CES and the Elasticity of Substitution.

Thomas Jørgensen

May 17, 2023

Motivated by the functional forms in Blundell, Pistaferri and Saporta-Eksten (2018), let child production be governed by the CES function

$$Q = -\frac{1}{1-\rho}\left[\alpha_1 T_1^{1-1/\phi_1} + \alpha_2 T_2^{1-1/\phi_2}\right]^{1-\rho}$$

where $\rho < 1$, $\phi_1 \in (0,1)$ (and potentially $\alpha_2 = 1-\alpha_1$). The authors state that if $\rho < 0$ the two inputs are substitutes and if $\rho > 0$ the two inputs are complements. We will look into this below through a simple static model.

The budget constraint is

$$C \leq (1-T_1)W_1 + (1-T_2)W_2$$

where C is consumption and hours worked is normalized to one. The budget constraint can be written as

$$Z \leq T_1 W_1 + T_2 W_2$$

where $Z = W_1 + W_2 - C$ is potential income, which we will condition on and thus ignore consumption maximization here.

The Lagrangian is (in which λ is the shadow price on potential income)

$$\mathcal{L} = -\frac{1}{1-\rho}\left[\alpha_1 T_1^{1-1/\phi_1} + \alpha_2 T_1^{1-1/\phi_2}\right]^{1-\rho} + \lambda[Z - T_1 W_1 - T_2 W_2]$$

with FOCS for $j \in \{1,2\}$

$$\frac{\partial \mathcal{L}}{\partial T_j} = -\left[\alpha_1 T_1^{1-1/\phi_1} + \alpha_2 T_1^{1-1/\phi_2}\right]^{-\rho}(1-1/\phi_j)\alpha_j T_j^{-1/\phi_j} - \lambda W_j = 0.$$

Combining the FOCS we get

$$\frac{W_1}{W_2} = \frac{\left[\alpha_1 T_1^{1-1/\phi_1} + \alpha_2 T_1^{1-1/\phi_2}\right]^{-\rho}(1-1/\phi_1)\alpha_1 T_1^{-1/\phi_1}}{\left[\alpha_1 T_1^{1-1/\phi_1} + \alpha_2 T_1^{1-1/\phi_2}\right]^{-\rho}(1-1/\phi_2)\alpha_2 T_2^{-1/\phi_2}}$$

which gives

$$T_1 = \left[\frac{(1-1/\phi_2)\alpha_2}{(1-1/\phi_1)\alpha_1}\right]^{-\phi_1} \left[\frac{W_1}{W_2}\right]^{-\phi_1} T_2^{\phi_1/\phi_2}. \tag{1}$$

Already here, we see that the relative inputs does not depend on ρ. Let the **elasticity of substitution** be the percentage change in the relative inputs from a percentage change in the (inverse) relative prices,

$$\varepsilon = \frac{\partial T_1/T_2}{\partial W_2/W_1}\frac{W_2/W_1}{T_1/T_2}$$

where eq. (1) gives us

$$\begin{aligned}\frac{\partial T_1/T_2}{\partial W_2/W_1} &= \frac{\partial}{\partial W_2/W_1}\left[\frac{(1-1/\phi_2)\alpha_2}{(1-1/\phi_1)\alpha_1}\right]^{-\phi_1} \left[\frac{W_2}{W_1}\right]^{\phi_1} T_2^{\phi_1/\phi_2-1} \\ &= \phi_1\left[\frac{(1-1/\phi_2)\alpha_2}{(1-1/\phi_1)\alpha_1}\right]^{-\phi_1} \left[\frac{W_2}{W_1}\right]^{\phi_1-1} T_2^{\phi_1/\phi_2-1}\end{aligned}$$

and we get

$$\varepsilon = \phi_1\left[\frac{(1-1/\phi_2)\alpha_2}{(1-1/\phi_1)\alpha_1}\right]^{-\phi_1} \left[\frac{W_2}{W_1}\right]^{\phi_1} T_2^{\phi_1/\phi_2-1}\frac{T_2}{T_1}$$

where inserting

$$\frac{T_2}{T_1} = \left[\frac{(1-1/\phi_1)\alpha_1}{(1-1/\phi_2)\alpha_2}\right]^{-\phi_2} \left[\frac{W_2}{W_1}\right]^{-\phi_2} T_1^{\phi_2/\phi_1-1}$$

finally gives

$$\varepsilon = \phi_1\left[\frac{(1-1/\phi_1)\alpha_1}{(1-1/\phi_2)\alpha_2}\frac{W_2}{W_1}\right]^{\phi_1-\phi_2} T_2^{\phi_1/\phi_2-1}T_1^{\phi_2/\phi_1-1} \tag{2}$$

which, if $\phi_1 = \phi_2 = \phi$, simplifies to

$$\varepsilon = \phi.$$

This is not ρ... It could be interesting to insert the estimated parameters from Blundell, Pistaferri and Saporta-Eksten (2018) in e.q. (2) for the parental time and the leisure time to see if it aligns with the story that leisure time is complements (they estimate $\rho_L > 0$) and parental time is substitutes ($\rho_T < 0$). I think it will because this is also what the simulations suggest.

References

Blundell, R., L. Pistaferri and I. Saporta-Eksten (2018): "Children, Time Allocation, and Consumption Insurance," *Journal of Political Economy*, 126(S1), S73–S115.