is true.
- Therefore, by induction, $\forall n \in N$, for $n > 1$ we have $n! < n^n$.

  QED