APPM2007
Lagrangian Mechanics
2021

Assignment

Assignment 02
Parametric Curves

Issued: 01 September 2021 **Total:** 45 points

Due: 17:00, 10 September 2021

Instructions

- Read all the instructions and questions carefully.
- Typeset the solution document using provided LaTeX Template 'Assignment.cls'. Submissions that have not used this template shall receive a zero grade.
- Use plain written English where necessary.
- Students are free to use whatever resources at their disposal to answer this assignment, including Computer Algebra and Graphing software. However, all necessary calculation steps and details should be include to obtain full credit.
- Students may use the supplementary `Mathematica` and LaTeX resources posted on the course Moodle page to complete this assignment.
- Students are encouraged to work in groups. However, this is to be individual work and each student must submit their own report.
- Plagiarized submissions shall receive a zero grade.
- No submissions after the cut-off date shall be considered.

Introduction

The objective of this assignment is to develop an understanding of the properties of parametric mappings. Students should use the computer algebra system `Mathematica` and the code provide below to perform computations on vectors and functions, and to construct graphical representations of parametric points, curves, surfaces and other objects.

Consider the parametric $(r.s)$-plane R at $z = 0$ in Euclidean 3-space $\mathbb{R}^3$. We construct a function taking the pair of numbers r and s as inputs and constructing a point in $\mathbb{R}^3$,

$$R(r,s) = \begin{pmatrix} r \\ s \\ 0 \end{pmatrix}.$$

The pair of numbers $(r.s)$ is mapped to R as a point $(r,s,0)$ in $\mathbb{R}^3$. Use the following code to define this function

```
R[r_, s_] := {r, s, 0};
```

Similarly, we construct a projective mapping from the R to the unit sphere S

$$S(r,s) = \begin{pmatrix} \frac{2r}{r^2+s^2+1} \\ \frac{2s}{r^2+s^2+1} \\ \frac{r^2+s^2-1}{r^2+s^2+1} \end{pmatrix}.$$

The pair of numbers $(r.s)$ is mapped to S as a point in $\mathbb{R}^3$. Use the following code to define this function

```
S[r_, s_] := {
              2 r / (r^2 + s^2 + 1),
              2 s / (r^2 + s^2 + 1),
              (r^2 + s^2 - 1) / (r^2 + s^2 + 1)
              };
```

Use the following code to construct a graphical representation of a portion of a 1-dimensional parametric functions f

```
ParametricPlot3D[ f[t], {t, mint, maxt}]
```

where `mint` and `maxt` are the minimum and maximum values of the parameter t, respectively. Similarly, use the following code to construct a graphical representation of a portion of R

```
ParametricPlot3D[ R[r, s], {r, minr, maxr}, {s, mins,
   maxs}]
```

where minr, maxr, mins and maxs are the minimum and maximum values of the parameter r, and the minimum and maximum values of the parameter s, respectively. A graphical representation of a portion of S may be constructed by replacing R[r, s] with S[r, s].

A white point marker can be placed at (a, b, c) in $\mathbb{R}^3$ by defining

```
coords = {a, b, c};
```

and then using the following code,

```
Graphics3D[ { White, Sphere[coords, 0.05] } ]
```

to place the marker in the scene. Similarly, a red arrow marker with initial point (a, b, c) and terminal point (d, e, f) can be generated by defining

```
start = {a, b, c};
end = {d, e, f};
```

and then using

```
Graphics3D[{Red, Arrow[{start, end}]}]
```

placed to place the arrow in the scene.

The Manipulate function is a useful directive for generating dynamic, interactive graphics with control parameters. Students should experiment using

```
Manipulate[Plot[Sin[s x], {x, minx, maxx}], {s, mins,
   maxs}]
```

where s in an interactively adjustable parameter running from mins to maxs. The Show function can be used to compose multiple graphics objects together in a single graphic, as follows

```
Show[
     Graphics3D[ { Red, Arrow[{start, end}] } ],
     Graphics3D[ { White, Sphere[coords, 0.05] } ],
     ParametricPlot3D[ f[t], {t, mint, maxt}]
     ]
```

The Show function can also be include inside a Manipulate function. A complete listing of options for all graphics related functions is available in the documentation and students should experiment with each function and option.

Use ImageSize -> 600 and add your student number to the background of each plot as a watermark using Epilog and the following formatting options

```
Style["1234567890", FontSize -> 84, Red, Opacity[0.1]]
```

Place this watermark at the center of the plot. Refer to the Mathematica documentation for complete information on Epilog. Figure 1 presents an example watermarked plot.

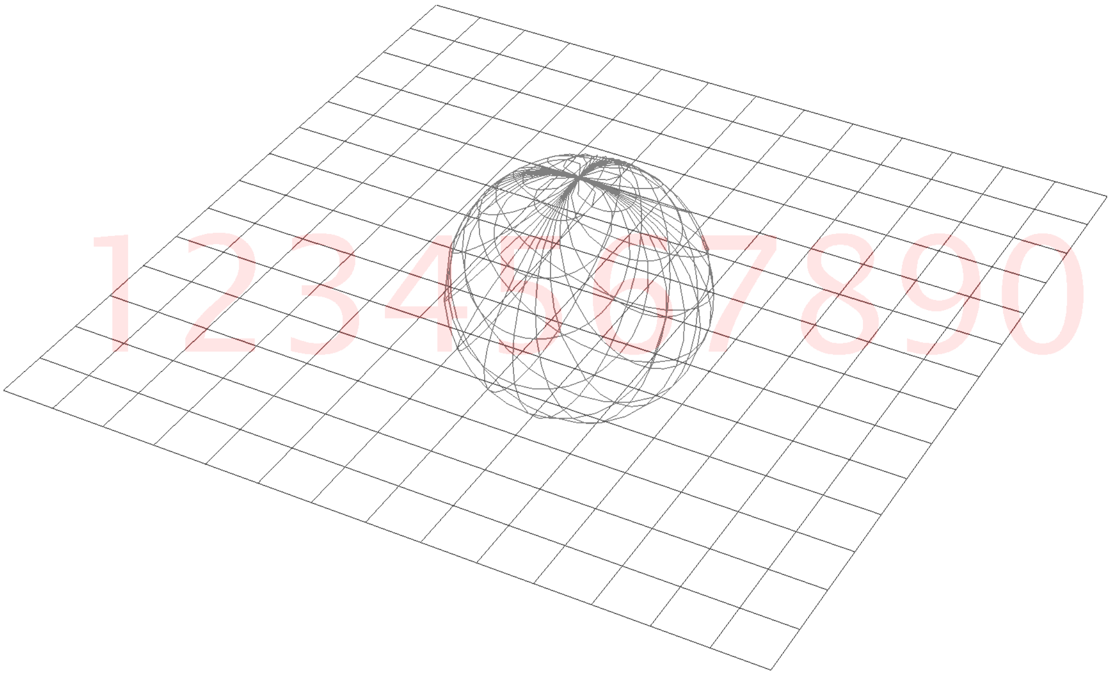

Figure 1: An example of a watermarked plot using Epilog.

The following parametric functions will be used in the questions that follow:

Parametric Curve $p_1 = (r_1, s_1)$: Define $\omega_1 = \frac{\pi}{4}$, then

$$\rho(t) = \tan\left(\frac{\pi}{2}t\right) \qquad r_1(t,\theta) = \rho(t)\cos(\theta) \qquad s_1(t,\theta) = \rho(t)\sin(\theta)$$

$$r_1(t) = r_1(t,\omega_1) \qquad s_1(t) = s_1(t,\omega_1)$$

Parametric Curve $p_2 = (r_2, s_2)$: Define $\omega_2 = 2\pi$, then

$$r_2(t,\theta) = 2\cos(\theta\, t) \qquad s_2(t,\theta) = 2\sin(2\theta\, t)$$

$$r_2(t) = r_2(t,\omega_2) \qquad s_2(t) = s_2(t,\omega_2)$$

Parametric Curve $p_3 = (r_3, s_3)$: Define $\omega_3 = 2\pi$, then

$$r_3(t,\theta) = (1+2t)\cos(2\theta\, t) \qquad s_3(t,\theta) = (1+2t)\sin(2\theta\, t)$$

$$r_3(t) = r_3(t,\omega_3) \qquad s_3(t) = s_3(t,\omega_3)$$

In each case, $t \in [0,1]$.

The derivative function D, the vector dot product Dot and the vector cross product Cross are useful function and students consult the Mathematica documentation for complete information on these functions.

Parametric Curves and Surfaces

Question 1 (5 Points)

Construct a single graphic containing the the following objects in to demonstrate the parametric mappings

1. The portion of R bounded by $r, s \in [-3, 3]$.
2. The entirety of S. (Hint: use an appropriately modified set of co-ordinates to include the entire sphere.)

Describe the relationship between the co-ordinate grids on the R and S. Include a picture of your output.

Question 2 (5 Points)

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The point $a_1(t) = p_1(t)$ on R.
2. The point $b_1(t) = p_1(t)$ on S.
3. The 1-dimensional parametric path $p_1(t)$ on R from from $p_1(0)$ until $a_1(t)$.
4. The 1-dimensional parametric path projection of the path $p_1(t)$ on S from $p_1(0)$ until $a_1(t)$.
5. The circle on S marking the height of $b_1(t)$ in $\mathbb{R}^3$.
6. The directed line segment (arrow) joining the north pole of S to the $b_1(t)$.
7. The directed line segment (arrow) joining the $a_1(t)$ to $b_1(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$.

Question 3 (5 Points)

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The point $a_2(t) = p_2(t)$ on R.
2. The point $b_2(t) = p_2(t)$ on S.

3. The 1-dimensional parametric path $p_2(t)$ on R from from $p_2(0)$ until $a_2(t)$.
4. The 1-dimensional parametric path projection of the path $p_2(t)$ on S from $p_2(0)$ until $a_2(t)$.
5. The circle on S marking the height of $b_2(t)$ in $\mathbb{R}^3$.
6. The directed line segment (arrow) joining the north pole of S to the $b_2(t)$.
7. The directed line segment (arrow) joining the $a_2(t)$ to $b_2(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$.

Question 4 (5 Points)

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The point $a_3(t) = p_3(t)$ on R.
2. The point $b_3(t) = p_3(t)$ on S.
3. The 1-dimensional parametric path $p_3(t)$ on R from from $p_3(0)$ until $a_3(t)$.
4. The 1-dimensional parametric path projection of the path $p_3(t)$ on S from $p_3(0)$ until $a_3(t)$.
5. The circle on S marking the height of $b_3(t)$ in $\mathbb{R}^3$.
6. The directed line segment (arrow) joining the north pole of S to the $b_3(t)$.
7. The directed line segment (arrow) joining the $a_3(t)$ to $b_3(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$.

Tangents to Curves

Question 5 (5 Points)

Compute the following path lengths

1. $p_1(t)$ on S.
2. $p_2(t)$ on S.
3. $p_3(t)$ on S.

In each case let $t \in [0, 1]$. Use `NIntegrate` to help in the computations.

Question 6 (5 Points)

Construct a local rectilinear co-ordinate system along the projection of the parametric curve $p_1(t)$ on S. Construct the following unit vectors

1. Unit tangent vector $\vec{T}_1(t)$.
2. Unit acceleration $\vec{N}_1(t)$.
3. Unit bi-normal $\vec{B}_1(t) = \vec{T}_1(t) \times \vec{N}_1(t)$.

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The directed line segment (arrow) depicting $\vec{T}_1(t)$.
2. The directed line segment (arrow) depicting $\vec{N}_1(t)$.
3. The directed line segment (arrow) depicting $\vec{B}_1(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$. Describe the relative orientation of $\vec{T}_1(t)$, $\vec{N}_1(t)$ and $\vec{B}_1(t)$. Describe the motion of $\vec{T}_1(t)$, $\vec{N}_1(t)$ and $\vec{B}_1(t)$ along the S.

Question 7 (5 Points)

Construct a local rectilinear co-ordinate system along the projection of the parametric curve $p_2(t)$ on S. Construct the following unit vectors

1. Unit tangent vector $\vec{T}_2(t)$.
2. Unit acceleration $\vec{N}_2(t)$.
3. Unit bi-normal $\vec{B}_2(t) = \vec{T}_2(t) \times \vec{N}_2(t)$.

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The directed line segment (arrow) depicting $\vec{T}_2(t)$.
2. The directed line segment (arrow) depicting $\vec{N}_2(t)$.
3. The directed line segment (arrow) depicting $\vec{B}_2(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$. Describe the relative orientation of $\vec{T}_2(t)$, $\vec{N}_2(t)$ and $\vec{B}_2(t)$. Describe the motion of $\vec{T}_2(t)$, $\vec{N}_2(t)$ and $\vec{B}_2(t)$ along the S.

Question 8 (5 Points)

Construct a local rectilinear co-ordinate system along the projection of the parametric curve $p_3(t)$ on S. Construct the following unit vectors

1. Unit tangent vector $\vec{T}_3(t)$.
2. Unit acceleration $\vec{N}_3(t)$.
3. Unit bi-normal $\vec{B}_3(t) = \vec{T}_3(t) \times \vec{N}_3(t)$.

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The directed line segment (arrow) depicting $\vec{T}_3(t)$.
2. The directed line segment (arrow) depicting $\vec{N}_3(t)$.
3. The directed line segment (arrow) depicting $\vec{B}_3(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$. Describe the relative orientation of $\vec{T}_3(t)$, $\vec{N}_3(t)$ and $\vec{B}_3(t)$. Describe the motion of $\vec{T}_3(t)$, $\vec{N}_3(t)$ and $\vec{B}_3(t)$ along the S.

Question 9 (5 Points)

Plot on a single set of axes the length of the tangent vectors $\vec{T}_1(t)$, $\vec{T}_2(t)$, $\vec{T}_3(t)$ on the interval $t \in [0, 1]$. Describe and explain the shape of each plot.