Lab 2: Eigenvalue and eigenvectors, Newton method and applications

Due: Friday 11 June 2021, 20:00

Instructions:

i) There are 7 questions in this assignment.

i) All your solutions must be in ONE file.

ii) The file name should be studentnumber.py, for example 87654321.py

Question 1. The Hilbert matrix H is an $n \times n$ matrix whose elements are given by

$$H_{ij} = \frac{1}{i+j-1}$$

Using numpy, write a function that creates a Hilbert matrix. Your function should take the form

```
def Hilbert(n):
    .......................
    return H
```

Test the function for n of your choice. [5]

Question 2. Write some code which calls the function you created in Question 1 to calculate the sum of

i) diagonal elements (red elements).

ii) upper diagonal elements (blue elements).

iii) lower diagonal elements (green elements),

for a Hilbert matrix H where $n = 1000$ and

$$H = \begin{pmatrix} {\color{red}*} & {\color{blue}*} & & \\ {\color{green}*} & {\color{red}\ddots} & {\color{blue}\ddots} & \\ & {\color{green}\ddots} & {\color{red}\ddots} & {\color{blue}*} \\ & & {\color{green}*} & {\color{red}*} \end{pmatrix}.$$

[2x3]

Question 3. Write a function to perform Power method to estimate the dominant eigenvalue for an $n \times n$ matrix A. Your function has to take the form

```
def Power_method(A, x0, n):
    ......................
    return eigenvalue, eigenvector
```

ie., has arguments A, the guess x_0 for x, number of iterations n and returns the eigenvalue and eigenvector. Test it using your choice of A, x_0 and n. (Create A and x_0 using numpy). [5]

Question 4. Write some code to approximate the dominant eigenvalue for a 123×123 Hilbert matrix using your answer to Question 3 where number of iterations is 10 and $x_0 = [1, 1, \cdots, 1]^T$. [5]

Question 5. Write a function to solve a system of nonlinear equations using Newton method. Your function has to take the form

```
def Newton(F, J, x0, tol):
    ......................
    return root_vector
```

using the stopping criteria

$$\frac{||\mathbf{x}_{n+1} - \mathbf{x}_n||_2}{||\mathbf{x}_n||_2} < tol$$

where $\mathbf{F}$ is system such that $\mathbf{F}(\mathbf{x}) = \mathbf{0}$, $\mathbf{x_0}$ is the guess and J is the Jacobian matrix.

Question 6. In this question you will solve a nonlinear boundary-value problem of the form

$$y'' = f(x, y, y')$$
$$y(a) = \alpha, \quad y(b) = \beta$$

using Newton method to find $y(x)$. (*You do not need any knowledge about boundary value problems or numerical methods for Ordinary differential equations. All you need is being able to use Newton method. This is a guided question. You will be led towards the solution*)

The equation above can be approximated by using finite differences as follows:

$$y_0 = \alpha, \quad y_6 = \beta$$
$$-\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right) = 0$$

where $x_i = a + ih$, $h = 1/3$. Your task is to find y_1, y_2, y_3, y_4, y_5 since y_0 and y_6 are given above. These can be found using Newton method by substituting $i = 1, \cdots, 5$ in the above approximation. For example if $i = 1$ we have equation

$$-\frac{y_2 - 2y_1 + y_0}{h^2} + f\left(x_1, y_1, \frac{y_2 - y_0}{2h}\right) = 0$$

Form the rest of the 4 equations by substituting $i = 2, 3, 4, 5$. Use your solution to Question 5 to write some code to solve the 5 equations for y_1, y_2, y_3, y_4, y_5 using Newton method for the problem:

$$y'' = \frac{1}{8}(32 + 2x^3 - yy')$$
$$y(1) = 17, \quad y(3) = 43/3.$$

Use the guess $yy0 = [17, 17, 17, 17, 17]^T$ and the stopping criteria given in Question 5. [5]

Question 7. Produce a graph showing a line plot of y_i against x_i for $i = 0, \cdots, 6$. Your plot should have labeled axes and a title. (*Please google to find out how to plot in python*) [5]