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Assignment 04
Smoothly Distributed Mass

APPM2007A
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Question 1

From our coordinates:

$$
\begin{aligned}
x &= \rho \sin(\theta) \cos(\phi) \\
y &= \rho \sin(\theta) \sin(\phi) \\
z &= \rho \cos(\theta)
\end{aligned}
$$

We have that the tangent vectors associated with these coordinates are:

$$
\begin{aligned}
\vec{e}_\rho &= \frac{\partial}{\partial \rho} \begin{pmatrix} \rho \sin(\theta) \cos(\phi) \\ \rho \sin(\theta) \sin(\phi) \\ \rho \cos(\theta) \end{pmatrix} = \begin{pmatrix} \sin(\theta) \cos(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\theta) \end{pmatrix} \\
\vec{e}_\theta &= \frac{\partial}{\partial \theta} \begin{pmatrix} \rho \sin(\theta) \cos(\phi) \\ \rho \sin(\theta) \sin(\phi) \\ \rho \cos(\theta) \end{pmatrix} = \begin{pmatrix} \rho \cos(\theta) \cos(\phi) \\ \rho \cos(\theta) \sin(\phi) \\ -\rho \sin(\theta) \end{pmatrix} \\
\vec{e}_\phi &= \frac{\partial}{\partial \phi} \begin{pmatrix} \rho \sin(\theta) \cos(\phi) \\ \rho \sin(\theta) \sin(\phi) \\ \rho \cos(\theta) \end{pmatrix} = \begin{pmatrix} -\rho \sin(\theta) \sin(\phi) \\ \rho \sin(\theta) \cos(\phi) \\ 0 \end{pmatrix}
\end{aligned}
$$

The area element expressed in terms of the first fundamental form with the assistance of Lagrange's identity:

$$
\begin{aligned}
dA &= \left| \vec{e}_\theta \times \vec{e}_\phi \right| d\theta d\phi \\
&= \sqrt{(\vec{e}_\theta \cdot \vec{e}_\theta)(\vec{e}_\phi \cdot \vec{e}_\phi) - (\vec{e}_\theta \cdot \vec{e}_\phi)^2}
\end{aligned}
$$

$$
\begin{aligned}
\vec{e}_\theta \cdot \vec{e}_\theta &= \rho^2 \cos^2(\theta)\cos^2(\phi) + \rho^2\cos^2(\theta)\sin^2(\phi) + \rho^2\sin^2(\theta) \\
&= \rho^2(\cos^2(\theta)\cos^2(\phi) + \cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)) \\
&= \rho^2((1-\sin^2(\theta))\cos^2(\phi) + (1-\sin^2(\theta))\sin^2(\phi) + \sin^2(\theta)) \\
&= \rho^2(\cos^2(\phi) - \cos^2(\phi)\sin^2(\theta) + \sin^2(\phi) - \sin^2(\phi)\sin^2(\theta) + \sin^2(\theta)) \\
&= \rho^2(\sin^2(\theta) - \cos^2(\phi)\sin^2(\theta) - \sin^2(\phi)\sin^2(\theta) + 1) \\
&= \rho^2(\sin^2(\theta)(1 - \cos^2(\phi) - \sin^2(\phi)) + 1) \\
&= \rho^2(\sin^2(\theta)(1-1) + 1) \\
&= \rho^2(1) \\
&= \rho^2
\end{aligned}
$$

$$
\begin{aligned}
\vec{e}_\phi \cdot \vec{e}_\phi &= \rho^2\sin^2(\theta)\sin^2(\phi) + \rho^2\sin^2(\theta)\cos^2(\phi) \\
&= \rho^2\sin^2(\theta)(\sin^2(\phi) + \cos^2(\phi)) \\
&= \rho^2\sin^2(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\vec{e}_\theta \cdot \vec{e}_\phi &= -\rho^2\sin(\theta)\sin(\phi)\cos(\theta)\cos(\phi) + \rho^2\sin(\theta)\sin(\phi)\cos(\theta)\cos(\phi) + 0 \\
&= 0
\end{aligned}
$$

Substituting these values into the formula for the area element:

$$
\begin{aligned}
\sqrt{(\rho^2)(\rho^2\sin^2(\theta)) - 0^2}\,d\theta d\phi &= \sqrt{\rho^4\sin^2(\theta)}\,d\theta d\phi \\
&= \rho^2\sin(\theta)\,d\theta d\phi
\end{aligned}
$$

Therefore:

$$
dA = \rho^2\sin(\theta)\,d\theta d\phi
$$

Question 2

$$
\begin{aligned}
dA &= \left|\vec{e}_\rho \times \vec{e}_\phi\right| d\rho d\phi \\
&= \sqrt{(\vec{e}_\rho \cdot \vec{e}_\rho)(\vec{e}_\phi \cdot \vec{e}_\phi) - (\vec{e}_\rho \cdot \vec{e}_\phi)^2}
\end{aligned}
$$

$$
\begin{aligned}
\vec{e}_\rho \cdot \vec{e}_\rho &= \sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + \cos^2(\theta) \\
&= \sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\sin^2(\phi) + (1 - \sin^2(\theta))
\end{aligned}
$$

$$
\begin{aligned}
&= \sin^2(\theta)(\cos^2(\phi) + \sin^2(\phi) - 1) + 1 \\
&= \sin^2(\theta)(1 - 1) + 1 \\
&= \sin^2(\theta)(0) + 1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\vec{e}_\phi \cdot \vec{e}_\phi &= \rho^2 \sin^2(\theta)\sin^2(\phi) + \rho^2 \sin^2(\theta)\cos^2(\phi) \\
&= \rho^2 \sin^2(\theta)(\sin^2(\phi) + \cos^2(\phi)) \\
&= \rho^2 \sin^2(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\vec{e}_\rho \cdot \vec{e}_\phi &= -\rho \sin^2(\theta)\sin(\phi)\cos(\phi) + \rho \sin^2(\theta)\sin(\phi)\cos(\phi) + 0 \\
&= 0
\end{aligned}
$$

$$
\sqrt{(\rho^2 \sin^2(\theta))(1) - 0^2} d\rho d\phi = \rho \sin(\theta) d\rho d\phi
$$

Therefore:

$$
dA = \rho \sin(\theta) d\rho d\phi
$$

Question 3

In a Euclidean coordinate space, the volume element = product of the Cartesian coordinates $dV = dxdydz$.The volume element changes by the Jacobian (determinant) of the coordinate change:

$$
dV = \left| \frac{\partial(x,y,z)}{\partial(\rho\theta\phi)} \right| d\rho d\theta d\phi
$$

$$
J = \begin{bmatrix} \sin(\theta)\cos(\phi) & \rho\cos(\theta)\cos(\phi) & -\rho\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\theta) & -\rho\sin(\theta) & 0 \end{bmatrix}
$$

Therefore the volume element is:

$$
\begin{aligned}
dV &= \left| J \right| d\rho d\theta d\phi \\
&= \left| \begin{bmatrix} \sin(\theta)\cos(\phi) & \rho\cos(\theta)\cos(\phi) & -\rho\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & \rho\cos(\theta)\sin(\phi) & \rho\sin(\theta)\cos(\phi) \\ \cos(\theta) & -\rho\sin(\theta) & 0 \end{bmatrix} \right| d\rho d\theta d\phi \\
&= \rho^2 \sin(\theta) d\rho d\theta d\phi
\end{aligned}
$$

Question 4

$$
\begin{aligned}
A &= \int_0^{\Theta}\int_0^{2\pi} \rho^2 \sin(\theta)\, d\theta\, d\phi \\
&= \rho^2 \int_0^{\Theta} \sin(\theta)\, d\theta \int_0^{2\pi} d\phi \\
&= \rho^2 (1-\cos(\Theta))(2\pi) \\
&= 2\pi\rho^2 (1-\cos(\Theta))
\end{aligned}
$$

Question 5

$$
\begin{aligned}
A &= \int_0^{R}\int_0^{2\pi} \rho \sin(\theta)\, d\rho\, d\phi \\
&= \sin(\theta) \int_0^{R} \rho\, d\rho \int_0^{2\pi} d\phi \\
&= (\sin(\theta))(\frac{R^2}{2})(2\pi) \\
&= \pi R^2 \sin(\theta)
\end{aligned}
$$

Set $\theta = \Theta$ (it's maximum value) as we need the maximum opening angle to get the entirety of the cone. Therefore:

$$A = \pi R^2 \sin(\Theta)$$

Question 6

$$
\begin{aligned}
V &= \int_0^{R}\int_0^{\Theta}\int_0^{2\pi} \rho^2 \sin(\theta)\, d\rho\, d\theta\, d\phi \\
&= \int_0^{R} \rho^2\, d\rho \int_0^{\Theta} \sin(\theta)\, d\theta \int_0^{2\pi} d\phi \\
&= (\frac{R^3}{3})(1-\cos(\Theta))(2\pi) \\
&= \frac{2\pi}{3} R^3 (1-\cos(\Theta))
\end{aligned}
$$

Question 7

$$
\begin{aligned}
\mu &= \frac{Mass}{Volume} \\
&= \frac{M}{\frac{4}{3}\pi R^3} \\
&= \frac{3M}{4\pi R^3}
\end{aligned}
$$

Question 8

Since the sphere is a uniform rigid body and is filled with a material of mass M which is smoothly and evenly distributed throughout the interior of the sphere,the sphere has constant mass density μ, and thus we can then take each integrals μ value to be the same which is constant and can therefore take it out of the integral.

Question 9

$$
\begin{aligned}
I_{zz} &= \mu \int_V (x^2 + y^2) dv \\
&= \mu \int_V (\rho^2 \sin^2(\theta)\cos^2(\phi) + \rho^2 \sin^2(\theta)\sin^2(\phi)) dv \\
&= \mu \int_V \rho^2 \sin^2(\theta)(\cos^2(\phi) + \sin^2(\phi)) dv \\
&= \mu \int_V \rho^2 \sin^2 dv \\
&= \mu \int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 \sin^2(\theta)(\rho^2 \sin(\theta)) d\rho d\theta d\phi \\
&= \mu \int_0^R \int_0^\pi \int_0^{2\pi} \rho^4 \sin^3(\theta) d\rho d\theta d\phi \\
&= \mu \int_0^R \rho^4 d\rho \int_0^\pi \sin^3(\theta) d\theta \int_0^{2\pi} d\phi \\
&= (\frac{3M}{4\pi R^3})(\frac{R^5}{5})(\frac{4}{3})(2\pi) \\
&= \frac{2}{5}R^2 M
\end{aligned}
$$

Question 10

A solid sphere has rotational symmetry and so it doesn't matter what axes you are rotating around it is going to be symmetrical. Therefore all moments of inertia are the same as they rotate around x y and z axes therefore they equal due to the symmetry around rotating axes.

Question 11

$$
\begin{aligned}
I_{xy} &= -\mu \int_V (xy)\,dv \\
&= -\mu \int_V \rho^2 \sin^2(\theta)(\cos(\phi)\sin(\phi))\,dv \\
&= -\mu \int_0^R \int_0^\pi \int_0^{2\pi} \rho^2 \sin^2(\theta)(\cos(\phi)\sin(\phi))(\rho^2 \sin(\theta))\,d\rho\, d\theta\, d\phi \\
&= -\mu \int_0^R \int_0^\pi \int_0^{2\pi} \rho^4 \sin^3(\theta)\cos(\phi)\sin(\phi)\,d\rho\, d\theta\, d\phi \\
&= -\mu \int_0^R \rho^4\, d\rho \int_0^\pi \sin^3(\theta)\, d\theta \int_0^{2\pi} \cos(\phi)\sin(\phi)\, d\phi \\
&= \left(\frac{3M}{4\pi R^3}\right)\left(\frac{R^5}{5}\right)\left(\frac{4}{3}\right)(0) \\
&= 0
\end{aligned}
$$

Question 12

The off diagonal elements of the inertial tensor are products of inertia which measure the imbalance of mass in the body, in this case a sphere, and since we were told that the mass of this sphere is smoothly and evenly distributed throughout the sphere it follows that there is no imbalances in the sphere and therefore all the off diagonal elements are the same and equal 0.