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Assignment 03
The Multi-Link Pendulum

APPM2007A
2021-10-15

Question 1

1.1

Potential energy under gravity, $V = mgh$. the height $h = r\cos\theta = -r\cos\theta$. The negative is because y is positive in the upward direction. Therefore:
$V = -mgr\cos\theta$

1.2

$x = r\sin(\theta)$
$y = r - r\cos(\theta)$
$V_x = \frac{dx}{dt} = r\dot{\theta}\cos\theta$
$V_y = \frac{dy}{dt} = r\dot{\theta}\sin(\theta)$

$$\begin{aligned}
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}m(V_x^2 + V_y^2) \\
&= \frac{1}{2}m(r^2\dot{\theta}^2\cos^2(\theta) + r^2\dot{\theta}^2\sin^2(\theta)) \\
&= \frac{1}{2}mr^2\dot{\theta}^2 \\
&= \frac{1}{2}m(r\dot{\theta})^2
\end{aligned}$$

1.3

$$\begin{aligned}
\mathcal{L} &= T - V \\
&= \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos(\theta)
\end{aligned}$$

1.4

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgr\sin(\theta)$$
$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$
$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) = mr^2\ddot{\theta}$$

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$
$$\rightarrow mr^2\ddot{\theta} + mgr\sin(\theta) = 0$$
$$\rightarrow r\ddot{\theta} + g\sin\theta = 0$$
$$\rightarrow \ddot{\theta} + \frac{g}{r}\sin(\theta) = 0$$
$$\rightarrow 0 = \ddot{\theta} + \frac{g}{r}\sin(\theta)$$

2.1

Question 2

2.1

$x_1 = r_1 \sin(\theta_1)$
$y_1 = -r_1 \cos(\theta_1)$
$\vec{\rho_1} = (x_1, y_1)$

$x_2 = r_1 \sin(\theta_1) + r_2 \sin(\theta_2)$
$y_2 = -r_1 \cos(\theta_1) - r_2 \cos(\theta_2)$
$\vec{\rho_1} = (x_2, y_2)$

find the velocities of each of the coordinates by taking the derivative with respect to time ($\frac{d}{dt}$):

$\dot{x_1} = \dot{\theta}_1 r_1 \cos(\theta_1)$
$\dot{y_1} = \dot{\theta}_1 r_1 \sin(\theta_1)$

$\dot{x_2} = \dot{\theta}_1 r_1 \cos(\theta_1) + \dot{\theta}_2 r_2 \cos(\theta_2)$
$\dot{y_2} = \dot{\theta}_1 r_1 \sin(\theta_1) + \dot{\theta}_2 r_2 \sin(\theta_2)$

$$\begin{aligned}
V &= m_1 g(y_1) + m_2 g(y_2) \\
&= m_1 g(-r_1 \cos(\theta_1)) + m_2 g(-r_1 \cos(\theta_1) - r_2 \cos(\theta_2)) \\
&= -(m_1 + m_2) g r_1 \cos(\theta_1) - m_2 g r_2 \cos(\theta_2) \\
&= -g[(m_1 + m_2) r_1 \cos(\theta_1) + m_2 r_2 \cos(\theta_2)]
\end{aligned}$$

2.2

$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$
Using the Velocities stated above:

$$\begin{aligned}
T &= \frac{1}{2} m_1 (\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2} m_2 (\dot{x_2}^2 + \dot{y_2}^2) \\
&= \frac{1}{2} m_1 (\dot{\theta}_1^2 r_1^2 \cos^2(\theta_1) + \dot{\theta}_1^2 r_1^2 \sin^2(\theta_1)) + \frac{1}{2} m_2 (\dot{x_2}^2 + \dot{y_2}^2) \\
&= \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 ([\dot{\theta}_1 r_1 \cos(\theta_1) + \dot{\theta}_2 r_2 \cos(\theta_2)]^2 + [\dot{\theta}_1 r_1 \sin(\theta_1) + \dot{\theta}_2 r_2 \sin(\theta_2)]^2)
\end{aligned}$$

$$\begin{aligned}
T = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (&\dot{\theta}_1^2 r_1^2 (\cos^2(\theta_1) + \sin^2(\theta_1)) \\
&+ 2\dot{\theta}_1 \dot{\theta}_2 r_1 r_2 (\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)) + \dot{\theta}_2^2 r_2^2 (\cos^2(\theta_2) + \sin^2 \theta_2))
\end{aligned}$$

$$T = \frac{1}{2} m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (\dot{\theta}_1^2 r_1^2 + \dot{\theta}_2^2 r_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 r_1 r_2 (\cos(\theta_1 - \theta_2))$$

Therefore:

$$T = \frac{1}{2} [(m_1 + m_2) r_1^2 \dot{\theta}_1^2 + m_2 (r_2^2 \dot{\theta}_2^2) + (2 m_2 \cos(\theta_2 - \theta_1)) r_1 r_2 \dot{\theta}_1 \dot{\theta}_2]$$

2.3

$$Y = \begin{bmatrix} r_1 \cos(\theta_1) \\ r_2 \cos(\theta_2) \end{bmatrix}, \; M = \begin{bmatrix} \mu_1 = m_1 + m_2 \\ \mu_2 = m_2 \end{bmatrix}, \; Y^T = \begin{bmatrix} r_1 \cos(\theta_1) & r_2 \cos(\theta_2) \end{bmatrix}$$

$$\begin{aligned}
V &= -g Y^T M \\
&= -g \begin{bmatrix} r_1 \cos(\theta_1) & r_2 \cos(\theta_2) \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \\
&= -g \begin{bmatrix} r_1 \cos(\theta_1)(\mu_1) + r_2 \cos(\theta_2)(\mu_2) \end{bmatrix} \\
&= -g \begin{bmatrix} r_1 \cos(\theta_1)(m_1 + m_2) + r_2 \cos(\theta_2)(m_2) \end{bmatrix} \\
&= -g[(m_1 + m_2) r_1 \cos(\theta_1) + m_2 r_2 \cos(\theta_2)]
\end{aligned}$$

2.4

$$X = \begin{bmatrix} r_1\theta_1 \\ r_2\theta_2 \end{bmatrix}, X^T = \begin{bmatrix} r_1\theta_1 & r_2\theta_2 \end{bmatrix}, \tilde{M} = \begin{bmatrix} \mu_1 = m_1 + m_2 & \mu_2 \cos(\theta_2 - \theta_1) \\ \mu_2 \cos(\theta_2 - \theta_1) & \mu_2 = m_2 \end{bmatrix}$$

$$\begin{aligned}
T &= \frac{1}{2}\dot{X}^T\tilde{M}\dot{X} \\
&= \frac{1}{2}\begin{bmatrix} r_1\dot\theta_1 & r_2\dot\theta_2 \end{bmatrix}\begin{bmatrix} \mu_1 & \mu_2\cos(\theta_2-\theta_1) \\ \mu_2\cos(\theta_2-\theta_1) & \mu_2 \end{bmatrix}\begin{bmatrix} r_1\dot\theta_1 \\ r_2\dot\theta_2 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} r_1\dot\theta_1\mu_1 + r_2\dot\theta_2\mu_2\cos(\theta_2-\theta_1) & r_1\dot\theta_1\mu_2\cos(\theta_2-\theta_1) + r_2\dot\theta_2\mu_2 \end{bmatrix}\begin{bmatrix} r_1\dot\theta_1 \\ r_2\dot\theta_2 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} r_1\dot\theta_1(r_1\dot\theta_1\mu_1 + r_2\dot\theta_2\mu_2\cos(\theta_2-\theta_1)) + r_2\dot\theta_2(r_1\dot\theta_1\mu_2\cos(\theta_2-\theta_1) + r_2\dot\theta_2\mu_2) \end{bmatrix} \\
&= \frac{1}{2}[(m_1+m_2)r_1^2\dot\theta_1^2 + m_2r_2^2\dot\theta_2^2 + 2m_2(\cos(\theta_2-\theta_1))r_1\dot\theta_1r_2\dot\theta_2]
\end{aligned}$$

2.5

$$\mathscr{L} = T - V$$

$$\mathscr{L} = \frac{1}{2}[(m_1+m_2)r_1^2\dot\theta_1^2 + m_2(r_2^2\dot\theta_2^2) + (2m_2\cos(\theta_2-\theta_1))r_1r_2\dot\theta_1\dot\theta_2] + g[(m_1+m_2)r_1\cos(\theta_1) - m_2r_2\cos(\theta_2)]$$

$$\mathscr{L} = \frac{1}{2}m_1r_1^2\dot\theta_1^2 + \frac{1}{2}m_2(\dot\theta_1^2r_1^2 + \dot\theta_2^2r_2^2 + 2\dot\theta_1\dot\theta_2r_1r_2(\cos(\theta_2-\theta_1))) + (m_1+m_2)gr_1\cos(\theta_1) + m_2gr_2\cos(\theta_2)$$

2.6

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot\theta_i}\right) - \frac{\partial\mathscr{L}}{\partial\theta_i}$$

$$\begin{aligned}
\frac{\partial\mathscr{L}}{\partial\dot\theta_1} &= m_1r_1^2\dot\theta_1 + m_2r_1^2\dot\theta_1 + m_2r_1r_2\cos(\theta_2-\theta_1)\dot\theta_2 \\
\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot\theta_1}\right) &= (m_1+m_2)r_1^2\ddot\theta_1 + m_2r_1r_2\ddot\theta_2\cos(\theta_2-\theta_1) - m_2r_1r_2\dot\theta_2\sin(\theta_2-\theta_1)(\dot\theta_2-\dot\theta_1) \\
\frac{\partial\mathscr{L}}{\partial\theta_1} &= -m_2r_1r_2\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1) - (m_1+m_2)gr_1\sin(\theta_1)
\end{aligned}$$

Substitute into $\frac{d}{dt}(\frac{\partial\mathscr{L}}{\partial\dot\theta_i}) - \frac{\partial\mathscr{L}}{\partial\theta_i}$:

$$\begin{aligned}
&= (m_1+m_2)r_1^2\ddot\theta_1 + m_2r_1r_2\ddot\theta_2\cos(\theta_2-\theta_1) - m_2r_1r_2\dot\theta_2\sin(\theta_2-\theta_1)(\dot\theta_2-\dot\theta_1) \\
&\qquad + m_2r_1r_2\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1) - (m_1+m_2)gr_1\sin(\theta_1)
\end{aligned}$$

$$= (m_1+m_2)r_1\ddot\theta_1 + m_2r_2\ddot\theta_2\cos(\theta_2-\theta_1) + m_2r_2\dot\theta_2^2\sin(\theta_2-\theta_1) + (m_1+m_2)g\sin(\theta_1) = 0$$

$$
\begin{aligned}
&= \ddot{\theta}_1 + \frac{m_2 r_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1)}{(m_1 + m_2) r_1} + \frac{m_2 r_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)}{(m_1 + m_2) r_1} + \frac{g \sin(\theta_1)}{r_1} = 0 \\
&= \ddot{\theta}_1 + \frac{m_2 r_2}{(m_1 + m_2) r_1} (\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)) + \frac{g \sin(\theta_1)}{r_1} = 0 \\
&\rightarrow \mu_1 = m_1 + m_2, \mu_2 = m_2 \\
&= \ddot{\theta}_1 + \frac{\mu_2}{\mu_1} \frac{r_2}{r_1} (\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)) + \frac{g \sin(\theta_1)}{r_1} = 0 \\
&= \ddot{\theta}_1 + \frac{g}{r_1} \sin(\theta_1) + \frac{\mu_2}{\mu_1} \frac{r_2}{r_1} (\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)) = 0
\end{aligned}
$$

$\frac{\partial \mathscr{L}}{\partial \dot{\theta}_2} = m_2 r_2^2 \dot{\theta}_2 + m_2 r_1 r_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1)$
$\frac{d}{dt}(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_2}) = m_2 r_2^2 \ddot{\theta}_2 + m_2 r_1 r_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) - m_2 r_1 r_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)$
$\frac{\partial \mathscr{L}}{\partial \theta_2} = m_2 r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - m_2 g r_2 \sin(\theta_2)$

Substitute into $\frac{d}{dt}(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_i}) - \frac{\partial \mathscr{L}}{\partial \theta_i}$:

$$
\begin{aligned}
= m_2 r_2^2 \ddot{\theta}_2 + m_2 r_1 r_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) - m_2 r_1 r_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1) - m_2 r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) \\
- m_2 g r_2 \sin(\theta_2)
\end{aligned}
$$

$$
\begin{aligned}
&= m_2 r_2 \ddot{\theta}_2 + m_2 r_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) - m_2 r_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + m_2 g \sin(\theta_2) = 0 \\
&= \ddot{\theta}_2 + \frac{r_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1)}{r_2} - \frac{r_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)}{r_2} + \frac{g \sin \theta_2}{r_2} = 0 \\
&= \ddot{\theta}_2 + \frac{g}{r_2} \sin(\theta_2) + \frac{r_1}{r_2} (\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)) = 0
\end{aligned}
$$

2.7

if $r_1 = r_2 = r$ and $m_1 = m_2 = m$ will result in the following equations of motion for a double pendulum:

$$
\begin{aligned}
0 &= \ddot{\theta}_1 + \frac{g}{r} \sin\theta_1 + m(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)) \\
0 &= \ddot{\theta}_2 + \frac{g}{r} \sin(\theta_2) + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_2 - \theta_1)
\end{aligned}
$$

as we can see in the equations above by allowing $r_1 = r_2 = r$ and $m_1 = m_2 = m$ we eliminate the multiple masses and rod lengths, however we are still left with two values of θ as we still have 2 different anchor points with two θ values. In addition we can see that in the first equation we have both values in the third term multiplied by m as even though both have the same mass the first pendulum still experiences the mass of the second and thus the mass does of the second still needs to be taken into account.

If we now compare with the above to the equation of motion of a simple pendulum, $0 = \ddot{\theta} + \frac{g}{r}\sin(\theta)$, we can see that the first two terms of equation 1 are almost identical to that of the simple pendulum except that in equation one we have θ_1 instead of θ this is due to the fact that we have two theta values in the double pendulum. We still need to take the second pendulum into account hence the remainder of the equation which does not appear for the simple pendulum as it only has 1 pendulum.

Similarly when comparing to equation 2 we again notice the similarities in the first 2 terms with the exception of it being θ_2 as we are evaluating at a second angle. The remainder of equation 2 which does not appear in the simple pendulum equation of motion is the taking into account of the first pendulum that the second pendulum is attached too.

3

3.1

$$X = \begin{bmatrix} r_1\theta_1 \\ r_2\theta_2 \\ \vdots \\ r_n\theta_n \end{bmatrix}, \quad Y = \begin{bmatrix} r_1\cos(\theta_1) \\ r_2\cos(\theta_2) \\ \vdots \\ r_n\cos(\theta_n) \end{bmatrix}, \quad M = \begin{bmatrix} \mu_1 = m_1 + m_2 + \cdots + m_n \\ \mu_2 = m_1 + m_2 + \cdots + m_{n-1} \\ \vdots \\ \mu_n = m_n \end{bmatrix},$$

$$\tilde{M} = \begin{bmatrix} \mu_1 & \mu_2\cos(\theta_2-\theta_1) & \mu_3\cos(\theta_3-\theta_1) & \dots & \mu_n\cos(\theta_n-\theta_1) \\ \mu_2\cos(\theta_2-\theta_1) & \mu_2 & \mu_3\cos(\theta_3-\theta_2) & \dots & \mu_n\cos(\theta_n-\theta_2) \\ \mu_3\cos(\theta_3-\theta_1) & \mu_3\cos(\theta_3-\theta_2) & \mu_3 & \dots & \mu_n\cos(\theta_n-\theta_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_n\cos(\theta_n-\theta_1) & \mu_n\cos(\theta_n-\theta_2) & \mu_n\cos(\theta_n-\theta_3) & \dots & \mu_n \end{bmatrix}$$

We know that $\mathscr{L} = T - V$ and therefore using the above matrices we can generalise the double pendulum to n-pendulums as follows:

$$V = -gY^{\mathsf{T}}M$$
$$T = \frac{1}{2}\dot{X}^{\mathsf{T}}\tilde{M}\ddot{X}$$
$$\mathscr{L} = T - V = \frac{1}{2}\dot{X}^{\mathsf{T}}\tilde{M}\ddot{X} + gY^{\mathsf{T}}M$$

Where X,Y,M,$\tilde{M}$ are the generalised version of the double pendulum matrices as shown above.

X is the matrix of the x-coordinates of each of the n masses in the pendulum. This general form takes values from 1 up to n as there are n masses.

Y is the matrix of the y-coordinates or heights of each of the n masses in the pendulum

M is a matrix that stores different sums of the n masses in n variables where the first entry (μ_1 is the sum of n the masses and the last entry (μ_n) is just the last mass m_n

$\tilde{M}$ is the mass matrix and is used to store. It has μ_1 to μ_n on its diagonal and a variation of $\mu_i \cos(\theta_i - \theta_x)$ in the rest of the matrix

7.1

3.2

We know that each element in the metric tensor is a dot product between 2 basis vectors of a particular coordinate space. It is used to calculate and scale different coordinate systems. They are also used to re-weight the sum of the vector components in a way that is appropriate and accurate for the coordinate system . It allows us to give vectors a length in different coordinate systems as well as measure angles of vectors. In addition it allows us to move between a vector space and a dual vector space (cotangent space). And lastly allows us to define distance in curved space. However in this case g is not being used for this, but is rather being used as the constant of gravity. We would expect this tensor to be a matrix of cotangent vectors defining gravity in this coordinate space, however what we find is just the constant of gravity which we then proceed to use

7.2

3.3

We see in the correct video of the chain that it moves smoothly with the momentum smoothly moving down to each link in pendulum(chain) until it finally reaches the bottom link. In the incorrect video we see how it initially moves smoothly and the momentum moves down the pendulum however when it reaches the bottom 2 links we see the consequences of an incorrect Lagrangian. The bottom two links start spinning on their own axis in a sped up and erratic manner which then disrupts the flow of the entire chain, this is due to the fact that angular dependence is not present in $\tilde{M}$, The angular dependence is what allows each mass to fall the correct amount of distance before pulling the next mass. This is because as each mass falls its anchor angle will become smaller while the pendulum that is anchored to its angle will increase until its pulled. If this angle is not taken into account the energy will move to quickly(change from kinetic to potential) through the chain resulting in a massive surge of energy at the bottom which causes the bottom two links to go into a an uncontrollable spiral

I would expect each mass to fall the maximum distance it can, after which it pulls the next mass. When the second mass is pulled and falls sufficiently far it will in turn pull the third mass and so on. Once the final mass has been pulled the chain will continue swinging in the direction of motion until its kinetic energy is 0 and it begins to fall back pulling the second to last mass which in turn pulls the mass above it and so on until the first mass is pulled and it has swung all the way back. This process will repeat until the system reaches equilibrium and comes to rest. The Lagrangian describes the change in energy over time (Kinetic-Potential=resultant energy) this change,which takes each angle into account, is what allows each mass to fall a sufficiently long distance before the next mass is pulled. Thus allowing a stable energy transfer from kinetic to potential until the kinetic energy reaches zero and the potential energy is maximum. At this point the masses will begin to gall in the opposite direction.

8.1

8.2