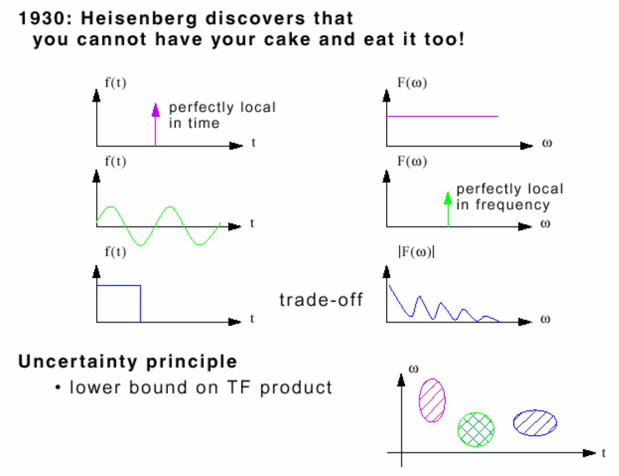
Wavelets in Pattern Recognition

Lecture Notes in Pattern Recognition by W.Dzwinel

Uncertainty principle



Uncertainty principle

Uncertainty principle

Joint time-frequency resolution is lower bounded by uncertainty principle

Define: time width $\Delta_t^2 = \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt$ frequency width $\Delta_\omega^2 = \int_{-\infty}^{\infty} \omega^2 |F(\omega)|^2 d\omega$

Theorem: If f(t) vanishes faster than $1/\sqrt{t}$ as $t \to \infty$, then

$$\Delta_t^2 \cdot \Delta_\omega^2 \geq \frac{\pi}{2}$$

Moreover, equality holds only for a Gaussian

$$f(t) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha t^2}$$

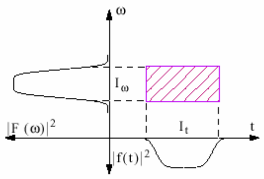
• can trade time for frequency resolution and vice-versa

Tiling

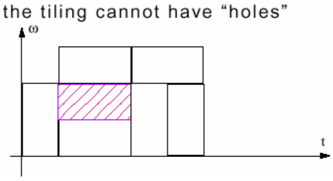
Time-frequency tiling

Basis functions have some spread in time and frequency

• leads to time-frequency tile or atom
• the area in which most of the signal's energy resides



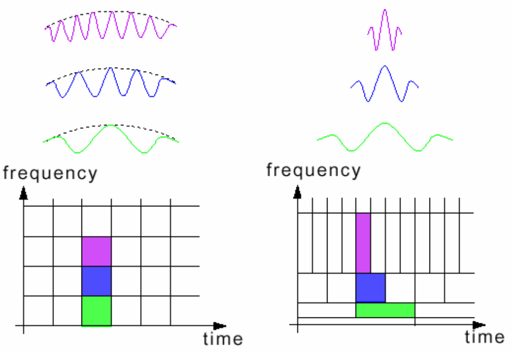
• to represent signals, the tiling cannot have "holes"



Windowed FT vs. WT

1945: Gabor localizes the Fourier transform ⇒ STFT

1980: Morlet proposes the continuous wavelet transform

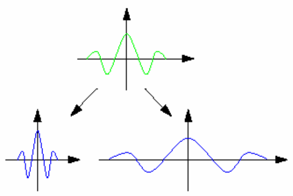


short-time Fourier transform   wavelet transform

Idea of “mother” wavelet

How do wavelets expand signals?

“mother” wavelet ψ



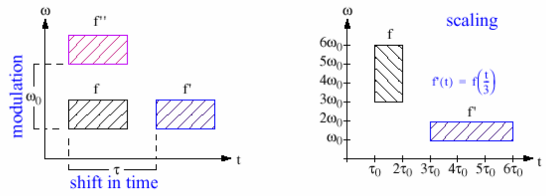
Who?
- families of functions obtained from “mother” wavelet by dilation and translation

Why?
- well localized in time and frequency
- it has the ability to “zoom”

Scale and resolution

Frequency, scale and resolution

Elementary operations on the basis functions



Scale $f'(t) = f(t/a)$ is as in maps:

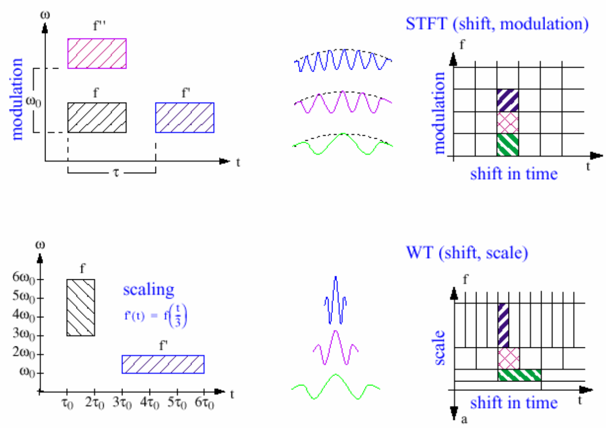
- large scale ⇔ less details, large area ⇔ long basis functions
- small scale ⇔ detail, small area ⇔ short basis functions

Frequency inversely proportional to scale

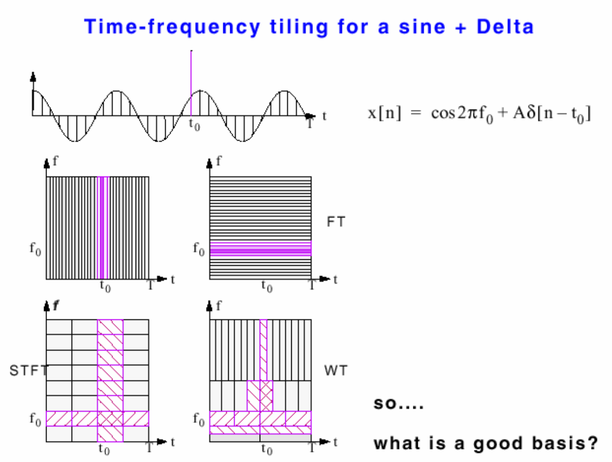
Resolution proportional to the amount of information

STFT vs. WT

STFT versus WT



Tiling



STFT vs. WT

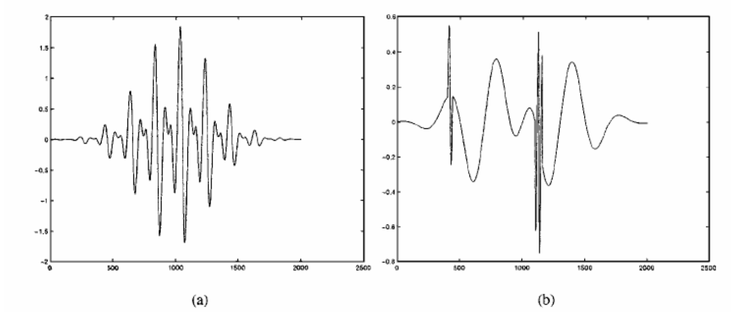


Figure 2 (*a*) A simple example of a signal for which a windowed Fourier transform is better suited than a wavelet transform. (*b*) A simple example of a signal with transients for which a wavelet transform is better suited than a windowed Fourier transform.

Wavelets – continuous transform

The *continuous* WT of a function $f(\mathbf{x})$, denoted for continuous functions $f(\mathbf{x})$ by $\hat{f}(a, \mathrm{b})$, is defined by

$$\hat{f}(a,\mathbf{b}) = \int_{-\infty}^{\infty} f(\mathbf{x})\psi_{a,b}(\mathbf{x})d\mathbf{x}, \qquad \int |\psi(x)|^2 dx = 1,$$

where

$$\psi_{a,b}(\mathbf{x}) = \frac{1}{\sqrt{a}}\psi\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right). \qquad \int_{-\infty}^{\infty} \psi(x)dx = 0,$$

Wavelets – continuous transform

$$\psi_{j,k}(x) = \frac{1}{\sqrt{a_0^j}}\psi\left(\frac{x - kb_0a_0^j}{a_0^j}\right) = a_0^{-j/2}\psi\left(a_0^{-j}x - kb_0\right),$$

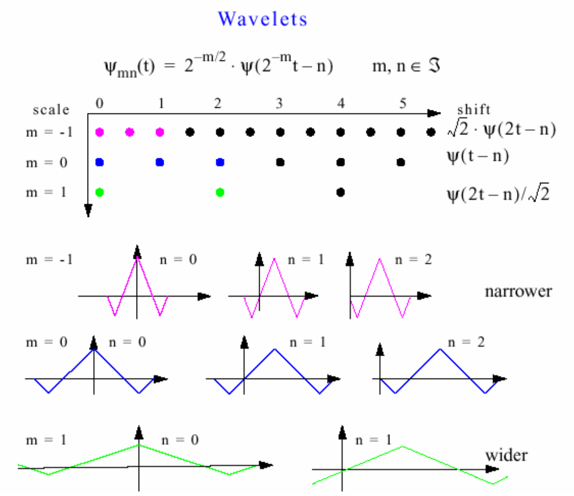
the resulting WT is given by

$$\hat{f}(j,k) = a_0^{-j/2}\int_{-\infty}^{\infty} f(x)\psi\left(a_0^{-j}x - kb_0\right)dx.$$

$$D_j(k) = 2^{-j/2}\int_{-\infty}^{\infty} f(x)\psi\left(2^{-j}x - k\right)dx,$$

$$f(x) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} D_j(k)\psi_{j,k}(x).$$

Wavelets – discrete transform



Scaling function

The most accurate approximation of a function at a fixed or given scale is obtained by using another function called the *scale-function* $\phi(x)$, which is orthogonal to $\psi(x)$. Whereas the mean value of ψ over the entire space is zero, the mean value of ϕ is unity over the same space, implying that $\phi(x) < \psi(x)$, thus $\phi(x)$ provides us with complementary information on the approximation to the function $f(x)$.

The wavelet *approximate* or *scale coefficients* are defined by

$$S_j(k) = \int_{-\infty}^{+\infty} \phi_{j,k}(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \, ,$$

Wavelet series

Any function f(t) can be represented by the series of wavelets expansion:

$$f(t) = \sum_{l \in \mathbf{Z}} c(l) j_{Jl}(t) + \sum_{j=l}^{J} \sum_{k \in \mathbf{Z}} d(j,k) y_{jk}(t), \qquad f(t) \in L^2(R)$$

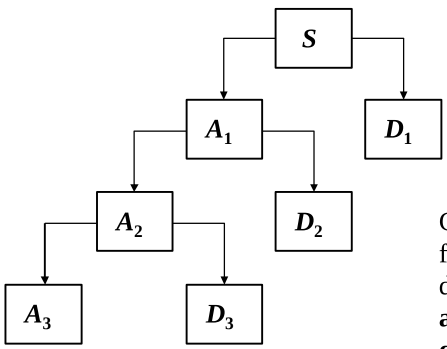
$$c(l) = \langle j_{Jl} \mid f \rangle$$

$$d(j,k) = \langle y_{jk} \mid f \rangle$$

c(l) – low frequency coefficients

d(j,k) – high frequency coefficients on different detail levels

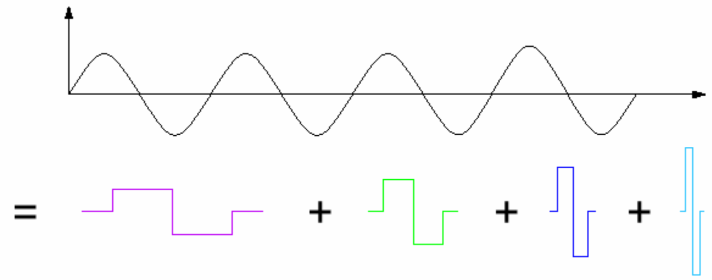
Wavelet decomposition



Consecutive iterations starting from a signal and decomposing it into **approximations** (A) and **details** (D).

Haar wavelet

1910: Alfred Haar discovers the Haar wavelet
dual to the Fourier construction



Haar wavelet

Haar system...
... scaling function and wavelet

The Haar scaling function
(indicator of unit interval)

$$\varphi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{else} \end{cases}$$

helps in the construction
of the wavelet, since

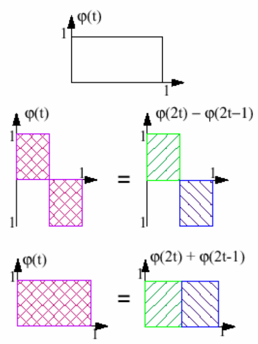
$$\psi(t) = \varphi(2t) - \varphi(2t-1)$$

and satisfies a
two-scale equation

$$\varphi(t) = \varphi(2t) + \varphi(2t-1)$$

Note:

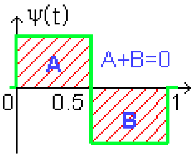
- Haar wavelet a bit too trivial to be useful...



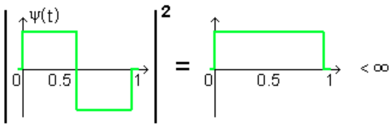
Wavelet transformation - conditions

Wavelet ? (t) has to fulfill a few conditions:

$$\int_{-\infty}^{+\infty} y(t)\,dt = 0$$

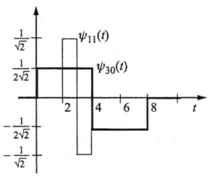


$$\int_{-\infty}^{\infty} |y(t)|^2\,dt < \infty$$



Wavelets represents a basis in the L^2 Hilbert Space which CAN be orthogonal and /or orthonormal:

$$\langle y_1 | y_2 \rangle = \int_{-\infty}^{+\infty} y_1(t) y_2(t)\,dt = 0 \qquad \|y\| = \int_{-\infty}^{+\infty} |y(t)|^2\,dt = 1$$



Haar wavelet

Haar system...

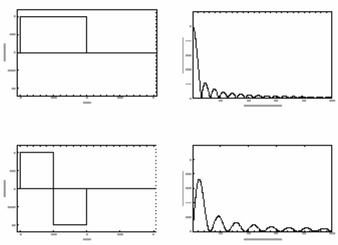
... scaling function and wavelet

time domain

frequency domain

scaling function

wavelet



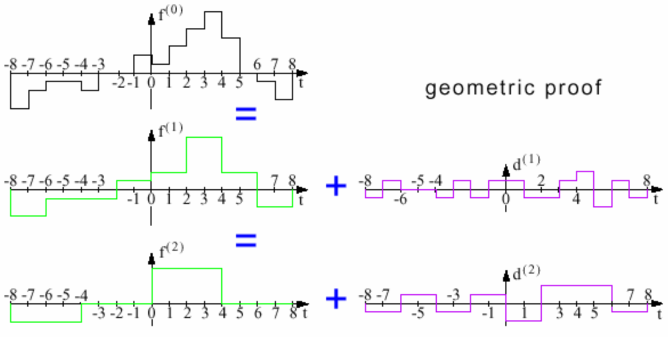
$$\Phi(\omega) = e^{-j\frac{\omega}{2}} \frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}$$

$$\Psi(\omega) = je^{-j\frac{\omega}{2}} \frac{\left(\sin\frac{\omega}{4}\right)^2}{\frac{\omega}{4}}$$

- well localized in time, as $m \to -\infty$, length goes to zero
- not well localized in frequency, FT decays as $1/\omega$ as $\omega \to \infty$

Haar decomposition

Haar system...
... as a basis for $L_2(\Re)$



geometric proof

Wavelets and images

Why expand signals?

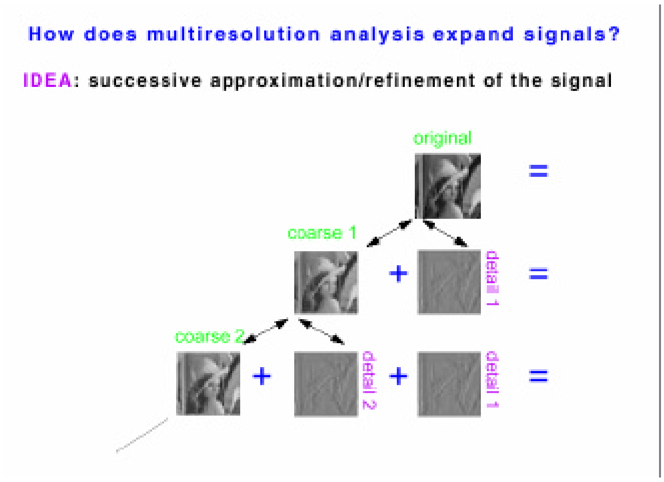
Suppose

original = coarse + detail

Advantages

- easier to analyze signal in pieces: “divide and conquer”
- extracts important features
- pieces can be treated in an independent manner

Wavelets and images

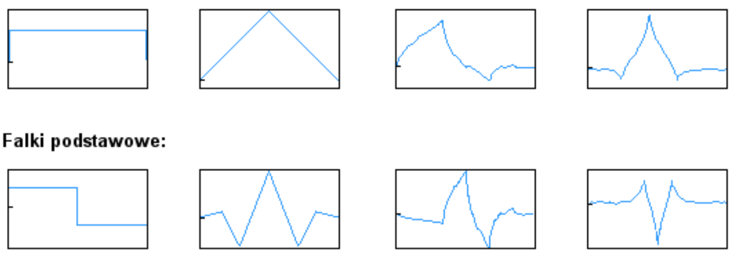


Wavelets – different bases

HAAR B-SPLINE DAUBECHIES COIFLET

Funkcje skalujące:

Falki podstawowe:



Multi-resolution

2.1. Multiresolution Analysis

Let us start by considering the decomposition of $L_2(\mathbf{R})$ into a set of nested function subspaces

$$\ldots V_{j-1} \subset V_j \subset V_{j+1} \ldots \quad j \in \mathcal{Z}, \tag{1}$$

where we associate with each subspace V_j, a set of points γ_j. These subspaces form a multiresolution analysis with the following properties:

1. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$.
2. $f(x) \in V_j \Leftrightarrow f(x+k) \in V_j : \forall k \in \gamma_j$.
3. $\cup_j V_j$ is dense in $L_2(\mathbf{R})$ and $\cap_j V_j = \{\emptyset\}$.
4. There exists for the scaling space V_j a *scaling function* $\phi_j(x) \in V_j$ such that the collection

$$\phi_j(x+k) : \forall k \in \gamma_j \tag{2}$$

forms a Riesz basis of V_j,

$$V_j = \operatorname{span}\{\phi_j(x+k) : k \in \gamma_j\}. \tag{3}$$

There also exists a *wavelet function* $\psi_j(x)$ which spans the detail space W_j, the complement of $V_j \in V_{j+1}$; i.e.,

$$V_{j+1} = W_j \oplus V_j, \quad V_j \perp W_j, \tag{4}$$

Wavelets construction

One important family of such wavelets contains the Daubechies wavelets of order M[19,22] (usually referred to as DBM). The first M moments of the DBM wavelets are zero. The scaling function $\phi(x)$ is related to those at the finer length scales by

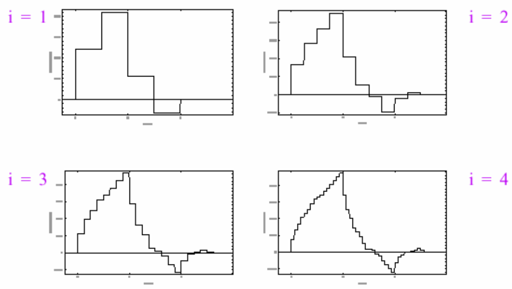
$$\phi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k \phi(2x-k)\,. \tag{15}$$

$$\psi(x) = \sqrt{2}\sum_{k=0}^{L-1} m_k \phi(2x-k)\,,$$

Wavelets – construction

Daubechies' construction...
... iteration algorithm

At ith **step associate piecewise constant approximation of length** $1/2^i$ **with** $g_0^{(i)}[n]$

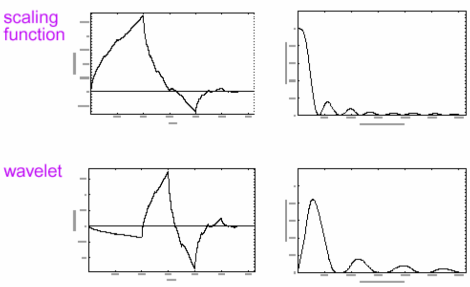


Fundamental link between discrete and continuous time!

Wavelets – construction

Daubechies' construction...
...scaling function and wavelet

- Haar and sinc systems: either good time OR frequency localization
- Daubechies system: good time AND frequency localization



Finite length, continuous $\varphi(t)$ **and** $\psi(t)$**, based on** L=4 **iterated filter**

Many other constructions: biorthogonal, IIR, multidimensional...

Wavelets – construction

Choice of the wavelet form

You can graph all scaling functions and wavelets that satisfy refinement relations with four coefficients. The scaling function is defined on [0,3] and satisfies

$$\varphi(x)=c_0\varphi(2x)+c_1\varphi(2x-1)+c_2\varphi(2x-2)+c_3\varphi(2x-3)$$

and

$$\int_0^3\varphi(x)dx=1.$$

The associated (QMF) wavelet is given by

$$\psi(x)=c_3\varphi(2x)-c_2\varphi(2x-1)+c_1\varphi(2x-2)-c_0\varphi(2x-3).$$

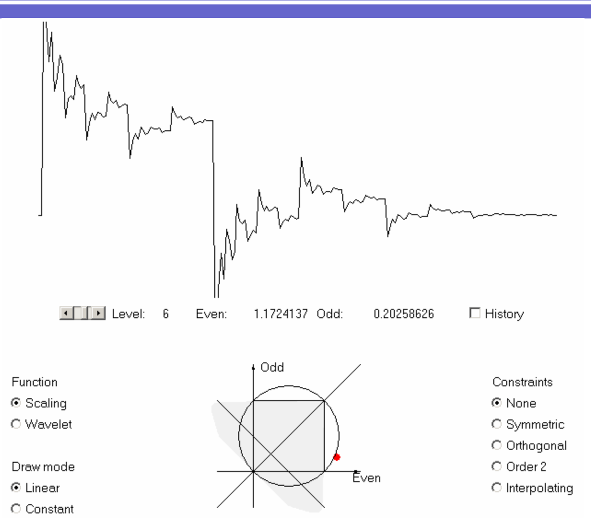
It is well known that a continuous solution can only exist in case the refinement coefficients satisfy:

$$c_0+c_2=c_1+c_3=1.$$

This leaves us with two degrees of freedom. We choose them to be the first coefficient *even* and the last one *odd*. We then have

$$c_0=\textit{even},\ c_1=1\text{-}\textit{odd},\ c_2=1\text{-}\textit{even},\ c_3=\textit{odd}.$$

Wavelets – construction



Wavelets in 2-D

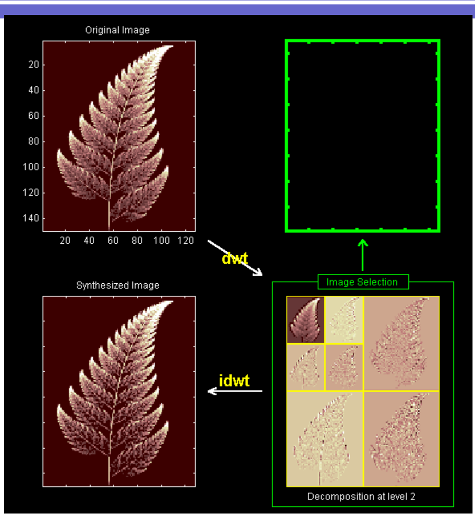
$$\phi_{j,k_1,k_2}(x,y) = \phi^j_{k_1}(x)\phi^j_{k_2}(y)\,,$$

$$\psi^{(1)}_{j,k_1,k_2}(x,y) = \phi^j_{k_1}(x)\psi^j_{k_2}(y)\,,$$

$$\psi^{(2)}_{j,k_1,k_2}(x,y) = \phi^j_{k_1}(x)\psi^j_{k_2}(y)\,,$$

$$\psi^{(3)}_{j,k_1,k_2}(x,y) = \phi^j_{k_1}(x)\psi^j_{k_2}(y)\,.$$

Two dimensional wavelets



Wavelets – in multiple dimensions

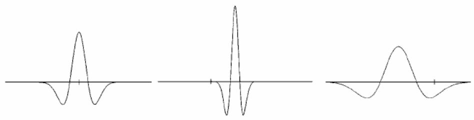
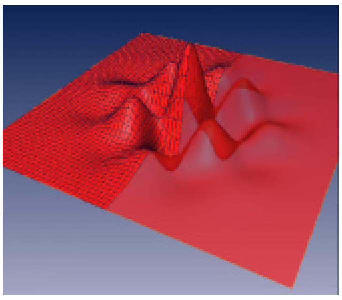


Figure 1 The "Mexican hat" wavelet $\psi(x) = (1 - x^2)e^{-x^2/2}$ (*left*), together with two examples of translates and dilates. (In each case, the tick corresponds to $x = 0$.) For this ψ, the $\{\psi_{j,k}\}$ values do not constitute an orthonormal basis.

2D 4th order wavelet on 64x64 grid

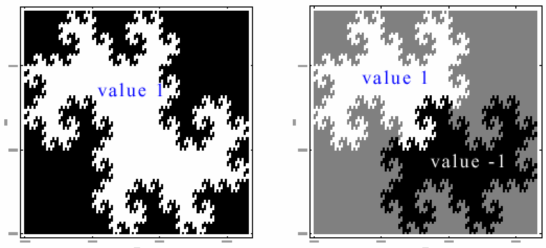
Mexican hat: 4th derivative of Gaussian



Wavelets – in multiple dimensions

Wavelet series...
... in multiple dimensions

Generalization of Haar



"twin dragon" system

Wavelets – in multiple dimensions

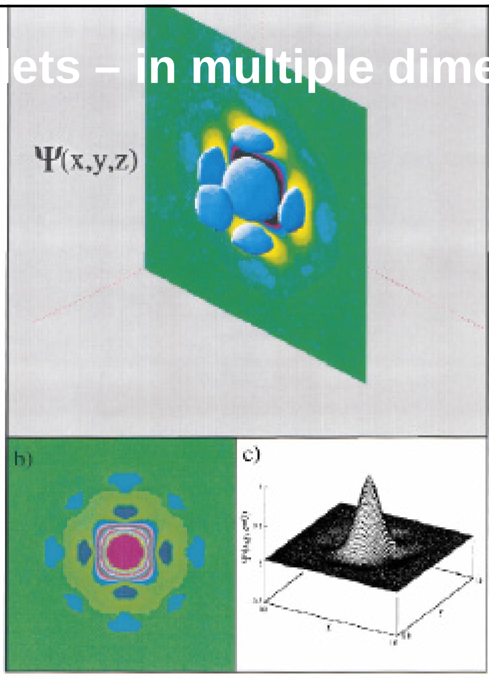


FIG. 9. (a) An iso-surface plot of the three-dimensional nonseparable multiwavelet $\psi^{\ast\ast}(x)$ for order [7, 7, 7]. Also shown, (b) and (c), a density and surface plot of the $z = 0$ plane for this wavelet.

Matlab and wavelets

Function	Description
dwt	One-dimensional single-level decomposition of a given signal
wavedec	multi-level signal decomposition
dwt2, wavedec2	Two-dimensional functions
idwt	Single-level reconstruction of 1D signal
waverec	Multi-level reconstruction
idwt2, waverec2	Two-dimensional functions

wavemenu – starts graphical interface

Noise removal

Select first:

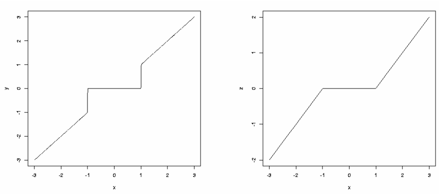
- Wavelet form
- Number of decomposition levels

1. Wavelet decomposition of signal S on level *N*.
2. Define the thresholds on all the levels from 1 to N and eliminate small wavelet coefficients of all the details.
3. Complete wavelet reconstruction by means of approximation and remaining coefficients of the details.

Thresholding and elimination

Two types (at least) of thresholding process:

- **Hard elimination** $$y_{tw}(t) = \begin{cases} y(t), & |y(t)| > d \\ 0, & |y(t)| \le d \end{cases}$$
- **Soft elimination** $$y_{mk}(t) = \begin{cases} \mathrm{sgn}(y(t))(|y(t)| - d), & |y(t)| > d \\ 0, & |y(t)| \le d \end{cases}$$
- Comparisons

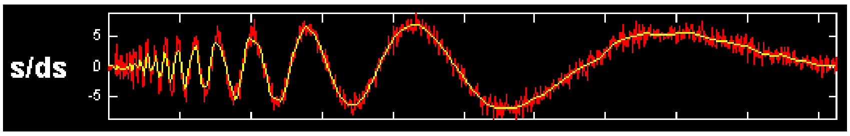


twarda miekka

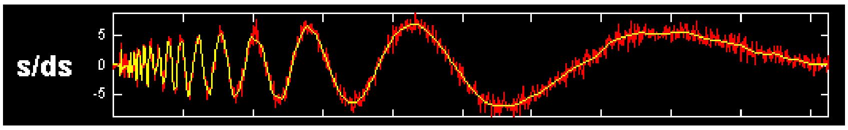
Noise removal

Symlets wavelets, and 4-level decomposition is used. The threshold values are the same.

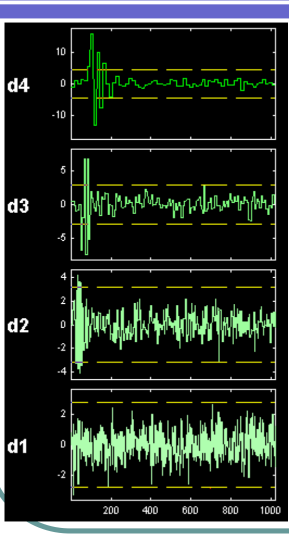
a) Soft



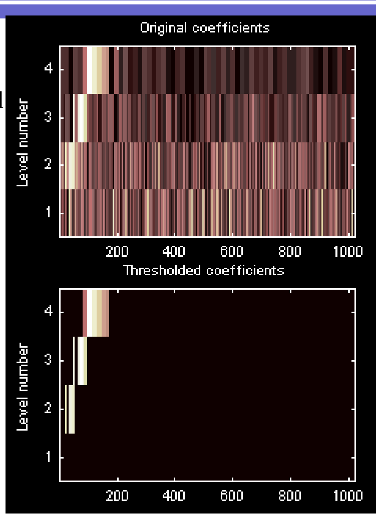
b) Hard



Noise removal (1D signal)

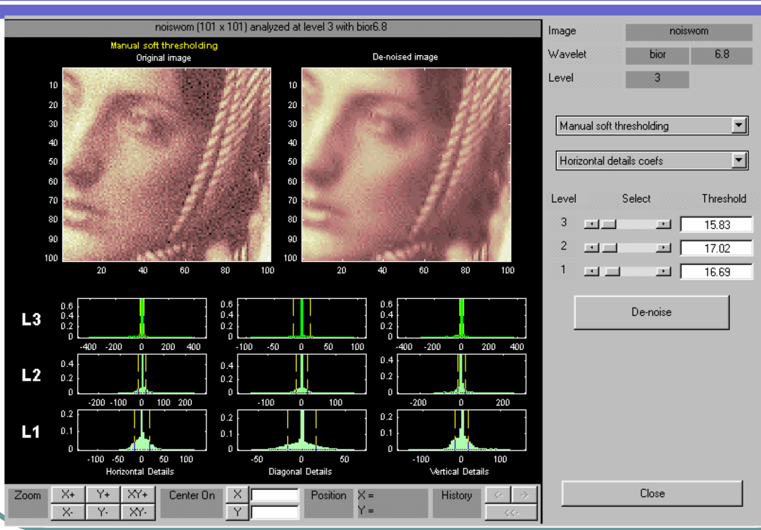


Details and threshold values



Decomposition coefficients before and after thresholding

Noise removal (2D signal)



Signal compression

1. Signal decomposition
2. Thresholding and elimination of coefficients
3. Reconstruction.

Ad. 1, 3 – similar as in noise removal

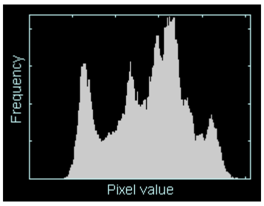
Ad. 2 – different approaches exist

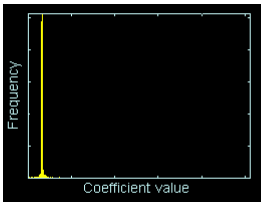
a) Fix the global threshold value and/or define a quality of compression parameter

b) Adaptive threshold setting on every decomposition level

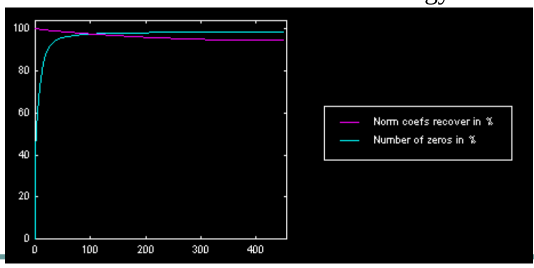
Signal compression

Histograms of some image before and after wavelet transform

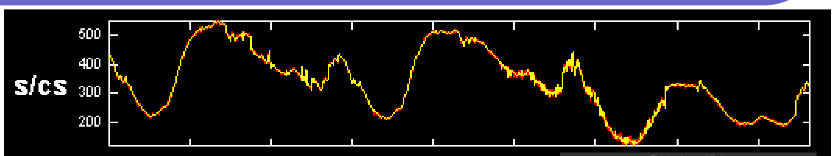




Number of eliminated coefficients vs. the energy of the signal kept



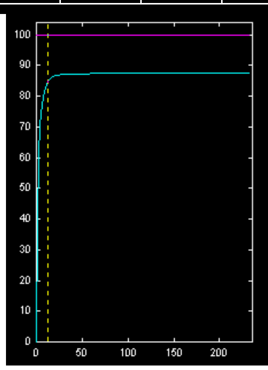
Signal compression (1D)

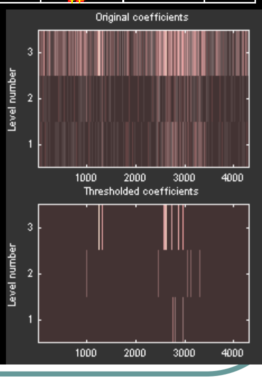


Daubechies 3 wavelets decomposed on 3 levels
<u>Result</u>:

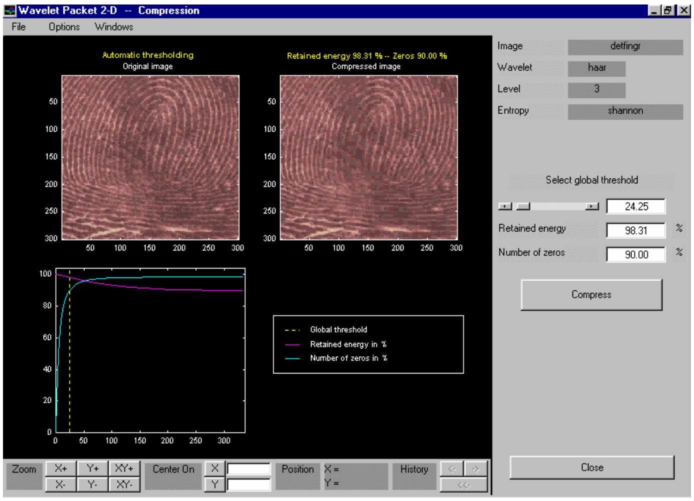
99.99% of signal energy preserved

Eliminated - 84.74% coefficients.

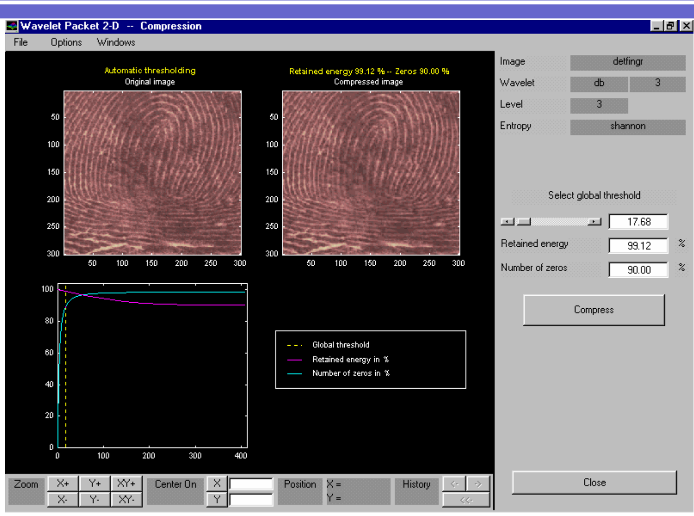




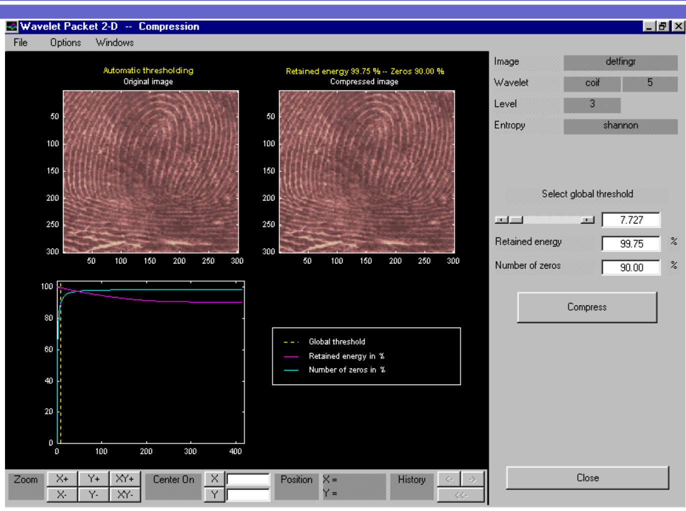
Signal compression-comparisons



Signal compression-comparisons



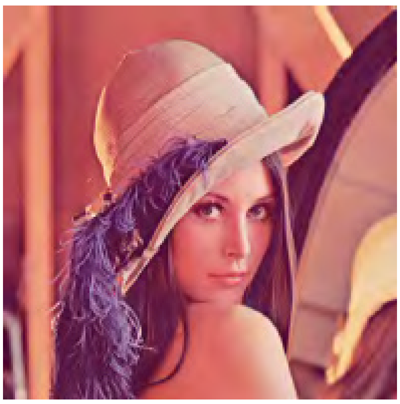
Signal compression-comparisons



Signal compression-comparisons

Wavelet compression vs JPEG:

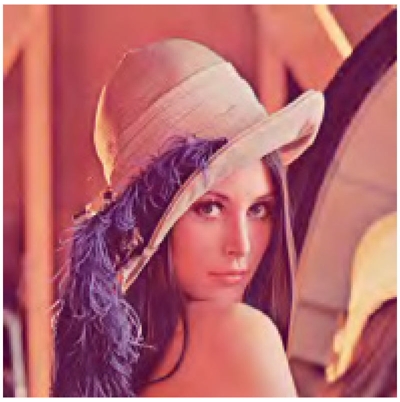
Original image - 786486 b



waveletcompression - 7812 b

Signal compression-comparisons

Wavelet compression vs JPEG:



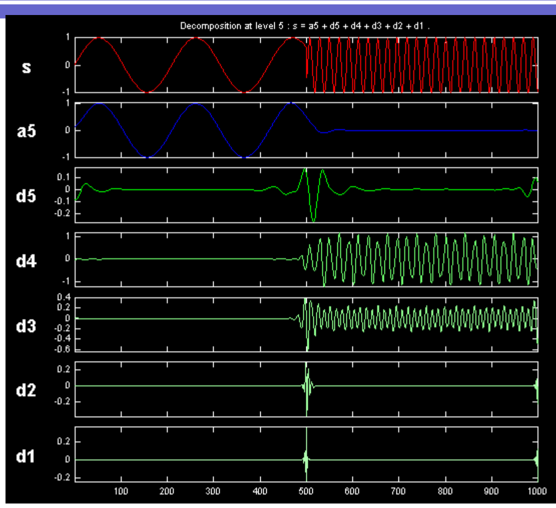
Wavelet compression - 7812 b

JPEG – 8071 b

Detection of singularities

(rapid change of frequency)

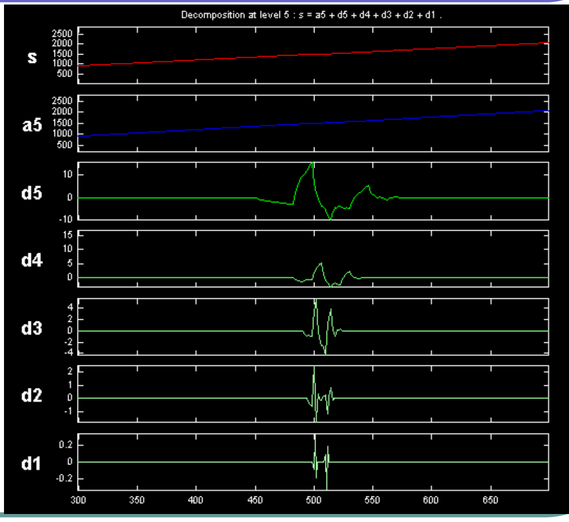
coiflet wavelet
order 5.



Detection of singularities
(singularity)

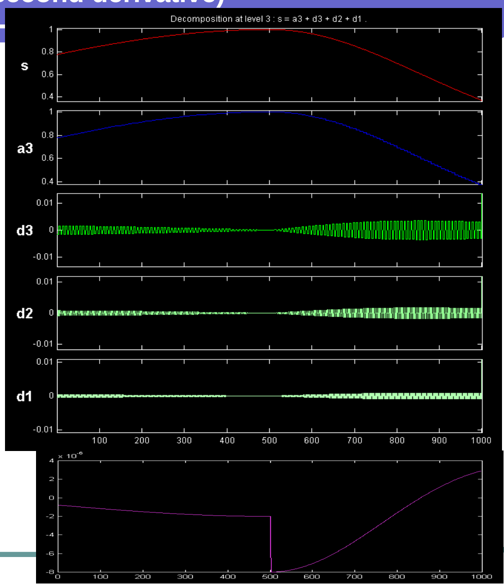
Two close discontinuities

Daubechies order 2.



Detection of singularities
(discontinuity of the second derivative)

Daubechies order 1:



Computational complexity

	sygnal 1D	obraz
DFT	N^2	N^3
FFT	$N \log_2 N$	$N^2 \log_2 N$
FWT	N	N^2

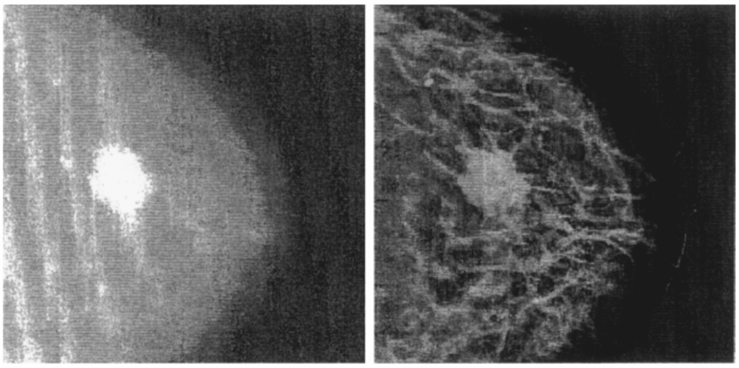
Which wavelets ???

- Continuous – very slow and redundant (overcompletness) but more reliable. The information cannot be lost easily.
- Biorthogonality – one set of wavelets for decomposition one for reconstruction (higher dimensions) are symmetric and have compact support but may amplify any error introduced on the coefficients
- Orthogonal – fast, concise but arbitrary scales – because orthogonal transformation are not translation invariant
- The number of vanishing moments determine what the wavelets do not see (first vanishing moment – linear function is not seen). More vanishing moments → search is focused on better selectivity in time but p → vanishing moments means that wavelet support must be at least 2p-1 larger support → more computations.

Which wavelets ????

- Image compression → 3-4 vanishing moments.
- A few large singularities → more vanishing moments, More singularities → smaller support → lesser number of vanishing moments
- Daubechies wavelets the most vanishing moments for the smallest possible support
- Regularity – regularity n → n+1 derivatives. Important for image encoding. Not important for audio. Most regular wavelets are splines.
- Frequency selectivity – not important for images, but important for audio.→ freq.select == many vanishing moments. The best trade off is using Gabor functions or B-spline wavelets

Mammograms and radiology

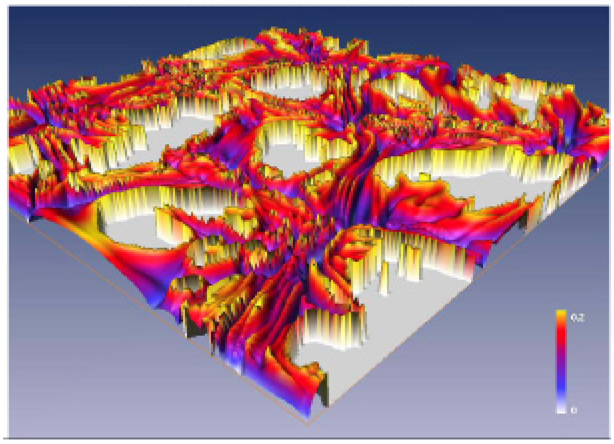


(a) (b)

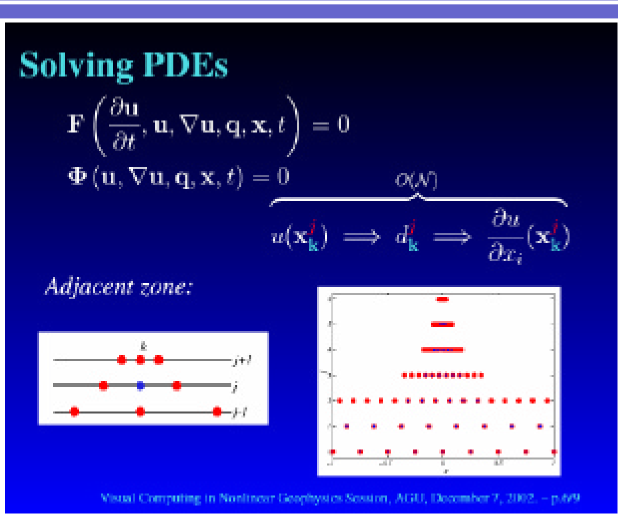
Figure 15 (*a*) Original mammogram containing a mass lesion. (*b*) Enhanced mammogram showing well-defined borders of the mass and clarity of subtle breast tissue and structures.

Wavelets in Sci. Visulization

Map of heat flux using wavelets



Wavelets transform in PDE solving



Ridglets

3.4. Ridgelets

The theory of ridgelets was developed in the Stanford Ph.D. Thesis of Emmanuel Candès (1998). In that work, Candès showed that one could develop a system of analysis based on ridge functions

$$\psi_{a,b,\theta}(x_1, x_2) = a^{-1/2}\psi((x_1 \cos(\theta) + x_2 \sin(\theta) - b)/a). \qquad (1)$$

He introduced a continuous ridgelet transform $R_f(a, b, \theta) = \langle \psi_{a,b,\theta}(x), f\rangle$ with a reproducing formula and a Parseval relation. The key point is the analysis by functions which are global in one direction and local in the other direction. For further

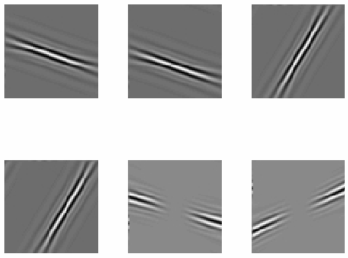
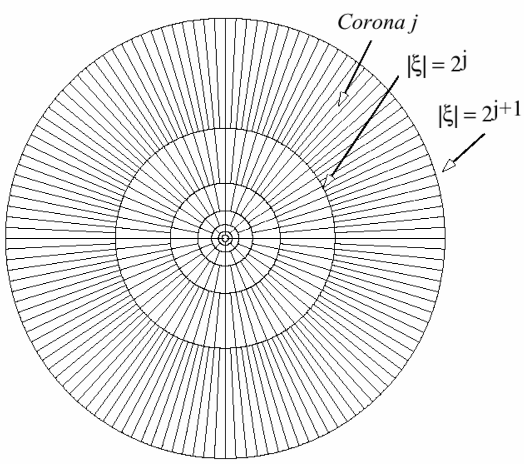
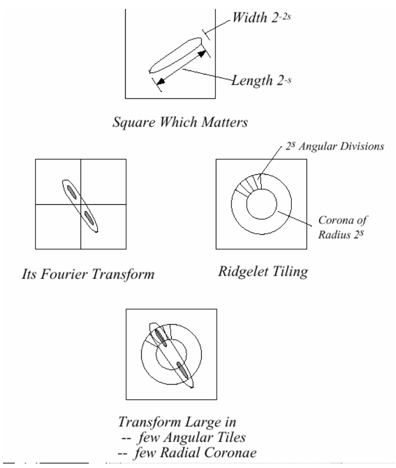


Figure 1. Several Ridgelets

Ridglets tiling



Ridglets



Curvelets

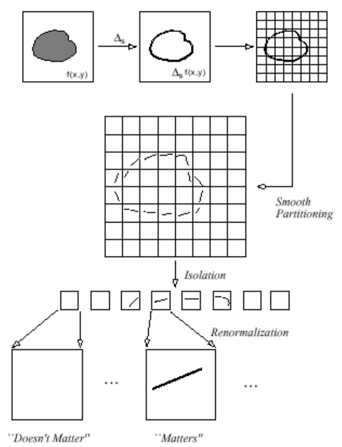
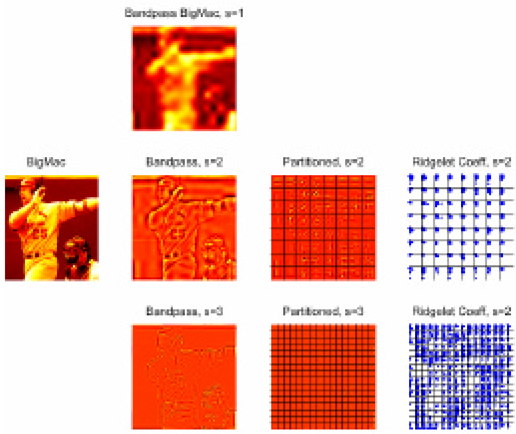


Figure 5. Decomposition at a Single Scale

Curvelets



. BigMac Image and stages of curvelet transform

Comparisons of different approaches

We propose to approach such questions by using *kurtosis* (a sort of normalized fourth moment) as a measure of nonGaussianity. This can be justified from the It follows, under the sparse coding hypothesis, that if two bases code essentially the same subband, the one achieving the sparser representation should yield coefficients with larger kurtosis.

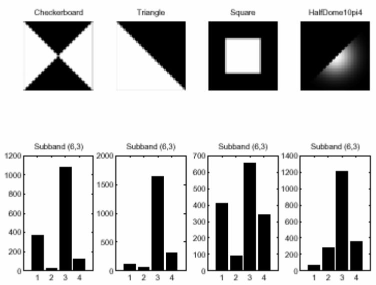


Figure 6. Kurtosis of Fourier(1), wavelet(2), ridgelet(3) and curvelet(4) coefficients; objects having only perfect straight edges, subband $2s = 6$, $j = i = 3$.

Comparisons of different approaches

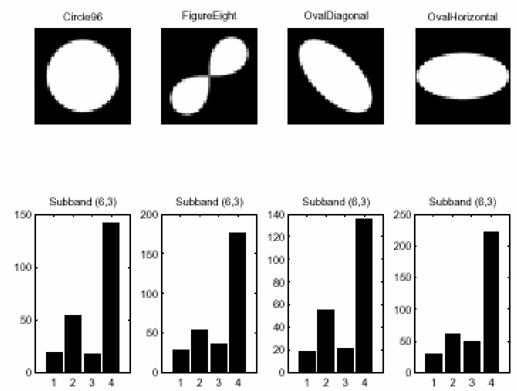


Figure 7. Kurtosis of Fourier (1), wavelet (2), ridgelet (3) and curvelet (4) coefficients; objects with curved edges, subband $2s = 6$, $j = i = 3$.

Comparisons of different approaches

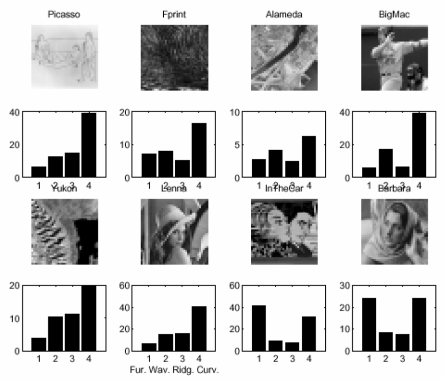


Figure 8. Kurtosis of Fourier, wavelet, ridgelet and curvelet coefficients of several non-synthetic images, subband $2s = 6$, $j = i = 3$.

sense, because it agrees with our sparse coding interpretation: we know that ridgelets are better than curvelets at representing discontinuities along perfectly straight lines.

Web pages


Welcome to

wavelet.org

The **Wavelet Digest** is a free monthly newsletter edited by Wim Sweldens which contains all kinds of information concerning wavelets: announcement of conferences, preprints, software, questions, etc. If you want to receive regular copies of the Wavelet Digest, click here.

Here you can also find:

- The latest issue: WD 10.4 (December 21, 2001)
- Calendar of events
- Detailed help information
- Back issues
- Keyword **search**
- Wavelet Digest team
- Wavelet Digest by country New
  tes

UniGuide Featured Site

Lucent Technologies
Bell Labs Innovations

Mathematical Sciences Research

References

1. Jan T. Bialasiewicz: *Falki i aproksymacje*, WNT Warszawa 2000.
2. B.B. Hubbard, The world according to waveletys, AK Peters Ltd, pp325
3. Andrew S. Glassner: *Principles of digital image synthesis*, Morgan Kaufmann Publishers 1995.
4. Wojciech Maziarz, Krystian Mikolajczyk: *A course on wavelets for beginners* - http://galaxy.uci.agh.edu.pl/~maziarz/Wavelets/, Kraków 1999.
5. The MathWorks Inc., *Developers of Matlab & Simulink* - http://www.mathworks.com.
6. Alain Fournier: *Wavelets and their applications in computer graphics*, SIGGRAPH'95 Course Notes, 1995.
7. Amara Graps: *An Introduction to wavelets, IEEE Computational Sciences and Engineering,* Vol. 2, Number 2, Summer 1995, pp 50-61 - http://www.amara.com/IEEEwave/IEEEwavelet.html.